第一题

☆线性多步法 ①形式: $\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ （化为标准形, 从 u_n, f_n 开始).

②解方程 $C_p = 0$ $C_{p+1} \neq 0$ ⇒ 整体误差 $E_n = O(h^p)$

$$
\begin{cases}
C_0 = \sum_{j=0}^{k} \alpha_j \\
C_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \\
C_p = \frac{1}{p!}\sum_{j=0}^{k} j^p \alpha_j - \frac{1}{(p-1)!}\sum_{j=0}^{k} j^{p-1}\beta_j
\end{cases}
$$

⇒ 局部截断误差 $R_{n,k} = C_{p+1} h^{p+1} u^{(p+1)}(t_n) + O(h^{p+2})$

③稳定性 第一特征 $\rho(\lambda) = \sum_{j=0}^{k} \alpha_j \lambda^j$ 第二 $\sigma(\lambda) = \sum_{j=0}^{k} \beta_j \lambda^j$

$\rho(\lambda)$ 满足根条件: $|\lambda| \le 1$

④相容性: $C_0 = 0$ $C_1 = 0$ ⇔ 至少一阶.

第二题

等距 ⇔ 二阶 $h_i = h$

非等距 ⇔ 一阶. $h_i = x_i - x_{i-1}$

☆两点边值问题 ①直接差分化: step 1. 得基本公式

$$u(x_{i+1}) = u(x_i) + h_{i+1} u'(x_i) + \frac{h_{i+1}^2}{2} u''(x_i) + \frac{h_{i+1}^3}{6} u'''(x_i) + O(h^4)$$

$$u(x_{i-1}) = u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2} u''(x_i) + \frac{h_i^3}{6} u'''(x_i) + O(h^4).$$

$$\therefore \boxed{\frac{u(x_{i+1}) - u(x_{i-1})}{h_{i+1} + h_i} = u'(x_i) + \frac{h_{i+1} - h_i}{2} u''(x_i) + O(h^2)}$$

☆若题上无"p" 可直接代入了 $qu \to q_i u(x_i)$ $f \to f_i$

step 2. 有 $\frac{d}{dx}[p\frac{du}{dx}]$ Taylor 之或+基本公式.

又似乎不用这个 反正就是求格式

$$u(x_i) = u(x_{i-\frac12}) + \frac{h_i}{2} u'(x_{i-\frac12}) + \frac12 \cdot \frac{h_i^2}{4} u''(x_{i-\frac12}) + \frac16 \cdot \frac{h_i^3}{8} u'''(x_{i-\frac12}) + O(h^4)$$

$$u(x_{i-1}) = u(x_{i-\frac12}) - \frac{h_i}{2} u'(x_{i-\frac12}) + \frac{h_i^2}{8} u''(x_{i-\frac12}) - \frac16 \cdot \frac{h_i^3}{8} u'''(x_{i-\frac12}) + O(h^4)$$

$$\Rightarrow u(x_i) - u(x_{i-1}) = h_i u'(x_{i-\frac12}) + \frac{h_i^3}{24} u'''(x_{i-\frac12}) + O(h^4).$$

$$\Rightarrow p(x_{i-\frac12}) \frac{u(x_i) - u(x_{i-1})}{h_i} = p u'(x_{i-\frac12}) + \frac{h_i^2}{24} u'''(x_{i-\frac12}) \cdot p + O(h^3).$$

$$= \left[p\frac{du}{dx}\right]_{i-\frac12} + \frac{h_i^2}{24}\left[p\frac{d^3u}{dx^3}\right]_{i-\frac12} + O(h^3)$$

i

再得 $p(x_{i+\frac12})$ 相减 $= \frac{2}{h_i + h_{i+1}}\left(\left[p\frac{du}{dx}\right]_{i+\frac12} - \left[p\frac{du}{dx}\right]_{i-\frac12}\right) + \triangle$ 一致代入. ✓

用基本公式 得 $\frac{d}{dx}\left[p\frac{du}{dx}\right]_i$

② 有限体积法 = 分部分离散化 + 五个公式.

Step1. 对方程两端在 $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ 积分

Step2. 令 $W(x) = p(x)\frac{du}{dx}$ $\therefore W(x_{i-\frac{1}{2}}) - W(x_{i+\frac{1}{2}}) = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{d}{dx}\left(p\frac{du}{dx}\right) dx$

☆ 化出来左边是离散, 右边是求导.

中矩形: $u_i - u_{i-1} = \int_{x_{i-1}}^{x_i} \frac{W(x)}{p(x)} dx = \frac{W_{i-\frac{1}{2}}}{p_{i-\frac{1}{2}}} h_i + O(h_i^3)$

导出 $W_{i-\frac{1}{2}}$, $W_{i+\frac{1}{2}}$

局部截断就是两边都写原形式, 右边去掉 Ln, 剩下的.

Step3. 令 $V(x) = r(x)\frac{du}{dx}$

对 $\int_{x_{i-1}}^{x_i} \frac{V(x)}{r(x)} dx = u(x_i) - u(x_{i-1})$ 右矩形 $= \frac{V_i}{r_i} h_i + O(h_i^2)$.

对 $\int_{x_i}^{x_{i+1}} \frac{V(x)}{r(x)} dx = u(x_{i+1}) - u(x_i)$ 左矩形 $= \frac{V_i}{r_i} h_{i+1} + O(h_{i+1}^2)$

相加. $V_i \Rightarrow \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} V(x) dx = V_i \cdot \frac{1}{2}(h_i + h_{i+1}) + O(h^2) = \frac{r_i}{2}(u_{i+1} - u_{i-1}) +$ 不想有二一步

全部代入就可以啦, Yeah!

No.

Date

第三题

☆一维热传导 + 各种格式 ① 形式：$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x)$ ，$0<t<T$ ，$a>0$ ，$f(x) \in C(-\infty,+\infty)$

② 向前差分格式：$\frac{u_j^{n+1}-u_j^n}{\tau} = a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + f_j^n$

背！☆

$$\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\tau} = \frac{\partial u(x_j,t_n)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j,t_n)}{\partial t^2} + O(\tau^2)$$

$$\frac{u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)}{h^2} = \frac{\partial^2 u(x_j,t_n)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_n)}{\partial x^4} + O(h^3)$$

$$= \frac{\partial^2 u}{\partial x^2} + O(h^2)$$

代入原方程，用代入的右边减 Lu.

☆ $R_j^n(u) = L_h u(x_j,t_n) - [Lu]_j^n$

$$= \frac{\tau}{2}\frac{\partial^2 u(x_j,t_n)}{\partial t^2} - \frac{ah^2}{12}\frac{\partial^4 u(x_j,t_n)}{\partial x^4} + O(\tau^2) + O(h^3).$$

$$= O(\tau + h^2).$$

③ $\frac{u_j^{n+1}-u_j^n}{\tau} = a\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + f_j^n$. 向后差分格式

还是基本分析，出现了 t_{n+1} 但依法减 ∴ 在 t_n 处 taylor 展开.

$$\frac{u(x_{j+1},t_{n+1}) - 2u(x_j,t_{n+1}) + u(x_{j-1},t_{n+1})}{h^2} = \frac{\partial^2 u(x_j,t_{n+1})}{\partial x^2} + \frac{h}{12}\frac{\partial^4 u(x_j,t_{n+1})}{\partial x^4} + O(h^3)$$

$$= \frac{\partial^2 u(x_j,t_n)}{\partial x^2} + \tau\frac{\partial}{\partial t}\left(\frac{\partial^2 u(x_j,t_n)}{\partial x^2}\right) + O(\tau^2) + \Delta + O(\tau h^2 + h^3).$$

再用 $a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - f$ 化简.

☆ ∴ $R_j^n(u) = -\frac{\tau}{2}\frac{\partial^2 u(x_j,t_n)}{\partial t^2} - \frac{\tau}{12r}\frac{\partial^2 u(x_j,t_n)}{\partial t^2} + O(\tau^2 + h^2)$

$$= O(\tau + h^2).$$

④ C-N 格式：$\frac{u_j^{n+1}-u_j^n}{\tau} = \frac{a}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right] + f_j^n$

看不懂！

老老实实默写吧

:)

$$\frac{u(x_{j+1},t_{n+1})-2u(x_j,t_{n+1})+u(x_{j-1},t_{n+1})}{h^2}=\frac{\partial^2 u(x_j,t_n)}{\partial x^2}+\tau\frac{\partial}{\partial t}\left(\frac{\partial^2 u(x_j,t_n)}{\partial x^2}\right)+O(\tau^2+h^2).$$

$R_j^n(u)=O(\tau^2+h^2).$

⑤ **Richardson 格式.** $\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}=a\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+f_j^n$

$R_j^n(u)=O(\tau^2+h^2)$ 真的考这个我就完蛋！555

$$L_h^{n+1}u_j^n=\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}-a\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}$$

⑥ **稳定性：** 1. $\rho(C)$ 为 $C(\tau)$ 的谱半径. $\rho(C)\le 1+M\tau$ $(\rho(C)\le 1+O(\tau))$

2. ① 令 $r=\frac{a\tau}{h^2}$ $\alpha=\frac{2p\pi}{L}$. 代入.

② 以 $u_j^n\sim V^n e^{i\alpha x_j}$ 代入得：$V^{n+1}=GV^n$

$|G|\le 1$ 稳定.

第四题.

☆ 迎风格式求双层.

① **格式**：还是前面那些基本公式 + 不是 $u_{j,n}$ 就在 $u_{j,n}$ 求导，哪个不一样求哪个.

$$u(x_j,t_{n+1})=u(x_j,t_n)+\tau\frac{\partial}{\partial t}u(x_j,x t_n)+\frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}u(x_j,t_n)+O(\tau^3)$$

代入式，能化成啥样算啥样吧

$$\frac{u_j^{n+1}-2u_j^n+2u_j^{n-1}}{h^2}=u_{tt}(x_j,t_n)+\frac{\tau^2}{12}\frac{\partial^2}{\partial t^4}u(x_j,t_n)+O(\tau^4)$$

② **误差**：相减了

· 迎风：$R_j^n(u)=\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}+\frac{a_j h}{2}\frac{\partial^2 u}{\partial x^2}+O(\tau^2+h^2)=O(\tau+h)$

· 双层：$R_j^n(u)=\frac{\tau^2}{12}\frac{\partial^4}{\partial t^4}u(x_j,t_n)+O(\tau^4)-\frac{ah^2}{12}\frac{\partial^4}{\partial t^4}u(x_j,t_n)+O(h^4)$

$=O(\tau^2+h^2).$

第五题

☆ 两点边值问题的变分问题 $\left\langle \begin{array}{l} \text{极小位能} \\ \text{虚功} \end{array} \right.$

(ps: 似乎差别不大). ↓以下真的广林化过程，不会两个原理，我好可怜我公

①列式 $\left\{ \begin{array}{ll} \text{方程} & x \in \text{范围}. \\ \text{边界} & \text{(最后积分分部的待时用)} \end{array} \right.$

② 两边积分，把加号利用分部积分合起来．得到 (Lu,v)

套出公式! $J(u) = \frac{1}{2}(Lu,u) - (f,u)$.

令 $a(u,v) =$ 前一半 $\quad J(u) = \frac{1}{2}a(u,v) - (f,u)$.