NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2021

Lecture: Filter Properties and Inverse Systems

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 5.4.2 Lowpass, highpass, and bandpass filters
  - 5.4.3 Digital resonators
  - 5.4.4 Notch filters
  - 5.4.5 Comb filters
  - 5.4.6 All-pass filters
  - 5.4.1 Ideal filter characteristics
  - 10.2.1 Symmetric and antisymmetric FIR filters
  - 5.5 Inverse systems and deconvolution

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Some simple filter properties
- Why linear phase?
- Minimum-phase and inverse systems

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Ideal filter characteristics

- Systems may be designed to amplify or attenuate parts of the input signal (e.g., remove noise, suppress interference)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$X(\omega) \qquad Y(\omega) = H(\omega)X(\omega)$$

$$X(z) \qquad Y(z) = H(z)X(z)$$

- Frequency response $H(\omega)$ shapes the spectrum of the input signal to have a desired form

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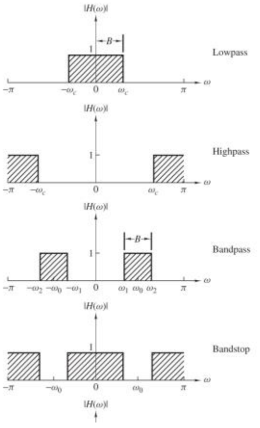
Linear time-invariant systems…



- Passband, stopband, cutoff frequencies
- Cannot get this kind of shapes using a causal impulse response with a finite number of coefficients (later)

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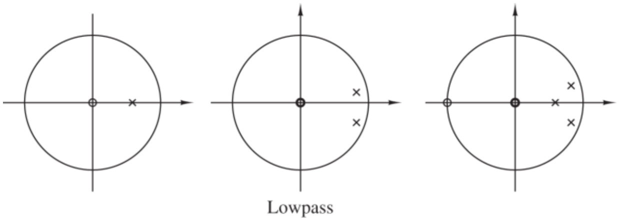
Lowpass

- Poles close(r) to $z = 1$ and zeros close(r) to $z = -1$. Why?



Lowpass

- Example: $H(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}}$

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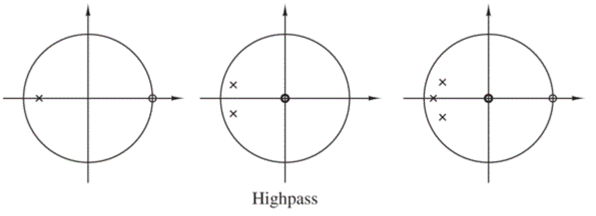
Highpass

- Poles close(r) to $z = -1$ and zeros close(r) to $z = 0$



- Reflect poles-zeros of lowpass around imaginary axis
- Frequency translation: $H_{\text{hp}}(\omega) = H_{\text{lp}}(\omega - \pi)$
- Example: $H_{\text{lp}}(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}} \rightarrow H_{\text{hp}}(z) = \frac{1-a}{2} \cdot \frac{1-z^{-1}}{1+az^{-1}}$

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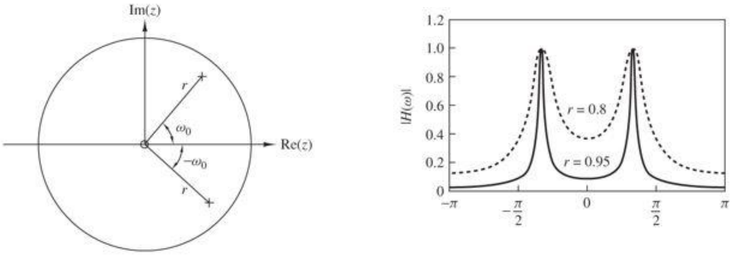
Digital resonator



- Complex-conjugate poles $p_{1,2} = re^{\pm j\omega_0}$ close to $|z| = 1$

$$H(z) = \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

- Resonant peak can be computed: $\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r}\cos\omega_0\right)$
- For $r \approx 1$, $\omega_r \approx \omega_0$

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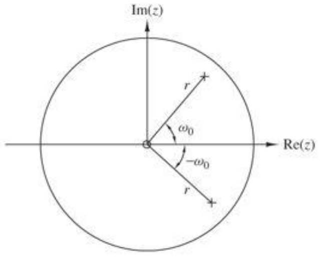
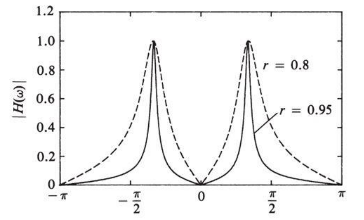
Digital resonator…





- Complex-conjugate poles $p_{1,2} = re^{\pm j\omega_0}$ and zeros $z_{1,2} = \pm 1$

$$H(z) = \frac{(1+z^{-1})(1-z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

- Exact location of resonant peak harder to find analytically

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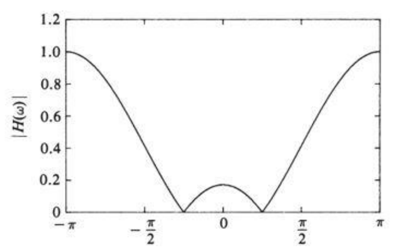
Notch filter



- A filter that contains deep notches in its frequency response
- Removing powerline frequency disturbance
- Create nulls by complex-conjugate zeros on the unit circle

$$H(z) = b_0\left(1 - e^{j\omega_0}z^{-1}\right)\left(1 - e^{-j\omega_0}z^{-1}\right)$$

- Large bandwidth is a problem with FIR notch filters

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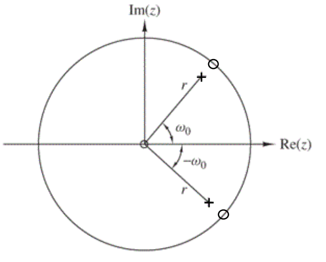
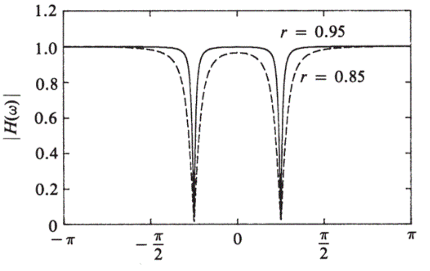
Notch filter…





- Introduce poles close to unit circle reduces notch bandwidth

$$H(z) = \frac{b_0(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

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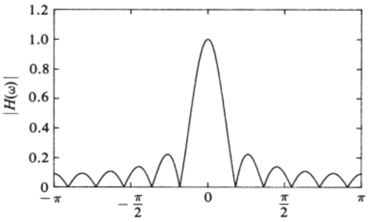
Comb filter



- Notch filter with nulls periodically spaced across frequency
- Simple moving average (FIR) filter

$$H(z) = \sum_{k=0}^{M-1} z^{-k} = \frac{1-z^{-M}}{1-z^{-1}}$$

- We may also construct a comb filter by replacing z with z^L

$$H_L(z) = \sum_{k=0}^{M} h[k]z^{-Lk} \Leftrightarrow H_L(\omega) = H(L\omega)$$

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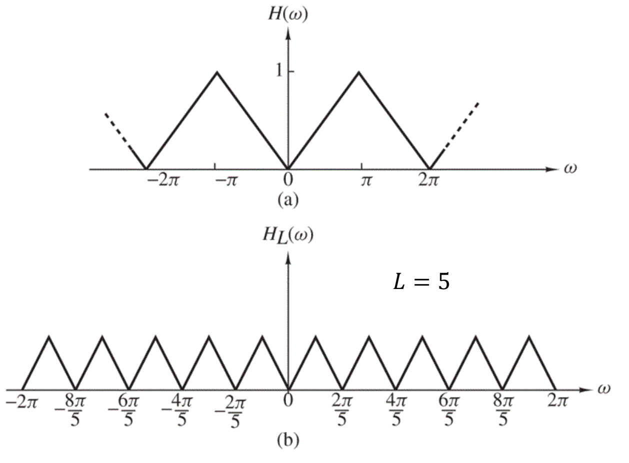
Comb filter…

$$H_L(z) = \sum_{k=0}^{M} h[k]z^{-Lk} \Leftrightarrow H_L(\omega) = H(L\omega)$$



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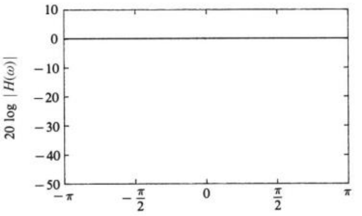
All-pass filters



- All-pass filter has constant magnitude response
- Can be used to compensate poor phase characteristics

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1}z^{-N+1} + a_N z^{-N}} = \frac{z^{-N}A(z^{-1})}{A(z)}$$

- Assuming real coefficients

$$H(z) = \frac{z^{-N}A(z^{-1})}{A(z)} \Leftrightarrow |H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = 1$$

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Linear phase filters…

- Why linear phase filters, i.e., $\measuredangle H(\omega) = a + b\omega$?

$$H(\omega) = |H(\omega)|e^{j\measuredangle H(\omega)}$$

- Compare the two ideal lowpass specifications

$$H_1(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| \le \omega_c \le \pi \end{cases}$$

$$H_2(\omega) = \begin{cases} e^{-jn_d\omega}, & |\omega| \le \omega_c \\ 0, & |\omega| \le \omega_c \le \pi \end{cases}$$

- How about the time-domain pulses?

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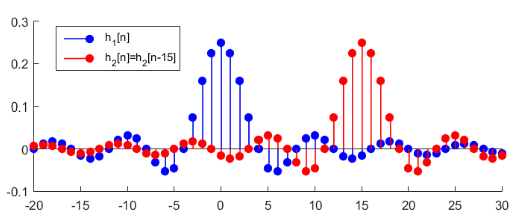
Linear phase filters…



Matlab

```
n  = -20:30;
wc = 2*pi*1/8;
x = wc/pi;
nd = 15;
stem(n,x*sinc(x*n),'r')
hold on
stem(n,x*sinc(x*(n-nd)))
```

- How about the time-domain pulses?

$$h_{1[n]} = \frac{w_c}{\pi}\frac{\sin[w_c n]}{w_c n}$$

$$h_2[n] = \frac{w_c}{\pi}\frac{\sin[w_c(n-n_d)]}{[w_c(n-n_d)]} \Longrightarrow h_2[n] = h_1[n-n_d]$$

- Delays the output signal with n_d samples, no signal distortion!

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Linear phase filters…

- Filter design in general (later in the course):

$$\min_{a,b} ||E(z)|| = \min_{a,b} \left\| H_{\text{des}}(z) - \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}} \right\|$$

- Consider FIR filters having a frequency response of the form

$$H(\omega) = H_{\text{r}}(\omega) e^{-j(\omega d + c)}, \; H_{\text{r}}(\omega) \text{ real-valued}$$

- We want a pure signal delay in passband
- Obtained by choosing $h[k]$ real and $h[k] = \pm h[M-1-k]$
  - Symmetric or antisymmetric

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Linear phase filters…

- Example: FIR with $M = 5 \Longrightarrow N = 2$

$$\begin{aligned} H(z) &= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} \\ &= z^{-2}\{h[0]z^2 + h[1]z + h[2] \pm h[1]z^{-1} \pm h[0]z^{-2}\} \\ &= z^{-2}\{h[2] + h[0][z^2 \pm z^{-2}] + h[1][z^1 \pm z^{-1}]\} \end{aligned}$$

- Frequency response symmetric filter (take the '+' signs):

$$H(z)\Big|_{e^{j\omega}} = e^{-j2\omega}\{h[2] + 2h[0]\cos 2\omega + 2h[1]\cos \omega\}$$

- Frequency response antisymmetric filter (take the '-' signs):

$$H(z)\Big|_{e^{j\omega}} = je^{-j2\omega}\{2h[0]\sin 2\omega + 2h[1]\sin \omega\}$$

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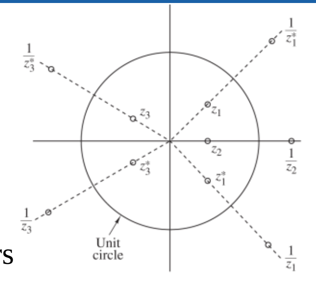
Linear phase filters…



- Zeros of $H(z)$ occur in reciprocal pairs
- Example (cont.): Symmetric FIR with $M = 5$ ($N = 2$)

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$$
$$= z^{-2}\{h[0]z^{2} + h[1]z + h[2] + h[1]z^{-1} + h[0]z^{-2}\}$$
$$= z^{-2}\{h[2] + h[0][z^{2} + z^{-2}] + h[1][z^{1} + z^{-1}]\}$$
$$= z^{-4}H(z^{-1})$$

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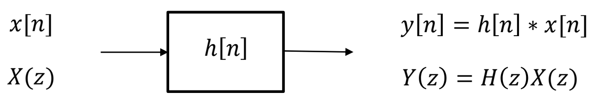
Inverse and minimum-phase systems



- What if we are given $y[n]$ and want to determine $x[n]$?
  - Information signal passing through communication channel

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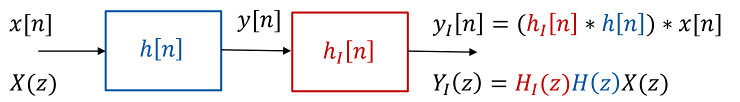
Inverse and minimum-phase systems



- If system $\mathcal{T}$ is invertible, $x[n]$ and can be recovered from $y[n]$

$$x[n]=\mathcal{T}^{-1}\{y[n]\} = \mathcal{T}^{-1}\{\mathcal{T}\ [x[n]]\}$$

- Linear time-invariant systems

$$h[n] * h_I[n] = \delta[n] \overset{z}{\leftrightarrow} H(z)H_I(z) = 1$$

- Solving for $h_I[n]$ usually simpler in z-domain, especially if $H(z)$ is rational, i.e., $H(z) = B(z)/A(z)$

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Inverse and minimum-phase systems…

- Example: Determine inverse system $h[n] = \delta[n] - \frac{1}{3}\delta[n-1]$
- Time-domain solution ($h_I[n]$ causal and stable)

$$h[n] * h_I[n] = \delta[n] \ \Leftrightarrow \sum_{k=0}^{n} h[k]h_I[n-k] = \delta[n]$$

- Solution via z-transform

$$H(z)H_I(z) = 1 \ \Leftrightarrow H_I(z) = 1/H(z)$$

Remember to specify ROC of $H_I(z)$ (two possibilities)!

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Inverse and minimum-phase systems…

- Solution via z-transform

$$H(z)H_I(z) = 1 \Leftrightarrow H_I(z) = 1/H(z)$$

Remember to specify ROC of $H_I(z)$ (two possibilities)!

$$H(z) = 1 - \frac{1}{3}z^{-1} \text{ ROC: } |z| \neq 0 \to H_I(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

- Corresponds to either

$$h_I[n] = -\left(\frac{1}{3}\right)^n u[-n-1], \text{ ROC: } |z| < \frac{1}{3} \text{ (anti-causal unstable) or}$$

$$h_I[n] = \left(\frac{1}{3}\right)^n u[n], \text{ ROC: } |z| > \frac{1}{3} \text{ (causal stable)}$$

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Inverse and minimum-phase systems…

- In general we have that if $H(z)$ is stable and causal then poles $|p_k| < 1\ \forall k$ and ROC: $|z| > \max_k |p_k|$

$\Longrightarrow$There exists a stable and causal inverse $H_I(z) = 1/H(z)$ if zeros of $H(z)$ are within the unit circle, i.e., $|z_k| < 1\ \forall k$

- Definition: A system is called minimum-phase if all zeros and poles are inside the unit circle

  $\Longrightarrow$ a stable pole-zero system that is minimum phase has a stable inverse that is also minimum phase

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Summary

Today:
- Some simple filter types and their properties
- Linear phase systems
- Inverse and minimum-phase systems

Next:
- Correlation and energy spectrum density

Inverse and minimum-phase systems…

- Example: Determine inverse system $h[n] = \delta[n] - \frac{1}{3}\delta[n-1]$
- Time-domain solution ($h_I[n]$ causal and stable)

$$h[n] * h_I[n] = \delta[n] \;\Leftrightarrow \sum_{k=-\infty}^{\infty} h[k]h_I[n-k] = \delta[n]$$

$$\sum_{k=0}^{n} h[k]h_I[n-k] = \delta[n]$$

$$n = 0\colon h[0]h_I[0] = 1 \;\Rightarrow h_I[0] = 1/h[0]$$

$$n = 1\colon h[0]h_I[1] + h[1]h_I[0] = 0 \Rightarrow h_I[1] = -h[1]h_I[0]/h[0]$$

$$n = 2\colon h[0]h_I[2] + h[1]h_I[1] + h[2]h_I[0] = 0 \Rightarrow$$

$$h_I[2] = -(h[1]h_I[1] + h[2]h_I[0])/h[0]$$

$$n \geq 1\colon h_I[n] = -\sum_{k=1}^{n} \frac{h[k]h_I[n-k]}{h[0]}$$