TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 5

Hand-out time:	Monday, October 14, 2013, at 12:00
Hand-in deadline:	Friday, October 25, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Minimal realizations and state estimators

Consider a system with the following state-space equation:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t), \end{aligned} \tag{1}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = 1.$$

The corresponding block diagram of the system is given in Fig. 1

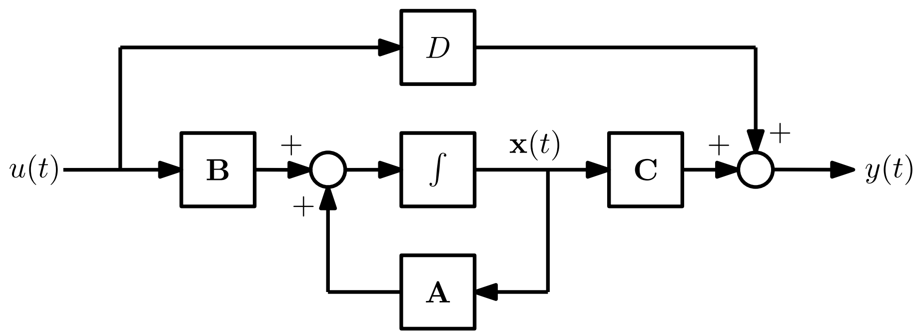

Fig. 1: Block diagram of the system.

a) Compute the transfer function $\hat{G}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ of the system in (1).

b) Use your answer in a) to determine if (1) is a minimal realization. Motivate your answer.

c) Is it possible to conclude from your answer in b) if the the system in (1) is observable. Motivate your answer.

d) Check if the system in (1) is observable by computing the observability matrix. Motivate your answer.

Consider the following state estimator for the system:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \hat{y}(t)),\\ \hat{y}(t) &= \mathbf{C}\hat{\mathbf{x}}(t) + Du(t),\end{aligned}$$

where $\mathbf{L}$ is a gain matrix that will be determined later.

e) Similar to Fig. 1, draw a block diagram of the system with state estimator.

f) Define the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ and show that $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t)$.

g) Determine the estimator-gain matrix $\mathbf{L}$ such that the poles of the state estimator (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$) are equal to -8 and -7, respectively.

We consider the feedback controller

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Pr(t),$$

where $r(t)$ is the reference for the output and the matrices $\mathbf{K}$ and P are given by

$$\mathbf{K} = \begin{bmatrix} 5 & -2 \end{bmatrix} \qquad \text{and} \qquad P = -3.$$

The system with controller and observer can be written in the form

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{E}\mathbf{z}(t) + \mathbf{F}r(t)\\ y(t) &= \mathbf{G}\mathbf{z}(t) + Hr(t),\end{aligned} \tag{2}$$

where the state is given by

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}.$$

h) Compute the matrices $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and H in (2) as a function of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D, $\mathbf{K}$, $\mathbf{L}$ and P. Moreover, calculate the values of $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and H by substituting the values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D, $\mathbf{K}$, $\mathbf{L}$ and P in the obtained expressions.

i) Is the system in (2) marginally stable, asymptotically stable, exponentially stable and/or unstable? Motivate your answer.

Problem 2: Process classification

Consider the following process:

$$X(t) = a\sin(\omega t + \Phi),$$

where a and ω are constants and the variable Φ is uniformly distributed in the interval $[-\pi, \pi]$ (i.e. $\Phi \sim \mathcal{U}(-\pi, \pi)$).

a) Calculate the mean $\mu_X(t) = E[X(t)]$.

b) Calculate the variance $\sigma_X^2(t) = E[X^2(t)]$.

c) Calculate the autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$. Write the autocorrelation function as $R_X(\tau) = E[X(t)X(t+\tau)]$ if possible.

d) Is the process deterministic? Motivate your answer.

e) Is the process wide-sense stationary? Motivate your answer.

f) Is the process ergodic (in wide sense)? Motivate your answer.

Problem 3: Linear system with white input noise

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}w(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned}$$

with state $\mathbf{x}(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The disturbance $w(t)$ is a white noise process with autocorrelation function

$$R_w(\tau) = 4\delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. Assume zero initial conditions for the state $\mathbf{x}(t)$ (i.e. $\mathbf{x}(0) = \mathbf{0}$).

a) Calculate the mean μ_w of the disturbance $w(t)$.

b) Calculate the variance σ_w^2 of the disturbance $w(t)$.

c) Calculate the power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$.

d) Calculate the transfer function $g(s) = \frac{y(s)}{w(s)}$.

Note that the transfer function can be written as $g(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, where λ_1 and λ_2 are the poles of the system and α_1 and α_2 are constants.

e) Calculate the impulse response of the system $g(t) = \mathcal{L}^{-1}\{g(s)\}$, where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Note that the output of the system is given by $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$.

f) Calculate the stationary mean $\bar{\mu}_y$ of the output $y(t)$ (i.e. $\bar{\mu}_y = \lim_{t\to\infty} \mu_y(t)$).

g) Calculate the stationary variance $\bar{\sigma}_y^2$ of the output $y(t)$ (i.e. $\bar{\sigma}_y^2 = \lim_{t\to\infty} \sigma_y^2(t)$).

h) Calculate the power spectral density function $S_y(j\omega)$ of the output $y(t)$.