NTNU – Trondheim
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Exercise 10

TTK4130 Modeling and Simulation

Problem 1 (Sliding stick (Exam 2010))

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass/gravity C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

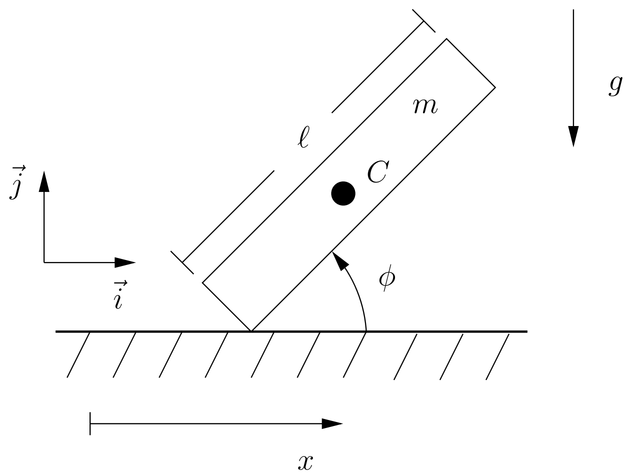

Figure 1: Stick sliding on frictionless surface

(a) Choose appropriate generalized coordinates (the figure should give you some hints). What are the corresponding generalized (actuator) forces?

Solution: A natural choice for generalized coordinates are the horizontal position of the center of mass/gravity (denoted x), and ϕ, the angle between the stick and the surface. An alternative to x could be the contact point between the stick and the surface.

There are no (generalized) actuator forces corresponding to these coordinates. (A candidate answering 'gravity' might get full score if he/she uses it correctly in the rest of the Problem.)

(b) What are the position, velocity, and angular velocity of the center of mass, as function of your chosen generalized coordinates (and/or their derivatives)?

Solution:

$$
\begin{aligned}
\vec{r}_c &= x\vec{i} + \frac{\ell}{2}\sin\phi\vec{j} \\
\vec{v}_c &= \dot{x}\vec{i} + \frac{\ell}{2}\dot{\phi}\cos\phi\vec{j} \\
\vec{w}_{ib} &= \dot{\phi}\vec{k}
\end{aligned}
$$

(Coordinate vectors also accepted for the position and velocity, scalar accepted for angular velocity.)

(c) Write up the kinetic and potential energy of the stick, as function of your chosen generalized coordinates (and/or their derivatives).

Solution: The kinetic energy for the rigid body is

$$T = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib}$$
$$= \frac{1}{2} m \left(\dot{x}^2 + \frac{\ell^2}{4} \dot{\phi}^2 \cos^2 \phi \right) + \frac{1}{2} I_z \dot{\phi}^2.$$

The potential energy due to gravity is

$$U = mg \frac{\ell}{2} \sin \phi.$$

(d) Derive the equations of motion for the stick.

Solution: It is probably easiest to use Lagrange's equation of motion. Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$

which reduces to

$$\ddot{x} = 0.$$

The second equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$$

which gives

$$\left(\frac{m\ell^2}{4} \cos^2 \phi + I_z \right) \ddot{\phi} - \frac{m\ell^2}{4} \dot{\phi}^2 \cos \phi \sin \phi + mg \frac{\ell}{2} \cos \phi = 0.$$

Problem 2 (Double inverted pendulum)

Figure 2 shows a double inverted pendulum on a cart. The two pendulums move independently of each other. The pendulum masses are m_2 and m_3, the rods have no mass. The cart mass is m_1 and τ is a force acting on the cart.

Find the equations of motion for the system, either using Kane's method (Ch. 7., only "borderline curriculum") or Lagrange's equations (Ch. 8).

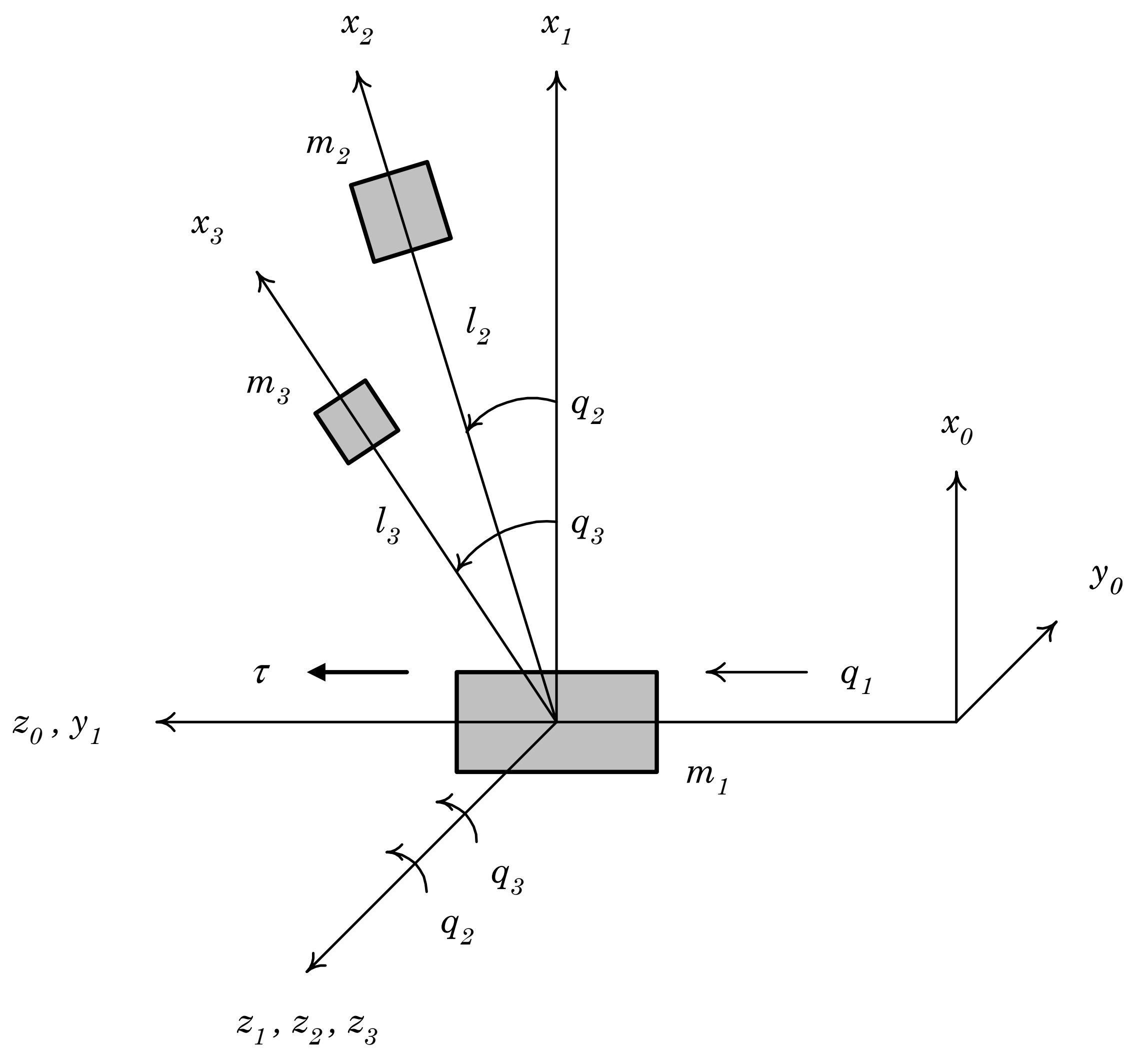

Figure 2: Double inverted pendulum

Solution: Here we use Kane's method. Using Lagrange's equations gives the same answer, perhaps a bit more straightforward.

Direction cosines:

$$
\begin{aligned}
&\vec{b}_3 = \vec{c}_3 = -\vec{a}_2 \\
&\vec{a}_1 \cdot \vec{b}_1 = \cos q_2, \qquad \vec{a}_1 \cdot \vec{b}_2 = -\sin q_2 \\
&\vec{a}_3 \cdot \vec{b}_1 = \sin q_2, \qquad \vec{a}_3 \cdot \vec{b}_2 = \cos q_2 \\
&\vec{a}_1 \cdot \vec{c}_1 = \cos q_3, \qquad \vec{a}_1 \cdot \vec{c}_2 = -\sin q_3 \\
&\vec{a}_3 \cdot \vec{c}_1 = \sin q_3, \qquad \vec{a}_3 \cdot \vec{c}_2 = \cos q_3
\end{aligned}
$$

Angular velocities become

$$
\begin{aligned}
\vec{\omega}_1 &= 0 \\
\vec{\omega}_2 &= \dot{q}_2 \vec{b}_3 \\
\vec{\omega}_3 &= \dot{q}_3 \vec{c}_3
\end{aligned}
$$

Mass center velocities:

$$\begin{aligned}\vec{v}_{c1} &= \dot{q}_1\vec{a}_3\\ \vec{v}_{c2} &= \dot{q}_1\vec{a}_3 + \dot{q}_2 l_2\vec{b}_2\\ \vec{v}_{c3} &= \dot{q}_1\vec{a}_3 + \dot{q}_3 l_3\vec{c}_2\end{aligned}$$

This gives

$$\begin{array}{lll}\vec{\omega}_{1,1} = \vec{0}, & \vec{\omega}_{1,2} = \vec{0}, & \vec{\omega}_{1,3} = \vec{0}\\ \vec{\omega}_{2,1} = \vec{0}, & \vec{\omega}_{2,2} = \vec{b}_3, & \vec{\omega}_{2,3} = \vec{0}\\ \vec{\omega}_{3,1} = \vec{0}, & \vec{\omega}_{3,2} = \vec{0}, & \vec{\omega}_{3,3} = \vec{c}_3\end{array}$$

and

$$\begin{array}{lll}\vec{v}_{c1,1} = \vec{a}_3 & \vec{v}_{c1,2} = \vec{0} & \vec{v}_{c1,3} = \vec{0}\\ \vec{v}_{c2,1} = \vec{a}_3 & \vec{v}_{c2,2} = l_2\vec{b}_2 & \vec{v}_{c2,3} = \vec{0}\\ \vec{v}_{c3,1} = \vec{a}_3 & \vec{v}_{c3,2} = \vec{0} & \vec{v}_{c3,3} = l_3\vec{c}_2\end{array}$$

The angular accelerations:

$$\begin{aligned}\vec{\alpha}_1 &= 0\\ \vec{\alpha}_2 &= \ddot{q}_2\vec{b}_3\\ \vec{\alpha}_3 &= \ddot{q}_3\vec{c}_3\end{aligned}$$

and

$$\begin{aligned}\vec{a}_{c1} &= \ddot{q}_1\vec{a}_3\\ \vec{a}_{c2} &= \ddot{q}_1\vec{a}_3 + \left(\ddot{q}_2\vec{b}_3 \times l_2\vec{b}_1\right) + \left[\dot{q}_2\vec{b}_3 \times \left(\dot{q}_2\vec{b}_3 \times l_2\vec{b}_1\right)\right]\\ &= \ddot{q}_1\vec{a}_3 + \left(\ddot{q}_2 l_2\vec{b}_2\right) + \left[\dot{q}_2\vec{b}_3 \times \dot{q}_2 l_2\vec{b}_2\right]\\ &= \ddot{q}_1\vec{a}_3 + \ddot{q}_2 l_2\vec{b}_2 - \dot{q}_2^2 l_2\vec{b}_1\\ \vec{a}_{c3} &= \ddot{q}_1\vec{a}_3 + (\ddot{q}_3\vec{c}_3 \times l_3\vec{c}_1) + [\dot{q}_3\vec{c}_3 \times (\dot{q}_3\vec{c}_3 \times l_3\vec{c}_1)]\\ &= \ddot{q}_1\vec{a}_3 + (\ddot{q}_3 l_3\vec{c}_2) + [\dot{q}_3\vec{c}_3 \times \dot{q}_3 l_3\vec{c}_2]\\ &= \ddot{q}_1\vec{a}_3 + \ddot{q}_3 l_3\vec{c}_2 - \dot{q}_3^2 l_3\vec{c}_1\end{aligned}$$

Furthermore,

$$\begin{aligned}\vec{M}_{c2} &= \vec{0}\\ \vec{M}_{c3} &= \vec{0}\end{aligned}$$

We insert this into Kane's equation

$$\sum_{i=1}^{3}\left[\vec{v}_{ci,j}\cdot m_i\vec{a}_{ci} + \vec{\omega}_{i,j}\cdot\left(\vec{M}_{ci}\cdot\vec{\alpha}_i + \vec{\omega}_i \times \left(\vec{M}_{ci}\cdot\vec{\omega}_i\right)\right)\right] = \tau_j$$

giving

$$\begin{aligned}&\vec{v}_{c1,1}\cdot m_1\vec{a}_{c1} + \vec{\omega}_{1,1}\cdot\left(\vec{M}_{c1}\cdot\vec{\alpha}_1 + \vec{\omega}_1 \times \left(\vec{M}_{c1}\cdot\vec{\omega}_1\right)\right)\\ &+\vec{v}_{c2,1}\cdot m_2\vec{a}_{c2} + \vec{\omega}_{2,1}\cdot\left(\vec{M}_{c2}\cdot\vec{\alpha}_2 + \vec{\omega}_2 \times \left(\vec{M}_{c2}\cdot\vec{\omega}_2\right)\right)\\ &+\vec{v}_{c3,1}\cdot m_3\vec{a}_{c3} + \vec{\omega}_{3,1}\cdot\left(\vec{M}_{c3}\cdot\vec{\alpha}_3 + \vec{\omega}_3 \times \left(\vec{M}_{c3}\cdot\vec{\omega}_3\right)\right) \quad = \tau_1\end{aligned}$$

$$
\begin{aligned}
&(\vec{a}_3) \cdot m_1 \left(\ddot{q}_1 \vec{a}_3\right) \\
&+ (\vec{a}_3) \cdot m_2 \left(\ddot{q}_1 \vec{a}_3 + \ddot{q}_2 l_2 \vec{b}_2 - \dot{q}_2^2 l_2 \vec{b}_1\right) \\
&+ (\vec{a}_3) \cdot m_3 \left(\ddot{q}_1 \vec{a}_3 + \ddot{q}_3 l_3 \vec{c}_2 - \dot{q}_3^2 l_3 \vec{c}_1\right) \quad = \tau_1
\end{aligned}
$$

$$
\boxed{\begin{aligned}
&(m_1 + m_2 + m_3)\, \ddot{q}_1 + m_2 l_2 \ddot{q}_2 \cos q_2 \\
&- m_2 l_2 \dot{q}_2^2 \sin q_2 + m_3 \ddot{q}_3 l_3 \cos q_3 - m_3 l_3 \dot{q}_3^2 \sin q_3 \quad = \tau_1
\end{aligned}}
$$

$$
\begin{aligned}
&\vec{v}_{c1,2} \cdot m_1 \vec{a}_{c1} + \vec{\omega}_{1,2} \cdot \left(\vec{M}_{c1} \cdot \vec{\alpha}_1 + \vec{\omega}_1 \times \left(\vec{M}_{c1} \cdot \vec{\omega}_1\right)\right) \\
&+ \vec{v}_{c2,2} \cdot m_2 \vec{a}_{c2} + \vec{\omega}_{2,2} \cdot \left(\vec{M}_{c2} \cdot \vec{\alpha}_2 + \vec{\omega}_2 \times \left(\vec{M}_{c2} \cdot \vec{\omega}_2\right)\right) \\
&+ \vec{v}_{c3,2} \cdot m_3 \vec{a}_{c3} + \vec{\omega}_{3,2} \cdot \left(\vec{M}_{c3} \cdot \vec{\alpha}_3 + \vec{\omega}_3 \times \left(\vec{M}_{c3} \cdot \vec{\omega}_3\right)\right) \quad = \tau_2
\end{aligned}
$$

$$
\vec{v}_{c2,2} \cdot m_2 \vec{a}_{c2} + \vec{\omega}_{2,2} \cdot \left(\vec{M}_{c2} \cdot \vec{\alpha}_2 + \vec{\omega}_2 \times \left(\vec{M}_{c2} \cdot \vec{\omega}_2\right)\right) \quad = \tau_2
$$

$$
\begin{aligned}
&l_2 \vec{b}_2 \cdot m_2 \left(\ddot{q}_1 \vec{a}_3 + \ddot{q}_2 l_2 \vec{b}_2 - \dot{q}_2^2 l_2 \vec{b}_1\right) \\
&+ \vec{b}_3 \cdot \left[\left(\vec{0}\right) \cdot \left(\ddot{q}_2 \vec{b}_3\right) + \left(\dot{q}_2 \vec{b}_3\right) \times \left(\left(\vec{0}\right) \cdot \dot{q}_2 \vec{b}_3\right)\right] \quad = \tau_2
\end{aligned}
$$

$$
\boxed{m_2 \ddot{q}_1 l_2 \cos q_2 + \boxed{m_2 l_2^2}\, \ddot{q}_2 = m_2 g l_2 \sin q_2}
$$

In the same way, we can find

$$
\boxed{m_3 \ddot{q}_1 l_3 \cos q_3 + \boxed{m_3 l_3^2}\, \ddot{q}_3 = m_3 g l_3 \sin q_3}
$$

The mass matrix becomes

$$
M = \begin{bmatrix} (m_1 + m_2 + m_3) & m_2 l_2 \cos q_2 & m_3 l_3 \cos q_3 \\ m_2 l_2 \cos q_2 & m_2 l_2^2 & 0 \\ m_3 l_3 \cos q_2 & 0 & m_3 l_3^2 \end{bmatrix}
$$

which we see is symmetrical, $M = M^{\mathsf{T}}$, and also positive definite, $x M x^{\mathsf{T}} > 0 \; \forall x$.

Problem 3 (Robotic manipulator)

We wish to model a robotic manipulator with the configuration shown in Figure 3.

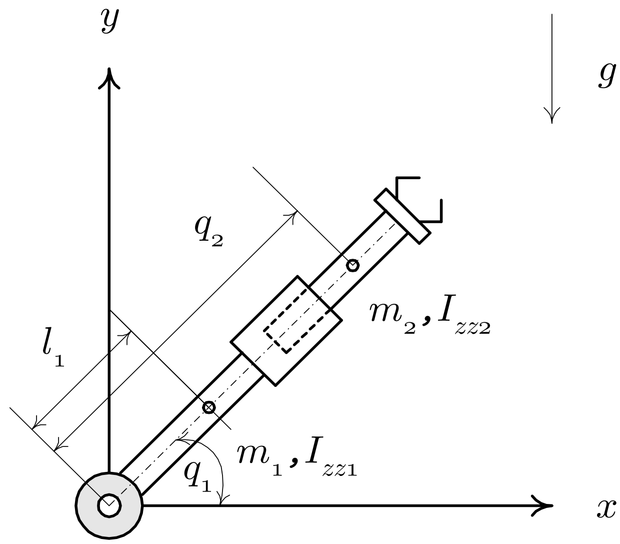

Figure 3: Manipulator

The manipulator has two degrees of freedom (that is, two generalized coordinates). We will use Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy} \tag{1}$$

and q_1 and q_2 are the generalized coordinates (see Figure 3). The axis x and y can be assumed fixed, that is, axes in an inertial system.

We will disregard mass and inertia of the motors in this problem. The moment of inertia of the first arm is denoted I_{zz1}, while the moment of inertia of the second arm is I_{zz2} (each referenced to the center of mass of the respective arm). The dots on the figure marks the centers of mass for each arm. The arrow marked g illustrates the direction of gravity.

(a) Find the total kinetic energy, T, for the manipulator, and show that it can be written on the form $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{q} = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^{\mathsf{T}}$ and

$$\mathbf{M}(\mathbf{q}) = \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Solution: The expression for kinetic energy for a rigid body (each arm) can be written

$$\frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib}.$$

We are only interested in motion in the plane, so we disregard (assume zero) velocity in the z-direction, and angular velocity not about the z-axis.

The velocity of center of mass (superscript i denotes the base (inertial) system) and angular velocity (about z-axis) for each body becomes:

- Arm 1:

$$\mathbf{r}^i_{c1} = \begin{pmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}^i_{c1} = \begin{pmatrix} -l_1 \sin q_1 \dot{q}_1 \\ l_1 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z1} = \dot{q}_1$$

$$\vec{v}_{c1} \cdot \vec{v}_{c1} = \left(\mathbf{v}^i_{c1}\right)^\mathsf{T} \mathbf{v}^i_{c1} = l_1^2 \sin^2 q_1 \dot{q}_1^2 + l_1^2 \cos^2 q_1 \dot{q}_1^2 = l_1^2 \dot{q}_1^2$$

- Arm 2:

$$\mathbf{r}^i_{c2} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}^i_{c2} = \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z2} = \dot{q}_1$$

$$\begin{aligned} \vec{v}_{c2} \cdot \vec{v}_{c2} &= \left(\mathbf{v}^i_{c2}\right)^\mathsf{T} \mathbf{v}^i_{c2} \\ &= \dot{q}_2^2 \cos^2 q_1 - \dot{q}_2 \cos q_1 q_2 \sin q_1 \dot{q}_1 + q_2^2 \sin^2 q_1 \dot{q}_1^2 \\ &\quad + \dot{q}_2^2 \sin^2 q_1 + \dot{q}_2 \sin q_1 q_2 \cos q_1 \dot{q}_1 + q_2^2 \cos^2 q_1 \dot{q}_1^2 \\ &= \dot{q}_2^2 + q_2^2 \dot{q}_1^2 \end{aligned}$$

The kinetic energy for each body becomes

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2, \\ T_2 &= \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2, \end{aligned}$$

and the total kinetic energy for the system is

$$T = T_1 + T_2.$$

Remark: For a simple system like the first body, one can "see" directly that the velocity is $|\vec{v}_{c1}| = l_1 \dot{q}_1$. This may also be possible for the second body, but as setups become more complicated, it is easy to make mistakes when using the "see"-method.

(b) Find the potential energy, U, for the manipulator.

Solution: The potential energy for each arm is

$$\begin{aligned} U_1 &= m_1 g l_1 \sin q_1, \\ U_2 &= m_2 g q_2 \sin q_1. \end{aligned}$$

The total potential energy is

$$U = U_1 + U_2.$$

(c) Derive the equations of motion for the manipulator by use of Lagrange's equation.

Solution: The Lagrangian of the manipulator is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= T_1 + T_2 - U_1 - U_2 \\ &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2 + \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2 - (m_1 l_1 + m_2 q_2) g \sin q_1 \end{aligned}$$

We use Lagrange's equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1,2$$

where

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \dot{q}_1 + I_{zz1}\dot{q}_1 + m_2 q_2^2 \dot{q}_1 + I_{zz2}\dot{q}_1 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \ddot{q}_1 + I_{zz1}\ddot{q}_1 + m_2 q_2^2 \ddot{q}_1 + I_{zz2}\ddot{q}_1 + 2m_2 q_2 \dot{q}_2 \dot{q}_1 \\
\frac{\partial \mathcal{L}}{\partial q_1} &= -\left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \dot{q}_2 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \ddot{q}_2 \\
\frac{\partial \mathcal{L}}{\partial q_2} &= m_2 q_2 \dot{q}_1^2 - m_2 g \sin q_1
\end{aligned}$$

which gives these equations of motion:

$$\begin{aligned}
\left(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2\right)\ddot{q}_1 + 2m_2 q_2 \dot{q}_2 \dot{q}_1 + \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 &= \tau_1 \\
m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + m_2 g \sin q_1 &= \tau_2
\end{aligned}$$

Here, τ_1 is the generalized force corresponding to q_1, that is, a motor torque giving rotation, and τ_2 is the generalized force corresponding to q_2, a motor force giving translational motion of arm 2.

(d) In this problem you should show that the equations of motion in (c) can be written

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}. \tag{2}$$

Explain how several choices are possible for $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$. Show that when you use the Christoffel symbols (cf. eq. (8.57)–(8.58) in the book), then

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}.$$

What is the vector $\mathbf{g}(\mathbf{q})$?

Solution: We first find the "mass matrix" $\mathbf{M}(\mathbf{q})$ and $\mathbf{g}(\mathbf{q})$,

$$\begin{aligned}
\mathbf{M}(\mathbf{q}) &= \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix} \\
\mathbf{g}(\mathbf{q}) &= \begin{pmatrix} \left(m_1 l_1 + m_2 q_2\right) g \cos q_1 \\ m_2 g \sin q_1 \end{pmatrix}
\end{aligned}$$

We see that we have term containing multiplications of derivatives of q_i, that is, $\dot{q}_1\dot{q}_2$. This term can be placed two places in $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$, depending on if we extract $\dot{q}_1$ or $\dot{q}_2$.
We find the Christoffel symbols c_{ijk} by using the equation

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{kj}}{q_i} + \frac{\partial m_{ik}}{q_j} - \frac{\partial m_{ij}}{q_k}\right)$$

which gives

$$
\begin{aligned}
c_{111} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_1} + \frac{\partial m_{11}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_1}\right) = 0 \\
c_{112} &= \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_2} + \frac{\partial m_{12}}{\partial q_1} - \frac{\partial m_{12}}{\partial q_1}\right) = m_2 q_2 \\
c_{122} &= \frac{1}{2}\left(\frac{\partial m_{12}}{\partial q_2} + \frac{\partial m_{12}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_1}\right) = 0 \\
c_{211} &= \frac{1}{2}\left(\frac{\partial m_{21}}{\partial q_1} + \frac{\partial m_{21}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_2}\right) = -m_2 q_2 \\
c_{212} &= \frac{1}{2}\left(\frac{\partial m_{21}}{\partial q_2} + \frac{\partial m_{22}}{\partial q_1} - \frac{\partial m_{12}}{\partial q_2}\right) = 0 \\
c_{222} &= \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_2} + \frac{\partial m_{22}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_2}\right) = 0
\end{aligned}
$$

where $c_{112} = c_{121}$ and $c_{212} = c_{221}$ due to the symmetry in the $\mathbf{M}(\mathbf{q})$ matrix.
We can now find the elements in the $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ matrix by

$$
c_{ij} = \sum_{k=1}^{n=2} c_{ijk} \dot{q}_k
$$

which gives

$$
\begin{aligned}
c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = m_2 q_2 \dot{q}_2 \\
c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = m_2 q_2 \dot{q}_1 \\
c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 q_2 \dot{q}_1 \\
c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0
\end{aligned}
$$

We then get

$$
\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}
$$

(e) What matrix properties do the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ possess?

Solution: The mass matrix $\mathbf{M}(\mathbf{q})$ is

$$
\begin{aligned}
&\text{symmetric :} \quad \mathbf{M} = \mathbf{M}^\mathsf{T} \\
&\text{positive definite :} \quad \mathbf{x}^\mathsf{T}\mathbf{M}(\mathbf{q})\mathbf{x} > 0 \qquad \forall \mathbf{x} \neq 0
\end{aligned}
$$

The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has no specific property, but as we will see next, we can choose it so $\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

(f) Show (using the matrices developed in this problem) that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has been defined by use of the Christoffel symbols.

Solution:

$$
\dot{\mathbf{M}}(\mathbf{q}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}
$$

which gives

$$\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \begin{pmatrix} 2m_2q_2\dot{q}_2 & 2m_2q_2\dot{q}_1 \\ -2m_2q_2\dot{q}_1 & 0 \end{pmatrix} - \begin{pmatrix} 2m_2q_2\dot{q}_2 & 2m_2q_2\dot{q}_1 \\ -2m_2q_2\dot{q}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2m_2q_2\dot{q}_1 \\ 2m_2q_2\dot{q}_1 & 0 \end{pmatrix}$$

We see that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)$ is skew symmetric.

(g) Show that the derivative of the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^{\mathsf{T}}\boldsymbol{\tau}.$$

Hint: Use $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and (2), do not insert the detailed model. Use that $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^{\mathsf{T}}(\mathbf{q})$. What can we say about passivity of the manipulator?

Solution:

$$\begin{aligned}\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}}\end{aligned}$$

Inserting (2) and $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^{\mathsf{T}}(\mathbf{q})$ gives

$$\begin{aligned}\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{\mathbf{q}}^{\mathsf{T}}\left(\boldsymbol{\tau} - \mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right)\dot{\mathbf{q}} - \mathbf{g}\left(\mathbf{q}\right)\right) + \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}^{\mathsf{T}}(\mathbf{q})\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^{\mathsf{T}}\boldsymbol{\tau} + \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^{\mathsf{T}}\boldsymbol{\tau}\end{aligned}$$

where we used that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

This almost implies that the manipulator is passive with applied generalized forces $\boldsymbol{\tau}$ (torque on arm 1 and force on arm 2) as input and generalized velocities $\dot{\mathbf{q}}$ as output (angular velocity of arm 1 and velocity of arm 2).

The only glitch is that the energy function that proves passivity should be positive, while the potential energy seems to be unbounded below. However, if we know that $U > U_{\mathrm{min}}$ for some constant U_{min}, we can use $V = E - U_{\mathrm{min}} = T + U - U_{\mathrm{min}}$ as energy function, which fulfill

$$\dot{V}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^{\mathsf{T}}\boldsymbol{\tau}.$$

The potential energy is bounded in this problem if $0 < q_2 < q_{2,\mathrm{max}}$.