Hybrid systems

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Outline

Hybrid systems

- Continuous and discrete states & dynamics co-exist
- Continuous states $\mathbf{x}(t)$ can jump
- Discontinuities can e.g. occur due to the state meeting some conditions

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Simple examples:

- Thermostat
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More difficult examples:

- Walk and running dynamics
- Take-off and landing
- Gripping (robots)

Example bouncing ball

Continuous dynamics (unique model):

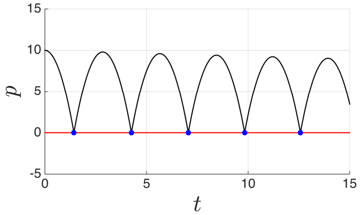
$$\mathbf{x} = \begin{bmatrix} p \\ v \end{bmatrix}, \qquad \dot{\mathbf{x}} = \begin{bmatrix} v \\ -g \end{bmatrix}$$

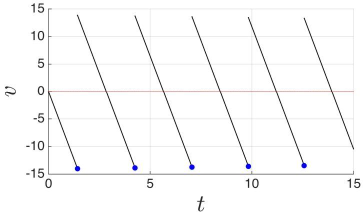
where $p,\ v \in \mathbb{R}$

Discontinuity (state condition):

$$v = -\gamma v \quad \text{if} \quad (p = 0 \quad \& \quad v < 0)$$

produces a "jump" in the continuous state $\mathbf{x}(t)$ when the ground is met





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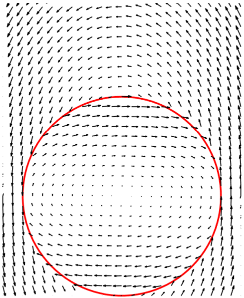

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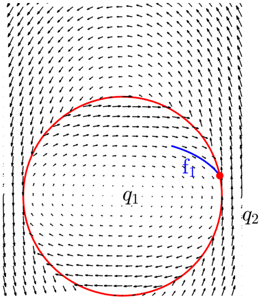

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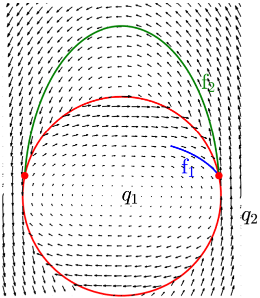

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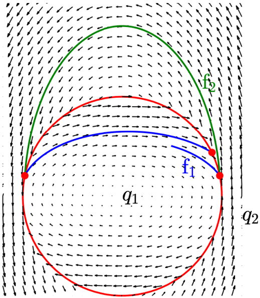

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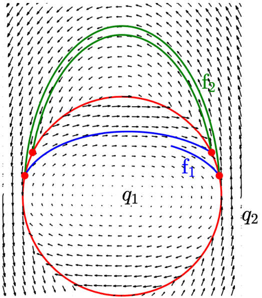

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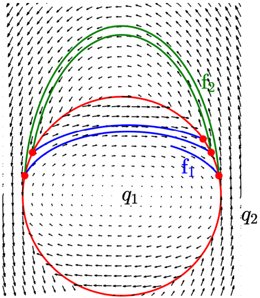

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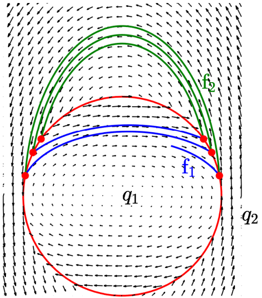

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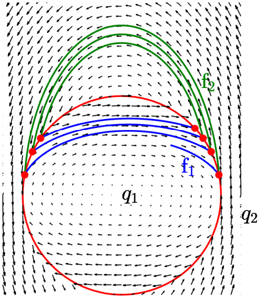

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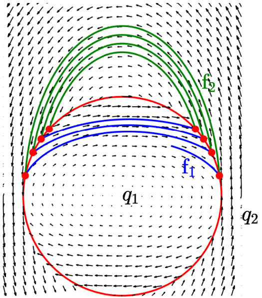

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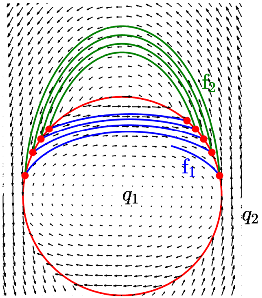

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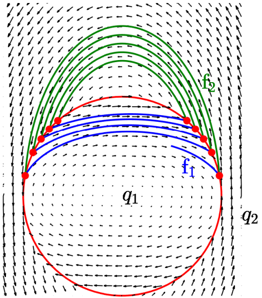

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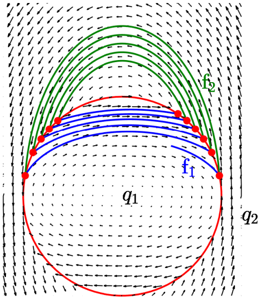

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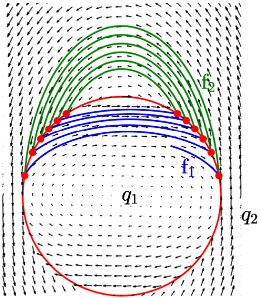

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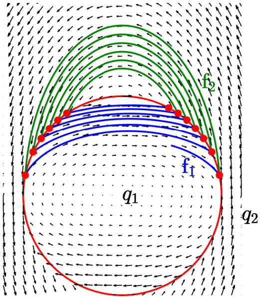

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E.g.:

$$G\left(1^{\mathrm{st}}\text{ gear},\ 2^{\mathrm{nd}}\text{ gear}\right) = \{\mathbf{x}\,|\,\mathrm{rpm} > 2000\}$$

means: "we shift from 1^{st} to 2^{nd} when the engine rpm is higher than 2000".

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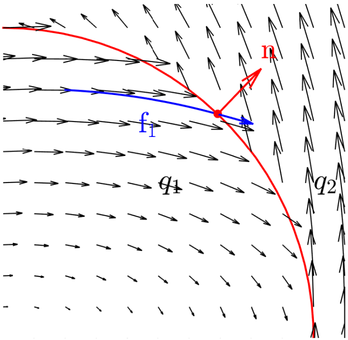

Guard condition:

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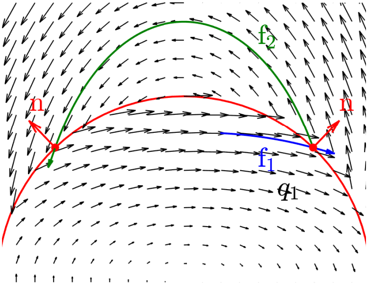

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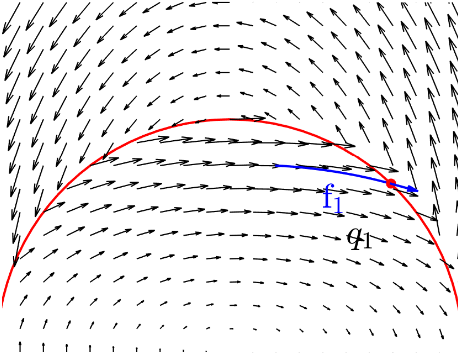

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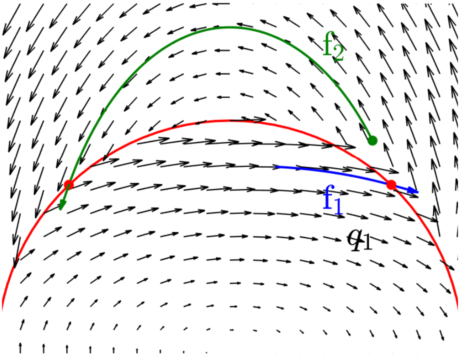

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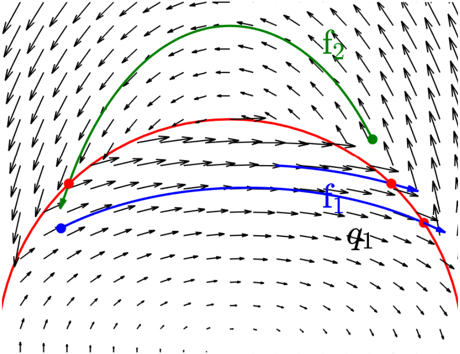

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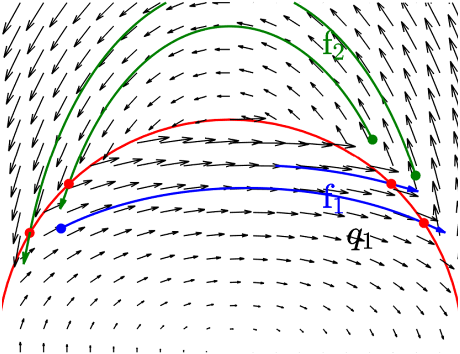

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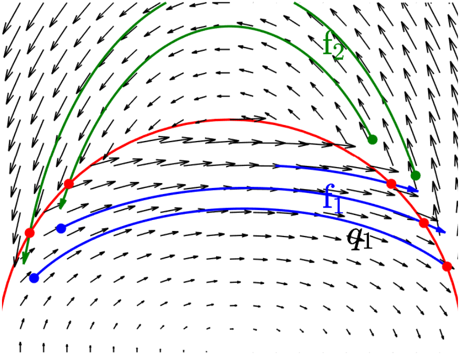

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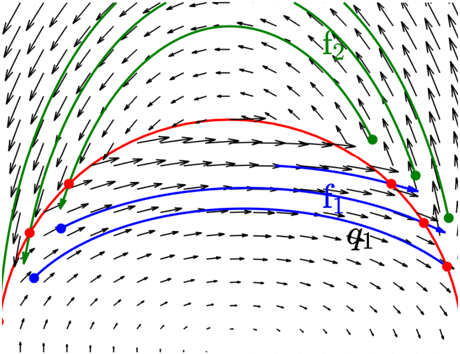

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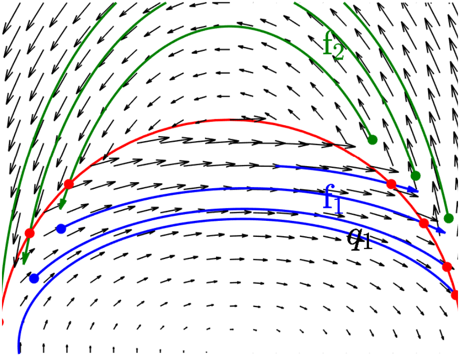

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Execution: is the collection of discrete and continuous trajectories resulting from the hybrid system, together with the time intervals

$$I_i = \left[\tau_i, \tau_i'\right]$$

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Example of Zeno execution

The bouncing ball has a Zeno execution:

$$\tau_i' - \tau_i = \frac{2|v_i|}{g} \quad \text{where } v_i \text{ is the velocity at bounce } i$$

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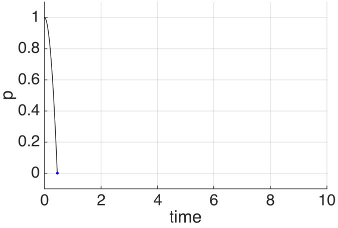

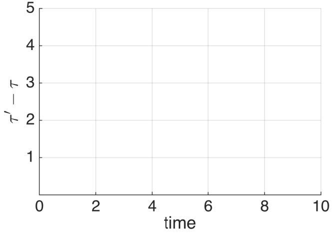

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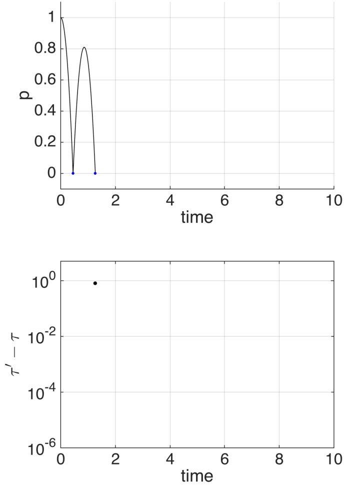

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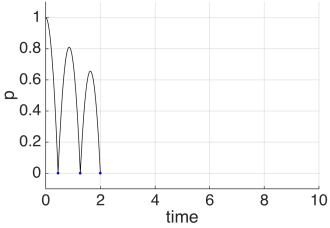

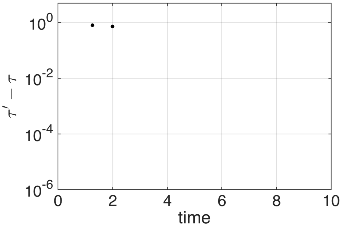

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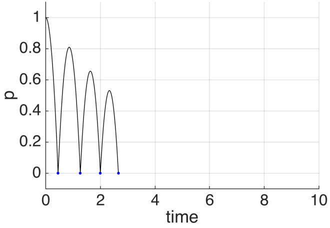

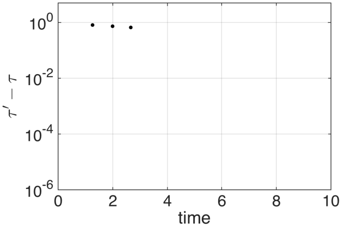

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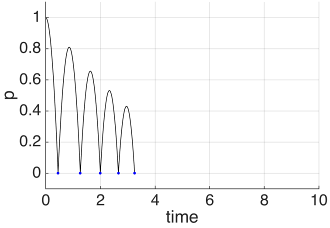

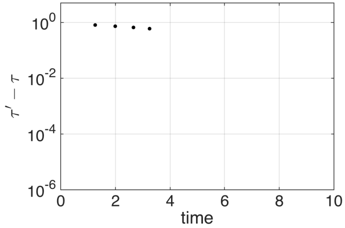

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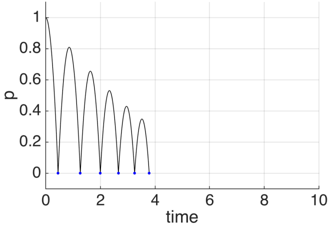

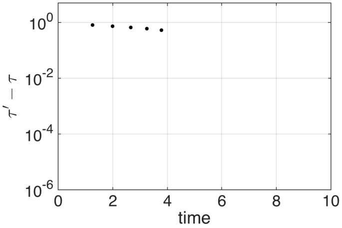

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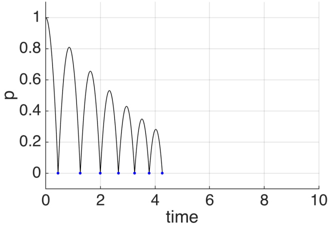

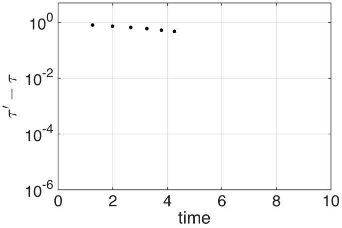

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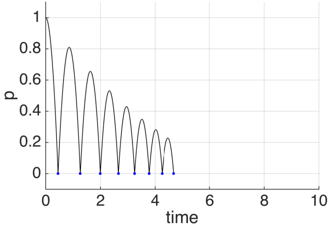

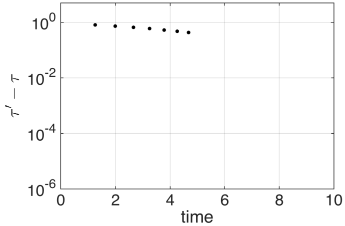

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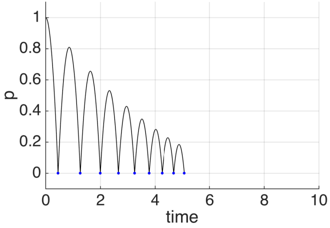

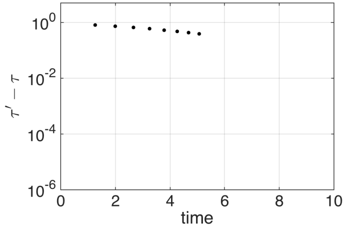

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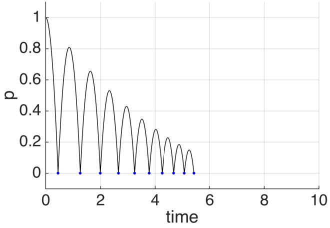

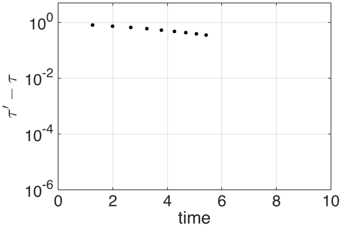

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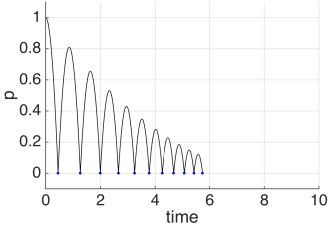

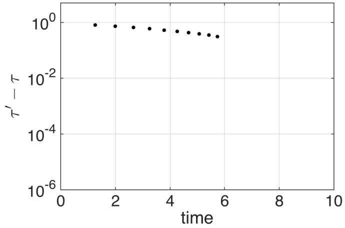

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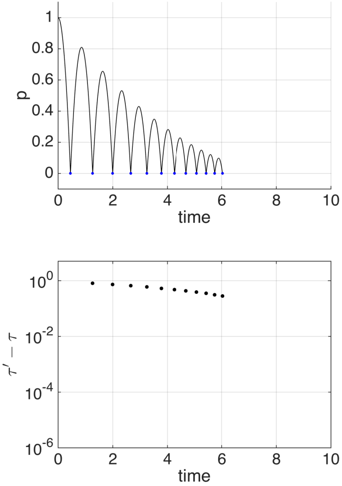

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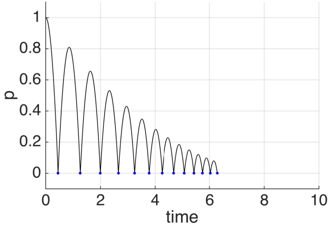

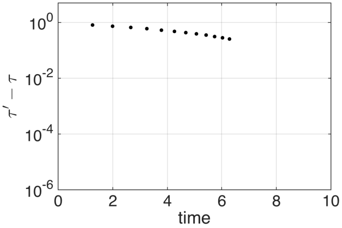

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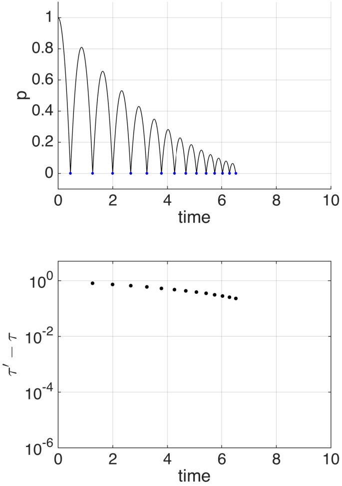

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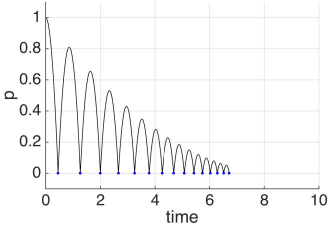

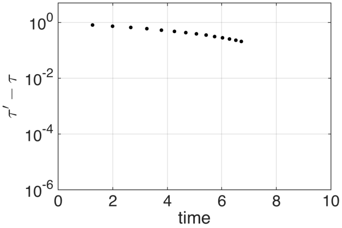

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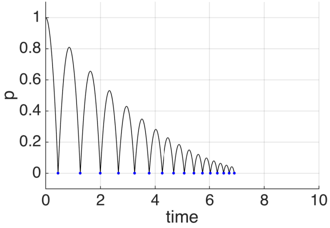

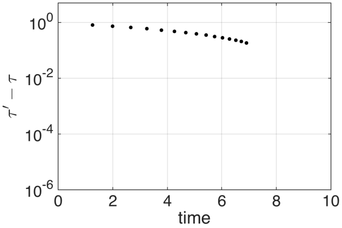

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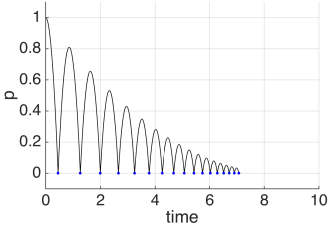

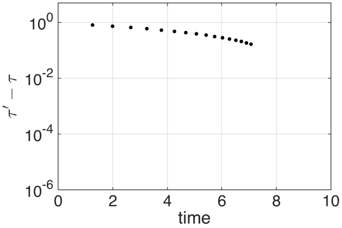

Example of Zeno execution

The bouncing ball has a Zeno execution:

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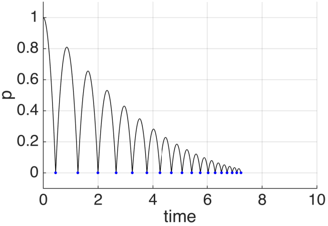

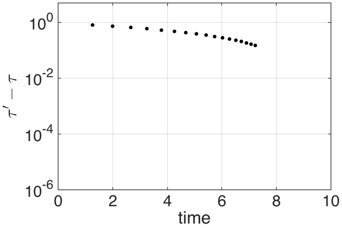

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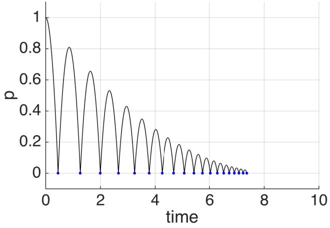

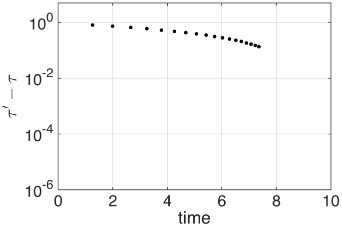

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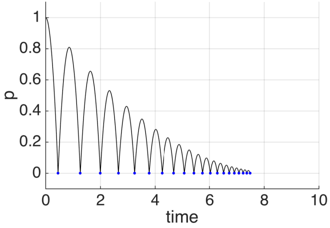

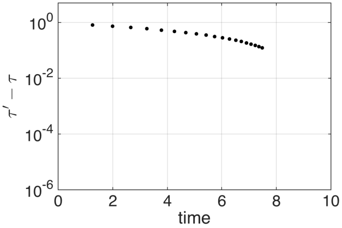

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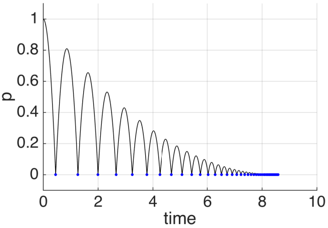

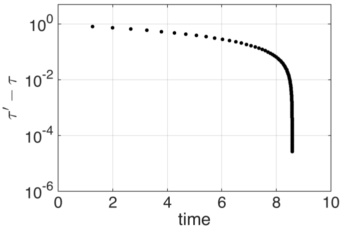

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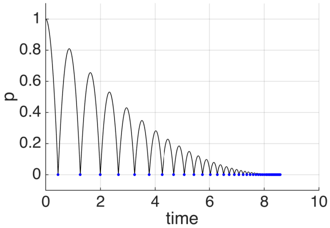
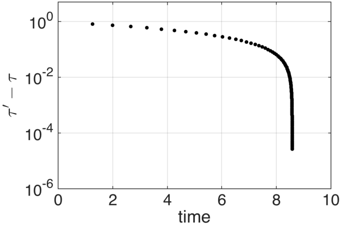
- Rebound velocity & time decay exponentially
- Assymptotically, ball rebounds ∞ fast





Example of a “degenerate” hybrid system

The mad thermostat:

- Temperature dynamics:

$$\dot{\theta} = \underbrace{\alpha\left(\theta_{\text{out}} - \theta\right)}_{\text{heat flux}} + \underbrace{u}_{\text{heating}}$$

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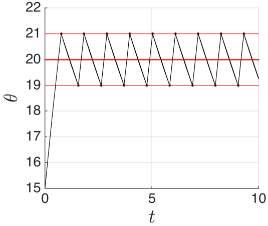

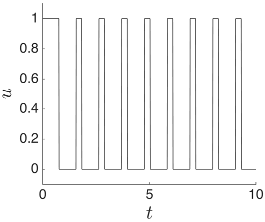

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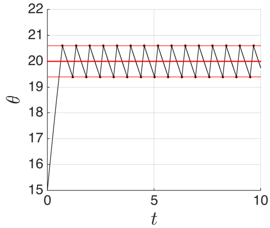

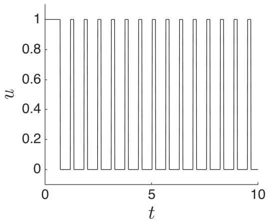

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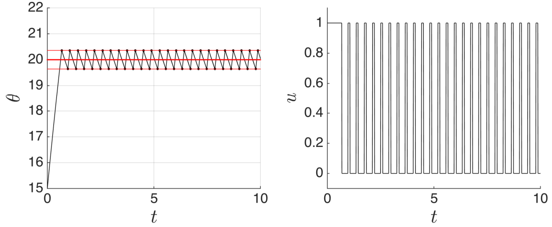

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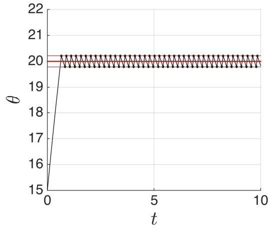

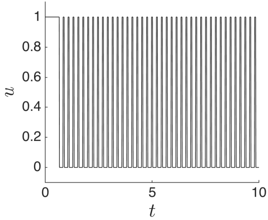

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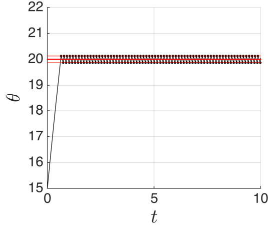

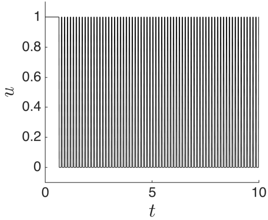

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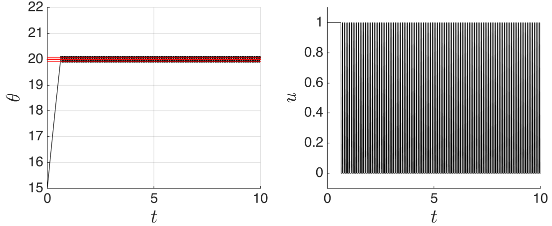

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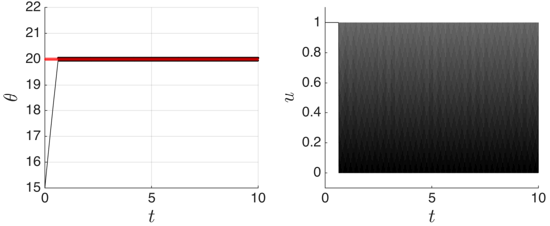

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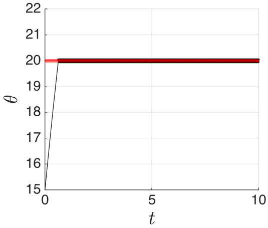
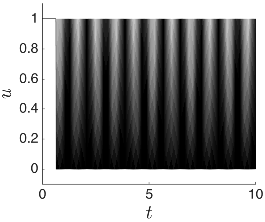


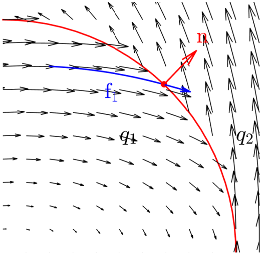


The temperature dynamics become ill-defined for $\theta_{\text{up}} - \theta_{\text{down}} \rightarrow 0$

Outline

What is an event?

Guard conditions: $G(.,.) : Q \times Q \mapsto X_G \subseteq X$ defines the set of $\mathbf{x}$ for which a jump from a discrete state q_i to a discrete state q_j is triggered



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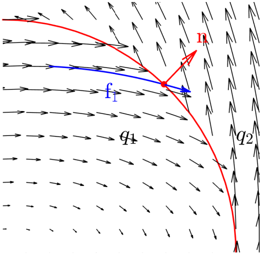

Formally an event is when the system is in state $(q_i, \mathbf{x}_i)$ meets a guard condition, i.e.

$$\mathbf{x}_i(t) \in G(q_i, q_j)$$

E.g.

- Ball touching the ground
- Thermostat hitting the "switching temperature"
- Engine reaches the RPM for a gear shift

How to simulate dynamics with events?

Can we just "ride through" the events without care?

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Explicit Euler on the Bouncing ball, with

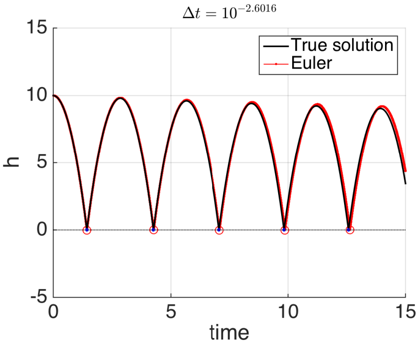
$$\mathbf{x} = \begin{bmatrix} p \\ v \end{bmatrix}$$

Input: Initial conditions $\mathbf{x}_0$
for $k = 0 : N-1$ **do**
  if $p_k \leq 0$ *and* $v_k < 0$ **then**

$$v_k = -\gamma v_k$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{f}(\mathbf{x}_k)$$

return $\mathbf{x}_{0,\dots,N}$



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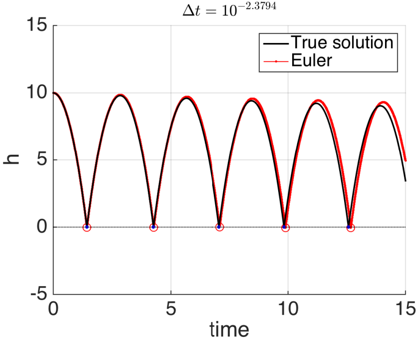

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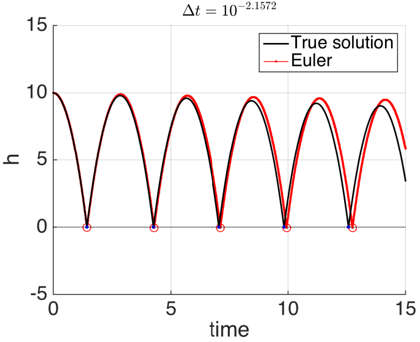

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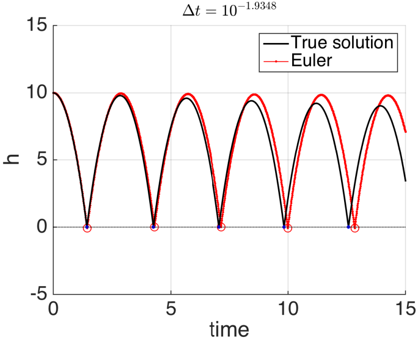

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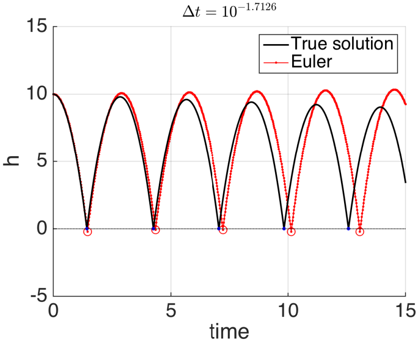

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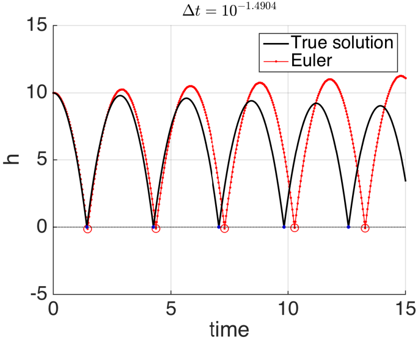

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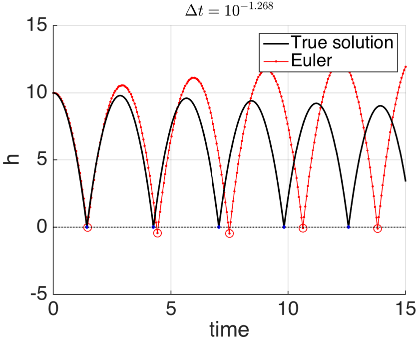

How to simulate dynamics with events?

Can we just "ride through" the events without care?

Explicit Euler on the Bouncing ball, with

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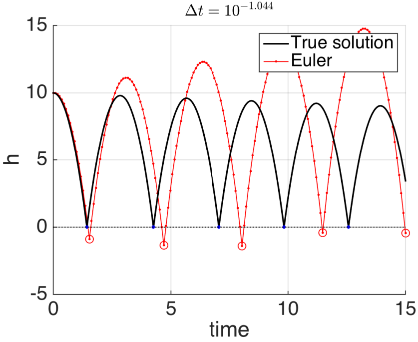

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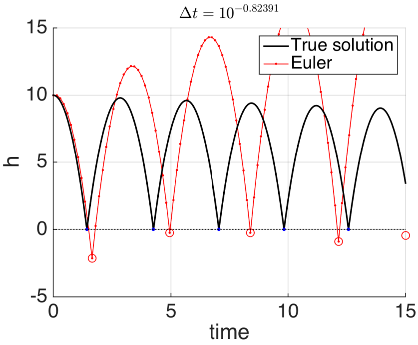

How to simulate dynamics with events?

- Simulations ignoring the events can be “risky”. Small step size are typically needed, expensive
- Problem was already observed with friction (non-smooth models), where $v = 0$ was the “event” triggering a change of dynamics (because the friction force changes sign)
- Clean simulations of dynamics with events (hybrid dynamics) require a proper treatment

How to simulate dynamics with events?

Event-based integrators

- Provide dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- Provide "event condition" (i.e. relevant guard conditions), typically in the form

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- Event-based integrators can take the condition:

$$e(\mathbf{x}) = 0, \qquad \dot{e}(\mathbf{x}) > 0$$

i.e. an event requires "crossing" the zero in a specific direction (up or down)

Example - Euler with event detection

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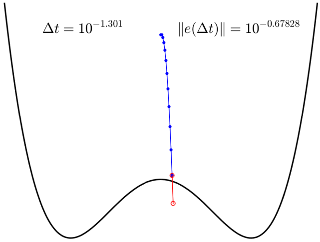

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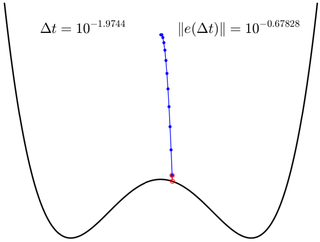

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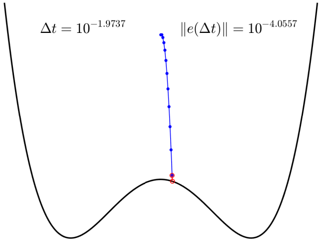

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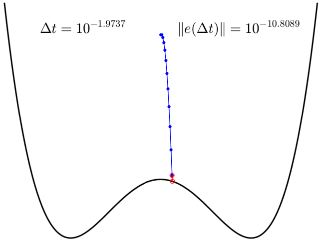

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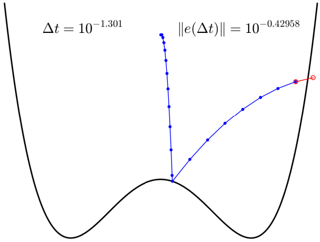

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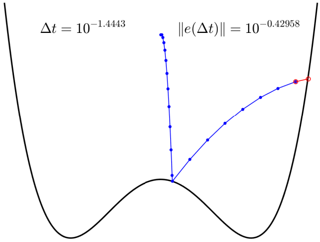

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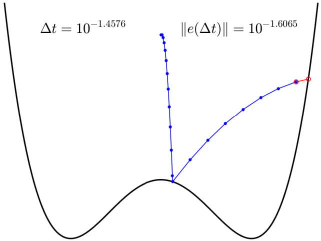

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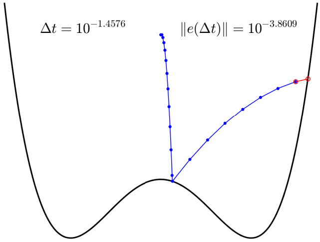

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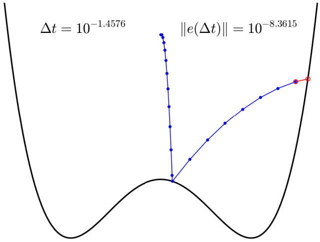

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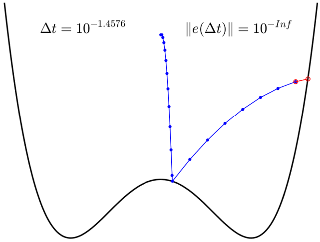

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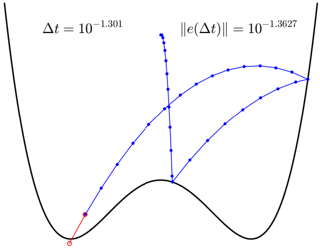

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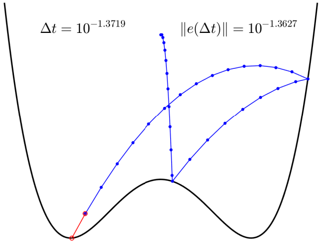

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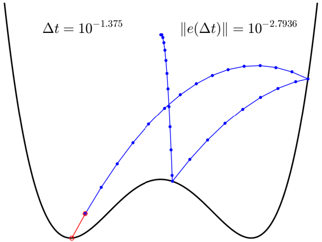

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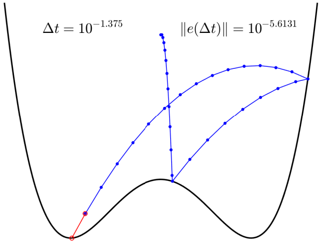

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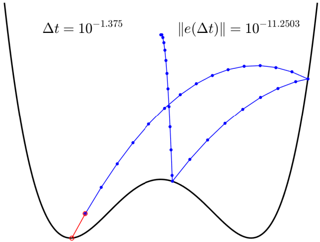

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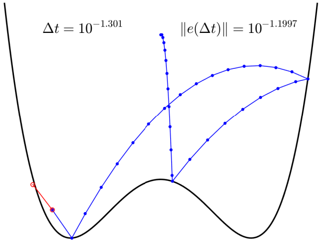

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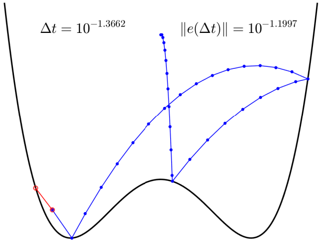

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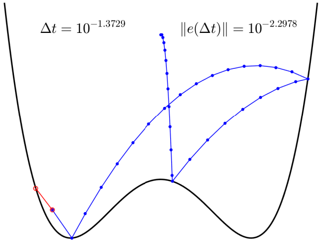
$$\Delta t_{\mathrm{e}} \leftarrow \Delta t_{\mathrm{e}} - \frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_{\mathrm{e}}}^{-1} e(\mathbf{x}_{k+1})$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t_{\mathrm{e}} \cdot \mathbf{f}(\mathbf{x}_k)$$

return $\mathbf{x}_{0,\dots,N}$



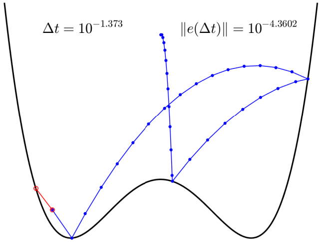
Example - Euler with event detection

Input: Initial conditions $\mathbf{x}_0$, Δt
for $k = 0 : N-1$ **do**
  $\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \cdot \mathbf{f}(\mathbf{x}_k)$
  if $e(\mathbf{x}_k)\, e(\mathbf{x}_{k+1}) < 0$ **then**
    $\Delta t_\mathrm{e} = \Delta t$
    while $|e(\mathbf{x}_{k+1})| > \mathrm{Tol}$ **do**

$$\Delta t_\mathrm{e} \leftarrow \Delta t_\mathrm{e} - \frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_\mathrm{e}}^{-1} e(\mathbf{x}_{k+1})$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t_\mathrm{e} \cdot \mathbf{f}(\mathbf{x}_k)$$

return $\mathbf{x}_{0,\dots,N}$



Example - Euler with event detection

Input: Initial conditions $\mathbf{x}_0$, Δt

for $k = 0 : N-1$ **do**

$\quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \cdot \mathbf{f}(\mathbf{x}_k)$

$\quad$ **if** $e(\mathbf{x}_k)\, e(\mathbf{x}_{k+1}) < 0$ **then**

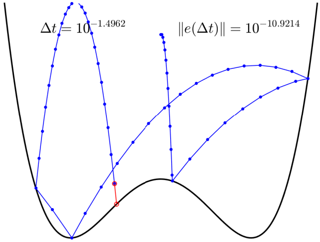
$\quad\quad \Delta t_{\mathrm{e}} = \Delta t$

$\quad\quad$ **while** $|e(\mathbf{x}_{k+1})| > \mathrm{Tol}$ **do**

$$\Delta t_{\mathrm{e}} \leftarrow \Delta t_{\mathrm{e}} - \frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_{\mathrm{e}}}^{-1} e(\mathbf{x}_{k+1})$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t_{\mathrm{e}} \cdot \mathbf{f}(\mathbf{x}_k)$$

return $\mathbf{x}_{0,\dots,N}$



Note:

- Derivative in the case of Euler

$$\frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_{\mathrm{e}}} = \left.\frac{\partial e}{\partial \mathbf{x}}\right|_{\mathbf{x}_{k+1}} \frac{\partial \mathbf{x}_{k+1}}{\partial \Delta t_{\mathrm{e}}} = \left.\frac{\partial e}{\partial \mathbf{x}}\right|_{\mathbf{x}_{k+1}} \mathbf{f}(\mathbf{x}_k)$$

- If $e(\mathbf{x})$ is linear, then Newton iteration converges in one full step (linear equation to solve)

Event detection with ERK integrators

Input: Initial conditions $\mathbf{x}_0$, Δt
for $k = 0 : N - 1$ **do**

$$\mathbf{K}_i = \mathbf{f}\left(\mathbf{x}_k + \Delta t \cdot \sum_{i=1}^{s} a_{ij}\mathbf{K}_j, \mathbf{u}(t_k + \Delta t c_i)\right)$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \cdot \sum_{i=1}^{s} b_i \mathbf{K}_i$$

if $e(\mathbf{x}_k)\, e(\mathbf{x}_{k+1}) < 0$ **then**
$\Delta t_\mathrm{e} = \Delta t$
while $|e(\mathbf{x}_{k+1})| > \mathrm{Tol}$ **do**

$$\Delta t_\mathrm{e} \leftarrow \Delta t_\mathrm{e} - \frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_\mathrm{e}}^{-1} e(\mathbf{x}_{k+1})$$

$$\mathbf{K}_i = \mathbf{f}\left(\mathbf{x}_k + \Delta t_\mathrm{e} \cdot \sum_{i=1}^{s} a_{ij}\mathbf{K}_j, \mathbf{u}(t_k + \Delta t_\mathrm{e} c_i)\right)$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t_\mathrm{e} \cdot \sum_{i=1}^{s} b_i \mathbf{K}_i$$

return $\mathbf{x}_{0,\dots,N}$

$$\frac{\partial e(\mathbf{x}_{k+1})}{\partial \Delta t_\mathrm{e}} = \left.\frac{\partial e}{\partial \mathbf{x}}\right|_{\mathbf{x}_{k+1}} \frac{\partial \mathbf{x}_{k+1}}{\partial \Delta t_\mathrm{e}}$$

$$\frac{\partial \mathbf{x}_{k+1}}{\partial \Delta t_\mathrm{e}} = \sum_{i=1}^{s} b_i \mathbf{K}_i + \Delta t_\mathrm{e} \cdot \sum_{i=1}^{s} b_i \frac{\partial \mathbf{K}_i}{\partial \Delta t_\mathrm{e}}$$

$$\frac{\partial \mathbf{K}_i}{\partial \Delta t_\mathrm{e}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \sum_{i=1}^{s} a_{ij}\mathbf{K}_j + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial t} c_i$$