TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Solution to Assignment 5

1. The system is given by

$$h(s) = 1.5\frac{(1+2s)(1+s)}{(1+3s)(1+0.5s)}$$

which has poles with real parts less than zero.

(a) An upper bound on the magnitude of $h(j\omega)$ is found as

$$\begin{aligned} |h(j\omega)| &= \left|1.5\frac{(1+j2\omega)(1+j\omega)}{(1+j3\omega)(1+j0.5\omega)}\right| \\ &= 1.5\left|\frac{1+j2\omega}{1+j3\omega}\right|\left|\frac{1+j\omega}{1+j0.5\omega}\right| \\ &= 3\left|\frac{1+j2\omega}{1+j3\omega}\right|\left|\frac{1+j\omega}{2+j\omega}\right| \end{aligned}$$

The absolute value of $h(j\omega)$ is given as

$$|h(j\omega)| = h(j\omega)\,h(-j\omega) \tag{1}$$

which results in

$$\left|\frac{1+2j\omega}{1+j3\omega}\right| = \frac{1+2^2\omega^2}{1+3^2\omega^2} \le 1$$

and

$$\left|\frac{1+j\omega}{2+j\omega}\right| = \frac{1+\omega^2}{2^2+\omega^2} \le 1$$

An upper bound on $|h(j\omega)|$ is therefore given by

$$|h(j\omega)| \le 3 = \frac{K_p\beta}{\alpha} \tag{2}$$

(b) We have that

$$\begin{aligned} h(j\omega) &= 1.5\frac{(1+j2\omega)(1+j\omega)}{(1+j3\omega)(1+j0.5\omega)} \\ &= 1.5\frac{(1+j2\omega)(1+j\omega)(1-j3\omega)(1-j0.5\omega)}{(1+3^2\omega^2)(1+0.5^2\omega^2)} \end{aligned}$$

where the numerator is calculated as

$$\begin{aligned} (1+j2\omega)(1+j\omega)(1-j3\omega)(1-j0.5\omega) &= \left(1+j3\omega+j^2 2\omega^2\right)\left(1-j3.5\omega+j^2 1.5\omega^2\right) \\ &= 1-j0.5\omega-j^2 7\omega^2-j^3 2.5\omega^3+j^4 3\omega^4 \\ &= \left(1+7\omega^2+3\omega^4\right)+j\left(2.5\omega^3-0.5\omega\right) \end{aligned}$$

The real value of $h(j\omega)$ is

$$\begin{aligned}
\mathrm{Re}\left[h(j\omega)\right] &= 1.5\frac{1+7\omega^2+3\omega^4}{\left(1+3^2\omega^2\right)\left(1+0.5^2\omega^2\right)} \\
&= \frac{1.5+10.5\omega^2+4.5\omega^4}{1+9.25\omega^2+1.5\omega^4} \\
&= \frac{1+9.25\omega^2+1.5\omega^4+1.25\omega^2+3\omega^4}{1+9.25\omega^2+1.5\omega^4}
\end{aligned}$$

and it can be recognized that to prove

$$\mathrm{Re}\left[h(j\omega)\right] \geq 1 = K_p \tag{3}$$

is the same as proving

$$\left(1.25\omega^2+3\omega^4\right) \geq 0$$

which is true.

(c) To prove that h is passive is the same as proving $\mathrm{Re}\left[h(j\omega)\right] \geq 0 \ \forall\omega$ (see Appendix A in Assignment 5). Since $\mathrm{Re}\left[h(j\omega)\right] \geq K_p > 0 \ \forall\omega$ we conclude that the control law is passive.

(d) To prove that h is input strictly passive is the same as proving that $\mathrm{Re}\left[h(j\omega)\right] \geq \delta \geq 0 \ \forall\omega$ for some positive δ (see Appendix A in Assignment 5). Since $\mathrm{Re}\left[h(j\omega)\right] \geq K_p > 0 \ \forall\omega$ we conclude that the control law is input strictly passive.

(e) To prove that the system is output strictly passive is the same as proving that $\mathrm{Re}\left[h(j\omega)\right] \geq \varepsilon\left|h(j\omega)\right|^2 \ \forall\omega$ for some positive ε (see Appendix A in Assignment 5). From the assumptions in the exercise we know that

$$\begin{aligned}
\left|h(j\omega)\right| &\leq \frac{K_p\beta}{\alpha} \\
\mathrm{Re}\left[h(j\omega)\right] &\geq K_p
\end{aligned}$$

Using these inequalities, an upper bound on $\left|h(j\omega)\right|^2$ is found as

$$\begin{aligned}
\left|h(j\omega)\right|^2 &\leq \left(\frac{K_p\beta}{\alpha}\right)^2 \\
&= \frac{K_p\beta^2}{\alpha^2}K_p \\
&\leq \frac{K_p\beta^2}{\alpha^2}\mathrm{Re}\left[h(j\omega)\right]
\end{aligned}$$

which is rewritten as

$$\begin{aligned}
\mathrm{Re}\left[h(j\omega)\right] &\geq \frac{\alpha^2}{K_p\beta^2}\left|h(j\omega)\right|^2 \\
&= \varepsilon\left|h(j\omega)\right|^2
\end{aligned}$$

and output strictly passivity of the control law is concluded.

(f) The system is given by

$$h(s) = \frac{u(s)}{e(s)} = K_p\beta\frac{(1+T_i s)(1+T_d s)}{(1+\beta T_i s)(1+\alpha T_d s)}$$

where e is the input and u is the output. When investigating if a system is zero-state observable, the system is analyzed with inputs set to zero, $e = 0$. This leads to the equation

$$\begin{aligned}
\frac{u(s)}{e(s)} &= K_p\beta\frac{(1+T_i s)(1+T_d s)}{(1+\beta T_i s)(1+\alpha T_d s)} \\
&\Leftrightarrow u(s)(1+\beta T_i s)(1+\alpha T_d s) = K_p\beta(1+T_i s)(1+T_d s)\,e(s) \\
&\Rightarrow u(s)(1+\beta T_i s)(1+\alpha T_d s) = 0 \text{ when } e(s) = 0 \\
&\Leftrightarrow u(s)\left(1+\beta T_i s+\alpha T_d s+\beta T_i\alpha T_d s^2\right) = 0 \\
&\Leftrightarrow u+\beta T_i\dot{u}+\alpha T_d\dot{u}+\beta T_i\alpha T_d\ddot{u} = 0
\end{aligned}$$

Let $z_1 = u$, $z_2 = \dot{u}$ and $y = z_1$, then the control law with zero input can be expressed as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \frac{1}{\beta T_i\alpha T_d}\left(-z_1-(\beta T_i+\alpha T_d)\,z_2\right) \\
y &= z_1
\end{aligned}$$

To show that the system is zero-state observable we require that no solution can stay identical in $y = 0$ other than the trivial solution $z \equiv 0$ (see Definition 6.5 on p. 243 in Khalil). This is done as

$$\begin{aligned}
y(t) &\equiv 0 \Leftrightarrow z_1(t) \equiv 0 \\
\dot{z}_1(t) &= 0 \Rightarrow z_2(t) \equiv 0 \\
\dot{z}_2 &= 0 \Rightarrow z_2 = \frac{1}{(\beta T_i+\alpha T_d)}z_1 = 0
\end{aligned}$$

by which we conclude that the PID control law is zero-state observable.

2. (Khalil 6.11) The system is given by

$$\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + u_3
\end{aligned}$$

where $u = [u_1\ u_2\ u_3]^T$ and $\omega = [\omega_1\ \omega_2\ \omega_3]$.

(a) Let $V(\omega) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(\omega) &= J_1\dot{\omega}_1\omega_1 + J_2\dot{\omega}_2\omega_2 + J_3\dot{\omega}_3\omega_3 \\
&= ((J_2 - J_3)\,\omega_2\omega_3 + u_1)\,\omega_1 \\
&\quad + ((J_3 - J_1)\,\omega_3\omega_1 + u_2)\,\omega_2 \\
&\quad + ((J_1 - J_2)\,\omega_1\omega_2 + u_3)\,\omega_3 \\
&= (J_2 - J_3)\,\omega_1\omega_2\omega_3 + (J_3 - J_1)\,\omega_1\omega_2\omega_3 + (J_1 - J_2)\,\omega_1\omega_2\omega_3 \\
&\quad + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\
&= (J_2 - J_3 + J_3 - J_1 + J_1 - J_2)\,\omega_1\omega_2\omega_3 + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\
&= u^T\omega
\end{aligned}$$

which shows that the map from u to ω is lossless with the storage function $V(\omega)$.

(b) With $u = -K\omega + v$ where $K = K^T$where we have that

$$\begin{aligned}
\dot{V}(\omega) &= u^T\omega \\
&= (-K\omega + v)^T\,\omega \\
&= -\omega^T K^T\omega + v^T\omega \\
&= v^T\omega - \omega^T K\omega \\
&\leq v^T\omega - \lambda_{\min}(K)\,\omega^T\omega \\
&\Rightarrow v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega
\end{aligned}$$

From the last equation it can be seen that the system is output strictly passive from v to ω with $v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega$. Hence, the map from v to ω is finite gain L_2 stable with L_2 gain less than or equal to $\frac{1}{\lambda_{\min}(K)}$ (Lemma 6.5).

(c) With $u = -K\omega$, we have that

$$\dot{V}(\omega) \leq -\lambda_{\min}(K)\,\omega^T\omega$$

for the system $\dot{\omega} = f(\omega, -K\omega) = f'(\omega)$. Since $V(\omega)$ is positive definite and radially unbounded and $\dot{V}(\omega)$ is negative definite, we conclude that the system is globally asymptotically stable.

3. (Khalil 6.14) Two systems

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases}$$

and

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

are connected as shown in Figure 6.11 in Khalil. The functions $h_i(\cdot)$ are locally Lipschitz and $h_i(\cdot) \in (0, \infty]$. Further, the function $h_2(z)$ satisfies $|h_2(z)| \geq \frac{|z|}{(1+z^2)}$.

(a) First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_1(x_1, x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2(-x_1 - h_1(x_2) + e_1) \\ &= x_1x_2 - x_1x_2 - h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(y_1)y_1 + e_1y_1 \\ &\Rightarrow e_1y_1 = \dot{V}_1(x_1, x_2) + y_1h_1(y_1) \end{aligned}$$

Since $h_1 \in (0, \infty]$, we know that $y_1h_1(y_1) > 0 \quad \forall y_1 \neq 0$ (See Definition 6.2 in Khalil on pp. 232–233). Thus, H_1 is output strictly passive.
The passivity properties of H_2 is investigated by using $V_2(x_3) = \int_0^{x_3} h_2(z)\,dz$ as a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_2(x_3) &= \frac{\partial}{\partial x_3}\left(\int_0^{x_3} h_2(z)\,dz\right)\dot{x}_3 \\ &= h_2(x_3)(-x_3 + e_2) \\ &= -x_3h_2(x_3) + h_2(x_3)e_2 \\ &= -x_3h_2(x_3) + y_2e_2 \\ &\Rightarrow y_2e_2 = \dot{V}_2(x_3) + x_3h_2(x_3) \end{aligned}$$

Since $h_2 \in (0, \infty]$, we know that $x_3h_2(x_3) > 0 \quad \forall x_3 \neq 0$ (See Definition 6.2 in Khalil on pp. 232–233). Thus, H_2 is strictly passive.
By Theorem 6.1 we conclude that the feedback connection is passive.

(b) Asymptotic stability of the unforced system is shown by using Theorem 6.3 from Khalil. Since we have one strictly passive system and one output strictly passive system, we need to show that the system which is output strictly passive also is zero-state observable. It can be recognized that no solution can stay identical in $S = \{x_2 = 0\}$ other than the trivial solution $(x_1, x_2) = (0, 0)$. That is

$$
\begin{aligned}
y_1 &\equiv 0 \Leftrightarrow x_2 \equiv 0 \\
\dot{x}_2 &= 0 \Rightarrow x_1 = -h_1(x_2) = 0
\end{aligned}
$$

Hence, the unforced system is asymptotically stable. To prove global results, we need to show that the storage functions are radially unbounded. The first storage function is given by

$$
\begin{aligned}
V_1(x_1, x_2) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) \\
&= \frac{1}{2}\left\|(x_1, x_2)\right\|_2
\end{aligned}
$$

which clearly is radially unbounded. The second storage function is given by

$$
\begin{aligned}
V_2(x_3) &= \int_0^{x_3} h_2(z)\, dz \\
&\geq \int_0^{x_3} \frac{|z|}{(1+z^2)} dz \\
&= \int_0^{x_3} \frac{z}{(1+z^2)} dz \\
&= \frac{1}{2}\ln\left(1 + x_3^2\right)
\end{aligned}
$$

where it can be recognized that $V_2(x_3) \to \infty$ as $|x_3| \to \infty$. Hence, the unforced system is globally asymptotically stable.

4. (Khalil 6.15) Two systems

$$
H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases}
$$

and

$$
H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}
$$

are connected as shown in Figure 6.11 in Khalil.

(a) First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}\dot{V}_1(x_1,x_2) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1^3(-x_1+x_2) + x_2\left(-x_1^3 - x_2 + e_1\right) \\ &= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + x_2e_1 \\ &= -x_1^4 - x_2^2 + e_1y_1 \\ &\Rightarrow\ e_1y_1 = \dot{V}_1(x_1,x_2) + x_1^4 + x_2^2\end{aligned}$$

Hence, H_1 is strictly passive with storage function $V_1(x_1,x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. The passivity properties of H_2 is investigated by using $V_2(x_3) = \frac{1}{4}x_3^4$ as a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}\dot{V}_2(x_3) &= x_3^3\dot{x}_3 \\ &= x_3^3(-x_3+e_2) \\ &= -x_3^4 + x_3^3e_2 \\ &= -x_3^4 + e_2y_2\end{aligned}$$

Hence, H_2 is strictly passive with storage function $V_2(x_3) = \frac{1}{4}x_3^4$, and the feedback connection is passive.

(b) Since both systems are strictly passive with radially unbounded storage functions, it follows from Theorem 6.3 that the origin of the unforced system is asymptotically stable.

5. (Khalil 14.43) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + v \\ y &= x_2\end{aligned}$$

With $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

$$\dot{V} = x_1^3x_2 - x_1^3x_2 + x_2v = yv$$

Hence the system is passive. With $v = 0$

$$y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Thus the system is zero-state observable. Therefore, we can globally stabilize the system by $v = -\phi(y)$ for any $\phi(y)$ such that

$$\begin{gathered}\phi(y) \text{ is locally Lipschitz} \\ \phi(0) = 0 \\ y\phi(y) > 0 \text{ for all } y \neq 0\end{gathered}$$

Pick $\phi(y) = -\psi(-y)$, then

$$\begin{gathered} \phi(y) \text{ is locally Lipschitz} \\ \phi(0) = -\psi(-0) = 0 \\ y\phi(y) = -y\psi(-y) > 0 \text{ for all } -y \neq 0 \end{gathered}$$

Thus $v = \psi(-y)$ is the stabilizing feedback which means $u = -y$.

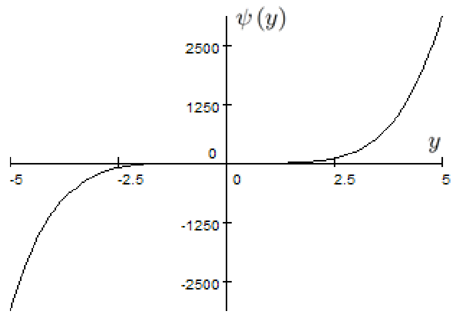

Figure 1: $\psi(y) = y^5$

6. (a) The nonlinear element is given by

$$\psi(y) = y^5$$

From Figure 1 it can be recognized that the nonlinearity is a time-invariant, memoryless and odd function. The describing function is thus calculated as

$$\begin{aligned} \Psi(a) &= \frac{2}{a\pi}\int_0^\pi \psi(a\sin(\theta))\sin(\theta)\,d\theta \\ &= \frac{2}{a\pi}\int_0^\pi (a\sin(\theta))^5\sin(\theta)\,d\theta \\ &= \frac{2}{a\pi}\int_0^\pi a^5\sin^6(\theta)\,d\theta \\ &= \frac{2a^4}{\pi}\cdot 2\int_0^{\pi/2}\sin^6(\theta)\,d\theta \\ &= \frac{2a^4}{\pi}\frac{5}{16}\pi \\ &= \frac{5a^4}{8} \end{aligned}$$

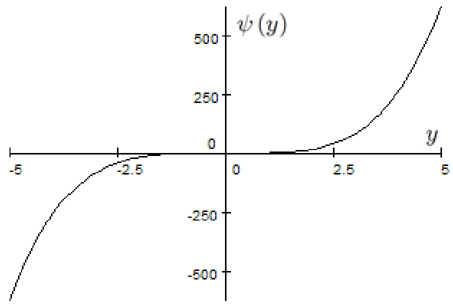

Figure 2: $\psi(y) = y^3 |y|$

(b) The nonlinear element is given by

$$\psi(y) = y^3 |y|$$

From Figure 2 it can be recognized that the nonlinearity is a time-invariant, memoryless and odd function. The describing function is calculated as

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi} \int_0^{\pi} \psi(a \sin(\theta)) \sin(\theta)\, d\theta \\
&= \frac{2}{a\pi} \int_0^{\pi} (a \sin(\theta))^3 |a \sin(\theta)| \sin(\theta)\, d\theta \\
&= \frac{2}{a\pi} \int_0^{\pi} a^3 \sin^3(\theta) |a| |\sin(\theta)| \sin(\theta)\, d\theta \\
&= \frac{2a^3}{\pi} \int_0^{\pi} |\sin(\theta)| \sin^4(\theta)\, d\theta \\
&= \frac{2a^3}{\pi} \cdot 2 \int_0^{\pi/2} \sin^5(\theta)\, d\theta \\
&= \frac{2a^3}{\pi} \frac{16}{15} \\
&= \frac{32a^3}{15\pi}
\end{aligned}$$

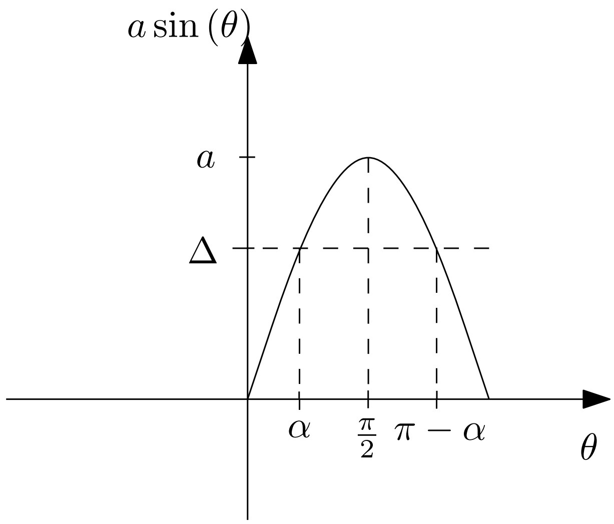

Figure 3: $a\sin(\theta)$

(c) The nonlinear element can be expressed as

$$\psi(y) = ky + Asgn(y)$$

and the describing function is found as

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi}\int_0^{\pi} \psi(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} (ka\sin(\theta) + Asgn(a\sin(\theta)))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} ka\sin(\theta)\sin(\theta)\,d\theta + \frac{2}{a\pi}\int_0^{\pi} Asgn(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2k}{\pi}\cdot 2\int_0^{\pi/2}\sin^2(\theta)\,d\theta + \frac{2A}{a\pi}\int_0^{\pi}\sin(\theta)\,d\theta \\
&= \frac{2k}{\pi}\frac{1}{2}\pi + \frac{2A}{a\pi}2 \\
&= k + \frac{4A}{a\pi}
\end{aligned}$$

(d) For $y \leq \Delta$ we have that $\psi(y) = 0$, so when $a \leq \Delta$ we have that $\psi(a\sin(\theta)) = 0$ and consequently $\Psi(a) = 0$. For $y > \Delta$ we have that $\psi(y) = A$, so when $a > \Delta$ we have that

$$\psi(a\sin(\theta)) = \begin{cases} 0 & \text{when } a\sin(\theta) \leq \Delta \\ A & \text{when } a\sin(\theta) > \Delta \end{cases}$$

which is equal to (see Figure 3)

$$\psi(a\sin(\theta)) = \begin{cases} 0 & \text{when } 0 \leq \theta \leq \alpha \text{ and } \pi-\alpha \leq \theta \leq \pi \\ A & \text{when } \alpha < \theta < \pi-\alpha \end{cases}$$

where

$$a \sin(\alpha) = \Delta \quad \text{i.e.} \quad \alpha = \arcsin \frac{\Delta}{a}$$

The describing function for $a > \Delta$ is found as

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi} \int_0^{\pi} \psi(a \sin(\theta)) \sin(\theta)\, d\theta \\
&= \frac{2}{a\pi} \cdot 2 \int_{\alpha}^{\pi/2} \psi(a \sin(\theta)) \sin(\theta)\, d\theta \\
&= \frac{4}{a\pi} \int_{\alpha}^{\pi/2} A \sin(\theta)\, d\theta \\
&= \frac{4A}{a\pi} \cos\alpha \\
&= \frac{4A}{a\pi} \sqrt{1 - \sin^2\alpha} \\
&= \frac{4A}{a\pi} \sqrt{1 - \frac{\Delta^2}{a^2}}
\end{aligned}$$

The entire describing function is hence given as

$$\Psi(a) = \begin{cases} 0 & \text{when } a \leq \Delta \\ \frac{4A}{a\pi} \sqrt{1 - \left(\frac{\Delta}{a}\right)^2} & \text{when } a > \Delta \end{cases}$$

7. The describing function of the nonlinearity $\psi(\cdot)$ is given by

$$\Psi(a) = \frac{5a^4}{8}$$

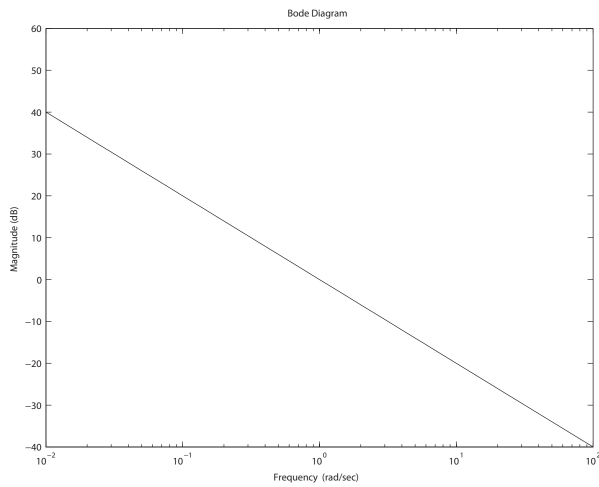

Figure 4: Bode diagram of $h(s)$

(a) Figure 4 shows a Bode diagram of the transfer function $h(s)$. As can be seen from the figure, the system has a low pass characteristic ($h(s)$ is strictly proper). This justifies the use of the describing function method (the simplification of ignoring higher order Fourier coefficients is valid since the effect of these will be reduced due to the low pass of the plant).

(b) If the harmonic balance equation

$$h(j\omega)\Psi(a) + 1 = 0$$

has a solution, then the closed loop system has a periodic solution (or at least this is a strong implication of a periodic solution). Using that $\Psi(a)$ is real, the complex harmonic balance equation is divided in to two real equations

$$\begin{aligned} \mathrm{Re}\left[h(j\omega)\right]\Psi(a) + 1 &= 0 \\ \mathrm{Im}\left[h(j\omega)\right] &= 0 \end{aligned}$$

where the second equation is solved first to determine possible frequencies of oscillations. For each solution of ω, if any, the first equation is used to determine the amplitude of the periodic solution a. In our case we have that

$$\begin{aligned}
h(j\omega) &= \frac{1-j\omega}{j\omega(j\omega+1)} \\
&= \frac{j(1-j\omega)(-j\omega+1)}{-\omega(\omega^2+1)} \\
&= \frac{j(1-j2\omega-\omega^2)}{-\omega(\omega^2+1)} \\
&= \frac{2\omega+j-j\omega^2}{-\omega(\omega^2+1)} \\
&= \frac{-2}{(\omega^2+1)}+j\frac{\omega^2-1}{\omega(\omega^2+1)}
\end{aligned}$$

Solving the harmonic balance equation with respect to ω results in

$$\begin{aligned}
\mathrm{Im}[h(j\omega)] &= \frac{\omega^2-1}{\omega(\omega^2+1)}=0 \\
&\Leftrightarrow \omega^2-1=0 \\
&\Rightarrow \omega=\pm 1
\end{aligned}$$

and solving the harmonic balance equation with respect to a results in

$$\begin{aligned}
\mathrm{Re}[h(j\omega)]\Psi(a)+1\big|_{\omega=1} &= \left.\frac{-2}{(\omega^2+1)}\right|_{\omega=1}\frac{5a^4}{8}+1 \\
&= \frac{-2}{2}\frac{5a^4}{8}+1 \\
&= -\frac{5a^4}{8}+1 \\
&= 0 \\
&\Rightarrow a=\sqrt[4]{\frac{8}{5}}=1.1247
\end{aligned}$$

Hence we have strong implications that a periodic solution exists in this system, and an estimate of the frequency and amplitude is given by

$$\begin{aligned}
\omega &= 1 \\
a &= 1.1247
\end{aligned}$$

(c) From the harmonic balance equation it can be seen that a solution exists if

$$h\,(j\omega) = -\frac{1}{\Psi\,(a)}$$

for some ω and a. This is usually investigated in a Nichols diagram where $h\,(j\omega)$ is plotted as a function of ω and $-\frac{1}{\Psi(a)}$ is plotted as a function of a. Such diagrams are shown in Figure 5–6 where it can be seen that the point of intersection is consistent with the result found when using an analytic approach.

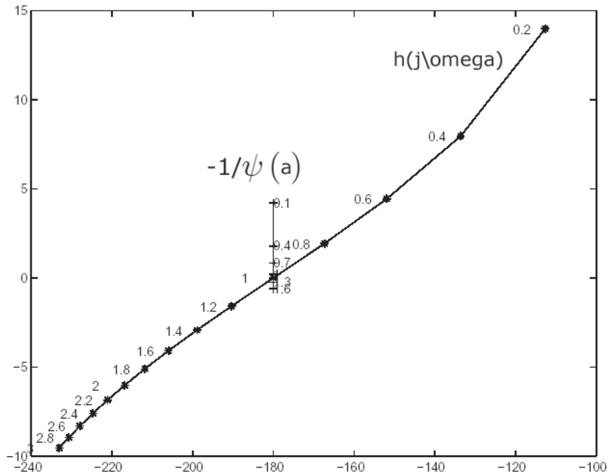

Figure 5: Nicols diagram of $h\,(j\omega)$ and $-\frac{1}{\psi(a)}$

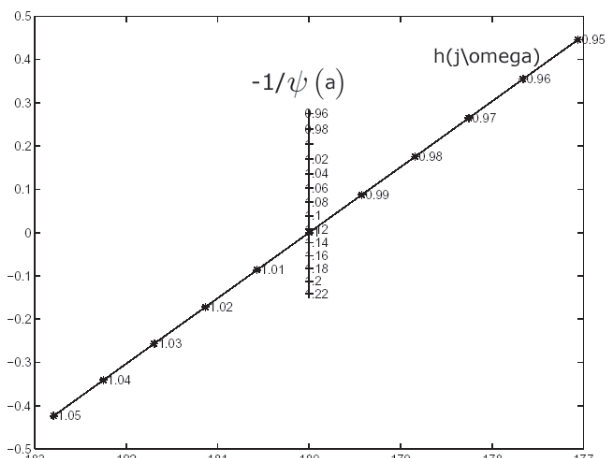

Figure 6: Nicols diagram of $h\,(j\omega)$ and $-\frac{1}{\psi(a)}$