Department of Engineering Cybernetics

TTK4215 System Identification and Adaptive Control

Solution 6

Problem 4.4 from I&S

We have from (4.3.29) in the book the following relation between the parameter estimation error and the estimation error

$$\epsilon = WL\left(-\tilde{\theta}^T\phi - \epsilon n_s^2\right). \tag{1}$$

At this point, the book assumes for simplicity that WL is strictly proper, so that WL can be realized on the form (4.3.30). In our case, WL is biproper, so any realization will have a direct feedthrough of u to y. That is, a state space realization will take the form

$$\begin{aligned} \dot{e} &= Ae + B\left(-\tilde{\theta}^T\phi - \epsilon n_s^2\right), && (2)\\ \epsilon &= C^Te + d\left(-\tilde{\theta}^T\phi - \epsilon n_s^2\right), && (3) \end{aligned}$$

with $d > 0$, and WL in terms of (A, B, C, d) is

$$WL = d + C^T(sI - A)^{-1}B. \tag{4}$$

Since WL is SPR, by Lemma 3.5.4 we have that for any $L = L^T > 0$, there exist a scalar $\nu > 0$, a vector q and a matrix $P = P^T > 0$ such that

$$\begin{aligned} A^TP + PA &= -qq^T - \nu L, && (5)\\ PB &= C \pm q\sqrt{2d}. && (6) \end{aligned}$$

Consider the function

$$V = \frac{1}{2}x^TPx + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}. \tag{7}$$

Its time derivative is

$$\dot{V} = \frac{1}{2}e^TP\dot{e} + \frac{1}{2}\dot{e}^TPe + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\dot{\theta}. \tag{8}$$

Defining $u = -\tilde{\theta}^T\phi - \epsilon n_s^2$ and using (2), we get

$$\begin{aligned} \dot{V} &= \frac{1}{2}e^TP(Ae + Bu) + \frac{1}{2}(Ae + Bu)^TPe + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\dot{\theta}\\ &= \frac{1}{2}e^T\left(PA + A^TP\right)e + e^TPBu + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\dot{\theta}. && (9) \end{aligned}$$

Using (5)–(6), we obtain

$$\begin{aligned}
\dot{V} &= \frac{1}{2}e^T\left(-qq^T-\nu L\right)e+e^T\left(C\pm q\sqrt{2d}\right)u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta} \\
&= \frac{1}{2}e^T\left(-qq^T-\nu L\right)e+e^TCu+e^Tq\left(\pm\sqrt{2d}\right)u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta} \\
&= \frac{1}{2}e^T\left(-qq^T-\nu L\right)e+C^Teu+e^Tq\left(\pm\sqrt{2d}\right)u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta} \\
&\quad +du^2-du^2 \\
&= \frac{1}{2}e^T\left(-qq^T-\nu L\right)e+\left(C^Te+du\right)u+e^Tq\left(\pm\sqrt{2d}\right)u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta}-du^2 \\
&= -\frac{1}{2}\nu e^TLe+\epsilon u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta} \\
&\quad -\frac{1}{2}\left(\left(e^Tq\right)^2-2e^Tq\left(\pm\sqrt{2d}\right)u+2du^2\right) \\
&= -\frac{1}{2}\nu e^TLe+\epsilon u+\tilde{\theta}^T\Gamma^{-1}\dot{\theta}-\frac{1}{2}\left(e^Tq\pm\sqrt{2d}u\right)^2. \qquad (10)
\end{aligned}$$

Inserting for u we have

$$\begin{aligned}
\dot{V} &= -\frac{1}{2}\nu e^TLe+\epsilon\left(-\tilde{\theta}^T\phi-\epsilon n_s^2\right)+\tilde{\theta}^T\Gamma^{-1}\dot{\theta}-\frac{1}{2}\left(e^Tq\pm\sqrt{2d}u\right)^2 \\
&= -\frac{1}{2}\nu e^TLe-\epsilon^2n_s^2+\tilde{\theta}^T\Gamma^{-1}\left(\dot{\theta}-\Gamma\epsilon\phi\right)-\frac{1}{2}\left(e^Tq\pm\sqrt{2d}u\right)^2. \qquad (11)
\end{aligned}$$

Selecting

$$\dot{\theta}=\Gamma\epsilon\phi, \qquad (12)$$

we have

$$\dot{V}=-\frac{1}{2}\nu e^TLe-\epsilon^2n_s^2-\frac{1}{2}\left(e^Tq\pm\sqrt{2d}u\right)^2. \qquad (13)$$

From (7) and (13), the usual arguments can be applied to conclude the statements of Theorem 4.3.1.

Problem 4.5 from I&S

Let $\Lambda(s)=s^n+\lambda_{n-1}s^{n-1}+...+\lambda_1s+\lambda_0$ be a Hurwitz polynomial of order n.

Case 1: $Z(s)$ is unknown while $R(s)$ is known.

We have that

$$R(s)y=Z(s)u. \qquad (14)$$

Defining

$$\theta^*=\left[\begin{array}{ccccc} b_{n-1} & b_{n-2} & \cdots & b_1 & b_0\end{array}\right]^T, \qquad (15)$$

we have

$$R(s)y=\theta^{*^T}\left[\begin{array}{ccccc} s^{n-1}u & s^{n-2}u & \cdots & su & u\end{array}\right]^T. \qquad (16)$$

Filtering each side with $1/\Lambda(s)$ we have

$$z=\theta^{*^T}\phi, \qquad (17)$$

where

$$z = \frac{R(s)}{\Lambda(s)} y, \tag{18}$$

$$\phi = \begin{bmatrix} \frac{s^{n-1}}{\Lambda(s)} u & \frac{s^{n-2}}{\Lambda(s)} u & \cdots & \frac{s}{\Lambda(s)} u & \frac{1}{\Lambda(s)} u \end{bmatrix}^T. \tag{19}$$

At this point, any method in the book can be applied.

Case 2: $Z(s)$ is known while $R(s)$ is unknown.

We have that

$$Z(s)\, u = R(s)\, y. \tag{20}$$

Defining

$$\theta^* = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}^T, \tag{21}$$

we have

$$R(s)\, y = s^n y + \theta^{*^T} \begin{bmatrix} s^{n-1} y & s^{n-2} y & \cdots & sy & y \end{bmatrix}^T. \tag{22}$$

Filtering each side of (20) with $1/\Lambda(s)$ and using (22) we have

$$z = \theta^{*^T} \phi, \tag{23}$$

where

$$z = \frac{Z(s)}{\Lambda(s)} u - \frac{s^n}{\Lambda(s)} y, \tag{24}$$

$$\phi = \begin{bmatrix} \frac{s^{n-1}}{\Lambda(s)} y & \frac{s^{n-2}}{\Lambda(s)} y & \cdots & \frac{s}{\Lambda(s)} y & \frac{1}{\Lambda(s)} y \end{bmatrix}^T. \tag{25}$$

At this point, any method in the book can be applied.

Problem 4.6 from I&S

The gradient of $J(\theta)$ is

$$\nabla J = \frac{\left(z - \theta^T \phi\right)}{m^2} (-\phi), \tag{26}$$

so we need to solve

$$\left(z - \theta^T \phi\right) \phi = 0. \tag{27}$$

Since $z - \theta^T \phi$ is scalar, we can exchange its order relative to ϕ, and since ϕ is nonzero we can multiply the equation with ϕ^T (from the left) to obtain

$$\phi^T \phi \left(z - \theta^T \phi\right) = 0. \tag{28}$$

Rearranging, we can write this as

$$\left(z\phi^T\right) \phi = \left(\phi^T \phi \theta^T\right) \phi. \tag{29}$$

It is now clear that a solution to this equation is given by

$$z\phi^T = \phi^T \phi \theta^T. \tag{30}$$

Transposing and dividing by $\phi^T \phi$ now gives the desired result.

Problem 4.7 from I&S

Let

$$H = \int_t^{t+T_0} \omega(\tau)\, \omega^T(\tau)\, d\tau. \tag{31}$$

Since ω is PE, there exists constants α_0, α_1, and T_0 so that

$$\alpha_0 I \leq H \leq \alpha_1 I. \tag{32}$$

In other words, H is positive definite. Let

$$H_0 = \int_t^{t+T_0} \omega_0(\tau)\, \omega_0^T(\tau)\, d\tau = \int_t^{t+T_0} F\omega(\tau)\, \omega_0^T(\tau)\, F^T d\tau = FHF^T. \tag{33}$$

If F has full rank, then $y = F^T x \in R^n$ is nonzero for any nonzero $x \in R^m$. Since H is positive definite, we have

$$0 < y^T H y = x^T F H F^T x = x^T H_0 x, \tag{34}$$

which proves that H_0 is positive definite. If F does not have full rank, there exists a nonzero $x \in R^m$ such that $F^T x = 0$. Thus,

$$x^T H_0 x = x^T F H F^T x = 0, \tag{35}$$

which proves that H_0 is not positive definite.