TTK4115 Linear System Theory
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Solution to homework assignment 2

Problem 1: Jordan form

a) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & -9 \\ 1 & -6-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalue $\lambda = -3$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A} - \lambda\mathbf{I})$:

$$\ker(\mathbf{A} - \lambda\mathbf{I}) = \ker\left(\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

where $\mathbf{q}$ is the corresponding eigenvector. Note that $\mathbf{A}$ has only one eigenvector associated with λ.

b) Because the eigenvalue $\lambda = -3$ of $\mathbf{A}$ has multiplicity 2, and $\mathbf{A}$ has only one eigenvector associated with λ, the eigenvalues of $\mathbf{A}$ are not (all) distinct. Therefore, the system cannot be transformed into a diagonal form using a similarity transformation.

c) In order to transform the system into a Jordan form, we have to find the generalized eigenvectors of $\mathbf{A}$. The chain of generalized eigenvectors satisfies the following equalities:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0},$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the generalized eigenvectors. Note that we can choose $\mathbf{v}_1 = \mathbf{q} = [3, 1]^T$, since $\mathbf{q}$ is an eigenvector associated with λ, and therefore $(\mathbf{A} - \lambda\mathbf{I})\mathbf{q} = \mathbf{0}$. The generalized eigenvector $\mathbf{v}_2$ can be obtained from the second equality:

$$\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that the choice for $\mathbf{v}_2$ is not unique. We define

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the similarity transformation $\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}}$, the system is transformed to Jordan form:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),\\ y(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t),\end{aligned}$$

with matrices

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}0 & -9\\ 1 & -6\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-3 & 1\\ 0 & -3\end{bmatrix},\\ \hat{\mathbf{B}} &= \mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}\begin{bmatrix}-2\\ 1\end{bmatrix} = \begin{bmatrix}1\\ -5\end{bmatrix},\\ \hat{\mathbf{C}} &= \mathbf{C}\mathbf{Q} = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\end{bmatrix},\\ \hat{\mathbf{D}} &= \mathbf{D} = \begin{bmatrix}2\end{bmatrix}.\end{aligned}$$

Problem 2: Stability

a) **BIBO stability** – A system is said to be *BIBO stable* (bounded-input bounded-output stable) if every bounded input excites a bounded output. For a linear system, this implies that a system is BIBO stable if there exists a finite constant g such that, for every input $\mathbf{u}(t)$, its (forced) response $\mathbf{y}(t)$ satisfies

$$\sup_{t\in[0,\infty)} \|\mathbf{y}(t)\| \leq g \sup_{t\in[0,\infty)} \|\mathbf{u}(t)\|.$$

b) The pole of the transfer function is zero and, therefore, does not have a strictly negative real part. Therefore, the system is not BIBO stable.

As an example, note that the transfer function $g(s)$ can be written as

$$g(s) = \frac{s+10}{2s} = \frac{1}{2} + \frac{5}{s}.$$

Hence, the transfer function $g(s)$ consists of a constant (i.e. $\frac{1}{2}$) and an integrator (i.e. $\frac{5}{s}$). For constant inputs unequal to zero, the amplitude of the output will go to infinity as time goes to infinity due to the integrator in the transfer function: by substituting $s = j\omega$ in the transfer function and calculating the absolute value of the resulting frequency response as $\omega \to 0$ (for constant inputs the frequency content of the input is only non-zero for $\omega = 0$), we obtain

$$\lim_{\omega\to 0} |g(j\omega)| = \lim_{\omega\to 0} \left|\frac{1}{2} + \frac{5}{j\omega}\right| = \infty.$$

Hence, for bounded constant inputs the input-to-output gain is infinite. Therefore, the output is not bounded for all bounded inputs and we conclude that the system in not BIBO stable.

c) **Marginal stability** – A system is said to be *marginally stable*, if for every initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$, the homogeneous state response

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0), \qquad \forall t \geq t_0$$

is uniformly bounded.

Asymptotic stability – A system is said to be *asymptotically stable*, if it is marginally stable, and if for every initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$, we have that $\mathbf{x}(t) \to 0$ as $t \to \infty$.

Exponential stability – A system is said to be *exponentially stable*, if it is asymptotically stable, and if there exist constants $c, \lambda > 0$ such that, for every initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$, we have

$$\|\mathbf{x}(t)\| \leq ce^{-\lambda(t-t_0)}\|\mathbf{x}(t_0)\|, \qquad \forall t \geq t_0.$$

Instability – A system is said to be *unstable*, if it is not marginally stable.

d) **Marginal stability** – The system in (3) is *marginally stable*, if and only if all the eigenvalues of $\mathbf{A}$ have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1.

Asymptotic stability – The system in (3) is *asymptotically stable*, if and only if all the eigenvalues of $\mathbf{A}$ have strictly negative real parts.

Exponential stability – The system in (3) is *exponentially stable*, if and only if all the eigenvalues of $\mathbf{A}$ have strictly negative real parts.

Instability – The system in (3) is *unstable*, if and only if at least one eigenvalue of A has a positive real part or zero real part, but the corresponding Jordan block is larger than 1×1.

e) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda + 3 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda + 3).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -3$ and $\lambda_{2,3} = 0$. Note that the matrix $\mathbf{A}$ is already in Jordan form. The Jordan block for the eigenvalue $\lambda_{2,3} = 0$ is of order two; the Jordan block has size 2×2. This implies that $\lambda_{2,3} = 0$ is not a simple root of the minimal polynomial Therefore, we conclude that the state-space equation is unstable and not marginally stable, asymptotically stable or exponentially stable.

f) To find the matrix $\mathbf{P}$, we solve the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}.$$

Note that $\mathbf{P}$ is a symmetric matrix, i.e. $\mathbf{P} = \mathbf{P}^T$. Let $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},$$

where p_1, p_2 and p_3 are constant that are yet to be determined. Substituting the matrices $\mathbf{A}$ and $\mathbf{P}$ in the Lyapunov equation, we obtain

$$\begin{bmatrix} -4 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} -4p_1 + p_2 & -4p_2 + p_3 \\ -2p_1 - 2p_2 & -2p_2 - 2p_3 \end{bmatrix} + \begin{bmatrix} -4p_1 + p_2 & -2p_1 - 2p_2 \\ -4p_2 + p_3 & -2p_2 - 2p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, we obtain the equations

$$\begin{aligned} -8p_1 + 2p_2 &= -1, \\ -2p_1 - 6p_2 + p_3 &= 0, \\ -4p_2 - 4p_3 &= -1, \end{aligned}$$

which can be written in the following form:

$$\begin{bmatrix} -8 & 2 & 0 \\ -2 & -6 & 1 \\ 0 & -4 & -4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

Solving for p_1, p_2 and p_3 yields $p_1 = \frac{1}{8}$, $p_2 = 0$ and $p_3 = \frac{1}{4}$. Hence, we obtain the matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

g) The eigenvalues of $\mathbf{A}$ have negative real parts if and only if the matrix $\mathbf{P}$ obtained in j) (which is the solution of the Lyapunov equation) is positive definite. The matrix $\mathbf{P}$ is positive definite if and only if all its leading principle minors are positive. The leading principle minors of $\mathbf{P}$ are

$$p_1 = \frac{1}{8} \qquad \text{and} \qquad \det(\mathbf{P}) = \begin{vmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{4} \end{vmatrix} = \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}.$$

Because all leading principle minors of $\mathbf{P}$ are positive, we conclude that the matrix $\mathbf{P}$ is positive definite and that the eigenvalues of $\mathbf{A}$ all have negative real parts. Because all eigenvalues of $\mathbf{A}$ have negative real parts, the system with system matrix $\mathbf{A}$ is asymptotically stable.

Problem 3: Linear quadratic regulator

a) Using the differential equation, we have

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}\dot{z}(t)\\ \ddot{z}(t)\end{bmatrix} = \begin{bmatrix}\dot{z}(t)\\ -2\dot{z}(t)+2u(t)\end{bmatrix} = \begin{bmatrix}x_2(t)\\ -2x_2(t)+2u(t)\end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}u(t).$$

Note that $y(t) = x_1(t)$. Therefore, we have

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}0\\ 2\end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix}1 & 0\end{bmatrix}.$$

b) Let the positive-definite matrix $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix},$$

where the constants p_1, p_2 and p_3 are obtained from the algebraic Riccati equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \frac{1}{\rho}\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C} = \mathbf{0}$. Substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ρ and $\mathbf{P}$ in the algebraic Riccati equation yields

$$\begin{aligned}&\begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} + \begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix} - \frac{1}{1}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}\begin{bmatrix}0 & 2\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & 0\\ p_1-2p_2 & p_2-2p_3\end{bmatrix} + \begin{bmatrix}0 & p_1-2p_2\\ 0 & p_2-2p_3\end{bmatrix} - 4\begin{bmatrix}p_2^2 & p_2p_3\\ p_2p_3 & p_3^2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}-4p_2^2+1 & p_1-2p_2-4p_2p_3\\ p_1-2p_2-4p_2p_3 & 2p_2-4p_3-4p_3^2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Hence, we obtain the equations

$$\begin{aligned}-4p_2^2+1 &= 0,\\ p_1-2p_2-4p_2p_3 &= 0,\\ 2p_2-4p_3-4p_3^2 &= 0.\end{aligned}$$

From the first equation, it follows that

$$p_2 = \pm\frac{1}{2}.$$

From the third equation, we obtain

$$2p_3^2 + 2p_3 - p_2 = 0 \qquad \Rightarrow \qquad p_3 = -\frac{1}{2} \pm \frac{\sqrt{1+2p_2}}{2}.$$

From the second equation, we get

$$p_1 = 2p_2 + 4p_2p_3 = 2p_2(1 + 2p_3).$$

The matrix $\mathbf{P}$ is positive definite if all the leading principal minors of $\mathbf{P}$ are positive. (Positive definiteness can also be checked by looking at the eigenvalues of $\mathbf{P}$, which should be positive.) The leading principle minors of $\mathbf{P}$ are

$$p_1 \qquad \text{and} \qquad \det(\mathbf{P}) = \begin{vmatrix} p_1 & p_2 \\ p_2 & p_3 \end{vmatrix} = p_1p_3 - p_2^2.$$

Therefore, we must have that

$$p_1 > 0 \qquad \text{and} \qquad p_1p_3 - p_2^2 > 0.$$

Next, we will determine the values of p_2, p_3 and p_1, respectively, by contradiction. Note that if $p_2 = -\frac{1}{2}$, we have that $p_3 = -\frac{1}{2} \pm \frac{\sqrt{1+2p_2}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1-1}}{2} = -\frac{1}{2}$, which implies that $p_1 = 2p_2(1 + 2p_3) = -1(1 - 1) = 0$, which contradicts $p_1 > 0$. Therefore, we conclude that $p_2 = \frac{1}{2}$. For $p_2 = \frac{1}{2}$, it follows that

$$p_1 = 1 + 2p_3 \qquad \text{and} \qquad p_3 = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}.$$

Note that if $p_3 = -\frac{1}{2} - \frac{\sqrt{2}}{2}$, we have that $p_1 = 1 + 2p_3 = -\sqrt{2}$, which again contradicts $p_1 > 0$. Therefore, we conclude that $p_3 = -\frac{1}{2} + \frac{\sqrt{2}}{2}$ (or alternatively written as $p_3 = \frac{1}{2}(\sqrt{2} - 1)$). Substituting $p_3 = \frac{1}{2}(\sqrt{2} - 1)$ in the expression for p_1, it follows that $p_1 = 1 + 2p_3 = 1 + \sqrt{2} - 1 = \sqrt{2}$. Note that for $p_1 = \sqrt{2}$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{2}(\sqrt{2} - 1)$, we have

$$p_1 = \sqrt{2} > 0 \qquad \text{and} \qquad p_1p_3 - p_2^2 = \frac{\sqrt{2}}{2}(\sqrt{2} - 1) - \frac{1}{4} = \frac{3}{4} - \frac{1}{\sqrt{2}} \approx 0.0429 > 0,$$

from which we conclude that the corresponding matrix $\mathbf{P}$ is positive definite. Substituting the values of p_1, p_2 and p_3 in $\mathbf{P}$ yields

$$\mathbf{P} = \begin{bmatrix} \sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}(\sqrt{2} - 1) \end{bmatrix}.$$

The corresponding gain matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \frac{1}{\rho}\mathbf{B}^T\mathbf{P} = \frac{1}{1}\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}(\sqrt{2} - 1) \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} - 1 \end{bmatrix}. \tag{1}$$

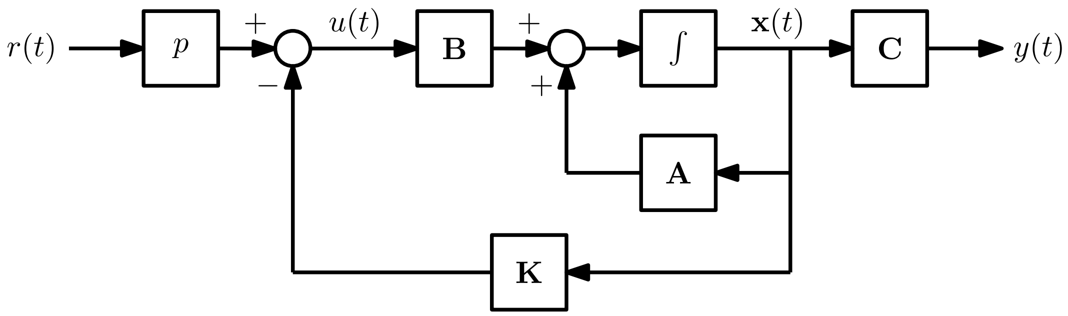

Fig. 1: Block diagram of closed-loop system.

c) The block diagram of the closed-loop system is depicted in Fig. 1.

d) Substituting the control law

$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr(t),$$

in the state-space equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

yields

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - p\mathbf{B}r(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

Hence, we obtain

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}, \qquad \bar{\mathbf{B}} = p\mathbf{B} \qquad \text{and} \qquad \bar{\mathbf{C}} = \mathbf{C}.$$

e) Using the finite value theorem, we have that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s).$$

The transfer function $\frac{y(s)}{r(s)}$ is given by

$$\frac{y(s)}{r(s)} = \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}.$$

The matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ are given by

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{2} - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{bmatrix},$$

$$\bar{\mathbf{B}} = p\mathbf{B} = p\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \bar{\mathbf{C}} = \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Substituting the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$, the transfer function $\frac{y(s)}{r(s)}$ is given by

$$\begin{aligned}\frac{y(s)}{r(s)} &= p\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s & -1\\ 2 & s+2\sqrt{2}\end{bmatrix}^{-1}\begin{bmatrix}0\\2\end{bmatrix}\\ &= p\begin{bmatrix}1 & 0\end{bmatrix}\frac{1}{s^2+2\sqrt{2}s+2}\begin{bmatrix}s+2\sqrt{2} & 1\\ -2 & s\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\\ &= \frac{2p}{s^2+2\sqrt{2}s+2}.\end{aligned}$$

Using the Laplace transform, we obtain that

$$r(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{r_c\} = r_c\mathcal{L}\{1\} = \frac{r_c}{s}.$$

Hence, we get

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sy(s) = \lim_{s\to 0} sr(s)\frac{y(s)}{r(s)}\\ &= \lim_{s\to 0} s\frac{r_c}{s}\cdot\frac{2p}{s^2+2\sqrt{2}s+2} = \lim_{s\to 0}\frac{2r_cp}{s^2+2\sqrt{2}s+2} = r_cp.\end{aligned}$$

Because we require that $\lim_{t\to\infty} y(t) = r_c$, it follows that $p = 1$.

f) In order to obtain a faster convergence of the output, we need to decrease the gain ρ in the cost function

$$J = \int_0^\infty \left[y^2(t) + \rho u^2(t)\right] dt.$$

By decreasing the gain ρ, the control input $u(t)$ is penalized less with respect to the output $y(t)$. Note that for tracking problems, $u(t)$ as denoted in the cost function represents the difference between the control input and the steady-state control input (which is equal to zero in this case) and $y(t)$ as denoted in the cost function represents the difference between the output and the steady-state output (which is equal to the reference value $r(t) = r_c$ in this case). Therefore, decreasing the gain ρ will penalize the difference between the output and the reference value $r(t) = r_c$ more, which will lead to a faster convergence of the output. "The price to pay" for a faster convergence is a larger control input, because the control input is penalized less.

To give an example, using the same conditions, for both $\rho = 1$ and $\rho = 0.25$ the output $y(t)$ and the control input $u(t)$ are depicted in Fig. 2. On the one hand, we see that for $\rho = 0.25$ the output $y(t)$ converges faster to the reference value $r(t) = 2$ than for $\rho = 1$. On the other hand, the control input $u(t)$ for $\rho = 0.25$ is overall larger than for $\rho = 1$.

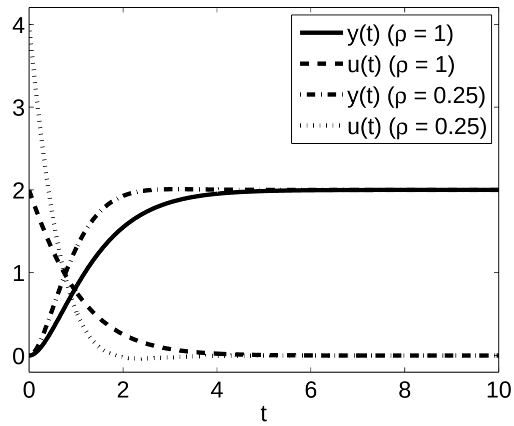

Fig. 2: Output $y(t)$ and control input $u(t)$ for $\rho = 1$ and $\rho = 0.25$.