TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 2

Hand-out time:	Monday, September 2, 2013, at 12:00
Hand-in deadline:	Friday, September 13, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Jordan forms

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t), \end{aligned} \tag{1}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 \end{bmatrix}.$$

a) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

b) Can we transform this system into a diagonal form using a similarity transform? Explain why.

c) Transform the system into a Jordan form using a similarity transform.

Problem 2: Stability

a) Give the definition of BIBO stability (bounded-input bounded-output stability).

Consider a system for which the input-output relation is given by the transfer function

$$g(s) = \frac{s+10}{2s}. \tag{2}$$

b) Is the system with transfer function $g(s)$ in (2) BIBO stable? Motivate your answer.

c) Give the definitions of marginal stability, asymptotic stability, exponential stability and instability.

Consider a state-space equation of the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{3}$$

with state $\mathbf{x}(t)$ and constant matrix $\mathbf{A}$.

d) Give the different conditions for the eigenvalues of $\mathbf{A}$ for which the state-space equation in (3) is marginally stable, asymptotically stable, exponentially stable and unstable.

Let the system matrix $\mathbf{A}$ be given by

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

e) Is the state-space equation in (3) with system matrix $\mathbf{A}$ in (4) marginally stable, asymptotically stable, exponentially stable and/or unstable? Motivate your answer.

Next, consider the system matrix:

$$\mathbf{A} = \begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix}. \tag{5}$$

f) Use the matrix $\mathbf{A}$ in (5) to compute the symmetric matrix $\mathbf{P}$ from the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q},$$

where $\mathbf{Q} = \mathbf{I}$ is the identity matrix.

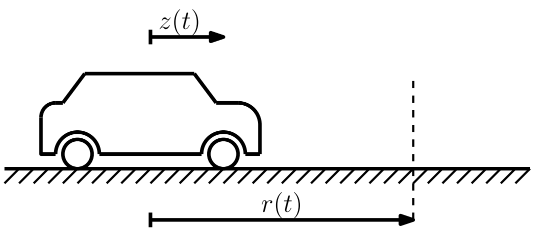
g) Conclude from your answer obtained in f) whether the state-space equation in (3) with system matrix $\mathbf{A}$ in (5) is asymptotically stable or not. Motivate your answer.

Problem 3: Linear quadratic regulator

Consider the following simplified car model:

$$\ddot{z}(t) + 2\dot{z}(t) = 2u(t),$$

where $z(t)$ is the distance of the car and the input $u(t)$ represents the force applied by the engine. In order to track a reference signal $r(t)$, a tracking controller needs to be designed.



a) Derive a state-space equation of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t),$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$ and output $y(t) = z(t)$.

We design a linear quadratic regulator for the system that minimizes the cost function

$$J = \int_0^\infty \left[y^2(t) + \rho u^2(t)\right] dt.$$

The control input that minimizes this cost function is given by

$$u(t) = -\mathbf{K}\mathbf{x}(t), \qquad \text{with} \qquad \mathbf{K} = \frac{1}{\rho}\mathbf{B}^T\mathbf{P},$$

where the positive-definite matrix $\mathbf{P}$ is the solution of the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \frac{1}{\rho}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C} = \mathbf{0}.$$

b) Let $\rho = 1$. Calculate the matrix $\mathbf{P}$ and show that $\mathbf{K} = [1, \sqrt{2} - 1]$.

Consider the following tracking controller:

$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr(t),$$

where $\mathbf{K} = [1, \sqrt{2} - 1]$ is the feedback-gain matrix of the linear quadratic regulator, p is a constant gain, and $r(t)$ is a reference signal for the output $y(t)$.

c) Draw a block diagram of the closed-loop system.

The closed-loop system can be written as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \bar{\mathbf{A}}\mathbf{x}(t) + \bar{\mathbf{B}}r(t), \\ y(t) &= \bar{\mathbf{C}}\mathbf{x}(t). \end{aligned}$$

d) Write the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ as a function of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{K}$ and p.

e) Let $r(t) = r_c$ for all $t \geq 0$, where r_c is an arbitrary constant. Calculate the gain p such that the output $y(t)$ asymptotically converges to r_c as time goes to infinity.

For $r(t) = 2$ and $z(0) = \dot{z}(0) = 0$, the output $y(t)$ and the control input $u(t)$ are depicted in Fig. 1.

f) How does the gain ρ of the linear quadratic regulator need to be tuned such that the output $y(t)$ converges faster to the reference value $r(t) = r_c$? How will this affect the control input $u(t)$?

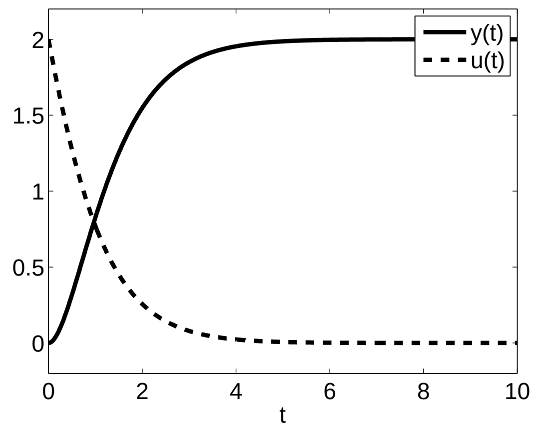


Fig. 1: Output $y(t)$ and control input $u(t)$.