NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2018

Discrete Random Signals

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 1.2.4 Deterministic versus random signals
  - 12.1 Random signals, correlation functions, and power spectra
- A comprehensive overview of topics treated in the lecture, see "Introdukjon til statistisk signalbehandling" on Blackboard

*Level of detail is defined by lectures and problem sets

Preliminary question

- What is the Fourier transform of a sequence of coin flips?

Contents and learning outcomes

- Models
- Stochastic process
- Statistical averages
- Stationarity and wide-sense stationarity
- Ergodicity
- Power spectral density

Introduction

- Signal analysis and processing require a mathematical description of the signal itself, or so-called signal model
- Deterministic signals uniquely described by an explicit mathematical expression, well-defined rule or a table of data

$$x[n] = 2e^{-4n}, n \geq 0$$

$$x[n] = \sin 2\pi f n$$

- All past, present and future values of the signals are known precisely without any uncertainty

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Introduction…

- Many signals cannot be described by explicit formulas
  - Speech signals, received noisy communication signals

  ⇒ Signals evolve in time in an unpredictable manner
- Stochastic signal is a sequence of random numbers
  - Signal value at instant n unknown and modeled as a stochastic variable $X[n]$ with probability density function $p_X(x[n])$

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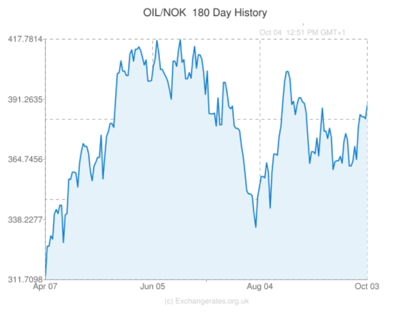
Introduction…



- Models derived are usually of statistical nature
  - Find a suitable model describing the random signal
  - Estimate model parameters

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Review stochastic variables

- First- and second-order moments
- Expected value: $m_X = E\{X\} = \int_{-\infty}^{\infty} x p_X(x) dx$
- Second-order moment: $E\{X^2\} = \int_{-\infty}^{\infty} x^2 p_X(x) dx$
- Variance: $\sigma_X^2 = E\{(X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x) dx$
  $= E\{X^2\} - m_X^2$
- Example: $X \sim N(m_X, \sigma_X^2) \Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m_X)^2}{2\sigma_X^2}}$

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Review stochastic variables…

- Study of several stochastic variables requires joint density function, e.g., variables X_1, and X_2 described by $p_{X_1,X_2}(x_1, x_2)$
- Stochastic variables independent if

$$p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

- Second-order moment:

$$E\{X_1X_2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2p_{X_1,X_2}(x_1, x_2)dx$$

- Covariance: $\sigma^2_{X_1,X_2} = E\{(X_1 - m_{X_1})(X_2 - m_{X_2})\}$
  $= E\{X_1X_2\} - m_{X_1}m_{X_2}$
- If $\sigma^2_{X_1,X_2} = 0 \Rightarrow X_1$ and X_2 are said to be uncorrelated

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Stochastic process

- Definition: A stochastic process is a family *or ensemble* of signals corresponding to every possible outcome of a certain signal measurement or experiment. Each signal in the ensemble is called a "realization" of the process.
- Notation: $X[n, S]$ is the ensemble of possible waveforms, where n represents time and $S = \{s_1, s_2 \ldots\}$ represents the set of all possible functions
- Single waveform in ensemble denoted $x[n, s]$ or $x[n]$
- Example 1: Toss a coin 35 times and assign 1 for head and 0 for tail. Repeat the experiment.

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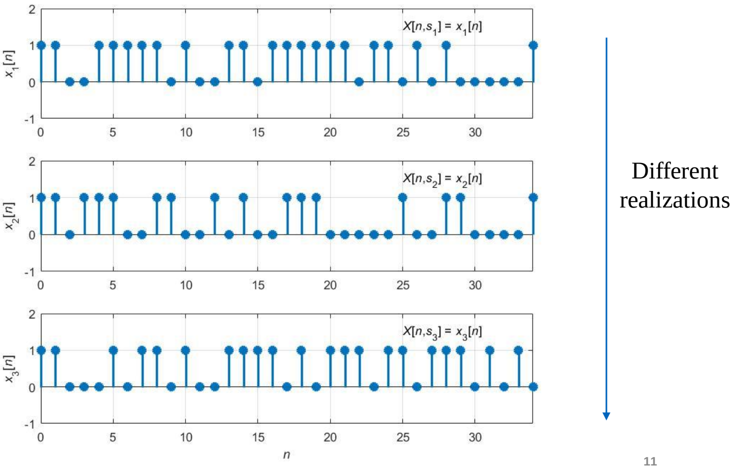
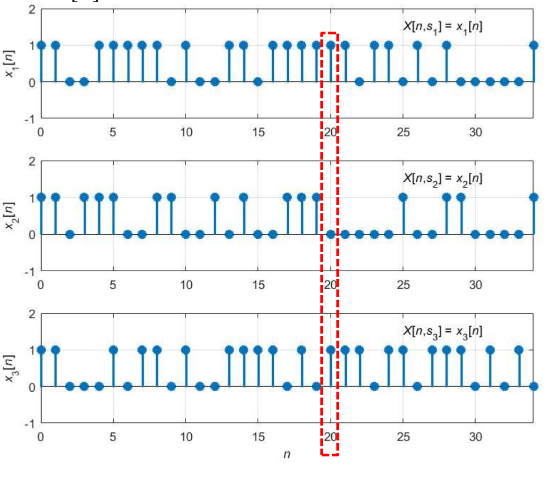
Stochastic process…

- Bernoulli process (coin flipping) with $p = 0.5$, $N = 35$



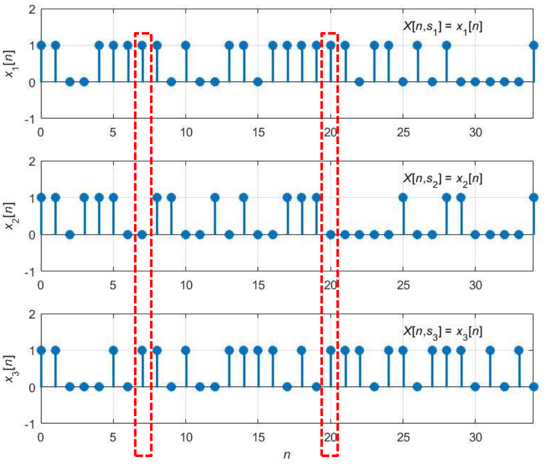
Stochastic process…

- Fixed time instant, e.g., $n = 20 \Rightarrow X(20, S)$ is a random variable defined by $p_{X[n]}(x)$



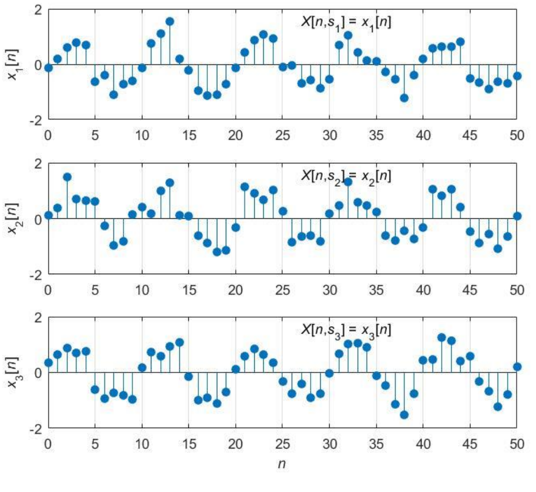
Stochastic process…

- Fixed time, e.g., $n_1 = 7$ and $n_2 = 20 \Rightarrow X(7, S)$ and $X(20, S)$ form a bivariate random vector defined by $p_{X[n_1],X[n_2]}(x_1, x_2)$



Stochastic process…

- Sinusoid with noise: $X(n) = \sin(2\pi f n) + W[n]$, $W[n] \sim N(0, \sigma_w^2)$

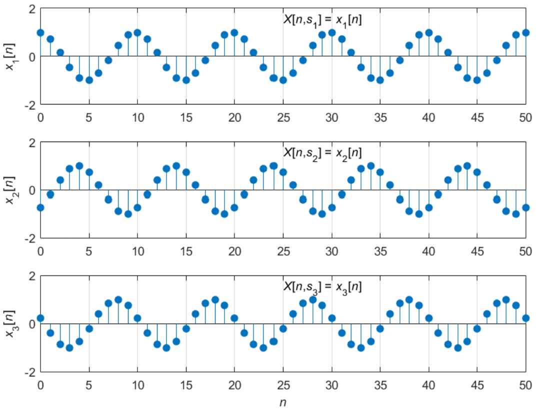



```
Matlab
nfigs = 4;
N = 51
n=(0:N-1);
x=sin(2*pi*0.1*n);

for i=1:nfigs,
   subplot(nfigs,1,i)
   w = 0.3*randn(1,N);
   stem(n,x+w),
end
```

Stochastic process…

- Sinusoid with random phase: $X(n) = \cos(2\pi f n + \Theta)$, $\Theta \sim \mathrm{U}[0,2\pi]$



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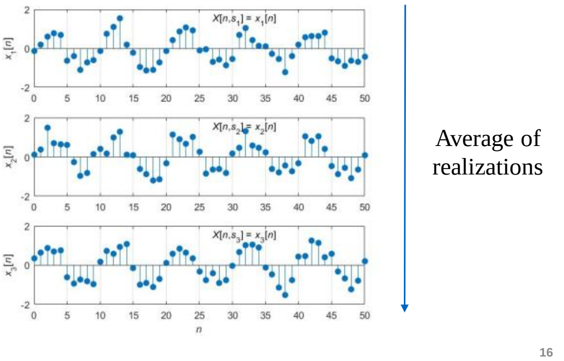
Statistical ensemble averages

- Definition: Mean of a stochastic process is the average of all realizations of the process

$$m_X[n] = E\{X[n]\} = \int_{-\infty}^{\infty} x p_{X[n]}(x) dx$$



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Statistical ensemble averages…

- Definition: Autocorrelation sequence of a stochastic process is the average product of a signal realization with a time-shifted version of itself

$$\gamma_{XX}(n, n+l) = E\{X[n]X[n+l]\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p_{X[n]X[n+l]}(x_1 x_2) dx_1 dx_2$$

- Measure of temporal similarity of a single stochastic process
- Related autocovariance sequence:

$$c_{XX}(n, n+l) = E\{(X[n] - m_X[n])(X[n+l] - m_X[n+l]\}$$
$$= \gamma_{XX}(n, n+l) \ - m_X[n]m_X[n+l]$$

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Statistical ensemble averages…

- Crosscorrelation sequence:

$$\gamma_{XY}(n, n+l) = E\{X[n]Y[n+l]\}$$

- Crosscovariance sequence:

$$c_{XY}(n, n+l) = E\{(X[n] - m_X[n])(Y[n+l] - m_Y[n+l]\}$$
$$= \gamma_{XY}(n, n+l) \ - m_X[n]m_Y[n+l]$$

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Statistical ensemble averages…

- Example: $X(n) = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$

  Calculate mean and covariance sequences

$$\mu_X[n] = \mathrm{E}[X(n)] = \mathrm{E}[\cos(2\pi f n + \Theta)]$$
$$= \int_0^{2\pi} \cos(2\pi f n + \theta) \frac{1}{2\pi} d\theta$$
$$= \frac{1}{2\pi} \sin(2\pi f n + \theta)|_{\theta=0}^{2\pi} = 0$$

- Mean is constant for all *n*

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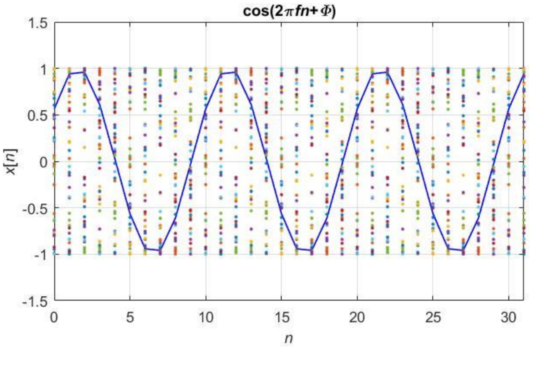
Statistical ensemble averages…

- Example: $X(n) = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$

  50 realizations



Matlab
```
n     = 0:31;
nreal = 50;
f     = 0.1;
zeros(nreal,length(n));

for i = 1:nreal
  phi = 2*pi*rand;
  x(i,:) =cos(2*pi*f*n+phi);
end

figure
plot(n,x(1,:)), grid
hold on
for i=2:nreal
    plot(n,x(i,:),'.')
end
```

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Statistical ensemble averages…

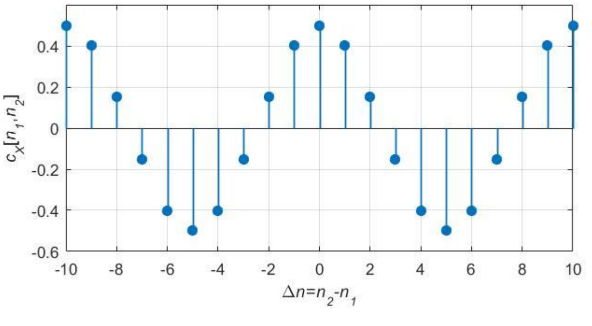
- Example: $X[n] = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$
  Calculate mean and covariance sequences

$$c_X[n, n+l] = \mathrm{E}\big[X[n]X[n+l]\big]$$
$$= \int_0^{2\pi} \cos(2\pi f n + \theta)\cos(2\pi f[n+l] + \theta)\,\frac{1}{2\pi}\,d\theta$$
$$= \int_0^{2\pi} \left\{\frac{1}{2}\cos(2\pi f l\,) + \frac{1}{2}\cos(2\pi f[2n+l] + 2\theta)\right\}\frac{1}{2\pi}\,d\theta$$
$$= \frac{1}{2}\cos(2\pi f l) + \frac{1}{8\pi}\sin(2\pi f[2n+l] + 2\theta)\Big|_{\theta=0}^{2\pi}$$
$$= \frac{1}{2}\cos(2\pi f l)$$

- Covariance sequence only depends on time difference l

Statistical ensemble averages…

- Example: $X[n] = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$
- Covariance sequence



- Covariance sequence only depends on $l = |n_2 - n_1|$

Stationarity

- A random process is said to be stationary in the strict sense if the statistical properties do not change over time
  $\Rightarrow$ Joint density function $p_{X[n_1],\dots,X[n_L]} = p_{X[n_1+l],\dots,X[n_L+l]}$
- Set of samples can be shifted in time, *with each one being shifted by the same amount*, without affecting the joint PDF
- Weakly stationary (or wide-sense) process:

$$m_X[n] = m_X \text{ (a constant independent of } n\text{)}$$

$$\gamma_{XX}[n, n+l] = \gamma_{XX}[l] = \gamma_{XX}[-l] \text{ (depends only on shift } l\text{)}$$

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Stationarity…

- Important example is white Gaussian noise (WGN) process
  - $W[n]$ are independent and zero-mean
  - Gaussian density function: $p_W(w) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{w^2}{2\sigma_W^2}}$

$$m_W = E\{W[n]\} = \int_{-\infty}^{\infty} w p_W(w) dw = 0$$

$$\sigma_W^2 = E\{W^2[n]\}$$

$$\gamma_{WW}[n, n+l] = E\{W[n]W[n+l]\} = \sigma_W^2 \delta[l]$$

- Samples are uncorrelated
- Is the GWN process wide-sense stationary? (Yes/No)

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Ergodicity

- Random process characterized in terms of statistical averages
- In practice we observe data from a single realization
- Definition: An ergodic process is one where time averages are equal to ensemble averages

$\Longrightarrow$ We can estimate the parameters of a stationary random process through measurements

- Mean-ergodic process:

$$m_X = E\{X[n]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x[n]$$

- Correlation-ergodic process:

$$\gamma_{XX}[l] = E\{X[n]X[n+l]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x[n]x[n+l]$$

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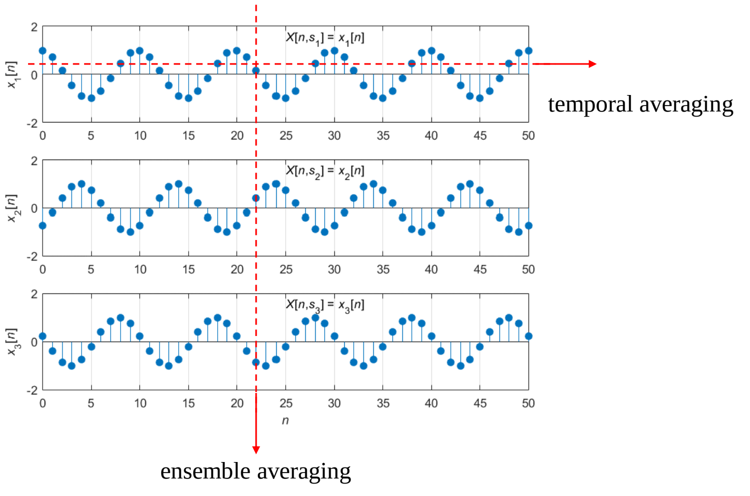
Ergodicity…



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Ergodicity…

- Revisit the example: $X[n] = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$
- Time average mean of single realization $x[n]$ of $X[n]$

$$\hat{m}_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos(2\pi f n + \theta)$$

$$= 0 \; (= m_X)$$

- Time average same as ensemble average

$\Longrightarrow X[n]$ is mean-ergodic

Ergodicity…

- Revisit the example: $X[n] = \cos(2\pi f n + \Theta)$ with $\Theta \sim \mathrm{U}[0,2\pi]$
- Time average autocorrelation of single realization $x[n]$ of $X[n]$

$$\hat{\gamma}_{xx}[l] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos(2\pi f n + \theta)\cos(2\pi f[n+l] + \theta)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2}\{\cos(2\pi f l) + \cos(2\pi f[2n+l] + 2\theta)\}$$

$$= \frac{1}{2}\cos(2\pi f l) \;\; (= \gamma_{XX}[l])$$

- Time average same as ensemble average

$\Longrightarrow X[n]$ is correlation-ergodic

Matlab
```
x = cos(2*pi*0.1*(0:10000)+2*pi*rand);
autocorr(x)
```

Power density spectrum

- For the rest of the course we assume wide-sense stationary processes that are both mean-ergodic and correlation-ergodic

- A stationary stochastic process is an infinite-energy signal $\Longrightarrow$ its Fourier transform does not exist
- How to measure frequency content in a random signal?
- Autocorrelation sequence measures similarity in time domain

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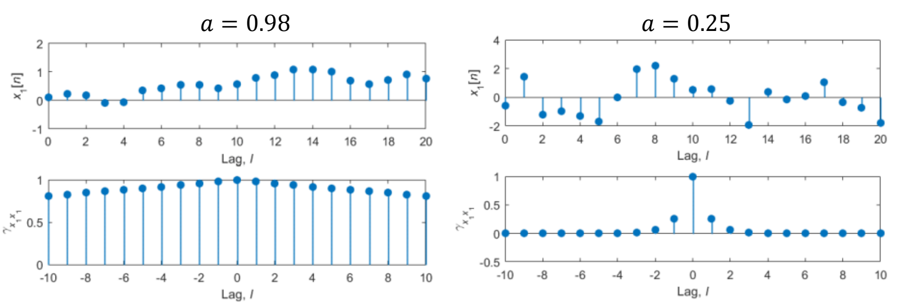
Power density spectrum…

- Example: $X[n] = aX[n-1] + W[n]$, $W[n] \sim N(0, \sigma_w^2)$



- Autocorrelation sequence related to the rate of change
  - Realization varies slowly, $\gamma_{XX}[l]$ decays slowly
  - Realization varies rapidly, $\gamma_{XX}[l]$ decays rapidly

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Power density spectrum…

- Autocorrelation sequence $\gamma_{XX}[l]$ reflects variability (frequency content) of random process
- We define the Fourier transform pair

$$\Gamma_{XX}(f) = \sum_{l=-\infty}^{\infty} \gamma_{XX}[l] e^{-j2\pi fl}$$

$$\gamma_{XX}[l] = \int_{-0.5}^{0.5} \Gamma_{XX}(f)\, e^{j2\pi fl} df$$

- Power density spectrum $\Gamma_{XX}(f)$ represents $\gamma_{XX}[l]$ in frequency
- Name power density spectrum comes from relation

$$P_X = E\{X^2[n]\} = \gamma_{XX}[0] = \int_{-0.5}^{0.5} \Gamma_{XX}(f)\, df$$

Power density spectrum…

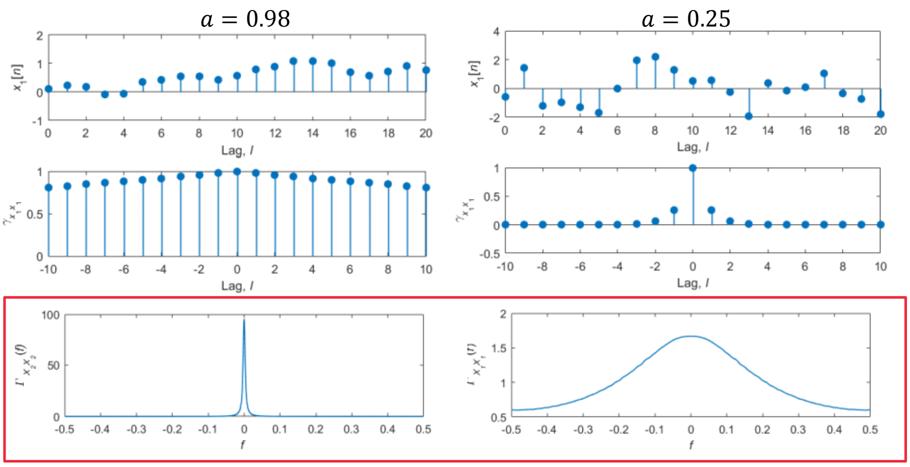
- Revisiting the case of white noise sequence $W[n]$
  - Zero mean: $m_W = 0$
  - Uncorrelated samples: $\gamma_{WW}[l] = \sigma_W^2 \delta[l]$

$$\Longrightarrow \Gamma_{WW}(f) = \sum_{l=-\infty}^{\infty} \gamma_{XX}[l] e^{-j2\pi fl} = \sigma_W^2 \text{ (constant } \forall f)$$

- Contains all frequencies (frequency-flat), hence the name white

Power density spectrum…

- Revisiting example: $X[n] = aX[n-1] + W[n]$, $W[n] \sim N(0, \sigma_w^2)$



Power density spectrum…

- Power density spectrum (PDS)
  - Frequency-domain interpretation of random signals
  - Information on how signal power is distributed in frequency
  - Fourier transform of the auto-correlation sequence
- Autocorrelation sequence (ACS)
  - Information of self-similarity of random signals in time-domain
  - Slow decay $\Longrightarrow$ most power is concentrated at low frequencies
  - Fast decay $\Longrightarrow$ power in high-frequency components
  - Inverse Fourier transform of the power density spectrum

Summary

- Today we discussed:
  - Stochastic processes and their statistical averages
  - Stationarity and wide-sense stationarity
  - Ergodicity
  - Power density spectrum

- Next time:
  - Filtering of stochastic processes (LTI systems)