



NTNU – Trondheim
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TTT4120 Digital Signal Processing Fall 2021

Lecture: Z-Transform - Introduction

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1

Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
 - 3.1 The z-transform
 - 3.2 Properties of the z-transform
 - 3.3 Rational z-transforms

*Level of detail is defined by lectures and problem sets

2

2

Contents and learning outcomes

- Definition of z-transform and its existence
- Some properties of the z-transform
- Rational z-transforms: poles and zeros

3

3

Motivation

- Linear time-invariant system:

$$\begin{array}{ccccc}
 x[n] & \longrightarrow & \boxed{h[n]} & \longrightarrow & y[n] = h[n] * x[n] \\
 e^{j\omega n} & & & & y[n] = e^{j\omega n} H(\omega) \\
 X(\omega) & & & & Y(\omega) = H(\omega)X(\omega)
 \end{array}$$

- What if $h[n] = 2^n u[n]$?
 - System is unstable $\sum |h[n]|$ not finite
 - DTFT of $h[n]$ **does not** exist
- Can we analyze such systems using a transform method while retaining the good properties of the DTFT?

4

4

Basic idea

- Capture the source of instability or inapplicability of the DTFT
- Apply the DTFT to the modified (captured) signal

5

5

Basic idea...

- Example: Suppose we have signal $x[n] = 2^n u[n]$
 - Problem is due to the exponential growth
 - Capture the signal by multiplying it by a decaying exponential **stronger** than the growing one, i.e., $r^{-n}x[n]$, $r > 0$
 - What values of r allow for a DTFT for $r^{-n}x[n]$?

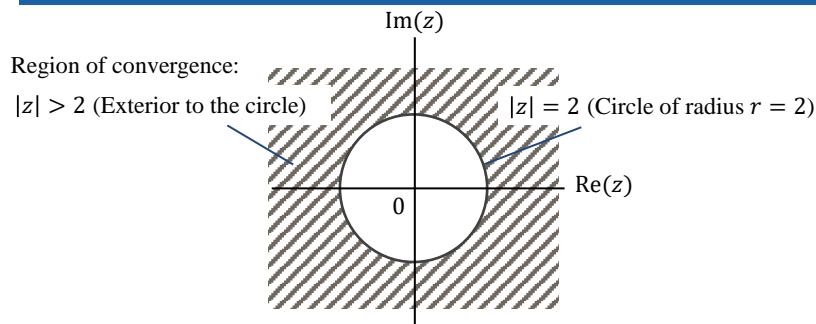
$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} &= \sum_{n=0}^{\infty} 2^n r^{-n} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (2r^{-1} e^{-j\omega})^n = \frac{1}{1 - 2r^{-1} e^{-j\omega}}\end{aligned}$$

Convergence if $|2r^{-1} e^{-j\omega}| < 1$ or $r > 2$

6

6

Basic idea...



- Define complex number $z = re^{j\omega}$ in previous expression

$$\sum_{n=0}^{\infty} 2^n r^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{1}{1-2z^{-1}}, \forall |z| > 2$$

- Convergence has only to do with $r = |z|$ and not ω
- We have a more general transform of the sequence $x[n]$

7

7

Definition of z-transform

- The z-transform of a discrete-time signal $x[n]$ is

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Notation: $x[n] \xleftrightarrow{Z} X(z)$ $x[n] = \mathcal{Z}^{-1}\{X(z)\}$
- Transforms $x[n]$ into its complex-plane representation $X(z)$
- Transform only exists whenever power series converges
- Region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value

8

8

Definition of z-transform...

- Example: Z-transforms of finite-length sequences

$$\begin{aligned}x_1[n] &= \{1, 2, 5, 0, 1\} \\ &= \delta[n] + 2\delta[n-1] + 5\delta[n-2] + \delta[n-4]\end{aligned}$$

$$x_2[n] = \{1, 2, 5, 0, 1\}$$

$$x_3[n] = 2\delta[n]$$

- ROC for **finite-length** signals is **entire z-plane**, except possibly when $z \rightarrow 0$ or $z \rightarrow \infty$
 - either z^k or z^{-k} grow unbounded

9

9

Definition of z-transform...

- Example: Compute z-transforms of infinite-length sequences

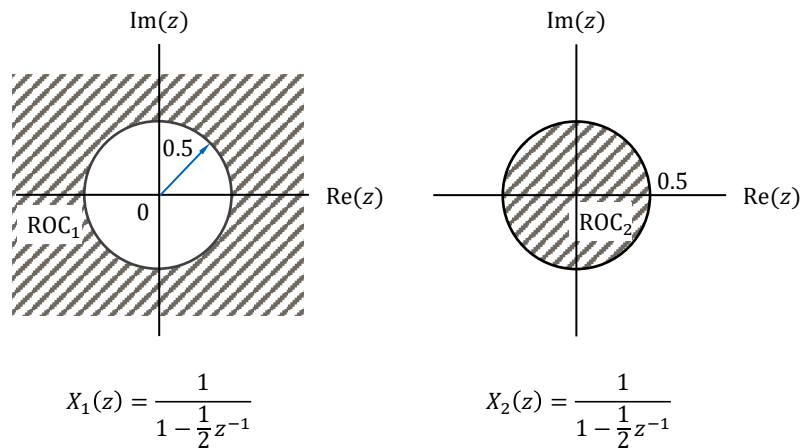
$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

10

10

Definition of z-transform...



11

11

Definition of z-transform...

- Observations for **infinite-duration** sequences:
 - z-transform **expression alone** does **not uniquely specify** the time-domain signal. ROC resolves ambiguity
 - ROC **causal sequence** is the **exterior of a circle**
 - ROC **anti-causal** sequence is the **interior of a circle**

12

12

ROC of z-transform

- In ROC of $X(z)$, we have $|X(z)| < \infty$
- Using polar form of z , i.e., $z = re^{j\omega}$, we get

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n] r^{-n} e^{-j\omega n}|$$

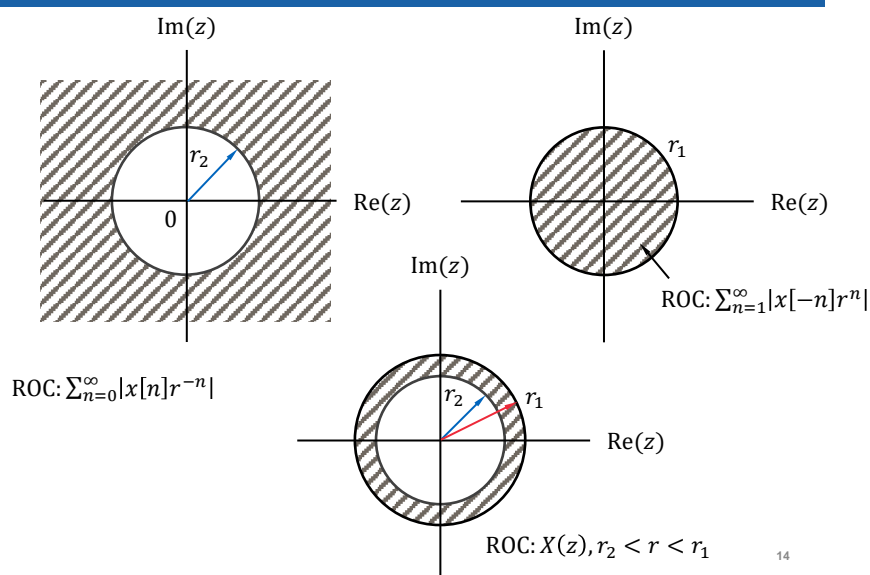
$$\leq \sum_{n=1}^{\infty} |x[-n] r^n| + \sum_{n=0}^{\infty} |x[n] r^{-n}|$$

- Observations:
 - both series should converge, $\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2$
 - for r sufficiently small, $r < r_1 < \infty$, first sum may converge
 - for r sufficiently large, $r > r_2$, second sum may converge

13

13

ROC of z-transform...



14

14

ROC of z-transform

- Example: Two-sided infinite-length sequences

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-n-1]$$

$$x_2[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X_1(z) = ?, X_2(z) = ?$$

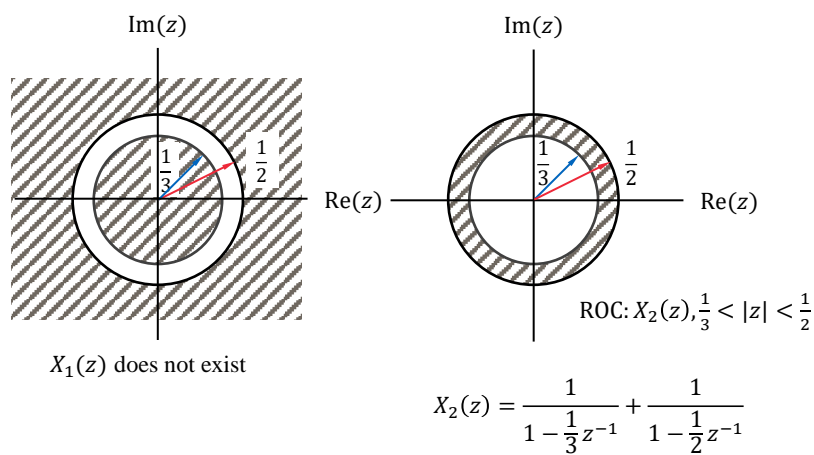
$$\text{From earlier: } \alpha^n u[n] \xleftrightarrow{Z} \frac{1}{(1-\alpha z^{-1})}, \text{ ROC: } |z| > \alpha$$

$$-\alpha^n u[-n-1] \xleftrightarrow{Z} \frac{1}{(1-\alpha z^{-1})}, \text{ ROC: } |z| < \alpha$$

15

15

ROC of z-transform...



16

16

Properties of the z-transform

- Linearity
- Time-shift
- Scaling
- Time-reversal
- Convolution
- Differentiation
- Initial value theorem

17

17

Properties of the z-transform...

- Linearity:

$$x_3[n] = a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{z} X_3(z) = a_1 X_1(z) + a_2 X_2(z)$$
 for any constants a_1 and a_2
- ROC of $X_3(z)$ at least $\mathcal{R}_{X_1} \cap \mathcal{R}_{X_2}$ **but can extend beyond** intersection
- Example: $x_1[n] = (3 \cdot 2^n - 4 \cdot 3^n)u[n]$

$$x_2[n] = (3 \cdot 2^n + 4 \cdot 3^n)u[n]$$

$$x_3[n] = x_1[n] + x_2[n]$$

18

18

Properties of the z-transform...

- Time-shift: $x[n - k] \xleftrightarrow{Z} z^{-k}X(z)$
- ROC of $z^{-k}X(z)$ same as $X(z)$ except at $z = 0$ and $z \rightarrow \infty$
- Coefficient of z^{-n} becomes $z^{-(n+k)}$
- Example: $x[n] = \{1, \underline{2}, -1, 0, 3\}$
 $x[n + 2]$
 $x[n - 2]$

19

19

Properties of the z-transform...

- Scaling: $a^n x[n] \xleftrightarrow{Z} X(a^{-1}z)$
- If ROC of $X(z)$ is $r_1 < |z| < r_2$, then ROC of $X(a^{-1}z)$ is $|a|r_1 < |z| < |a|r_2$
- Example: $x[n] = 2^n u[n]$

20

20

Properties of the z-transform...

- Time reversal: $x[-n] \xleftrightarrow{z} X(z^{-1})$
- If ROC of $X(z)$ is $r_1 < |z| < r_2$, then ROC of $X(z^{-1})$ is $1/r_2 < |z| < 1/r_1$
- Example: $x[n] = u[-n]$

21

21

Properties of the z-transform...

- Convolution: $x[n] = x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)X_2(z)$
- ROC at least the intersection of that of $X_1(z)$ and $X_2(z)$
- Many cases much easier to carry out in z-domain
- Example: $x_1[n] = \{\underline{1}, -1\}$
 $x_2[n] = \{\underline{1}, 1\}$

22

22

Properties of the z-transform...

- Differentiation: $nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$
- ROC convergence stays the same
- Example: $x[n] = na^n u[n]$
- Initial value theorem: $x[0] = \lim_{z \rightarrow \infty} X(z)$, $x[n]$ causal

23

23

Rational z-transforms

- Family of transforms where $X(z)$ can be represented as the ratio of two polynomials in z^{-1} (or z)

$$\begin{aligned}
 X(z) &= \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + b_N z^{-N}} \\
 &= \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \\
 &= \frac{b_0 \sum_{k=0}^M (b_k/b_0) z^{-k}}{a_0 \sum_{k=0}^N (a_k/a_0) z^{-k}} \quad (a_0, b_0 \neq 0) \\
 &= \frac{b_0 \prod_{k=1}^M (1 - z_k z^{-1})}{a_0 \prod_{k=1}^N (1 - p_k z^{-1})}
 \end{aligned}$$

24

24

Rational z-transforms...

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 \prod_{k=1}^M (1 - z_k z^{-1})}{a_0 \prod_{k=1}^N (1 - p_k z^{-1})}$$

- The **zeros** of $X(z)$: values of z for which $X(z) = 0$, $B(z) = 0$
- The **poles** of $X(z)$: values of z for which $X(z) \rightarrow \infty$, $A(z) = 0$
- If a_k and b_k real-valued \Rightarrow poles (zeros) are either real-valued or must occur in conjugate pairs
- Example: $x_1[n] = \{\underline{1}, 1, 1\}$
 $x_2[n] = \{-\underline{1}, -1, 1\}$

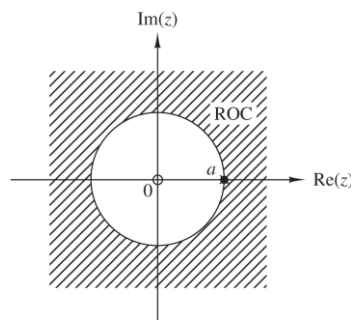
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25

Rational z-transforms...

- Example (pole-zero plot):

$$x[n] = a^n u[n], a > 0 \xleftrightarrow{z} X(z) = \frac{1}{1 - az^{-1}}$$



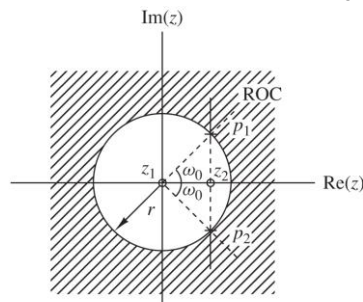
26

26

Rational z-transforms...

- Example (pole-zero plot):

$$x[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n] \stackrel{Z}{\leftrightarrow} X(z) = \frac{\frac{5}{6} \sin\left(\frac{\pi}{3}\right) z^{-1}}{\left(1 - \frac{5}{6} e^{j\frac{\pi}{3}} z^{-1}\right) \left(1 - \frac{5}{6} e^{-j\frac{\pi}{3}} z^{-1}\right)}$$



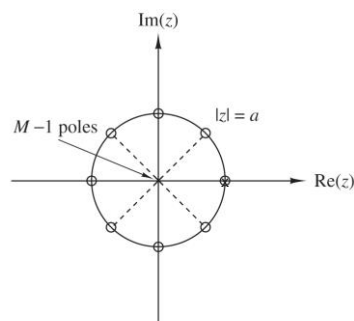
27

27

Rational z-transforms...

- Example (pole-zero plot):

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq M-1 \\ 0, & \text{elsewhere} \end{cases} \stackrel{Z}{\leftrightarrow} X(z) = \frac{1 - (az^{-1})^M}{1 - az^{-1}}$$



28

28

Summary

Today:

- Z-transform and its existence (ROC)
- Properties of the z-transform
- Rational z-transforms: poles and zeros

Next:

- LTI systems: The system function, stability and causality
- Computation and sketching of frequency response function

29