NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2019

Filtering of Discrete Random Signals

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 12.1 Random signals, correlation functions, and power spectra
  - 5.3 Correlation functions and spectra at the output of LTI systems
- A comprehensive overview of topics treated in the lecture, see "Introdukjon til statistisk signalbehandling" on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Filtering of stochastic signals in time-domain
- Frequency-domain interpretation
- Example: Power density spectrum of AR(1) process

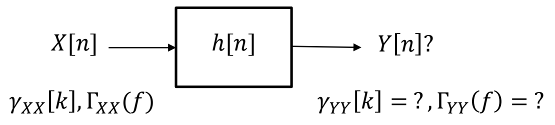
Filtering of stochastic signals

$$X[n] \longrightarrow \boxed{h[n]} \longrightarrow Y[n]?$$

$$\gamma_{XX}[k], \Gamma_{XX}(f) \qquad \gamma_{YY}[k] = ?, \Gamma_{YY}(f) = ?$$

- Let $X[n]$ be a wide-sense stationary process
- Linear time-invariant filter described by $h[n]$, $H(z)$, or $H(f)$
- Can we relate output signal $Y[n]$ to input signal $X[n]$?

Filtering of stochastic signals…



- Consider a single realization $x[n]$ of process $X[n]$
- Each input realization $x[n]$ produces output realization $y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

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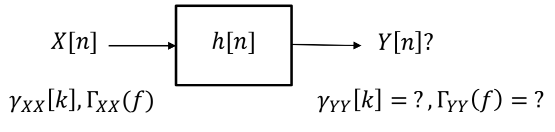
Filtering of stochastic signals…



- Since $x[n]$ is a realization of $X[n]$, $y[n]$ is a realization of the random process $Y[n]$

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k]X[n-k]$$

- We want to relate the statistical properties of output process $Y[n]$ to the statistical properties of input process $X[n]$

$$E\{Y[n]\} = ?,\ \gamma_{YY}[k] = ?, \Gamma_{YY}(f) = ?$$

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Filtering of stochastic signals…

- Expected value of output process $Y[n]$:

$$m_Y = E\{Y[n]\} = E\{\sum_{k=-\infty}^{\infty} h[k]X[n-k]\}$$
$$= \sum_{k=-\infty}^{\infty} h[k]E\{X[n-k]\}$$
$$= m_X \sum_{k=-\infty}^{\infty} h[k]$$
$$= m_X \sum_{k=-\infty}^{\infty} h[k]e^{j2\pi 0 k}$$
$$= m_X H(0)$$

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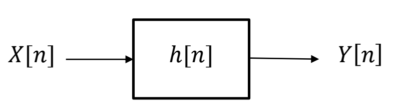
Filtering of stochastic signals…



- Example: WSS signal $X[n]$ with mean $m_X = 3$ is filtered by LTI system $H(f) = \frac{1}{1-0.5e^{-j2\pi f}}$. Compute the mean of output process $Y[n]$.

$$m_Y = E\{Y[n]\} = m_X H(0) = \frac{3}{1-0.5e^{-j2\pi 0}} = 6$$

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Filtering of stochastic signals…

- Autocorrelation sequence of output process $Y[n]$:

$$\gamma_{YY}[l] = E\{Y[n]Y[n+l]\} = h[-l] * h[l] * \gamma_{XX}[l]$$
$$= r_{hh}[l] * \gamma_{XX}[l]$$

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Filtering of stochastic signals…

- Proof:

$$\gamma_{YY}[l] = E\left\{\left(\sum_{i=-\infty}^{\infty} h[i]X[n-i]\right)\left(\sum_{j=-\infty}^{\infty} h[j]X[n+l-j]\right)\right\}$$
$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i]h[j]E\{X[n-i]X[n+l-j]\}$$
$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i]h[j]\gamma_{XX}[l-j+i]$$
$$= \sum_{i=-\infty}^{\infty} h[i] \sum_{j=-\infty}^{\infty} h[j]\gamma_{XX}[(l+i)-j]$$
$$= \sum_{i=-\infty}^{\infty} h[i]g[l+i] \text{ with } g[l] = h[l] * \gamma_{XX}[l]$$
$$= \sum_{k=-\infty}^{\infty} h[-k]g[l-k] = h[-l] * g[l]$$

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Filtering of stochastic signals…

- Power density spectrum of $Y[n]$:

$$\Gamma_{YY}(f) = \mathcal{F}\{\gamma_{YY}[k]\} = \mathcal{F}\{r_{hh}[k] * \gamma_{XX}[k]\}$$

$$= \mathcal{F}\{r_{hh}[k]\}\mathcal{F}\{\gamma_{XX}[k]\}$$

$$= S_{hh}(f)\Gamma_{XX}(f) = |H(f)|^2\Gamma_{XX}(f)$$

- The output PDS is the input PDS multiplied by the magnitude-squared of the frequency response!

Filtering of stochastic signals…

- Example: What is the output power $E\{Y^2[n]\}$ of a linear system $H(f)$ when the input $X[n]$ is WGN?

$$\Gamma_{YY}(f) = |H(f)|^2\Gamma_{XX}(f) = |H(f)|^2\mathcal{F}\{\sigma_X^2\,\delta[n]\}$$

$$= \sigma_X^2|H(f)|^2$$

$$\sigma_Y^2 = E\{Y^2[n]\} = \gamma_{YY}[0]$$

$$= \int_{-0.5}^{0.5}\Gamma_{YY}(f)e^{j2\pi f 0}df = \sigma_X^2\int_{-0.5}^{0.5}|H(f)|^2 df$$

Filtering of stochastic signals…

- Crosscorrelation sequence of processes $Y[n]$ and $X[n]$:

$$\gamma_{YX}[l] = E\{(\sum_{k=-\infty}^{\infty} h[k]X[n-k])X[n-l]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]E\{X[n-k]X[n-l]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\gamma_{XX}[l-k] = h[l] * \gamma_{XX}[l]$$

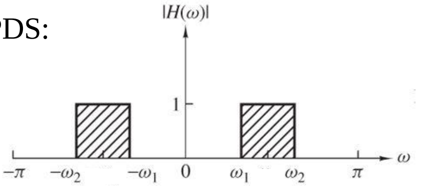
- Crosscorrelation spectrum of processes $Y[n]$ and $X[n]$:

$$\Gamma_{YX}(f) = \mathcal{F}\{h[k] * \gamma_{XX}[l]\} = \mathcal{F}\{h[k]\}\mathcal{F}\{\gamma_{XX}[k]\}$$

$$= H(f)\Gamma_{XX}(f)$$

Frequency domain interpretation

Interpretation PDS:



- Filter with narrow frequency band ⇒ $\Gamma_{YY}(f) = |H(f)|^2\Gamma_{XX}(f)$
- Average output power

$$E\{Y^2[n]\} = \gamma_{YY}[0] = \int_{-0.5}^{0.5} \Gamma_{YY}(f)df$$

$$= \int_{-\frac{\omega_2}{2\pi}}^{\frac{-\omega_1}{2\pi}} |H(f)|^2\Gamma_{XX}(f)df + \int_{\frac{\omega_1}{2\pi}}^{\frac{\omega_2}{2\pi}} |H(f)|^2\Gamma_{XX}(f)df$$

- Area under $\Gamma_{XX}(f)$ for $\omega_1 \le |\omega| \le \omega_2$ is the average power for that frequency band ⇒ $\Gamma_{XX}(f)$ can be viewed as density function for power in frequency domain

Example: PDS of AR(1) process

- Example: Calculate $\Gamma_{XX}(f)$ for the random process

$$X[n] = aX[n-1] + W[n],\ W[n] \sim N(0, \sigma_W^2)$$

$$\text{WGN: } E\{W[n]W[n+l]\} = \sigma_W^2 \delta[l]$$

- Approach 1: Calculate $\gamma_{XX}[l]$ and take its Fourier transform
- Approach 2: Use the idea of LTI systems

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Example: PDS of AR(1) process…

- Approach 1: Calculate $\gamma_{XX}[l]$ and take its Fourier transform
- Consider lag $l = 0$:

$$\begin{aligned} E\{X[n]X[n]\} = \gamma_{XX}[0] &= E\{(aX[n-1] + W[n])^2\} \\ &= E\{a^2X^2[n-1] + 2aX[n-1]W[n] + W^2[n]\} \\ &= a^2E\{X^2[n-1]\} + 2aE\{X[n-1]W[n]\} + E\{W^2[n]\} \\ &= a^2\gamma_{XX}[0] + 0 + \sigma_W^2 \end{aligned}$$

$$\Rightarrow \boxed{\gamma_{XX}[0] = \frac{1}{1-a^2}\sigma_W^2}$$

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Example: PDS of AR(1) process…

- Consider lag $l \geq 1$:

$$\begin{aligned}\gamma_{XX}[l] &= E\{X[n]X[n+l]\} \\ &= E\{X[n](aX[n+l-1]+W[n+l])\} \\ &= E\left\{X[n]\left(a^l X[n] + \textstyle\sum_{j=0}^{l-1} a^j W[n+l-j]\right)\right\} \\ &= a^l E\{X^2[n]\} + \textstyle\sum_{j=0}^{l-1} a^j E\{X[n]W[n+l-j]\} \\ &= a^l \gamma_{XX}[0] + 0 = \frac{a^l}{1-a^2}\sigma_W^2\end{aligned}$$

- Symmetry $\gamma_{XX}[l] = \gamma_{XX}[-l]$ provides the final answer

$$\boxed{\gamma_{XX}[l] = \frac{a^{|l|}}{1-a^2}\sigma_W^2}$$

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Example: PDS of AR(1) process…

- Take the Fourier transform:

$$\begin{aligned}\Gamma_{XX}(f) &= \mathcal{F}\{\gamma_{XX}[l]\} \\ &= \textstyle\sum_{l=-\infty}^{\infty} \frac{a^{|l|}\sigma_W^2}{1-a^2} e^{-j2\pi f l} \\ &= \frac{\sigma_W^2}{1-a^2} \textstyle\sum_{l=-\infty}^{\infty} a^{|l|} e^{-j2\pi f l} \\ &= \cdots = \frac{\sigma_W^2}{1-a^2} \frac{1-a^2}{\left|1-ae^{-j2\pi f}\right|^2} \\ &= \frac{\sigma_W^2}{\left|1-ae^{-j2\pi f}\right|^2}\end{aligned}$$

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Example: PDS of AR(1) process…

- Approach 2: Model the problem with a linear system

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = aX[n-1] + W[n]$$

$$\gamma_{WW}[k], \Gamma_{WW}(f) \qquad \Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f)$$

- WGN process $W[n] \Rightarrow \gamma_{WW}[k] = ?, \Gamma_{WW}(f) = ?$
- What is the system frequency response $H(f)$?

Example: PDS of AR(1) process…

- Find the frequency response $H(f)$:

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = aX[n-1] + W[n]$$

$$\gamma_{WW}[k], \Gamma_{WW}(f) \qquad \Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f)$$

- For any realization $x[n]$: $X(z) \ = az^{-1}X(z) + W(z)$

$$\Longrightarrow H(z) = \frac{1}{1-az^{-1}}$$

- Consequently we obtain the frequency response

$$H(f) = \frac{1}{1 - ae^{-j2\pi f}}$$

Example: PDS of AR(1) process…

- Power density spectrum of $X[n]$:

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = aX[n-1] + W[n]$$

$\gamma_{WW}[k], \Gamma_{WW}(f)$ $\qquad$ $\Gamma_{XX}(f) = |H(f)|^2\Gamma_{WW}(f)$

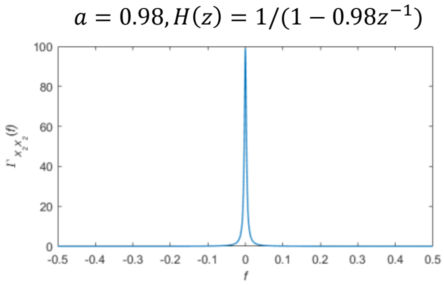
$$\Gamma_{XX}(f) = |H(f)|^2\Gamma_{WW}(f)$$

$$\boxed{= \frac{1}{\left|1-ae^{-j2\pi f}\right|^2}\sigma_W^2}$$

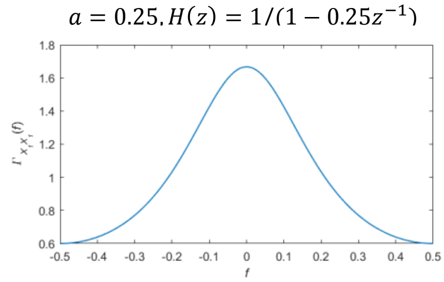
Example: PDS of AR(1) process…

- Power density spectrum of $X[n]$:

$a = 0.98, H(z) = 1/(1 - 0.98z^{-1})$



$a = 0.25, H(z) = 1/(1 - 0.25z^{-1})$



Matlab

```
[H,W]=freqz(1,[1 -0.98],(-pi:pi/500:pi))
plot(W/2/pi,(1-0.98^2)*abs(H).^2)
```

Summary

- Today we discussed:
  - Linear filtering of stochastic processes
  - Power density and cross-spectra
- Next:
  - Basics of parameter estimation