TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
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Fall 2014 - Solution to Assignment 4

1. (a) Substitution of the equation

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x \, \mathrm{d}\sigma$$

into

$$x^{\mathrm{T}} P f(x) + f^{\mathrm{T}}(x) P x$$

yields

$$x^{\mathrm{T}} P \int_0^1 \left[\frac{\partial f}{\partial x}(\sigma x)\right] x \, \mathrm{d}\sigma + \int_0^1 x^{\mathrm{T}} \left[\frac{\partial f}{\partial x}(\sigma x)\right]^{\mathrm{T}} \mathrm{d}\sigma \; P x =$$
$$= x^{\mathrm{T}} \int_0^1 \left\{ P \left[\frac{\partial f}{\partial x}(\sigma x)\right] + \left[\frac{\partial f}{\partial x}(\sigma x)\right]^{\mathrm{T}} P \right\} \mathrm{d}\sigma x \le -x^{\mathrm{T}} \int_0^1 \mathrm{d}\sigma x = -x^{\mathrm{T}} x$$

(b) Since $P > 0$ (positive definite), $V(x)$ must be positive semi-definite, $V(x) \ge 0$. To show that it is positive definite, we need to show that $f(x) = 0$ only when $x = 0$. But the equality in (a) shows that if $f(p) = 0$ then

$$0 \le -p^{\mathrm{T}} p.$$

Suppose that f is bounded, i.e. that $\|f(x)\| \le c$ for all x. Then

$$\|x^{\mathrm{T}} P f + f^{\mathrm{T}} P x\| \le 2c\|P\|\|x\|.$$

This contradicts the inequality in (a) as $\|x\| \to \infty$.

(c) V is positive definite and radially unbounded. The total derivative of V is

$$\dot{V} = \dot{x}^{\mathrm{T}} \left[\frac{\partial f}{\partial x}\right]^{\mathrm{T}} P f + f^{\mathrm{T}} P \left[\frac{\partial f}{\partial x}\right] \dot{x} = f^{\mathrm{T}} \left\{ P \left[\frac{\partial f}{\partial x}\right] + \left[\frac{\partial f}{\partial x}\right]^{\mathrm{T}} P \right\} f \le -\|f\|^2.$$

Hence $\dot{V} < 0, \ \forall \ x \neq 0$, thus $x = 0$ is globally asymptotically stable.

2. (a) The function $V_1(x_1, x_2, t)$ is given by

$$V_1(x_1, x_2, t) = x_1^2 + \left(1 + e^t\right) x_2^2$$

Since $e^t \to \infty$ when $t \to \infty$, the term $(1 + e^t)$ may not be upper bounded uniformly in t. Hence, the function is not decrescent. The function may however be lower bounded by

$$\begin{aligned} V_1(x_1, x_2, t) &= x_1^2 + \left(1 + e^t\right) x_2^2 \\ &\ge x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

where $W_1(x)$ is positive definite. This implies that the function $V_1(x_1, x_2, t)$ is positive definite.

(b) The function $V_2(x_1, x_2, t)$ is given by

$$V_2(x_1, x_2, t) = \frac{x_1^2 + x_2^2}{1 + t}$$

Since $\frac{1}{1+t} \to 0$ when $t \to \infty$ the function $V_2(x_1, x_2, t)$ may not be lower bounded uniformly in t. Hence, the function is not positive definite. The function may however be upper bounded by

$$\begin{aligned} V_2(x_1, x_2, t) &= \frac{x_1^2 + x_2^2}{1 + t} \\ &\leq x_1^2 + x_2^2 \\ &= W_2(x) \end{aligned}$$

where $W_2(x)$ is positive definite. This implies that the function $V_2(x_1, x_2, t)$ is decrescent.

(c) The function $V_3(x_1, x_2, t)$ is given by

$$V_3(x_1, x_2, t) = \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right)$$

Since $1 \leq (1 + \cos^4 t) \leq 2$, the function may be lower and upper bounded according to

$$\begin{aligned} V_3(x_1, x_2, t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\geq x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

and

$$\begin{aligned} V_3(x_1, x_2, t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\leq 2\left(x_1^2 + x_2^2\right) \\ &= W_2(x) \end{aligned}$$

Since $W_1(x)$ and $W_2(x)$ are both positive definite, we conclude that the function $V_3(x_1, x_2, t)$ is positive definite and decrescent.

3. The system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - c(t)\, x_2 \end{aligned}$$

A Lyapunov function candidate is taken as

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \tag{1}$$

The time derivative of $V(x)$ along the trajectories of the system is

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1x_2 + x_2(-x_1 - c(t)\,x_2) \\
&= x_1x_2 - x_1x_2 - c(t)\,x_2^2 \\
&= -c(t)\,x_2^2 \\
&\leq -k_1x_2^2 \qquad (2)
\end{aligned}$$

and it can be seen that $\dot{V}(x)$ is negative semidefinite. By Theorem 4.8 we conclude that the origin is uniformly stable ($V(x)$ is positive definite and decrescent). In order to prove that $x_2 \to 0$ as $t \to \infty$ we apply Barbalat's lemma. Since $\dot{V}(x) = -c(t)\,x_2^2$ where $c(t)$ is some bounded value greater than zero, $\dot{V}(x) = 0 \Leftrightarrow x_2 = 0$. Following the notation of Lemma 8.2, let $\phi(t) = \dot{V}(t)$. $\dot{V}(t)$ is uniformly continuous in t if $\ddot{V}(t)$ is bounded

$$\begin{aligned}
\ddot{V}(t) &= -\dot{c}(t)\,x_2^2 - 2c(t)\,x_2\dot{x}_2 \\
&= -\dot{c}(t)\,x_2^2 - 2c(t)\,x_2(-x_1 - c(t)\,x_2) \\
&= -\dot{c}(t)\,x_2^2 + 2c(t)\,x_1x_2 + 2c^2(t)\,x_2^2
\end{aligned}$$

Since the time derivative of $\dot{V}(t)$ is negative semidefinite, it follows that $V(t) \leq V(t_0)$, and since $V(x_1, x_2, t)$ is radially unbounded in x this again implies that x_1 and x_2 are bounded. Since x_1 and x_2 are bounded and it is given that $c(t)$ and $\dot{c}(t)$ are bounded, it follows that $\ddot{V}(t)$ is bounded. The bound on $\ddot{V}(t)$ guarantees that $\dot{V}(t)$ is uniformly continuous. In order to conclude by Barbalat's lemma we also need to prove that $\lim_{t\to\infty}\int_0^t \dot{V}(\tau)\,d\tau$ exists and is finite. This is proven according to

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t \dot{V}(\tau)\,d\tau &= \lim_{t\to\infty}(V(t) - V(0)) \\
&= \lim_{t\to\infty} V(t) - V(0)
\end{aligned}$$

where we know that $\lim_{t\to\infty} V(t) = V_\infty$ is a finite number since $V(t) \geq 0\forall t$ and $\dot{V}(t) \leq 0\forall t$. By Lemma 8.2 it is shown that $\dot{V}(t) \to 0$ as $t \to \infty$, and hence that $x_2 \to 0$ as $t \to \infty$.

4. (Khalil 4.54)

   (1) The system is not input-to-state stable (ISS) since with $u(t) \equiv c < -1$ and $x(0) > 1$ we have $x(t) \to \infty$ for $t \to \infty$.

   (2) Let $V(x) = \frac{1}{2}x^2$ which is positive definite and decrescent. Then

$$\begin{aligned}
\dot{V} &= -x^4 - ux^4 - x^6 \leq -x^6 + |u|\,x^4 \\
\dot{V} &\leq -(1-\theta)\,x^6 - \theta x^6 + |u|\,x^4
\end{aligned}$$

where $0 < \theta < 1$, and

$$\dot{V} \leq -(1-\theta)\, x^6 \quad \forall \ \ \|x\| \geq \rho(|u|) > 0$$

where

$$\rho(|u|) = \sqrt{\frac{|u|}{\theta}}$$

By Theorem 4.19 in Khalil, the system is ISS.

(3) The system is not ISS since with $u(t) \equiv 1$ and $x(0) > 0$ we have $x(t) \to \infty$ for $t \to \infty$.

(4) With $u(t) \equiv 0$, the origin of $\dot{x} = x - x^3$ is unstable. Hence, the system is not ISS.

5. (Khalil 4.55)

(1) The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 + u \end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

which is a $\mathcal{K}_\infty$ function. The time derivative along the trajectories of the system is

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1\left(-x_1 + x_1^2 x_2\right) + x_2\left(-x_1^3 - x_2 + u\right) \\ &= -x_1^2 + x_1^3 x_2 - x_1^3 x_2 - x_2^2 + u x_2 \\ &= -x_1^2 - x_2^2 + u x_2 \\ &= -\|x\|_2^2 + u x_2 \end{aligned}$$

and upper bounded as

$$\begin{aligned} \dot{V}(x) &\leq -\|x\|_2^2 + |u x_2| \\ &= -\|x\|_2^2 + |u|\,|x_2| \\ &\leq -\|x\|_2^2 + |u|\,\|x\|_2 \\ &= -\|x\|_2^2 + |u|\,\|x\|_2 + \theta\|x\|_2^2 - \theta\|x\|_2^2 \\ &= -(1-\theta)\|x\|_2^2 + |u|\,\|x\|_2 - \theta\|x\|_2^2 \\ &= -(1-\theta)\|x\|_2^2 - (\theta\|x\|_2 - |u|)\|x\|_2 \\ &\leq -(1-\theta)\|x\|_2^2 \quad \forall \ \ \theta\|x\|_2 - |u| \geq 0 \\ &= -(1-\theta)\|x\|_2^2 \quad \forall \ \ \|x\|_2 \geq \frac{|u|}{\theta} \end{aligned}$$

where $\theta \in (0,1)$. Hence, by Theorem 4.19, the system is input-to-state stable (ISS) with $\rho(|u|) = \frac{|u|}{\theta}$.

(2) The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 + u\end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

The time derivative along the trajectories of the system is calculated as

$$\begin{aligned}\dot{V}(x) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1^3(-x_1 + x_2) + x_2\left(-x_1^3 - x_2 + u\right) \\ &= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + ux_2 \\ &= -x_1^4 - x_2^2 + ux_2\end{aligned}$$

and upper bounded as

$$\begin{aligned}\dot{V}(x) &= -x_1^4 - (1-\theta)x_2^2 + ux_2 - \theta x_2^2 \\ &\leq -x_1^4 - (1-\theta)x_2^2 \ \forall |x_2| \geq \frac{|u|}{\theta}\end{aligned} \tag{3}$$

where $\theta \in (0,1)$. When $|x_2| \leq \frac{|u|}{\theta}$ have that

$$\begin{aligned}\dot{V}(x) &= -x_1^4 - x_2^2 + ux_2 \\ &\leq -x_1^4 - x_2^2 + |x_2||u| \\ &\leq -x_1^4 - x_2^2 + \frac{|u|^2}{\theta} \\ &= -(1-\theta)x_1^4 - x_2^2 - \left(\theta x_1^4 - \frac{|u|^2}{\theta}\right) \\ &\leq -(1-\theta)x_1^4 - x_2^2 \ \forall |x_1| \geq \sqrt{\frac{|u|}{\theta}}\end{aligned} \tag{4}$$

By using (3) and (4) it follows that

$$\dot{V}(x) \leq -(1-\theta)\left(x_1^4 + x_2^2\right) \ \forall \|x\|_\infty \geq \rho(|u|)$$

where

$$\rho(r) = \max\left(\frac{r}{\theta}, \sqrt{\frac{r}{\theta}}\right)$$

Hence, the system is ISS.

(4) With $u = 0$ the system is given by

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 - 1\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 - 1\right)\end{aligned}$$

and it can be seen that it has an equilibrium set $\{x_1^2 = 1\}$. Hence, the origin is not globally asymptotically stable. It follows that the system is not ISS.

(5) The unforced system ($u = 0$) has equilibrium points $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Hence, the origin is not globally asymptotically stable. Consequently, the system is not ISS.

6. (Khalil 4.56) The system is given by

$$\dot{x}_1 = -x_1^3 + x_2 \tag{5}$$

$$\dot{x}_2 = -x_2^3 \tag{6}$$

We first show that the system $\dot{x}_1 = -x_1^3 + u$ is ISS using Theorem 4.19 with $V = \frac{1}{2}x_1^2$.

$$\begin{aligned}\dot{V} &= x_1\left(-x_1^3 + u\right) = -x_1^4 + x_1 u\\ &= -(1-\theta)\,x_1^4 - \theta x_1^4 + x_1 u\\ &\le -(1-\theta)\,x_1^4 - \theta x_1^4 + |x|\,|u|\\ &\le -(1-\theta)\,x_1^4 \;\;\forall\;\; |x_1| \ge \left(\frac{|u|}{\theta}\right)^{1/3}\end{aligned}$$

where $0 < \theta < 1$. The system $\dot{x}_1 = -x_1^3 + u$ is thus ISS.

Next we show that the system $\dot{x}_2 = -x_2^3$ is GAS at the origin using Theorem 4.2 with $V = \frac{1}{2}x_2^2$.

$$\dot{V} = -x_2^4 < 0 \;\;\forall\;\; x \neq 0$$

Since V also is radially unbounded in x_2, the system $\dot{x}_2 = -x_2^3$ is GAS at the origin. Hence, by Lemma 4.7 the cascade system (5) - (6) is GAS at the origin.

7. (Khalil 5.3)

(a) Let $\alpha(r) = r^{1/3}$; α is a class $\mathcal{K}_\infty$ function. We have

$$|y| \le |u|^{1/3} \Longrightarrow \|y_\tau\|_{\mathcal{L}_\infty} \le \left(\|u_\tau\|_{\mathcal{L}_\infty}\right)^{1/3} \Longrightarrow \|y_\tau\|_{\mathcal{L}_\infty} \le \alpha\left(\|u_\tau\|_{\mathcal{L}_\infty}\right).$$

Hence the system is $\mathcal{L}_\infty$ stable with zero bias.

(b) The two curves $|y| = |u|^{1/3}$ and $|y| = a\,|u|$ intersect at the point $|u| = (1/a)^{3/2}$. See Figure 1. Therefore, for $|u| \le (1/a)^{3/2}$ we have

$$|y| \le |u|^{1/3} \le (1/a)^{3/2 \cdot 1/3} = (1/a)^{1/2}$$

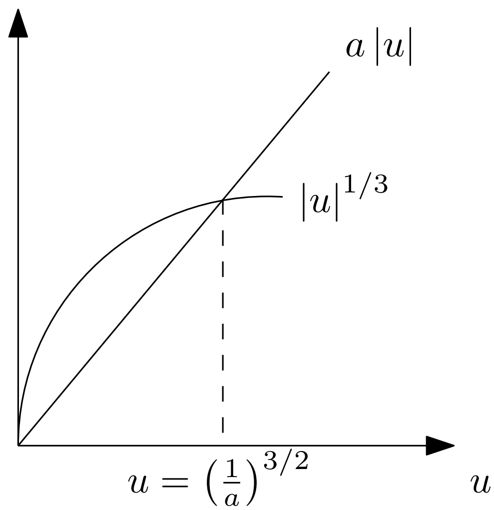

Figure 1: $y = a\,|u|$ and $y = |u|^{1/3}$.

while for $|u| > (1/a)^{3/2}$ we have

$$|y| \leq a\,|u|$$

Thus

$$|y| \leq a\,|u| + (1/a)^{1/2}\,, \quad \forall\,|u| \geq 0.$$

Setting $\gamma = a$ and $\beta = (1/a)^{1/2}$ we obtain

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \gamma\,\|u_\tau\|_{\mathcal{L}_\infty} + \beta.$$

(c) To show finite-gain stability we must use nonzero bias. This example shows that a nonzero bias term may be used to achieve finite-gain stability in situations where it is not possible to have finite-gain stability with zero bias.

8. (Khalil 5.4)

(1) $h(0) = 0 \Longrightarrow |h(u)| \leq L\,|u|\,,\ \forall u$. For $p = \infty$ we have

$$\sup_{t\geq 0}|y(t)| \leq L\sup_{t\geq 0}|u(t)|$$

which shows that the system is finite-gain $\mathcal{L}_\infty$ stable with zero bias. For $p \in [1, \infty)$ we have

$$\int_0^\tau |y(t)|^p\,dt \leq L^p \int_0^\tau |u(t)|^p\,dt \Longrightarrow \|y_\tau\|_{\mathcal{L}_p} \leq L\,\|u_\tau\|_{\mathcal{L}_p}\,.$$

Hence for each $p \in [1, \infty)$ the system is finite gain $\mathcal{L}_p$ stable with zero bias.

(2) Let $|h(0)| = k > 0$. Then $|h(u)| \leq L|u| + k$. For $p = \infty$ we have

$$\sup_{t\geq 0} |y(t)| \leq L \sup_{t\geq 0} |u(t)| + k$$

which shows that the system is finite gain $\mathcal{L}_\infty$ stable. For $p \in [1, \infty)$ the integral $\int_0^\tau (L|u(t)| + k)^p \, dt$ diverges as $\tau \to \infty$. The system is not $\mathcal{L}_p$ stable for $p \in [1, \infty)$ as it can be seen by taking $u(t) \equiv 0$.

9. (Khalil 5.20) The closed-loop transfer functions are given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+2} & \frac{-1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & \frac{-(s+1)}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

The closed-loop transfer function from (u_1, u_2) to (y_1, y_2) (or (e_1, e_2)) has four components. Due to pole-zero cancellation of the unstable pole $s = 1$, three of these components do not contain the unstable pole; thus, each component by itself is input-output stable. If we restrict our attention to any one of these components, we miss the unstable hidden mode. By studying all four components we will be sure that unstable hidden modes must appear in at least one component.

10. (Khalil 6.2) Using $V(x) = a\int_0^x h(\sigma)$, we have

$$\dot{V} = ah(x)\dot{x} = h(x)\left[-x + \frac{1}{k}h(x) + u\right] = \frac{1}{k}h(x)\left[h(x) - kx\right] + h(x)u$$

By using the sector condition $h \in [0, k]$ and Definition 6.2 (in Khalil on p. 232), we know that

$$h(x)\left[h(x) - kx\right] \leq 0$$

which leads to

$$\dot{V} = \frac{1}{k}h(x)\left[h(x) - kx\right] + h(x)u \leq yu$$

Thus, by definition 6.3 the system is passive.

11. (Khalil 6.4) The system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 + u \\ y &= kx_2 + u \end{aligned}$$

where

$$\begin{aligned} a &> 0 \\ k &> 0 \\ h &\in [\alpha_1, \infty] \\ &\Rightarrow zh(z) \geq \alpha_1 z^2 \\ \alpha_1 &> 0 \end{aligned}$$

A storage function is given by

$$V(x) = k\int_0^{x_1} h(z)\,dz + x^T P x$$

where $p_{11} = ap_{12}$, $p_{22} = \frac{k}{2}$ and $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$. The time derivative of the storage functions along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(x) &= k\frac{\partial}{\partial x_1}\left(\int_0^{x_1} h(z)\,dz\right)\dot{x}_1 + \dot{x}^T P x + x^T P \dot{x} \\
&= kh(x_1)\dot{x}_1 + 2x^T P\dot{x} \\
&= kh(x_1)x_2 - h(x_1)kx_2 + kux_2 - 2h(x_1)x_1p_{12} + 2ux_1p_{12} - akx_2^2 + 2x_2^2p_{12} \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + kux_2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + (kx_2 + u)u - u^2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 - u^2 + yu
\end{aligned}$$

By rewriting this last expression it can be seen that

$$\begin{aligned}
yu &= \dot{V}(x) + akx_2^2 - 2p_{12}x_2^2 + 2p_{12}h(x_1)x_1 - 2p_{12}ux_1 + u^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}h(x_1)x_1 + (u - p_{12}x_1)^2 - p_{12}^2x_1^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}\alpha_1x_1^2 - p_{12}^2x_1^2 + (u - p_{12}x_1)^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 + (u - p_{12}x_1)^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 \\
&= \dot{V}(x) + \psi(x)
\end{aligned}$$

where

$$\psi(x) = (ak - 2p_{12})x_2^2 + p_{12}(2\alpha_1 - p_{12})x_1^2$$

Since $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$ we have that $\psi(x)$ is positive definite. Hence, by definition 6.3 the system is strictly passive.

12. (Duckmaze)

(a) We have

$$\begin{aligned}
\dot{V} &= \tilde{x}_1\dot{\tilde{x}}_1 + m\tilde{x}_2\dot{\tilde{x}}_2 \\
&= \tilde{x}_1\tilde{x}_2 + \tilde{x}_2\left[-f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u}\right]
\end{aligned}$$

Selecting $\tilde{u}$ as

$$\tilde{u} = f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] + f_1\tilde{x}_1 - \tilde{x}_1 + v$$

yields

$$\dot{V} = -d\tilde{x}_2^2 + \tilde{x}_2 v \tag{7}$$

This means that the system is passive from the input v to the output $y = \tilde{x}_2$.

(b) The zero state observability is checked:

$$y = 0 \implies \tilde{x}_2 = 0 \implies \dot{\tilde{x}}_2 = 0 \implies -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 = 0 \quad (8)$$

This means that $\tilde{x}_1 = 0$, so the system is zero state observable.

(c) We have shown that the system is passive and zero-state observable, and it is clear that the storage function $V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$ is radially unbounded and positive definite. Hence, according to Theorem 14.4 in Khalil, the origin can be globally stabilized by $v = -\phi(y)$ where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y\phi(y) > 0$ for all $y \neq 0$.
The function ϕ is selected as $\phi = k_2 y = k_2\tilde{x}_2$ which gives the controller

$$v = -k_2\tilde{x}_2 \quad (9)$$

This controller makes the origin globally asymptotically stable. The coordinates are transformed back to the original coordinates x_1, x_2 by using the relationships $\tilde{x}_1 = x_1 - x_{1d}, \tilde{x}_2 = x_2$. Recall from assignment 2 that $u = u_0 + \tilde{u}$ where u_0 was found to be

$$u_0 = f_3 x_{1d}^3 + f_1 x_{1d} + mg \quad (10)$$

The result is

$$\begin{aligned} u &= u_0 + \tilde{u} \\ &= f_3 x_{1d}^3 + f_1 x_{1d} + mg + f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] + f_1\tilde{x}_1 - \tilde{x}_1 + v \\ &= f_3 x_{1d}^3 + f_1 x_{1d} + mg + f_3\left(\tilde{x}_1 + x_{1d}\right)^3 - f_3 x_{1d}^3 + f_1\tilde{x}_1 - \tilde{x}_1 + v \\ &= f_3\left(\tilde{x}_1 + x_{1d}\right)^3 + f_1\left(\tilde{x}_1 + x_{1d}\right) - \tilde{x}_1 + v \\ &= f_x x_1^3 + f_1 x_1 - (x_1 - x_{1d}) - k_2 x_2 \end{aligned} \quad (11)$$

(d) In Assignment 2 the input u was biased to move the equilibrium point to a desired equilibrium point. The disturbance w can be seen on as a contribution to this bias. This means that the input u is now described by $u = (u_0 + w) + \tilde{u}$, not $u = u_0 + \tilde{u}$ as before and the equilibrium point will not be moved to $(x_{1d}, 0)$ but to another equilibrium point which will be called $(x_{1d,w}, 0)$. A similar change of coordinate as in Assignment 2 (Exercise 1b) will show that the controller $u = (u_0 + w) + \tilde{u}$ makes the equilibrium point $(x_{1d,w}, 0)$ globally asymptotically stable. The conclusion is therefore that in the presence of the constant disturbance $w \neq 0$ the controller (11) will give a stationary deviation in the position x_1 from the reference value x_{1d} since $x_{1d,w} \neq x_{1d}$ when $w \neq 0$.

13. (Khalil 6.6) A parallel connection, as seen in Figure 2, is characterized by

$$\begin{aligned} u &= u_1 = u_2 \\ y &= y_1 + y_2 \end{aligned}$$

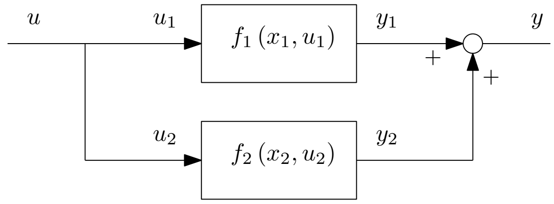

Figure 2: Parallel connected systems.

where the two systems is given by

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, u_1)\\ \dot{x}_2 &= f_2(x_2, u_2)\end{aligned}$$

with the storage functions $V_1(x_1)$ and $V_2(x_2)$.
Suppose the overall storage function $V(x) = V_1(x) + V_2(x)$ where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. Then

$$\begin{aligned}\dot{V} &= \frac{\partial}{\partial x} V(x) f(x,u) = \begin{bmatrix} \frac{\partial V_1(x_1)}{x_1} & \frac{\partial V_2(x_2)}{x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1, u_1) \\ f_2(x_2, u_2) \end{bmatrix}\\ &= \frac{\partial V_1(x_1)}{x_1} f_1(x_1, u_1) + \frac{\partial V_2(x_2)}{x_2} f_2(x_2, u_2) \qquad (12)\end{aligned}$$

We know that the passivity properties of the interconnected systems may be expressed as

$$\dot{V}_1 = \frac{\partial V_1(x_1)}{x_1} f_1(x_1, u_1) \le u_1^T y_1 - u_1^T \varphi_1(u_1) - y_1^T \rho_1(y_1) - \psi_1(x_1) \qquad (13)$$

$$\dot{V}_2 = \frac{\partial V_2(x_2)}{x_2} f_2(x_2, u_2) \le u_2^T y_2 - u_2^T \varphi_2(u_2) - y_2^T \rho_2(y_2) - \psi_2(x_2) \qquad (14)$$

Inserting (13) and (14) into (12) leads to

$$\begin{aligned}\dot{V} = \frac{\partial}{\partial x} V(x) f(x,u) &\le u_1^T y_1 - u_1^T \varphi_1(u_1) - y_1^T \rho_1(y_1) - \psi_1(x_1)\\ &+ u_2^T y_2 - u_2^T \varphi_2(u_2) - y_2^T \rho_2(y_2) - \psi_2(x_2)\\ &= u^T y - u^T \varphi(u) + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi(x) \qquad (15)\end{aligned}$$

where

$$\varphi(u) = \varphi_1(u_1) + \varphi_2(u_2) = \varphi_1(u) + \varphi_2(u) \qquad (16)$$

$$\psi(x) = \psi_1(x_1) + \psi_2(x_2) \qquad (17)$$

From (15)–(17) it can be seen that the parallel connection keeps the passivity properties of passive, input strictly passive and stricly passive from the interconnected systems.

For the output strictly passive property, we assume

$$y_i^T \rho_i(y_i) \geq \delta_i y_i^T y_i \tag{18}$$

for some positive δ_i. Using (18) and $\delta = \min\{\delta_1,\ \delta_2\}$ we may rewrite $y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2)$ according to

$$\begin{aligned}
y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) &\geq \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \\
&\geq \delta y_1^T y_1 + \delta y_2^T y_2 \\
&= \delta\left(y_1^T y_1 + y_2^T y_2\right) \\
&\geq \delta\left(\frac{1}{2}(y_1 + y_2)^T (y_1 + y_2)\right) \\
&= \frac{1}{2}\delta y^T y
\end{aligned}$$

where we used the fact that

$$(y_1 + y_2)^T (y_1 + y_2) \leq 2\left(y_1^T y_1 + y_2^T y_2\right)$$

Then (15) will be expressed as

$$\dot{V} = \frac{\partial}{\partial x} V(x) f(x,u) \quad \leq \quad u^T y - u^T \varphi(u) + \frac{1}{2}\delta y^T y + \psi(x) \tag{19}$$

We see that the parallel connection also keeps the property of output strictly passive from the interconnected systems.

14. (Khalil 6.1) Let the input to the system be denoted $\tilde{u}$ and the output be denoted $\tilde{y}$. From the block diagram we have the following relations

$$\begin{aligned}
\tilde{y} &= h(t,u) - K_1 u \\
\tilde{u} + \tilde{y} &= K u
\end{aligned}$$

From the sector condition we have that

$$\begin{gathered}
(h(t,u) - K_1 u)^T (h(t,u) - K_2 u) \leq 0 \\
K = K_2 - K_1 = K^T > 0
\end{gathered} \tag{20}$$

Evaluating the block diagram it can be seen that

$$h(t,u) - K_1 u = \tilde{y} \tag{21}$$

and that

$$\begin{aligned}
h(t,u) - K_2 u &= h(t,u) - K_2 u - K_1 u + K_1 u \\
&= \tilde{y} - (K_2 - K_1) u \\
&= \tilde{y} - K u \\
&= \tilde{y} - \tilde{u} - \tilde{y} \\
&= -\tilde{u}
\end{aligned} \tag{22}$$

Using (21), (22) and the sector condition (20) we have

$$\begin{aligned}
\left(h\left(t,u\right)-K_1u\right)^T\left(h\left(t,u\right)-K_2u\right) &= \tilde{y}^T\left(-\tilde{u}\right) \\
&= -\tilde{u}^T\tilde{y} \\
&\leq 0 \\
&\Rightarrow \tilde{u}^T\tilde{y} \geq 0
\end{aligned}$$

which implies that the system is passive from $\tilde{u}$ to $\tilde{y}$, which corresponds to being in sector $[0,\infty]$.