Department of Engineering Cybernetics

TTK4215 SYSTEM IDENTIFICATION AND ADAPTIVE CONTROL
SOLUTION OF ASSIGNMENT 5

Problem 2.7:

Part (a)

The plant can be expressed by a differential equation

$$\dddot{y} = -a_2\ddot{y} - a_1\dot{y} - a_0 y + b_2\ddot{u} + b_1\dot{u} + b_0 u \tag{1}$$

Equation (1) is expressed as

$$\dddot{y} = \theta^T Y \tag{2}$$

where

$$\theta^T = \left[b_0, b_1, b_2, a_2, a_1, a_0\right] \tag{3}$$

and

$$Y = \left[\ddot{u}, \dot{u}, u, -\ddot{y}, -\dot{y}, -y\right]^T \tag{4}$$

If we define the operator vector

$$\alpha_i(s) = \left[s^i, s^{i-1}, \cdots, 1\right]^T \tag{5}$$

we can write (4) in the form

$$Y = \left[\alpha_2^T(s)u, -\alpha_2^T(s)y\right]^T \tag{6}$$

To avoid differentiation, we filter both sides of (2) with an 3-order stable filter

$$\begin{aligned} &\frac{1}{\Lambda(s)} \\ &\text{where} \quad \Lambda(s) = s^3 + \lambda^T \alpha_2(s) \\ &\text{where} \quad \lambda^T = [\lambda_1, \lambda_0] \end{aligned} \tag{7}$$

Therefore, the parametric model of the system is expressed as

$$z = \theta^T \phi \tag{8}$$

where

$$z = \frac{s^3}{\Lambda(s)} y \tag{9}$$

and

$$\begin{aligned} &\phi = \frac{Y}{\Lambda(s)} \\ &\Rightarrow \phi = \left[\frac{\alpha_2^T(s)}{\Lambda(s)} u, -\frac{\alpha_2^T(s)}{\Lambda(s)} y \right]^T \end{aligned} \tag{10}$$

The filter can be chosen, for example $\Lambda(s) = (s+a)^3$ in which $a > 0$ is a constant design parameter we can choose arbitrarily. Consequently, we can write

$$\frac{\alpha_2^T(s)}{\Lambda(s)} = \left[\frac{s^2}{(s+a)^3}, \frac{s}{(s+a)^3}, \frac{1}{(s+a)^3} \right] \tag{11}$$

You can use Parameterization 2, of course.

Part (b)

Since the parameters on the denominator are known, the vector of unknown parameters is

$$\theta^T = \left[b_0, b_1, b_2 \right] \tag{12}$$

We can write (1) as

$$\begin{aligned} &\dddot{y} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{u} + b_1 \dot{u} + b_0 u \\ &\rightarrow (s^3 + a_2 s^2 + a_1 s + a_0) y = \theta^T Y, \quad Y = \alpha_2(s) u \end{aligned} \tag{13}$$

that can be written as

$$\begin{aligned} &y = \theta^T \phi \\ &\text{in which } \phi^T = \frac{\alpha_2^T(s)}{\Gamma(s)} u = \frac{\left[s^2 \quad s \quad 1 \right] u}{s^3 + a_2 s^2 + a_1 s + a_0} \end{aligned} \tag{14}$$

In this case, we do not need to filter the signals with a proper stable filter since we can use the inherent filter of the system which is available.

Part (c)

In this part, we have:

$$\dddot{y} = -a_2 \ddot{y} - a_1 \dot{y} - a_0 y + u \tag{15}$$

Defining $\theta^T = [a_2, a_1, a_0]$, we have

$$\dddot{y} - u = \theta^T Y, \quad Y^T = -\alpha_2^T(s) y \tag{16}$$

Filtering with $\frac{1}{\Lambda(s)}$ will result in

$$z = \theta^T \phi \tag{17}$$

with

$$z = \frac{s^3}{\Lambda(s)} y - \frac{1}{\Lambda(s)} u \tag{18}$$

and

$$\phi^T = -\frac{\alpha_2^T(s)}{\Lambda(s)} y \tag{19}$$

Notice that $\frac{1}{\Lambda(s)}$ is a stable filter of order three.

Problem 2.8:

Part (a)

Choosing the states of the system as

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \tag{20}$$

the state-space representation of the system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= M^{-1}\left(-kx_1 - fx_2 + u\right) \end{aligned} \tag{21}$$

The matrices of the system are

$$A = \begin{bmatrix} 0 & 1 \\ -M^{-1}k & -M^{-1}f \end{bmatrix}, B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tag{22}$$

Part (b)

Since we have to find the relation between *x* and *u*, so we choose the output as

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{23}$$

The transfer function of the system is obtained using

$$G(s) = C\left(sI_2 - A\right)^{-1} B \tag{24}$$

Definitely, you can use a simpler method by using Laplace transformation

$$G(s) = \frac{1}{\alpha_2^T(s)\theta}, \theta = \left[M, f, k\right]^T \tag{25}$$

Part (c)

The differential equation describing the system is

$$\ddot{x} = \frac{1}{M}u - \left(\frac{f}{M}, \frac{k}{M}\right)\alpha_1^T(s)x \tag{26}$$

Thus, we have

$$\begin{aligned} \ddot{x} &= \theta^T Y \\ \theta^T &= \left[\frac{1}{M}, \frac{f}{M}, \frac{k}{M}\right] \\ Y^T &= \left[u, -\alpha_1^T(s)x\right] \end{aligned} \tag{27}$$

To avoid use of differentiators, we pass signals through a stable filter, for example

$$\frac{1}{\Lambda(s)}=\frac{1}{(s+a)^2},a>0 \tag{28}$$

and then we obtain

$$\begin{aligned} z &= \theta^T \phi \\ z &= \frac{s^2}{(s+a)^2}x \\ \phi &= \left[\frac{1}{(s+a)^2}u, -\frac{\alpha_1^T(s)}{(s+a)^2}x\right] \end{aligned} \tag{29}$$

NOTE: In case we like to have

$$\theta^* = [M, f, k]^T \tag{30}$$

similar to what has been mentioned in the problem description, we can have

$$\begin{aligned} &M\ddot{x} + f\dot{x} + kx = u \\ &\rightarrow u = \theta^{*T}\alpha_2(s)x \end{aligned} \tag{31}$$

Since derivatives of the signals are not measured, we filter both sides of (31)

$$\underbrace{\frac{1}{\Lambda(s)}u}_{z} = \theta^{*T}\underbrace{\frac{\alpha_2(s)x}{\Lambda(s)}}_{\phi} \tag{32}$$

Problem 2.10:

To refresh your memory of definitions

$$\Lambda(s) = s^n + \lambda^T \alpha_{n-1}(s) \tag{33}$$

where $\lambda^T = [\lambda_{n-1}, \cdots, 1]$.

$$\alpha_n(s) = \begin{bmatrix} s^n \\ s^{n-1} \\ \vdots \\ 1 \end{bmatrix} \tag{34}$$

Also, it is recommended to review pages 36-37 of the course book.

For canonical controller form, we define Λ_c and l similar to page 50 of the course book

Part (a)

The system is strictly proper

$$\begin{aligned} \dot{x} &= \Lambda_c x + lu \\ \phi &= I_n x \end{aligned} \tag{35}$$

The matrix I_n is the identity matrix of order *n*.

Part (b)

The system is not strictly proper; that is, it is proper.

$$\Lambda_1(s) = s^{n-1} + \lambda_1^T \alpha_{n-2}(s), \quad \text{where} \quad \lambda_1 \in \Re^{n-1} \tag{36}$$

Thus,

$$\phi = \begin{bmatrix} \dfrac{s^{n-1}}{\Lambda_1(s) = s^{n-1} + \lambda_1^T \alpha_{n-2}(s)} \\ \dfrac{s^{n-2}}{\Lambda_1(s)} \\ \vdots \\ \dfrac{1}{\Lambda_1(s)} \end{bmatrix} u = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u + \begin{bmatrix} -\lambda_1^T \\ I_{n-1} \end{bmatrix} \frac{\alpha_{n-2}(s)}{\Lambda_1(s)} u \tag{37}$$

If Λ_{1c} is the corresponding controller form for Λ_1, we find the state-space representation as

$$
\begin{aligned}
\dot{x} &= \Lambda_{1c} x + l u \\
\phi &= \begin{bmatrix} -\lambda_1^T \\ I_{n-1} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u
\end{aligned}
\tag{38}
$$

Part (c)

In case *m<n-1* the system is strictly proper whereas if *m=n-1* the system is proper

Problem 3.3

Part (a)

The sequence $u(t)$ belongs to $\mathcal{L}_p$ if $\|u\|_p$ exists.

$$\|u\|_1 = \int_0^\infty |u(\tau)| d\tau = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| < \infty \Rightarrow u \in \mathcal{L}_1 \tag{39}$$

$$\|u\|_2 = \left(\int_0^\infty |u(\tau)|^2 d\tau \right)^{1/2} = \left(\sum_{n=1}^{\infty} \left| \frac{n^2}{n^3} \right| \right)^{1/2} = \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \right| \right)^{1/2} = \infty \Rightarrow u \notin \mathcal{L}_2 \tag{40}$$

$$\begin{aligned} &\|u\|_\infty = \sup_{t>0} |u(\tau)| = \infty \Rightarrow u \notin \mathcal{L}_\infty \\ &u(t) \to \infty \end{aligned} \tag{41}$$

Part (b)

The impulse response of the system is given by the inverse Laplace transform of the transfer function; thus,

$$h(t) = \mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{-t} \tag{42}$$

since

$$\int_0^\infty h(t)dt = \left[-e^{-t} \right]_{t=0}^{t=\infty} = 1 \to h(t) \in \mathcal{L}_1 \tag{43}$$

From Corollary 3.3.1, it follows that $y \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, $\dot{y} \in \mathcal{L}_1$, y is continuous, and $\lim_{t\to\infty} y = 0$

Problem 3.13

G_3 is not PR since it is non-minimum-phase and its inverse is not PR. (Cololllary 3.5.1)

G_4 is not PR since the relative degree is 2. (Cololllary 3.5.1)

From Theorem 3.2.1 at page 127, it follows that G_1 is not SPR while it is PR. All poles and zeros are stable, and

$$\mathrm{Re}\{G_1\} = \frac{20}{\left(4-\omega^2\right)^2 + 25\omega^2} > 0$$

However, for all frequencies and

$$\lim_{|\omega|\to\infty} \omega^2 \,\mathrm{Re}\left[G_1(j\omega)\right] = 0 \tag{44}$$

Therefore, because the condition (iiia) of Theorem 3.5.1 is violated, it is not SPR. You can see a general statement about a special class of systems at the top of page 128.

G_2 is PR since all conditions of Lemma 3.5.1 hold. It is stable and min-phase. The relative degree is one. For all frequencies

$$\mathrm{Re}\left[G_2(j\omega)\right] = \frac{4\omega^2}{\left(4-\omega^2\right)^2 + 16\omega^2} \tag{45}$$

This function does not satisfy condition (ii) of Theorem 3.5.1 since $\mathrm{Re}\left[G_2(j\omega)\right] = 0$ for $\omega = 0$ although condition (iiia) holds.

- (OPTIONAL) For study of positive realness, we can use Theorem 3.5.1 and Lemma 3.5.1. Definition 3.5.1 is not recommended to verify even though it leads to the correct outcome. For instance, for G_2, one can use Definition 3.5.2 and form $G_2(s-\varepsilon)$ for some $\varepsilon > 0$, but you cannot show that $G_2(s-\varepsilon)$ is PR since condition (iii) of Lemma 3.5.1 does not hold **for all frequencies**.

Problem 3.17

According to Lemma 3.5.3, since W_m is SPR, for any given $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$\begin{aligned} &A^T P + PA = -Q \\ &PB = C \end{aligned} \tag{46}$$

We can choose this Lyapunov function candidate

$$V = e^T Pe + \phi^T \phi \tag{47}$$

which is positive definite and radially unbounded. The time derivative of that is given by

$$\begin{aligned} &\dot{V} = \dot{e}^T Pe + e^T P\dot{e} + 2\phi^T \dot{\phi} \\ &\to \dot{V} = \left(e^T A^T + \phi^T B^T \sin t\right) Pe + e^T P\left(Ae + B\phi \sin t\right) + 2\phi^T \left(-C^T e \sin t\right) \\ &\to \dot{V} = e^T \left(A^T P + PA\right) e + \phi^T B^T Pe \sin t + e^T PB\phi \sin t - 2\phi^T C^T e \sin t \\ &\to \dot{V} = -e^T Qe + 2\phi^T B^T Pe \sin t - 2\phi^T C^T e \sin t \\ &\to \dot{V} = -e^T Qe + 2\phi^T C^T e \sin t - 2\phi^T C^T e \sin t \\ &\to \dot{V} = -e^T Qe \end{aligned} \tag{48}$$

Therefore, the time derivative of V is negative semi-definite since ϕ is missing. It implies that $e, \phi \in \mathcal{L}_\infty$; consequently $e_1, \dot{e}, \dot{\phi} \in \mathcal{L}_\infty$. Since $V > 0$ and $\dot{V} \le 0$, $\lim_{t\to\infty} V(t) = V_\infty$ exists. Thus,

$$-\int_0^\infty e^T Qe d\tau < V_\infty - V_0 \tag{49}$$

implies that $e \in \mathcal{L}_2$. As $\left|\dot{\phi}\right| \le |e|$, we can say $\dot{\phi} \in \mathcal{L}_2$. Accordingly, $\dot{\phi} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Because $e \in \mathcal{L}_2$ and $\dot{e} \in \mathcal{L}_\infty$, according to the Barbalat lemma, we conclude $e \to 0$; as a result, $\dot{\phi} \to 0$.

See also Theorem 3.5.2.