Norwegian University of Science and Technology

TTK4135 – Lecture 18
Sequential Quadratic Programming (SQP)

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Outline

- Recap: Newton's method for solving nonlinear equations
- Recap: Equality-constrained QPs

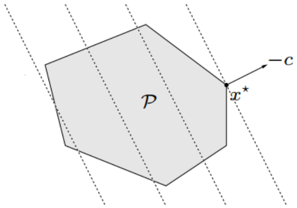
- SQP for *equality-constrained* nonlinear programming problems
  - Next time: SQP for general

Reference: N&W Ch.18-18.1

Types of Constrained Optimization Problems

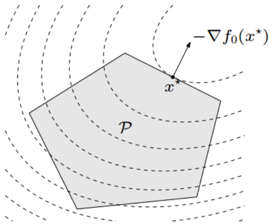
- Linear programming
  - Convex problem
  - Feasible set polyhedron

$$\begin{array}{rl} \min & c^\top x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$



- Quadratic programming
  - Convex problem if $P \geq 0$
  - Feasible set polyhedron

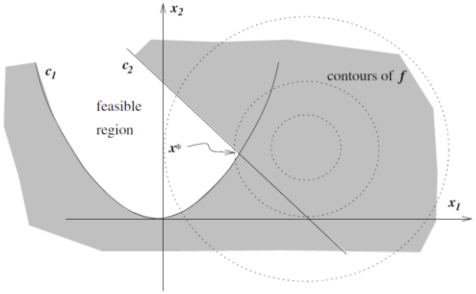
$$\begin{array}{rl} \min & \frac{1}{2}x^\top Px + q^\top x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$



- Nonlinear programming
  - In general non-convex!

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & g(x) = 0 \\ & h(x) \geq 0 \end{array}$$

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

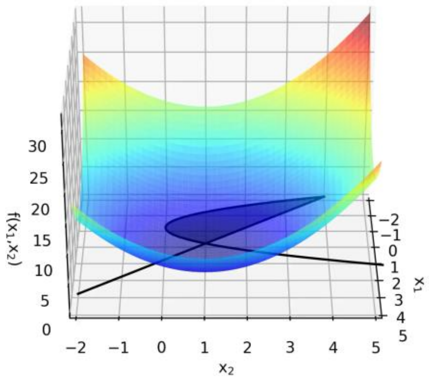


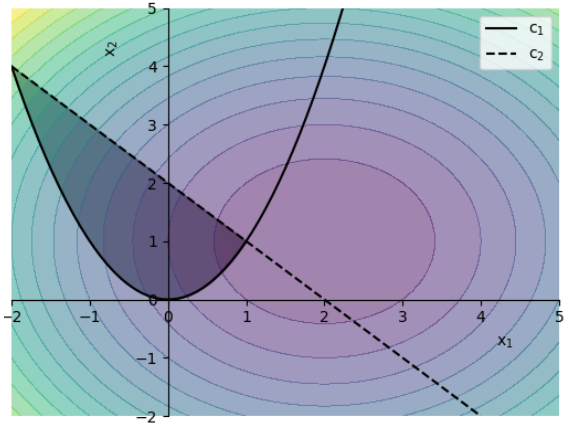
General Optimization Problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

- Example:

$$\min \, (x_1 - 2)^2 + (x_2 - 1)^2 \quad \text{subject to} \quad \begin{array}{l} x_1^2 - x_2 \leq 0, \\ x_1 + x_2 \leq 2. \end{array}$$





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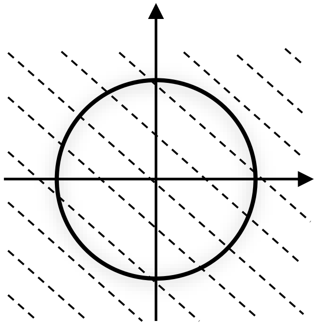
Today: Only equality constraints



The Lagrangian

For constrained optimization problems, introduce modification of objective function:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

- Multipliers for *equality* constrains may have both signs in a solution
- Multipliers for *inequality* constraints cannot be negative (cf. shadow prices)
- For (inequality) constraints that are *inactive*, multipliers are zero

KKT conditions (Theorem 12.1)

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

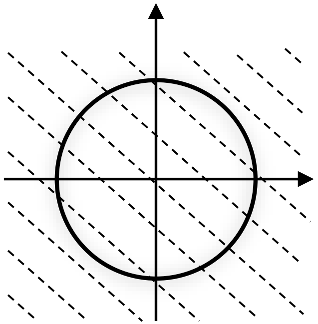
KKT conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, & & \text{(stationarity)} \\
c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, & & \\
c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, & & \text{(primal feasibility)} \\
\lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, & & \text{(dual feasibility)} \\
\lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. & & \text{(complementarity condition/ complementary slackness)}
\end{aligned}$$

Either $\lambda_i^* = 0$ *or* $c_i(x^*) = 0$

Example KKT system

$$\min_{x \in \mathbb{R}^2} \; -x_1 - x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 = 0$$



Today: Equality-constrained NLP

Newton's method for solving nonlinear equations (Ch. 11)

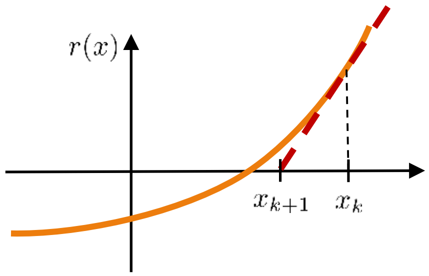
- Solve equation system $r(x) = 0, \quad r(x) : \mathbb{R}^n \to \mathbb{R}^n$
- Assume Jacobian $J(x) \in \mathbb{R}^{n \times n}$ exists and is continuous
- Taylor: $r(x+p) = r(x) + J(x)p + O(\|p\|^2)$

$$J(x) = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \cdots \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Algorithm 11.1 (Newton's Method for Nonlinear Equations).
Choose x_0;
for $k = 0, 1, 2, \ldots$
  Calculate a solution p_k to the Newton equations

$$J(x_k)p_k = -r(x_k);$$

  $x_{k+1} \leftarrow x_k + p_k$;
end (for)



- Convergence rate (Thm 11.2): Quadratic convergence if $J(x)$ is invertible
  (quadratic convergence is very good, but only holds close to the solution)

- If we set $r(x) = \nabla f(x)$, then this method corresponds to Newton's method for minimizing $f(x)$

$$p_k = -J(x_k)^{-1} r(x_k) \quad \longleftrightarrow \quad p_k = -\left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$$

Newton's method to solve $\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}$

$$F(x, \lambda) = \begin{pmatrix} \nabla f(x) - A^\top(x)\lambda \\ c(x) \end{pmatrix}$$

Equality-constrained QP (EQP)

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{2}x^\top Gx + c^\top x$$
$$\text{subject to} \quad Ax = b, \quad A \in \mathbb{R}^{m\times n}$$

Basic assumption:
A full row rank

- KKT-conditions (KKT system, KKT matrix):

$$\begin{pmatrix} G & -A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \quad \text{or, if we let } x^* = x + p, \quad \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} c + Gx \\ Ax - b \end{pmatrix}$$

- Solvable when $Z^\top GZ > 0$ (columns of *Z* basis for nullspace of *A*)

- That is: QP with only equality constraints is solved by a solving a set of linear equations (*linear system*)

Alternative “derivation” of KKT-system

$$\min_{x\in\mathbb{R}^n} \; f(x) \quad \text{s.t.} \quad c(x) = 0$$

Local SQP-algorithm for solving equality-constrained NLPs

$$\min f(x)$$
$$\text{subject to } c(x) = 0$$

Algorithm 18.1 (Local SQP Algorithm for solving (18.1)).

Choose an initial pair (x_0, λ_0); set $k \leftarrow 0$;
repeat until a convergence test is satisfied
  Evaluate f_k, ∇f_k, $\nabla^2_{xx}\mathcal{L}_k$, c_k, and A_k;
  Solve (18.7) to obtain p_k and l_k;
  Set $x_{k+1} \leftarrow x_k + p_k$ and $\lambda_{k+1} \leftarrow l_k$;
end (repeat)

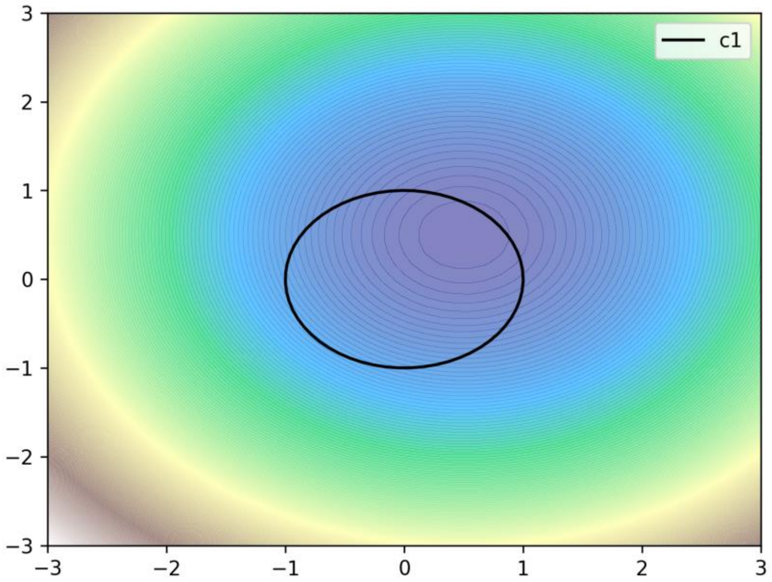
EQP:

$$\min_p \quad f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla^2_{xx}\mathcal{L}_k p$$
$$\text{subject to} \quad A_k p + c_k = 0.$$

Local SQP

$$f(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$$

$$c_1(x) = x_1^2 + x_2^2 - 1 = 0$$



Local SQP

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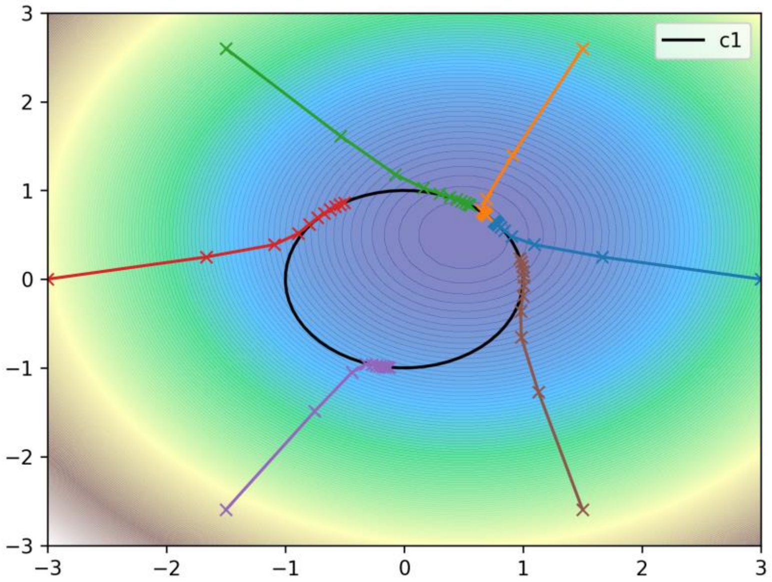

$$
\begin{aligned}
\min_{p} \quad & f_k + \nabla f_k^T p + \tfrac{1}{2} p^T \nabla_{xx}^2 \mathcal{L}_k p \\
\text{subject to} \quad & A_k p + c_k = 0.
\end{aligned}
$$