[AI2613 Lecture 8] Optional Stopping Theorem

April 17, 2023

1 Stopping Time

我们引入stopping time来刻画SP中完成某事的时刻这一r.v.，在martingale中，τ记为停止时间。

Suppose $Z_0, Z_1, \ldots, Z_n, \ldots$ is a martingale with respect a certain filtration $\{\mathcal{F}_t\}$. We know that for any t, $\mathbf{E}[Z_t] = \mathbf{E}[Z_0]$. However, does $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0]$ still hold if τ is a random variable?

之所以会有 $E[Z_t] = E[Z_0] \neq E[Z_\tau]$，是因为 $E[Z_\tau]$ 在 τ 取 t 时能利用 out-t 时刻的信息，其源于stopping time的定义。因此 → 在赌徒例子中 $E[Z_\tau]=1 > E[Z_t]=0$

Consider the following gambling strategy in a fair game. At the first round, the gambler bet \$1. Then he simply double his stake until he wins

The strategy was called martingale!

Let τ be the first time he wins. Then expected money he win at time τ is 1, which is not equal to 0, his initial money. In order to understand the phenomenon, let us first formally introduce *stopping time*.

- If $\tau = 1$, he wins 1 dollar.
- If $\tau = 2$, he wins $-1 + 2 = 1$ dollar.
- If $\tau = 3$, he wins $-1 - 2 + 4 = 1$ dollar.
- $\ldots$

Definition 1 (Stopping Time) *Let $\tau \in \mathbb{N} \cup \{\infty\}$ be a random variable. We say τ is a* stopping time *if for all $t \geq 0$, the event "$\tau \leq t$" is $\mathcal{F}_t$-measurable.*

For example, the first time that a gambler wins five games in a row is a stopping time, since for a given t, this can be determined by looking at the outcomes of all the previous games, and therefore the time is $\mathcal{F}_t$-measurable. However, the *last* time the gambler wins five games in a row is *not* a stopping time, since determining whether the time is t cannot be done without knowing $X_{t+1}, X_{t+2}, \ldots$

但是我们研究何种条件下 $E[Z_\tau] = E[Z_0]$

1.1 Optional Stopping Theorem(OST)

The optional stopping theorem provides sufficient condition for $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0]$ to hold.

Theorem 2 (Optional Stopping Theorem) *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if* at least one of *the following conditions holds:*

OST定理：满足如下条件，则 $E[X_\tau]=E[X_0]$

注意不是 $E[\tau]$，是 $E[X_\tau]$

1. *τ is bounded almost surely, that is, $\exists n \in \mathbb{N}$ such that $\mathbf{Pr}[\tau \leq n] = 1$;* → $\tau \leq n$ 的概率为1.
2. *$\mathbf{Pr}[\tau < \infty] = 1$, and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$;* → $\tau<\infty$ & $|X_t| (t<\tau)$ 有界
3. *$\mathbf{E}[\tau] < \infty$, and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.* → $E[\tau]<\infty$ & ΔX_t 幅度期望有界

※ 注意 $E[\tau]<\infty$ 强于 $\tau<\infty$ (概率符号略)

但 $E[\Delta X \mid \mathcal{F}_t] \leq c$ 弱于 X_t 有界

We will prove the theorem next time. Let us look back at the boy-or-girl example mentioned in the first class.

Example 1 (Boy or Girl) *Suppose there is a country in which people only want boys. What is the sex ratio of the country in the following three scenarios?*

C_1

- *Each family continues to have children until they have a boy.*

- C_2 • *Each family continues to have children until there are more boys.*
- C_3 • *Each family continues to have children until there are more boys or there are 10 children.*

We can model the problem as a random walk. Suppose there is a man walking randomly on a one-dimensional axis. Let $\{X_t\}_{t\geq 0}$ be the positions of the man at each time where X_t stands for the number of boys minus the number of girls *in the first t children of a family. Starting at $X_0 = 0$, at time 0, the man takes a step $c_t \in_R \{-1, 1\}$ and reach X_{t+1}, i.e., $X_{t+1} = X_t + c_t$. It is easy to verify that $\{X_t\}_{t\geq 0}$ is a martingale. The three scenarios mentioned correspond to the following three different definitions of a stopping time τ. The identity $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ means that the sex ratio is balanced. We will check respectively whether it is the case using OST.*

- *Let τ be the first time t such that $c_t = 1$. Then $\mathbf{E}[\tau] < \infty$ since by definition $\tau \sim \mathrm{Geom}\left(\frac{1}{2}\right)$, and $|X_{t+1} - X_t| \leq 1$ for all $t < \tau$. Therefore from the 3rd condition of OST we have $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$. In other words, if the man stops at the first time of $c_t = 1$, then the expected final position is 0.*

C_1 满足 3^{rd} 条件 + 2^{nd} 条件.

- *Let τ be the first time t such that $X_t = 1$, then of course $\mathbf{E}[X_\tau] = 1 \neq \mathbf{E}[X_0]$. This process is called "1-d random walk with one absorbing barrier" and it is well-known that $\mathbf{E}[\tau] = \infty$. No condition in OST is satisfied.*

C_2 无一满足 (1-d RW 为 null recurrent)
($\mathbf{E}[T_{0\to 1}] = \infty$, X_t 可 $\to -\infty$)

The property $\mathbf{E}[\tau] = \infty$ of the random work is called "null recurrent". You can find more on this from my lecture on stochastic processes.

- *Let τ be the minimum between 10 and the first time t such that $X_t = 1$. In this case, τ is at most 10, which satisfies the first condition of OST. Therefore we have $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$.*

C_3 满足 1^{st} 条件

2 *Applications of OST*

2.1 *Doobs martingale inequality*

With OST, we can obtain concentration property of the maximum element in a seuqence of random variables.

Claim 3 *Let $\{X_t\}_{t\geq 0}$ be a martingale with respect to itself where $X_t \geq 0$ for every t. Prove that for every $n \in \mathbb{N}$,*

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \frac{\mathbf{E}[X_0]}{\alpha}.$$

☆ mk Ineq: $\Pr[X \geq \alpha] \leq \frac{E[X]}{\alpha}$.

Proof. We define a stopping time τ when the first element that is greater that α occurs, and otherwise set $\tau = n$. Formally, define

$$\tau \triangleq \min\left(n, \min_{t\leq n}\{t \mid X_t \geq \alpha\}\right).$$

By definition of τ, we have

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] = \mathbf{Pr}[X_\tau \geq \alpha].$$

Since τ is bounded, we apply Optional Stopping Theorem to obtain that $E[X_\tau] = E[X_0]$. Therefore, by Markov's Inequality,

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \mathbf{Pr}\left[X_\tau \ge \alpha\right] \le \frac{\mathbf{E}\left[X_\tau\right]}{\alpha} = \frac{\mathbf{E}\left[X_0\right]}{\alpha}$$

Markov ineq

□

2.2 One-dimensional Random Walk with Two Absorbing Barriers

We've discussed one-dimensional random walk with one absorbing barrier before

We consider another problem in one-dimensional random walk. Let $a, b > 0$ be two integers. A man starts the random walk from 0 and stops when he arrives at $-a$ or b. Let τ be the time when the man first reaches $-a$ or b, i.e., the first time t that $X_t = -a$ or $X_t = b$. The model is called "one-dimensional random walk with two absorbing barriers". We want to compute the expected value of $\mathbf{E}[\tau]$, that is, the average stopping time of the walk.

建模.

目标.

We want to construct a martingale $\{Y_t\}_{t\ge 0}$ such that OST can be applied to $\{Y_t\}_{t\ge 0}$ and τ and thereby we can derive an equality related to $\mathbf{E}[\tau]$. Before calculating $\mathbf{E}[\tau]$, we first determine $\mathbf{Pr}[X_\tau = -a]$, the probability that the man stops at position $-a$. Let $P_a \triangleq \mathbf{Pr}[X_\tau = -a]$. We want to apply OST to show $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$. Therefore, we verify that some of conditions in OST is satisfied.

构造 martingale, 证明OST条件以获得 $E[X_\tau]=0$ 的信息,

In a time period of length $T = a + b$, if the man walks towards the same direction, he must have stopped, either at $-a$ or b, which happens with probability $2^{-(a+b)}$. Therefore, if we divide the time into consecutive periods in this manner, in expected finite time, we can meet some period when the event happened. Hence, $\mathbf{E}[\tau] < \infty$. Moreover, we clearly have $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] < 1$ for every $0 \le t < \tau$, so the third condition of OST holds, which implies that $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$. On the other hand, we have $\mathbf{E}[X_\tau] = P_a \cdot (-a) + (1 - P_a) \cdot b$. These two equalities give $P_a = \frac{b}{a+b}$.

3rd OST 条件成立 → $E[X_\tau]=0$

→ 由OST结论计算 P_a、P_b

Then for all $t \ge 0$, we define a new random variable $Y_t \triangleq X_t^2 - t$ which involves the time t. The following fact is easy to verify by definition.

Claim 4 *$\{Y_t\}_{t\ge 0}$ is a martingale.*

再用一次相同方法

Proof. First we have

$$\begin{aligned}\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] &= \mathbf{E}\left[X_{t+1}^2 - (t+1) \,\middle|\, \mathcal{F}_t\right]\\ &= \mathbf{E}\left[(X_t + c_t)^2 - (t+1) \,\middle|\, \mathcal{F}_t\right]\\ &= \mathbf{E}\left[X_t^2 \,\middle|\, \mathcal{F}_t\right] + 2\mathbf{E}\left[X_t c_t \mid \mathcal{F}_t\right] + \mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] - (t+1)\,.\end{aligned}$$

Since X_t is $\mathcal{F}_t$-measurable, $\mathbf{E}[c_t \mid \mathcal{F}_t] = 0$ and $\mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] = 1$, we can further derive that

$$\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] = X_t^2 + 0 + 1 - (t+1) = X_t^2 - t = Y_t.$$

Hence $\{Y_t\}_{t\ge 0}$ is a martingale. □

Sometimes one can use OST in a reverse way. Consider the random walk with only one barrier at $-a$. The fact that $\mathbf{E}[\tau] = \infty$ can be proved in the following way (due to Biaoshuai Tao): If $\mathbf{E}[\tau] < \infty$, then by (cond 3 of) OST, $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$. On the other hand, we know $X_\tau = -a \ne 0$. Therefore it must be that $\mathbf{E}[\tau] = \infty$.

Note that $X_t \in [-a, b]$ for all $t \geq 0$. Thus $|Y_{t+1} - Y_t| = \left|X_{t+1}^2 - (t+1) - X_t^2 + t\right| = \left|X_{t+1}^2 - X_t^2 - 1\right|$ is bounded by some constant. We can apply OST again to obtain $\mathbf{E}[Y_\tau] = \mathbf{E}[Y_0] = 0$. On the other hand, we have $\mathbf{E}[Y_\tau] = \mathbf{E}\left[X_\tau^2\right] - \mathbf{E}[\tau]$ by definition, and thus

$$\mathbf{E}[\tau] = \mathbf{E}\left[X_\tau^2\right] = a^2 P_a + b^2(1 - P_a) = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = ab.$$

QED .

(理论上也可以用 RW递推式做).

2.3 Pattern Matching

之前我们研究的是 n长序列中出现某k长序列的次数期望.
现在是求第一次抛出该k长序列的时间

Suppose that there is a $\{H, T\}$-string P of length ℓ (H for "head" and T for "tail"). We flip a coin consecutively until the last ℓ results form exactly the same string as P. How many times do we flip the coin?

Note that if we flip the coin N times and observe the string S consisting of N results. No matter which pattern we choose, by the linearity of expectation, the expected number of occurrence [1] is

$$\mathbf{E}[\# \text{ of occurrence of } P \text{ in } S] = \sum_{i=1}^{n-\ell+1} \mathbf{E}\left[\mathbb{1}[S_{i,i+1,\ldots,i+\ell-1} = P]\right] = (n - l + 1) \cdot 2^{-\ell}.$$

[1] That means the expected number of substrings exactly the same as P in the resulting string S.

However, if we would like to compute the first time that pattern P occurs, the pattern itself has an impact on the expected time. Intuitively, lets consider two patterns HT and HH. Assume that the first flipping result is H. Then we consider what happens if the second result fails. Suppose that the desired pattern is HT and H appears. Although we fail, we obtain an H. However, if the desired pattern is HH and the second flipping result is T, then we obtain nothing and the first two flips are a waste. So we should believe that the expected times of the first occurrence of HT is smaller than HH.

⇐ 二者的不同点

We now use the optional stopping theorem to solve this problem. Let $P = p_1 p_2 \ldots p_\ell$. For every $n \geq 0$, assume that before $n+1$-th flipping there is a new gambler G_{n+1} coming with 1 unit of money to bet that the following ℓ result (i.e., the $n+1$-th to $n+\ell$-th results) are exactly the same as P. At the $n+k$-th flipping, G_{n+1} will bet that the result is p_k by an all in strategy, that is, if the $n+k$-th result is p_k then G_{n+1} will have twice as much money as before; otherwise they will lose all. Suppose that the patter P = HTHTH and the flipping results are HTHHTHTH. The following table shows the total money of each gambler after flipping.

我们采用的方法是将pattern matching 等效为Gambler 问题,
目的是得到新的式子计算 $E[\tau]$

Let X_t be the result of t-th flipping, $M_i(t)$ denote the money that G_i has after t-th flipping, and $Z_t \triangleq \sum_{i=1}^{t}(M_i(t) - 1)$ be the total income of all gamblers after t-th flipping. It is easy to verify that $\{M_i(t)\}_{t\geq 0}$ is a martingale with respect to $\{X_t\}$ since

$$\mathbf{E}\left[M_i(t+1) \mid \bar{X}_{0,t}\right] = \frac{1}{2} \cdot 2M_i(t) + \frac{1}{2} \cdot 0 = M_i(t). \Leftarrow \text{fair game}$$

Then by the linearity of expectation we conclude that $\{Z_t\}_{t\geq 0}$ is a martingale with respect to the flipping results $\{X_t\}$ since $\mathbf{E}[M_i(t)] = 1$. Let

$$E[Z_{t+1} \mid \bar{X}_{0,t}] = E\left[\sum_{i=1}^{t+1}(M_i(t+1) - 1) \mid \bar{X}\right]$$
$$= \sum^{t+1}\{E[M_i(t+1) \mid \bar{X}] - 1\}$$
$$= \sum_{i=1}^{t}\{M_i(t) - 1\} + M_{t+1}(t+1) - 1, \quad M_{t+1}(t+1) = 1$$

Gambler	H	T	H	H	T	H	T	H	Money	
1	H	T	H	T					0	1⟶2⟶4⟶8⟶0
2		H							0	1⟶0
3			H	T					0	1⟶2⟶0
4				H	T	H	T	H	32	1⟶2⟶4⟶8⟶16⟶32
5					H				0	1⟶0
2						H	T	H	8	1⟶2⟶4⟶8
5							H		0	1⟶0
5								H	2	1⟶2

τ be the stopping time defined by the first time that some gambler wins, namely, the first time that P occurs in the flipping results. Applying Condition 2 of OST we obtain that $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0] = 0$. Sequentially we have $\mathbf{E}\left[\sum_{i=1}^{\tau} M_i(\tau) - \tau\right] = 0$ and $\boxed{\mathbf{E}[\tau] = \sum_{i=1}^{\tau} \mathbf{E}[M_i(\tau)].}$ 关键等式

Note that $M_i(t) = 0$ for $i \le \tau - \ell$ and $M_i(t) = 2^{\tau-i+1}\chi_{\tau-i+1}$ for $i > \tau - \ell$ where χ_j is defined by

$$\chi_j = \mathbb{1}[p_1 p_2 \dots p_j = p_{\ell-j+1} \dots p_{\ell-1} p_\ell].$$

Hence,

$$\mathbf{E}[\tau] = \sum_{i=\tau-\ell+1}^{\tau} \mathbf{E}[M_i(\tau)] = \sum_{i=1}^{\ell} 2^i \chi_i.$$

⇐ 通过转换为新问题并用OST解决.

Recall the example of HH and HT. If P is HH, $\mathbf{E}[\tau] = 2 + 4 = 6$. If P is HT, $\mathbf{E}[\tau] = 4$. This confirms our hypothesis that $\mathbf{E}[\tau]$ for HH is larger than $\mathbf{E}[\tau]$ for HT.

2.4 *Wald's Equation*

In practice, we often need to analyze the (expected) running time of following procedure where both *cond* and *compute()* are random.

```
while cond do
    compute();
end while
```

当算法中出现循环,但循环次数是随机变量怎么办?

假设循环进行T轮,每轮时间 $X_i \sim X$

则期望时间为 $\mathbb{E}[\sum_{i=1}^{T} X_i]$

Wald Equation 证明在上述条件

以及 [$\mathbb{E}(T) < \infty$, $\mathbb{E}(X) < \infty$,

X_i iid X, T为stopping time] 时,

$\mathbb{E}[\sum_{i=1}^{T} X_i] = \mathbb{E}(T) \cdot \mathbb{E}(X)$

Assume the i-th call to *compute()* costs X_i time and the algorithm terminates after T iterations. Then the total running time is $N \triangleq \sum_{i=1}^{T} X_i$. Suppose X_is are independently and identically distributed as a random variable X. The Wald's equation gives a formula for $\mathbf{E}[N]$.

Theorem 5 (Wald's Equation) *If we have*

- $X_1, X_2, \dots$ *are non-negative, independent, identically distributed random variables with the same distribution as* X.
- T *is a stopping time for* $X_1, X_2, \dots$.

- $\mathbf{E}[T], \mathbf{E}[X] < \infty$,

then

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X].$$

Proof. For $i \geq 1$, let $Z_i := \sum_{j=1}^{i}(X_j - \mathbf{E}[X])$. Clearly the sequence $Z_i, Z_2, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and $\mathbf{E}[Z_1] = 0$. And we have

$$\begin{aligned}\mathbf{E}[|Z_{i+1} - Z_i| \mid \mathcal{F}_i] &= \mathbf{E}[|X_{i+1} - \mathbf{E}[X]| \mid \mathcal{F}_i] \\ &\leq \mathbf{E}[X_{i+1} + \mathbf{E}[X] \mid \mathcal{F}_i] \\ &\leq 2\mathbf{E}[X].\end{aligned}$$

We know that $\mathbf{E}[T], \mathbf{E}[X] < \infty$, and therefore applying OST derives $\mathbf{E}[Z_T] = \mathbf{E}[Z_1] = 0$. Then

$$\begin{aligned}\mathbf{E}[Z_t] &= \mathbf{E}\left[\sum_{j=1}^{T}(X_j - \mathbf{E}[X])\right] \\ &= \mathbf{E}\left[\sum_{i=1}^{T} X_i - T\mathbf{E}[X]\right] \\ &= \mathbf{E}\left[\sum_{i=1}^{T} X_i\right] - \mathbf{E}[T]\,\mathbf{E}[X] = 0.\end{aligned}$$

□

证明过程如左

因为 X_i 为 iid，且 $E\{X_j - E[X]\} = 0$

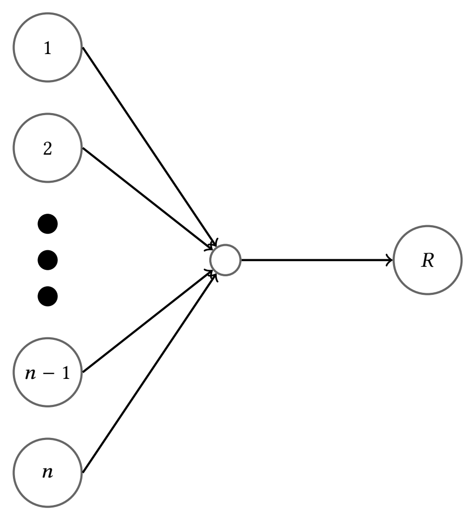
故用定义很容易验证.

构成 OST 3rd condition.

故 $E[Z_T] = 0$,

代入 $Z_T = \sum_{i=1}^{T} X_i - E[X]$, 得证

An Application of Wald's Equation: A Routing Problem Let us consider an application of Wald's equation. There are n senders and one receiver. In each round, each sender sends a packet to the receiver with probability $\frac{1}{n}$. Since all senders share the same channel, if there are multiple packets sent at the same time, all of them will fail. The question is, on average, how many rounds are required so that each sender can successfully send at least one packet.



We let X_i be the variable indicating how long the receiver needs to get another packet after he has received $i-1$ ones (counting packets from repeated sender). And let T be the number of packets received when first time the receiver receives at least one packet from each sender. The quantity we are interested in is

$$N \triangleq \sum_{i=1}^{T} X_i.$$

Clearly $X_1, X_2, \ldots$ are independently and identically distributed, and $\mathbf{E}[T]$ is finite. Therefore $\mathbf{E}[N] = \mathbf{E}[T] \cdot \mathbf{E}[X_1]$ by Wald's equation.

Note that by the definition, T is the number of coupons in the coupon collector's problem we met before. So $\mathbf{E}[T] = nH_n = \Theta(n \log n)$.

On the otherhand, $X_1 \sim \mathtt{Geom}(p)$ with

$$p = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \approx e^{-1}$$

一个用 Wald Equation 分析的例子如左所示.

将收到 n 个不同(一套)信息的时刻数建模为 T 轮 + 1 次发出信息平均用时

coupon collector

仅一人发时才成功的概率为:

$p = \underset{\text{人数}}{n} \cdot \underset{\text{单发概率}}{\frac{1}{n}} \cdot \underset{\text{其余均不发}}{(1-\frac{1}{n})^{n-1}}$, 然后再套几何分布

实现解耦后 证 Wald Eq 条件:

1. X_i ~ iid, 2. $E[T] < \infty$.

分开算 $E[T]$、$E[X]$, 后套 Wald Eq, 得证

which implies $E[X_1] = e$. Therefore,

$$\mathbf{E}[N] = \mathbf{E}[T] \cdot \mathbf{E}[X_1] \approx enH_n.$$

套Wald Eq 相乘得证

3 Proof of Optional Stopping Theorem

Let us restate the theorem.

Theorem 6 (Optional Stopping Theorem) *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if* at least one of *the following conditions holds:*

1. *τ is bounded almost surely, that is, $\exists n \in \mathbb{N}$ such that $\mathbf{Pr}[\tau \leq n] = 1$;*
2. *$\mathbf{Pr}[\tau < \infty] = 1$, and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$;*
3. *$\mathbf{E}[\tau] < \infty$, and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.*

本质上是证

$E[\sum_1^\infty X_i] = E[X_\tau] = \sum_i^\infty E[X_i]$

$\{Z_t\}$ is martingle, $Z_t = \sum_{i=1}^t X_i$

$= E[Z_0] = E[Z_t]$

故类似绝数分证明，略.

See video 2023/4/13/15:08 起

Proof. It is obvious that for every $n \in \mathbb{N}$, $\mathbf{E}[X_n] = \mathbf{E}[X_0]$. So first we show that for every $n \in \mathbb{N}$, $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[X_0]$. Define $Z_n \triangleq X_{\min\{n,\tau\}} = X_0 + \sum_{i=0}^{n-1}(X_{i+1} - X_i)\mathbb{1}[\tau > i]$. We verify that $\{Z_n\}_{n\geq 0}$ is a martingale. By definition

$$\begin{aligned}
\mathbf{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbf{E}[Z_n + (X_{n+1} - X_n)\mathbb{1}[\tau > n] \mid \mathcal{F}_n] \\
&= Z_n + \mathbb{1}[\tau > n]\left(\mathbf{E}[X_{n+1} \mid \mathcal{F}_n] - X_n\right) \\
&= Z_n.
\end{aligned}$$

So we have $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[Z_n] = \mathbf{E}[Z_0] = \mathbf{E}[X_0]$.

Therefore, this motivates us to decompose X_τ into two terms:

$$\forall n \in \mathbb{N},\ X_\tau = X_{\min\{n,\tau\}} + \mathbb{1}[\tau > n] \cdot (X_\tau - X_n).$$

Taking expectation and letting n tend to infinity, we obtain

$$\mathbf{E}[X_\tau] = \mathbf{E}[X_0] + \lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)].$$

Therefore, we only need to verify that each of the three conditions in the statement guarantee $\lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$.

1. If τ is bounded almost surely, then clearly $\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$ for sufficiently large n.
2. In this case,

$$\begin{aligned}
\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\leq \mathbf{E}[\mathbb{1}[\tau > n] \cdot (|X_\tau| + |X_n|)] \\
&\leq 2M \cdot \mathbf{E}[\mathbb{1}[\tau > n]] \\
&= 2M \cdot \mathbf{Pr}[\tau > n] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

3. In order to apply our bounds on the gap between consecutive X_t, we write

$$\begin{aligned}\mathbb{1}[\tau > n] \cdot (X_\tau - X_n) &= \sum_{t=n}^{\tau-1} (X_{t+1} - X_t) \\ &\le \sum_{t=n}^{\tau-1} |X_{t+1} - X_t| \\ &= \sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t].\end{aligned}$$

Taking expectation on both sides, we have

$$\begin{aligned}\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\le \mathbf{E}\left[\sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]\right] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t] \mid \mathcal{F}_t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \cdot \mathbb{1}[\tau > t]] \\ &\le \sum_{t=n}^{\infty} c \cdot \mathbf{Pr}[\tau > t],\end{aligned}$$

where the first equality follows from the monotone convergence theorem.

On the other hand, we know $\mathbf{E}[\tau] = \sum_{t=0}^{\infty} \mathbf{Pr}[\tau > t] < \infty$. Therefore, the tail of this sequence, $\sum_{t=n}^{\infty} \mathbf{Pr}[\tau > t] \to 0$ as $n \to \infty$.

□