TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Solution to homework assignment 1

Problem 1: State-space equation, transfer function and impulse response

a) Using the differential equation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} - \dot{u} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -2\dot{y} + 4u \end{bmatrix} = \begin{bmatrix} \dot{y} - u + u \\ -2(\dot{y} - u) + 2u \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -2x_2 + 2u \end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u.$$

Note that $y = x_1$. Therefore, we have

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du, \end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

b) First, we compute $(s\mathbf{I} - \mathbf{A})^{-1}$ as follows:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

Using $\hat{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$, it follows that

$$\hat{G}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s} + \frac{2}{s^2 + 2s} = \frac{s+4}{s^2 + 2s}.$$

c) Applying the Laplace transform to the differential equation while assuming zero initial conditions yields

$$s^2 y(s) + 2sy(s) = su(s) + 4u(s).$$

It follows that

$$\hat{G}(s) = \frac{y(s)}{u(s)} = \frac{s+4}{s^2+2s},$$

which is the same result as obtained in the previous question.

d) To compute the constants α_1 and α_2, note that

$$\hat{G}(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{s(s+2)} + \frac{s\alpha_2}{s(s+2)} = \frac{(\alpha_1+\alpha_2)s + 2\alpha_1}{s^2+2s} = \frac{s+4}{s^2+2s}.$$

From this, we obtain the equations

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 = 4.$$

Solving for α_1 and α_2 yields $\alpha_1 = 2$ and $\alpha_2 = -1$. Therefore, we have

$$\hat{G}(s) = \frac{2}{s} + \frac{-1}{s+2}.$$

The corresponding impulse response is given by

$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)] = \mathcal{L}^{-1}\left[\frac{2}{s} + \frac{-1}{s+2}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2 - e^{-2t}.$$

Problem 2: Realizations

a) A transfer matrix is realizable if and only if it is proper and rational.

- **Proper:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is proper if the degree of its denominator $d(s)$ is larger than or equal to the degree of its numerator $n(s)$, i.e. $\deg d(s) \geq \deg n(s)$. A transfer matrix is proper if all its elements (i.e. transfer functions) are proper.
- **Rational:** A transfer function $\hat{G}(s) = \frac{n(s)}{d(s)}$ is rational if the degrees of the numerator $n(s)$ and the denominator $d(s)$ are finite. A transfer matrix is rational if all its elements (i.e. transfer functions) are rational.

b) The transfer matrix $\hat{\mathbf{G}}(s)$ can be written as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \hat{G}_{11}(s) & \hat{G}_{12}(s) \\ 0 & \hat{G}_{22}(s) \end{bmatrix},$$

with

$$\begin{aligned} \hat{G}_{11}(s) &= \frac{n_{11}(s)}{d_{11}(s)} = \frac{s^2+4s+2}{s^2+2s}, \\ \hat{G}_{12}(s) &= \frac{n_{12}(s)}{d_{12}(s)} = \frac{3}{s+2}, \\ \hat{G}_{22}(s) &= \frac{n_{22}(s)}{d_{22}(s)} = \frac{2s^2}{s^2-4}. \end{aligned}$$

The degrees of the numerator and denominator polynomials of the transfer functions are

$$\begin{aligned}
\deg n_{11}(s) &= 2, & \deg d_{11}(s) &= 2,\\
\deg n_{12}(s) &= 0, & \deg d_{12}(s) &= 1,\\
\deg n_{22}(s) &= 2, & \deg d_{22}(s) &= 2.
\end{aligned}$$

Because the degrees of the denominators of the transfer functions $\hat{G}_{11}(s)$, $\hat{G}_{12}(s)$ and $\hat{G}_{22}(s)$ are larger than or equal to the degrees of the corresponding numerators, the transfer matrix is proper. Moreover, because the degrees of the numerators and denominators of each transfer function are finite, the transfer matrix is rational. Hence, it follows that the transfer matrix is realizable.

c) The constant matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \lim_{s\to\infty} \hat{\mathbf{G}}(s) = \begin{bmatrix} \lim\limits_{s\to\infty} \frac{s^2+4s+2}{s^2+2s} & \lim\limits_{s\to\infty} \frac{3}{s+2} \\ 0 & \lim\limits_{s\to\infty} \frac{2s^2}{s^2-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, the strictly proper transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ can be calculated as

$$\begin{aligned}
\hat{\mathbf{G}}_{sp}(s) = \hat{\mathbf{G}}(s) - \mathbf{D} &= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - 1 & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - 2 \end{bmatrix}\\
&= \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} - \frac{s^2+2s}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - \frac{2s^2-8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix}.
\end{aligned}$$

Hence, we obtain $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, with $\hat{\mathbf{G}}_{sp}(s)$ and $\mathbf{D}$ defined above.

d) For notational convenience, we write

$$\hat{\mathbf{G}}_{sp}(s) = \begin{bmatrix} \hat{G}_{sp11}(s) & \hat{G}_{sp12}(s) \\ 0 & \hat{G}_{sp22}(s) \end{bmatrix},$$

with transfer functions

$$\begin{aligned}
\hat{G}_{sp11}(s) &= \frac{n_{sp11}(s)}{d_{sp11}(s)} = \frac{2s+2}{s^2+2s},\\
\hat{G}_{sp12}(s) &= \frac{n_{sp12}(s)}{d_{sp12}(s)} = \frac{3}{s+2},\\
\hat{G}_{sp22}(s) &= \frac{n_{sp22}(s)}{d_{sp22}(s)} = \frac{8}{s^2-4}.
\end{aligned}$$

To find the least common denominator for the transfer functions of the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$, we write the denominator of each transfer function as a product of first-order factors:

$$\begin{aligned}
d_{sp11}(s) &= s^2 + 2s = s(s+2),\\
d_{sp12}(s) &= s + 2,\\
d_{sp22}(s) &= s^2 - 4 = (s+2)(s-2).
\end{aligned}$$

The least common denominator is given by

$$d(s) = s(s+2)(s-2) = s^3 - 4s.$$

From this, we obtain that

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3,$$

with $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = -4$.

Next, the transfer matrix $\hat{\mathbf{G}}_{sp}(s)$ is written as

$$\begin{aligned}
\hat{\mathbf{G}}_{sp}(s) &= \begin{bmatrix} \frac{2s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} & \frac{3}{s+2} \\ 0 & \frac{8}{(s+2)(s-2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+2}{s(s+2)} \cdot \frac{s-2}{s-2} & \frac{3}{s+2} \cdot \frac{s(s-2)}{s(s-2)} \\ 0 & \frac{8}{(s+2)(s-2)} \cdot \frac{s}{s} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2s^2-2s-4}{s^3-4s} & \frac{3s^2-6s}{s^3-4s} \\ 0 & \frac{8s}{s^3-4s} \end{bmatrix} = \frac{1}{s^3-4s} \begin{bmatrix} 2s^2-2s-4 & 3s^2-6s \\ 0 & 8s \end{bmatrix} \\
&= \frac{1}{s^3-4s} \left\{ \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} s + \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Hence, we obtain

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)} \left[\mathbf{N}_1 s^2 + \mathbf{N}_2 s + \mathbf{N}_3 \right],$$

with

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{N}_3 = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

e) Substitution of $\mathbf{D}$, α_1, α_2, α_3, $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ yields

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} 2 & 3 & -2 & -6 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{u}(t).
\end{aligned}$$

Problem 3: Similarity transforms and equivalent state-space equations

a) Using the equations of the coordinate transformation (2) and the system (1), we obtain

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{T}\mathbf{A}\mathbf{x} + \mathbf{T}\mathbf{B}u = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{T}\mathbf{B}u$$

and

$$y = \mathbf{C}\mathbf{x} + Du = \mathbf{C}\mathbf{T}^{-1}\bar{\mathbf{x}} + Du.$$

Hence, we get

$$\begin{aligned}\dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}u\\ y &= \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{D}u,\end{aligned}$$

with

$$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \bar{\mathbf{B}} = \mathbf{T}\mathbf{B}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad \text{and} \quad \bar{D} = D.$$

Substituting the values for **A**, **B**, **C**, D and **T** yields

$$\begin{aligned}\bar{\mathbf{A}} &= \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}\begin{bmatrix}-2 & 4\\ -1 & 3\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & 1\\ -\frac{1}{2} & 0\end{bmatrix} = \begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}\\ \bar{\mathbf{B}} &= \mathbf{T}\mathbf{B} = \begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}\begin{bmatrix}8\\ 2\end{bmatrix} = \begin{bmatrix}-4\\ 6\end{bmatrix}\\ \bar{\mathbf{C}} &= \mathbf{C}\mathbf{T}^{-1} = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & 1\\ -\frac{1}{2} & 0\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}\\ \bar{D} &= D = 2.\end{aligned}$$

b) Because $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$ is a similarity transformation, the systems (1) and (3) are algebraically equivalent.

c) Because the systems (1) and (3) are algebraically equivalent, they are also zero-state equivalent.

d) Because the dimensions of the states of the systems (1) and (4) are different, there exists no similarity transform for the systems, i.e. there exists no invertible matrix **S** such that $\tilde{x} = \mathbf{S}\mathbf{x}$. Therefore, the systems (1) and (4) are not algebraically equivalent.

e) To check if the systems (1) and (4) are zero-state equivalent, we have to check if the systems have the same transfer function (or impulse response). The transfer function of system (1) is given by

$$\begin{aligned}\hat{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\\ &= \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}s+2 & -4\\ 1 & s-3\end{bmatrix}^{-1}\begin{bmatrix}8\\ 2\end{bmatrix} + 2\\ &= \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\frac{s-3}{s^2-s-2} & \frac{4}{s^2-s-2}\\ \frac{-1}{s^2-s-2} & \frac{s+2}{s^2-s-2}\end{bmatrix}\begin{bmatrix}8\\ 2\end{bmatrix} + 2\\ &= \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.\end{aligned}$$

The transfer function of system (4) is given by

$$\hat{\tilde{G}}(s) = \tilde{C}(s - \tilde{A})^{-1}\tilde{B} + \tilde{D} = 3(s+1)^{-1}2 + 2 = \frac{6}{s+1} + 2 = \frac{2s+8}{s+1}.$$

Hence, because the systems (1) and (4) have the same transfer function, they are zero-state equivalent.

Problem 4: Solutions of state-space equations

a) First, $(s\mathbf{I} - \mathbf{A})^{-1}$ can be computed as

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+3s}\begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+3s} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Note that

$$\frac{1}{s^2+3s} = \frac{1}{3s} - \frac{1}{3(s+3)}.$$

Therefore, we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{3s} - \frac{1}{3(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Taking the inverse Laplace transform leads to

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s}\right] & \mathcal{L}^{-1}\left[\frac{1}{3s} - \frac{1}{3(s+3)}\right] \\ 0 & \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}.$$

b) The output $y(t)$ is given by

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t).$$

Substitution yields

$$\begin{aligned} y(t) &= \begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix}\int_0^t \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} d\tau + 1 \\ &= \begin{bmatrix} 3 & 1 - e^{-3t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix}\int_0^t \begin{bmatrix} -2e^{-3(t-\tau)} \\ 6e^{-3(t-\tau)} \end{bmatrix} d\tau + 1 \\ &= 3x_1(0) + (1 - e^{-3t})x_2(0) + \begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{3}e^{-3(t-\tau)} - \frac{2}{3} \\ 2 - 2e^{-3(t-\tau)} \end{bmatrix} + 1 \\ &= 3x_1(0) + (1 - e^{-3t})x_2(0) + 2e^{-3t} - 1. \end{aligned}$$

c) From the solution of question d), we have

$$\begin{aligned} y(1) &= 3x_1(0) + (1 - e^{-3})x_2(0) + 2e^{-3} - 1, \\ y(2) &= 3x_1(0) + (1 - e^{-6})x_2(0) + 2e^{-6} - 1. \end{aligned}$$

By substituting $y(1) = y(2) = 4$, we obtain

$$\begin{aligned} 3x_1(0) + (1 - e^{-3})x_2(0) &= 5 - 2e^{-3}, \\ 3x_1(0) + (1 - e^{-6})x_2(0) &= 5 - 2e^{-6}. \end{aligned}$$

Solving for $x_1(0)$ and $x_2(0)$ yields $x_1(0) = 1$ and $x_2(0) = 2$, or equivalently $\mathbf{x}(0) = [1, 2]^T$.