NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2021

Lecture: Correlation and Energy Spectral Density

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 2.6.1 Crosscorrelation and autocorrelation sequences
  - 2.6.2 Properties of crosscorrelation and autocorrelation…
  - 2.6.4 Input-output correlation sequences
  - 4.2.5 Energy density spectrum of aperiodic signals
  - 5.3.1 Input-output correlation functions and spectra

*Level of detail is defined by lectures and problem sets

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Contents and learning outcomes

- Cross- and autocorrelation sequences
- Properties of cross- and autocorrelation sequences
- Linear time-invariant systems
- Energy spectral density

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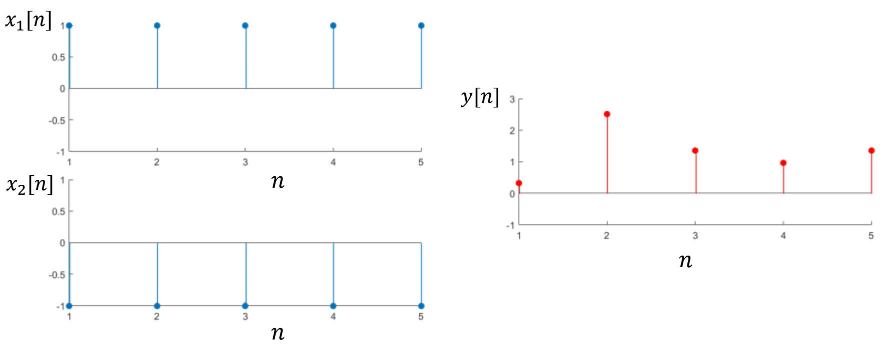
Introduction



- Which of the sequences $x_1[n]$ and $x_2[n]$ resembles $y[n]$?
- How to measure similarity between signal sequences

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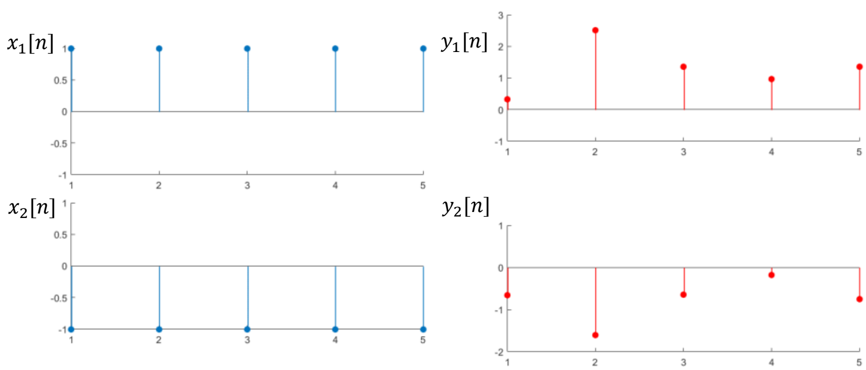
Introduction…

- Signals transmitted over some medium, e.g., wireless channels, experience delays, echoes, and noise
  - Difficult to recognize/or detect the signals at the receiving end
- Suppose that $x_1[n]$ or $x_2[n]$ is transmitted and $y[n]$ is received
  - If $y[n]$ is more similar to $x_1[n]$ than to $x_2[n]$, we decide that $x_1[n]$ was transmitted
  - If $y[n]$ is more similar to $x_2[n]$ than to $x_1[n]$, we decide that $x_2[n]$ was transmitted

- Correlation is a measure of similarity

Introduction…

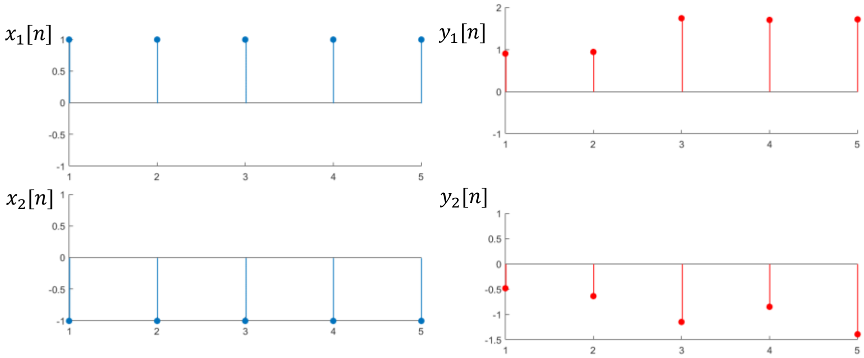
- Digital communication example: $y_i[n] = x_i[n] + w[n]$



- Noise can make received signal fluctuate significantly

Introduction…

- Digital communication example: $y_i[n] = x_i[n] + w[n]$



- Noise can make received signal fluctuate significantly

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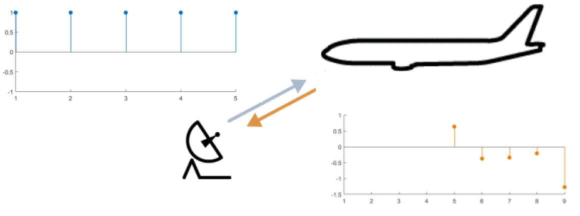
Introduction…

- Radar example: $y[n] = \alpha x[n - D] + w[n]$, find D?



- Here we need a similarity measure that gives a maximum for D, considering all possible delays

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Crosscorrelation and autocorrelation

- Crosscorrelation of real-valued sequences $x[n]$ and $y[n]$

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$
$$= \sum_{n=-\infty}^{\infty} x[n+l]y[n]\,, l = \pm 1, \pm 2, \ldots$$

- Measure of similarity between signals $x[n]$ and $y[n]$
- Reverse role $r_{yx}[l] \neq r_{xy}[l]$

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l] = r_{xy}[-l]$$

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Crosscorrelation and autocorrelation…

- Similarity to convolution of $x[n]$ and $y[n]$

$$x[l] * y[l] = \sum_{k=-\infty}^{\infty} x[k]y[l-k]$$

$$r_{xy}[l] = \sum_{k=-\infty}^{\infty} x[k]y[k-l] = x[l] * y[-l]$$

- Relation can be exploited for efficient computation
- Autocorrelation sequence (self-similarity), $y[n] = x[n]$

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$
$$= \sum_{n=-\infty}^{\infty} x[n+l]x[n]$$

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Crosscorrelation and autocorrelation…

- Finite causal sequences $x[n] = y[n] = 0, n < 0, n \geq N$

$$x[n] = \{\underline{x[0]}, x[1], x[2], \dots x[N-1]\}$$

$$y[n] = \{\underline{y[0]}, y[1], y[2], \dots y[N-1]\}$$

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

- Try a couple of values of l and search for pattern $\Rightarrow$

$$r_{xy}[l] = \sum_{n=l}^{N-1} x[n]y[n-l]\,, l \geq 0$$

$$r_{xy}[l] = \sum_{n=0}^{N-|l|-1} x[n]y[n-l]\,, l < 0$$

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Properties of autocorrelation

- Energy of sequences $x[n]$

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] = r_x[0] \geq 0$$

- Autocorrelation is maximum at lag l=0

$$|r_{xx}[l]| \leq r_{xx}[0] = E_x$$

- Autocorrelation is even $\Rightarrow$ only compute values for $l \geq 0$

$$r_{xy}[l] = r_{xy}[-l] \Rightarrow r_{xx}[l] = r_{xx}[-l]$$

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Properties of autocorrelation…

- Normalized versions

$$\varrho_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]} \Rightarrow |\varrho_{xx}[l]| \leq 1$$

$$\varrho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]r_{yy}[0]}} \Rightarrow \left|\varrho_{xy}[l]\right| \leq 1$$

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Properties cross- and autocorrelation…

- Example: Compute the autocorrelation of $x[n] = \alpha^n u[n]$

- Solution:

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n+l]x[n]$$

$$= \sum_{n=-\infty}^{\infty} \alpha^{n+l} u[n+l]\alpha^n u[n]$$

$$= \alpha^l \sum_{n=0}^{\infty} \alpha^{2n} = \frac{\alpha^l}{1-\alpha^2}, \, l \geq 0$$

Since $r_{xx}[-l] = r_{xx}[l]$, we get the final expression

$$r_{xx}[l] = \frac{\alpha^{|l|}}{1-\alpha^2}, \forall l$$

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Example 1

- Let $y[n] = Ax[n-D]$. Show that $D = \arg\max_{l} |r_{yx}[l]|$

- Solution:

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l] = \sum_{n=-\infty}^{\infty} y[n+l]x[n]$$
$$= \sum_{n=-\infty}^{\infty} Ax[n+l-D]x[n] = Ar_{xx}[D-l]$$

From properties of autocorrelation sequences, we know

$$\left|r_{xy}[l]\right| = |A||r_{xx}[D-l]| \leq |A||r_{xx}[0]|, \forall l$$

$\therefore \left|r_{xy}[l]\right|$ reach its maximum for $l = D$

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Example 2

- Let $y[n] = x[n] + Rx[n-D]$, i.e., received signal contains an echo. How to estimate R, D using the autocorrelation of $y[n]$?

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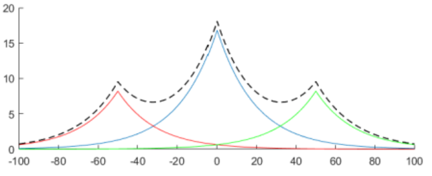
Example 3

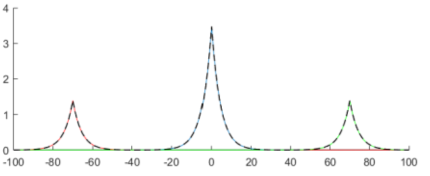
- Let $x[n] = \alpha^n u[n]$ and $y[n] = x[n] + Rx[n - D]$

$$\Rightarrow r_{yy}[l] = (1 + R^2) r_{xx}[l] + R r_{xx}[l + D] + R r_{xx}[l - D]$$

$\alpha = 0.95, R = 0.8, D = 50$



$\alpha = 0.8, R = 0.5, D = 70$



- Shape of $r_{yy}[l]$ depends on R, D

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Example 3...

- Details on how to obtain $r_{yy}[l]$ and R in previous slide
- From Slide 14: $r_{xx}[l] = \frac{\alpha^{|l|}}{1-\alpha^2}$
- $r_{yy}[l] = \sum_{n=-\infty}^{\infty} y[n + l] y[n]$

$$= \sum_{n=-\infty}^{\infty} (x[n + l] + Rx[n + l - D])(x[n] + Rx[n - D])$$
$$= \sum_{n=-\infty}^{\infty} x[n + l] x[n] + R \sum_{n=-\infty}^{\infty} x[n + l - D] x[n]$$
$$+ R \sum_{n=-\infty}^{\infty} x[n + l] x[n - D] + R^2 \sum_{n=-\infty}^{\infty} x[n - D] x[n + l - D]$$
$$= r_{xx}[l] + R r_{xx}[l - D] + R r_{xx}[l + D] + R^2 r_{xx}[l]$$

- Look at the following values (corresponding to the peaks in figure)

$$r_{yy}[0] = (1 + R^2) r_{xx}[0] + R r_{xx}[-D] + R r_{xx}[D] \approx (1 + R^2) r_{xx}[0]$$
$$r_{yy}[D] = (1 + R^2) r_{xx}[D] + R r_{xx}[0] + R r_{xx}[2D] \approx R r_{xx}[0]$$

- Given values $r_{yy}[0]$ and $r_{yy}[D]$, we can solve for R

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Example 2…

Matlab

```
l = (-100:100);
a=0.8;    % decay rate
R = 0.5; % echo strength
D = 70;  % delay of echo

rxx_l   = a.^abs(l)/(1-a^2);
rxx_lpD = a.^abs(l+D)/(1-a^2);
rxx_lmD = a.^abs(l-D)/(1-a^2);
ryy     = (1+R^2)*rxx_l+R*rxx_lpD+R*rxx_lmD;

figure
plot(l,(1+R^2)*rxx_l); hold on
plot(l,R*rxx_lpD,'r');
plot(l,R*rxx_lmD,'g')
plot(l,ryy,'k--','LineWidth',1)
```


Energy spectral density

- Quantity $S_{xx}(\omega) \geq 0$ is the energy density spectrum of $x[n]$

$$r_{xx}[l] = x[l] * x[-l] \overset{\mathcal{F}}{\leftrightarrow} S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$$

- Energy of complex-valued sequence $x[n]$

$$E_x = r_{xx}[0] = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

- Quantity $S_{xy}(\omega)$ is the cross-energy density spectrum

$$r_{xy}[l] = x[l] * y[-l] \overset{\mathcal{F}}{\leftrightarrow} S_{xy}(\omega) = X(\omega)Y^*(\omega)$$

Input-output correlations

$x[n]$ → LTI → $y[n] = h[n] * x[n]$

$r_{xx}[l]$ $\quad h[n] \quad$ $r_{yy}[l]?\ r_{yx}[l]?$

- Input-output correlations

$$r_{yx}[l] = h[l] * r_{xx}[l]$$
$$r_{yy}[l] = r_{hh}[l] * r_{xx}[l]$$
$$E_y = r_{yy}[0] = \sum_{k=-\infty}^{\infty} r_{hh}[k] r_{xx}[k]$$

- Crosscorrelation between $x[n]$ and $y[n]$ can be seen as the output signal of an LTI system when input signal is $r_{xx}[n]$

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Input-output correlations and energy spectrum

$x[n]$ → LTI → $y[n] = h[n] * x[n]$

$r_{xx}[l]$ $\quad h[n] \quad$ $r_{yx}[l] = h[l] * r_{xx}[l]$

$r_{yy}[l] = r_{hh}[l] * r_{xx}[l]$

- In z-transform domain

$$h[l] * h[-l] \overset{z}{\leftrightarrow} H(z)H(z^{-1})$$
$$r_{yx}[l] = h[l] * r_{xx}[l] \overset{z}{\leftrightarrow} H(z)S_{xx}(z)$$
$$r_{yy}[l] = r_{hh}[l] * r_{xx}[l] \overset{z}{\leftrightarrow} H(z)H(z^{-1})S_{xx}(z)$$

- Output- and cross-energy density spectra

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = |H(\omega)|^2 |X(\omega)|^2$$
$$S_{yx}(\omega) = H(\omega)S_{xx}(\omega)$$

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Input-output correlations and energy …

- We have the following relation Fourier transform pair

$$r_{yy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega) e^{j\omega l} d\omega$$

- Energy of output sequence (of an LTI system)

$$r_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega) d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 S_{xx}(\omega) d\omega$$

- Determine impulse response by signal with flat spectrum

$$h[n] = \frac{1}{S_{xx}} r_{yx}[n]$$

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Summary

Today:

- Crosscorrelation and autocorrelation sequences
- Linear time invariant systems
- Energy spectrum

Next:

- Inverse z-transform

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Example 2 (modified)

- Let $y[n] = x[n] + Rx\,[n - D]$ be an audio signal corrupted by an echo. We would like to estimate R, D using the autocorrelation of $y[n]$, and design a filter to remove the echo.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$X(z) \qquad\qquad Y(z) = H(z)X(z)$$

```
Matlab files on BB:
      Correlation.m
```


Example 2 (modified)…

- Model the problem using an LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] + Rx[n-D]$$

$$X(z) \qquad\qquad Y(z) = H(z)X(z) = (1 + Rz^{-D})X(z)$$

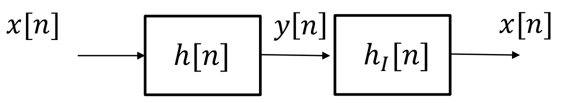
- Estimate R and D using autocorrelation sequence $r_{yy}[n]$
- Find the inverse system $H_I(z)$ such that (see previous lecture)

$$h[n] * h_I[n] = \delta[n] \overset{Z}{\leftrightarrow} H(z)H_I(z) = 1$$

$$\Rightarrow H_I(z) = \frac{1}{1 + Rz^{-D}}$$

Example 2 (modified)…

- If $R < 1$, $H(z)$ is minimum phase and so is $H_I(z)$
- We can find a causal and stable filter $h_I[n]$



- We get D from inspecting the peaks of $r_{yy}[l]$
- We obtain an estimate R from the relation

$$\frac{r_{yy}[0]}{r_{yy}[D]} = \frac{1+R^2}{R}$$

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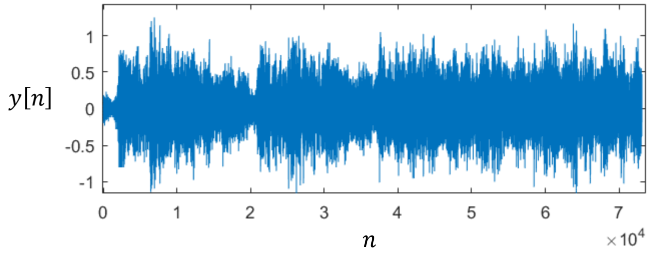
Example 2 (modified)…

- Plot of received signal $y[n]$ $(R = 0.98,\ D = 4196)$:



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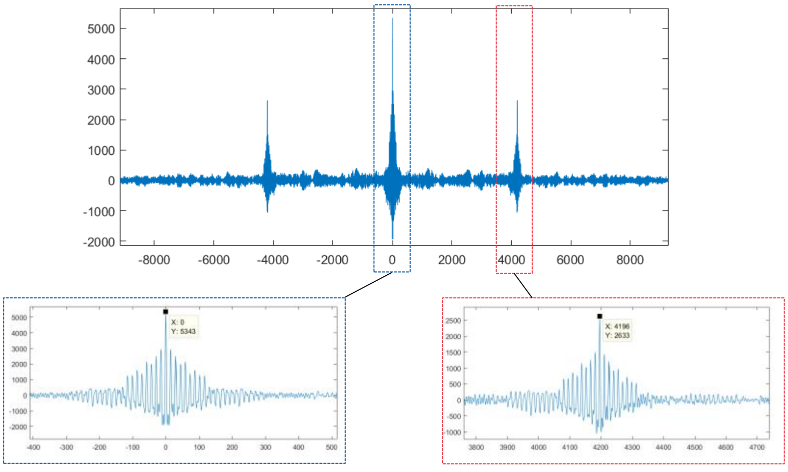
Example 2 (modified)…

- Autocorrelation of $y[n]$:



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Example 2 (modified)…

- From figure we get:

$$D = 4196, \quad \frac{r_{yy}[0]}{r_{yy}[D]} = \frac{1+R^2}{R} = \frac{5343}{2633} \Rightarrow R = 0.8430$$

- Delay is correct but parameter estimate of R is not exact. Listen to the equalized signal and judge whether the echo is removed (or suppressed)

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