[AI2613 Lecture 4] Metropolis Algorithm, Countable Infinite Markov Chain

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本节我们研究如何设计MC，使得其稳态分布等于待求分布π.

1 *Reversible Markov Chains*

A Markov chain P over state space $[n]$ is *(time) reversible* if iff there exists some distribution π satisfying

$$\forall i, j \in [n],\ \pi(i)P(i,j) = \pi(j)P(j,i).$$

细致平稳条件.

Reversible 可以理解为对于稳态分布π来说,
P matrix 看起来是对称的.
也可以理解为对π来说, MC的图是无向的 (有 i→j 边就有 j→i 且 $P_{ij} = P_{ji}$)

This family of identities is called *detailed balance conditions.* Moreover, the distribution π must be a stationary distribution of P. To see this, note that

$$\pi^{\mathsf{T}}P(j) = \sum_{i\in[n]} \pi(i)P(i,j) = \sum_{i\in[n]} \pi(j)P(j,i) = \pi(j).$$

⇐ (其实逻辑是满足这种条件的都是稳态分布)

※ 细致平衡条件只需通过验证某时刻每一对 i,j 即可, 也是验π为稳态解的一个方便方法.

* Reversible Chain 可 cover 大部分 MC 应用

The name *reversible* comes from the fact that for any sequence of variables $X_0, X_1, \ldots, X_t$ following the chain which start from the stationary distribution, the distribution of $(X_0, X_1, \ldots, X_{t-1}, X_t)$ is identical to the distribution of $(X_t, X_{t-1}, \ldots, X_1, X_0)$, namely for all $x_0, x_2, \ldots x_t \in [n]$,

$$\begin{aligned}
&\mathbf{Pr}_{X_0\sim\pi}\left[X_0 = x_0, X_1 = x_1, \ldots, X_t = x_t\right] \\
&= \pi(x_0)P(x_0,x_1)\cdots P(x_{t-1},x_t) \\
&= \pi(x_t)P(x_t,x_{t-1})\cdots P(x_1,x_0) \\
&= \mathbf{Pr}_{X_0\sim\pi}\left[X_0 = x_t, X_1 = x_{t-1}, \ldots, X_t = x_0\right]
\end{aligned}$$

We will study reversible chains since their transition matrices are essentially *symmetric* in some sense, so many powerful tools in linear algebra apply. We will also see that reversible chains are general enough for most of our (algorithmic) applications. You can verify that the the random walks on the hypercube is reversible Markov chains with respect to uniform distribution.

Recall the two conditions of FTMC: irreducibility and aperiodicity. Since the transition graph is undirected if we only consider the connectivity, irreducibility is equivalent to the connectivity of the transition graph. Aperiodicity, on the other hand, is equivalent to that the graph is *not* bipartite.

2 *The Metropolis Algorithm*

Given a distribution π over a state space Ω, how can we design a Markov chain P so that π is the stationary distribution of P? The *Metropolis algorithm* provides a way to achieve the goal as long as the transition graph G is connected and undirected.

Let Δ be the maximum degree of the transition graph except selfloop (that is $\Delta \triangleq \max_{u\in[n]} \sum_{v\neq u\in[n]} \mathbb{1}[(u,v) \in E]$). We describe the following process to construct a transition matrix P: Choose $k \in [\Delta+1]$ uniformly at random. For any $i \in [n]$, let $\{j_1, j_2, \dots, j_d\}$ be the d neighbours of i. We consider the transition at state i:

- If $d+1 \le k \le \Delta+1$, do nothing. step 1
- If $k \le d$, step 2
  - propose to move from i to j_k.
  - accept the proposal with probability $\min\left\{\frac{\pi(j_k)}{\pi(i)}, 1\right\}$.

  Then the transition matrix is, for $i, j \in [n]$, 先投骰子，再尝试转移

$$P(i,j) = \begin{cases} \frac{1}{\Delta+1}\min\left\{\frac{\pi(j)}{\pi(i)}, 1\right\}, & \text{if } i \neq j; \\ 1 - \sum_{k\neq i} P(i,k), & \text{if } i = j. \end{cases}$$

We can verify that P is reversible with respect to π:

$\forall i, j \in \Omega:$

$$\pi(i)P(i,j) = \pi(i)\cdot\frac{1}{\Delta+1}\min\left\{\frac{\pi(j)}{\pi(i)}, 1\right\} = \frac{\min\{\pi(i), \pi(j)\}}{\Delta+1} = \pi(j)P(j,i).$$

i=j 显然成立，Pij=0 时也显然成立.

Example 1 *We give a toy example to show how the algorithm works. Consider a graph with 3 vertices $\{a,b,c\}$. There are undirected edges between (a,b), (b,c) and (a,c) and selfloops for each vertex. In this situation, $\Delta = 2$. If we want to design a transition matrix P with stationary distribution $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, by Metropolis algorithm we have*

$$P(a,b) = \frac{1}{2+1}\cdot\frac{2}{3} = \frac{2}{9},$$
$$P(a,c) = \frac{1}{2+1}\cdot\frac{1}{3} = \frac{1}{9},$$
$$P(a,a) = 1 - \frac{1}{9} - \frac{2}{9} = \frac{2}{3}.$$

3 *Sample Proper Coloring*

Let's consider the problem of sampling proper colorings. Given a graph $G = (V, E)$, we want to color the vertices using q colors under the condition that no two adjacent vertices share the same color. More formally, a coloring of G is a mapping $c : V \mapsto [q]$, and we call it *proper* iff $\forall \{u,v\} \in E, c(u) \neq c(v)$. The proper coloring problem is **NP**-hard in general. However, for $q > \Delta$ there always exists a proper coloring that can be easily obtained by a greedy algorithm, where Δ is the maximum degree of the graph.

If we want to count the number of proper colorings, then the problem becomes harder. It is known that for every $q \ge \Delta$, the problem is #**P**-hard. On the other hand, we can use a uniform sampler to obtain an algorithm to

Metropolis 方法简介

① 对一个 MC 图上点 V_i 投 $[\Delta+1]$ 面骰子，$\Delta = \max\deg(V_i)$，$V_i \in Ver$，并编号出顶点。抛到谁决定去谁，抛到没编到的数进行自环并 Exit

② 若抛到 $V_k < \deg(V_i)$，以概率 $\min\{1, \frac{\pi(V_i)}{\pi(V_k)}\}$ 去 V_k.

这么做的话，转移概率矩阵为：

用 detailed balanced condition 验证知，代入 π，π 确为稳态解.

再由 irreducible（强连通子图条件），π 为唯一稳态解.

The advantage of the Metropolis algorithm is that we do not need to know π in order to implement the algorithm. We only need to know the quantity $\frac{\pi(j)}{\pi(i)}$, which is much easier to compute in many applications.

实际应用中将设计 MC 的条件变为已知 $\frac{\pi(i)}{\pi(j)}$，已经十分实用了.

再由取 Δ+1，可知所有点允许自环，AP 成立，故可收敛，QED.

接下来我们以三色问题来看 Metrop. 算法如何用其设计一个 desired MC.

※ 三色问题本身是想求所给 n 点图上用 q 色 (q>Δ) 有多少种上色方法.

可证只需求出所有可行方案上的均匀分布，其种类数量将也可算出.

但是暂且不证，我们先看是如何设计 MC，并用 MCMC 求出 π（均匀）的

approximately counting the number of proper coloring, at an arbitrarily low cost in the precision.

In fact, it is known that an approximate counting algorithm is equivalent to an uniform sampler in many cases (for example, sampling proper coloring). We only show one direction here: a sampler implies an algorithm for approximate counting. Given a graph $G = (V, E)$ with $V = [n]$, let C be the set of proper colorings and $Z = |C|$. Suppose we have an oracle that can uniformly generate a proper coloring from C. Fix a proper coloring σ. We have

$$\begin{aligned}\frac{1}{Z} &= \mathbf{Pr}_{x\sim C}\left[x = \sigma\right]\\ &= \mathbf{Pr}_{x\sim C}\left[x(1) = \sigma(1) \wedge x(2) = \sigma(2) \wedge \dots\right]\\ &= \prod_{i=1}^{n} \mathbf{Pr}\left[x(i) = \sigma(i) \,\middle|\, \bigcap_{j<i} x(j) = \sigma(j)\right].\end{aligned}$$

The above probability can be estimated by taking a number of samples from the oracle, and computing the ratio between colorings such that $x(j) = \sigma(j)$ for $j \le i$ and ones that $x(j) = \sigma(j)$ for $j < i$. Moreover, the ratio we just estimated is bounded below by an inverse polynomial and therefore polynomial number of sample suffices to estimate ratio accurately. The strategy works even if the sampler is an approximate one. Hence one can approximately compute Z. See [JVV86] for more details.

Now we use MCMC to do sampling. Consider the following Markov chain to sample proper colorings:

- Pick $v \in V$ and $c \in [q]$ uniformly at random. ①
- Recolor v with c if possible. ②

The chain is aperiodic since self-loops exist in the walk. For $q \ge \Delta + 2$, the chain is irreducible. The bound $q \ge \Delta + 2$ is tight for irreducibility since when $q = \Delta + 1$, each proper coloring of complete graph is frozen. It is still an open problem if the mixing time of the chain is polynomial in the size of the graph under the condition $q \ge \Delta + 2$. The best bound so far requires that $q \ge (\frac{11}{6} - \varepsilon)\Delta$ for a certain constant $\varepsilon > 0$. Here, we shall give a rapid mixing proof when $q > 4\Delta$ using the method of coupling.

The coupling we used is simple: Both players pick same v and c to move. However, we are not able to reduce the analyze the coupling to *coupon collector* as we did before. We introduce a more general method to analyze couplings. We define a certain distance $d(x, y)$ for any two configurations $x, y \in \Omega$. We can assume without loss of generality that if $x \ne y$ then $d(x, y) \ge 1$ since Ω is finite. Consider a coupling ω_t of μ_t, ν_v. Then for every $t \ge 0$ and $(X_t, Y_t) \sim \omega_t$, we try to establish

$$\mathbf{E}\left[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)\right] \le (1-\alpha) d(X_t, Y_t)$$

※ 用求 uniform distribution sampling 进而求 $|\Omega|$ 原因很简单，由频率逼近概率，多次实验后用频率求概率中的参数 $|\Omega|$ 即可。

三色问题的解空间可以用 $\Omega \subseteq [q]^n$ 表示(q色n点)，这种解空间类似

? 超立方体。有意思的是，三色问题的MC构造与超立方体上随机游走的MC几乎相似。我们接下来将证明这种MC可看作Metrop-算法的结果，故而验证其正确性。

※ 可由细致平衡条件验证 π 为稳态解，因为 设 $\sigma_1, \sigma_2 \in \Omega$ 是两个相邻解，则证 $\pi(\sigma_1)\cdot P(\sigma_1,\sigma_2) = \pi(\sigma_2)\cdot P(\sigma_2,\sigma_1)$

proof ∵ $P(\sigma_1,\sigma_2)$ = 先取点 σ_1，再选 σ_2 该点的颜色

∴ $p(\sigma_1,\sigma_2) = p(\sigma_2,\sigma_1) = \frac{1}{nq}$

均匀使得 $\pi(\sigma_1) = \pi(\sigma_2)$

∴ $\pi(\sigma_1)p(\sigma_1,\sigma_2) = \pi(\sigma_2)p(\sigma_2,\sigma_1)$

其余也都是trivial情况

故 π 为稳态解。

It is indeed a Metropolis algorithm. Let

$$\sigma^{v\leftarrow c}(u) = \begin{cases} \sigma(u) & u \ne v \\ c & u = v. \end{cases}$$

$\sigma^{v\leftarrow c}$ is a neighbor of σ on the transition graph, and we accept it if $\sigma^{v\leftarrow c}$ is a proper coloring, i.e. $\frac{\pi(\sigma^{v\leftarrow c})}{\pi(\sigma)} = 1$.

我们现在说明为什么该MC就是Metrop的结果：

t时刻的情况 σ_t 可看作 $[q]^n$ 上一点，则 step ① 可视为与 metrop step 1 对应，step ② 中，impossible的地 $\pi(j) = 0$，possible的点 $\frac{\pi(j)}{\pi(i)} = 1$ (unif. distrib) 故 step ② 可行就走的策略也与 metrop step 2 策略相同，证毕。

for some $\alpha \in (0, 1]$. In other words, $\{d(X_t, Y_t)\}_{t\geq 0}$ is a supermartingale. This implies that for every $t \geq 1$,

$$\mathbf{E}\left[d(X_t, Y_t)\right] \leq (1-\alpha)\mathbf{E}\left[d(X_{t-1}, Y_{t-1})\right] \leq (1-\alpha)^t d(X_0, Y_0).$$

If we have a universal upper bound for $d(X_0, Y_0)$, say n, then by coupling lemma

$$\boxed{\begin{aligned} D_{\mathrm{TV}}(\mu_t, \nu_t) &\leq \mathbf{Pr}_{(X_t,Y_t)\sim\omega_t}\left[X_t \neq Y_t\right] \\ &= \mathbf{Pr}\left[d(X_t, Y_t) \geq 1\right] \\ &\leq \mathbf{E}\left[d(X_t, Y_t)\right] \\ &\leq (1-\alpha)^t \cdot n. \end{aligned}}$$

我们接下来分析这种MC的收敛情况

我们将证明 $q>4\Delta$ 时，问题收敛，即MC很快能得到一个均匀分布
颜色 maxdeg

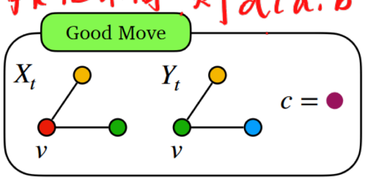
我们使用coupling证明，则仍然去证 $\Pr_{X,Y\sim\omega}[X_t \neq Y_t] \leq \varepsilon$ 再推出 $D_{TV}(\mu_t, \nu_t) \to 0$

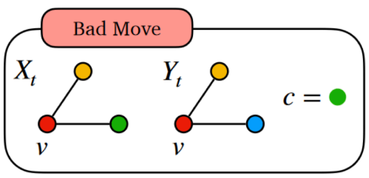
之前直接（用coupon collector）求解Pr很方便，然而大部分时候，Pr求解具有困难，

这里我们使用了通过超立方体上距离d来bound $\Pr[X_t \neq Y_t]$ 的方法：
将着色方案视为一个n维向量 $\in [q]^n$，第i位即i区域的颜色，若两向量有k位不同，则 $d(\vec{a}, \vec{b}) = k$（见后）

Now come back to our problem of sampling proper colorings. Suppose X_t, Y_t are two proper colorings. We define the distance $d(X_t, Y_t)$ as their Hamming distance, i.e. the number of vertices colored differently in two colorings. Our coupling of two chains is that we always choose the same v, c in each step. The distance between two colorings can change at most 1 since only v is affected. The possible changes can be divided into two kinds:

- Good move: $X_t(v) \neq Y_t(v)$, and both change into c successfully. It will decrease distance by 1.
- Bad move: $X_t(v) = Y_t(v)$, one succeeds and one fails in the changing. It will increase distance by 1.





Consider the probabilities of two types of moves. For good moves, w.p. $\frac{d(X_t,Y_t)}{n}$, $X_t(v) \neq Y_t(v)$, and there are at least $q - 2\Delta$ choices of c to make it a good move. So

$$\begin{aligned} \mathbf{Pr}\left[d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) - 1\right] =& \mathbf{Pr}_{(v,c)\in V\times[q]}\left[(v, c) \text{ is a good move}\right] \\ &\geq \frac{d(X_t, Y_t)}{n} \cdot \frac{q-2\Delta}{q}. \end{aligned}$$

For bad moves, there exists a neighbor w of v such that its color is different in two colorings, and in one coloring w is of color c. By a counting argument, we have

$$\mathbf{Pr}\left[d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) + 1\right] = \mathbf{Pr}_{(v,c)\in V\times[q]}\left[(v, c) \text{ is a bad move}\right] \leq \frac{\Delta d(X_t, Y_t)}{n} \cdot \frac{2}{q}.$$

Therefore,

$$\begin{aligned} \mathbf{E}\left[d(X_{t+1}, Y_{t+1}) | (X_t, Y_t)\right] &= d(X_t, Y_t) + \mathbf{Pr}\left[d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) + 1\right] - \mathbf{Pr}\left[d(X_{t+1}, Y_{t+1}) = d(X_t, Y_t) - 1\right] \\ &\leq d(X_t, Y_t) + \frac{\Delta d(X_t, Y_t)}{n} \cdot \frac{2}{q} - \frac{d(X_t, Y_t)}{n} \cdot \frac{q-2\Delta}{q} \\ &\leq d(X_t, Y_t)\left(1 - \frac{q-4\Delta}{nq}\right). \quad (*) \end{aligned}$$

这里我们详细阐述该方法：

Step 1：第一步转换证明对象

$$
\begin{aligned}
D_{TV}(\mu_t, \nu_t) &\le \mathbf{Pr}_{(X_t,Y_t)\sim\omega_t}[X_t \ne Y_t] \\
&= \mathbf{Pr}[d(X_t, Y_t) \ge 1] \\
&\le \mathbf{E}[d(X_t, Y_t)]
\end{aligned}
$$

$X_t \ne Y_t$ eq. to $d(X_t, Y_t) \ge 1$

by calculation.

r.v 的期望

Step 2：bound $E[d(X_t, Y_t)]$，方法是先推：$E[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)]$ 与 $d(X_t, Y_t)$ 关系

我们分析 (X_t, Y_t) 条件下 $d(X_{t+1}, Y_{t+1})$ 从 $d(X_t, Y_t)$ 发生的变化。

依据我们设计的 MC，可以知道何时 $d+1$，何时 $d-1$，何时不变。

将数量 / 总选择数 nq，即得 $Pr[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)]$ 表达式，进而得期望。

具体方法见 Note 前页蓝框区域。

Step 3：使用 $E[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t)]$ 与 $d(X_t, Y_t)$ 关系 bound $E[d(X_t, Y_t)]$。

我们对(米)式左右取期望，可将 $E[d \mid (X, Y)]$ 变为 $E[d(X_{t+1}, Y_{t+1})]$，得下①式。

$$
\underbrace{\mathbf{E}[d(X_t, Y_t)] \le (1-\alpha)\mathbf{E}[d(X_{t-1}, Y_{t-1})]}_{①} \le \underbrace{(1-\alpha)^t d(X_0, Y_0)}_{②}.
$$

iterate 得②式，成功 bound 了 $E[d(X_t, Y_t)]$。

由 step ① 所言，至此已得证。

In the case $q > 4\Delta$, if we want

$$D_{\mathrm{TV}} \le \left(1 - \frac{1}{nq}\right)^t n \le \varepsilon,$$

we have the mixing time is bounded by

$$\tau_{\mathrm{mix}}(\varepsilon) \le nq \log \frac{n}{\varepsilon}.$$

最终收敛步数数量级在 nlogn 附近

4 Countably Infinite Markov Chains

我们接下来开始对无限状态的MC进行讨论。

We have proved that finite Markov chain must have a stationary distribution using Perron-Frobenius Theorem. However, when the Markov chain has infinite states, even it's countable infinite, there is something going wrong.

Consider the following random walk on $\mathbb{N}$. The state space is $\mathbb{N}$ and at each state i, go to $i+1$ w.p. p and go to $i-1$ w.p $1-p$ (if $i = 0$, w.p. $1-p$ stay put).

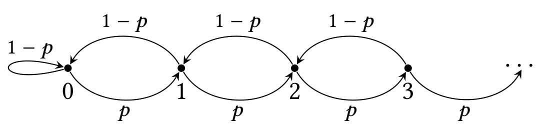
无限MC不一定有稳态解，

本页展示了不收敛的一个例子，

下面将证明结论：

IR + Recurrent on one point i

$\Rightarrow$ Inf MC convergence .



Let π be the stationary distribution of this Markov chain (if there exists a stationary distribution). We have that

$$\pi(0) = \pi(0)(1-p) + \pi(1)(1-p) \qquad \Longrightarrow \pi(1) = \frac{p}{1-p}\pi(0),$$

$$\pi(1) = \pi(0)p + \pi(2)(1-p) \qquad \Longrightarrow \pi(2) = \frac{p}{1-p}\pi(1),$$

$$\cdots$$

$$\pi(i) = \pi(i-1)p + \pi(i+1)(1-p) \qquad \Longrightarrow \pi(i+1) = \frac{p}{1-p}\pi(i).$$

$$\cdots$$

Note that π is a distribution, so $\sum_{i=0}^{\infty} \pi(i) = 1$. Then, we have

- If $p < \frac{1}{2}$, that is, $\frac{p}{1-p} < 1$, then $\sum_{i=0}^{\infty} \left(\frac{p}{1-p}\right)^i \pi(0) = 1$. By direct calculation we have $\pi(0) = \frac{1-2p}{1-p}$ and $\pi(i) = \left(\frac{p}{1-p}\right)^i \frac{1-2p}{1-p}$ for $i \in \mathbb{N}$.
- If $p > \frac{1}{2}$, then $\frac{p}{1-p} > 1$. When $i \to \infty$, if $\pi(0) \ne 0$, $\pi(i) \to \infty$. This yields that $\pi(0) = \pi(1) = \cdots = \pi(i) = \cdots = 0$. The Markov chain doesn't have a stationary distribution in this case.
- If $p = \frac{1}{2}$, $\frac{p}{1-p} = 1$. Then $\pi(0) = \pi(1) = \cdots = \pi(i) = \cdots$ and $\sum_{i=0}^{\infty} \pi(0) = 1$. This yields that $\pi(0) = 0$ and there is no stationary distribution in this case.

4.1 Recurrence

Definition 1 *For $i \in \Omega$, let $T_i > 0$ be the first hitting time of state i. Let $\mathbf{P}_i = \mathbf{Pr}[\cdot | X_0 = i]$. We say a state i is* recurrent *if $\mathbf{P}_i[T_i < \infty] = 1$, o.w. we say the state is* transient.

$T_i \triangleq \min\{t > 0 | X_t = i\}$.

Recall the probability space of a stochastic process. One can view the outcomes of the probability space is the set of infinite sequence of real numbers between $[0, 1]$, namely $\hat{\Omega} = [0, 1]^{\mathbb{N}}$. The sigma-algebra can be defined in a way similar to the problem 1 in our first homework. Therefore, the random variable T_i is therefore a function $\hat{\Omega} \to \mathbb{R}$.

Let $N_i \triangleq \sum_{t=0}^{\infty} \mathbb{1}[X_t = i]$, then we have the following propositions.

Proposition 2 *If i is recurrent, then $\mathbf{P}_i[N_i = \infty] = 1$.*

Proof. Assume that $\mathbf{P}_i[N_i = \infty] < 1$. Then there exists $\Omega' \subseteq \hat{\Omega}$ such that $N_i < \infty$ on Ω' and $\mathbf{P}_i[\Omega'] > 0$. This means that with probability larger than 0, we will never reach state i after the last time we visit it. This is in contradiction with the fact that i is recurrent. □

首先介绍了Recurrence(常返态)定义为i点出发代入MC，有限步回到i状态的概率为1.

只要<1，即认为是瞬态. transient

ProP2意为常返 ⟺ 无限步回到i次数为无限次

Proposition 3 *If i is recurrent and there exists a finite path from i to j, then*

- $\mathbf{P}_i[T_j < \infty] = 1$.
- $\mathbf{P}_j[T_i < \infty] = 1$.
- *j is recurrent.*

Proof.

- Let $q \triangleq \mathbf{P}_i[\text{reach } j \text{ before returning to } i]$. Since there is a finite path from i to j, we have $q > 0$ and $\mathbf{P}_i[\text{visit } i\ n \text{ times before reaching } j] = (1-q)^n$.

  Assume that $\mathbf{P}_i[T_j = \infty] = \alpha > 0$. Then we have $\mathbf{P}_i[T_j = \infty | N_i = \infty] = \alpha$ since $\mathbf{P}_i[N_i = \infty] = 1$. Let T_i^n be the n^{th} time that the chain visits state i. Then

  $$\forall n > 0,\ \mathbf{P}_i[T_j > T_i^n | N_i = \infty] \geq \mathbf{P}_i[T_j = \infty | N_i = \infty] = u$$

  On the otherhand, we have $\lim_{n\to\infty} \mathbf{P}_i[T_j > T_i^n | N_i = \infty] = \lim_{n\to\infty} \mathbf{P}_i[T_j > T_i^n] = \lim_{n\to\infty}(1-q)^n = 0$. This is a contradiction. Thus, $\mathbf{P}_i[T_j = \infty] = 0$.

- If $\mathbf{P}_j[T_i = \infty] = p > 0$, then we have that $\mathbf{P}_i[T_i = \infty] \geq q \cdot p > 0$. This is in contradiction with the fact that i is recurrent.

- If $\mathbf{P}_j[T_j = \infty] = r > 0$, then $\mathbf{P}_i[T_j = \infty] \geq q \cdot r > 0$. This is in contradiction with the first item of this proposition.

□

$\mathbf{P}_i[T_j = \infty] = \mathbf{P}_i[T_j = \infty \mid N_i = \infty] \cdot \mathbf{P}_i[N_i = \infty] + \mathbf{P}_i[T_j = \infty \mid N_i < \infty] \cdot \mathbf{P}_i[N_i < \infty] = \mathbf{P}_i[T_j = \infty \mid N_i = \infty]$.

ProP3 意为 i常返 + i到j连通

⟹ j常返.

由于 Inf convergence ⟺ all state i 是常返的(否则走着走着就回不来了)

而由ProP3可知 ∃点i Recurrent + IR(强连通性) ⟹ all state i 常返

⟹ Inf convergence.

故证毕

某一维在t时刻为0的概率

References

[JVV86] Mark R Jerrum, Leslie G Valiant, and Vijay V Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theoretical computer science*, 43:169–188, 1986. 3

RW维数 d

又可证: $E_0[N_0] = \sum_{t\geq 1} Pr[X_t = 0] = \sum_{t\geq 1} Pr[Y_t = 0]$

且可求 $Pr[Y_t = 0] \approx \frac{1}{\sqrt{\pi t}}$

则知在 $d \leq 2$ 时 $E_0[N_0] \to \infty$, 即此时常返

Why E_0: 不妨设从原点开始 RW

※在课堂中还证明了 Random Walk 在 x 维上的收敛性:

通过证明 $E_0[N_0] = \infty$, 由 ProP2 推常返