Department of Engineering Cybernetics

TTK4215 System Identification and Adaptive Control
Solution 4

Problem 1
Since $V(t) \geq 0$, $V(t)$ is bounded from below and thus has an infimum given by

$$V_m = \inf_{t\in[0,\infty)} V(t). \tag{1}$$

Therefore, for any $\varepsilon > 0$, there exists a $\bar{t} \in [0, \infty)$ such that

$$V_m \leq V(\bar{t}) < V_m + \varepsilon. \tag{2}$$

Since $V(t)$ is non-increasing, $V(t) \leq V(\bar{t})$ for all $t \geq \bar{t}$. It follows that

$$V_m \leq V(t) < V_m + \varepsilon, \text{ for all } t \geq \bar{t}. \tag{3}$$

Since ε can be chosen arbitrarily, it follows that

$$\lim_{t\to\infty} V(t) = V_m. \tag{4}$$

Problem 4.1 from I&S

The simple scalar system

$$\dot{\tilde{\theta}} = -\gamma u^2 \tilde{\theta} \tag{5}$$

can be solved in closed form. The solution is

$$\tilde{\theta}(t) = e^{-\gamma \int_0^t u^2(\tau)d\tau} \tilde{\theta}(0), \tag{6}$$

which is equation (4.2.10) in the book. Let n be an integer such that $nT_0 \leq t < (n+1)T_0$. Then the fact

$$\int_t^{t+T_0} u^2(\tau)\, d\tau \geq \alpha_0 T_0 \tag{7}$$

gives

$$\int_0^t u^2(\tau)\, d\tau = \sum_{j=1}^{n} \int_{(j-1)T_0}^{jT_0} u^2(\tau)\, d\tau + \int_{nT_0}^{t} u^2(\tau)\, d\tau \geq n\alpha_0 T_0 > \alpha_0 (t - T_0). \tag{8}$$

Thus,

$$-\alpha_0 T_0 - \int_0^t u^2(\tau)\, d\tau < -\alpha_0 t \tag{9}$$

and

$$\left|\tilde{\theta}(t)\right| = e^{-\gamma\int_0^t u^2(\tau)d\tau}\left|\tilde{\theta}(0)\right| = e^{\gamma\alpha_0 T_0} e^{\gamma\left(-\alpha_0 T_0 - \int_0^t u^2(\tau)d\tau\right)}\left|\tilde{\theta}(0)\right| \le e^{\gamma\alpha_0 T_0}\left|\tilde{\theta}(0)\right| e^{-\gamma\alpha_0 t}, \quad (10)$$

which proves that (7) is a sufficient condition for $\tilde{\theta}(t)$ to converges exponentially fast to zero. Suppose (7) does not hold. Then, for any choice of T_0,

$$\lim_{t\to\infty}\int_t^{t+T_0} u^2(\tau)\,d\tau \to 0, \quad (11)$$

since otherwise, we would be able to find an α_0 that would satisfy (7). Thus, on the one hand, for any $\gamma > 0$ and $T > 0$ there exists $t_1 \in [0,\infty)$ such that

$$e^{-\gamma\int_{t_1}^{t_1+T} u^2(\tau)d\tau} > \frac{1}{2}. \quad (12)$$

On the other hand, given any $\alpha_0 > 0$, $\gamma_0 > 0$, we can pick $T > 0$ such that

$$\alpha_0 e^{-\gamma_0 T} < \frac{1}{2}. \quad (13)$$

Thus, t_1 can be picked sufficiently large to obtain

$$\alpha_0 e^{-\gamma_0 T} < e^{-\gamma\int_{t_1}^{t_1+T} u^2(\tau)d\tau}. \quad (14)$$

In other words, unless (7) holds, $\tilde{\theta}(t)$ will decay slower than any exponentially decaying function we pick.

Problem 4.2 from I&S

We choose the parallel model to solve the problem, but the same procedure can be taken for the series-parallel model. The update laws are given on page 160 of I&S (4.2.29). Download and run the simulink files from It's Learning.

Problem 4.3 from I&S

By defining

$$\theta^* = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \end{bmatrix}^T \quad (15)$$
$$\psi = \begin{bmatrix} f_1(x) & f_2(x) & g_1(x)u & g_2(x)u \end{bmatrix}^T \quad (16)$$

we can rewrite the system as

$$\dot{x} = \theta^{*^T}\psi. \quad (17)$$

Next, we filter the equation with a strictly proper filter $G(s)$

$$G(s)\dot{x} = \theta^{*^T}(G(s)), \quad (18)$$

and define

$$\begin{aligned} y &= sG(s)\,x, & (19)\\ \phi &= G(s)\,\psi, & (20)\end{aligned}$$

so that

$$y = {\theta^*}^T \phi. \tag{21}$$

The prediction is

$$\hat{y} = \theta^T \phi, \tag{22}$$

and the estimation error is

$$\varepsilon = \hat{y} - y = \tilde{\theta}^T \phi \tag{23}$$

where

$$\tilde{\theta} = \theta - \theta^*. \tag{24}$$

We select the gradient method, using the cost function

$$J(\theta) = \frac{1}{2}\varepsilon^2, \tag{25}$$

which is a convex function of θ, and therefore has a global minimum that can be reached by moving in the opposite direction of the gradient, that is

$$\begin{aligned} \dot{\theta} &= -\gamma \nabla J \\ &= -\gamma \varepsilon \frac{\partial \varepsilon}{\partial \theta} \\ &= -\gamma \varepsilon \phi. \end{aligned} \tag{26}$$

Selecting the Lyapunov function

$$V(\theta) = \frac{1}{2\gamma}\tilde{\theta}^T \tilde{\theta}, \tag{27}$$

we have

$$\begin{aligned} \dot{V} &= \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} \\ &= \frac{1}{\gamma}\tilde{\theta}^T (-\gamma \varepsilon \phi) \\ &= \frac{1}{\gamma}\tilde{\theta}^T \left(-\gamma \tilde{\theta}^T \phi \phi\right) \\ &= -\tilde{\theta}^T \left(\phi \phi^T \tilde{\theta}\right) \\ &= -\tilde{\theta}^T \phi \phi^T \tilde{\theta} \end{aligned} \tag{28}$$

which is negative semi-definite, proving that the estimate is bounded. It can be proven that $\tilde{\theta} \to 0$ if ϕ satisfies a persistent excitation condition.