Probabilistic Modelling of Sequences

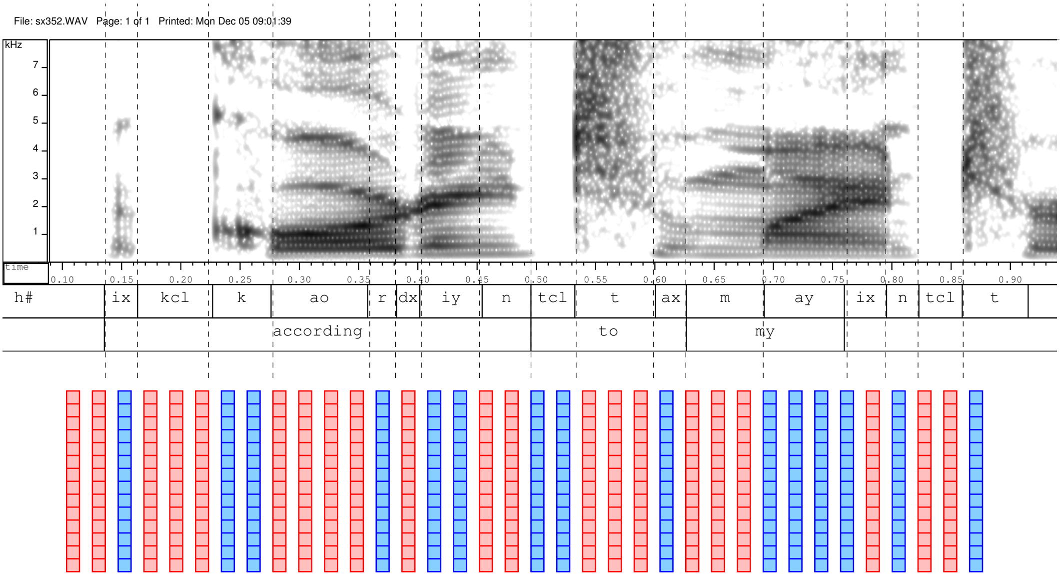
TTT4185 Machine Learning for Signal Processing

Giampiero Salvi

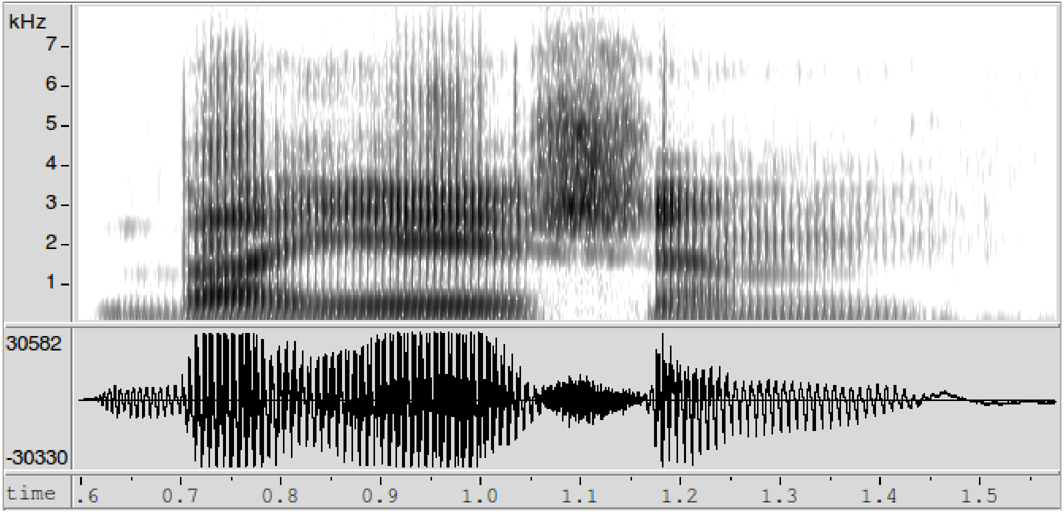
Department of Electronic Systems
NTNU

HT2020

Frame-Based Processing



Sequences in Statistical Terms



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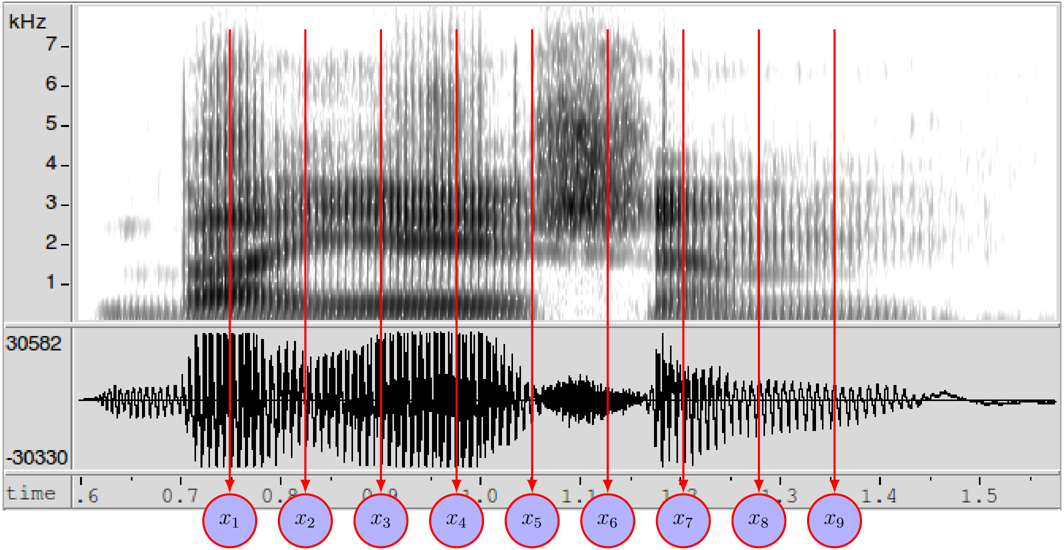
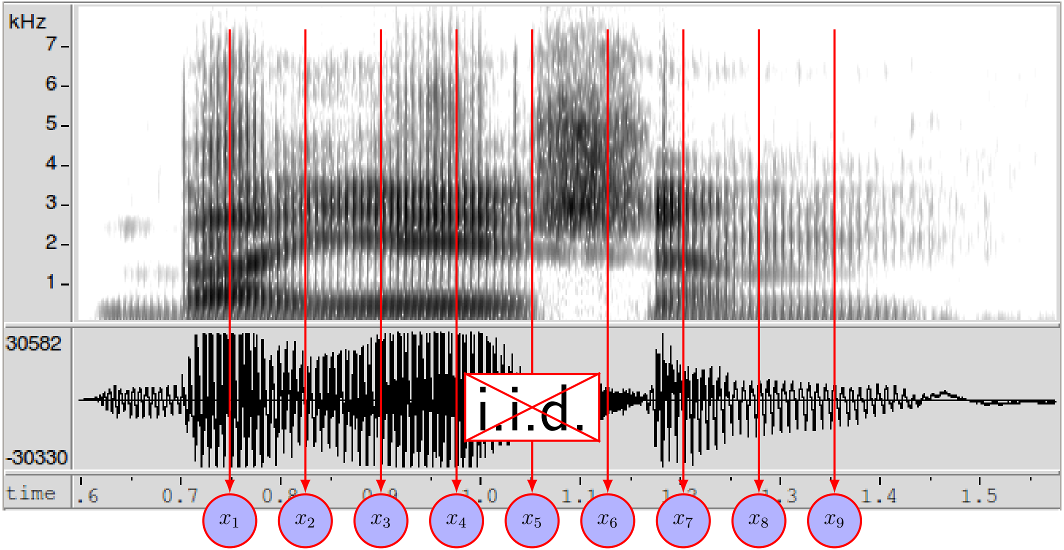


Sequences in Statistical Terms

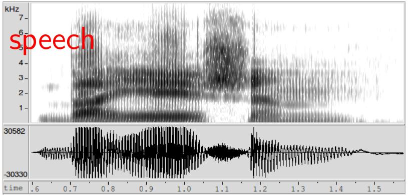


Sequential Data: Not Only Speech

Time sequences

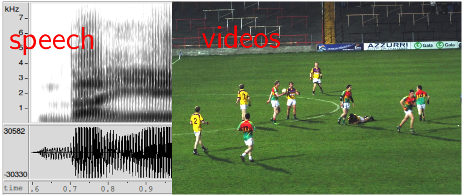
Sequential Data: Not Only Speech

Time sequences



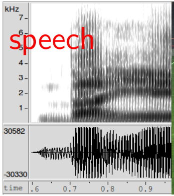
Sequential Data: Not Only Speech

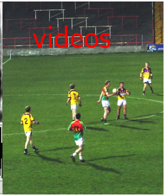
Time sequences



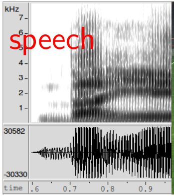
Sequential Data: Not Only Speech

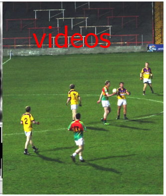
Time sequences

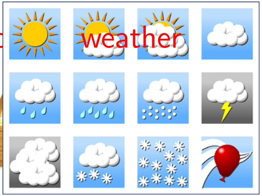




Time sequences







Sequential Data: Not Only Speech

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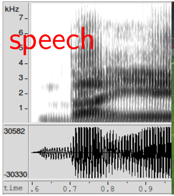
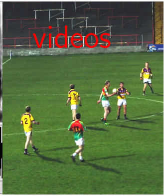
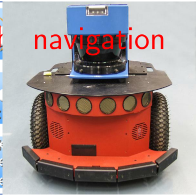
Time sequences



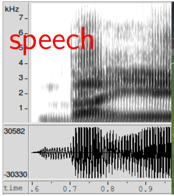


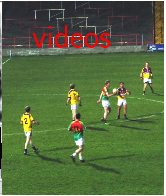


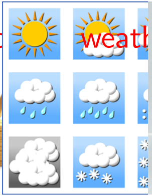
Timeless sequences

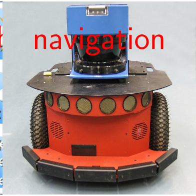
Sequential Data: Not Only Speech

Time sequences

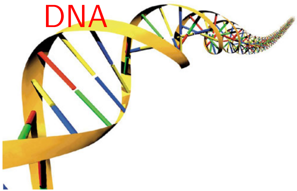








Timeless sequences



text

Nel mezzo del cammin di nostra vita
mi ritrovai per una selva oscura,
ché la diritta via era smarrita.
Ahi quanto a dir qual era è cosa dura
esta selva selvaggia e aspra e forte
che nel pensier rinova la paura!
Tant' è amara che poco è più morte;
ma per trattar del ben ch'i' vi trovai,
dirò de l'altre cose ch'i' v'ho scorte.
Io non so ben ridir com' i' v'intrai,
tant' era pien di sonno a quel punto
che la verace via abbandonai.

Historical Perspective

- Hidden Markov Models first studied in the '60s[1][2]
- applied to ASR in the mid '70s[3]
- later seen as special case of Bayesian Networks[4]

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[1] R. Stratonovich. "Conditional Markov Processes". In: *Theory of Probability and its Applications* 5.2 (1960), pp. 156–178.

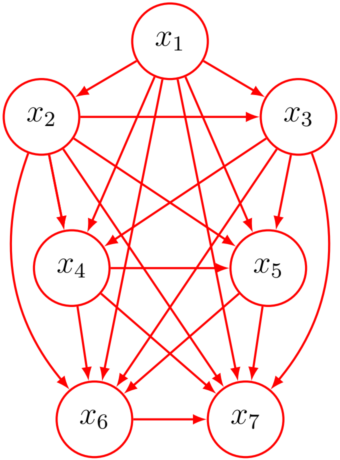
[2] L. E. Baum, T. Petrie, G. Soules, and N. Weiss. "A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains". In: *Ann. Math. Statist.* 41.1 (1970), pp. 164–171.

[3] J. Baker. "The DRAGON system—An overview". In: *IEEE Trans. Acoust., Speech, Signal Process.* 23 (1975), pp. 24–29.

[4] J. Pearl. "Bayesian networks: a model of self-activated memory for evidential reasoning". In: *Proceedings of the 7th Conference of the Cognitive Science Society*. University of California, Irvine, Aug. 1985, pp. 329–334.

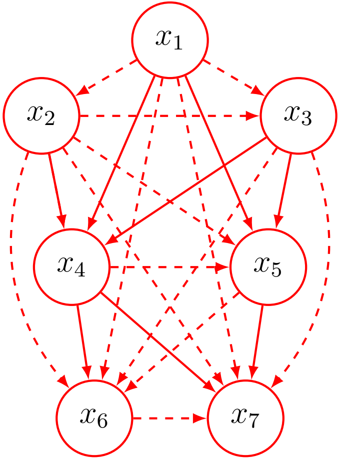
Bayesian Networks (reminder)

$$
\begin{aligned}
p(x_1, \ldots, x_7) = \\
& p(x_1) \\
& p(x_2|x_1) \\
& p(x_3|x_1, x_2) \\
& p(x_4|x_1, x_2, x_3) \\
& p(x_5|x_1, x_2, x_3, x_4) \\
& p(x_6|x_1, x_2, x_3, x_4, x_5) \\
& p(x_7|x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}
$$



Bayesian Networks (reminder)

$$
\begin{aligned}
p(x_1, \ldots, x_7) = \\
&p(x_1) \\
&p(x_2 | x_1) \\
&p(x_3 | x_1, x_2) \\
&p(x_4 | x_1, x_2, x_3) \\
&p(x_5 | x_1, x_2, x_3, x_4) \\
&p(x_6 | x_1, x_2, x_3, x_4, x_5) \\
&p(x_7 | x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}
$$



Bayesian Networks (reminder)

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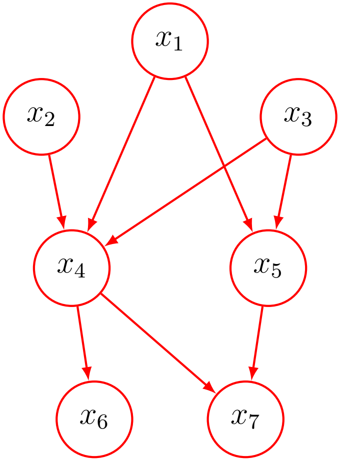
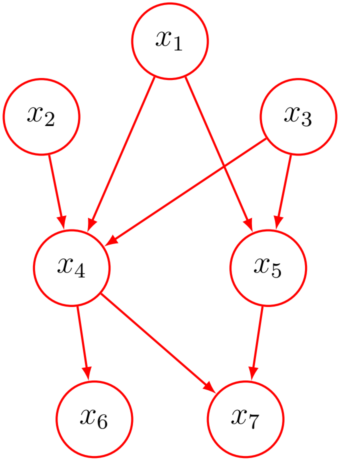
\begin{aligned}
p(x_1, \ldots, x_7) = \\
& p(x_1) \\
& p(x_2) \\
& p(x_3) \\
& p(x_4 | x_1, x_2, x_3) \\
& p(x_5 | x_1, x_3) \\
& p(x_6 | x_4) \\
& p(x_7 | x_4, x_5)
\end{aligned}
$$



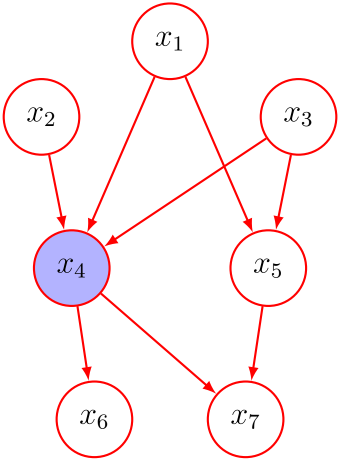
Bayesian Networks (reminder)

$$p(x_1, \ldots, x_7) = \prod_{k=1}^{K} p(x_k | \mathsf{pa}_k)$$



Bayesian Networks (reminder)

If we observe x_4...



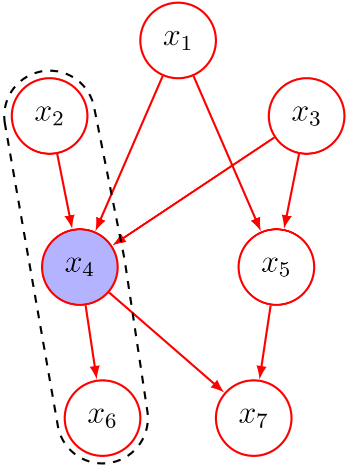
Bayesian Networks (reminder)

If we observe $x_4 \ldots$

d-separation:

Head-to-tail:

x_2 and x_6 conditionally independent



Bayesian Networks (reminder)

If we observe $x_4 \ldots$

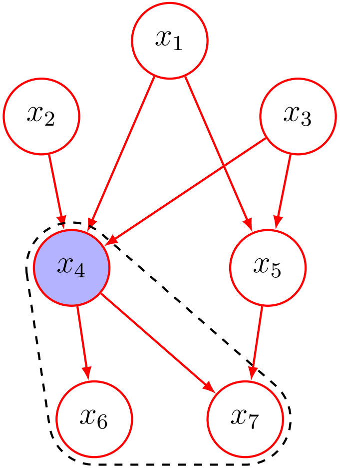
d-separation:

Head-to-tail:

x_2 and x_6 conditionally independent

Tail-to-tail:

x_6 and x_7 conditionally independent



Bayesian Networks (reminder)

If we observe $x_4 \ldots$

d-separation:

Head-to-tail:

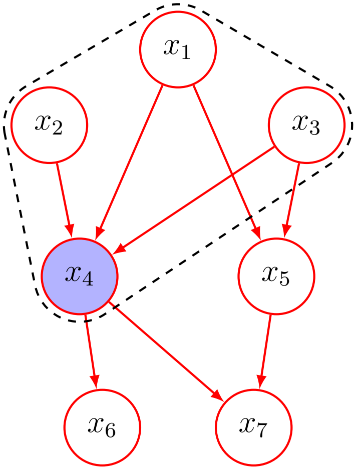
x_2 and x_6 conditionally independent

Tail-to-tail:

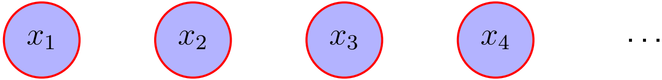
x_6 and x_7 conditionally independent

Head-to-head:

x_1, x_2 and x_3 dependent (explaining away)

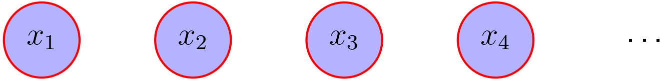


(Dynamic) Bayesian Networks



independence assumption (e.g. i.i.d) not satisfactory

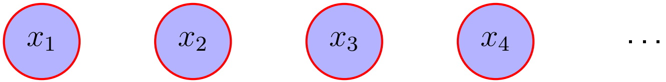
(Dynamic) Bayesian Networks



Most general case, applying chain rule recursively ($p(a, b) = p(a)p(b|a)$)

$$p(x_1, \ldots, x_N) = p(x_1)p(x_2, \ldots, x_N|x_1)$$

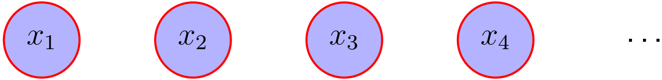
(Dynamic) Bayesian Networks



Most general case, applying chain rule recursively ($p(a, b) = p(a)p(b|a)$)

$$p(x_1, \ldots, x_N) = p(x_1)p(x_2|x_1)p(x_3, \ldots, x_N|x_1, x_2)$$

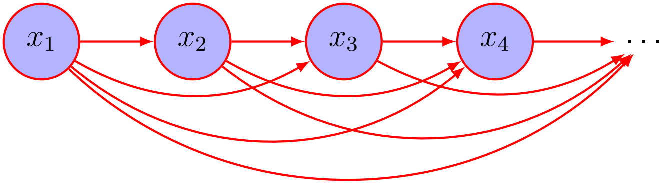
(Dynamic) Bayesian Networks



Most general case, applying chain rule recursively ($p(a, b) = p(a)p(b|a)$)

$$\begin{aligned} p(x_1, \ldots, x_N) =& p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)\cdots \\ & \cdots p(x_N|x_1, \ldots, x_{N-1}) \end{aligned}$$

(Dynamic) Bayesian Networks



Most general case, applying chain rule recursively ($p(a, b) = p(a)p(b|a)$)

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\begin{aligned}
p(x_1, \ldots, x_N) =& p(x_1)p(x_2|x_1)p(x_3|x_1, x_2) \cdots \\
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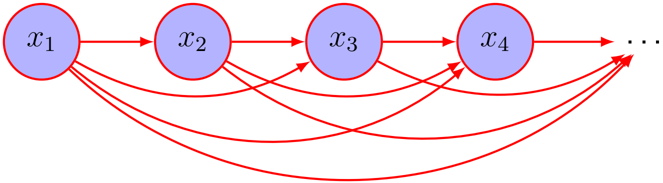
(Dynamic) Bayesian Networks



Most general case, applying chain rule recursively ($p(a, b) = p(a)p(b|a)$)

$$
\begin{aligned}
p(x_1, \ldots, x_N) = & p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)\cdots \\
& \cdots p(x_N|x_1, \ldots, x_{N-1})
\end{aligned}
$$

Grows quadratically with sequence length (N)!!!

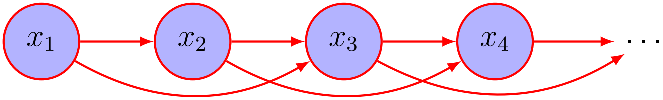
Markov assumption

$$x_1 \to x_2 \to x_3 \to x_4 \to \cdots$$

First order Markov assumption: $p(x_n|x_1,\ldots,x_{n-1}) \approx p(x_n|x_{n-1})$

$$p(x_1,\ldots,x_N) = p(x_1)\prod_{n=2}^{N} p(x_n|x_{n-1})$$

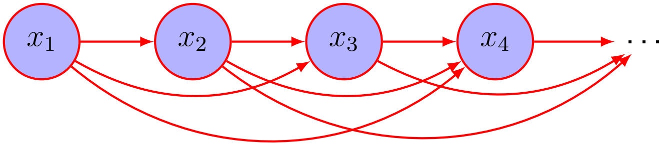
Markov assumption



Second order Markov assumption:

$$p(x_1, \ldots, x_N) = p(x_1)p(x_2|x_1) \prod_{n=3}^{N} p(x_n|x_{n-2}, x_{n-1})$$

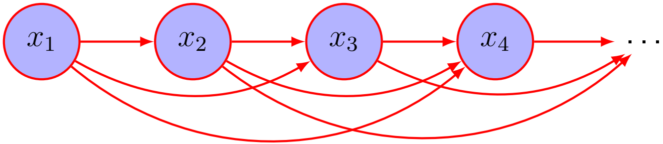
Markov assumption



Third order Markov assumption:

$$p(x_1, \ldots, x_N) \quad = \quad p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)$$
$$\prod_{n=4}^{N} p(x_n|x_{n-3}, x_{n-2}, x_{n-1})$$

Markov assumption



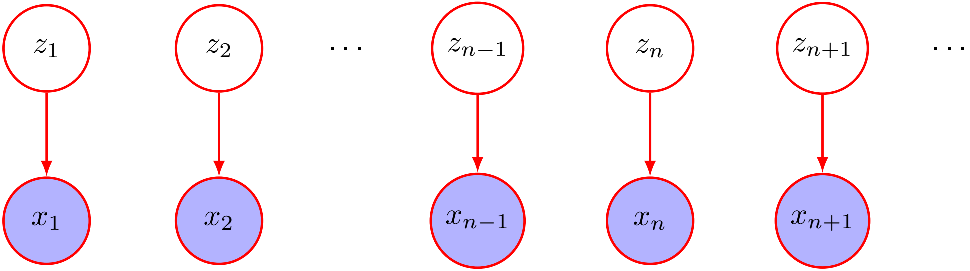
Third order Markov assumption:

$$p(x_1, \ldots, x_N) = p(x_1)p(x_2|x_1)p(x_3|x_1, x_2) \prod_{n=4}^{N} p(x_n|x_{n-3}, x_{n-2}, x_{n-1})$$

Grows quadratically with order!!!

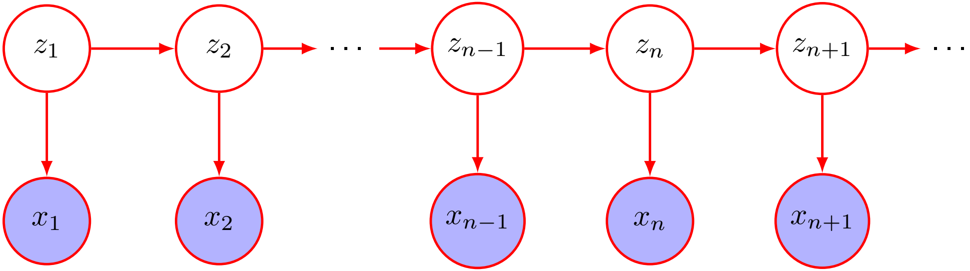
Mixture Models

Adding latent variables z_n

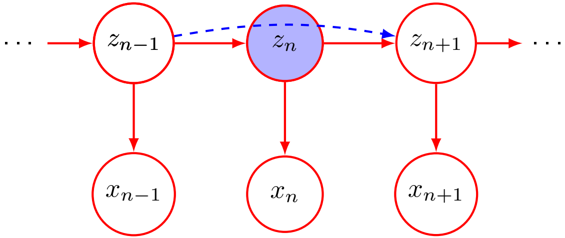


State Space Models

Adding latent variables z_n

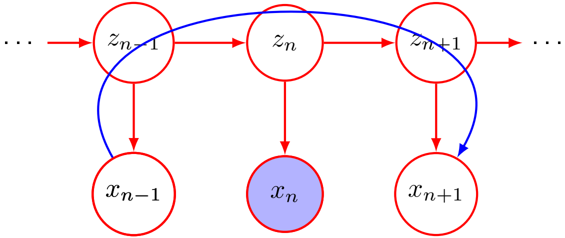


State Space Models: Properties



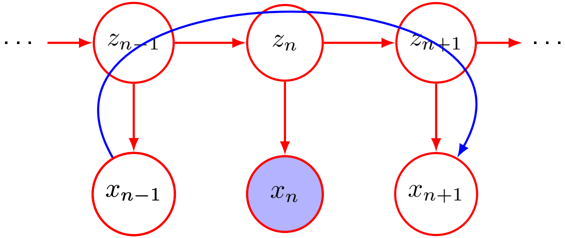
- given z_n, z_{n+1} is independent of $z_1, \ldots, z_{n-1}$
  $p(z_{n+1}|z_1, \ldots, z_n) = p(z_{n+1}|z_n)$

State Space Models: Properties



- given z_n, z_{n+1} is independent of $z_1, \ldots, z_{n-1}$
  $p(z_{n+1}|z_1, \ldots, z_n) = p(z_{n+1}|z_n)$
- $p(x_{n+1}|x_1, \ldots, x_n)$ does not simplify

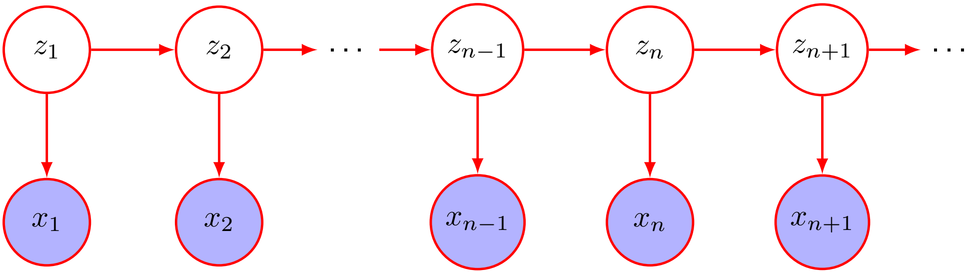
State Space Models: Properties



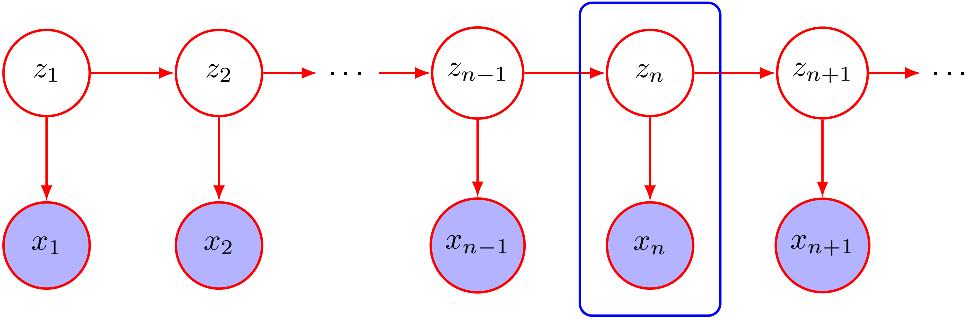
- given z_n, z_{n+1} is independent of $z_1, \ldots, z_{n-1}$
  $p(z_{n+1}|z_1, \ldots, z_n) = p(z_{n+1}|z_n)$
- $p(x_{n+1}|x_1, \ldots, x_n)$ does not simplify

We have modelled indefinitely long dependencies with a limited set of parameters!

Stationary State Space Models

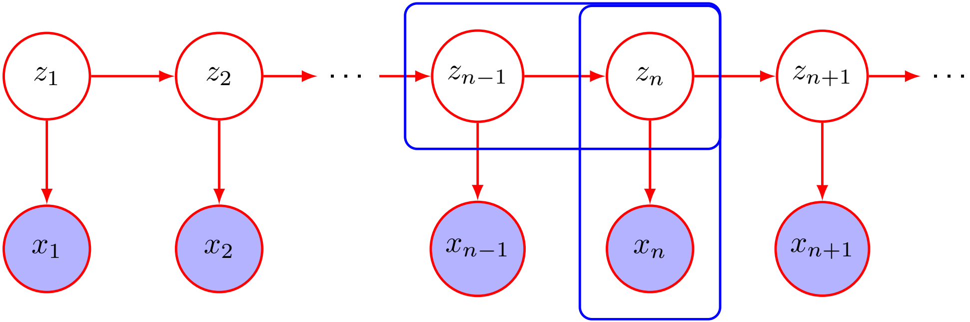


Stationary State Space Models



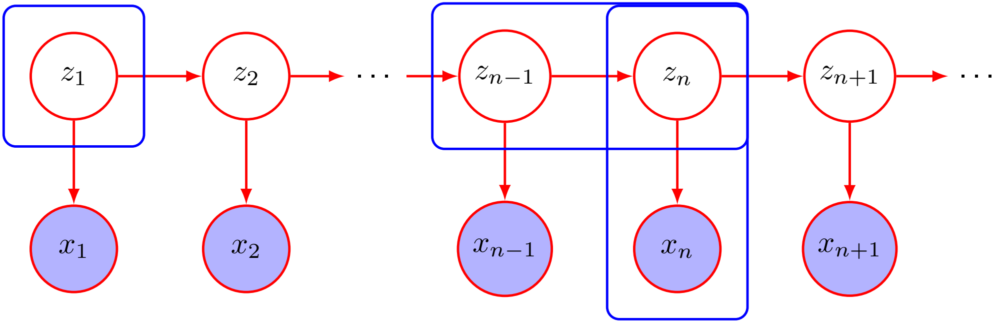
- Emission: $p(x_n|z_n)$

Stationary State Space Models



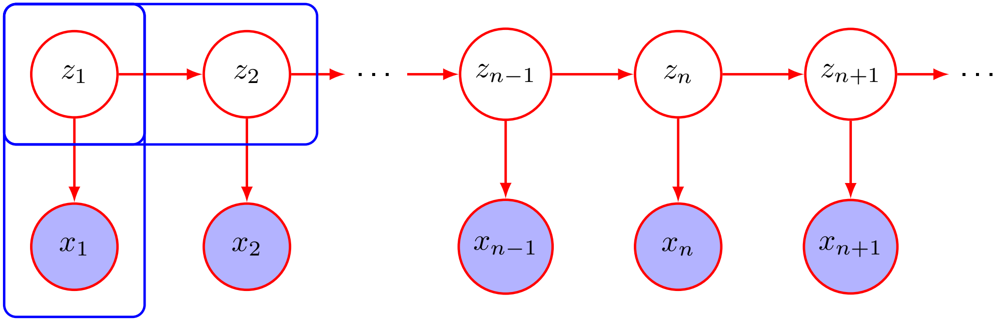
- Emission: $p(x_n|z_n)$
- Transition: $p(z_n|z_{n-1})$

Stationary State Space Models



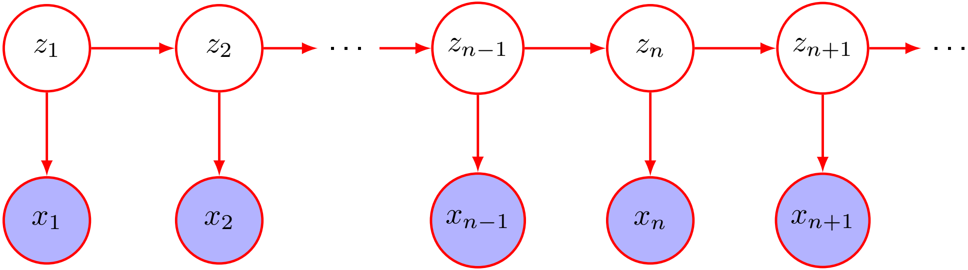
- Emission: $p(x_n|z_n)$
- Transition: $p(z_n|z_{n-1})$
- Initial: $p(z_1)$

Stationary State Space Models



- Emission: $p(x_n|z_n)$
- Transition: $p(z_n|z_{n-1})$
- Initial: $p(z_1)$

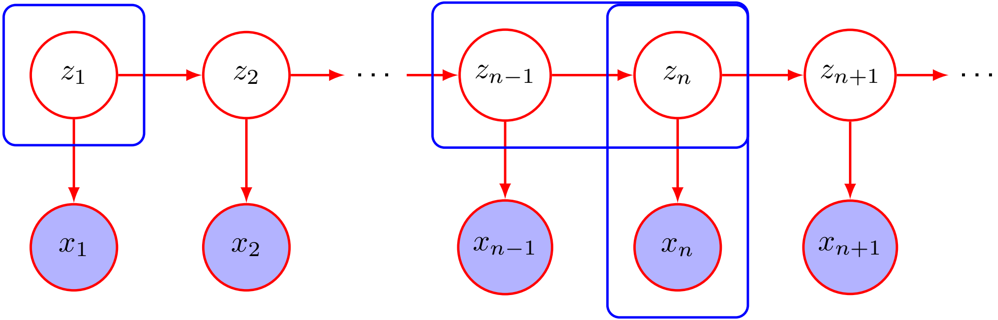
State Space Models Instances



- if z_n are discrete: Hidden Markov Models
- if z_n are continuous: Linear Dynamical Systems

Hidden Markov Models

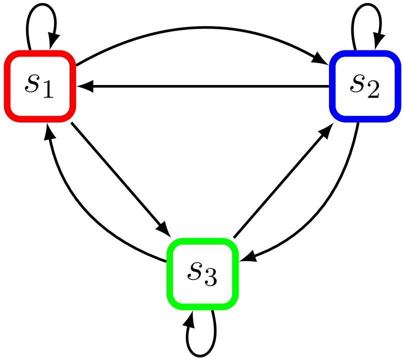
State space models with discrete z_n



- Emission: $p(x_n|z_n) = p(x_n|z_n, \phi)$
  equivalent to Mixture Model
- Transition: $p(z_n|z_{n-1}) = p(z_n|z_{n-1}, A)$
- Initial: $p(z_1) = p(z_1|\pi)$

Hidden Markov Models (HMMs)

Ergodic HMM



Elements:	
set of states:	$S = \{s_1, s_2, s_3\}$
transition probabilities:	$A(s_a, s_b) = P(s_b, t \mid s_a, t-1)$
prior probabilities:	$\pi(s_a) = P(s_a, t_0)$
state to observation probs:	$\phi(o, s_a) = P(o \mid s_a)$

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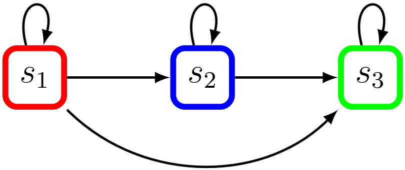
Left-to-right HMM



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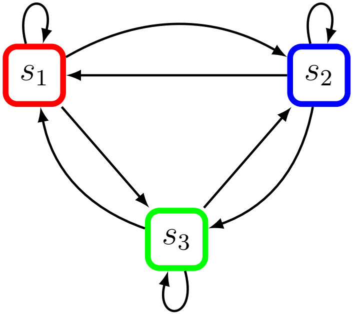
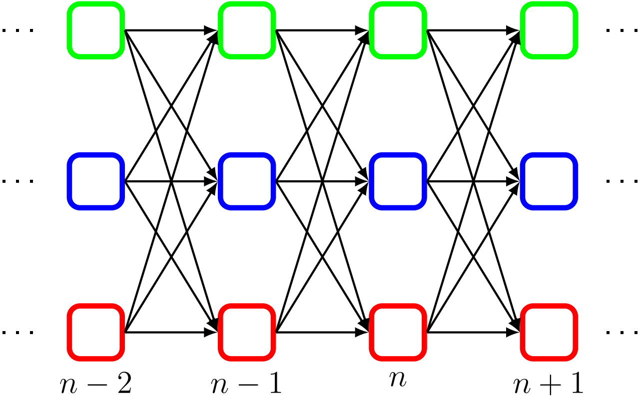
HMMs: Trellis (Lattice)





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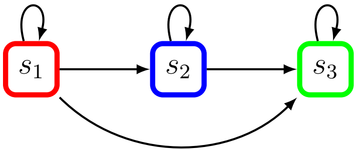

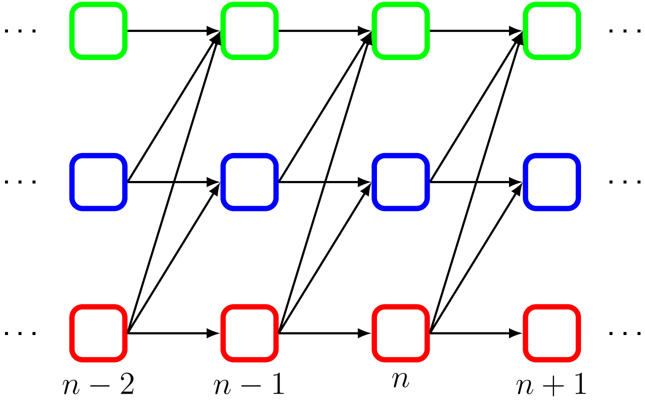

A probabilistic perspective: Bayes' rule

$$P(\text{words}|\text{sounds}) = \frac{P(\text{sounds}|\text{words})P(\text{words})}{P(\text{sounds})}$$

- $P(\text{sounds}|\text{words})$ can be estimated from training data and transcriptions
- $P(\text{words})$: *a priori* probability of the words (Language Model)
- $P(\text{sounds})$: *a priori* probability of the sounds (constant, can be ignored)

Probabilistic Modelling

Problem: How do we model $P(\text{sounds}|\text{words})$?

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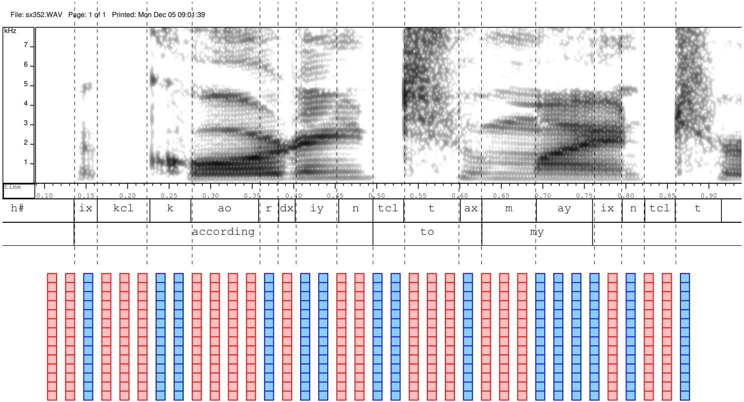
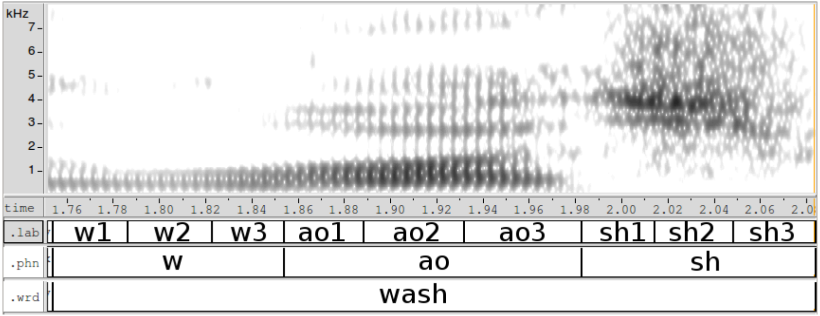


Every feature vector (observation at time t) is a continuous stochastic variable (e.g. MFCC)

Stationarity

Problem: speech is not stationary

- we need to model short segments independently
- the fundamental unit can not be the word, but must be shorter
- usually we model three segments for each phoneme



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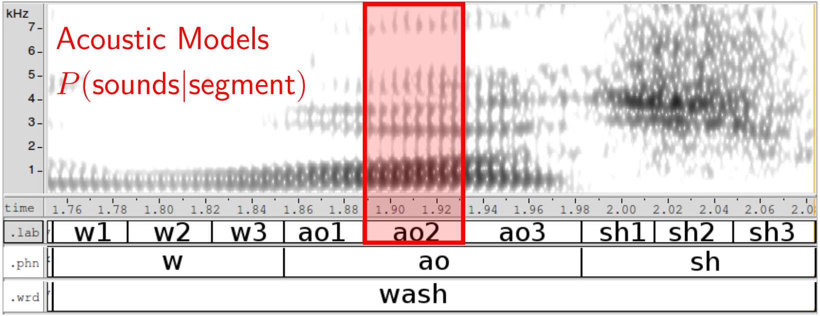

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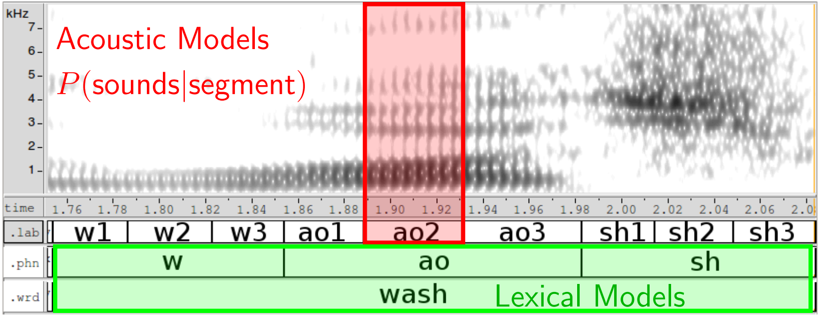


Local probabilities (frame-wise)

If segment sufficiently short

$$P(\text{sounds}|\text{segment})$$

can be modelled with standard probability distributions

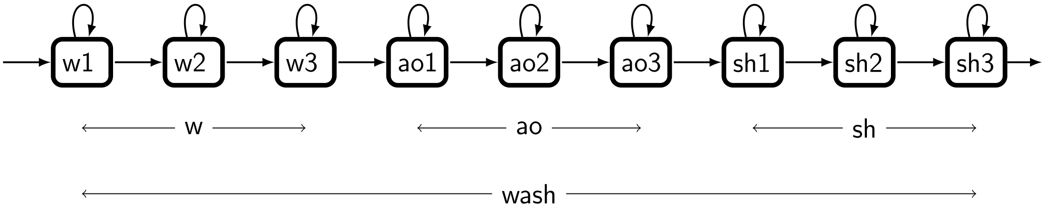
$$\phi_{s_a}(x) = P(x|s_a)$$

Usually Gaussian or Gaussian Mixture

Global Probabilities (utterance)

Problem: How do we combine the different $P(\text{sounds}|\text{segment})$ to form $P(\text{sounds}|\text{words})$?

Answer: Hidden Markov Model (HMM)



HMM-questions (Inference)

1. what is the probability that the model has generated the sequence of observations? (isolated word recognition)

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1. what is the probability that the model has generated the sequence of observations? (isolated word recognition) forward algorithm

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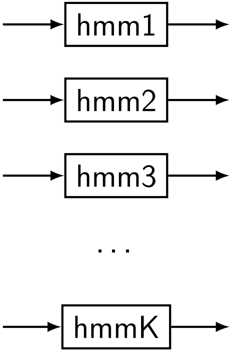
Isolated Words Recognition

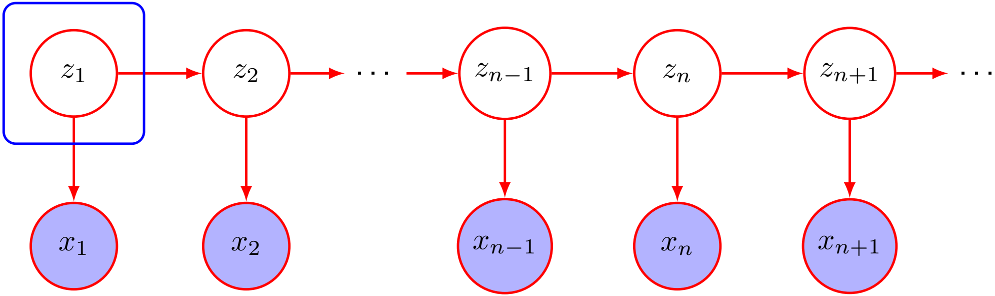


Compare Likelihoods (forward algorithm)

HMM Inference: Joint Distribution

$$X = \{x_1, \ldots, x_N\}$$
$$Z = \{z_1, \ldots, z_N\}$$

$$P(X, Z|\theta) = p(z_1|\pi) \left[\prod_{n=2}^{N} p(z_n|z_{n-1}, A)\right] \prod_{m=1}^{N} p(x_m|z_m, \phi)$$



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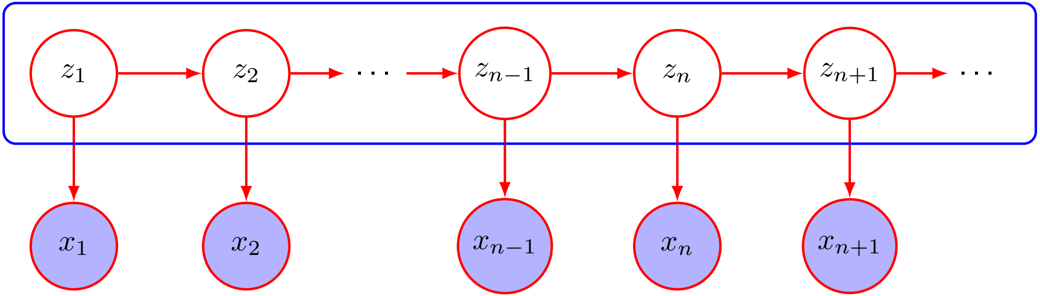

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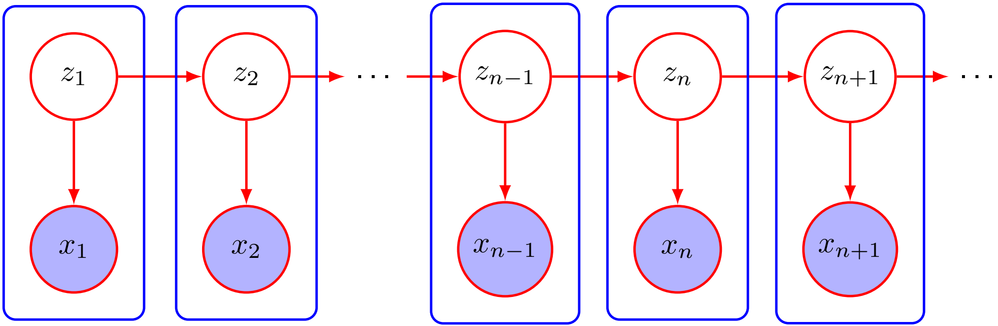
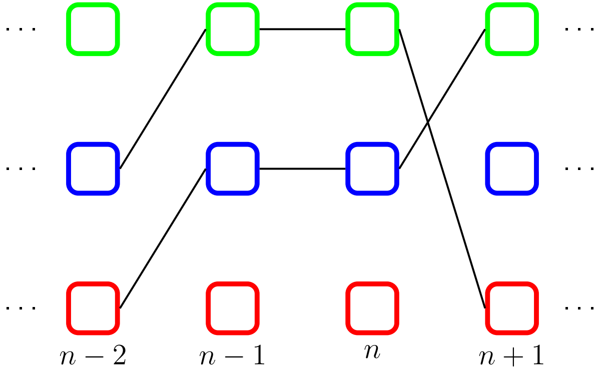


HMM Inference: Likelihood Function

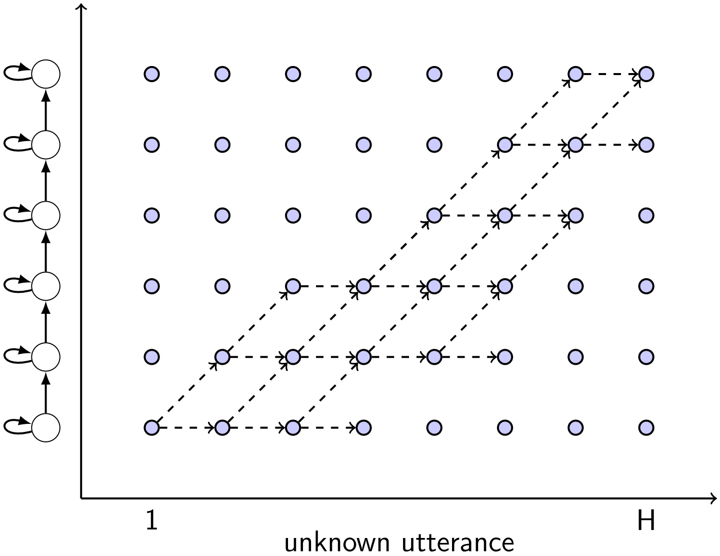
marginalise joint distribution over Z:

$$P(X|\theta) = \sum_Z p(X, Z|\theta)$$

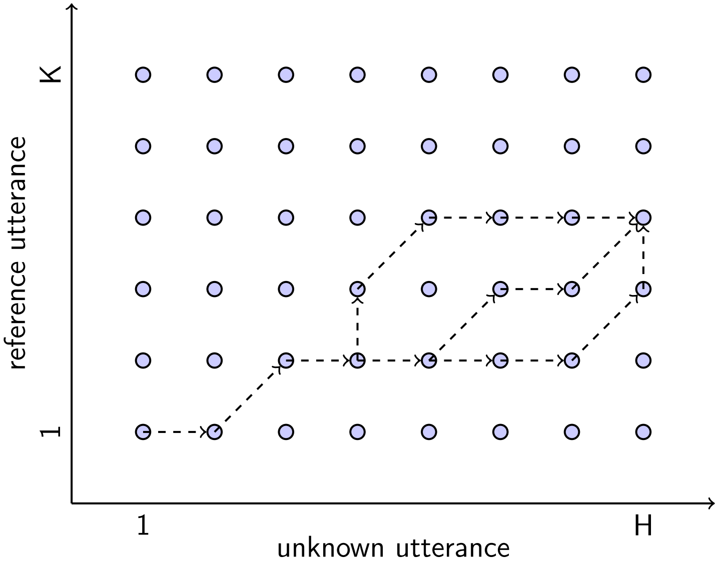
Problem: there are K^N possible sequences for Z



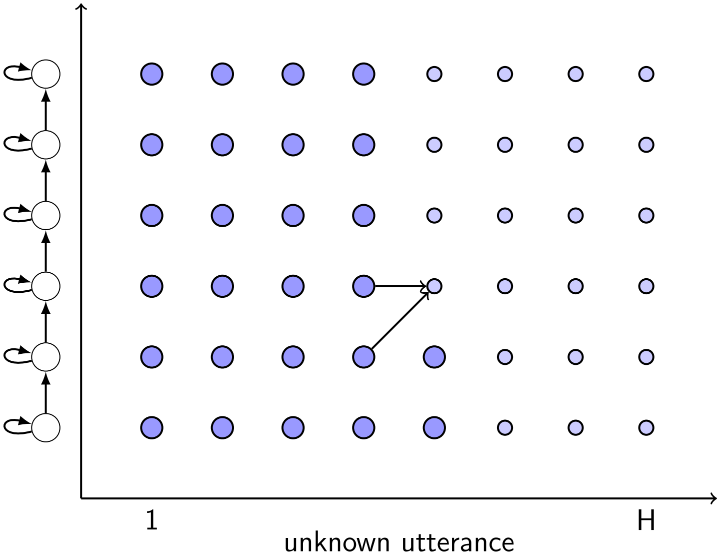
HMM Likelihood



Very Similar to Template Matching



Same Solution: Dynamic Programming



Solution: Forward algorithm

Instead of AccD[h,k] (Template Matching)

$$\alpha_n(j) \equiv p(x_1, \ldots, x_n, z_n = s_j | \theta)$$

At the end, instead of AccD[H,K]:

$$P(X|\theta) = \sum_{i=1}^{M} \alpha_N(i)$$

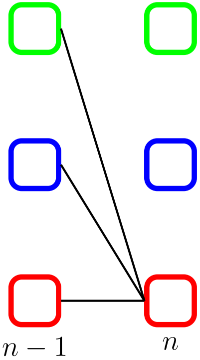
Forward Probability

Initialization:

$$\alpha_1(j) = \pi_j \phi_j(x_1)$$

Recursion:

$$\alpha_n(j) = \left[\sum_{i=1}^{M} \alpha_{n-1}(i) a_{ij} \right] \phi_j(x_n)$$



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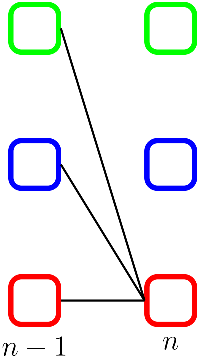
equivalent to sum-product in Bayesian Networks



Backward probability

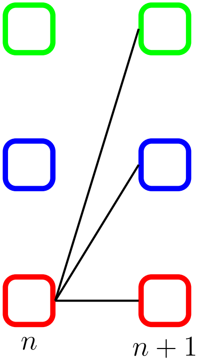
$$\beta_n(i) \equiv p(x_{n+1}, \ldots, x_N | z_n = s_i)$$

Initialization:

$$\beta_N(i) \equiv p(?|z_n = s_i) \equiv 1$$

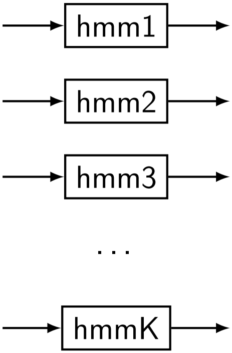
Recursion:

$$\beta_n(i) = \left[\sum_{j=1}^{M} a_{ij} \phi_j(x_{n+1}) \beta_{n+1}(j) \right]$$



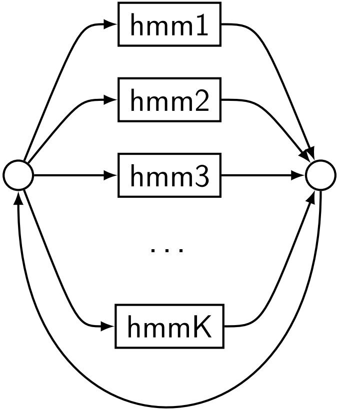
Find best sequence of states: why?

Isolated Words



(Likelihood)

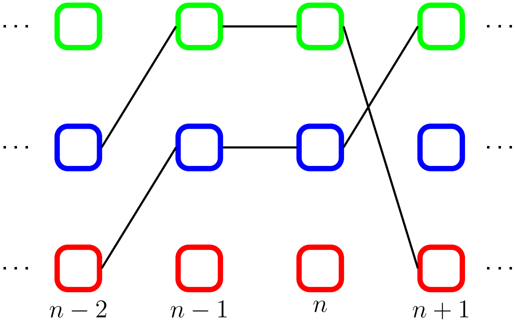
Continuous Speech



(best path)

Find best sequence of states: how?

- Viterbi algorithm[7]
- equivalent to max-sum in Bayesian Networks



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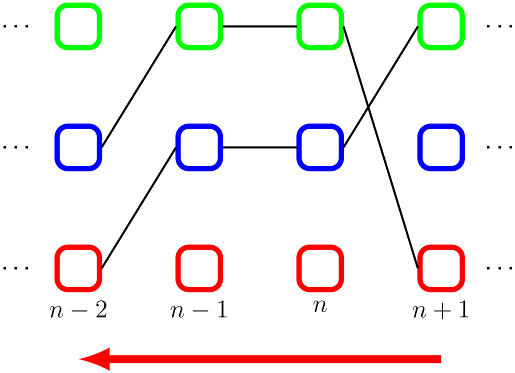

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Summary: update rules

Forward algorithm (sum-product):

$$\alpha_n(j) = \left[\sum_{i=1}^{M} \alpha_{n-1}(i) a_{ij}\right] \phi_j(x_n)$$

Viterbi algorithm (max-sum):

$$V_n(j) = \max_{i=1}^{M} \left[V_{n-1}(i) a_{ij}\right] \phi_j(x_n)$$

$$B_n(j) = \arg\max_{i=1}^{M} \left[V_{n-1}(i) a_{ij}\right]$$