The goal of this document is to estimate the motion (rotation and translation), given 2 images and at least one image depth for the first image.

Disclaimer: There are no new concepts in this document, this is more an attempt to carry one formalism all the way to the implementation in order to understand as much as possible. See citations at the end.

We're looking to find the function **w** that warps the pixels from one image to the other (knowing some info about the depth).

Let **w** be a warp parametrized by the "twist" ξ vector in R6 representing the 6 degrees of freedom of a rigid body in the Lie group SE(3), defined as :

$$w(x,\xi) \;=\; KP(\xi)K^{-1}(x,z(x))^T \quad (1)$$

(x, z(x))' represents a homogenous 3d point, with 4 coordinates. z(x) being the function that maps an image point x=(u,v) to a depth (using a depth map, such as one given by a Kinect).

K is a projection matrix, sometimes called "intrinsics" matrix which maps a point in 3d space onto the camera plane. (K is dependent on the focal length and the camera center in the camera pinhole model).

P is a 4x4 matrix composed of a Rotation and a Translation, modeling the motion described by a camera between two consecutive frames.
The function P(ξ) maps given twist coordinates to a Rotation and Translation.
It is convenient to use twist as the Lie algebra associated with them, allows us to linearize pose changes in the case of infinitesimal variations.

Let's define an Energy that minimizes the warp by using the brightness constancy constraint.
i-e: **one pixel in the first image, warped to the same pixel in the second pixel should have roughly the same brightness**.
This is in essence the Lucas-Kanade / Horn-Schunk premises.

$$E(\xi) \;=\; \int_{\Omega} |I_2(w(x,\xi)) - I_1(x)|^2 \, dx \quad (2)$$

The problem is that this energy is non-linear and cannot be minimized easily.

We're now going to linearize this energy for small pose changes.

By doing a first order Taylor expansion we get :

$$I_2(w(x,\xi)) \approx I_2(x)+(w(x,\xi)-x).\nabla I_2 \quad (3)$$

We can generally assume that for a small increment of $\Delta\xi$, this general first order taylor expansion :

(Note: probably need more of an explanation on SE(3) algebra ? Need to define + and - operations in Lie algebra).

$$w(x,\xi\boxplus\Delta\xi) \approx w(x,\xi)+\frac{dw}{d\xi}.((\xi\boxplus\Delta\xi)\boxminus\xi) = w(x,\xi)+\frac{dw}{d\xi}.\Delta\xi \quad (4)$$

Therefore we can write:

$$w(x,0\boxplus\Delta\xi) \approx w(x,0)+\frac{dw}{d\xi}.\Delta\xi = x+\frac{dw}{d\xi}.\Delta\xi \quad (5)$$

By plugging (3) and (5) Into (2) we get :

$$E(\xi) = \int_\Omega \left| I_2(x)-I_1(x)+\nabla I_2.\frac{dw}{d\xi}.\Delta\xi \right|^2 dx \quad (6)$$

Unfortunately the approximation in (5) is wrong, as the expression in (4) in only linearizable for infinitesimal variations of $\Delta\xi$

We have to solve iteratively using (4), starting with $\xi_1 = 0$, and I2(w(x, ξ_1)) = I2(x).

$$E(\xi_k) = \int_\Omega \left| I_2(w(x,\xi_k)-I_1(x)+\nabla I_2(x,\xi_k).\frac{dw}{d\xi}.\Delta\xi_{k+1} \right|^2 dx$$

(We'll explicit later what is dw/dξ).

Let's simplify the notation with :

$$I_2^*=I_2(w(x,\xi_k)); \quad \nabla I_2^*=\nabla I_2(w(x,\xi_k)); \quad I_1=I_1(x)$$

the * denoting the warped image, or image gradient at each iteration k.

We want to calculate $\Delta\xi$ here, at each iteration.

The transformation $\Delta\xi$ should be constant for all the image, therefore the optimality constraint dictates that $dE(\xi)/\Delta\xi$ must be equal to 0.

$$\frac{d(E_k)}{d\Delta\xi_{k+1}} = 2\int_{\Omega} \left|\nabla I_2^*. \frac{dw}{d\xi}\right|^T \left|I_2^* - I_1 + \nabla I_2^*. \frac{dw}{d\xi}\Delta\xi_{k+1}\right| dx$$

(7)

Since (7) must be equal to 0, we have :

$$\left[\int_{\Omega} \left|\nabla I_2^*. \frac{dw}{d\xi}\right|^T \left|\nabla I_2^*. \frac{dw}{d\xi}\right| dx\right].\Delta\xi_{k+1} = -\int_{\Omega} \left|\nabla I_2^*. \frac{dw}{d\xi}\right|^T |I_2^* - I_1|\, dx$$

(8)

Which can be expressed using Jacobian matrixes :

$$\Delta\xi_{k+1} = -((J_i(\xi_k)J_w(\xi_k))^T(J_i(\xi_k)J_w(\xi_k)))^{-1}.(J_i(\xi_k)J_w(\xi_k))^T.|I_2^* - I_1|$$

Total derivative with respect to time

Staring back from (3) we can also note that in the discrete setting :

$$w(x,\xi)-x \;=\; w(x,\xi)-w(x,0) \;=\; \frac{dw}{dt}$$

(w(x) is the transformation of x at time t+1)

So we can write :

$$E(\xi) \;=\; \int_{\Omega} \left| I_2(x)-I_1(x)+\nabla I_2(x).\,\frac{dw}{dt} \right|^2 dx$$

We can simplify further by noting that in the discrete setting
I2(x) - I1(x) = dI/dt, therefore :

$$E(\xi) \;=\; \int_{\Omega} \left| \frac{\partial I}{\partial t}+\nabla I_2(x).\,\frac{dw}{dt} \right|^2 dx$$

We get back the “total derivation” with respect to time.

Calculating dw/dξ

Now let's rewrite (1) using functions, instead of matrix to make it clearer when applying a chain rule.

P(ξ) is the function that maps a 6-dof vector ξ from its algebra to its manifold, a 4x4 matrix composed of a Rotation and a Translation

g(P(ξ), X) = P(ξ)X = RX + t

Writing everything with functions we get :

$$w(x,\xi;s) \;=\; k(g(p(\xi),s(x))$$

with:

$$k(X)=KX=\begin{pmatrix} fx & 0 & cx \\ 0 & fy & cy \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$g(P,X) \;=\; PX$$

$$p(\xi)=e^{\xi}$$

$$s(x) \;=\; (x,\;\; depth(x))^T \quad x \in \Omega$$

Applying the chain rule we now get :

TODO, expression of the Jacobian Jw(ξ)

Citations :

Real-Time Visual Odometry from Dense RGB-D Images
Robust Odometry Estimation for RGB-D Cameras

Lucas-Kanade 20 Years On: A Unifying Framework

TechRep: A tutorial on SE(3) transformation parameterizations and on-manifold optimization

WIP latex formulas :

Warp func:
w(x,\xi)\quad=\quad KP(\xi)K^{-1}(x, z(x))^T

Taylor 1:
I_2(w(x,\xi))\quad\approx\quad I_2(x)+(w(x,\xi)-x).\nabla I_2

Taylor 2:
w(x,\xi \boxplus \Delta \xi)\quad \approx\quad w(x,\xi)+\frac { dw }{ d\xi }.((\xi \boxplus\Delta\xi\) \boxminus \xi)\quad =\quad w(x,\xi)+\frac { dw }{ d\xi } .\Delta \xi

with identity:
w(x,0\boxplus \Delta \xi)\quad \approx \quad w(x,0)+\left \frac { dw }{ d\xi } \right| _{ \xi =0 }.\Delta \xi \quad =\quad x+\frac{dw}{d\xi}.\Delta\xi

discrete dw/dt :
w(x,\xi)-x\quad =\quad w(x,\xi) -w(x,0)\quad =\quad \frac { dw }{ dt }

Energy :
E(\xi)\quad =\quad \int _{ \Omega }{ \left| I_{ 2 }(w(x,\xi))-I_{ 1 }(x)\right| ^{ 2 } } dx

(6)
E(\xi)\quad =\quad \int _{ \Omega }{ \left| I_{ 2 }(x)-I_{ 1 }(x)+\nabla I_{ 2 }.\frac { dw }{ d\xi } .\Delta\xi \right| ^{ 2 } } dx

(7)
\frac{d(E_k)}{d\Delta\xi_k_+_1}\quad =\quad 2\int _{ \Omega } \left| \nabla I_{ 2 }^{ * }.\frac { dw }{ d\xi } \right| ^{ T }\left| I_{ 2 }^{ * }-I_{ 1 }+\nabla I_{ 2 }^{ * }.\frac { dw }{ d\xi } \Delta \xi _{ k }_{ + }_{ 1 } \right| dx

(8)
\left[\int _{ \Omega } \left| \nabla I_{ 2 }.\frac { dw }{ d\xi } \right| ^{ T }\left| \nabla I_{ 2 }.\frac { dw }{ d\xi } \right| dx \right] .\Delta \xi \quad =\quad -\int _{ \Omega } \left| \nabla I_{ 2 }.\frac { dw }{ d\xi } \right| ^{ T }\left| I_{ 2 }(x)-I_{ 1 }(x) \right| dx

(9)
\Delta \xi _{ k }_{ + }_{ 1 }=-((J_{ i }(\xi _{ k })J_{ w }(\xi _{ k }))^{ T }(J_{ i }(\xi _{ k })J_{ w }(\xi _{ k })))^{ - }^{ 1 }.(J_{ i }(\xi _{ k })J_{ w }(\xi _{ k }))^{ T }.|I_{ 2 }^{ * }-I_{ 1 }|

expansion of dw/dxi:
w(x,\xi ;s)\quad =\quad k(g(p(\xi), s(x))

$$k(X)=KX=\begin{pmatrix} fx & 0 & cx \\ 0 & fy & cy \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}x\\y\\z \end{pmatrix}$$