[AI2613 Lecture 7] Doob Martingale, Azuma-Hoeffding, McDiarmid

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1 Hoeffding's Inequality

在上一讲中我们从 Hoeffding 的缺点引出了 Martingale.

先review 一下 Hoeffding Ineq，并举例阐明

We introduced the following Hoeffding's inequality to bound the concentration for the sum of a sequence independent random variables.

Theorem 1 (Hoeffding's Inequality) *Let $X_1, \ldots, X_n$ be independent random variables where each $X_i \in [a_i, b_i]$ for certain $a_i \le b_i$ with probability 1. Let $X = \sum_{i=1}^n X_i$ and $\mu \triangleq \mathbf{E}[X] = \sum_{i=1}^n \mathbf{E}[X_i]$, then*

$$\mathbf{Pr}[|X-\mu| \ge t] \le 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i-a_i)^2}\right)$$

for all $t \ge 0$.

Before proving Theorem 1 in Section 3, we see a practical application of Hoeffding's inequality.

Example 1 (Meal Delivery) *During the quarantine of our campus, the professors deliver meals for students using their private cars or trikes. Then a practical problem is how to estimate the amount of meals on a trike conveniently (See the news).*

Assume there are n boxes of meal on the trike ($n \ge 200$ and is unknown for us). Let X_i be the weight of the i-th box of meal. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, each $X_i \in [250, 350]$ (unit: gram) and $\mu = \mathbf{E}[X_i] = 300$. Let S be the total weight of the meal boxes on the trike, that is, $S = \sum_{i=1}^n X_i$. We can weigh the meal boxes and use $\hat{n} = \frac{S}{\mu}$ as an estimator for n. Now we compute how accurate this estimator is.

Let $\delta \ge 0$ be a constant. By Hoeffding's inequality,

$$\mathbf{Pr}[|\hat{n}-n| > \delta n] = \mathbf{Pr}[|S-\mu n| > \delta\mu n] \le 2\exp\left\{-\frac{2\delta^2\mu^2n^2}{\sum_{i=1}^n (350-250)^2}\right\}. \quad (1)$$

想研究的变量是 $\hat{n}=\frac{S}{\mu}$，
但 S 才是符合 Hoeffding，故这里使用 concentration 的常用技巧.
改造随机变量.

Plugging $\mu = 300$, $\delta = 0.05$ and $n \ge 200$ into Equation (1), by direct calculation, we have

$$\mathbf{Pr}[\hat{n} \in [0.95n, 1.05n]] \ge 1 - 2.4682 \times 10^{-4}.$$

2 Concentration on Martingale

We consider the balls-in-a-bag problem. There are g green balls and r red balls in a bag. These balls are the all same except for the color. We want to estimate the ratio $\frac{r}{r+g}$ by drawing balls. There are two scenarios.

- Draw balls with replacement. Let $X_i = \mathbf{1}[\text{the } i\text{-th ball is red}]$. Let $X = \sum_{i=1}^{n} X_i$. Then clearly each $X_i \sim \mathrm{Ber}\left(\frac{r}{r+g}\right)$ and $\mathbf{E}[X] = n \cdot \frac{r}{r+b}$.

  Since all X_i's are independent, we can directly apply Hoeffding's inequality and obtain

  $$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq t] \leq 2\exp\left(-\frac{2t^2}{n}\right).$$

- Draw balls without replacement. Again we let $Y_i = \mathbf{1}[\text{the } i\text{-th ball is red}]$, then unlike the case of drawing with replacement, variables in $\{Y_i\}$ are dependent. Let $Y = \sum_{i=1}^{n} Y_i$. We first calculate $\mathbf{E}[Y]$.

  For every $i \geq 1$, $\mathbf{E}[Y_i]$ is the probability that the i-th draw is a red ball. Note that drawing without replacement is equivalent to first drawing a uniform permutation of $r+g$ balls and drawing each ball one by one in that order. Therefore, the probabilty of $Y_i = 1$ is $\frac{r\cdot(r+g-1)!}{(r+g)!} = \frac{r}{r+g}$. So we have $\mathbf{E}[Y] = n \cdot \frac{r}{r+g}$.

  However, since $\{Y_i\}$ are dependent, we cannot apply Hoeffding's inequality directly. This motivate us to generalize it by removing the requirement of independence.

X_i 的独立性是很高的要求

2.1 Azuma-Hoeffding's Inequality

Theorem 2 (Azuma-Hoeffding's Inequality) *Let $\{Z_n\}_{n\geq 0}$ is a martingale with respect to a filtration $\{\mathcal{F}_n\}$. If for every $i \geq 1$, $|Z_i - Z_{i-1}| \leq c_i$ with probability 1, then*

$$\mathbf{Pr}[|Z_n - Z_0| \geq t] \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2}\right).$$

因此发展了泛化的 Hoeffding Ineq，要求从独立性变为要求 r.v. 构成一个 martingale wrt $\{\mathcal{F}_n\}$.

★ $Z_n = Z_0 + \sum_{0}^{n-1} X_t$，则 $\{Z_n\}$ is a mtg wrt X_t

$\iff \mathbb{E}\{X_t \mid \overline{X}_{0,t-1}\} = 0$

$\iff \mathbb{E}\{Z_{n+1} \mid \overline{X}_{0,t-1}\} = Z_n$

$\iff \mathbb{E}\{Z_n\} = \mathbb{E}\{Z_0\}$

Azuma-Hoeffding's inequality generalizes Hoeffding's inequality by letting $Z_n = \sum_{i=1}^{n}(X_i - \mathbf{E}[X_i])$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

The proof of this theorem is in Section 3. The requirement of martingale in Theorem 2 seems to be even harder to satisfy than the requirement of independence. However, in many cases, we can construct a doob martingale to apply the Azuma-Hoeffding's inequality.

Definition 3 (Doob Martingale, Doob Sequence) *Let $X_1, \ldots, X_n$ be a sequence of (unnecessarily independent) random variables and $f(\overline{X}_{1,n}) = f(X_1, \ldots, X_n) \in \mathbb{R}$ be a function. For $i \geq 0$, Let $Z_i \triangleq \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i}\right]$. Then we call $\{Z_n\}_{n\geq 0}$ a Doob martingale or a Doob sequence.*

Doob Martingale 阐明了，给定 n 个 r.v. 按 Doob Martingale 方法就能构造出一个 Martingale，结合上面方法，极大地提高了实用性.

It is easy to verify that $\{Z_n\}_{n\geq 0}$ in Definition 3 is indeed a martingale w.r.t. $\{X_n\}$ by seeing

$$\mathbf{E}\left[Z_i \,\middle|\, \overline{X}_{1,i-1}\right] = \mathbf{E}\left[\mathbf{E}[f(\overline{X}_{1,n}) \mid \overline{X}_{1,i}] \,\middle|\, \overline{X}_{1,i-1}\right] = \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}\right] = Z_{i-1}.$$

大概可理解为：基于 i-1 信息看基于 1~i 信息的 r.v. 的期望，仍只有 i-1 的信息可用.

Doob Sequence 对f几乎没有要求.

Let $\mathcal{F} = \sigma(\overline{X}_{1,i})$. We can see that Z_i is $\mathcal{F}_i$ measurable by definition. Moreover, we know that $Z_0 = \mathbf{E}\left[f(\overline{X}_{1,n})\right]$ and $Z_n = f(\overline{X}_{1,n})$.

使用Doob martingale 来证明无放回抽样的变动范围.

Recall the balls-in-a-bag problem we discussed above. Define $f : \mathbb{R}^n \to \mathbb{R}$ by letting $f(y_1, y_2, \dots, y_n) = \sum_{i=1}^n y_i$. Then in the drawing without replacement scenario, $Y = \sum_{i=1}^n Y_i = f(Y_1, Y_2, \dots, Y_n)$. Now we construct the Doob martingale for f.

→ 首先建模问题，既然是求r.v. Σy_i 的变动范围，就将其构造为 Doob martingale 中$\{Z_t\}$的 Z_n

Let $Z_i = \mathbf{E}\left[f(\overline{Y}_{1,n}) \,\middle|\, \overline{Y}_{1,i}\right]$. We know that $Z_0 = \mathbf{E}\left[f(\overline{Y}_{1,n})\right] = \mathbf{E}[Y] = n \cdot \frac{r}{r+g}$ and $Z_n = f(\overline{Y}_{1,n})$. In order to apply Azuma-Hoeffding, we need to bound the *width* of the martingale $|Z_i - Z_{i-1}|$.

只需再求 martingale的 width，即 $Z_t - Z_{t-1} \in [a_i, b_i]$，就可套用 AH定理，

By definition,

$$Z_i - Z_{i-1} = \mathbf{E}\left[f(\overline{Y}_{1,n}) \,\middle|\, \overline{Y}_{1,i}\right] - \mathbf{E}\left[f(\overline{Y}_{1,n}) \,\middle|\, \overline{Y}_{1,i-1}\right].$$

意义是已取前i个球预测总红球数和取前i-1个就预测的期望结果的差距.

If we use S_i to denote the number of red balls among the first i balls, namely $S_i = \sum_{j=1}^i Y_j$, then

运用结论：不放回取

$$\mathbf{E}\left[f(\overline{Y}_{1,n}) \,\middle|\, \overline{Y}_{1,i}\right] = \mathbf{E}\left[f(\overline{Y}_{1,n}) \,\middle|\, S_i\right] = S_i + (n-i) \cdot \frac{r - S_i}{g + r - i}.$$

Therefore $S_i = S_{i-1} + Y_i$ and

球的概率期望与放回时相同

$$\begin{aligned} Z_i - Z_{i-1} &= \left(S_i + (n-i) \cdot \frac{r - S_i}{g+r-i}\right) - \left(S_{i-1} + (n-i+1) \cdot \frac{r - S_{i-1}}{g+r-i+1}\right) \\ &= \frac{g+r-n}{g+r-i}\left(Y_i + \frac{S_{i-1} - r}{g+r-i+1}\right). \end{aligned}$$

Note that $r \geq S_{i-1}$ and $g \geq (i-1) - S_{i-1}$, we have

$$Z_i - Z_{i-1} \leq \frac{g+r-n}{g+r-i}\left(1 + \frac{S_{i-1} - r}{g+r-i+1}\right) \leq \frac{g+r-n}{g+r-i} \leq 1,$$

$$Z_i - Z_{i-1} \geq \frac{g+r-n}{g+r-i}\left(\frac{S_{i-1} - r}{g+r-i+1}\right) \geq -\frac{g+r-n}{g+r-i} \geq -1.$$

⇐ 左侧就计算了该范围

Therefore $-1 \leq X_i \leq 1$ and we can apply Azuma-Hoeffding to $Z_n - Z_0$ to obtain

$$\mathbf{Pr}\left[|Y - \mathbf{E}[Y]| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2n}\right).$$

2.2 McDiarmids Inequality

The Doob sequence we used in the balls-in-a-bag example is a very powerful and general tool to obtain concentration bounds. For a model defined by n random variables $X_1, \dots, X_n$ and any quantity $f(X_1, \dots, X_n)$ that we want to estimate, we can apply the Azuma-Hoeffding inequality to the Doob sequence of f. As shown in the previous example, the quality of the bound relies on the *width* of the martingale, that is, the magnitude of $|Z_i - Z_{i-1}|$. To determine the width of each $|Z_i - Z_{i-1}|$ is relatively easy if the function f and the variables $\{X_i\}_{1 \leq i \leq n}$ enjoy certain nice properties.

由于我们经常 Doob 和 AH 连用，因此不如将 $f(X_1 \cdots X_n)$ 代替 Z_n

将 $E[f(X_1, \dots, X_n)]$ 代替 Z_0，

然而我们还需 width of martingale $\triangleq |Z_i - Z_{i-1}| \triangleq E[f \mid X_{1,i}] - E[f \mid X_{1,i-1}]$

$\in [a_i, b_i]$，若有关于 X_i 的更多条件，width 将更好算

结合以上两点，诞生了 McDiamids Ineq

Definition 4 (c-Lipschitz Function) *A function $f(x_1, \cdots, x_n)$ satisfies c-Lipschitz condition if*

$$\forall i \in [n], \forall x_1, \cdots, x_n, \forall y_i : \quad |f(x_1, \cdots, x_i, \cdots, x_n) - f(x_1, \cdots, y_i, \cdots, x_n)| \le c.$$

The McDiarmid's inequality is the application of Azuma-Hoeffding inequality to Lipschitz f and independent $\{X_i\}$.

Theorem 5 (McDiarmid's Inequality) *Let f be a function on n variables satisfying $\underbrace{\text{c-Lipschitz condition and}}_{1}$ $\underbrace{X_1, \cdots, X_n \text{ be } n \text{ independent}}_{2}$ variables. Then we have*

$$\Pr\left[\left|f(X_1, \cdots, X_n) - \mathbf{E}\left[f(X_1, \cdots, X_n)\right]\right| \ge t\right] \le 2e^{-\frac{2t^2}{nc^2}}.$$

⇐ 相比于 Doob Martingale，这里直接用c代替了 width ai、bi，原因在于 c-Lipschitz + independence 可以推出 width ≤ C 证明见左

Proof. We use f and $\{X_i\}_{i\ge1}$ to define a Doob martingale $\{Z_i\}_{i\ge1}$:

$$\forall i : Z_i = \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i}\right].$$

Then

$$Z_i - Z_{i-1} = \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i}\right] - \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}\right].$$

Next we try to determine the width of $Z_i - Z_{i-1}$. Clearly

$$Z_i - Z_{i-1} \ge \inf_x \left\{\mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}, X_i = x\right] - \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}\right]\right\},$$

and

$$Z_i - Z_{i-1} \le \sup_y \left\{\mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}, X_i = y\right] - \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}\right]\right\}.$$

The gap between the upper bound and the lower bound is

$$\sup_{x,y} \left\{\mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}, X_i = y\right] - \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \overline{X}_{1,i-1}, X_i = x\right]\right\}.$$

For every x, y and $\sigma_1, \ldots, \sigma_{i-1}$,

$$\begin{aligned}
&\mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \bigwedge_{1\le j\le i-1} X_j = \sigma_j, X_i = y\right] - \mathbf{E}\left[f(\overline{X}_{1,n}) \,\middle|\, \bigwedge_{1\le j\le i-1} X_j = \sigma_j, X_i = x\right] \\
&= \sum_{\sigma_{i+1},\ldots,\sigma_n} \left(\Pr\left[\bigwedge_{i+1\le j\le n} X_j = \sigma_j \,\middle|\, \bigwedge_{1\le j\le i-1} X_j = \sigma_j, X_i = y\right] \cdot f(\sigma_1, \ldots, \sigma_{i-1}, y, \sigma_{i+1}, \ldots, \sigma_n)\right. \\
&\quad \left. - \Pr\left[\bigwedge_{i+1\le j\le n} X_j = \sigma_j \,\middle|\, \bigwedge_{1\le j\le i-1} X_j = \sigma_j, X_i = x\right] \cdot f(\sigma_1, \ldots, \sigma_{i-1}, x, \sigma_{i+1}, \ldots, \sigma_n)\right) \\
&\overset{(\heartsuit)}{=} \sum_{\sigma_{i+1},\ldots,\sigma_n} \Pr\left[\bigwedge_{i+1\le j\le n} X_j = \sigma_j\right] \cdot \left(f(\sigma_1, \ldots, \sigma_{i-1}, y, \sigma_{i+1}, \ldots, \sigma_n) - f(\sigma_1, \ldots, \sigma_{i-1}, x, \sigma_{i+1}, \ldots, \sigma_n)\right) \\
&\overset{(\clubsuit)}{\le} c.
\end{aligned}$$

条件期望定义.

代入 Xj=σj后，j=1~i-1 的条件可去除. 见 Xi=x, Xi=y 条件可去是因为Xi独立(

c-Lipschitz

where ($\heartsuit$) uses independence of $\{X_i\}$ and ($\clubsuit$) uses the c-Lipsichitz property of f.

Applying Azuma-Hoeffding, we have

$$\mathbf{Pr}\left[|Z_n - Z_0| \geq t\right] = \mathbf{Pr}\left[\left|f\left(X_1, \cdots, X_n\right) - \mathbf{E}\left[f\left(X_1, \cdots, X_n\right)\right]\right| \geq t\right] \leq 2e^{-\frac{2t^2}{nc^2}}.$$

将到width ≤c 后代入AH公式即得证.

□

Then we examine two applications of McDiarmid's inequality.

下举两例运用Martingale的例子

Example 2 (Pattern matching) *Let $P \in \{0,1\}^k$ be a fixed string. For a random string $X \in \{0,1\}^n$, what is the expected number of occurrences of P in X?* 该例想计算n长序列中有m长子序列的次数期望

The expectation of occurrence times can be easily calculated using the linearity of expectations. We define n independent random variables $X_1, \cdots, X_n$, where X_i denotes i-th character of X. Let $Y = f(X_1, \cdots, X_n)$ be the number of occurrences of P in X. Note that there are at most $n-k+1$ occurrences of P in X, and we can enumerate the first position of each occurrence. By the linearity of expectation, we have

$$\mathbf{E}[f] = \frac{n-k+1}{2^k}. \rightarrow \frac{1}{2^k}\cdot(n-k+1)$$

想利用martingale做concentration只需以下三步.

①：将事件建模成随机过程

②：设计f使$f(X_1, \cdots X_n)$为欲证的r.v.，并算出$E[f(\cdots)]$

③ 算$|Z_i - Z_{i-1}|$，使用Doob+AH.

或证X_i独立，+ x-Lipchitz，用McDiamid

We can then use McDarmid's inequality to show that f is well-concentrated. To see this, we note that variables in $\{X_i\}$ are independent and the function f is k-Lipschitz: If we change one bit of X, the number of occurrences changes at most k.

Therefore

$$\mathbf{Pr}\left[|Z_n - Z_0| \geq t\right] = \mathbf{Pr}\left[|f - \mathbf{E}[f]| \geq t\right] \leq 2e^{-\frac{2t^2}{nk^2}}.$$

Another application of McDiarmid's Inequality is to establish the concentration of chromatic number for Erdős-Rényi random graphs $\mathcal{G}(n,p)$.

本例希望用concentration证随机图色数的集中度

Example 3 (Chromatic Number of $\mathcal{G}(n,p)$) *Recall the notation $\mathcal{G}(n,p)$ specifies a distribution over all undirected simple graphs with n vertices. In the model, each of the $\binom{n}{2}$ possible edges exists with probability p independently.* 建模

For a graph $G \sim \mathcal{G}(n,p)$, we use $\chi(G)$ to denote its chromatic number, the minimum number q so that G can be properly colored using q colors. There are different ways to represent G using random variables.

The most natural way is to introduce a variable X_e for every pair of vertices $e = \{u,v\} \subseteq V$ where $X_e = \mathbf{1}[\text{the edge } e \text{ exists in } G]$. Then $\{X_e\}$ are independent and the chromatic number can be written as a function $\chi(X_{e_1}, X_{e_2}, \ldots, X_{e_{\binom{n}{2}}})$. It is easy to see that χ is 1-Lipschitz as removing to adding one edge can only change the chromatic number by at most one. So by McDarmid's inequality, we have

建模随机过程方一

$$\mathbf{Pr}\left[|\chi - \mathbf{E}[\chi]| \geq t\right] \leq 2e^{-2t^2\boxed{\binom{n}{2}^{-1}}}.$$

However, this bound is not satisfactory as we need to set $t = \Theta(n)$ in order to upper bound the RHS by a constant.

为了得到好的bound，我们需要.

1. Lipchitz 数不大.

2. #r.v. 不大.

才能bound住范围't'

We can encode the graph G in a more efficient way while reserving the Lipschitz and the independence property. Suppose the vertex set of G is $\{v_1, \ldots, v_n\}$. We define n random variables $Y_1, \cdots, Y_n$, where Y_i encodes the edges between v_i and $\{v_1, \cdots, v_{i-1}\}$. Once $Y_1, \cdots, Y_n$ are given, the graph is known, so the chromatic number can be written as a function $\chi(Y_1, \ldots, Y_n)$. Since Y_i only involves the connections between v_i and $v_1, \cdots, v_{i-1}$, the n variables are independent.

因此设计合适的r.v. 用于martingale/McDiarmid很重要.

本例中，另一种方法为边r.v.法，特点是将随机边作为r.v. 还可以用点r.v.法，具体构造例子如左.

It is also easy to see that if Y_i changes, the chromatic number changes at most one. Hence χ is 1-Lipschitz as well. Applying McDiarmid's inequality we have

Lipchitz bound.

$$\mathbf{Pr}\left[|\chi - \mathbf{E}[\chi]| \geq t\right] \leq 2e^{-\frac{2t^2}{n}}.$$

Apply McDiamid.

In this way, we only need $t = \Theta(\sqrt{n})$ to bound the RHS.

3 *Proof*

略

3.1 *Proof of Theorem 1*

First, we prove the following Hoeffding's lemma which will be the main technical ingredient to prove the inequality.

Lemma 6 *Let X be a random variable with $\mathbf{E}[X] = 0$ and $X \in [a, b]$. Then it holds that*

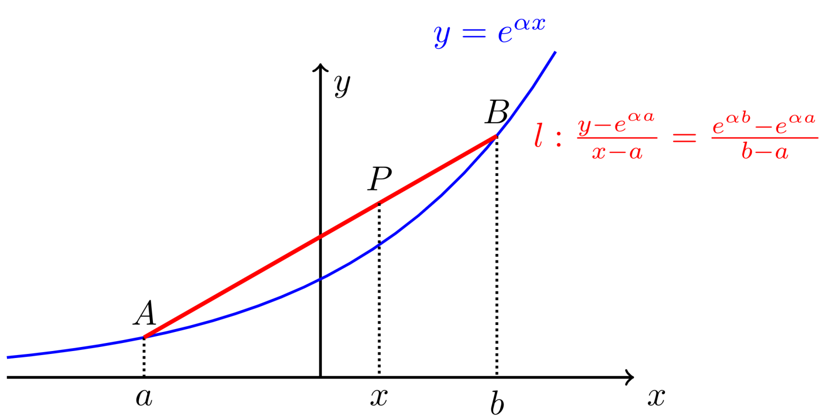
$$\mathbf{E}\left[e^{\alpha X}\right] \leq \exp\left(\frac{\alpha^2(b-a)^2}{8}\right) \quad \textit{for all } \alpha \in \mathbb{R}.$$

Proof.

We first find a linear function to upper bound $e^{\alpha x}$ so that we could apply the linearity of expectation to bound $\mathbf{E}\left[e^{\alpha X}\right]$. By the convexity of the exponential function and as illustrated in the figure below, we have

$$e^{\alpha x} \leq \frac{e^{\alpha b} - e^{\alpha a}}{b - a}(x - a) + e^{\alpha a}, \quad \text{for all } a \leq x \leq b.$$

Thus,



$$
\begin{aligned}
\mathbf{E}\left[e^{\alpha x}\right] &\le \frac{e^{\alpha b}-e^{\alpha a}}{b-a}(-a)+e^{\alpha a} = \frac{-a}{b-a}e^{\alpha b}+\frac{b}{b-a}e^{\alpha a} \\
&= e^{\alpha a}\left(\frac{b}{b-a}-\frac{a}{b-a}e^{\alpha(b-a)}\right) \\
&= e^{-\theta t}(1-\theta+\theta e^{t}) && \left(\theta = -\frac{a}{b-a}, t=\alpha(b-a)\right) \\
&\triangleq e^{g(t)},
\end{aligned}
$$

where

$$
g(t) = -\theta t + \log\left(1-\theta+\theta e^{t}\right).
$$

By Taylor's theorem, for every real t there exists a δ between 0 and t such that,

$$
g(t) = g(0) + tg'(0) + \frac{1}{2}g''(\delta)t^2
$$

Note that,

$$
\begin{aligned}
g(0) &= 0; \\
g'(0) &= -\theta + \left.\frac{\theta e^{t}}{1-\theta+\theta e^{t}}\right|_{t=0} \\
&= 0; \\
g''(\delta) &= \frac{\theta e^{t}(1-\theta+\theta e^{t})-\theta e^{t}}{(1-\theta+\theta e^{t})^2} \\
&= \frac{(1-\theta)\theta e^{t}}{(1-\theta+\theta e^{t})^2} \\
&= \frac{(1-\theta)\theta}{\theta^2 z + 2(1-\theta)\theta + \frac{(1-\theta)^2}{z}} && (z = e^{t}) \\
&\le \frac{(1-\theta)\theta}{2\theta(1-\theta)+2(1-\theta)\theta} && (z > 0) \\
&= \frac{1}{4}.
\end{aligned}
$$

Thus

$$
g(t) \le 0 + t\cdot 0 + \frac{1}{2}t^2\cdot\frac{1}{4} = \frac{1}{8}t^2 = \frac{1}{8}\alpha^2(b-a)^2.
$$

Therefore, $\mathbf{E}\left[e^{\alpha x}\right] \le \exp\left(\frac{\alpha^2(b-a)^2}{8}\right)$ holds. □

Armed with Hoeffding's lemma, it is routine to prove Hoeffding's inequality.

Proof. [Proof of Theorem 1]

First note that we can assume $\mathbf{E}\left[X_i\right] = 0$ and therefore $\mu = 0$ (if not so, replace X_i by $X_i - \mathbf{E}\left[X_i\right]$). By symmetry, we only need to prove that $\mathbf{Pr}\left[X \ge t\right] \le \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right)$. Since

$$
\mathbf{Pr}\left[X \ge t\right] \overset{\alpha\ge 0}{=} \mathbf{Pr}\left[e^{\alpha X} \ge e^{\alpha t}\right] \le \frac{\mathbf{E}\left[e^{\alpha X}\right]}{e^{\alpha t}}
$$

and

$$
\mathbf{E}\left[e^{\alpha X}\right] = \mathbf{E}\left[e^{\alpha\sum_{i=1}^{n}X_i}\right] = \prod_{i=1}^{n}\mathbf{E}\left[e^{\alpha X_i}\right],
$$

applying Hoeffding's lemma for each $\mathbf{E}\left[e^{\alpha X_i}\right]$ yields

$$\mathbf{E}\left[e^{\alpha X_i}\right] \le \exp\left(\frac{\alpha^2(b_i-a_i)^2}{8}\right).$$

Let $\alpha = \frac{4t}{\sum_{i=1}^n (b_i-a_i)^2}$, we have,

$$\begin{aligned}\mathbf{Pr}\left[X \ge t\right] \le \frac{\prod_{i=1}^n \mathbf{E}\left[e^{\alpha X_i}\right]}{e^{\alpha t}} &\le \exp\left(-\alpha t + \frac{\alpha^2}{8}\sum_{i=1}^n (b_i-a_i)^2\right)\\ &= \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i-a_i)^2}\right).\end{aligned}$$

□

3.2 *Proof of Theorem 2*

Now we will sketch a proof of the Azuma-Hoeffding, which is quite similar to our proof of the Hoeffding inequality.

Proof. [Proof of Theorem 2]

Recall when we were trying to prove the Hoeffding inequality, the most difficult part is to estimate the term

$$\mathbf{E}\left[e^{\alpha Z_n}\right] = e^{\alpha Z_0} \cdot \mathbf{E}\left[\prod_{i=1}^n e^{\alpha(Z_i - Z_{i-1})}\right].$$

In the case of Azuma-Hoeffding, we can use the property of martingales instead of independence to obtain a bound of this term. To see this, we have

$$\begin{aligned}\mathbf{E}\left[\prod_{i=1}^n e^{\alpha Z_i - Z_{i-1}}\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^n e^{\alpha Z_i - Z_{i-1}} \middle| \mathcal{F}_{n-1}\right]\right]\\ &= \mathbf{E}\left[\prod_{i=1}^{n-1} e^{\alpha Z_i - Z_{i-1}} \mathbf{E}\left[e^{\alpha Z_n - Z_{n-1}} \mid \mathcal{F}_{n-1}\right]\right].\end{aligned}$$

The bounds then follows by an induction argument and a conditional expectation version of Hoeffding lemma:

$$\mathbf{E}\left[e^{\alpha(Z_n - Z_{n-1})} \middle| \mathcal{F}_{n-1}\right] \le e^{-\frac{\alpha c_i^2}{8}}.$$

The proof is almost the same as our proof of Hoeffding lemma in the last lecture. □