SOLUTION TO EXAM IN TEP4100 FLUID MECHANICS
29.05.2012

Problem 1

a)

Force pushing downwards:

Pressure on top of the sphere:

$$P_{\text{top}} = P_{\text{atm}} + \rho_w g(L - 2R)$$

Weight of the fluid block around the top of the sphere:

$$F_{\text{wt}} = \rho_w g(\pi R^3 - (\frac{4}{6}\pi R^3)) = (\frac{\pi}{3})\rho_w g R^3$$

Forces exerted by air and fluid pushing down on top of the sphere:

$$\begin{aligned} F_{\text{down}} &= F_{\text{top pressure}} + F_{\text{wt}} = P_{\text{top}} \cdot A + F_{\text{wt}} \\ &= (P_{\text{atm}} + \rho_w g(L - 2R))\pi R^2 + \left(\frac{\pi}{3}\rho_w g R^3\right) \end{aligned}$$

Weight of the sphere:

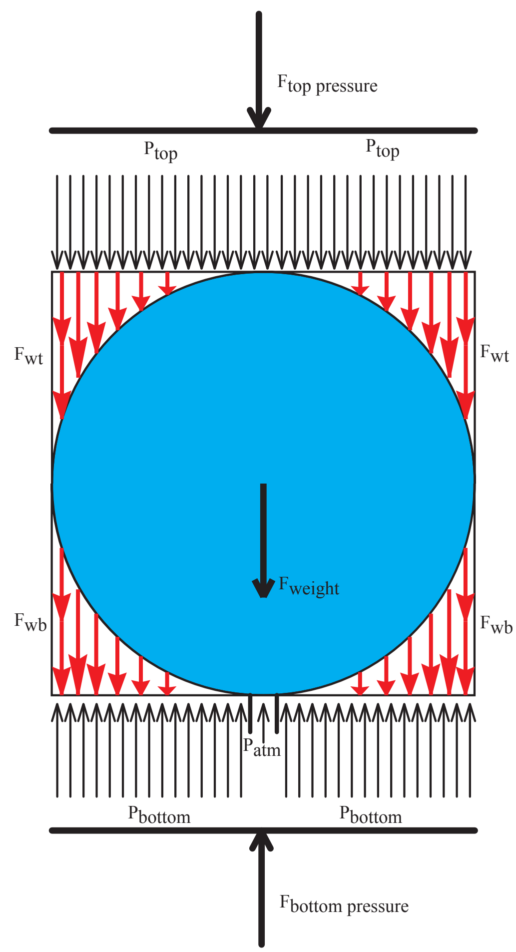
$$F_{\text{weight}} = mg = SG_S \rho_w (\frac{4}{3})\pi R^3 g$$

Forces exerted by air and fluid pushing upwards:

The assumption that the hole radii $a \ll R$ leads us to assume that the fluid block around the bottom of the sphere is approximately the same size as the one around the top of the sphere, $F_{\text{wt}} \cong F_{\text{wb}}$

Forces pushing the sphere upwards:

$$\begin{aligned} F_{\text{up}} &= F_{\text{bottom pressure}} - F_{\text{wb}} \\ &= \left(P_{\text{atm}}\pi a^2 + (P_{\text{atm}} + \rho_w g L)\pi(R^2 - a^2)\right) - \left(\frac{\pi}{3}\rho_w g R^3\right) \end{aligned}$$



For the sphere to remain on the bottom of the tank, the weight of the sphere must be greater than or equal to the net force (or buoyancy force) exerted by the surrounding fluid upwards.

Force balance:

$$SG_S{\cdot}\rho_w g\left(\frac{4}{3}\right)\pi R^3 \geq P_{atm}\pi a^2{+}(P_{atm}{+}\rho_w gL)\pi(R^2{-}a^2){-}\left(\frac{\pi}{3}\right)\rho_w gR^3{-}(P_{atm}{+}\rho_w g(L{-}2R))\pi R^2{-}\left(\frac{\pi}{3}\right)\rho_w gR^3$$

Group terms:

$$SG_S{\cdot}\rho_w g\left(\frac{4}{3}\right)\pi R^3 \geq P_{atm}\pi(a^2{+}(R^2{-}a^2){-}R^2){+}\rho_w gL\pi(R^2{-}a^2){-}\left(\frac{2\pi}{3}\right)\rho_w gR^3{-}\rho_w g(L{-}2R)\pi R^2$$

The Terms including P_{atm} cancel each other out. Divide by $\rho_w g(\frac{4}{3})\pi R^3$ leaves:

$$SG_S \geq \frac{3}{4}\left(\frac{L(R^2-a^2)}{R^3} - \frac{(L-2R)}{R}\right) - \frac{1}{2}$$

Which reduces to:

$$SG_S \geq 1 - \frac{3}{4}\frac{L}{R}\left(\frac{a}{R}\right)^2$$

Alternative solution of 1 a)
Condition for sphere to lie at rest on the bottom:

$$F_{down} > F_{up}, \tag{1}$$

where

$$F_{down} = \frac{4}{3}\pi R^3 \cdot SG_s \cdot \rho g \tag{2}$$

is the weight of the sphere and F_{up} is the upward force being the difference between the ordinary buoyancy force $(4/3)\pi R^3\rho g$ and the hydrostatic pressure on the area πa^2 cut out at the bottom of the sphere. As the hydrostatic pressure at this position is ρgL, it follows that

$$F_{up} = \frac{4}{3}\pi R^3\rho g - \rho gL\pi a^2. \tag{3}$$

Since the pressure is atmospheric outside of the tank, the atmospheric pressure can be omitted to obtain (3).
Inserting (2) and (3) into (1), we obtain the answer,

$$SG_s = 1 - \frac{3La^2}{4R^3}. \tag{4}$$

b)

If the tank has depth $L = 80cm$, the sphere has a radius $R = 2cm$, the hole has a radius $a = 0.2cm$, determine the minimum specific gravity, SG_S, for which the sphere will remain at the bottom of the tank.

$$SG_S = 1 - \frac{3}{4}\left(\frac{80}{2}\right)\left(\frac{0.2}{2}\right)^2 = 0.7$$

Explain briefly in words what will happen if the level of water in the tank drops below a height of $80cm$ for the specific gravity determined above?

The inequality tells us that as the water level decreases, SG_S must increase to remain at the bottom of the tank. If the water level is reduces with SG_S as determined above, the sphere will float to the surface.

c)

The sphere is now removed from the base of the tank and placed in a new tank of water. Show that the height (h) of the sphere protruding above the surface of the water can be represented by the following cubic equation:

$$2h^3 - 6Rh^2 + 8R^3(1 - SG_s) = 0$$

For the sphere to float the weight of the sphere must be equal to the weight of the water displaced, providing $SG_S < 1$.

$$SG_S \cdot \rho_w g V_{\text{sphere}} = \rho_w g V_w$$

So,

$$V_{\text{w}} = SG_S \cdot V_{\text{sphere}} = V_{\text{sphere}} - V_{\text{cap}}$$
$$\Rightarrow V_{\text{cap}} = V_{\text{sphere}}(1 - SG_S)$$

Therefore:

$$\frac{1}{6}\pi h(3a^2 + h^2) = \frac{4}{3}\pi R^3(1 - SG_S)$$

For a sphere $a^2 = h(2R - h)$, as informed in the question sheet. Therefore:

$$\frac{1}{6}\pi(3h(2R - h) + h^2) = \frac{4}{3}\pi R^3(1 - SG_S)$$

Which reduces to a cubic equation with regard to h as follows:

$$2h^3 - 6Rh^2 + 8R^3(1 - SG_S) = 0$$

Utilizing the relationship derived above and using the data for the sphere, $R = 2cm$ and with SG_S as defined in 2 above , select the correct height h, the sphere protrudes above the water line from:

(a) $1.253cm$

(b) $1.453cm$

(c) $2cm$

By computing the cubic equation with h set as the different alternatives one finds that for $h = 1.453cm$ the equation is approximately zero. Hence, the correct answer is **(b)** $h = 1.453cm$

Problem 2

PART I

a) Conservation of mass implies:

$$\dot{\mathcal{V}}_1 = \dot{\mathcal{V}}_2 \qquad \Rightarrow \qquad V_1 h_1 w = V_2 h_2 w \,.$$

Thus:

$$V_2 = \frac{h_1}{h_2} V_1 \,. \tag{5}$$

Since the flow is steady and inviscid, we can apply the Bernoulli equation (5-48) along the streamline on the bottom from section 1 upstream of the dam to section 2 downstream of the dam. We choose the reference level $z = 0$ at the bottom. With the hydrostatic pressure $P = P_o + \rho g h$, we get:

$$P_o + \rho g h_1 + \rho \frac{V_1^2}{2} + 0 = P_o + \rho g h_2 + \rho \frac{V_2^2}{2} + 0 \tag{6}$$

Replacing V_2 in (6) using (5), we get

$$\rho \frac{V_1^2}{2} \left(\frac{h_1^2}{h_2^2} - 1 \right) = \rho g (h_1 - h_2)\,.$$

Using $\frac{h_1^2}{h_2^2} - 1 = \frac{h_1^2 - h_2^2}{h_2^2}$ and canceling $h_1 - h_2$, we obtain:

$$V_1 = h_2 \sqrt{\frac{2g}{h_1 + h_2}}\,. \tag{7}$$

Inserting (7) into (5) yields:

$$V_2 = h_1 \sqrt{\frac{2g}{h_1 + h_2}}\,. \tag{8}$$

b) We choose a control volume to contain sections 1 and 2 and to cut the dam and the bottom, cf. Figure 2(a) showing the control surface CS. Let the direction from section 1 to section 2 be the x-direction. The x-component of the linear momentum equation (6-7) reads:

$$F_x + F_{press_x} + F_{gravity_x} = \frac{d}{dt} \int_{CV} \rho u \, d\mathcal{V} + \int_{CS} \rho u (\vec{V} \cdot \vec{n}) \, dA\,, \tag{9}$$

where F_x is the x-component of the force, which the dam exerts on the water. $F_{press_x} = -\int_{CS} P_{gage} n_x \, dA$ is the x-component of the pressure force on the control surface and $F_{gravity_x} = \int_{CV} \rho g_x \, d\mathcal{V}$ is the is the x-component of the gravity force on the control volume.

Since the pressure is hydrostatic at sections 1 and 2, we can integrate the pressure or just use (3-19) to get:

$$F_{press_x} = \rho g \frac{h_1}{2} h_1 w - \rho g \frac{h_2}{2} h_1 w = \frac{\rho g w}{2} (h_1^2 - h_2^2)\,. \tag{10}$$

Since $g_x = 0$, $F_{gravity_x} = 0$.

As the flow is steady and we have one inlet and one outlet with constant velocities, the right hand side of (9) can be expressed as:

$$\rho V_1 h_1 w (V_2 - V_1)\,.$$

Inserting this expression, F_{press_x} and $F_{gravity_x}$ discussed above into (9), we get:

$$F_x = \frac{\rho g w}{2} (h_2^2 - h_1^2) + \rho V_1 h_1 w (V_2 - V_1)\,. \tag{11}$$

Inserting (7) and (8) into (11), we get:

$$F_x = \frac{\rho g w}{2} (h_2^2 - h_1^2) + \rho h_1 h_2 w \frac{2g}{h_1 + h_2} (h_1 - h_2)\,.$$

Ordering the right hand side yields:

$$F_x = \frac{\rho g w}{2} \frac{(h_2 - h_1)^3}{h_1 + h_2} \, . \tag{12}$$

Due to Newton's 3rd law actio = reactio, the horizontal force which the water exerts on the dam becomes:

$$F_h = -F_x = \frac{\rho g w}{2} \frac{(h_1 - h_2)^3}{h_1 + h_2} \, . \tag{13}$$

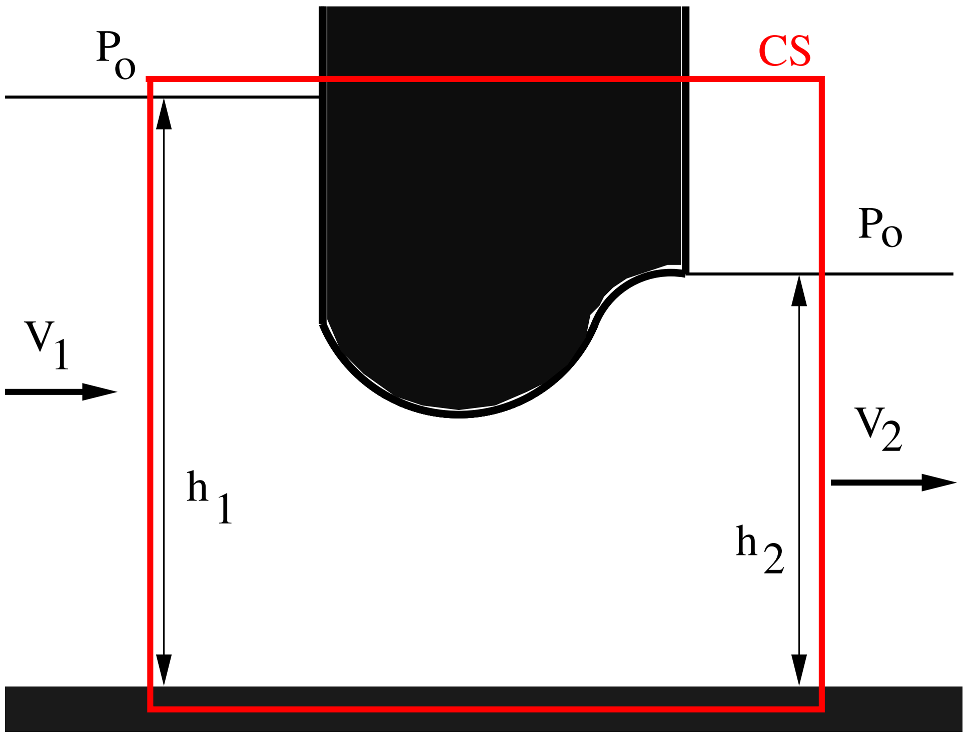


Figure 2(a)

PART II

c) The pump efficiency η is the ratio of the useful power delivered by the pump $\dot{W}_u$ and the actual shaft power delivered to the pump $\dot{W}$:

$$\eta = \frac{\dot{W}_u}{\dot{W}} \, . \tag{14}$$

Due to losses in the pump, we have $\eta < 1$. The useful power delivered by the pump is obtained from:

$$\dot{W}_u = \rho g h_{pump,u} \dot{\mathcal{V}} \, . \tag{15}$$

The useful pump head $h_{pump,u}$ is obtained from the energy equation for steady flow with one inlet and one outlet (5-74):

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 + h_{pump,u} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L \, . \tag{16}$$

We choose points 1 and 2 at the water surfaces of reservoirs A and B, respectively. For there we have atmospheric pressure P_{atm} and the velocity is approximately zero, because the volumes of the reservoirs are much larger than the pumped water volume. Thus, the useful pump head becomes:

$$h_{pump,u} = z_2 - z_1 + h_L = h + h_L\,. \tag{17}$$

The head loss h_L is determined from the Darcy-Weisbach equation (8-24):

$$h_L = f\frac{L}{D}\frac{V_{avg}^2}{2g}\,. \tag{18}$$

The average velocity is obtained from:

$$V_{avg} = \frac{\dot{\mathcal{V}}}{(\pi D^2)/4)}\,. \tag{19}$$

Darcy's friction factor f for laminar pipe flow is given by (8-23):

$$f = \frac{64}{Re}\,, \tag{20}$$

where

$$Re = \frac{\rho V_{avg} D}{\mu} = \frac{4\rho\dot{\mathcal{V}}}{\pi D\mu} \tag{21}$$

is the Reynolds number expressed in terms of the volume flow.

Combining equations (14 - 21), we get for the shaft power that must be applied to a pump with efficiency η and volume flow $\dot{\mathcal{V}}$ for laminar flow:

$$\dot{W} = \frac{1}{\eta}\rho g(h + \frac{128\mu L\dot{\mathcal{V}}}{\pi\rho g D^4})\dot{\mathcal{V}}\,. \tag{22}$$

The higher the pump is placed in the pipeline, the higher becomes the risk that the pressure upstream of the pump falls below the vapor pressure and the water evaporates. Thus, the pump is best placed in the pipeline close to reservoir A such that the pump lies below the water surface of reservoir A. Then, the pipe is in contact with water at a pressure larger than atmospheric pressure. Then, there is no risk that the pressure in the pipeline drops below the vapor pressure.

d) We choose a control volume to contain the pipe section cutting intersection point O and the outlet, cf. Figure 2(b) showing the control surface CS. We choose intersection point O as the origin of the coordinate system with x to the right, y up, and z out of the paper. The angular momentum equation (6-48) reads:

$$\vec{M}_O + \vec{M}_{press} + \vec{M}_{gravity} = \frac{d}{dt}\int_{CV}(\vec{r}\times\vec{V})\rho\,d\mathcal{V} + \int_{CS}(\vec{r}\times\vec{V})\rho(\vec{V}\cdot\vec{n})\,dA\,, \tag{23}$$

where $\vec{M}_O$ is the moment, which the intersection point exerts on the water. $\vec{M}_{press}$ is the moment exerted by the pressure on the control volume. As the gage pressure P_{gage} is zero at the outlet and the vector $\vec{r}$ is zero at the intersection point, we have:

$$\vec{M}_{press} = 0\,. \tag{24}$$

The gravity force acts at the center of gravity of the pipe segment, i.e. at the distance $\frac{b}{2}$ from O, in the negative y-direction. The moment arm is $\frac{b}{2}\cos\beta$. Thus, the moment exerted by the gravity force on the pipe segment becomes:

$$\vec{M}_{gravity} = \frac{b}{2}\cos\beta G\vec{k} \tag{25}$$

contributing a moment in the counter-clockwise direction around O.

As the flow is steady and the only momentum flow that yields a moment is the outlet stream as $\vec{r} = 0$ at O, the right hand side of (14) simplifies to:

$$\int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \vec{n}) \, dA = b \rho V^2 \frac{\pi D^2}{4} \vec{k} \,. \tag{26}$$

Inserting (24 - 26) into (23), we get:

$$\vec{M}_O = b \left(\rho V^2 \frac{\pi D^2}{4} - \frac{1}{2} \cos \beta G \right) \vec{k} \,. \tag{27}$$

Using (19), the average velocity V can be expressed in terms of the volume flow leading to:

$$\vec{M}_O = b \left(\rho \dot{\mathcal{V}}^2 \frac{4}{\pi D^2} - \frac{1}{2} \cos \beta G \right) \vec{k} \,. \tag{28}$$

The moment exerted by the water on the intersection point $\vec{T}_O$ is just the opposite vector of $\vec{M}_O$, i.e.:

$$\vec{T}_O = -\vec{M}_O = b \left(\frac{1}{2} \cos \beta G - \rho \dot{\mathcal{V}}^2 \frac{4}{\pi D^2} \right) \vec{k} \,. \tag{29}$$

If $G < \frac{8 \rho \dot{\mathcal{V}}^2}{\pi D^2 \cos \beta}$, $\frac{1}{2} \cos \beta G - \rho \dot{\mathcal{V}}^2 \frac{4}{\pi D^2} < 0$. Thus, $\vec{T}_O$, the moment exerted by the water on the intersection point has a negative z-component, i.e. the moment is in the clockwise direction around O. Therefore, if $G < \frac{8 \rho \dot{\mathcal{V}}^2}{\pi D^2 \cos \beta}$ and the intersection point O loosens and becomes a hinge, the pipe segment will turn in the clockwise direction around O.

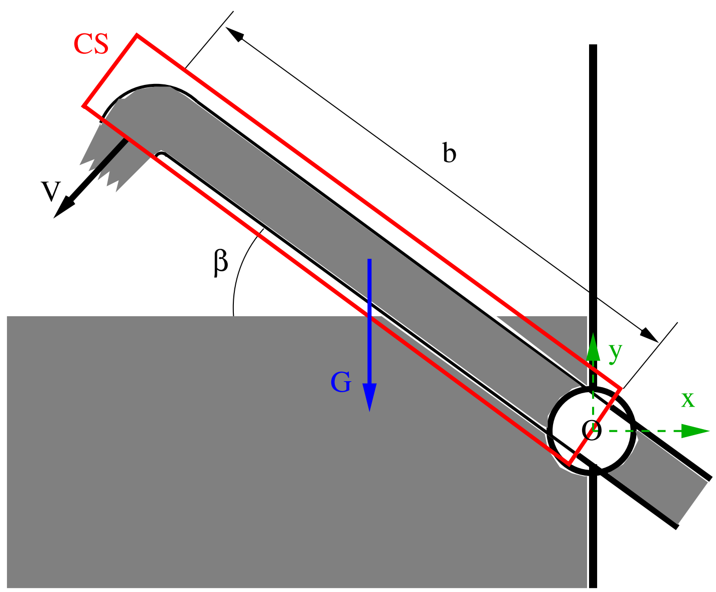


Figure 2(b)

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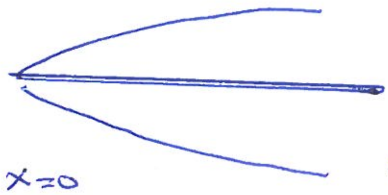
PROBLEM 3



$$u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right), \quad 0 \le y \le \delta(x)$$

a) $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$. Since $\frac{\partial u}{\partial y} = U\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)$ one gets

$$\underline{\tau_w = \mu \frac{2U}{\delta(x)}}$$

Momentum thickness $\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$.

As $u = U$ outside the boundary layer, the upper limit is replaced by δ. Thus

$$\theta = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy$$

$$\underline{\theta} = \delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta = \underline{\frac{2}{15}\delta(x)}$$

b) Given relationship $\tau_w = \rho U^2 \frac{d\theta}{dx}$.

Put the two expressions for τ_w equal:

$$\rho U^2 \cdot \frac{2}{15} \cdot \frac{d\delta(x)}{dx} = \mu \cdot \frac{2U}{\delta(x)}$$

Multiply with $\delta(x)$, and use that $\mu = \rho\nu$:

$$\delta(x)\frac{d\delta(x)}{dx} = \frac{15\nu}{U}dx. \quad \text{Integrate from 0 to x:}$$

$$\frac{1}{2}\Big|_0^x \delta^2(x) = \frac{15\nu}{U}\cdot x$$

Use that $\delta(0)=0$:

$$\frac{1}{2}\delta^2(x) = \frac{15\nu}{U}\cdot x \implies \frac{\delta(x)}{x} = \sqrt{\frac{30\nu}{U\cdot x}} = \frac{\sqrt{30}}{\sqrt{Re_x}} = \frac{5{,}48}{\sqrt{Re_x}}$$

Comparison with $\frac{\delta(x)}{x} = \frac{C}{\sqrt{Re_x}}$ yields $\underline{C = 5{,}48}$

From formula sheet: $\frac{\delta}{x} = \frac{4{,}91}{\sqrt{Re_x}}$, for laminar boundary layer.

The present model gives thus a laminar layer, approximately.

c) Total drag force follows by integrating τ_w over the area, multiplying with a factor 2 (upper/lower sides):

$$F_D = 2\int_{Area} \tau_w \cdot dA = 2L\int_0^L \tau_w\, dx$$

From above is $\tau_w = \frac{2\mu U}{\delta(x)} = \frac{2\mu U^{3/2}}{\sqrt{30\nu}}\cdot\frac{1}{\sqrt{x}}$

Thus $F_D = 2L\frac{2\mu U^{3/2}}{\sqrt{30\nu}}\int_0^L \frac{dx}{\sqrt{x}}$

Integral $\int_0^L \frac{dx}{\sqrt{x}} = \Big|_0^L 2\sqrt{x} = 2\sqrt{L}$, thus

$$\underline{F_D} = 2L\frac{2\mu U^{3/2}}{\sqrt{30\nu}}\cdot 2\sqrt{L} = \underline{8\rho\sqrt{\frac{\nu}{30}}(UL)^{3/2}}$$

Drag coefficient defined by $F_D = C_D\cdot\frac{1}{2}\rho U^2\cdot \underbrace{A}_{L^2}$

$$\implies C_D\cdot\frac{1}{2}\rho U^2\cdot L^2 = 8\rho\sqrt{\frac{\nu}{30}}(UL)^{3/2}$$

Thus

$$\underline{C_D = \frac{16}{\sqrt{30}}\frac{1}{\sqrt{Re_L}}, \quad \text{where } Re_L = \frac{UL}{\nu}}$$

En litt mer elegant løsning er å bruke at $\tau_w = \rho U^2 \frac{d\theta}{dx}$:

$$\frac{1}{2}F_D = L\int_0^L \tau_w = \rho U^2 L \int_0^L \frac{d\theta}{dx}dx = \rho U^2 L \int_{\theta(0)}^{\theta(L)} d\theta = \rho U^2 L[\theta(L) - \underbrace{\cancel{\theta(0)}}_{=0}]$$

$$= \rho U^2 L\theta(L) = \frac{2}{15}\rho U^2 L\delta(L).$$

Fra 3a)

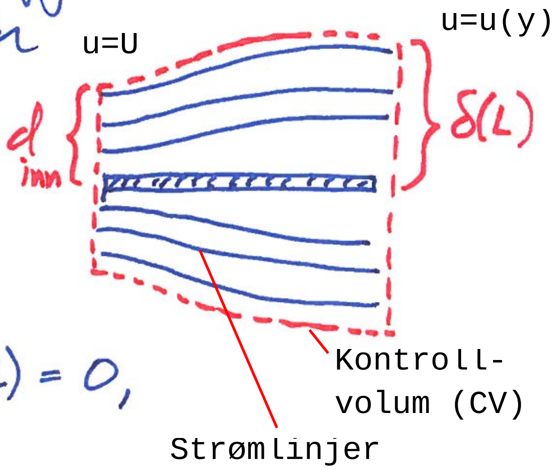
Merk: nye grenser som følge av substitusjonen!

Fra a) og b) har vi at $\theta(L) = \frac{2}{15}\delta(L) = \frac{2}{15}\frac{\sqrt{30}\,L}{\sqrt{Re_L}} = \frac{4}{\sqrt{30}}\frac{L}{\sqrt{Re_L}}$.

Dermed: $\underline{\underline{F_D = \frac{8}{\sqrt{30}}\frac{\rho U^2 L^2}{\sqrt{Re_L}}}}$, og $C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L^2} = \underline{\underline{\frac{16}{\sqrt{30}\sqrt{Re_L}}}}$.

Alternativ løsning v.h.a. kontrollvolumanalyse

3c) kan også løses med Newtons 2. lov for kontrollvolum. Med den informasjonen vi har bør (må) vi legge CV slik at det følger en strømlinje. Bredde på innløpsflaten ved massebevarelse: $-\dot{Q}_{inn} + \dot{Q}_{ut} = 0$:



$$-\frac{1}{2}\dot{Q}_{inn} + \frac{1}{2}\dot{Q}_{ut} = \frac{1}{2}\sum \dot{Q} = \underset{(inn)}{-U d_{inn}} + \underset{(ut)}{\int_0^{\delta(L)} u(y)\,dy}$$

$$= -U d_{inn} + \int_0^{\delta(L)}\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)dy = -U d_{inn} + \frac{2}{3}U\delta(L) = 0,$$

$$\Rightarrow \quad \underline{d_{inn} = \frac{2}{3}\delta(L).}$$

Vi kan nå beregne kreftene på CV i x-retning:

$$\frac{1}{2}\sum F_x = \oint_{CS} \rho u(\vec{u}\cdot\vec{n})dS = \underset{(innløp)}{-\rho L U^2 \cdot \frac{2}{3}\delta(L)} + \underset{(utløp)}{L\int_0^{\delta(L)}\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^2 dy \cdot U^2\rho}$$

$$= -\frac{2}{3}\rho L U^2\delta(L) + \rho U^2 L\int_0^{\delta(L)}\left(\frac{4y^2}{\delta^2} - \frac{4y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)dy = \frac{-2}{15}\rho L U^2\delta(L).$$

Krafta fra vannet på plata er like stor og motsatt retta. Dermed er svaret det samme som før.