Master Thesis

Numerical investigation of quantum channel capacities using symmetric states

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1. Introduction

1.1. Quantum channels

A major goal of quantum information theory is to understand what happens to a quantum mechanical system that interacts with some environment and how this interaction affects our ability to save and transmit quantum information. A general framework for this scenario is the *quantum channel*. A quantum channel, also called a *quantum operation*, is a map $\mathcal{N}$ that describes how a mixed state ρ evolves to $\rho' = \mathcal{N}(\rho)$. We will only discuss the most elementary properties of quantum channels here; a great introduction to this topic can be found in chapter 8.1 of [1], on which we will base this section.

There are three equivalent ways to characterize the class of maps that are called quantum operations. The first way is motivated by the scenario mentioned above, interaction with an environment. Thus, we assume that the environment is in some state $|0\rangle$ and our system is in state ρ and some unitary time evolution acts on the combined system. After the interaction, we trace out the environment to find the state of the system

$$\mathcal{N}(\rho) = \mathrm{Tr}_{\mathrm{Env}}\, U(\rho \otimes |0\rangle\langle 0|)U^\dagger \,. \tag{1.1}$$

When we extend this definition by allowing the environment to be in a mixed state, we do not change the resulting class of maps and it is also sufficient to consider environments with dimension d^2 if the dimension of the system is d. If the basis of the environment is

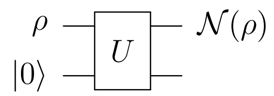


Figure 1.1.: System and environment evolution

given by vectors $\{|k\rangle\}$, we have

$$\mathcal{N}(\rho) = \sum\nolimits_k \langle k|U(\rho \otimes |0\rangle\langle 0|)U^\dagger|k\rangle = \sum\nolimits_k A_k \rho A_k^\dagger \tag{1.2}$$

with the operators $A_k \equiv \langle k|U|0\rangle$ that are called the *Kraus operators*. Because our map is trace-preserving, it follows that $\sum_k A_k^\dagger A_k = \mathbb{1}$. The situation where measurements are performed on the environment after the interaction is described similarly with $\sum_k A_k^\dagger A_k \leq \mathbb{1}$. This description gives the second way to characterize quantum channels and is called the *operator-sum representation*. The third description is that a quantum channels is a linear map that is completely positive and trace-preserving. The equivalence of this criterion to the others is called the Choi-Kraus theorem, but we will not use this description in this work.

Bit flip and depolarizing channel

We will now describe two important quantum channels acting on qubits that we will later use in our examples. The *bit flip channel* (see [1], section 8.3.3) flips the states $|0\rangle \leftrightarrow |1\rangle$ with probability p. Clearly, the Kraus operators for this channel are then given by

$$A_0 = \sqrt{1-p} \cdot \mathbb{1} = \sqrt{1-p} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \sqrt{p} \cdot X = \sqrt{p} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

The *depolarizing channel* (see [1], section 8.3.4) changes the qubit state to the completely mixed state $\frac{\mathbb{1}}{2}$ with probability p, that is

$$\mathcal{N}(\rho) = \frac{p}{2} \cdot \mathbb{1} + (1-p)\rho\,. \tag{1.4}$$

The Kraus operators of the depolarizing channel can be shown (see [1], section 8.3.4) to be

$$E_0 = \sqrt{1 - \frac{3p}{4}}\mathbb{1}\,, \quad E_1 = \frac{\sqrt{p}}{2}X\,, \quad E_2 = \frac{\sqrt{p}}{2}Y\,, \quad E_3 = \frac{\sqrt{p}}{2}Z\,. \tag{1.5}$$

1.2. Quantum capacity and coherent information

Analogously to the Shannon entropy for classical systems, the *von Neumann entropy* of a quantum state ρ is defined as

$$S(\rho) = \mathrm{Tr}\left[-\rho \log_2(\rho)\right]. \tag{1.6}$$

Schumacher's quantum noiseless channel theorem gives a justification for this definition: For large n, the state $\rho^{\otimes n}$ can be reliably encoded in a state of a 2^{nR}-dimensional system if $R > S(\rho)$, but not if $R < S(\rho)$ (see [1], section 12.2.2).

In this work, we are interested in the quantum capacity of a quantum channel. To compute the quantum capacity, we will need a quantity called the *coherent information* of a bipartite state ρ_{AB} (see [2], section 11.5):

$$\mathcal{I}\left(\rho_{AB}\right) = S(\rho_B) - S(\rho_{AB})\,, \tag{1.7}$$

where $\rho_B = \mathrm{tr}_A(\rho_{AB})$.

Next, consider a pure state $|\phi_{AA'}\rangle$ of two systems A and A'. We apply our channel $\mathcal{N} \colon \mathcal{H}_{A'} \to \mathcal{H}_B$ to this state to find the state $\rho_{AB} = (\mathcal{N} \otimes \mathbb{1})(|\phi_{AA'}\rangle\langle\phi_{AA'}|)$. The coherent information of the quantum channel $\mathcal{N}$ is then given by (see [2], section 13.5):

$$\mathcal{I}(\mathcal{N}) = \max_{\phi_{AA'}} \mathcal{I}(\rho_{AB}) \,. \tag{1.8}$$

Now, the *quantum capacity theorem* (see [2], section 24.3) tells us that the quantum capacity of a quantum channel $\mathcal{C}_Q$ is given by

$$\mathcal{C}_Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \mathcal{I}\left(\mathcal{N}^{\otimes n}\right) . \tag{1.9}$$

The quantity on the right-hand side of this equation is also called the *regularized* coherent information.

We can give an intuitive explanation why the regularized coherent information yields the quantum capacity: The entropy $S(\rho_B)$ measures the amount of information about the initial state and the amount of noise that is the result of the interaction with the environment. This amount of noise is measured by the entanglement of system AB with its environment. As the whole system of AB and environment is in a pure state, this entanglement is measured by $S(\rho_{AB})$. Thus, we need to subtract $S(\rho_{AB})$ from $S(\rho_B)$ to find a measure of the transmitted amount of quantum information.

The goal of this work is to find a method to compute lower bounds of the quantum capacity of arbitrary channels. Our approach is to simplify to the maximization in (1.8) by considering only symmetric states (see next section) and thus calculating lower bounds for $\mathcal{I}(\mathcal{N}^{\otimes n})$ for n as large as possible. A lower bound for $\mathcal{I}(\mathcal{N}^{\otimes n})$ is also a lower bound for $\mathcal{C}_Q(\mathcal{N})$, as the state $|\phi_{AA'}\rangle$ can be repeated, i.e. the state $|\phi_{AA'}\rangle^{\otimes m}$ is a possible solution for the maximization for larger n.

1.3. Symmetric states

1.3.1. Pure states

We now consider pure states of n qudits, namely $|\psi\rangle \in \mathcal{H}^{\otimes n} = \left(\mathbb{C}^d\right)^{\otimes n}$. The symmetric group $G = S_n$ acts on this space by permuting the qudits, i.e.

$$\sigma(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle) = \left|\psi_{\sigma^{-1}(1)}\right\rangle \otimes \left|\psi_{\sigma^{-1}(2)}\right\rangle \otimes \ldots \otimes \left|\psi_{\sigma^{-1}(n)}\right\rangle \tag{1.10}$$

for $\sigma \in G$.[1] By linear extension, σ becomes a unitary operator on $\mathcal{H}^{\otimes n}$, but we will not use a different notation for σ as an element of the symmetric group and σ as an operator.

[1]The inverse of σ appears on the right side because we map qudit i to qudit $\sigma(i)$, and thus qudit $\sigma^{-1}(i)$ will be mapped to qudit i.

We call a pure state *symmetric* iff it is invariant under this action, i.e. if for all $\sigma \in G$, we have $\sigma |\psi\rangle = |\psi\rangle$. These states form a subspace, designated $\mathrm{Sym}^n(\mathcal{H})$, which we will now investigate.

First, we find a projection operator onto the space of symmetric states:

Theorem 1.1. *The operator* $P = \frac{1}{n!} \sum_{\sigma \in G} \sigma$ *is an orthogonal projection onto* $\mathrm{Sym}^n(\mathcal{H})$.

Proof. The operator is an orthogonal projection because

$$P^2 = \frac{1}{n!^2} \sum_{\sigma,\sigma' \in G} \sigma\sigma' = \frac{n!}{n!^2} \sum_{\sigma \in G} \sigma = P\,,$$
$$P^\dagger = \frac{1}{n!} \sum_{\sigma \in G} \sigma^{-1} = P\,.$$

Symmetric states are in the image of P as they are unaffected by permutations and conversely we have

$$\sigma' P |x\rangle = \sum_{\sigma \in G} \sigma'\sigma |x\rangle = \sum_{\sigma \in G} \sigma |x\rangle = P |x\rangle$$

and so $P|x\rangle$ is symmetric. $\square$

We will now construct a basis of $\mathrm{Sym}^n(\mathcal{H})$ using this projection. If $|y\rangle = \sigma|x\rangle$ for some permutation σ, applying the projection will yield the same state, i.e. $P|x\rangle = P|y\rangle$. Thus, the order of the entries x_i is irrelevant when we apply the projection to computational basis states $|x\rangle = |x_1, \ldots, x_n\rangle$. The equivalence class of these states can then be specified by *occupation numbers* $\mathbf{n} = (n_1, \ldots, n_d)$ that describe how often the entries take on values in $\{1, \ldots, d\}$. We now define standard states

$$|x(\mathbf{n})\rangle = |\underbrace{1\ldots1}_{n_1}\ldots\underbrace{d\ldots d}_{n_d}\rangle \tag{1.11}$$

for all possible occupation numbers. An orthogonal basis for the symmetric subspace is then given by applying the projection P to these states:

$$|\mathbf{n}\rangle = P|x(\mathbf{n})\rangle = \frac{1}{n!} \sum_{\sigma \in G} \sigma |x(\mathbf{n})\rangle\ . \tag{1.12}$$

As not all of the summands on the right side of this equation are different, it is useful to consider the stabilizer of the standard states

$$G_{\mathbf{n}} = \left\{\sigma \in G \;\middle|\; \sigma|x(\mathbf{n})\rangle = |x(\mathbf{n})\rangle\right\} \cong S_{n_1} \times \ldots \times S_{n_d} \tag{1.13}$$

to understand why summands appear repeatedly. The isomorphism in (1.13) holds because we can arbitrarily permute the first n_1 qudits, the following n_2 qudits and so on.

Thus, the states of (1.12) are up to a constant given by a sum over representatives of the left cosets of the stabilizer and we redefine:[2]

$$|\mathbf{n}\rangle = \sum_{\sigma \in G/G_{\mathbf{n}}} \sigma\,|x(\mathbf{n})\rangle = \frac{1}{|G_{\mathbf{n}}|}\sum_{\sigma\in G}\sigma\,|x(\mathbf{n})\rangle\ . \tag{1.14}$$

It is easily possible to normalize these states so that we obtain an orthonormal basis, but we will not need this. The normalizing constant would simply be the square root of the number of cosets, which, by Lagrange's theorem, is given by the multinomial coefficient:

$$|G/G_{\mathbf{n}}| = \frac{n!}{\prod_{i=1}^{d} n_i!} = \binom{n}{n_1,\ldots,n_d}. \tag{1.15}$$

Finally, we determine the dimension of this subspace: Each basis state is given by a partition of the number n into d parts, so the dimension is the number of these partitions. A well-known trick to determine this number is to distribute $d-1$ barriers among $n+d-1$ available positions, such that the partition is given by the number of empty positions between the barriers. Thus, we conclude

$$\dim \mathrm{Sym}^n(\mathcal{H}) = \#\mathrm{Partitions}(n,d) = \binom{n+d-1}{d-1}. \tag{1.16}$$

1.3.2. Mixed states

We will now consider mixed states and define an analogous notion of symmetry. Here, the symmetric group acts on an operator as:

$$\rho \mapsto \sigma\rho\sigma^{-1}\,. \tag{1.17}$$

Thus, we call a mixed state ρ *symmetric* iff $\rho = \sigma\rho\sigma^{-1}$ for all $\sigma \in G$, which is the same as saying that it commutes with all permutations. In this way, the operator $|\psi\rangle\langle\psi|$ corresponding to a symmetric state $|\psi\rangle$ is again symmetric.

The space of symmetric operators can now be described analogously to the pure state case, where the role of the computational basis states will be taken by rank-one operators $|i_1,\ldots,i_n\rangle\langle j_1,\ldots,j_n|$. Here, the order of pairs of indices (i_k, j_k) is irrelevant and we specify the symmetrized states by giving occupation numbers for these pairs. It is convenient to organize these occupation numbers in what we call a *transition matrix*:

$$\mathbf{T} = \begin{pmatrix} n_{|1\rangle\langle 1|} & \cdots & n_{|1\rangle\langle d|} \\ \vdots & & \vdots \\ n_{|d\rangle\langle 1|} & \cdots & n_{|d\rangle\langle d|} \end{pmatrix} = \begin{pmatrix} t_{1,1} & \cdots & t_{1,d} \\ \vdots & & \vdots \\ t_{d,1} & \cdots & t_{d,d} \end{pmatrix}. \tag{1.18}$$

[2] In the equation, we slightly abuse the notation by not distinguishing between the cosets and their representatives.

Again, we define standard states

$$\rho_{\mathbf{T}} = \bigotimes_{i,j=1}^{d} |i\rangle\langle j|^{\otimes t_{i,j}} \tag{1.19}$$

and symmetrize them to obtain basis states

$$\hat{\rho_{\mathbf{T}}} = \frac{1}{\prod_{i,j} t_{i,j}!} \sum_{\sigma \in G} \sigma \rho_{\mathbf{T}} \sigma^{-1} . \tag{1.20}$$

Here, we divide by $\prod_{i,j} t_{i,j}!$, analogously to (1.14), to avoid having duplicate terms. The dimension of the subspace of symmetric operator is given by the number of transition matrices. Hence we need to count possible partitions of n into d^2 parts, and so the dimension is given by

$$\dim \mathrm{SymOp}^n(\mathcal{H}) = \#\mathrm{Partitions}(n, d^2) = \binom{n + d^2 - 1}{d^2 - 1} . \tag{1.21}$$

Writing the occupation numbers for operators in a matrix can be motivated by observing

$$\hat{\rho_{\mathbf{T}}} |\mathbf{c}\rangle \sim |\mathbf{r}\rangle \ , \tag{1.22}$$

$$\hat{\rho_{\mathbf{T}}} |\mathbf{c}'\rangle = 0 \, , \tag{1.23}$$

where $\mathbf{c}$ is the sum of the columns of $\mathbf{T}$, $\mathbf{r}$ is the sum of the rows of $\mathbf{T}$ and $\mathbf{c}' \neq \mathbf{c}$. The factor of proportionality missing in (1.22) is easily determined: Each of the $\binom{n}{(t_{i,j})}$ terms represented by $\hat{\rho_{\mathbf{T}}}$ produces one of the $\binom{n}{(r_i)}$ terms belonging to $|\mathbf{r}\rangle$. The resulting state is clearly symmetric, again. Thus, we have

$$\hat{\rho_{\mathbf{T}}} |\mathbf{c}\rangle = \frac{\binom{n}{(t_{i,j})}}{\binom{n}{(r_i)}} |\mathbf{r}\rangle = \frac{\prod_i r_i!}{\prod_{i,j} t_{i,j}!} |\mathbf{r}\rangle \ . \tag{1.24}$$

Example

If $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$\begin{aligned} \hat{\rho_{\mathbf{T}}} = {} & |0\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 0| \\ & + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 0| \ . \end{aligned}$$

Then, for $|(2,1)\rangle = |001\rangle + |010\rangle + |100\rangle$, we find

$$\begin{aligned} \hat{\rho_{\mathbf{T}}} |(2,1)\rangle &= (|0\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0|) \, |010\rangle + \ldots \text{(six terms in total)} \\ &= 2\,|001\rangle + 2\,|010\rangle + 2\,|100\rangle = 2\,|(2,1)\rangle \ . \end{aligned}$$

2. Evaluating channels on symmetric states

We will now describe how to calculate the output of a channel $\mathcal{N}^{\otimes n}(|\psi\rangle\langle\psi|)$ when it is applied to a symmetric input state $|\psi\rangle$. In the first section, we will use the occupation number representation described in the previous chapter for this purpose. In the second section, we will describe an alternative approach using outer product states $|\phi\rangle^{\otimes n}$.

2.1. Using the occupation number representation

Here, we will assume that the channel is specified by its Kraus operators A_k and that the input is given by its coefficients in the basis defined in (1.14). The output will then be given as coefficients in the basis of symmetric operators, (1.20). Thus, we need to compute

$$\mathcal{N}^{\otimes n}\big(|\psi\rangle\langle\psi|\big) = \sum_{k_1,\ldots,k_n} \left(\bigotimes_{i=1}^{n} A_{k_i}\right) |\psi\rangle\langle\psi| \left(\bigotimes_{i=1}^{n} A_{k_i}^{\dagger}\right), \tag{2.1}$$

but we will start by considering the slightly simpler problem of evaluating $A^{\otimes n}|\mathbf{n}\rangle$ for an arbitrary operator A and then describe two methods that use this to compute (2.1). So let A be an operator on $\mathcal{H}$ and let $|\mathbf{n}\rangle$ be a symmetric state. Then

$$A^{\otimes n}|\mathbf{n}\rangle = \frac{1}{|G_{\mathbf{n}}|}\sum_{\sigma\in G} A^{\otimes n}\sigma\,|x(\mathbf{n})\rangle = \frac{1}{|G_{\mathbf{n}}|}\sum_{\sigma\in G} \sigma A^{\otimes n}\,|x(\mathbf{n})\rangle\ , \tag{2.2}$$

as clearly $[A^{\otimes n},\sigma]=0$. We will now use the matrix elements of A to expand

$$A^{\otimes n}|x(\mathbf{n})\rangle = \sum_{y} A_{y,x}\,|y\rangle\ , \quad \text{with} \quad A_{y,x} = \prod_{i=1}^{n} A_{y_i,x_i}\ , \tag{2.3}$$

where the sum over $y=(y_1,\ldots,y_n)$ is a sum over all computational basis states and $(x_1,\ldots,x_n)$ are the coefficients of the standard state $x(\mathbf{n})$. Inserting this into (2.2), we have

$$
\begin{aligned}
A^{\otimes n}|\mathbf{n}\rangle &= \frac{1}{|G_{\mathbf{n}}|}\sum_y A_{y,x}\sum_{\sigma\in G}\sigma|y\rangle \\
&= \sum_y \frac{|G_{\mathbf{n}(y)}|}{|G_{\mathbf{n}}|}A_{y,x}|\mathbf{n}(y)\rangle \\
&= \sum_y \frac{\prod_i n_i(y)!}{\prod_i n_i!}A_{y,x}|\mathbf{n}(y)\rangle\,,
\end{aligned} \tag{2.4}
$$

where $|\mathbf{n}(y)\rangle$ is the symmetrized state corresponding to $|y\rangle$ and $n_i(y)$ are its occupation numbers.

This equation tells us how symmetric states are mapped to symmetric states by the operator $A^{\otimes n}$, but unfortunately it involves a sum over all computational basis states. To improve upon this, we group terms that contain the same factor $A_{y,x}$. As this factor depends on the number of pairs (y_i, x_i), we have to sum over transition matrices with given column- and row sum. Consequently, we define

$$
\mathbf{T}\in \mathrm{ON}_{dd}[\mathbf{n},\mathbf{m}] \;\;\Leftrightarrow\;\; \left(n_j = \sum\nolimits_i t_{i,j} \;\wedge\; m_i = \sum\nolimits_j t_{i,j}\right). \tag{2.5}
$$

The number of times the standard state with occupation numbers n can be mapped to a state with occupation numbers m by a $\mathbf{T}\in \mathrm{ON}_{dd}[\mathbf{n},\mathbf{m}]$ is clearly given by the product of multinomial coefficients

$$
\prod_{j=1}^{d}\binom{n_j}{t_{1,j},\dots,t_{d,j}} = \binom{n}{(t_{i,j})_{i,j\in\{1,\dots,d\}}}\Big/\binom{n}{n_1,\dots,n_d}\,. \tag{2.6}
$$

Thus, we combine the terms in (2.4) to find

$$
\begin{aligned}
A^{\otimes n}|\mathbf{n}\rangle &= \sum_{\mathbf{m}}\left(\prod_{i=1}^{d}\frac{m_i!}{n_i!}\right)\sum_{\mathbf{T}\in\mathrm{ON}_{dd}[\mathbf{n},\mathbf{m}]}\prod_{j=1}^{d}\binom{n_j}{t_{1,j},\dots,t_{d,j}}\prod_{i=1}^{d}A_{i,j}^{t_{i,j}}|\mathbf{m}\rangle \\
&= \sum_{\mathbf{m}}\sum_{\mathbf{T}\in\mathrm{ON}_{dd}[\mathbf{n},\mathbf{m}]}\frac{\prod_i m_i!}{\prod_{i,j}t_{i,j}!}\prod_{i,j=1}^{d}A_{i,j}^{t_{i,j}}|\mathbf{m}\rangle\,.
\end{aligned} \tag{2.7}
$$

It is not surprising that a sum over transition matrices occurs here, as these matrices represent a basis for symmetric maps. Effectively, we have expanded $A^{\otimes n}$ in terms of the operators $\hat{\rho_{\mathbf{T}}}$ with coefficients $\prod_{i,j}A_{i,j}^{t_{i,j}}$ (cf. equation (1.24)).

If we want to calculate all matrix elements $\langle\mathbf{m}|A^{\otimes n}|\mathbf{n}\rangle$, we simply sum over *all* transition matrices where the entries sum to n and add up the appropriate contributions to the matrix elements. The number of these terms is given by (1.21) and is thereby bounded by a polynomial in n.

Let us now return to the original problem of computing (2.1). A direct approach is treating mixed states as vectors in the space $(\mathbb{C}^{d^2})^{\otimes n}$. Evaluation of the channel is then of the form $A^{\otimes n}\,|\mathbf{n}\rangle$, where A is given by a sum over the (modified) Kraus operators. Alas, the number of terms to be summed over is now $\#\mathrm{Partitions}(n, d^4)$.

A second, more efficient, approach is inspired by purification. We extend the space $\mathcal{H}$ by adding an environment of dimension p, where p is the number of Kraus operators. On the space $\mathcal{H}' = \mathcal{H} \otimes \mathbb{C}^p$ we define new versions of the Kraus operators by

$$A'_k\big(|x\rangle \otimes |0\rangle\big) = A_k\,|x\rangle \otimes |k\rangle\ . \tag{2.8}$$

Now, we need only compute

$$|\psi'\rangle = \Big(\sum\nolimits_k A'_k\Big)^{\otimes n}\Big[|\psi\rangle \otimes |0\rangle^{\otimes n}\Big] \tag{2.9}$$

and use

$$\mathcal{N}^{\otimes n}\big(|\psi\rangle\langle\psi|\big) = \mathrm{Tr}_{\mathrm{Env}}\,|\psi'\rangle\langle\psi'| \tag{2.10}$$

to evaluate the channel.

Symmetric vectors in $\mathcal{H}'^{\otimes n}$ are written in a basis analogous to (1.14), but now the occupation numbers count states $|i,k\rangle$ with pairs of indices $(i,k) \in \{1,\dots,d\}\times\{1,\dots,p\}$. The computations for (2.9) are done analogously to (2.7). Luckily, as we need only the terms where the environment is in state $|0\rangle$, we need to sum over $\#\mathrm{Partitions}(n, d^2p)$ terms, only.

Tracing out the environment

We now describe how to compute (2.10). As before, the state $|\psi'\rangle \in \mathrm{Sym}^n(\mathcal{H}')$ is given in the occupation number representation with occupation numbers $\mathbf{n} = (n_{i,k})$ counting states $|i,k\rangle$. The operator $|\psi'\rangle\langle\psi'|$ is then expanded in terms of operators $|\mathbf{n}\rangle\langle\mathbf{m}|$. The tracing operation maps all terms that contain tensor factors of $|i,k\rangle\langle j,k'|$ with $k \neq k'$ to zero. To find all non-zero contributions to the trace, it is thus convenient to consider three-dimensional integer arrays $\mathbf{A} = (a_{i,j,k})$ of dimension $d \times d \times p$ that give occupation numbers for the states $|i,k\rangle\langle j,k|$. We denote

$$\mathbf{A} \in \mathrm{ON}_{ddp}\,[\mathbf{m},\mathbf{n},\mathbf{T}] \;\;\Leftrightarrow\;\; \Big(m_{j,k} = \sum\nolimits_i a_{i,j,k} \wedge n_{i,k} = \sum\nolimits_j a_{i,j,k} \wedge t_{i,j} = \sum\nolimits_k a_{i,j,k}\Big)\,.$$

Each array $\mathbf{A} \in \mathrm{ON}_{ddp}\,[\mathbf{m},\mathbf{n},\mathbf{T}]$ describes $\binom{n}{(a_{i,j,k})}$ terms from $|\mathbf{n}\rangle\langle\mathbf{m}|$ that will be mapped to one of $\binom{n}{(t_{i,j})}$ terms from $\hat{\rho_{\mathbf{T}}}$. Thus, we find

$$\begin{aligned}\mathrm{Tr}_{\mathrm{Env}}\,|\mathbf{n}\rangle\langle\mathbf{m}| &= \sum_{\mathbf{T}}\;\sum_{\mathbf{A}\in\mathrm{ON}_{ddp}\,[\mathbf{m},\mathbf{n},\mathbf{T}]} \binom{n}{(a_{i,j,k})}\binom{n}{(t_{i,j})}^{-1}\hat{\rho_{\mathbf{T}}} \\ &= \sum_{\mathbf{T}}\;\sum_{\mathbf{A}\in\mathrm{ON}_{ddp}\,[\mathbf{m},\mathbf{n},\mathbf{T}]} \frac{\prod_{i,j} t_{i,j}!}{\prod_{i,j,k} a_{i,j,k}!}\hat{\rho_{\mathbf{T}}}\,.\end{aligned} \tag{2.11}$$

To evaluate (2.10), we simply iterate over all integer arrays $\mathbf{A}$ with entries summing to n, of which there are $\#\text{Partitions}(n, d^2 p)$, and add up the contributions according to (2.11).

Multiplication of density operators

We will later need to be able to multiply two density operators given in the occupation number basis. The procedure is surprisingly similar to the computation of the partial trace that was just described. Consider two operators $\hat{\rho_{\mathbf{R}}}$ and $\hat{\rho_{\mathbf{S}}}$ specified by transition matrices $\mathbf{R}$ and $\mathbf{S}$. Here, we need three-dimensional integer arrays $\mathbf{A} = (a_{i,j,k})$ of dimension $d\times d\times d$ that specify occupation numbers for factors $|i\rangle\langle j|j\rangle\langle k|$. Using

$$\mathbf{A} \in \mathrm{ON}_{ddd}\left[\mathbf{R}, \mathbf{S}, \mathbf{T}\right] \;\;\Leftrightarrow\;\; \left(r_{j,k} = \sum\nolimits_i a_{i,j,k} \wedge s_{i,k} = \sum\nolimits_j a_{i,j,k} \wedge t_{i,j} = \sum\nolimits_k a_{i,j,k}\right),$$

we find

$$\begin{aligned} \hat{\rho_{\mathbf{S}}} \cdot \hat{\rho_{\mathbf{R}}} &= \sum_{\mathbf{T}} \sum_{\mathbf{B}\in \mathrm{ON}_{ddd}\left[\mathbf{R},\mathbf{S},\mathbf{T}\right]} \binom{n}{(b_{i,j,k})} \binom{n}{(t_{i,j})}^{-1} \hat{\rho_{\mathbf{T}}} \\ &= \sum_{\mathbf{T}} \sum_{\mathbf{B}\in \mathrm{ON}_{ddd}\left[\mathbf{R},\mathbf{S},\mathbf{T}\right]} \frac{\prod_{i,j} t_{i,j}!}{\prod_{i,j,k} b_{i,j,k}!} \hat{\rho_{\mathbf{T}}}\,. \end{aligned} \tag{2.12}$$

Working with bipartite states

We are ultimately interested in the quantum capacity of the channel and thus need to be able to compute the coherent information which is defined using a bipartite states (see (1.7)). We will always let the symmetric group act on system and environment simultaneously and states will be considered to be symmetric in this sense. Thus, to evaluate $\mathcal{I}\left(\mathcal{N}^{\otimes n}\right)$, we will use an input state $|\phi_{AA'}\rangle \in \mathrm{Sym}^n(\mathcal{H}_A \otimes \mathcal{H}_{A'})$ and the resulting output state will be $\rho_{AB} = \mathcal{N}^{\otimes n}(|\phi_{AA'}\rangle) \in \mathrm{SymOp}^n(\mathcal{H}_A \otimes \mathcal{H}_B)$. We will now describe how to compute $\rho_B = \mathrm{tr}_A\left(\rho_{AB}\right)$ in the case that both system and environment are two-dimensional: The state ρ_{AB} is given in the basis defined in (1.20), we will thus consider a state $\hat{\rho_{\mathbf{T}}}$ with

$$\mathbf{T} = \begin{pmatrix} n_{|00\rangle\langle 00|} & n_{|00\rangle\langle 01|} & n_{|00\rangle\langle 10|} & n_{|00\rangle\langle 11|} \\ n_{|01\rangle\langle 00|} & n_{|01\rangle\langle 01|} & n_{|01\rangle\langle 10|} & n_{|01\rangle\langle 11|} \\ n_{|10\rangle\langle 00|} & n_{|10\rangle\langle 01|} & n_{|10\rangle\langle 10|} & n_{|10\rangle\langle 11|} \\ n_{|11\rangle\langle 00|} & n_{|11\rangle\langle 01|} & n_{|11\rangle\langle 10|} & n_{|11\rangle\langle 11|} \end{pmatrix}. \tag{2.13}$$

Then, because we trace out the A-system for each qubit and $\mathrm{tr}_A\left(|ij\rangle\langle kl|\right) = \delta_{ik}\,|j\rangle\langle l|$, we clearly have $\mathrm{tr}_A\left(\hat{\rho_{\mathbf{T}}}\right) \sim \hat{\rho_{\mathbf{T}'}}$ with

$$\mathbf{T}' = \begin{pmatrix} n_{|00\rangle\langle 00|} + n_{|10\rangle\langle 10|} & n_{|00\rangle\langle 01|} + n_{|10\rangle\langle 11|} \\ n_{|01\rangle\langle 00|} + n_{|11\rangle\langle 10|} & n_{|01\rangle\langle 01|} + n_{|11\rangle\langle 11|} \end{pmatrix} \tag{2.14}$$

if the other terms (e.g. $n_{|00\rangle\langle 10|}$) are zero or, equivalently, the entries in $\mathbf{T}'$ sum to n. If this is not the case, clearly $\operatorname{tr}_A(\hat{\rho_{\mathbf{T}}}) = 0$. The missing factor of proportionality can be found by considering the number of terms represented by $\hat{\rho_{\mathbf{T}}}$ and $\hat{\rho_{\mathbf{T}'}}$ and is thus given by $\frac{\prod_{i,j} t_{i,j}!}{\prod_{i,j} t'_{i,j}!}$, using the obvious labeling of the entries of the matrices $\mathbf{T}$ and $\mathbf{T}'$.

The computation of the entropy of these states will be the topic of chapter 3.

2.2. Using product states

We will now consider states $|\psi\rangle \in \operatorname{Sym}^n(\mathcal{H})$ that are written in form that we will call *outer product decomposition*:

$$|\psi\rangle = \sum_{i=1}^{r} a_i |\psi_i\rangle^{\otimes n} . \tag{2.15}$$

An overview of the theory of the outer product decomposition can be found in [3], where we find the following definition and result.

Definition 2.1. The smallest number r for which there are states $|\psi_i\rangle$ such that (2.15) holds is called the *symmetric tensor rank* of $|\psi\rangle$.

Theorem 2.1. *Every symmetric state $|\psi\rangle$ can be written as in* (2.15)*, thus the symmetric tensor rank is well-defined for every symmetric state.*

It is not clear, how states $|\psi_i\rangle$ can be chosen so that their outer products $|\psi_i\rangle^{\otimes n}$ form a spanning set for $\operatorname{Sym}^n(\mathcal{H})$. Nevertheless, the outer product decomposition has some advantages in the context of our current task, the computation of the output of a quantum channel. To wit, we have

$$\mathcal{N}^{\otimes n}\big(|\psi\rangle\langle\psi|\big) = \sum_{i,j=1}^{r} a_i \overline{a_j} \big[\mathcal{N}\big(|\psi_i\rangle\langle\psi_j|\big)\big]^{\otimes n} . \tag{2.16}$$

Thus, it is sufficient to evaluate the channel on the space $\mathcal{H}$ for all pairs $(i,j) \in \{1,\ldots,n\}$ and we can compute the matrix elements of the output state as

$$\langle x|\mathcal{N}^{\otimes n}\big(|\psi\rangle\langle\psi|\big)|y\rangle = \sum_{i,j=1}^{r} a_i \overline{a_j} \prod_{k=1}^{n} \langle x_k|\mathcal{N}\big(|\psi_i\rangle\langle\psi_j|\big)|y_k\rangle . \tag{2.17}$$

3. Computing the entropy of symmetric states

To evaluate the coherent information, we need to be able to compute the von Neumann entropy of symmetric states ρ that are specified either in the occupation number representation as described in section 1.3 or as product states as in section 2.2. We will try to find a fast method that avoids the need to diagonalize the operator ρ in the full space $\mathcal{H}^{\otimes n}$ by using the symmetry of ρ.

The entropy can be calculated approximately using a polynomial $P(x) = \sum_n a_i x^n$ that is close to the function

$$\eta(x) = -x \log_2(x) \tag{3.1}$$

on the interval $[0, 1]$. If $x_i \in \mathbb{R}$ are the eigenvalues and n_i their geometric multiplicities with $i \in \{1, \ldots, k\}$, the entropy is given by

$$S(\rho) = \mathrm{Tr}\left[-\rho \log_2(\rho)\right] = \sum_{i=1}^{k} -n_i \eta(x_i)\,. \tag{3.2}$$

Thus, if $P(x) \approx \eta(x)$,

$$S(\rho) \approx \sum_{i=1}^{k} -n_i P(x_i) = \sum_n a_n \,\mathrm{Tr}\, \rho^n\,. \tag{3.3}$$

The expression can then be evaluated using the procedure to multiply density operators given in the occupation number basis described in section 2.1.

Unfortunately, it is difficult to approximate the function η, especially close to zero, where it becomes arbitrarily steep. This problem is exacerbated by that fact that we will often encounter states with very small eigenvalues with large degeneracy.

Thus, we will now try to find the complete spectrum, which will allow us to compute the entropy precisely using (3.2). To this end, we will present two methods. The first approach is completely our own, but only succeeds at determining the spectrum using many multiplications. The second, more efficient, approach is based on representation theory.

3.1. Naive approach

3.1.1. Determining the eigenvalues

An obvious starting point for determining the spectrum is to look at the properties of eigenvectors of the symmetric operator ρ. Let $x \in \mathbb{R}$ be an eigenvalue of ρ and

$$\rho |\psi\rangle = x |\psi\rangle \ . \tag{3.4}$$

Then, because ρ commutes with all permutations $\sigma \in S_n$, we will necessarily have

$$\rho\sigma |\psi\rangle = x\sigma |\psi\rangle \ , \tag{3.5}$$

too, and therefore

$$\rho \sum_{\sigma\in S_n} \sigma |\psi\rangle = x \sum_{\sigma\in S_n} \sigma |\psi\rangle \ . \tag{3.6}$$

Thus, it appears on first glance that for each eigenvalue there exists a symmetric eigenvector. Unfortunately, it is also possible that an eigenvector $|\psi\rangle$ is antisymmetric in the sense that $\sum_{\sigma\in S_n} \sigma |\psi\rangle = 0$. To find all eigenvalues and still reduce the dimension of the space in which the operator needs to be diagonalized, we try to find subgroups of S_n to partially symmetrize the states. For this purpose, we consider d blocks of qudits, with $d = \dim \mathcal{H}$, for all non-increasing partitions of n , $\lambda = (n_1, \dots, n_d)$. We symmetrize each block separately, that is, we define the subgroup $S_\lambda = S_{\{1,\dots,n_1\}} \times S_{\{n_1+1,\dots,n_1+n_2\}} \times \cdots \times S_{\{n-n_d+1,\dots,n\}}$ and apply the corresponding projection

$$P_\lambda = \frac{1}{n_1! \cdots n_d!} \sum_{\sigma\in S_\lambda} \sigma \ . \tag{3.7}$$

Basis states for the corresponding partially symmetrized subspace can be specified, analogously to section 1.3.1, by occupation numbers for each of the d blocks. We can now find all eigenvalues of ρ by computing the matrix representation of ρ in all partially symmetrized subspaces and diagonalizing those matrices:

Theorem 3.1. *If x is an eigenvalue of ρ, then there exists a non-increasing partition of n, $\lambda = (n_1, \dots, n_d)$, such that there is an eigenvector of ρ with eigenvalue x in the image of P_λ.*

Proof. Let $|\psi\rangle$ be an eigenvector of ρ with eigenvalue x. Choose a computational basis state $|i_1, \dots, i_n\rangle$ such that the component of $|\psi\rangle$ for it is non-zero. Next, find a permutation σ such that $\sigma |i_1, \dots, i_n\rangle$ groups the identical indices in blocks of non-increasing length, e.g. $|020210\rangle$ is reordered as $|000221\rangle$. The length of the blocks yields a non-increasing partition of n, $\lambda = (n_1, \dots, n_d)$, and $P_\lambda\sigma |\psi\rangle$ is an eigenvector of ρ with eigenvalue x in the image of P_λ. ☐

Sadly, the method described so far does not enable us to determine the geometric multiplicities of the eigenvalues as an eigenvector can have non-zero components in the image of different P_λ. Thus, we are still unable to compute the entropy. But, now that we know all the eigenvalues, it is possible to improve upon the approximation approach described at the beginning of this chapter. Instead of trying to approximate η on the whole interval $[0, 1]$, we only look for polynomials that approximate η at the eigenvalues. For reasons already mentioned, this approach does not succeed and thus we will try to determine the multiplicities of the eigenvalues next.

3.1.2. Determining the multiplicities

To determine the multiplicities, we consider the trace of powers of ρ:

$$\operatorname{Tr} \rho^j = \sum_{i=1}^{k} n_i x_i^j \, . \tag{3.8}$$

Thus, by computing the traces of ρ^j for $j \in \{1, \ldots, k\}$, we can set up a system of linear equations with variables n_i that has a unique solution. This simple idea is sadly not successful, as the resulting system of equations is usually numerically ill-conditioned. The main problem is posed by the existence of small eigenvalues that contribute little to $\operatorname{Tr} \rho^j$ for larger values of j.

A different idea, inspired by the attempt to approximate the entropy using polynomials, is to find polynomials that help us determine the multiplicities: If $P(x) = \sum_n a_i x^n$ is chosen such that $P(x_i) = 1$ for one eigenvalue x_i and $P(x_j) = 0$ for all the others, then

$$n_i = \sum_n a_n \operatorname{Tr} \rho^n \tag{3.9}$$

(cf. equation (3.3)). Thus, if we find polynomials that approximate this property, we can determine the multiplicity of the corresponding eigenvalue. Additionally, once we know some multiciplities, we can remove the condition that $P(x_j) = 0$ for these eigenvalues as we can substract out the terms that they contribute to $\sum_n a_n \operatorname{Tr} \rho^n$. This method is often successfull in determining the multiciplities of the larger eigenvalues, but tends to fail when there are many eigenvalues that are very close to zero. Ultimately, we have to accept that information about these eigenvalues is not contained in the numerical values of $\operatorname{Tr} \rho^n$.

We can resolve this issue by instead working with $\mathbb{1} - \rho$ and solving

$$\operatorname{Tr}(\mathbb{1} - \rho)^j = \sum_{i=1}^{k} n_i (1 - x_i)^j \, , \tag{3.10}$$

instead. Here, even high values of j continue to give useful information and $\frac{\operatorname{Tr}(\mathbb{1}-\rho)^j}{1-x_l}$ converges reliably to n_l if l is the index of the smallest eigenvalue. A relatively fast way to set up a well conditioned system is to repeatedly square the operator $(\mathbb{1}-\rho)$ until $\frac{\mathrm{Tr}(\mathbb{1}-\rho)^j}{1-x_l}$ becomes approximately integer and use the resulting traces.

To summarize, we have found a method to compute all eigenvalues in time polynomial in n. The number of multiplications required to determine the degeneracies is at least the number of the eigenvalues that can not be neglected, and is thus potentially exponentially large. Even for small n, multiplication using (2.12) is slow as it involves a sum over $\#\mathrm{Partitions}(n, d^3)$ terms.

3.1.3. Example

In this section, we will give some numbers to illustrate the methods described in the previous sections. We consider a randomly chosen symmetric qubit state of five qubits, sent through the bitflip-channel with $p = 0.05$. The eigenvalues of the resulting state ρ_{AB} are

$$0.0, 8.453 \cdot 10^{-8}, 1.533 \cdot 10^{-6}, 2.196 \cdot 10^{-6}, 2.867 \cdot 10^{-5}, 3.277 \cdot 10^{-5}, 5.551 \cdot 10^{-5},\\ 5.706 \cdot 10^{-4}, 6.706 \cdot 10^{-4}, 1.876 \cdot 10^{-3}, 0.01243, 0.04461, 0.8978\,.$$

Here, all eigenvalues smaller than 10^{-12} have been lumped together to 0.0.

n	0	1	2	3	4	5	6	7
$\mathrm{Tr}\,\rho^n$	1024.0	1.0	0.8087	0.7238	0.6497	0.5833	0.5237	0.4702
n	8	9	10	11	12	13	14	15
$\mathrm{Tr}\,\rho^n$	0.4222	0.3790	0.3403	0.3055	0.2743	0.2463	0.2211	0.1985
n	16	17	18	19				
$\mathrm{Tr}\,\rho^n$	0.1782	0.1600	0.1437	0.1290				

n	0	1	2	4	8	16	32	64
$\mathrm{Tr}(\mathbb{1}-\rho)^n$	1024.0	1023.0	1022.8	1022.6	1022.3	1021.6	1020.7	1019.4
n	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
$\mathrm{Tr}(\mathbb{1}-\rho)^n$	1017.9	1016.4	1014.8	1012.6	1009.9	1007.4	1005.6	1003.7
n	2^{15}	2^{16}	2^{17}	2^{18}				
$\mathrm{Tr}(\mathbb{1}-\rho)^n$	1001.2	998.7	997.2	996.2				

As the biggest eigenvalue is greater than $\frac{1}{2}$, its degeneracy is clearly 1. Using peaked polynomials, we find that the degeneracies of the next three eigenvalues are 1, 4 and 1, but further values can not be reliably determined from the values of $\mathrm{Tr}\,\rho^n$. Trying to numerically solve (3.8) does not yield the correct integer values, even though the linear system of equations is overdetermined. Indeed, we can see that traces for larger powers are completely determined by the biggest eigenvalue and looking to use higher powers is fruitless.

Using a polynomial that approximates η at the remaining eigenvalues with unknown degeneracy, we find an approximate value of $S(\rho_{AB}) \approx 0.7423$.

In contrast, using the values of $\mathrm{Tr}(\mathbb{1} - \rho)^n$ and solving (3.10), we easily find all the degeneracies[1]:

$$992, 1, 4, 1, 5, 4, 1, 4, 5, 1, 4, 1, 1$$

Inserting these values into (3.2), we find $S(\rho_{AB}) \approx 0.7365$. Indeed, the numerical result is up to 12 digits identical to the result found by complete diagonalization.

3.2. Using representation theory

In this section, we will describe how to compute the spectrum of a symmetric state using representation theory, the famous Schur-Weyl duality in particular.[2] A good explanation of Schur-Weyl duality can be found in [4] as well as in various standard textbooks (e.g. [5]). A particularly gentle introduction to the representation theory of finite groups is [6], on which we base the next subsection.

3.2.1. Basics of representation theory

Definition 3.1 (Representation). A *representation* of a (finite) group G on a (finite-dimensional, complex) vector space V is a homomorphism $\varphi\colon G \to \mathrm{GL}(V)$. The *degree* of the representation is just the dimension of V.

It is common to refer to V as the representation when it is hoped to be clear which map φ is being used. We usually write φ_g for $\varphi(g)$ or even completely neglect mentioning φ, as in section 1.3 where we already encountered a representation of the symmetric group.

Definition 3.2 (G-linear map). A linear map $T\colon V \to W$ between two representations $\varphi\colon G \to \mathrm{GL}(V)$ and $\psi\colon G \to \mathrm{GL}(W)$ is called a *G-linear map* if $T\varphi_g = \psi_g T$ for all $g \in G$, or, in other words, the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"] & W \end{tikzcd}$$

[1]The result from the numerical solver is floating-point numbers that are very close to the correct integer values.

[2]We thank David Gross for clarifying the use of this result.

commutes.

Definition 3.3 (Equivalence)**.** We call two representations $\varphi\colon G \to \mathrm{GL}(V)$ and $\psi\colon G \to \mathrm{GL}(W)$ *equivalent* if there is an invertible G-linear map $T\colon V \to W$. We sometimes write this as $\varphi \sim \psi$.

Two representations can be combined by using the direct sum of their vector spaces:

Definition 3.4 (Direct product)**.** If $\varphi\colon G \to \mathrm{GL}(V)$ and $\psi\colon G \to \mathrm{GL}(W)$ are representations, then their *direct product* is the representation $\varphi \oplus \psi\colon G \to V \oplus W$ defined by

$$(\varphi \oplus \psi)_g(v, w) \mapsto (\varphi_g(v), \psi_g(w)) .$$

If a representation can be non-trivially written as a direct sum, we will call it *decomposable*. We now aim to understand when a representation is not decomposable. This leads us directly to the next two definitions.

Definition 3.5 (Invariant subspace)**.** If $\varphi\colon G \to \mathrm{GL}(V)$ is a representation, we call a subspace $W \leq V$ *G-invariant* if $w \in W \Rightarrow \varphi_g w \in W$ for all $g \in G$.

Definition 3.6 (Irreducible representation)**.** A representation $\varphi\colon G \to \mathrm{GL}(V)$ is called *irreducible* if the only G-invariant subspaces are the trivial ones, $\{0\}$ and V.

It turns out that every (non-trivial) representation of a finite group is either irreducible or decomposable. Using this, one can proof via induction the following powerful theorem.

Maschke's theorem. *Every representation $\varphi\colon G \to \mathrm{GL}(V)$ of a finite group G can be written as*

$$V = \bigoplus_{i=1}^{n} V_i \,, \tag{3.11}$$

where all V_i are G-invariant subspaces and all $\varphi\big|_{V_i}$ are irreducible.

The following lemma is "probably the most frequently used result in representation theory" ([4], pg. xxii) and gives strong restrictions on the set of G-linear maps between irreducible representations.

Schur's lemma. *If V and W are irreducible representations of G and T is a G-linear map, then either T is an isomorphism, or $T = 0$. It follows directly that if $V \nsim W$, we must have $T = 0$. Furthermore, if $V = W$, then $T = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$.*

Proof. The first part follows directly from the fact that both $\operatorname{Ker} T$ and $\operatorname{Im} T$ are G-invariant subspaces. For the second part, we need to use the fact that we work with complex vector spaces and thus T always has an eigenvalue $\lambda \in \mathbb{C}$. Then, clearly, $T - \lambda\mathbb{1}$ is a G-linear map with $\operatorname{Ker}(T - \lambda\mathbb{1}) \neq 0$. Hence, T can not be invertible and we have $T = 0$. $\square$

Applying Schur's lemma to the identity map $I\colon V \to V$, we find that the decomposition given in Maschke's theorem is unique up to isomorphism. From now on, we will index the equivalence classes of the irreducible representations of a group G by the set $\hat{G}$. Using this notation, (3.11) becomes

$$V \cong \bigoplus_{\alpha \in \hat{G}} V_\alpha^{\oplus n_\alpha}\,. \tag{3.12}$$

It is possible to rewrite direct sums of identical representations $V_\alpha^{\oplus n_\alpha}$ using a dummy vector space U_α of dimension n_α by defining an isomorphism

$$\begin{aligned} \varphi\colon V_\alpha^{\oplus n_\alpha} &\to U_\alpha \otimes V_\alpha\,, \\ (v_1, \ldots, v_n) &\mapsto \sum_{i=1}^{d} e_i \otimes v_i\,. \end{aligned}$$

In this way, we have

$$V \cong \bigoplus_{\alpha \in \hat{G}} U_\alpha \otimes V_\alpha\,. \tag{3.13}$$

We now introduce two closely related objects, the group algebra and the regular representation, that we will use in the next section to understand the representations of the symmetric group. The following material is based on the 'preliminaries' chapter of [4].

Definition 3.7 (Group algebra)**.** We construct a complex vector space by taking the elements of a group G as basis elements. Then, we define a (bilinear) product by linear extension of the group operation. The resulting algebra is called the *group algebra* $\mathcal{A}(G)$ of G.

It is often useful to broaden a given group representation to a representation of the group algebra by linear extension. The elements of the group algebra can alternatively be thought of as functions $f\colon G \to \mathbb{C}$, where the function maps each group element to its coefficient. In this way, the product of the algebra takes the form of an involution:

$$(a \cdot b)(g) = \sum_{h \in G} a(gh^{-1})b(h)\,. \tag{3.14}$$

A scalar product on $\mathcal{A}(G)$ can be defined by

$$\langle a, b\rangle = \frac{1}{|G|} \sum_{g \in G} \overline{a(g)}b(g)\,. \tag{3.15}$$

If, for a given representation $\varphi\colon G \to \mathrm{GL}(V)$, we fix a basis and write the operators $\varphi(g)$ as matrices, their entries $\varphi_{ij}(g)$ become elements of the group algebra. For every irreducible representation, it is possible to find a *unitary* representation (i.e. $\varphi(g)$ is unitary for all $g \in G$) that is equivalent to it. The matrix entries of the unitary representations obey the so-called *orthogonality relations.*

Theorem 3.2 (Orthogonality relations)**.** *For all $\alpha \in \hat{G}$, let $\varphi^\alpha \colon G \to \mathrm{U}(V_\alpha)$ be unitary representations and $d_\alpha = \dim V_\alpha$. Then*

$$\langle \varphi^\alpha_{ij}, \varphi^\beta_{kl} \rangle = \frac{1}{d_\alpha} \delta_{\alpha,\beta} \delta_{i,k} \delta_{j,l} \,. \tag{3.16}$$

Using these relations, we can deduce

$$\left(\frac{d_\alpha}{|G|} \varphi^\alpha_{ij} \cdot \frac{d_\beta}{|G|} \varphi^\beta_{kl} \right)(g) = \frac{d_\alpha}{|G|} \delta_{\alpha,\beta} \delta_{j,k} \varphi^\alpha_{il} \,. \tag{3.17}$$

Furthermore, the group algebra gives rise to a representation:

Definition 3.8 (Regular representation)**.** The representation of G given by multiplication from the left on $\mathcal{A}(G)$ is called the *regular representation.*

Using the orthogonality relations and characters of representations (see chapter 4 of [6]), one can show that the irreducible representations correspond to conjugacy classes of G and that the regular representation is a direct sum of *all* irreducible representations.

Theorem 3.3. *Let V_α be the irreducible representations of G and $d_\alpha = \dim V_\alpha$. Then*

$$\mathcal{A}(G) \cong \bigoplus_\alpha V_\alpha^{\oplus d_\alpha} \,. \tag{3.18}$$

The isomorphism implies that $|G| = \dim \mathcal{A}(G) = \sum_\alpha d_\alpha^2$.

The orthogonality relations now tell us that the functions φ^α_{ij} form an orthogonal basis of $\mathcal{A}(G)$, and further, using (3.17), we find that

$$\mathcal{A}(G) \cong \bigoplus_\alpha \mathrm{End}(\mathbb{C}^{d_\alpha}) \,. \tag{3.19}$$

Here, the isomorphism is an an algebra isomorphism and the multiplication on the right side is component-wise matrix multiplication (cf. [4], proof of theorem 1.3).

3.2.2. Representation theory of the symmetric group

We now present some results that will allow us to work with the irreducible representations of the symmetric group. This subsection is mainly based on chapter 1 of [4]. Our goal will be to construct a *minimal* projection in the group algebra associated with each irreducible representation.

Definition 3.9 (Minimal projection)**.** A projection p is *minimal* iff it can not be written as $p = q + r$ with projections $q, r \neq 0$.

The irreducible representations of the symmetric group S_n can be indexed by partitions of n into a non-increasing sequence of integers $\lambda = (n_1, \ldots, n_k)$. These partitions are usually represented by so-called *Young diagrams* which are just collections of empty boxes arranged in rows such that the number of boxes in row i is n_i.

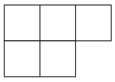


Figure 3.1.: The Young diagram corresponding to the partition $(3, 2)$

A Young diagram λ becomes a *Young tableau* T_λ by filling in the boxes with numbers. A *standard* Young tableau is a diagram filled with the numbers $\{1, \ldots, n\}$ such that each number is used exactly once and numbers are increasing when moving to the right in a row or down in a column. In a *semistandard* Young tableau we are allowed to use numbers repeatedly and they may only weakly increase in each row, but must strictly increase in each column.

1	2	3
4	5	

(a) A standard tableau

1	1	2
2	2	

(b) A semistandard tableau

The symmetric group acts on standard tableaus in the obvious way. Given such a tableau T, we can define two subgroups of the symmetric group:

Definition 3.10 (Row- and column stabilizer)**.** The *row stabilizer* $\mathcal{R}(T)$ of the Young tableau T is the set of permutations that leave the set of numbers in each row invariant. The *column stabilizer* $\mathcal{C}(T)$ is analogously defined for the columns of T.

Using these subgroups, we now construct two elements of the group algebra:

$$c_T = \sum_{\sigma \in \mathcal{C}(T)} \operatorname{sgn}(\sigma)\sigma \,, \qquad r_T = \sum_{\sigma \in \mathcal{R}(T)} \sigma \,. \tag{3.20}$$

Now, we can define the object we were looking for:

Definition 3.11 (Young symmetrizer)**.** The *Young symmetrizer* e_T corresponding to the standard tableau T is given by

$$e_T = r_T c_T = \sum_{\substack{\sigma \in \mathcal{R}(T) \\ \sigma' \in \mathcal{C}(T)}} \operatorname{sgn}(\sigma')\sigma\sigma' \,. \tag{3.21}$$

Theorem 3.4. *The Young symmetrizer e_{T_λ} is proportional to a minimal projection associated with the irreducible representation V_λ of S_n. (see [4], Theorem 1.14)*

We will never have to worry about the proportionality constant and will pretend from now on that the symmetrizer is itself a projection. As a consequence of minimality, the symmetrizer e_{T_λ} is only non-zero in one component of $\mathcal{A}(S_n) \cong \bigoplus_{\lambda'} \mathrm{End}(\mathbb{C}^{d_{\lambda'}})$, namely $\mathrm{End}(\mathbb{C}^{d_\lambda})$. We will now proof that the symmetrizer acts as a minimal projection on the irreducible representation V_λ.

Theorem 3.5. *The irreducible representation $\varphi^\lambda\colon \mathcal{A}(S_n) \to \mathrm{End}(V_\lambda)$ restricted to $\mathrm{End}(\mathbb{C}^{d_\lambda})$ is an isomorphism.*

Proof. As we know from Theorem 3.3, the functions $\varphi^{\lambda'}_{kl}$ span $\mathcal{A}(S_n)$ and so do their complex conjugates $\overline{\varphi^{\lambda'}_{kl}}$. When we apply the representation to these group elements and look at the components of the matrices in $\mathrm{End}(V_\lambda)$, we find

$$\begin{aligned} \varphi^{\lambda'}_{ij}\left(\overline{\varphi^{\lambda'}_{kl}}\right) &= \sum_{g\in G} \overline{\varphi^{\lambda'}_{kl}}(g)\varphi^{\lambda'}_{ij}(g) \\ &= G\langle \varphi^{\lambda'}_{kl}, \varphi^{\lambda}_{ij}\rangle = \frac{G}{d_\alpha}\delta_{\lambda,\lambda'}\delta_{i,k}\delta_{j,l} \end{aligned} \tag{3.22}$$

using the orthogonality relations. Thus, the elements $\overline{\varphi^{\lambda'}_{kl}}$ with $\lambda' \neq \lambda$ are mapped to zero and the elements $\overline{\varphi^{\lambda}_{ij}}$ are each mappped to a different matrix in $\mathrm{End}(V_\lambda)$ with only one non-zero entry. Hence, $\mathrm{End}(\mathbb{C}^{d_\lambda}) \cong \mathrm{End}(V_\lambda)$. □

Using this result, it is immediately clear that $\varphi^\lambda(e_{T_\lambda})$ is a minimal projection in $\mathrm{End}(V_\lambda)$. The Young symmetrizer is generally not an orthogonal projection and thus not Hermitian, but the Hermitian conjugate $e_T^\dagger = c_T r_T$ projects onto an isomorphic subspace [7]. As matrices, minimal projections are easily understood to be of rank one and can thus be written in the form $|\psi\rangle\langle\psi'|$ (see [4], proof of theorem 1.3).

Schur-Weyl duality. *We have*

$$\mathcal{H}^{\otimes n} \cong \bigoplus_\lambda U_\lambda \otimes V_\lambda\,, \tag{3.23}$$

where the sum is over Young diagrams, U_λ are irreducible representations of the unitary group $U(d)$ and V_λ are irreducible representations of the symmetric group S_n.

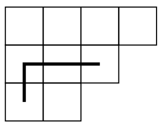
The decomposition into terms $U_\lambda \otimes V_\lambda$ is familiar from (3.13), but this theorem gives a new interpretation to the spaces U_λ. We will not need this result itself, but the following formulae that are based on it.

Theorem 3.6 (Dimension formulae)**.**

$$\dim V_\lambda = \left|\left\{T \mid T \textit{ standard tableau for Young diagram } \lambda\right\}\right| \tag{3.24}$$

$$\dim U_\lambda = \left|\left\{T \mid T \textit{ semistandard tableau for Young diagram } \lambda \right.\right. \tag{3.25}$$
$$\left.\left. \textit{with numbers in } \{1,\dots,d\}\right\}\right|$$

It is important to note here that the dimension of U_λ depends on $d = \dim \mathcal{H}$. Furthermore, diagrams λ with a number of rows greater than d always have $\dim U_\lambda = 0$ and thus do not play a role in the decomposition (3.23). To determine the dimensions of the irreducible representations, the number of the (semi-)standard tableaux can be conveniently calculated using the so-called hook formulae. The hook lenght $h(i,j)$ of box (i,j) in a Young diagram is the number of boxes to its right plus the number of boxes below from it, including the box itself.



(a) Hook of box $(1,2)$

6	5	3	1
4	3	1	
2	1		

(b) Hook lengths for all boxes

Theorem 3.7 (Hook length formulae)**.**

$$\dim V_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)} \tag{3.26}$$

$$\dim U_\lambda = \prod_{(i,j)\in\lambda} \frac{d+j-i}{h(i,j)} \tag{3.27}$$

3.2.3. Application to symmetric states

We now apply what we have learned so far to the task of calculating the spectrum of a symmetric mixed state ρ as in section 1.3. The representation of the symmetric group S_n defined in (1.10), also called the *natural representation*, will play a major role here.

As the operator ρ commutes with each permutation (see (1.17)), it is a S_n-linear map in the sense of definition 3.2. While using the decompostion $\mathcal{H}^{\otimes n} \cong \bigoplus_\lambda U_\lambda \otimes V_\lambda$, we restrict ρ to one of the (irreducible) representations $e_i \otimes V_\lambda$ and project onto another representation $e_j \otimes V_{\lambda'}$. The resulting map $\rho_{i,j}^{\lambda,\lambda'} : e_i \otimes V_\lambda \to e_j \otimes V_{\lambda'}$ is a S_n-linear map between irreducible representations, and so Schur's lemma applies. Thus, $\rho_{i,j}^{\lambda,\lambda'} : e_i \otimes V_\lambda \to e_j \otimes V_{\lambda'}$ is a multiple of identity if $\lambda = \lambda'$ and zero otherwise. It follows that $\rho_{i,j}^{\lambda,\lambda}$ maps

$$e_i \otimes v \mapsto e_j \otimes (a_{i,j} v) = (a_{i,j} e_j \otimes v)$$

and ρ decomposes as

$$\rho \cong \bigoplus_{\lambda} \left(A^{\lambda} \otimes \mathbb{1} \right). \tag{3.28}$$

We will now proceed to find a basis of each U_λ using the Young symmetrizer. For each diagram λ, we choose to work with the *ordered* standard tableau T_λ and define $e_\lambda = e_{T_\lambda}$. As the Young symmetrizer is a rank one projection on the irreducible representations in the group algebra and the natural representation itself is a S_n-linear map, Schur's lemma implies that the symmetrizer acts analogously on the irreducible representations in the decomposition of $\mathcal{H}^{\otimes n}$. Hence we have

$$e_\lambda \cong \bigoplus_{\lambda'} e_{\lambda,\lambda'}, \quad \text{with} \quad e_{\lambda,\lambda'} = \begin{cases} \mathbb{1} \otimes |\psi_\lambda\rangle\langle\psi'_\lambda| & \text{if } \lambda = \lambda' \\ 0 & \text{if } \lambda \neq \lambda' \end{cases} \tag{3.29}$$

where we think of e_λ as an operator on $\mathcal{H}^{\otimes n}$. We can thus find states in $U_\lambda \otimes |\psi_\lambda\rangle$ by applying the symmetrizer to arbitrary states in $\mathcal{H}^{\otimes n}$. When applying the symmetrizer to computational basis states $|x\rangle = |x_1, \ldots, x_n\rangle$, we note the following two points:

- If two indices i, j are in the same column of the ordered standard tableau T_λ and $x_i = x_j$, then $e_\lambda |x\rangle = 0$.[3]
- If $|x'\rangle = \sigma |x\rangle$ for $\sigma \in \mathcal{C}(T)$, then $e_\lambda |x'\rangle = \operatorname{sgn}(\sigma) e_\lambda |x\rangle$.

Consequently, when constructing a basis for U_λ, it would be redundant to use states other than those corresponding to the semistandard tableaux for the diagram λ. On the other hand, the number of these states is just the dimension of the space (see (3.25)), so we must have found a basis.

We can now compute the spectrum of a symmetric operator in the following manner:

1. For each diagram λ, apply the Young symmetrizer e_λ to the computational basis states corresponding to the semistandard tableaux for the diagram to construct a basis of $U_\lambda \otimes |\psi_\lambda\rangle$, $\left\{ \left| \varphi_i^\lambda \right\rangle \right\}$.
2. Use the Gram-Schmid procedure to find an orthonormal basis $\left\{ \left| \psi_i^\lambda \right\rangle \right\}$.
3. Calculate the matrix elements $A_{i,j}^\lambda = \left\langle \psi_i^\lambda \middle| \rho \middle| \psi_j^\lambda \right\rangle$.
4. Determine the spectrum of A^λ using some numerical method.
5. Multiply the degeneracies of the eigenvalues by $\dim V_\lambda$.

[3] This is a more general form of the Pauli exclusion principle.

Implementation

It is convenient to symmetrize computational basis states, as this yields a basis consisting of *sparse* vectors. Applying the Gram-Schmid procedure reduces this sparsity. We avoid this by specifying the orthonormal basis $\left\{\left|\psi_i^\lambda\right\rangle\right\}$ by its coefficients in the basis $\left\{\left|\varphi_i^\lambda\right\rangle\right\}$. These coefficients can be derived using scalar products $\left\langle\psi_i^\lambda\middle|\psi_j^\lambda\right\rangle$ as in the Gram-Schmidt procedure.

Summary

Using the method described in this chapter, it is possible to compute the complete spectrum of symmetric operators. Computing the basis states for the irreducible representations involves as many as $n!$ steps for each representation, as we need to evaluate the Young symmetrizer. This might be improved upon by describing these states as sums over block-symmetric states and not explicitely evaluating the sum over the row stabilizer.

Additionally, we know that for the dimensions of the irreducible subspaces we have $n! = \sum_\alpha d_\alpha^2$. Thus, the time to diagonalize the operator in the appropriate subspaces must scale exponentially.

Nevertheless, for small n, this method is much faster then the method of 3.1.

4. Example: Quantum capacity of the depolarizing channel

In this chapter, we will use the methods developed so far to investigate the quantum capacity of the depolarizing channel and try to improve upon the numerical results given in section 24.8 of [2].

Applying the depolarizing channel to a single copy of the state $|\phi_{AA'}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the coherent information is given by (see [2], eq. 24.159)

$$\mathcal{I}\left(\rho_{AB}\right) = 1 + \left(1 - \frac{3p}{4}\right)\log_2\left(1 - \frac{3p}{4}\right) + \frac{3}{4}p\log_2\left(\frac{p}{4}\right) \tag{4.1}$$

where $\rho_{AB} = (\mathcal{N} \otimes \mathbb{1})(|\phi_{AA'}\rangle\langle\phi_{AA'}|)$. We will next consider n-qubit repetition codes for $n \in \{3, 5, 7\}$. Thus, the input to our channel will be the state

$$|\phi^{(n)}_{AA'}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle^{\otimes n} + |11\rangle^{\otimes n}\right). \tag{4.2}$$

We compute the output state of the channel as described in section 2.2 and the entropies using the methods of section 3.2. The resulting regularized coherent information $\mathcal{I}(\rho^{(n)}_{AB})/n$ is displayed in Fig. 4.1 for $p \in [0.245, 0.254]$. We can clearly see the superadditivity of the coherent information: For $p \approx 0.2524$, the single-copy rate (4.1) becomes negative, but the rate for the repetition codes is still positive.[1]

The five-qubit repetition code remains positive for slightly larger p than the three-qubit code (for $p \in [0.2535, 0.2538]$, approximately). Interestingly, the seven-qubit code is always worse than the five-qubit code and becomes negative for smaller p ($p \approx 0.25365$).

We also considered input states of the form

$$\begin{aligned}|\psi^{(n)}_{AA'}\rangle \sim a_0\,|00\rangle^{\otimes n} + a_1\,|11\rangle^{\otimes n} + a_2\,|x+,x+\rangle^{\otimes n} + a_3\,|x-,x-\rangle^{\otimes n} \\ + a_4\,|y+,y+\rangle^{\otimes n} + a_5\,|y-,y-\rangle^{\otimes n}\end{aligned} \tag{4.3}$$

and tried to optimize the resulting regularized coherent information over the parameters a_i. Here, a litte care must be taken to normalize the state $|\psi^{(n)}_{AA'}\rangle$ correctly. The results of

[1] Superadditivity is also shown for $n = 5$ in Fig. 24.4 of [2], but it already happens for $n = 3$.

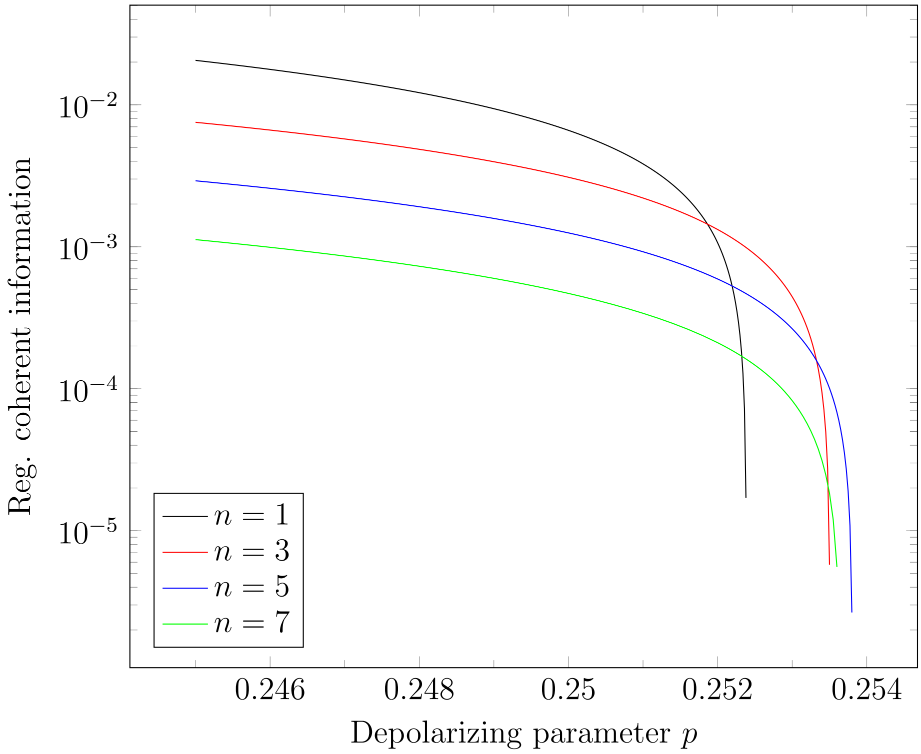


Figure 4.1.: Regularized coherent information of n-qubit repetition codes $|\phi^{(n)}_{AA'}\rangle$.

the numerical optimization of the parameters for $n = 5$ and $n = 7$ are shown in Fig. 4.2. The optimization is difficult and unreliable, even when multiple random inital values for the parameters are tried.[2] For both values of n we observe a similar behavior: For smaller values of p, the states $|\phi^{(n)}_{AA'}\rangle$ are slightly surpassed, but for larger p the same state (up to unitary transformation) seems to be found, even if it results in a lower value of the coherent information. For even higher values of p, often only negative values for coherent information are achieved, resulting in omitted data points. Finally, the optimization reliably finds the states $|\phi^{(n)}_{AA'}\rangle$ (again up to unitary transformation) and achieves the same values of the coherent information. It is especially disappointing that in the region $p > 0.2524$ we can not improve upon the repetition codes and the value of p where the coherent information becomes zero can not be increased.

For $n = 5$, we additionally tried optimizing over input states of the form $\left|\varphi^{(n)}\right\rangle = \sum_{i=1}^{r} a_i \left|\varphi_i\right\rangle^{\otimes n}$ with randomly chosen states $|\varphi_i\rangle$ and $r \in \{6, 10\}$ (see Fig. 4.3). Here, the optimized states only improve upon the five-qubit repetition code while it is worse than the single-copy rate, but only negative values are achieved for larger p.

[2]We tried 100 initial values for $n = 5$ and 20 initial values for $n = 7$.

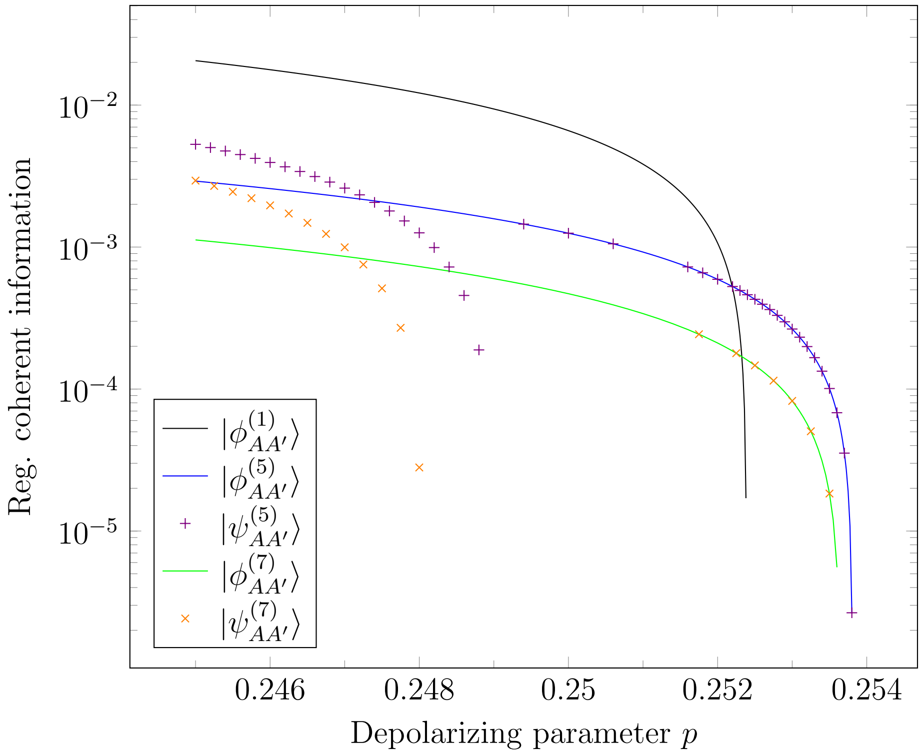


Figure 4.2.: Reg. coherent information of optimized states $|\psi^{(n)}_{AA'}\rangle$ compared to $|\phi^{(n)}_{AA'}\rangle$.

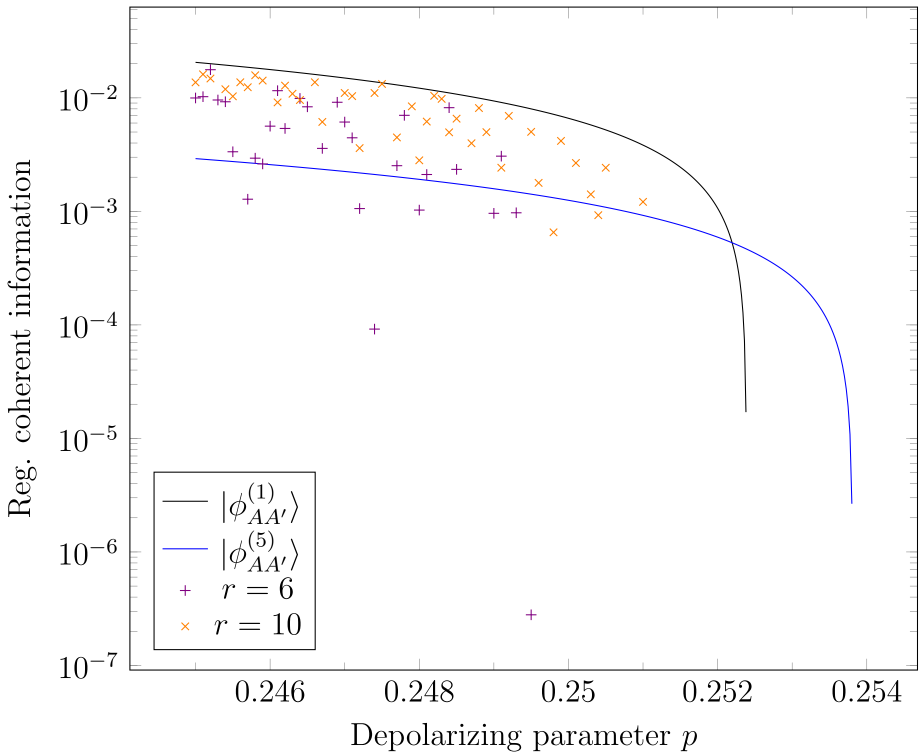


Figure 4.3.: Reg. coherent information of optimized states $|\varphi^{(n)}_{AA'}\rangle$ compared to $|\phi^{(n)}_{AA'}\rangle$.

5. Summary and outlook

Our goal in this work was to develop methods to compute lower bounds of quantum capacities for arbitrary quantum channels by evaluating coherent information on symmetric states. To this end, we described two methods to evaluate quantum channels on symmetric states and two methods to compute the complete spectrum, and thereby the entropy, of symmetric states.

In section 1.3, we explained the occupation-number representation of symmetric pure and mixed states. In section 2.1, we used this representation to evaluate arbitrary channels, specified by their Kraus operators, on symmetric states.

A different method, using product states, was introduced in section 2.2. Here, it remains an open question how to find a basis of product states for the space of all symmetric states.

In chapter 3, we first considered approximating the von Neumann entropy of symmetric mixed states. A precise determination of the entropy requires the complete spectrum, which we discussed in the remainder of the chapter.

In section 3.1, we used block-symmetric states to determine all eigenvalues of symmetric operators and found that the degeneracies of the eigenvalues can be determined by calculating $\mathrm{Tr}(\mathbb{1}-\rho)^j$ for multiple j using repeated squaring of $(\mathbb{1}-\rho)$.

An alternative approach, using representation theory, was the subject of section 3.2.

We implemented all the methods described in this work using the Python programming language. Numerical diagonalization and optimization were done using the standard tools of the NumPy and SciPy libraries. Finally, we applied our software to the depolarizing channel, with the results given in chapter 4. Additionally, the software was used to create the data discussed in section 3.1.3.

An analysis of the running time of the various methods was not done in depth, as it is heavily dependent on the implementation. Generally, using product states is faster than using the occupation number representation, and the representation theory approach is superior to our approach of section 3.1.

In future work, the software implementations should be optimized and the open problem of the systematic use of product states should be investigated.

A. Example: Decomposition of $\left(\mathbb{C}^2\right)^{\otimes 3}$

To illustrate Schur-Weyl duality and the Young symmetrizer, typically the symmetric and antisymmetric cases are mentioned. Here, we describe how the state space for three qubits, $\mathcal{H}^{\otimes 3} = \left(\mathbb{C}^2\right)^{\otimes 3}$, decomposes into irreducible representations as a minimal example that includes a further case. We will denote the natural representation as $\varphi\colon S_3 \to \mathrm{GL}\left(\mathcal{H}^{\otimes 3}\right)$.

In this case, only two diagrams contribute to the decomposition and we will use the following standard tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

(a) $\lambda_1 = (3)$ (b) $\lambda_2 = (2,1)$

In the first case, the possible semistandard tableaux, using the digits $\{0,1\}$, are

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$$

and the Young symmetrizer e_{λ_1} is just the symmetrizer P well-known from section 1.3. Thus, the four semistandard tableaux describe just the four symmetric states

$$\begin{aligned} |(3,0)\rangle &= |000\rangle & |(2,1)\rangle &= |001\rangle + |010\rangle + |100\rangle \\ |(1,2)\rangle &= |110\rangle + |101\rangle + |011\rangle & |(0,3)\rangle &= |111\rangle \end{aligned}$$

As the group action leaves all these states unchanged, the group representation of S_3 is just the trivial one, in agreement with $\dim V_{\lambda_1} = 1$ and $\dim U_{\lambda_1} = 4$.

The second case is more interesting. Here, the semistandard tableaux are

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

and the Young symmetrizer is given by

$$\begin{aligned} e_{\lambda_2} &= (\mathbb{1} + (12))(\mathbb{1} - (13)) \\ &= \mathbb{1} + (12) - (13) - (321)\,. \end{aligned} \tag{A.1}$$

This is an example of a non-orthogonal projection, as

$$\begin{aligned} e_{\lambda_2}^\dagger &= (\mathbb{1} - (13))(\mathbb{1} + (12)) \\ &= \mathbb{1} + (12) - (13) - (123) \neq e_{\lambda_2} \,. \end{aligned} \tag{A.2}$$

The projection acts on the states $\{|100\rangle\,, |010\rangle\,, |001\rangle\}$ as

$$\varphi(e_{\lambda_2}) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \tag{A.3}$$

and analogously in the complementary basis $\{|011\rangle\,, |101\rangle\,, |110\rangle\}$. Proceeding as described in section 3.2.3, we find

$$e_{\lambda_2} |001\rangle = 2\,|001\rangle - |100\rangle - |010\rangle \equiv |A\rangle \ , \tag{A.4}$$

$$e_{\lambda_2} |011\rangle = |011\rangle + |101\rangle - 2\,|110\rangle \equiv |C\rangle \ . \tag{A.5}$$

Defining $|B\rangle \equiv (13)\,|A\rangle$ and $|D\rangle \equiv (13)\,|C\rangle$, we find that in the $\{|A\rangle\,, |B\rangle\}$ / $\{|C\rangle\,, |D\rangle\}$ basis the group elements of S_3 are represented by

$$\varphi(\mathbb{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \varphi((13)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \varphi((12)) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi((23)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \varphi((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \varphi((321)) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

We can thus conclude that we have found two isomorphic two-dimensional representations, in agreement with $\dim V_{\lambda_2} = 2$ and $\dim U_{\lambda_2} = 2$. We have also found a basis that matches the decomposition of $(\mathbb{C}^2)^{\otimes 3}$.

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