波动 $u_{tt}=a^2u_{xx}+f$
热传导 $u_t=a^2u_{xx}+f$
调和 $-\Delta u=0$
(Possion $-\Delta u=f$)

大 连 理 工 大 学

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院 系:__________

_____级_____班

课 程 名 称: 数学物理方程 试卷: 线上 考试形式: 闭卷

授课院(系): 数学科学学院 考试日期: 2020年8月24日 试卷共 6 页

题 号	一	二	三	四	五	六	七	八	九	总 分
标准分	20	25	15	15	10	15	/	/	/	100
得 分							/	/	/	

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得分	

一、(20分) 求解波动方程的初值问题

$$\begin{cases} u_{tt}-u_{xx}=t\sin x, & x\in\mathbb{R},\ t>0, \\ u(x,0)=0,\ u_t(x,0)=x+\sin x, & x\in\mathbb{R}. \end{cases}$$

(I) $\begin{cases} u_{tt}-u_{xx}=0 & x\in\mathbb{R},\ t>0 \\ u(x,0)=0,\ u_t(x,0)=x+\sin x & x\in\mathbb{R} \end{cases}$

由达朗贝尔公式 $u(x,t)=\dfrac{\varphi(x-at)+\varphi(x+at)}{2}+\displaystyle\int_{x-at}^{x+at}\frac{1}{2a}\psi(\alpha)\,d\alpha$

其中 $\varphi=0$ $\psi=x+\sin x$ $a=1$

$\therefore u_1(x,t)=\dfrac{1}{2}\displaystyle\int_{x-t}^{x+t}(\alpha+\sin\alpha)\,d\alpha=\frac{1}{2}\left(\frac{1}{2}\alpha^2-\cos\alpha\right)\Big|_{x-t}^{x+t}$

$=\dfrac{1}{2}\left[\dfrac{1}{2}(x+t)^2-\cos(x+t)-\dfrac{1}{2}(x-t)^2+\cos(x-t)\right]=tx+\sin x\sin t$

(II) $\begin{cases} u_{tt}-u_{xx}=t\sin x & x\in\mathbb{R},\ t>0. \\ u(x,0)=u_t(x,0)=0 & x\in\mathbb{R} \end{cases}$

$\therefore u_2(x,t)=\displaystyle\int_0^t W(x,t;\tau)\,d\tau=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\,d\xi\,d\tau$

其中 $a=1$ $f(\xi,\tau)=\tau\sin\xi$

$\therefore u_2(x,t)=\dfrac{1}{2}\displaystyle\int_0^t\int_{x-t+\tau}^{x+t-\tau}\tau\sin\xi\,d\xi\,d\tau=\frac{1}{2}\int_0^t\tau\left[-\cos(x+t-\tau)+\cos(x-t+\tau)\right]dt$

$=\dfrac{1}{2}\displaystyle\int_0^t\tau\left[-2\sin x\sin(\tau-t)\right]d\tau=\sin x\int_0^t\tau\sin(t-\tau)\,d\tau$

$=\sin x\,(t-\sin t)$

$\therefore u(x,t)=u_1(x,t)+u_2(x,t)=tx+t\sin x$

得分	

二、 (25分) 利用分离变量法求解初边值问题

$$\begin{cases} u_t - u = u_{xx}, & 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0, & t > 0, \\ u(x,0) = \sin\frac{\pi}{l}x, & 0 \le x \le l. \end{cases}$$

分离变量法 令 $u(x,t) = X(x)T(t)$

由 $u_t - u = u_{xx}$ 得 $XT' - XT = X''T$ 即 $\frac{T'-T}{T} = \frac{X''}{X} \triangleq -\lambda$

则 $X'' + \lambda X = 0$ 且 $T' + (\lambda - 1)T = 0$

$T' - T = -\lambda T$

$T' + (\lambda-1)T$

针对 $X'' + \lambda X = 0$

当 $\lambda \le 0$ 时 只有平凡解 $X \equiv 0$

当 $\lambda > 0$ 时 $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

由 $0 = X(0) = A$

$0 = X(l) = B\sin\sqrt{\lambda}l \Rightarrow \sqrt{\lambda}l = k\pi \Rightarrow \lambda_k = \left(\frac{k\pi}{l}\right)^2 \quad k = 1, 2, \cdots$

则 $X_k(x) = B_k \sin\frac{k\pi x}{l}$

$T' + a^2\lambda T = 0 \Rightarrow T_k(t) = C_k e^{-a^2\lambda_k t}$

代入 $T' + (\lambda - 1)T = 0$ 得 $T_k(t) = C_k \cdot e^{-(\lambda_k - 1)t}$

$\Rightarrow u_k(x,t) = T_k(t)X_k(x) = A_k e^{-(\lambda_k-1)t}\sin\frac{k\pi x}{l}$

$\therefore u(x,t) = \sum_{k=1}^{\infty} A_k e^{-\left[\left(\frac{k\pi}{l}\right)^2 - 1\right]t}\sin\frac{k\pi x}{l}$

由 $u(x,0) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{l} = \sin\frac{\pi}{l}x$

$\int_0^l \sin^2\frac{\pi}{l}x\,dx = \int_0^l \frac{1-\cos\frac{2\pi}{l}x}{2}dx = \frac{l}{2}$ ($k=1$时)

$\int_0^l \sin\frac{\pi}{l}x \sin\frac{k\pi}{l}x\,dx$

$= -\frac{1}{2}\int_0^l \cos\left(\frac{\pi}{l}x + \frac{k\pi}{l}x\right) - \cos\left(\frac{\pi}{l}x - \frac{k\pi}{l}x\right) dx$

$= -\frac{1}{2}\left[\frac{l}{(k+1)\pi}\sin\frac{(k+1)\pi}{l}x\Big|_0^l - \frac{l}{(1-k)\pi}\sin\frac{(1-k)\pi}{l}x\Big|_0^l\right]$

$= 0$ ($k \ge 2$时)

(正交性)

即 $A_k = \frac{1}{M_k}\int_0^l \sin\frac{\pi}{l}x \sin\frac{k\pi}{l}x\,dx$

其中 $M_k = \int_0^l \sin^2\sqrt{\lambda_k}x\,dx = \int_0^l \frac{1-\cos 2\sqrt{\lambda_k}x}{2}dx = \frac{l}{2} - \frac{\sin 2\sqrt{\lambda_k}l}{4\sqrt{\lambda_k}} = \frac{l}{2}$

$\therefore u(x,t) = A_1 e^{-\left[\left(\frac{\pi}{l}\right)^2 - 1\right]t}\sin\frac{\pi x}{l} = e^{\left(1 - \left(\frac{\pi}{l}\right)^2\right)t}\sin\frac{\pi}{l}x$

#2.4

得分	

三、 (15分) 证明初边值问题

$$\begin{cases} u_t = a^2 u_{xx} - \lambda u + f(x,t), & (x,t) \in \Omega_T := (0,l) \times (0,T], \\ u(0,t) = 0,\ u_x(l,t) + u(l,t) = 0, & t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [0,l] \end{cases}$$

的解 $u(x,t)$ 在 $\bar{\Omega}_T$ 上满足

$$u(x,t) \le \max\left\{0,\ \sup_{x\in(0,l)} \varphi(x),\ \frac{1}{\lambda} \sup_{(x,t)\in\Omega_T} f(x,t)\right\},$$

其中 $\lambda > 0$ 为常数.

① 若 $u<0$ 则恒成立.

② 若 $u \geq 0$

则在 R_T 中某点 (x^*, t^*) 取得极大值，此时 $u_t \geq 0$, $u_{xx} \leq 0$, $u>0$

进而 $\lambda u = a^2 u_{xx} + f(x,t) - u_t \leq f(x,t) \Rightarrow u \leq \frac{1}{\lambda} f(x,t)$

则 $\sup\limits_{\Omega_T} u \leq \frac{1}{\lambda} \sup\limits_{R_T} f(x,t)$

又 极值定理 $\sup\limits_{\bar{\Omega}_T} u(x,t) \leq \max\{0, \underbrace{\sup\limits_{t\in[0,T]} \mu_1(t)}_{0}, \sup\limits_{x\in[0,l]} \varphi(x), \underbrace{\sup\limits_{t\in[0,T]} \frac{1}{\lambda}\mu_2(t)}_{0}\}$

综上 $u(x,t) \leq \max\{0, \sup\limits_{x\in(0,l)} \varphi(x), \frac{1}{\lambda} \sup\limits_{(x,t)\in\Omega_T} f(x,t)\}$ 得证

一维分部积分
二维/三维 Green公式

#15

得分	

四、(15分) 设 $u = u(x,t) \in C^2([0,l]\times[0,\infty))$ ($l>0$ 是常数) 满足

$$\begin{cases} u_{tt} = 4u_{xx}, & x\in(0,l),\ t>0, \\ u(0,t) = u_x(l,t) = 0, & t>0, \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x), & x\in[0,l]. \end{cases}$$

试证明: 对任意 $t\ge 0$, 都有 $\int_0^l u_t^2 + 4u_x^2 dx = \int_0^l \psi^2 + 4|\varphi'|^2 dx$.

能量不等式

$$E(t) = \int_0^l u_t^2 + 4u_x^2 dx$$

物理角度 没有外力 总能量守恒 故 $\frac{dE(t)}{dt} = 0$

(二维 数学角度: $\frac{dE(t)}{dt} = \int_0^l 2u_tu_{tt} + 8u_xu_{xt}dx = \int_0^l 2u_tu_{tt} + 8u_xu_t\cos(n,x) - 8u_{xx}u_tdx$

$$= \int_0^l 2u_t(\cancel{u_{tt}-4u_{xx}})^{0}dx + \int_0^l 8u_x\cancel{u_t}^{0}\cos(n,x)dx = 0 \ )$$

数学角度: $\frac{dE(t)}{dt} = \int_0^l 2u_tu_{tt} + 8u_xu_{xt}dx = \int_0^l 2u_t(\cancel{u_{tt}-4u_{xx}})^{0}dx + 8\int_0^l u_tu_{xx} + u_xu_{xt}dx$

$$= \int_0^l (u_tu_x)_x dx = u_tu_x\Big|_0^l = 0$$

$\therefore E(t)$ 与 t 无关 即 $E(t) = E(0)$

$\therefore \int_0^l u_t^2 + 4u_x^2 dx = \int_0^l \psi^2 + 4|\varphi'|^2 dx$ 得证

※ w_n ≙ 单位球体积
nw_n ≙ 单位球面积

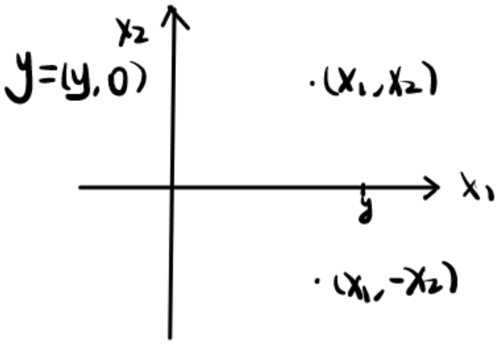
得分	

五、(10分) 求半平面

$$\mathbb{R}^2_+ = \{(x_1, x_2) : x_1 \in \mathbb{R},\ x_2 > 0\}$$

上的Green函数. [注意: 平面上调和方程的基本解为$\Gamma(x_1, x_2) = -\frac{1}{2\pi}\log\sqrt{x_1^2 + x_2^2}$, $(x_1, x_2) \neq (0, 0)$.]

由右图 $x(x_1, x_2)$ $\tilde{x} = (x_1, -x_2)$



$$\Gamma(x-y) = \Gamma(x_1-y, x_2) = -\frac{1}{2\pi}\log\sqrt{(x_1-y)^2+x_2^2}$$

$$\Gamma(\tilde{x}-y) = -\frac{1}{2\pi}\log\sqrt{(x_1-y)^2+x_2^2}$$

$$\therefore G(x,y) = \Gamma(x,y) - \Gamma(\tilde{x}-y) = -\frac{1}{2\pi}(\log|x-y| - \log|\tilde{x}-y|)$$

基本解 $\Gamma(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & n=2 \\ \frac{1}{n(n-2)w_n}|x|^{2-n} & n\geq 3 \end{cases}$

求$B_R(0)$上的Green函数

$$\frac{|x|}{R} = \frac{R}{k|x|} \Rightarrow k = \frac{R^2}{|x|^2}$$

则 $\tilde{x} = kx = \frac{R^2}{|x|^2}x$ 为点x关于$\partial B_R(0)$的反演点.

$n\geq 3$时 $\Gamma(x-y) = e\Gamma(\tilde{x}-y)$ $\quad y \in \partial B_R(0)$

$$e = \frac{\Gamma(x-y)}{\Gamma(\tilde{x}-y)} = \left(\frac{|x-y|}{|\tilde{x}-y|}\right)^{2-n} = \left(\frac{|x|}{R}\right)^{2-n}$$

$$\therefore G(x,y) = \Gamma(x-y) - \left(\frac{|x|}{R}\right)^{2-n}\Gamma(\tilde{x}-y) = \Gamma(x-y) - \Gamma\left(\frac{|x|}{R}(\tilde{x}-y)\right)$$

$$\therefore G(x,y) = \begin{cases} \frac{1}{n(n-2)w_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right) & n\geq 3 \\ -\frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right) & n=2 \end{cases}$$

得分	

六、(15分) 设 $\Omega = \{(x,y) : x \in \mathbb{R},\ y > 0\}$.

(i) 举例说明: 边值问题

$$\begin{cases} -(u_{xx}+u_{yy}) = 0, & (x,y)\in\Omega, \\ u(x,y) = 0, & (x,y)\in\partial\Omega \end{cases} \tag{$\star$}$$

的解不唯一.

(ii) 证明: 若 $u \in C^2(\Omega)\cap C(\bar{\Omega})$ 是 $(\star)$ 的解, 且满足 $\lim\limits_{\sqrt{x^2+y^2}\to\infty} u(x,y) = 0$, 则 $u \equiv 0$ 于 $\bar{\Omega}$.

(1) $\Omega = \{(x,y): x\in\mathbb{R}, y>0\}$ $\begin{cases} -\Delta u = 0 & (x,y)\in\Omega \\ u=0 & (x,y)\in\partial\Omega \end{cases}$

反例 $u=0$

$\tilde{u} = y$

反例2.0 $\Omega = \{x\in\mathbb{R}^n \mid |x|>1\}$

$u=0$

$\tilde{u} = \begin{cases} \log|x| & n=2 \\ |x|^{2-n}-1 & n\geq 3 \end{cases}$

(2) 反证 若 $u \not\equiv 0$ 于 $\bar{\Omega}$ 即 $\exists (x_0,y_0)\in\bar{\Omega}$ s.t. $u(x_0,y_0)>0$

由 $\lim\limits_{\sqrt{x^2+y^2}\to\infty} u(x,y)=0$ 知 $|u(x,y)| < \varepsilon = u(x_0,y_0)$ 于 $\partial B_R(0)$

其中 $B_R(0)$ 为上半圆.

则 $\begin{cases} -\Delta u = 0 & 于\ B_R'(0) \\ |u(x,y)| < u(x_0,y_0) & 于\ \partial B_R'(0) \\ u=0 & 于\ \partial\Omega \end{cases}$ $\Rightarrow$ $u(x_0,y_0) > u(x',y')$

其中 (x',y') 取上半圆的边界

即半圆弧与边界.

此时与最大值原理矛盾

故 $u\equiv 0$ 于 $\bar{\Omega}$ 得证

$X(0) = A = 0$

$X(\pi) = B\sin\sqrt{\lambda}\pi = 0 \Rightarrow \underline{\lambda_k = k^2}$

$\therefore X_k(x) = B_k \sin\sqrt{\lambda_k}\, x = B_k \sin kx$

代入得 $T_k(t) = C_k e^{-9\pi t}$

$T_k(t) = C_k e^{-a^2\lambda_k t}$

$= C_k e^{-9k^2 t}$

$\therefore u_k(x,t) = T_k(t) X_k(x) = A_k e^{-9\pi t} \sin kx$

$\therefore u(x,t) = \sum_{k=1}^{\infty} A_k e^{-9\pi t} \sin kx$

又 $u(x,0) = \sum_{k=1}^{\infty} A_k \sin kx = \varphi(x)$

$\therefore A_k = \frac{1}{M_k}\int_0^{\pi} \varphi(x) \sin kx\, dx$ 其中 $M_k = \int_0^{\pi} \sin^2 kx\, dx = \frac{\pi}{2}$

即 $u(x,t) = \sum_{k=1}^{\infty} \frac{2}{\pi}\int_0^{\pi} \varphi(x) \sin kx\, dx \cdot e^{-9\pi t} \cdot \sin kx$

又 $e^{-9\pi t}$ 的阶数大于有界权或 t^{α} 的阶 且无论求导 u 几次都有 $e^{-9\pi t}$ 项

故易得 $u \in C^{\infty}([0,\pi] \times [0,\infty))$

三. $x := (x_1, x_2)$ 及 $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ 设 $B = \{x \in R^2 \mid |x|^2 < 1\}$ 定义 $C(x) = \frac{1}{1-x_1^2}\sin\frac{1}{1-x_1^2}$

(1)