Modeling and Control of Robots

Lecture 1: Kinematics of a Point

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Learning outcomes: Formulas for velocity and acceleration of a point in curvilinear coordinates. Moving frame for a curve in $\mathbb{R}^3$. Frenet-Serret formulas for velocity and acceleration of a point.

Outline

1. Notations

2. Kinematics of a Point in $\mathbb{R}^3$
   - Velocity and Acceleration of a Point in Curvilinear Coordinates
   - Frenet–Serret formulas for Computing Point's Velocity/Acceleration

Notations

Basic Objects and Operations

A vector $\vec{b} \in \mathbb{R}^n$ is a column $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

For a $(n \times m)$ matrix $B = [b_{ij}]$ its transpose B^T is $(m \times n)$ matrix $[b_{ji}]$

The inner product of vectors $\vec{a}$, $\vec{b} \in \mathbb{R}^n$ is a number:

$$(\vec{a})^T \vec{b} := a_1 \cdot b_1 + \ldots + a_n \cdot b_n$$

The cross product of vectors $\vec{a}$, $\vec{b} \in \mathbb{R}^3$ is a vector $\vec{a} \times \vec{b} =: \vec{c} \in \mathbb{R}^3$

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} : \quad \begin{array}{rcl} c_1 & = & a_2 \cdot b_3 - a_3 \cdot b_2 \\ c_2 & = & a_3 \cdot b_1 - a_1 \cdot b_3 \\ c_3 & = & a_1 \cdot b_2 - a_2 \cdot b_1 \end{array}$$

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$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} : \quad \begin{aligned} c_1 &= a_2 \cdot b_3 - a_3 \cdot b_2 \\ c_2 &= a_3 \cdot b_1 - a_1 \cdot b_3 \\ c_3 &= a_1 \cdot b_2 - a_2 \cdot b_1 \end{aligned}$$

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$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \quad \Rightarrow \quad B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{bmatrix}$$

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$$c_1 = a_2 \cdot b_3 - a_3 \cdot b_2$$

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$$(\vec{a})^{\mathsf{T}} \vec{b} := a_1 \cdot b_1 + \ldots + a_n \cdot b_n = |\vec{a}| \cdot |\vec{b}| \cdot \cos\left(\widehat{\vec{a}, \vec{b}}\right)$$

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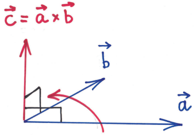

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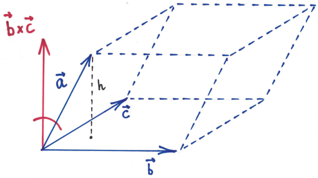
Basic Objects and Operations

The mixed product of vectors $\vec{a},\ \vec{b},\ \vec{c} \in \mathbb{R}^3$ is a number

$$\left[\vec{a},\ \vec{b},\ \vec{c}\right] \ \mapsto\ \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$$

equal to the (signed) volume of a parallelepiped defined by these vectors



The mixed product can be computed as the matrix made of the vectors

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

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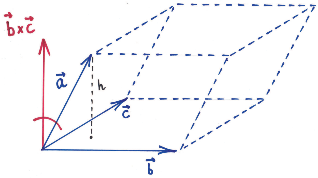

The mixed product is unchanged under a circular shift of its operands

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$$

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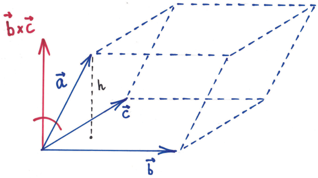

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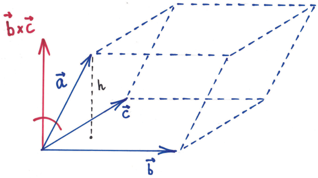

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Basic Objects and Operations

The vector triple product of vectors $\vec{a}$, $\vec{b}$, $\vec{c} \in \mathbb{R}^3$ is equal to

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b} \cdot \varepsilon + \vec{c} \cdot \delta$$

where ε and δ are scalars

Exercise: Compute the vector triple product $\vec{r} \times \left(\vec{\Omega} \times \vec{r}\right)$.

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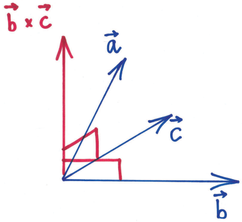

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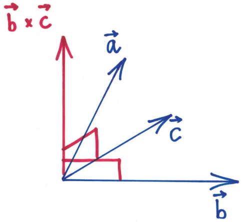

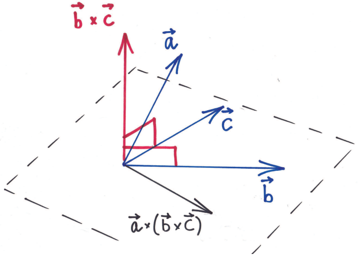

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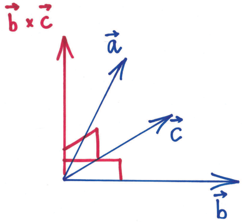

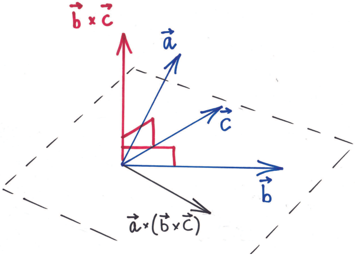

$$\varepsilon = (\vec{a})^T \vec{c}, \qquad \delta = -(\vec{a})^T \vec{b}$$

Exercise: Compute the vector triple product $\vec{r} \times \left(\vec{\Omega} \times \vec{r}\right)$.

Basic Objects and Operations

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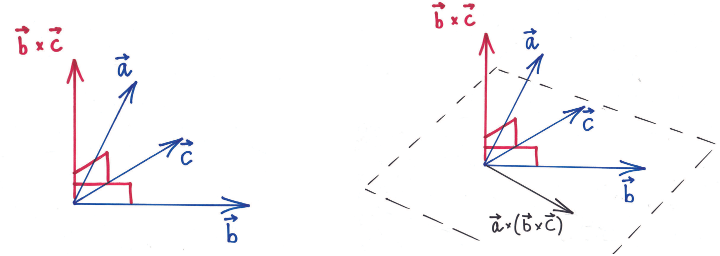

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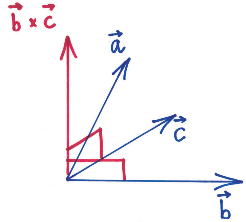

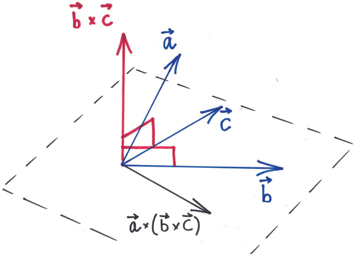

Exercise: Compute the vector triple product $\vec{r} \times \left(\vec{\Omega} \times \vec{r}\right)$.

$$\vec{r} \times \left(\vec{\Omega} \times \vec{r}\right) = \vec{\Omega} \cdot |\vec{r}|^2 - \vec{r} \cdot \left(\vec{\Omega}^T \vec{r}\right)$$

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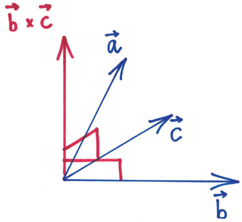

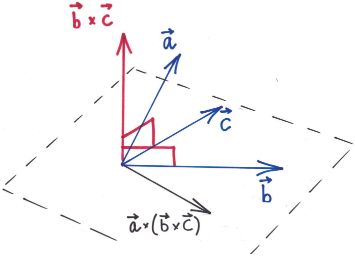

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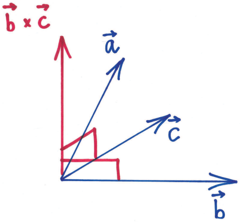

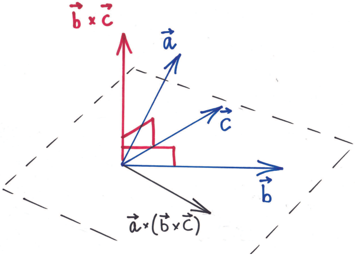

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$$\vec{r} \times \left(\vec{\Omega} \times \vec{r}\right) = \vec{\Omega} \cdot |\vec{r}|^2 - \vec{r} \cdot \left(\vec{\Omega}^T \vec{r}\right) = \vec{\Omega} \cdot |\vec{r}|^2 - \vec{r} \cdot \left(\vec{r}^T \vec{\Omega}\right) = \left(|\vec{r}|^2 \cdot I_3 - \vec{r}\,\vec{r}^T\right) \vec{\Omega}$$

Basic Objects and Operations

Let a smooth function $f(\cdot)$ map a vector of $\mathbb{R}^n$ into a set of real numbers

$$\vec{x} = \left[x_1;\, x_2;\, \ldots;\, x_n\right] \in \mathbb{R}^n \quad \mapsto \quad f(\vec{x}) \in \mathbb{R}^1,$$

the gradient of this function $\mathrm{grad}_x f$ is the vector function

$$\mathrm{grad}_x f = \left[\frac{\partial f}{\partial x_1};\, \frac{\partial f}{\partial x_2};\, \ldots;\, \frac{\partial f}{\partial x_n}\right]$$

Let a smooth function $f(\cdot)$ map a vector of $\mathbb{R}^n$ into a vector of $\mathbb{R}^m$

$$\vec{x} = \left[x_1;\, x_2;\, \ldots;\, x_n\right] \in \mathbb{R}^n \quad \mapsto \quad f(\vec{x}) = \left[f_1(\vec{x});\, f_2(\vec{x});\, \ldots;\, f_m(\vec{x})\right] \in \mathbb{R}^m,$$

the Jacobian matrix $J(\cdot)$ of this vector function is defined as

$$J(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

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the gradient of this function $\mathrm{grad}_x f$ is the vector function

$$\mathrm{grad}_x f = \left[\frac{\partial f}{\partial x_1};\, \frac{\partial f}{\partial x_2};\, \ldots;\, \frac{\partial f}{\partial x_n}\right]$$

Exercise: Compute the gradient of the function $f(\vec{x}) = a_1 x_1 + \cdots + a_n x_n$

Let a smooth function $f(\cdot)$ map a vector of $\mathbb{R}^n$ into a vector of $\mathbb{R}^m$

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$$\mathrm{grad}_x f = \left[a_1;\, a_2;\, \ldots;\, a_n\right]$$

Let a smooth function $f(\cdot)$ map a vector of $\mathbb{R}^n$ into a vector of $\mathbb{R}^m$

$$\vec{x} = \left[x_1;\, x_2;\, \ldots;\, x_n\right] \in \mathbb{R}^n \quad \mapsto \quad f(\vec{x}) = \left[f_1(\vec{x});\, f_2(\vec{x});\, \ldots;\, f_m(\vec{x})\right] \in \mathbb{R}^m,$$

the Jacobian matrix $J(\cdot)$ of this vector function is defined as

$$J(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Basic Objects and Operations

Let a smooth function $f(\cdot)$ map a vector of $\mathbb{R}^n$ into a set of real numbers

$$\vec{x} = \left[x_1;\, x_2;\, \ldots;\, x_n\right] \in \mathbb{R}^n \quad \mapsto \quad f(\vec{x}) \in \mathbb{R}^1,$$

the gradient of this function $\mathrm{grad}_x f$ is the vector function

$$\mathrm{grad}_x f = \left[\frac{\partial f}{\partial x_1};\, \frac{\partial f}{\partial x_2};\, \ldots;\, \frac{\partial f}{\partial x_n}\right]$$

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Basic Objects and Operations

Given a C^1-smooth vector function $\vec{x}(\cdot)$

$$\mathbb{R}^1 \ni t \mapsto \vec{x}(t) = \left[x_1(t);\, x_2(t);\, \ldots;\, x_n(t)\right] \in \mathbb{R}^n$$

and a C^1-smooth scalar function $f(\cdot)$

$$\mathbb{R}^n \times \mathbb{R}^1 \ni \left\{\vec{y},\, t\right\} \mapsto f\left(\vec{y},\, t\right) \in \mathbb{R}^1,$$

the time derivative of their composition $g(t) = f\left(\vec{y},\, t\right)\Big|_{\vec{y}=\vec{x}(t)}$ is then

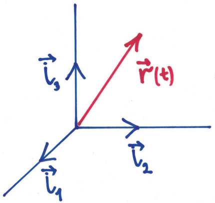
$$\frac{dg(t)}{dt} = \frac{\partial f\left(\vec{y},\, t\right)}{\partial y_1}\Bigg|_{\vec{y}=\vec{x}(t)} \cdot \frac{dx_1}{dt} + \cdots + \frac{\partial f\left(\vec{y},\, t\right)}{\partial y_n}\Bigg|_{\vec{y}=\vec{x}(t)} \cdot \frac{dx_n}{dt} + \frac{\partial f\left(\vec{y},\, t\right)}{\partial t}\Bigg|_{\vec{y}=\vec{x}(t)}$$

Kinematics of a Point in $\mathbb{R}^3$

Curvilinear coordinates in $\mathbb{R}^3$

Let a point of $\mathbb{R}^3$ change its position in time by the rule

$$\vec{r}(t) = \big[x(t);\ y(t);\ z(t)\big] = x(t)\cdot\vec{i_1} + y(t)\cdot\vec{i_2} + z(t)\cdot\vec{i_3}$$



For the coordinates $\big[x;\ y;\ z\big]$ the answer is obvious:

$$\frac{d}{dt}\vec{r}(t) = \Big[\frac{d}{dt}x(t);\ \frac{d}{dt}y(t);\ \frac{d}{dt}z(t)\Big], \qquad \frac{d^2}{dt^2}\vec{r}(t) = \Big[\frac{d^2}{dt^2}x(t);\ \frac{d^2}{dt^2}y(t);\ \frac{d^2}{dt^2}z(t)\Big]$$

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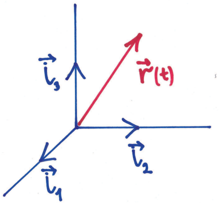

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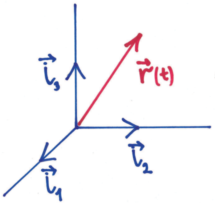

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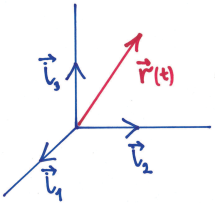

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The answer for other choices of coordinates $\big[q_1;\ q_2;\ q_3\big]$ requires arguments

Curvilinear coordinates in $\mathbb{R}^3$

Consider the case when the behavior of the point is written as

$$\vec{r}(t) = \vec{r}\big(q_1(t);\, q_2(t);\, q_3(t)\big) \in \mathbb{R}^3$$

with $\big[q_1;\, q_2;\, q_3\big]$ being a new set of coordinates instead of $\big[x;\, y;\, z\big]$.

Such change literally means that we can express one set of variables as

$$x = x\big(q_1;\, q_2;\, q_3\big), \quad y = y\big(q_1;\, q_2;\, q_3\big), \quad z = z\big(q_1;\, q_2;\, q_3\big)$$

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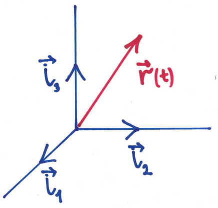

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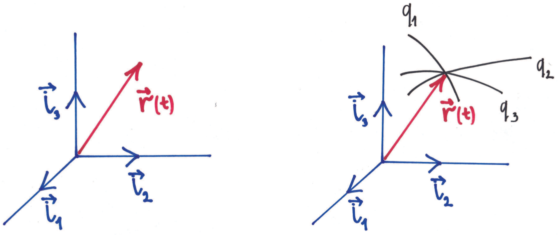

Curvilinear coordinates in $\mathbb{R}^3$

New format of representation of the point behavior

$$\vec{r}(t) = \vec{r}\big(q_1(t);\ q_2(t);\ q_3(t)\big) \in \mathbb{R}^3$$

and the chain rule allow computing the point's velocity as

$$\frac{d}{dt}\vec{r} = \frac{\partial \vec{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}}{\partial q_2}\dot{q}_2 + \frac{\partial \vec{r}}{\partial q_3}\dot{q}_3$$

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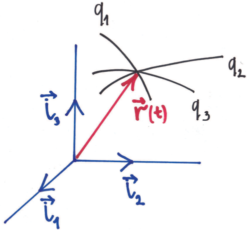

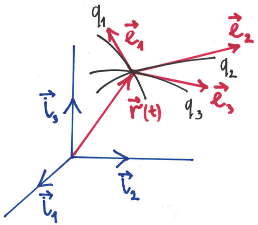

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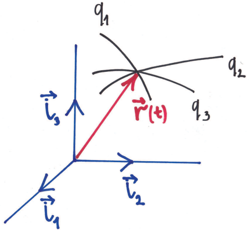

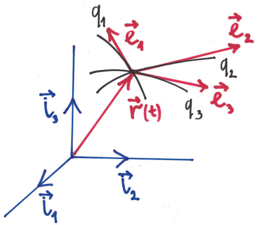

What are the properties of the vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ and the new frame?

Curvilinear coordinates in $\mathbb{R}^3$

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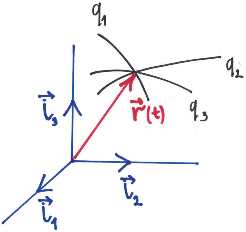

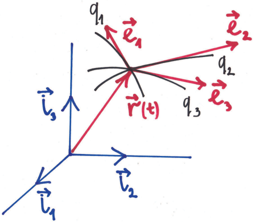

In general, new frame is not orthogonal and the vectors $\vec{e}_i$ are not normalized. The scalars $|\vec{e}_i|$ are known as Lamé coefficients.

Important observation: The velocity of the point (as any other vector) written with help of new frame's axes

$$\dot{\vec{r}} = \dot{q}_1 \cdot \vec{e}_1 + \dot{q}_2 \cdot \vec{e}_2 + \dot{q}_3 \cdot \vec{e}_3$$

can be also characterized by projections of this vector on the axes $\vec{e}_1$-$\vec{e}_3$.

Curvilinear coordinates in $\mathbb{R}^3$

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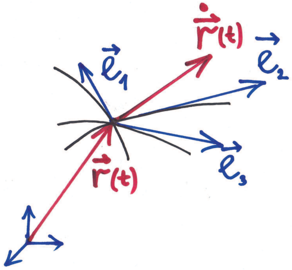

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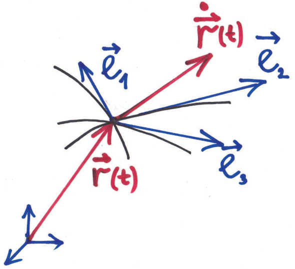

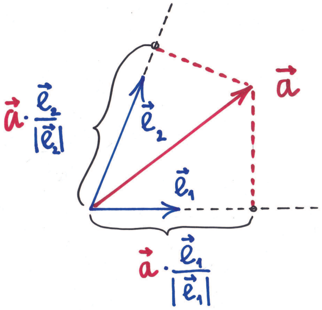

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\frac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\frac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\frac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad \Rightarrow \quad = \vec{e}_1$$

$$= \frac{\partial}{\partial q_1}\left[\frac{\partial \vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial \vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial \vec{r}}{\partial q_1}\right]\cdot\dot{q}_3$$

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&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3
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\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} \quad &= \quad \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3 \\
&= \quad \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3 \\
&= \quad \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right]
\end{aligned}$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad\Rightarrow\quad \boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}} = \vec{e}_1 = \boxed{\frac{\partial\vec{r}}{\partial q_1}}$$

$$\begin{aligned}
\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} &= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right] = \boxed{\frac{\partial}{\partial q_1}\left[\frac{d}{dt}\vec{r}\right]}
\end{aligned}$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad \Rightarrow \quad \boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}} = \vec{e}_1 = \boxed{\frac{\partial\vec{r}}{\partial q_1}}$$

$$\begin{aligned}
\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} &= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3\\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3\\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right] = \boxed{\frac{\partial}{\partial q_1}\left[\frac{d}{dt}\vec{r}\right]}
\end{aligned}$$

$$\frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial r}{\partial q_1}\right) = \left(\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_1}\right) + \left(\dot{\vec{r}}\cdot\frac{d}{dt}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad \Rightarrow \quad \boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}} = \vec{e}_1 = \boxed{\frac{\partial\vec{r}}{\partial q_1}}$$

$$\begin{aligned}
\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} &= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right] = \boxed{\frac{\partial}{\partial q_1}\left[\frac{d}{dt}\vec{r}\right]}
\end{aligned}$$

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 = \left(\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_1}\right) = \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_1}\right) - \left(\dot{\vec{r}}\cdot\frac{d}{dt}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad\Rightarrow\quad \boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}} = \vec{e}_1 = \boxed{\frac{\partial\vec{r}}{\partial q_1}}$$

$$\begin{aligned}
\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} &= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right] = \boxed{\frac{\partial}{\partial q_1}\left[\frac{d}{dt}\vec{r}\right]}
\end{aligned}$$

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 = \left(\ddot{\vec{r}}\cdot\frac{\partial r}{\partial q_1}\right) = \frac{d}{dt}\left(\dot{\vec{r}}\cdot\boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}}\right) - \left(\dot{\vec{r}}\cdot\frac{d}{dt}\left[\frac{\partial r}{\partial q_1}\right]\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Observations:

$$\dot{\vec{r}} = \dot{q}_1\cdot\vec{e}_1 + \dot{q}_2\cdot\vec{e}_2 + \dot{q}_3\cdot\vec{e}_3 \quad\Rightarrow\quad \boxed{\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}} = \vec{e}_1 = \boxed{\frac{\partial\vec{r}}{\partial q_1}}$$

$$\begin{aligned}
\boxed{\frac{d}{dt}\left[\frac{\partial}{\partial q_1}\vec{r}\right]} &= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_2}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_3}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\right]\cdot\dot{q}_1 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_2}\right]\cdot\dot{q}_2 + \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_3}\right]\cdot\dot{q}_3 \\
&= \frac{\partial}{\partial q_1}\left[\frac{\partial\vec{r}}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial\vec{r}}{\partial q_2}\cdot\dot{q}_2 + \frac{\partial\vec{r}}{\partial q_3}\cdot\dot{q}_3\right] = \boxed{\frac{\partial}{\partial q_1}\left[\frac{d}{dt}\vec{r}\right]}
\end{aligned}$$

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 = \left(\ddot{\vec{r}}\cdot\frac{\partial r}{\partial q_1}\right) = \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}\right) - \left(\dot{\vec{r}}\cdot\boxed{\frac{\partial}{\partial q_1}\left[\frac{d\vec{r}}{dt}\right]}\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Finally,

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 \;=\; \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}\right)-\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q_1}\right)$$

Example: Consider a set of new coordinates $[\rho;\ \phi;\ \tilde{z}]$ such that

$$x=\rho\cdot\cos\phi, \quad y=\rho\cdot\sin\phi, \quad z=\tilde{z}.$$

Then

$$\dot{x}=\dot{\rho}\cdot\cos\phi-\rho\cdot\sin\phi\cdot\dot{\phi}, \quad \dot{y}=\dot{\rho}\cdot\sin\phi+\rho\cdot\cos\phi\cdot\dot{\phi}, \quad \dot{z}=\dot{\tilde{z}}$$

$$K=\tfrac{1}{2}\cdot\left(\dot{\rho}^2+\rho^2\cdot\dot{\phi}^2+\dot{\tilde{z}}^2\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Finally,

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 \;=\; \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}\right)-\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q_1}\right) \;=\; \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_1}\right)-\frac{\partial K}{\partial q_1}}$$

where the scalar function $K(\cdot)$ is defined as

$$K=\tfrac{1}{2}\cdot\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right)=\tfrac{1}{2}\cdot\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right).$$

Example: Consider a set of new coordinates $[\rho;\ \phi;\ \tilde{z}]$ such that

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$$\dot{x}=\dot{\rho}\cdot\cos\phi-\rho\cdot\sin\phi\cdot\dot{\phi}, \quad \dot{y}=\dot{\rho}\cdot\sin\phi+\rho\cdot\cos\phi\cdot\dot{\phi}, \quad \dot{z}=\dot{\tilde{z}}$$

$$K=\tfrac{1}{2}\cdot\left(\dot{\rho}^2+\rho^2\cdot\dot{\phi}^2+\dot{\tilde{z}}^2\right)$$

Curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Finally,

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 \;=\; \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}\right) - \left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q_1}\right) \;=\; \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_1}\right) - \frac{\partial K}{\partial q_1}}$$

where the scalar function $K(\cdot)$ is defined as

$$K = \tfrac{1}{2}\cdot\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \tfrac{1}{2}\cdot\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right).$$

Example: Consider a set of new coordinates $[\rho;\,\phi;\,\tilde{z}]$ such that

$$x = \rho\cdot\cos\phi, \quad y = \rho\cdot\sin\phi, \quad z = \tilde{z}.$$

Then

$$\dot{x} = \dot{\rho}\cdot\cos\phi - \rho\cdot\sin\phi\cdot\dot{\phi}, \quad \dot{y} = \dot{\rho}\cdot\sin\phi + \rho\cdot\cos\phi\cdot\dot{\phi}, \quad \dot{z} = \dot{\tilde{z}}$$

$$K = \tfrac{1}{2}\cdot\left(\dot{\rho}^2 + \rho^2\cdot\dot{\phi}^2 + \dot{\tilde{z}}^2\right)$$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

Finally,

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 = \frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_1}\right) - \left(\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q_1}\right) = \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_1}\right) - \frac{\partial K}{\partial q_1}}$$

where the scalar function $K(\cdot)$ is defined as

$$K = \tfrac{1}{2}\cdot\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \tfrac{1}{2}\cdot\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right).$$

Example: Consider a set of new coordinates $[\rho;\ \phi;\ \tilde{z}]$ such that

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Then

$$\dot{x} = \dot{\rho}\cdot\cos\phi - \rho\cdot\sin\phi\cdot\dot{\phi}, \quad \dot{y} = \dot{\rho}\cdot\sin\phi + \rho\cdot\cos\phi\cdot\dot{\phi}, \quad \dot{z} = \dot{\tilde{z}}$$

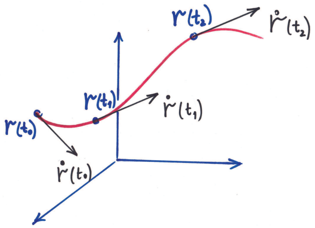
$$K = \tfrac{1}{2}\cdot\left(\dot{\rho}^2 + \rho^2\cdot\dot{\phi}^2 + \dot{\tilde{z}}^2\right)$$

Moving Frame for a Curve in $\mathbb{R}^3$

Given a point's motion

$$\vec{r}(t) = \big[x(t);\, y(t);\, z(t)\big] \in \mathbb{R}^3, \quad t \in [t_0, T]$$

it can be locally seen as a curve in $\mathbb{R}^3$ equipped with a velocity vector assigned to each position on such curve.



The "natural" parametrization of such motion is then its representation

$$\vec{r} = \vec{r}(s(t)) = \big[x(s(t));\, y(s(t));\, z(s(t))\big]$$

where $s(\cdot)$ is the distance along the curve from the initial point at $t = t_0$

Since the distance between two neighboring points is approximated as

$$\big[s(t+dt) - s(t)\big]^2 \approx \big[\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2\big]\, dt^2,$$

then

$$s(t) = \int_{t_0}^{t} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt$$

Moving Frame for a Curve in $\mathbb{R}^3$

Given a point's motion

$$\vec{r}(t) = \big[x(t);\, y(t);\, z(t)\big] \in \mathbb{R}^3, \quad t \in [t_0, T]$$

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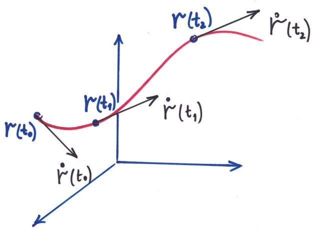


The "natural" parametrization of such motion is then its representation

$$\vec{r} = \vec{r}\big(s(t)\big) = \Big[x\big(s(t)\big);\, y\big(s(t)\big);\, z\big(s(t)\big)\Big]$$

where $s(\cdot)$ is the distance along the curve from the initial point at $t = t_0$

Since the distance between two neighboring points is approximated as

$$\big[s(t+dt) - s(t)\big]^2 \approx \big[\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2\big]\, dt^2,$$

then

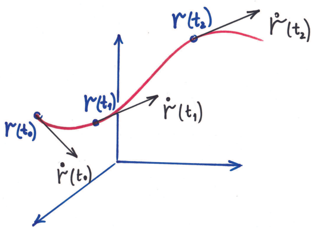
$$s(t) = \int_{t_0}^{t} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt$$

Moving Frame for a Curve in $\mathbb{R}^3$

Given a point's motion

$$\vec{r}(t) = \big[x(t);\ y(t);\ z(t)\big] \in \mathbb{R}^3, \quad t \in [t_0, T]$$

it can be locally seen as a curve in $\mathbb{R}^3$ equipped with a velocity vector assigned to each position on such curve.



The "natural" parametrization of such motion is then its representation

$$\vec{r} = \vec{r}\big(s(t)\big) = \Big[x\big(s(t)\big);\ y\big(s(t)\big);\ z\big(s(t)\big)\Big]$$

where $s(\cdot)$ is the distance along the curve from the initial point at $t = t_0$

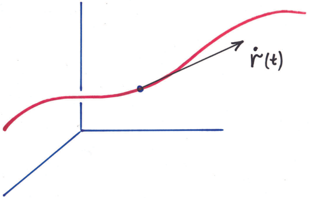
Since the distance between two neighboring points is approximated as

$$\big[s(t+dt) - s(t)\big]^2 \approx \big[\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2\big]\,dt^2,$$

then

$$s(t) = \int_{t_0}^{t} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\,dt$$

Moving Frame for a Curve in $\mathbb{R}^3$



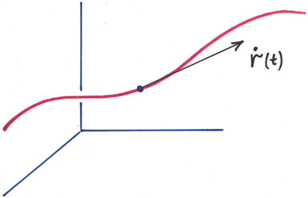
Consider a point $\vec{r}(t)$ on the curve, it has the velocity $\dot{\vec{r}}(t)$

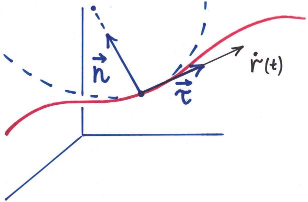
Denote the normalized velocity vector as

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

$$= \quad \frac{d}{dt}\dot{\vec{r}} = \frac{d}{dt}\left[V \cdot \vec{\tau}\right]$$

Moving Frame for a Curve in $\mathbb{R}^3$



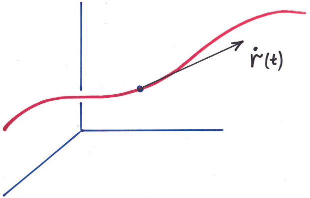


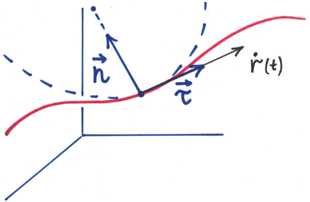
Consider a point $\vec{r}(t)$ on the curve, it has the velocity $\dot{\vec{r}}(t)$

Denote the normalized velocity vector as

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

Moving Frame for a Curve in $\mathbb{R}^3$



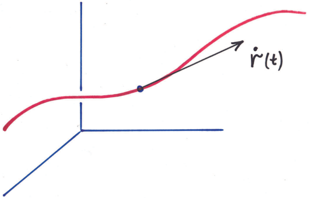


Consider a point $\vec{r}(t)$ on the curve, it has the velocity $\dot{\vec{r}}(t)$

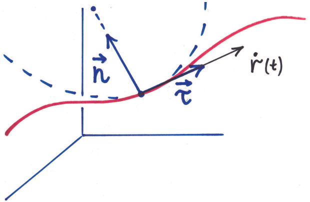
Denote the normalized velocity vector as

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \Rightarrow \boxed{\dot{\vec{r}} = V \cdot \vec{\tau}}, \quad V = |\dot{\vec{r}}|$$

$$= \frac{d}{dt}\dot{\vec{r}} = \frac{d}{dt}\left[V \cdot \vec{\tau}\right]$$



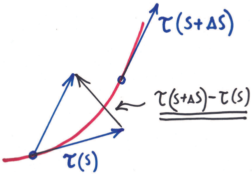
Consider a point $\vec{r}(t)$ on the curve, it has the velocity $\dot{\vec{r}}(t)$



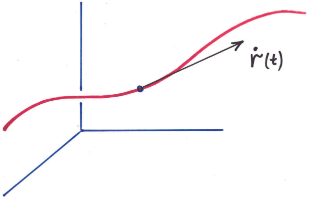
Denote the normalized velocity vector as

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \;\Rightarrow\; \dot{\vec{r}} = V \cdot \vec{\tau}, \quad V = |\dot{\vec{r}}|$$

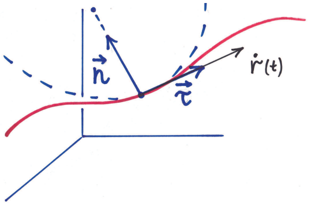
$$\lim_{\triangle s \to 0} \frac{1}{\triangle s}\left[\vec{\tau}(s+\triangle s) - \vec{\tau}(s)\right] = \frac{d}{ds}\vec{\tau} = \frac{1}{\rho}\vec{n}$$



Moving Frame for a Curve in $\mathbb{R}^3$



Consider a point $\vec{r}(t)$ on the curve, it has the velocity $\dot{\vec{r}}(t)$



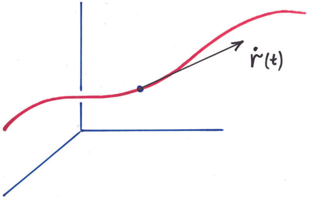
Denote the normalized velocity vector as

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \Rightarrow \boxed{\dot{\vec{r}} = V \cdot \vec{\tau}}, \quad V = |\dot{\vec{r}}|$$

$$\frac{d}{ds}\vec{\tau} = \frac{1}{\rho} \cdot \vec{n} = \kappa \cdot \vec{n}$$

$$\boxed{\ddot{\vec{r}}} \quad = \quad \tfrac{d}{dt}\dot{\vec{r}} = \tfrac{d}{dt}\left[V \cdot \vec{\tau}\right]$$

Moving Frame for a Curve in $\mathbb{R}^3$



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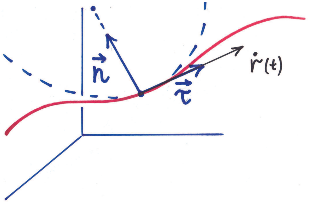

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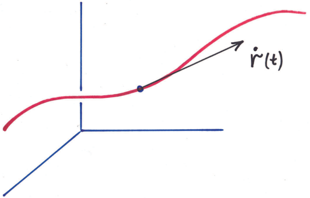
$$\boxed{\ddot{\vec{r}}} \quad = \quad \tfrac{d}{dt}\dot{\vec{r}} = \tfrac{d}{dt}\left[V \cdot \vec{\tau}\right] = \dot{V} \cdot \vec{\tau} + V \cdot \left[\tfrac{d}{dt}\vec{\tau}\right]$$

Moving Frame for a Curve in $\mathbb{R}^3$



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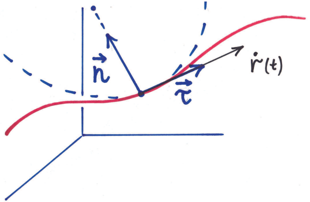

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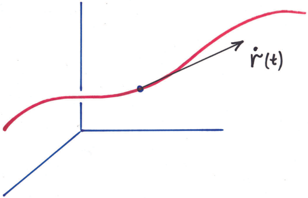

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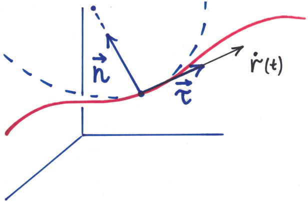

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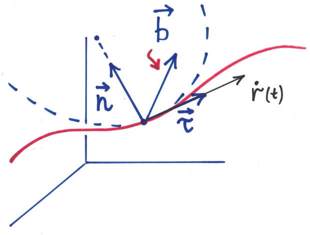
$$\begin{aligned} \boxed{\ddot{\vec{r}}} \;&=\; \tfrac{d}{dt}\dot{\vec{r}} = \tfrac{d}{dt}\left[V \cdot \vec{\tau}\right] = \dot{V} \cdot \vec{\tau} + V \cdot \left[\tfrac{d}{dt}\vec{\tau}\right] = \dot{V} \cdot \vec{\tau} + V \cdot \left[\tfrac{d}{ds}\vec{\tau} \cdot \tfrac{ds}{dt}\right] \\ &=\; \dot{V} \cdot \vec{\tau} + \frac{V^2}{\rho} \cdot \vec{n} = \boxed{\dot{V} \cdot \vec{\tau} + V^2 \cdot \kappa \cdot \vec{n}} \end{aligned}$$



To complete the vectors $\vec{\tau}$ and $\vec{n}$ to the basis of the frame, introduce new vector

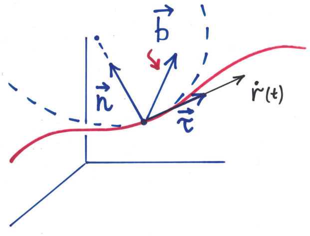
$$\vec{b} := \vec{\tau} \times \vec{n}$$

Frenet–Serret formulas for the basis vectors are

$$\frac{d}{ds}\begin{bmatrix} \vec{\tau} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \delta \\ 0 & -\delta & 0 \end{bmatrix}\begin{bmatrix} \vec{\tau} \\ \vec{n} \\ \vec{b} \end{bmatrix}$$

where κ and δ are known as curvature and torsion of the curve

Frenet–Serret formulas for a Curve in $\mathbb{R}^3$



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Then these vectors $\left[\vec{\tau},\, \vec{n},\, \vec{b}\right]$ are of unit size and pairwise orthogonal. Hence they define the local frame at each point of the curve

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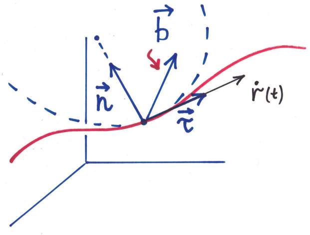

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Then these vectors $\left[\vec{\tau}, \vec{n}, \vec{b}\right]$ are of unit size and pairwise orthogonal. Hence they define the local frame at each point of the curve

---

How do they change if the point moves along the curve?

Frenet–Serret formulas for the basis vectors are

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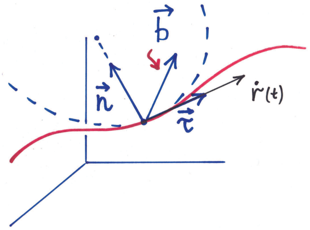
Frenet–Serret formulas for a Curve in $\mathbb{R}^3$



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Modeling and Control of Robots

Lecture 2: Kinematics of a Rigid Body.

Anton Shiriaev

January 12, 2020

Learning outcomes: A concept of rigid body and angular velocity of its motion. Basic properties of angular velocity of a rigid body. Euler, Rivals and Chasles formulas.

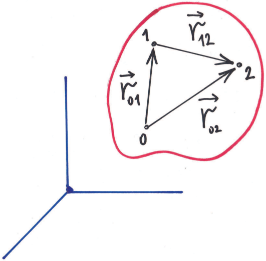
Outline

1. Rotation in $\mathbb{R}^2$
2. Angular velocity of a rigid body motion
3. Euler formula and kinematic invariants
4. Rivals formula
5. Screw motion of a rigid body

Rigid Body

A rigid body is a system of points such that along any of its motion

- the distance between any two points of the system is preserved

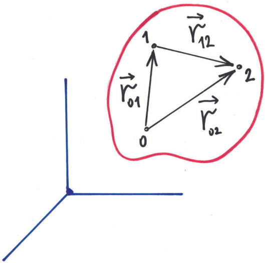


$$\begin{aligned} |\vec{r}_{01}(t)| &\equiv const \\ |\vec{r}_{02}(t)| &\equiv const \qquad \forall\, t \\ |\vec{r}_{12}(t)| &\equiv const \end{aligned}$$

Rigid Body

A rigid body is a system of points such that along any of its motion

- the distance between any two points of the system is preserved



$$
\begin{aligned}
|\vec{r}_{01}(t)| &\equiv const \\
|\vec{r}_{02}(t)| &\equiv const \qquad \forall\, t \\
|\vec{r}_{12}(t)| &\equiv const
\end{aligned}
$$

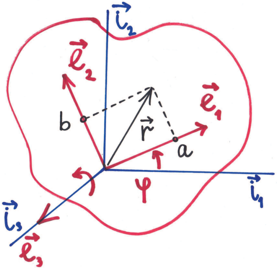
- the scalar product of any two vectors connecting points of the system is preserved, e.g.

$$\vec{r}_{02}(t) \cdot \vec{r}_{01}(t) \equiv const \qquad \forall\, t$$

Rotation in $\mathbb{R}^2$

Rotation in $\mathbb{R}^2$

Consider the simplest motion of rigid body: rotation around a fixed axis



- $[\vec{i}_1,\ \vec{i}_2,\ \vec{i}_3]$ is the basis of the world frame
- $[\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3]$ is the basis of the body frame
- the origins of both frames coincide
- the basis vectors $\vec{i}_3$ and $\vec{e}_3$ coincide
- the rigid body rotates around $\vec{i}_3$ by angle $\phi(t)$
- $\vec{r}$ do not change in the body frame

If the angle $\phi(\cdot)$ varies with time, then the vector

varies as well. Indeed, basis vectors are changing

$$\vec{e}_1(t) = \begin{bmatrix} \cos\phi(t) \\ \sin\phi(t) \end{bmatrix}, \quad \vec{e}_2(t) = \begin{bmatrix} -\sin\phi(t) \\ \cos\phi(t) \end{bmatrix}$$

Rotation in $\mathbb{R}^2$

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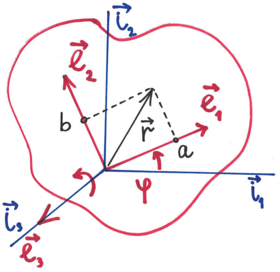

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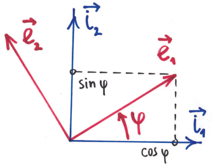

If the angle $\phi(\cdot)$ varies with time, then the vector

$$\boxed{\vec{r}(t) = a \cdot \vec{e}_1(t) + b \cdot \vec{e}_2(t)}$$

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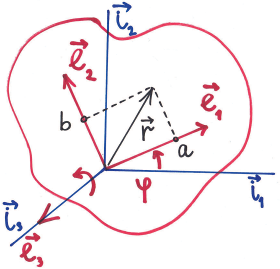

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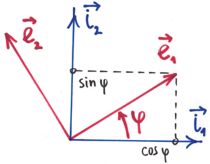

If the angle $\phi(\cdot)$ varies with time, then the vector

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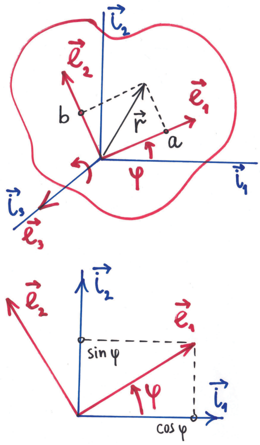
How to compute $\frac{d}{dt}\vec{r}$ and $\frac{d^2}{dt^2}\vec{r}$?

Rotation in $\mathbb{R}^2$



Given the relations

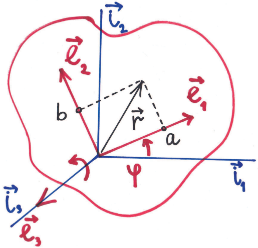
$$\vec{r}(t) = a \cdot \vec{e}_1(t) + b \cdot \vec{e}_2(t)$$

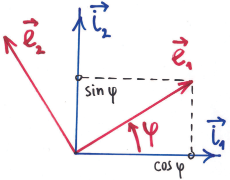
$$\vec{e}_1(t) = \begin{bmatrix} \cos\phi(t) \\ \sin\phi(t) \end{bmatrix}, \quad \vec{e}_2(t) = \begin{bmatrix} -\sin\phi(t) \\ \cos\phi(t) \end{bmatrix}$$

the velocity of the point is by definition

$$\frac{d}{dt}\vec{r}(t) = a \cdot \frac{d}{dt}\vec{e}_1(t) + b \cdot \frac{d}{dt}\vec{e}_2(t)$$

Rotation in $\mathbb{R}^2$





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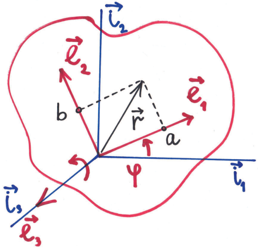
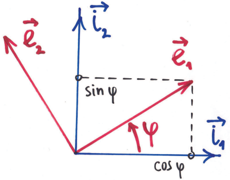
$$
\begin{aligned}
\vec{r}(t) &= a \cdot \vec{e}_1(t) + b \cdot \vec{e}_2(t) \\
\vec{e}_1(t) &= \begin{bmatrix} \cos\phi(t) \\ \sin\phi(t) \end{bmatrix}, \quad \vec{e}_2(t) = \begin{bmatrix} -\sin\phi(t) \\ \cos\phi(t) \end{bmatrix}
\end{aligned}
$$

the velocity of the point is by definition

$$
\begin{aligned}
\frac{d}{dt}\vec{r}(t) &= a \cdot \frac{d}{dt}\vec{e}_1(t) + b \cdot \frac{d}{dt}\vec{e}_2(t) \\
&= a \cdot \begin{bmatrix} -\sin\phi(t) \\ \cos\phi(t) \end{bmatrix} \cdot \dot{\phi}(t) \\
&\qquad + b \cdot \begin{bmatrix} -\cos\phi(t) \\ -\sin\phi(t) \end{bmatrix} \cdot \dot{\phi}(t)
\end{aligned}
$$

Rotation in $\mathbb{R}^2$





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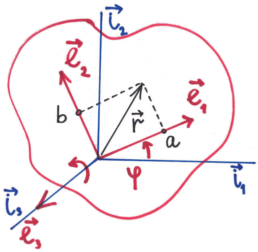
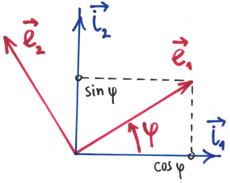
&\qquad + b \cdot \begin{bmatrix} -\cos\phi(t) \\ -\sin\phi(t) \end{bmatrix} \cdot \dot{\phi}(t) \\
&= \Big[a \cdot \vec{e}_2(t) - b \cdot \vec{e}_1(t)\Big] \cdot \dot{\phi}(t)
\end{aligned}
$$

Rotation in $\mathbb{R}^2$





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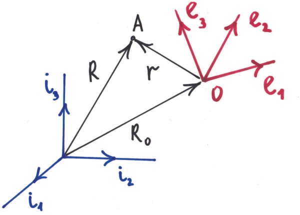
&\qquad + b \cdot \begin{bmatrix} -\cos\phi(t) \\ -\sin\phi(t) \end{bmatrix} \cdot \dot{\phi}(t) \\
&= \Big[a \cdot \vec{e}_2(t) - b \cdot \vec{e}_1(t) \Big] \cdot \dot{\phi}(t) \\
&= \boxed{\dot{\phi}(t) \cdot \vec{e}_3(t) \times \vec{r}(t)}
\end{aligned}
$$

Angular velocity of a rigid body motion



The coordinates of the point A at any time moment can be written in the i-frame as

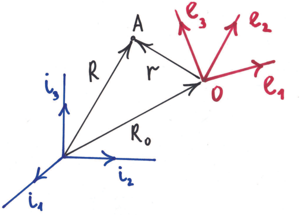
$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$

Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the e-frame

$$\vec{r}(t) \cdot \vec{e}_1(t) \equiv r_1, \qquad \vec{r}(t) \cdot \vec{e}_2(t) \equiv r_2, \qquad \vec{r}(t) \cdot \vec{e}_3(t) \equiv r_3$$

At the same time $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right] \cdot \vec{e}_i(t) + \vec{r}(t) \cdot \frac{d}{dt}\left[\vec{e}_i(t)\right]$

Angular Velocity of a Rigid Body Motion



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$$\begin{aligned}\vec{R}(t) &= \vec{R}_0(t) + \vec{r}(t)\\ &= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)\end{aligned}$$

Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the e-frame

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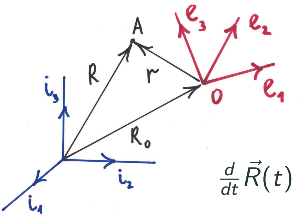
At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right]$

Angular Velocity of a Rigid Body Motion



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$
$$= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$

$$\Downarrow\Downarrow\Downarrow$$

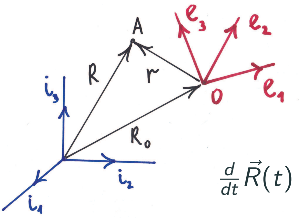
$$\tfrac{d}{dt}\vec{R}(t) = \tfrac{d}{dt}\vec{R}_0(t) + \tfrac{d}{dt}\vec{r}(t)$$
$$= \tfrac{d}{dt}\vec{R}_0(t) + r_1 \cdot \tfrac{d}{dt}\vec{e}_1(t) + r_2 \cdot \tfrac{d}{dt}\vec{e}_2(t) + r_3 \cdot \tfrac{d}{dt}\vec{e}_3(t)$$

Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the e-frame

$$\vec{r}(t)\cdot\vec{e}_1(t) \equiv r_1, \qquad \vec{r}(t)\cdot\vec{e}_2(t) \equiv r_2, \qquad \vec{r}(t)\cdot\vec{e}_3(t) \equiv r_3$$

At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right]$

Angular Velocity of a Rigid Body Motion



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$

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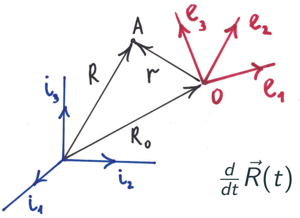
Angular Velocity of a Rigid Body Motion



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$$\Downarrow\Downarrow\Downarrow$$
$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
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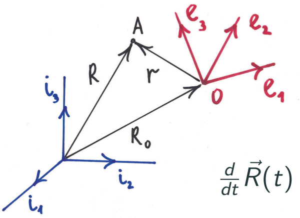
At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right]$

Angular Velocity of a Rigid Body Motion



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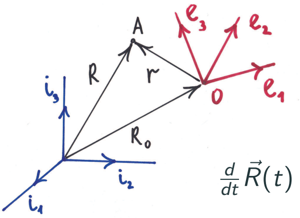
$$= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$
$$\Downarrow\Downarrow\Downarrow$$
$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
$$= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the e-frame

$$\begin{array}{ccccccccc} \vec{r}(t)\cdot\vec{e}_1(t) & \equiv & r_1, & \vec{r}(t)\cdot\vec{e}_2(t) & \equiv & r_2, & \vec{r}(t)\cdot\vec{e}_3(t) & \equiv & r_3 \\ & \Downarrow & & & \Downarrow & & & \Downarrow & \\ \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_1(t)\right] & \equiv & 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_2(t)\right] & \equiv & 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_3(t)\right] & \equiv & 0 \end{array}$$

At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right]$

Angular Velocity of a Rigid Body Motion



The coordinates of the point A at any time moment can be written in the i-frame as

$$\begin{aligned}\vec{R}(t) &= \vec{R}_0(t) + \vec{r}(t) \\ &= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)\end{aligned}$$

$$\Downarrow\Downarrow\Downarrow$$

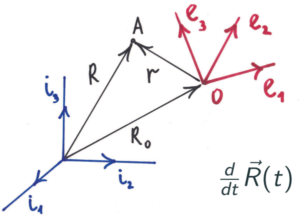
$$\begin{aligned}\frac{d}{dt}\vec{R}(t) &= \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) \\ &= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)\end{aligned}$$

Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the e-frame

$$\begin{array}{ccc} \vec{r}(t)\cdot\vec{e}_1(t) \equiv r_1, & \vec{r}(t)\cdot\vec{e}_2(t) \equiv r_2, & \vec{r}(t)\cdot\vec{e}_3(t) \equiv r_3 \\ \Downarrow & \Downarrow & \Downarrow \\ \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_1(t)\right] \equiv 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_2(t)\right] \equiv 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_3(t)\right] \equiv 0 \end{array}$$

At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right] = 0$

Angular Velocity of a Rigid Body Motion



The coordinates of the point A at any time moment can be written in the *i*-frame as

$$\begin{aligned}\vec{R}(t) &= \vec{R}_0(t) + \vec{r}(t)\\ &= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)\end{aligned}$$

$$\Downarrow\Downarrow\Downarrow$$

$$\begin{aligned}\tfrac{d}{dt}\vec{R}(t) &= \tfrac{d}{dt}\vec{R}_0(t) + \tfrac{d}{dt}\vec{r}(t)\\ &= \tfrac{d}{dt}\vec{R}_0(t) + r_1 \cdot \tfrac{d}{dt}\vec{e}_1(t) + r_2 \cdot \tfrac{d}{dt}\vec{e}_2(t) + r_3 \cdot \tfrac{d}{dt}\vec{e}_3(t)\end{aligned}$$

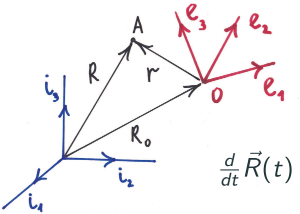
Observation: Coordinates of $\vec{r}(\cdot)$ does not change in the *e*-frame

$$\begin{array}{ccccccccc} \vec{r}(t)\cdot\vec{e}_1(t) & \equiv & r_1, & \vec{r}(t)\cdot\vec{e}_2(t) & \equiv & r_2, & \vec{r}(t)\cdot\vec{e}_3(t) & \equiv & r_3 \\ & \Downarrow & & & \Downarrow & & & \Downarrow & \\ \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_1(t)\right] & \equiv & 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_2(t)\right] & \equiv & 0, & \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_3(t)\right] & \equiv & 0 \end{array}$$

At the same time $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{e}_i(t)\right] = \frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) + \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right] = 0$

$$\Rightarrow \quad \boxed{\frac{d}{dt}\left[\vec{r}(t)\right]\cdot\vec{e}_i(t) \equiv -\vec{r}(t)\cdot\frac{d}{dt}\left[\vec{e}_i(t)\right]} \quad \forall\, t,\ i = 1, 2, 3$$

Angular Velocity of a Rigid Body Motion (cont'd)



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$
$$= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$

$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
$$= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

What are the coordinates $\left[a;\ b;\ c\right]$ of the vector $\frac{d}{dt}\vec{r}(t)$ in the e-frame

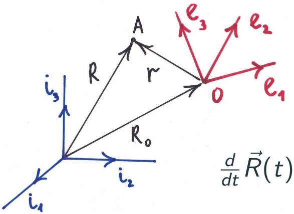
$$\dot{\vec{r}}(t) = a \cdot \vec{e}_1(t) + b \cdot \vec{e}_2(t) + c \cdot \vec{e}_3(t)\,?$$

Since the e-frame is made of orthogonal vectors of length one, then

$$a = \dot{\vec{r}}(t) \cdot \vec{e}_1(t), \quad b = \dot{\vec{r}}(t) \cdot \vec{e}_2(t), \quad c = \dot{\vec{r}}(t) \cdot \vec{e}_3(t)$$

Taking into account $\frac{d}{dt}\left[\vec{r}(t)\right] \cdot \vec{e}_i(t) = -\vec{r}(t) \cdot \frac{d}{dt}\left[\vec{e}_i(t)\right]$, we have

Angular Velocity of a Rigid Body Motion (cont'd)



The coordinates of the point A at any time moment can be written in the i-frame as

$$\begin{aligned}\vec{R}(t) &= \vec{R}_0(t) + \vec{r}(t)\\ &= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\vec{R}(t) &= \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)\\ &= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)\end{aligned}$$

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$$a = \dot{\vec{r}}(t) \cdot \vec{e}_1(t), \quad b = \dot{\vec{r}}(t) \cdot \vec{e}_2(t), \quad c = \dot{\vec{r}}(t) \cdot \vec{e}_3(t)$$

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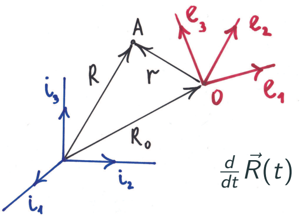
Angular Velocity of a Rigid Body Motion (cont'd)



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$
$$= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$

$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
$$= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

What are the coordinates $\left[a;\ b;\ c\right]$ of the vector $\frac{d}{dt}\vec{r}(t)$ in the e-frame

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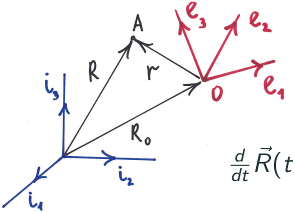
$$a = \dot{\vec{r}}(t) \cdot \vec{e}_1(t), \quad b = \dot{\vec{r}}(t) \cdot \vec{e}_2(t), \quad c = \dot{\vec{r}}(t) \cdot \vec{e}_3(t)$$

Taking into account $\frac{d}{dt}\left[\vec{r}(t)\right] \cdot \vec{e}_i(t) \equiv -\vec{r}(t) \cdot \frac{d}{dt}\left[\vec{e}_i(t)\right]$, we have

$$\frac{d}{dt}\vec{r}(t) = -\left[\vec{r} \cdot \frac{d}{dt}\vec{e}_1(t)\right] \cdot \vec{e}_1(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_2(t)\right] \cdot \vec{e}_2(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_3(t)\right] \cdot \vec{e}_3(t)$$

Angular Velocity of a Rigid Body Motion (cont'd)



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t)$$
$$= \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$

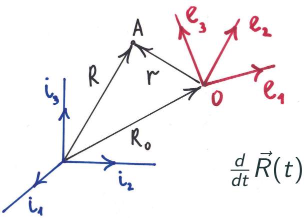
$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
$$= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

$$\frac{d}{dt}\vec{r}(t) = -\left[\vec{r} \cdot \frac{d}{dt}\vec{e}_1(t)\right] \cdot \vec{e}_1(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_2(t)\right] \cdot \vec{e}_2(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_3(t)\right] \cdot \vec{e}_3(t)$$

There is only one function $\vec{\omega}(\cdot)$ such that

Consider the candidate $\vec{\omega} = \frac{1}{2}\left[\vec{e}_1 \times \dot{\vec{e}}_1 + \vec{e}_2 \times \dot{\vec{e}}_2 + \vec{e}_3 \times \dot{\vec{e}}_3\right]$

Angular Velocity of a Rigid Body Motion (cont'd)



The coordinates of the point A at any time moment can be written in the i-frame as

$$\vec{R}(t) = \vec{R}_0(t) + \vec{r}(t) = \vec{R}_0(t) + r_1 \cdot \vec{e}_1(t) + r_2 \cdot \vec{e}_2(t) + r_3 \cdot \vec{e}_3(t)$$

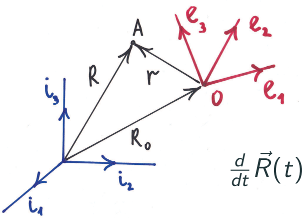
$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

$$\boxed{\frac{d}{dt}\vec{r}(t) = -\left[\vec{r}\cdot\frac{d}{dt}\vec{e}_1(t)\right]\cdot\vec{e}_1(t) - \left[\vec{r}\cdot\frac{d}{dt}\vec{e}_2(t)\right]\cdot\vec{e}_2(t) - \left[\vec{r}\cdot\frac{d}{dt}\vec{e}_3(t)\right]\cdot\vec{e}_3(t)}$$

There is only one function $\vec{\omega}(\cdot)$ such that $\boxed{\frac{d}{dt}\vec{r}(t) = \vec{\omega}(t)\times\vec{r}(t),\ \forall\vec{r}}$

Consider the candidate $\vec{\omega} = \frac{1}{2}\left[\vec{e}_1\times\dot{\vec{e}}_1 + \vec{e}_2\times\dot{\vec{e}}_2 + \vec{e}_3\times\dot{\vec{e}}_3\right]$

Angular Velocity of a Rigid Body Motion (cont'd)



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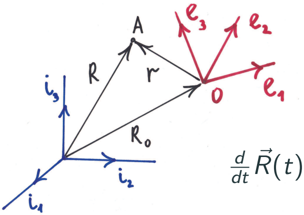
$$\frac{d}{dt}\vec{R}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$
$$= \frac{d}{dt}\vec{R}_0(t) + r_1 \cdot \frac{d}{dt}\vec{e}_1(t) + r_2 \cdot \frac{d}{dt}\vec{e}_2(t) + r_3 \cdot \frac{d}{dt}\vec{e}_3(t)$$

$$\frac{d}{dt}\vec{r}(t) = -\left[\vec{r} \cdot \frac{d}{dt}\vec{e}_1(t)\right] \cdot \vec{e}_1(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_2(t)\right] \cdot \vec{e}_2(t) - \left[\vec{r} \cdot \frac{d}{dt}\vec{e}_3(t)\right] \cdot \vec{e}_3(t)$$

There is only one function $\vec{\omega}(\cdot)$ such that $\frac{d}{dt}\vec{r}(t) = \vec{\omega}(t) \times \vec{r}(t), \ \forall \vec{r}$

Consider the candidate $\vec{\omega} = \frac{1}{2}\left[\vec{e}_1 \times \dot{\vec{e}}_1 + \vec{e}_2 \times \dot{\vec{e}}_2 + \vec{e}_3 \times \dot{\vec{e}}_3\right]$



The coordinates of the point A at any time moment can be written in the i-frame as

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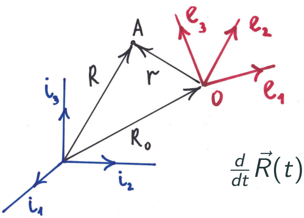
Angular Velocity of a Rigid Body Motion (cont'd)



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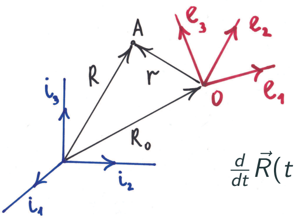

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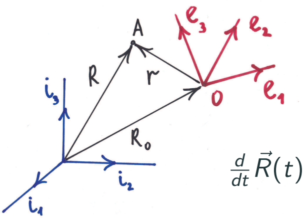
Angular Velocity of a Rigid Body Motion (cont'd)



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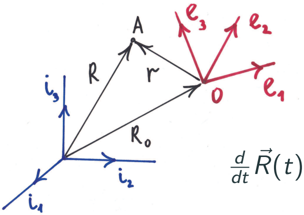
Angular Velocity of a Rigid Body Motion (cont'd)



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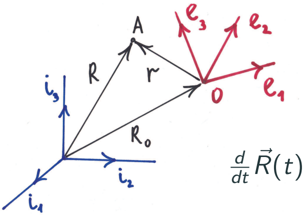
Angular Velocity of a Rigid Body Motion (cont'd)



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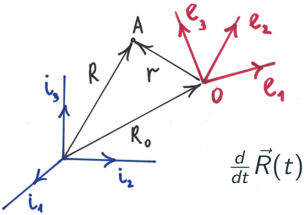
Angular Velocity of a Rigid Body Motion (cont'd)



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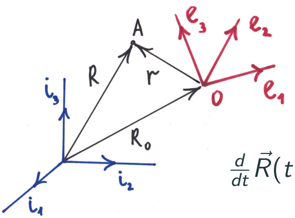
&= -\frac{1}{2}\,\vec{r} \times \left[\vec{e}_1 \times \dot{\vec{e}}_1\right] - \dots \\
&= -\frac{1}{2}\,\vec{e}_1(\vec{r} \cdot \dot{\vec{e}}_1) + \frac{1}{2}\,\dot{\vec{e}}_1(\vec{r} \cdot \vec{e}_1) - \dots = -\frac{1}{2}(\vec{r} \cdot \dot{\vec{e}}_1) \cdot \vec{e}_1 + \frac{1}{2}\,r_1 \cdot \dot{\vec{e}}_1 - \dots \\
&= \frac{d}{dt}\vec{r}
\end{aligned}$$

Angular Velocity of a Rigid Body Motion (cont'd)



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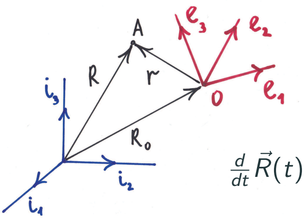

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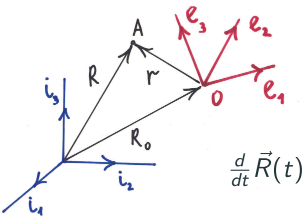
Angular Velocity of a Rigid Body Motion (cont'd)



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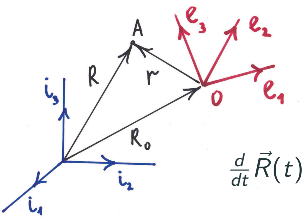
then

$$\forall \vec{r}: \ [\vec{\omega}_1(t) - \vec{\omega}_2(t)] \times \vec{r} \equiv 0$$



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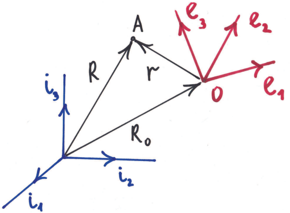
then

$$\forall \vec{r}: \ [\vec{\omega}_1(t) - \vec{\omega}_2(t)] \times \vec{r} \equiv 0 \quad \Rightarrow \quad \vec{\omega}_1(t) = \vec{\omega}_2(t)$$

Euler formula and kinematic invariants

Euler formula



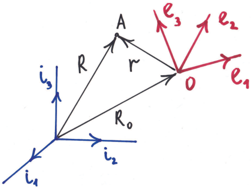
Coordinates of a point A of a rigid body at any time moment can be written in the i-frame as

$$\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)$$

Velocity of the point A of a rigid body in the i-frame is

$$\frac{d}{dt}\vec{R}_A(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}(t) \times \vec{r}(t)$$

Kinematic invariants



Given a motion of a rigid body written as a time evolution of e-frame

$$t \mapsto \left[R_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right],$$

we search for invariants of such motion, which means functions of coordinates and velocities of points of the rigid body that do not change in time

Velocity of the point A of a rigid body in the i-frame is

$$\boxed{V_A(t) = V_0(t) + \vec{\omega}(t) \times \vec{r}(t)}$$

One of invariants is

$$V_A(t) \cdot \vec{r}(t) = \Big(V_0(t) + \vec{\omega}(t) \times \vec{r}(t)\Big) \cdot \vec{r}(t) = V_0(t) \cdot \vec{r}(t)$$

Another invariant is

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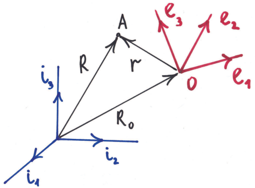

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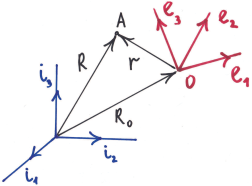
Kinematic invariants



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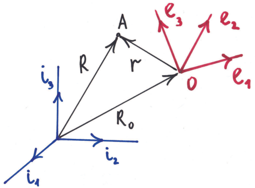
Kinematic invariants



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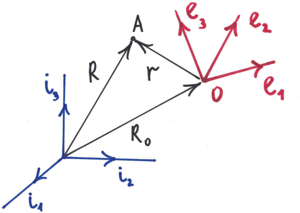
Rivals formula

Rivals formula for acceleration



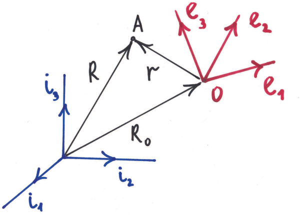
Coordinates of a point A of a rigid body at any time moment can be written in the i-frame as

$$\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)$$

How to compute an acceleration of the point A?

$$= \frac{d^2}{dt^2}\left[\vec{R}_0(t) + \vec{r}(t)\right]$$

Rivals formula for acceleration



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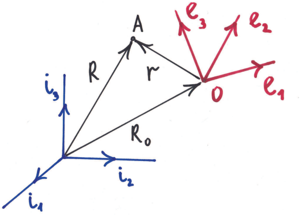
$$\boxed{\frac{d^2}{dt^2}\vec{R}_A(t)} \quad = \quad \frac{d^2}{dt^2}\left[\vec{R}_0(t) + \vec{r}(t)\right]$$

Rivals formula for acceleration



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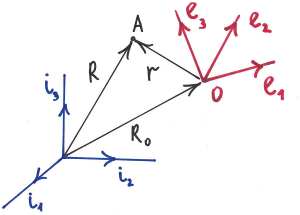
$$\boxed{\frac{d^2}{dt^2}\vec{R}_A(t)} \quad = \quad \frac{d^2}{dt^2}\left[\vec{R}_0(t) + \vec{r}(t)\right] = \frac{d}{dt}\left[\frac{d}{dt}\left[\vec{R}_0(t) + \vec{r}(t)\right]\right]$$



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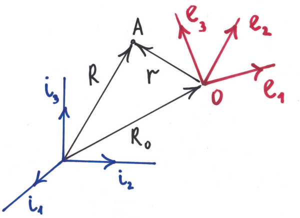
How to compute an acceleration of the point A?

$$\begin{aligned}
\boxed{\tfrac{d^2}{dt^2}\vec{R}_A(t)} \quad &= \quad \tfrac{d^2}{dt^2}\left[\vec{R}_0(t) + \vec{r}(t)\right] = \tfrac{d}{dt}\left[\tfrac{d}{dt}\left[\vec{R}_0(t) + \vec{r}(t)\right]\right] \\
&= \quad \tfrac{d}{dt}\left[\tfrac{d}{dt}\vec{R}_0(t) + \vec{\omega}(t) \times \vec{r}(t)\right]
\end{aligned}$$

Rivals formula for acceleration



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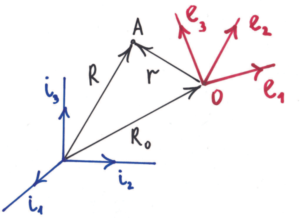
&= \frac{d}{dt}\left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}(t) \times \vec{r}(t)\right] \\
&= \frac{d^2}{dt^2}\vec{R}_0(t) + \frac{d}{dt}\left[\vec{\omega}(t)\right] \times \vec{r}(t) + \vec{\omega}(t) \times \frac{d}{dt}\left[\vec{r}(t)\right]
\end{aligned}$$

Rivals formula for acceleration



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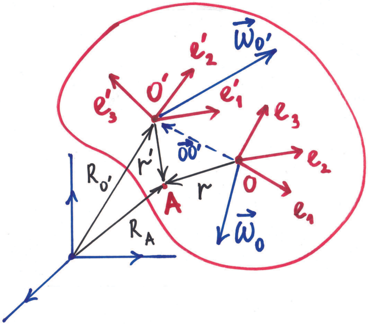
How to compute an acceleration of the point A?

$$\begin{aligned}
\boxed{\tfrac{d^2}{dt^2}\vec{R}_A(t)} &= \tfrac{d^2}{dt^2}\left[\vec{R}_0(t)+\vec{r}(t)\right] = \tfrac{d}{dt}\left[\tfrac{d}{dt}\left[\vec{R}_0(t)+\vec{r}(t)\right]\right] \\
&= \tfrac{d}{dt}\left[\tfrac{d}{dt}\vec{R}_0(t) + \vec{\omega}(t)\times\vec{r}(t)\right] \\
&= \tfrac{d^2}{dt^2}\vec{R}_0(t) + \tfrac{d}{dt}\left[\vec{\omega}(t)\right]\times\vec{r}(t) + \vec{\omega}(t)\times\tfrac{d}{dt}\left[\vec{r}(t)\right] \\
&= \boxed{\tfrac{d^2}{dt^2}\vec{R}_0(t) + \vec{\varepsilon}(t)\times\vec{r}(t) + \vec{\omega}(t)\times\left[\vec{\omega}(t)\times\vec{r}(t)\right]}
\end{aligned}$$

The vector $\vec{\varepsilon} = \frac{d}{dt}\vec{\omega}$ is known as angular acceleration of the rigid body.

Screw motion of a rigid body

Angular velocity and a choice of the body frame



We have seen that for a given body frame the vector of angular velocity is unique.

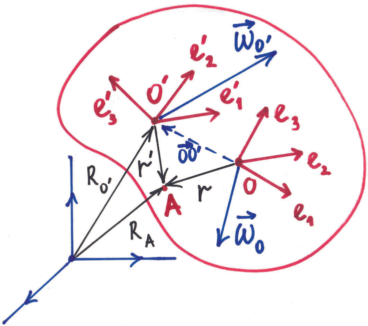
Since the functions $\vec{r}\,'(\cdot)$, $\vec{r}(\cdot)$ and $\vec{R}_0(\cdot)$, $\vec{R}_{0'}(\cdot)$ satisfy the identities

$$\vec{r}\,'(t) = \vec{r}(t) - \overrightarrow{00'}, \qquad \frac{d}{dt}\vec{R}_{0'}(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t) \times \overrightarrow{00'}$$

then

$$\vec{\omega}_0(t) \times \left[\vec{r}(t) - \overrightarrow{00'}\right] = \vec{\omega}_{0'}(t) \times \left[\vec{r}(t) - \overrightarrow{00'}\right]$$

Angular velocity and a choice of the body frame



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Since the functions $\vec{r}(\cdot)$, $\vec{r}'(\cdot)$ and $\vec{R}_0(\cdot)$, $\vec{R}_{0'}(\cdot)$ satisfy the identities

$$\vec{r}'(t) = \vec{r}(t) - \overrightarrow{00'}, \qquad \frac{d}{dt}\vec{R}_{0'}(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t) \times \overrightarrow{00'}$$

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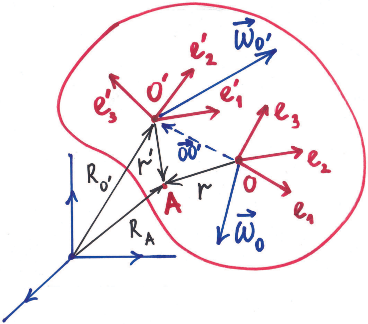

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Given two body frames $\left[\vec{R}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\right]$ and $\left[\vec{R}_{0'}, \vec{e'}_1, \vec{e'}_2, \vec{e'}_3\right]$, the velocity of any point of the body (let say of the point A) can be computed in two different ways

$$\frac{d}{dt}\vec{R}_A(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t)\times\vec{r}(t) \quad \text{and} \quad \frac{d}{dt}\vec{R}_A(t) = \frac{d}{dt}\vec{R}_{0'}(t) + \vec{\omega}_{0'}(t)\times\vec{r'}(t)$$

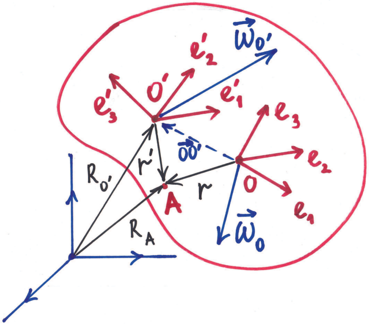
Since the functions $\vec{r}(\cdot)$, $\vec{r'}(\cdot)$ and $\vec{R}_0(\cdot)$, $\vec{R}_{0'}(\cdot)$ satisfy the identities

$$\vec{r'}(t) = \vec{r}(t) - \overrightarrow{00'}, \qquad \frac{d}{dt}\vec{R}_{0'}(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t)\times\overrightarrow{00'}$$

then

$$\vec{\omega}_0(t)\times\left[\vec{r}(t) - \overrightarrow{00'}\right] = \vec{\omega}_{0'}(t)\times\left[\vec{r}(t) - \overrightarrow{00'}\right]$$

Angular velocity and a choice of the body frame



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Since the functions $\vec{r}(\cdot)$, $\vec{r}(\cdot)$ and $\vec{R}_0(\cdot)$, $\vec{R}_{0'}(\cdot)$ satisfy the identities

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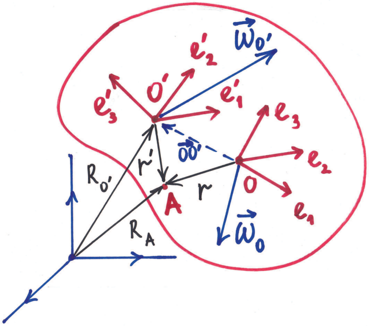
then

$$\tfrac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t)\times\vec{r}(t) \equiv \tfrac{d}{dt}\vec{R}_0(t) + \vec{\omega}_0(t)\times\overrightarrow{00'} + \vec{\omega}_{0'}(t)\times\left[\vec{r}(t) - \overrightarrow{00'}\right]$$

Angular velocity and a choice of the body frame



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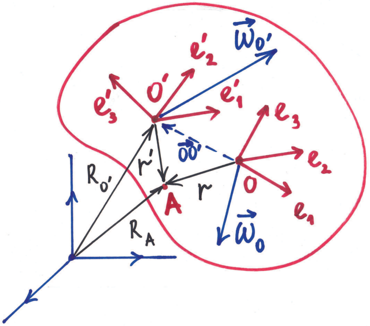
Angular velocity and a choice of the body frame



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Since the functions $\vec{r}(\cdot)$, $\vec{r}(\cdot)$ and $\vec{R}_0(\cdot)$, $\vec{R}_{0'}(\cdot)$ satisfy the identities

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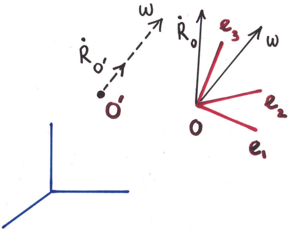
then

$$\vec{\omega}_0(t) \times \left[\vec{r}(t) - \overrightarrow{00'}\right] \equiv \vec{\omega}_{0'}(t) \times \left[\vec{r}(t) - \overrightarrow{00'}\right] \quad \Rightarrow \quad \boxed{\vec{\omega}_0(t) \equiv \vec{\omega}_{0'}(t)}$$

Screw motion of a rigid body. Chasles theorem



Given a body frame $\left[\vec R_0, \vec e_1, \vec e_2, \vec e_3\right]$ and the vectors of body angular velocity $\vec\omega \neq 0$ and of velocity of its pole $\frac{d}{dt}\vec R_0 \neq 0$, can we choose new pole $\vec R_{0'}$ such that the vectors $\vec\omega$, $\frac{d}{dt}\vec R_{0'}$ lie on the same line?

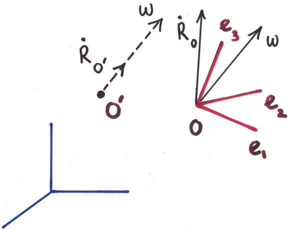
If two vectors $\vec\omega$, $\frac{d}{dt}\vec R_{0'}$ lie on the same line, then $\vec\omega \times \frac{d}{dt}\vec R_{0'} = \vec 0$

Let us search for new pole, i.e the vector $\overrightarrow{00'}$, that solves the task, then

$$\frac{d}{dt}\vec R_{0'} = \frac{d}{dt}\vec R_0 + \vec\omega \times \overrightarrow{00'}, \qquad \overrightarrow{00'} = \vec H + \vec L, \quad \vec H \perp \vec\omega, \quad \vec L \parallel \vec\omega$$

$$\vec 0 \;=\; \vec\omega \times \frac{d}{dt}\vec R_{0'} = \vec\omega \times \frac{d}{dt}\vec R_0 + \vec\omega \times \left[\vec\omega \times \left(\vec H + \vec L\right)\right]$$

Screw motion of a rigid body. Chasles theorem



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If so, then such line is known as a screw axis and the motion is instantaneously combined translation and rotation around this axis

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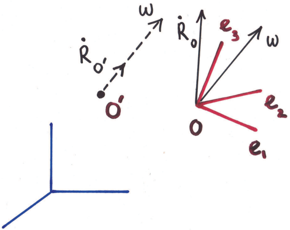

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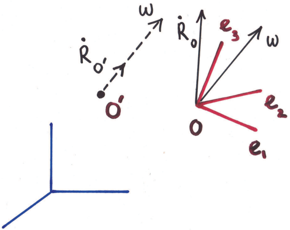
$$\frac{d}{dt}\vec{R}_{0'} = \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \overrightarrow{00'}, \qquad \overrightarrow{00'} = \vec{H} + \vec{L}, \quad \vec{H} \perp \vec{\omega}, \quad \vec{L} \parallel \vec{\omega}$$

$$\vec{0} \;=\; \vec{\omega} \times \frac{d}{dt}\vec{R}_{0'} = \vec{\omega} \times \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \left[\vec{\omega} \times \left(\vec{H} + \vec{L}\right)\right]$$

Screw motion of a rigid body. Chasles theorem



Given a body frame $\left[\vec{R}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\right]$ and the vectors of body angular velocity $\vec{\omega} \neq 0$ and of velocity of its pole $\frac{d}{dt}\vec{R}_0 \neq 0$, can we choose new pole $\vec{R}_{0'}$ such that the vectors $\vec{\omega}$, $\frac{d}{dt}\vec{R}_{0'}$ lie on the same line?

If so, then such line is known as a screw axis and the motion is instantaneously combined translation and rotation around this axis

If two vectors $\vec{\omega}$, $\frac{d}{dt}\vec{R}_{0'}$ lie on the same line, then $\vec{\omega} \times \frac{d}{dt}\vec{R}_{0'} = \vec{0}$

Let us search for new pole, i.e the vector $\overrightarrow{00'}$, that solves the task, then

$$\frac{d}{dt}\vec{R}_{0'} = \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \overrightarrow{00'}, \qquad \overrightarrow{00'} = \vec{H} + \vec{L}, \quad \vec{H} \perp \vec{\omega}, \quad \vec{L} \parallel \vec{\omega}$$

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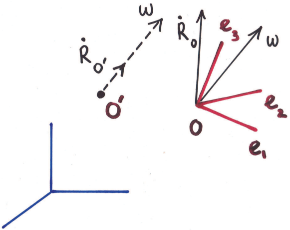
Screw motion of a rigid body. Chasles theorem



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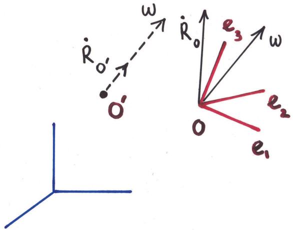
$$\begin{aligned} \vec{0} &= \vec{\omega} \times \frac{d}{dt}\vec{R}_{0'} = \vec{\omega} \times \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \left[\vec{\omega} \times \left(\vec{H} + \vec{L}\right)\right] \\ &= \vec{\omega} \times \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \left[\vec{\omega} \times \vec{H}\right] \end{aligned}$$

Screw motion of a rigid body. Chasles theorem



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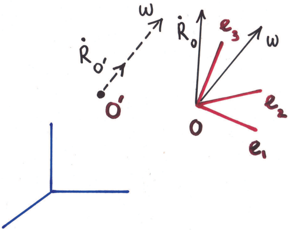
$$\begin{aligned} \vec{0} &= \vec{\omega} \times \frac{d}{dt}\vec{R}_{0'} = \vec{\omega} \times \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \left[\vec{\omega} \times \left(\vec{H} + \vec{L}\right)\right] \\ &= \vec{\omega} \times \frac{d}{dt}\vec{R}_0 + \vec{\omega} \times \left[\vec{\omega} \times \vec{H}\right] \qquad \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \end{aligned}$$

Screw motion of a rigid body. Chasles theorem



Given a body frame $\left[\vec{R}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\right]$ and the vectors of body angular velocity $\vec{\omega} \neq 0$ and of velocity of its pole $\frac{d}{dt}\vec{R}_0 \neq 0$, can we choose new pole $\vec{R}_{0'}$ such that the vectors $\vec{\omega}$, $\frac{d}{dt}\vec{R}_{0'}$ lie on the same line?

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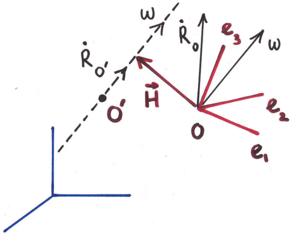
Screw motion of a rigid body. Chasles theorem



Given a body frame $\left[\vec{R}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\right]$ and the vectors of body angular velocity $\vec{\omega} \neq 0$ and of velocity of its pole $\frac{d}{dt}\vec{R}_0 \neq 0$, if one chooses new pole $\vec{R}_{0'}$ for the body frame as

$$\vec{R}_{0'} = \vec{R}_0 + \vec{H} + \vec{L},$$

where $\vec{L}$ is any vector parallel to $\vec{\omega}$ and

$$\vec{H} = \frac{\vec{\omega} \times \frac{d}{dt}\vec{R}_0}{|\vec{\omega}|^2},$$

then the vectors $\vec{\omega}$, $\frac{d}{dt}\vec{R}_{0'}$ lie on the same line

Modeling and Control of Robots

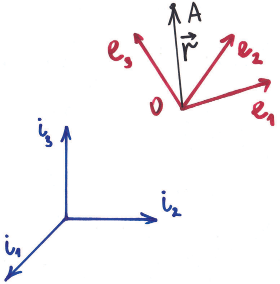
Lecture 3: Kinematics of a Rigid Body (cont'd).

Anton Shiriaev
January 18, 2020

Learning outcomes: Complex motion of a point and of a rigid body.

Conceptual Problems

As discussed, a motion of a rigid body is characterized by behavior of a frame $\left[\vec{R}_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right]$ attached to the body



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Conceptual Problems

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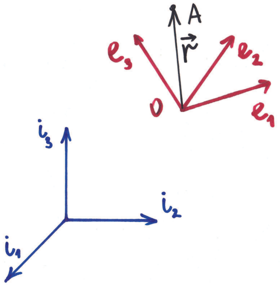


Computing velocities and accelerations of points of the rigid body then requires

$\frac{d}{dt}\vec{R}_0(\cdot)$ - velocity of the frame's origin

$\frac{d^2}{dt^2}\vec{R}_0(\cdot)$ - acceleration of the frame's origin

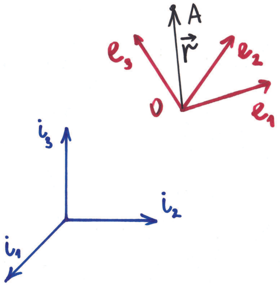
$\vec{\omega}(\cdot)$ - angular velocity of the body

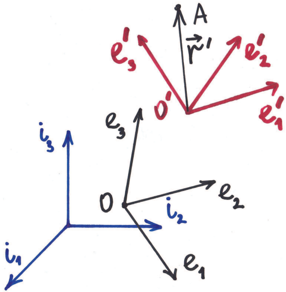
$\vec{\varepsilon}(\cdot)$ - angular acceleration of the body

As discussed, a motion of a rigid body is characterized by behavior of a frame $\left[\vec{R}_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right]$ attached to the body

Conceptual Problems

As discussed, a motion of a rigid body is characterized by behavior of a frame $\left[\vec{R}_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right]$ attached to the body



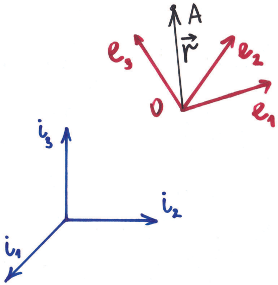


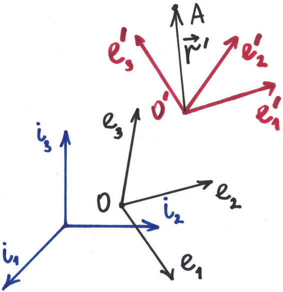
Often there is one or more moving frames introduced in-between the world and body frames $\left[i_1, i_2, i_3\right] \mapsto \left[e_1, e_2, e_3\right] \mapsto \ \dots \ \mapsto \ \left[e_1, e_2, e_3\right]$

- Computing kinematics of a point (its absolute velocity and acceleration) using relative info on several consecutive frames
- Computing ω, ε of the body expressed in the world frame

Conceptual Problems

As discussed, a motion of a rigid body is characterized by behavior of a frame $\left[\vec{R}_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right]$ attached to the body



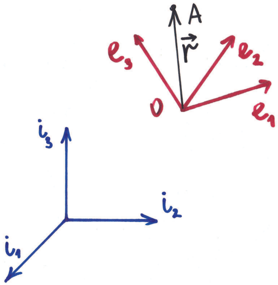


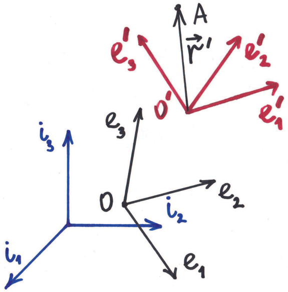
Often there is one or more moving frames introduced in-between the world and body frames $\left[i_1, i_2, i_3\right] \mapsto \left[e_1, e_2, e_3\right] \mapsto \ \ldots\ \mapsto \ \left[e_1, e_2, e_3\right]$

How to compute $\frac{d}{dt}\vec{R}_{0'}$, $\frac{d^2}{dt^2}\vec{R}_{0'}$, $\vec{\omega}$, $\vec{\varepsilon}$ of the body expressed in the world frame?

Conceptual Problems

As discussed, a motion of a rigid body is characterized by behavior of a frame $\left[\vec{R}_0(t),\ \vec{e}_1(t),\ \vec{e}_2(t),\ \vec{e}_3(t)\right]$ attached to the body





- Computing kinematics of a point (its absolute velocity and acceleration) using relative info on several consecutive frames
- Computing $\vec{\omega}$, $\vec{\varepsilon}$ of the body expressed in the world frame

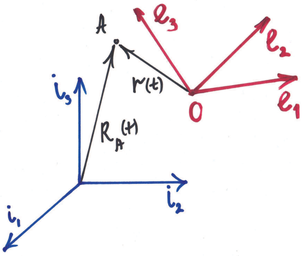
Outline

1. Complex motion of a point

2. Complex motion of a rigid body

Complex motion of a point

Complex motion of a point



Suppose the point A experiences a complex motion: it moves with respect to the frame $\left[\vec{R}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right]$ and given functions describing relative motion

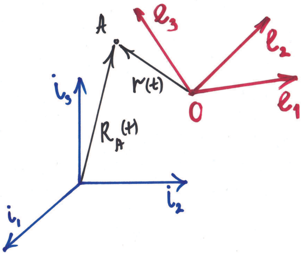
$$
\begin{aligned}
\vec{r}(t) &= r_1(t)\,\vec{e}_1(t) + r_2(t)\,\vec{e}_2(t) + r_3(t)\,\vec{e}_3(t)\\
\tfrac{d}{dt}\vec{r}(t) &= \dot{r}_1(t)\,\vec{e}_1(t) + \dot{r}_2(t)\,\vec{e}_2(t) + \dot{r}_3(t)\,\vec{e}_3(t)\\
\tfrac{d^2}{dt^2}\vec{r}(t) &= \ddot{r}_1(t)\,\vec{e}_1(t) + \ddot{r}_2(t)\,\vec{e}_2(t) + \ddot{r}_3(t)\,\vec{e}_3(t)
\end{aligned}
$$

Given functions describing behavior of $\left[\vec{R}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right]$ with respect to the world frame, i.e. the vector functions

$$\vec{R}_0(t), \quad \tfrac{d}{dt}\vec{R}_0(t), \quad \tfrac{d^2}{dt^2}\vec{R}_0(t), \quad \vec{e}_1(t), \quad \vec{e}_2(t), \quad \vec{e}_3(t),$$

its angular velocity $\vec{\omega}^{rel}(t) = \sum_{k=1}^{3} \Omega_k(t)\vec{i}_k$ and acceleration $\vec{\varepsilon}^{rel}(t) = \sum_{k=1}^{3} \dot{\Omega}_k(t)\vec{i}_k$

Complex motion of a point



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\tfrac{d^2}{dt^2}\vec{r}(t) &= \ddot{r}_1(t)\,\vec{e}_1(t) + \ddot{r}_2(t)\,\vec{e}_2(t) + \ddot{r}_3(t)\,\vec{e}_3(t)
\end{aligned}$$

Given functions describing behavior of $\left[\vec{R}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right]$ with respect to the world frame, i.e. the vector functions

$$\vec{R}_0(t), \quad \tfrac{d}{dt}\vec{R}_0(t), \quad \tfrac{d^2}{dt^2}\vec{R}_0(t), \quad \vec{e}_1(t), \quad \vec{e}_2(t), \quad \vec{e}_3(t),$$

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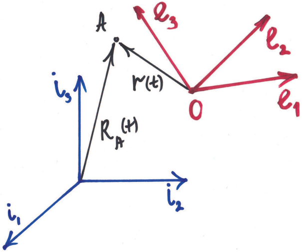
Complex motion of a point



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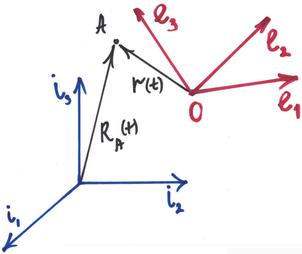
\end{aligned}$$

Given functions describing behavior of $\left[\vec{R}_0(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right]$ with respect to the world frame, i.e. the vector functions

$$\vec{R}_0(t), \quad \frac{d}{dt}\vec{R}_0(t), \quad \frac{d^2}{dt^2}\vec{R}_0(t), \quad \vec{e}_1(t), \quad \vec{e}_2(t), \quad \vec{e}_3(t),$$

its angular velocity $\vec{\omega}^{rel}(t) = \sum_{k=1}^{3} \Omega_k(t)\vec{i}_k$ and acceleration $\vec{\varepsilon}^{rel}(t) = \sum_{k=1}^{3} \dot{\Omega}_k(t)\vec{i}_k$

The task is to find $\vec{R}_A(t)$, $\frac{d}{dt}\vec{R}_A(t)$, and $\frac{d}{dt}\vec{R}_A(t)$

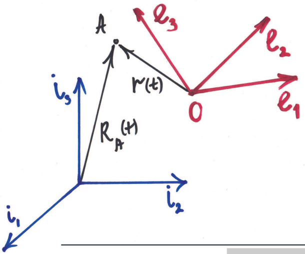


Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

$$\vec{R}_A(t), \quad \frac{d}{dt}\vec{R}_A(t), \quad \frac{d^2}{dt^2}\vec{R}_A(t)$$

It is clear that

$$\Rightarrow \quad \frac{d}{dt}\vec{R}_A(t) \quad = \quad \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$



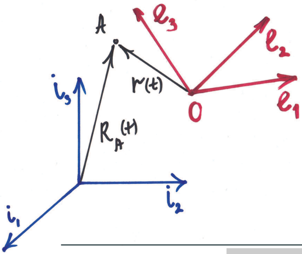
Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

$$\vec{R}_A(t), \quad \frac{d}{dt}\vec{R}_A(t), \quad \frac{d^2}{dt^2}\vec{R}_A(t)$$

It is clear that $\boxed{\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)}$

$$\Rightarrow \quad \frac{d}{dt}\vec{R}_A(t) \quad = \quad \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$

Complex motion of a point

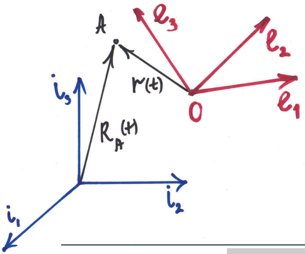


Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

$$\vec{R}_A(t), \quad \frac{d}{dt}\vec{R}_A(t), \quad \frac{d^2}{dt^2}\vec{R}_A(t)$$

It is clear that $\boxed{\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)}$

$$\Rightarrow \quad \frac{d}{dt}\vec{R}_A(t) \quad = \quad \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t)$$



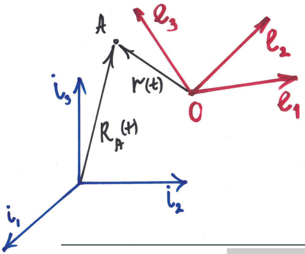
Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

$$\vec{R}_A(t), \quad \frac{d}{dt}\vec{R}_A(t), \quad \frac{d^2}{dt^2}\vec{R}_A(t)$$

It is clear that $\boxed{\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)}$

$$\Rightarrow \frac{d}{dt}\vec{R}_A(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right]$$

Complex motion of a point

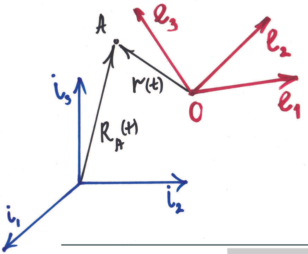


Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

$$\vec{R}_A(t), \quad \frac{d}{dt}\vec{R}_A(t), \quad \frac{d^2}{dt^2}\vec{R}_A(t)$$

It is clear that $\boxed{\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)}$

$$\begin{aligned} \Rightarrow \frac{d}{dt}\vec{R}_A(t) &= \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right] \\ &= \frac{d}{dt}\vec{R}_0(t) + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) + \sum_{k=1}^{3} r_k(t)\frac{d}{dt}\left[\vec{e}_k(t)\right] \end{aligned}$$



Given all the vector functions describing relative characteristics of the pont/frame motion: $\vec{r}(t)$, $\frac{d}{dt}\vec{r}(t)$, $\frac{d^2}{dt^2}\vec{r}(t)$, $\vec{R}_0(t)$, $\frac{d}{dt}\vec{R}_0(t)$, $\frac{d^2}{dt^2}\vec{R}_0(t)$, $\vec{e}_1(t)$, $\vec{e}_2(t)$, $\vec{e}_3(t)$, $\vec{\omega}^{rel}(t)$ and $\vec{\varepsilon}^{rel}(t)$, the task is to find

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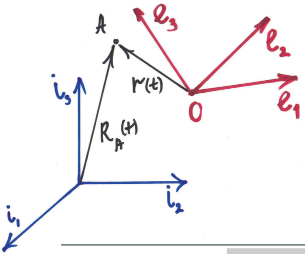
Complex motion of a point



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$$\begin{aligned}
\Rightarrow \quad \frac{d}{dt}\vec{R}_A(t) &= \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right] \\
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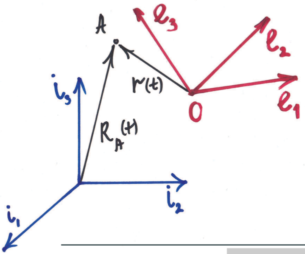
&= \frac{d}{dt}\vec{R}_0(t) + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) + \vec{\omega}^{rel} \times \left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right]
\end{aligned}$$

Complex motion of a point



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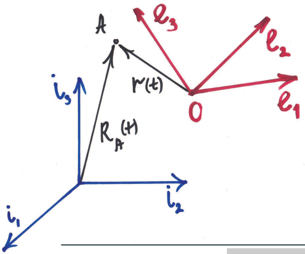
\Rightarrow \quad \frac{d}{dt}\vec{R}_A(t) \quad &= \quad \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\vec{R}_0(t) + \frac{d}{dt}\left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right] \\
&= \quad \frac{d}{dt}\vec{R}_0(t) + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) + \sum_{k=1}^{3} r_k(t)\frac{d}{dt}\left[\vec{e}_k(t)\right] \\
&= \quad \frac{d}{dt}\vec{R}_0(t) + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) + \vec{\omega}^{rel} \times \vec{r}(t)
\end{aligned}$$

Complex motion of a point



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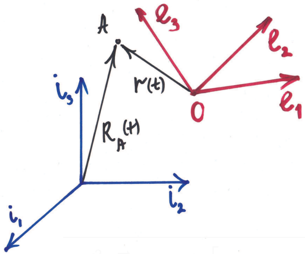
&= \underbrace{\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)}_{= V_{frame\ \ motion}} + \underbrace{\sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)}_{= V_{relative\ \ motion}}
\end{aligned}$$

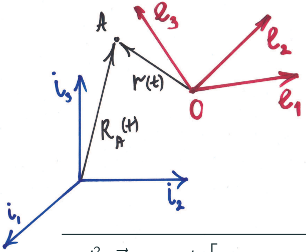


As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$
$$= V_{frame\ \ motion} + V_{relative\ \ motion}$$

$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$

Complex motion of a point



As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$
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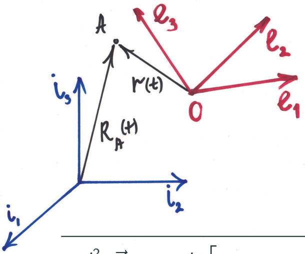
Complex motion of a point

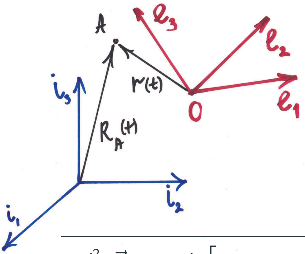


As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$
$$= V_{frame\ \ motion} + V_{relative\ \ motion}$$

$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$
$$= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]$$

Complex motion of a point

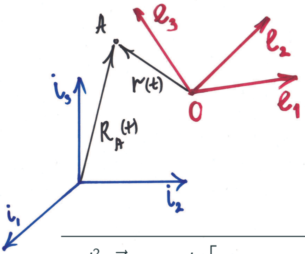


As derived

$$
\begin{aligned}
\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) \\
&= V_{frame\ \ motion} + V_{relative\ \ motion}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2}{dt^2}\vec{R}_A &= \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right] \\
&= \frac{d^2}{dt^2}\vec{R}_0 + \underbrace{\frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right]}_{\frac{d}{dt}\left[\vec{\omega}^{rel}\right] \times \vec{r} + \vec{\omega}^{rel} \times \frac{d}{dt}\left[\vec{r}\right]} + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]
\end{aligned}
$$

Complex motion of a point



As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$

$$= V_{frame\ \ motion} + V_{relative\ \ motion}$$

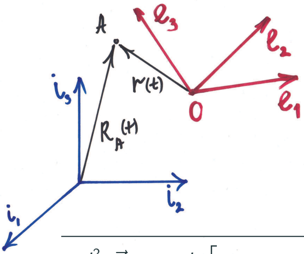
$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$

$$= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]$$

$$\overbrace{\vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \frac{d}{dt}\left[\vec{r}\right]}$$

$$\overbrace{\frac{d}{dt}\left[\sum_{k=1}^{3} r_k(t)\vec{e}_k(t)\right]}$$

Complex motion of a point



As derived

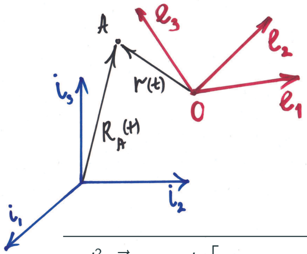
$$\begin{aligned}\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\color{red}{\vec{e}_k(t)} \\ &= V_{frame\ \ motion} + V_{relative\ \ motion}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt^2}\vec{R}_A &= \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right] \\ &= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \color{red}{\vec{e}_k}\right]\end{aligned}$$

$$\overbrace{\vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \frac{d}{dt}\left[\vec{r}\right]}$$

$$\overbrace{\sum_{k=1}^{3} \dot{r}_k \color{red}{\vec{e}_k} + \sum_{k=1}^{3} r_k \frac{d}{dt}\left[\color{red}{\vec{e}_k}\right]}$$

Complex motion of a point



As derived

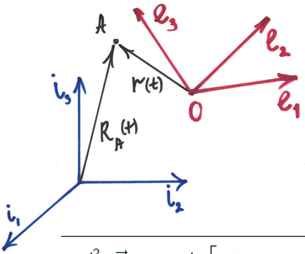
$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$

$$= V_{frame\ \ motion} + V_{relative\ \ motion}$$

$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$

$$= \frac{d^2}{dt^2}\vec{R}_0 + \overbrace{\frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right]}^{\vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \frac{d}{dt}\left[\vec{r}\right]} + \overbrace{\frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]}^{V_{rel.motion} + \sum_{k=1}^{3} r_k\left[\vec{\omega}^{rel} \times \vec{e}_k\right]}$$

Complex motion of a point



As derived

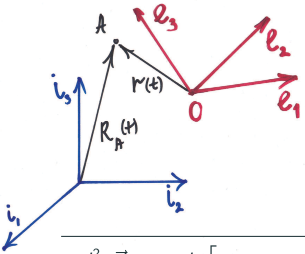
$$\begin{aligned} \frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) \\ &= V_{frame\ \ motion} + V_{relative\ \ motion} \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dt^2}\vec{R}_A &= \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right] \\ &= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k\vec{e}_k\right] \end{aligned}$$

$$\overbrace{\vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \frac{d}{dt}\left[\vec{r}\right]}$$

$$\overbrace{V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}}$$

Complex motion of a point

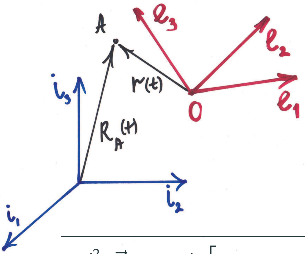


As derived

$$
\begin{aligned}
\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) \\
&= V_{frame\ \ motion} + V_{relative\ \ motion}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2}{dt^2}\vec{R}_A &= \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right] \\
&= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right] \\
&= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]
\end{aligned}
$$

Complex motion of a point



As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$
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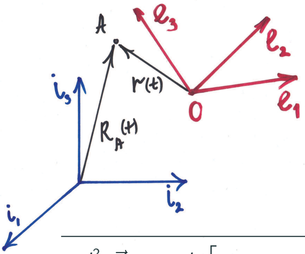
$$= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \overbrace{\frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]}$$

$$\sum_{k=1}^{3} \ddot{r}_k \vec{e}_k + \sum_{k=1}^{3} \dot{r}_k \frac{d}{dt}\left[\vec{e}_k\right]$$

Complex motion of a point



As derived

$$\frac{d}{dt}\vec{R}_A(t) = \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)$$

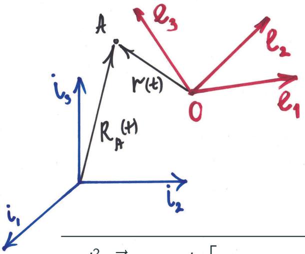
$$= V_{frame\ \ motion} + V_{relative\ \ motion}$$

$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$

$$= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]$$

$$= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \underbrace{\frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]}_{\sum_{k=1}^{3} \ddot{r}_k \vec{e}_k + \sum_{k=1}^{3} \dot{r}_k \left[\vec{\omega}^{rel} \times \vec{e}_k\right]}$$

Complex motion of a point

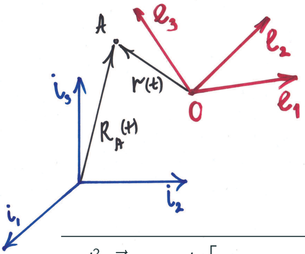


As derived

$$\begin{aligned}\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t) \\ &= V_{frame\ \ motion} + V_{relative\ \ motion}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}&\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right] \\ &= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3}\dot{r}_k\vec{e}_k\right] \\ &= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \underbrace{\frac{d}{dt}\left[\sum_{k=1}^{3}\dot{r}_k\vec{e}_k\right]}_{\sum_{k=1}^{3}\ddot{r}_k\vec{e}_k + \vec{\omega}^{rel} \times \left[\sum_{k=1}^{3}\dot{r}_k\vec{e}_k\right]}\end{aligned}$$

Complex motion of a point

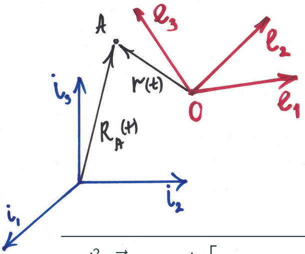


As derived

$$\begin{aligned}\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)\\ &= V_{frame\ \ motion} + V_{relative\ \ motion}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}&\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]\\ &= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]\\ &= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]\\ &= \ddot{\vec{R}}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.mot.} + \vec{\omega}^{rel} \times \vec{r}\right] + \ddot{\vec{r}} + \vec{\omega}^{rel} \times V_{rel.mot.}\end{aligned}$$

Complex motion of a point



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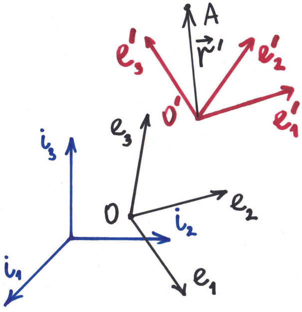
$$\begin{aligned}\frac{d}{dt}\vec{R}_A(t) &= \left[\frac{d}{dt}\vec{R}_0(t) + \vec{\omega}^{rel} \times \vec{r}(t)\right] + \sum_{k=1}^{3} \dot{r}_k(t)\vec{e}_k(t)\\ &= V_{frame\ \ motion} + V_{relative\ \ motion}\end{aligned}$$

$$\frac{d^2}{dt^2}\vec{R}_A = \frac{d}{dt}\left[V_{frame\ \ motion} + V_{relative\ \ motion}\right]$$

$$\begin{aligned}&= \frac{d^2}{dt^2}\vec{R}_0 + \frac{d}{dt}\left[\vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]\\ &= \frac{d^2}{dt^2}\vec{R}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.motion} + \vec{\omega}^{rel} \times \vec{r}\right] + \frac{d}{dt}\left[\sum_{k=1}^{3} \dot{r}_k \vec{e}_k\right]\\ &= \ddot{\vec{R}}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[V_{rel.mot.} + \vec{\omega}^{rel} \times \vec{r}\right] + \ddot{\vec{r}} + \vec{\omega}^{rel} \times V_{rel.mot.}\\ &= \underbrace{\ddot{\vec{R}}_0 + \vec{\varepsilon}^{rel} \times \vec{r} + \vec{\omega}^{rel} \times \left[\vec{\omega}^{rel} \times \vec{r}\right]}_{W_{frame\ \ motion}} + \underbrace{\ddot{\vec{r}}}_{W_{relative\ \ motion}} + \underbrace{2 \cdot \vec{\omega}^{rel} \times V_{rel.mot.}}_{Coriolis\ \ term}\end{aligned}$$

Complex motion of a rigid body

Complex motion of a rigid body



Suppose a rigid body experiences a complex motion: it moves w.r.t. to the frame $\left[\vec{R_0}(t), \vec{e_1}(t), \vec{e_2}(t), \vec{e_3}(t)\right]$, which, in turn, moves with respect to the world frame & the following functions describe such relative motions:

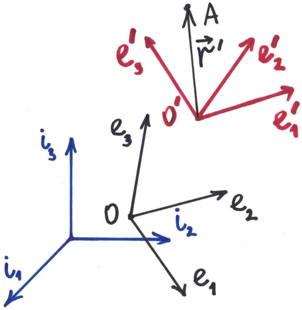
[World frame $\mapsto$ 0$-$frame] :

$$\vec{R_0}(t), \qquad \vec{\omega}^e = \sum_{k=1}^{3} \Omega_k \vec{i_k}, \qquad \vec{\varepsilon}^e = \sum_{k=1}^{3} \dot{\Omega}_k \vec{i_k}$$

[0$-$frame $\mapsto$ 0′$-$frame] :

$$\vec{R_{00'}}(t), \quad \vec{\omega}^{e'} = \sum_{k=1}^{3} \omega_k \vec{e}_k, \quad \vec{\varepsilon}^{e'} = \sum_{k=1}^{3} \dot{\omega}_k \vec{e}_k$$

Complex motion of a rigid body



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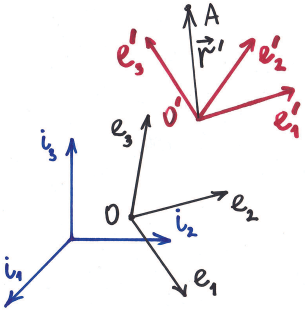
[0−frame $\mapsto$ 0′−frame] :

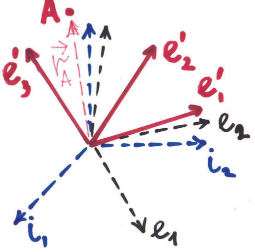
$$\vec{R}_{00'}(t), \quad \vec{\omega}^{e'} = \sum_{k=1}^{3} \omega_k \vec{e}_k, \quad \vec{\varepsilon}^{e'} = \sum_{k=1}^{3} \dot{\omega}_k \vec{e}_k$$

We need to find both angular velocity $\vec{\omega}^{abs}$ and acceleration $\vec{\varepsilon}^{abs}$

of the body with respect to the world frame

Complex motion of a rigid body

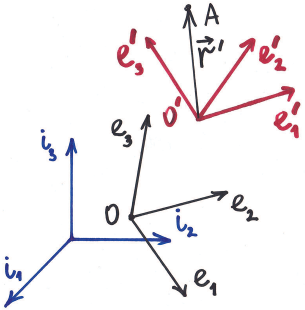


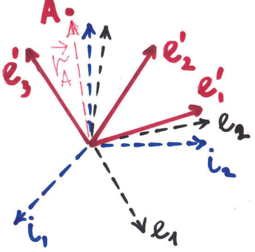


To compute an angular velocity of the body in the world frame, pick up a point A of the body,compute it velocity. It is induced by two motions

$$\frac{d}{dt}\vec{r}^{\,i}_A \quad =$$

Complex motion of a rigid body





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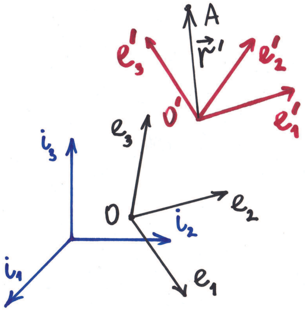

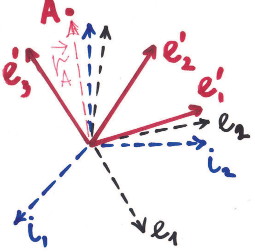

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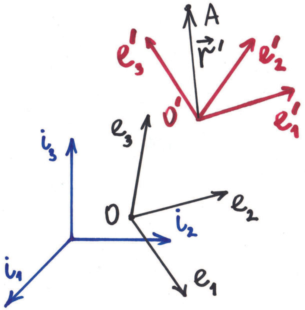
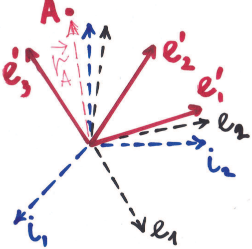
$$\frac{d}{dt}\vec{r}^{\,i}_A \quad = \quad \frac{d}{dt}\vec{r}^{\,(i\mapsto e)}_A + \frac{d}{dt}\vec{r}^{\,(e\mapsto e')}_A$$

Complex motion of a rigid body

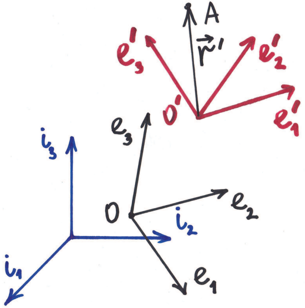


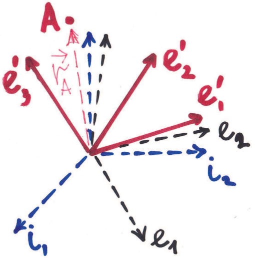


To compute an angular velocity of the body in the world frame, pick up a point A of the body,compute it velocity. It is induced by two motions

$$\frac{d}{dt}\vec{r}^{\,i}_{A} \quad = \quad \frac{d}{dt}\vec{r}^{\,(i\mapsto e)}_{A} + \frac{d}{dt}\vec{r}^{\,(e\mapsto e')}_{A} = \vec{\omega}^{\,(i\mapsto e)} \times \vec{r}_A + \vec{\omega}^{\,(e\mapsto e')} \times \vec{r}_A$$

Complex motion of a rigid body





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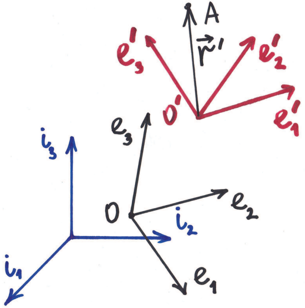
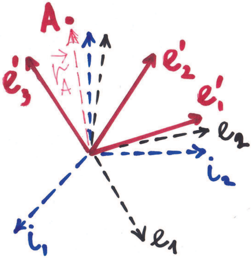
Complex motion of a rigid body

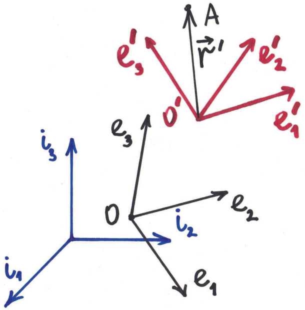


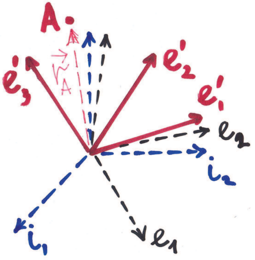


To compute an angular velocity of the body in the world frame, pick up a point A of the body,compute it velocity. It is induced by two motions

$$\frac{d}{dt}\vec{r}^{\,i}_{A} \quad = \quad \frac{d}{dt}\vec{r}_{A}^{\,(i\mapsto e)} + \frac{d}{dt}\vec{r}_{A}^{\,(e\mapsto e')} = \left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right] \times \vec{r}_{A}$$

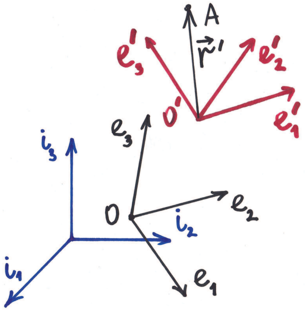
Complex motion of a rigid body

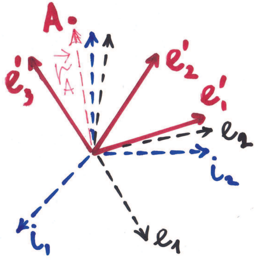




To compute an angular velocity of the body in the world frame, pick up a point A of the body,compute it velocity. It is induced by two motions

$$\begin{aligned} \frac{d}{dt}\vec{r}_A^{\,i} &= \frac{d}{dt}\vec{r}_A^{\,(i\mapsto e)} + \frac{d}{dt}\vec{r}_A^{\,(e\mapsto e')} = \left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right] \times \vec{r}_A \\ &= \vec{\omega}^{\,abs} \times \vec{r}_A \end{aligned}$$





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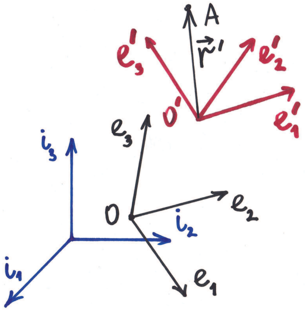
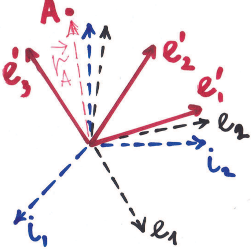
$$\frac{d}{dt}\vec{r}^{\,i}_{A} = \frac{d}{dt}\vec{r}^{\,(i\mapsto e)}_{A} + \frac{d}{dt}\vec{r}^{\,(e\mapsto e')}_{A} = \left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right] \times \vec{r}_{A}$$

$$= \vec{\omega}^{\,abs} \times \vec{r}_{A} \qquad \Rightarrow \qquad \boxed{\vec{\omega}^{\,abs} \equiv \vec{\omega}^{\,e} + \vec{\omega}^{\,e'}}$$

Complex motion of a rigid body

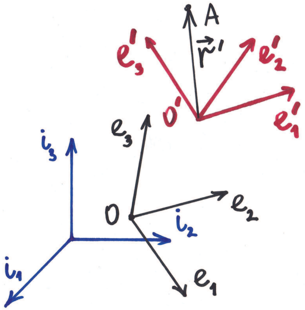


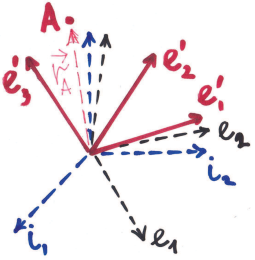


To compute an angular acceleration of the body in the world frame, let us differentiate the vector of angular velocity

$$\vec{\varepsilon}^{\,abs} \;=\; \frac{d}{dt}\vec{\omega}^{\,abs} = \frac{d}{dt}\left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right]$$

Complex motion of a rigid body





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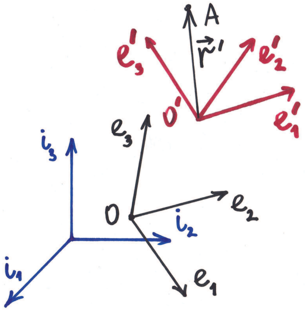

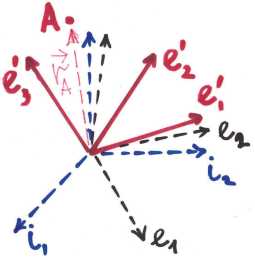

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$$
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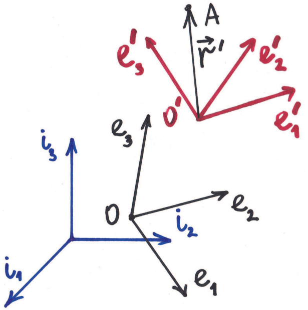
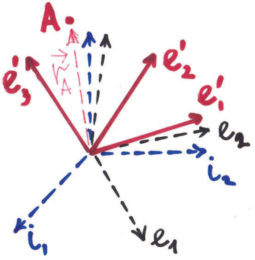
\vec{\varepsilon}^{\,abs} &= \frac{d}{dt}\vec{\omega}^{\,abs} = \frac{d}{dt}\left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right] = \frac{d}{dt}\left[\sum_{k=1}^{3}\Omega_k \vec{i}_k + \sum_{k=1}^{3}\omega_k \vec{e}_k\right] \\
&= \sum_{k=1}^{3}\dot{\Omega}_k \vec{i}_k + \sum_{k=1}^{3}\dot{\omega}_k \vec{e}_k + \sum_{k=1}^{3}\omega_k \frac{d}{dt}\left[\vec{e}_k\right]
\end{aligned}
$$





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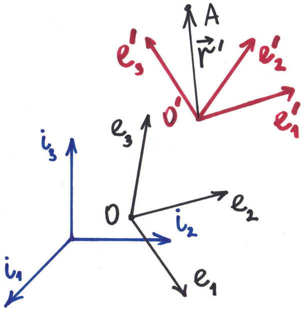
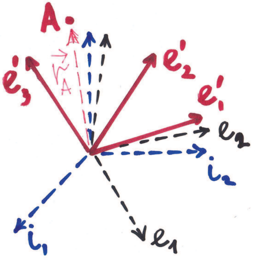
\end{aligned}
$$





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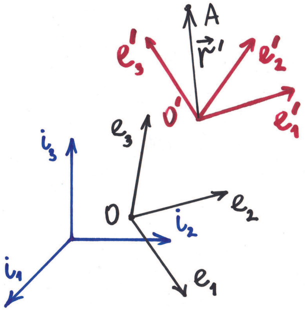
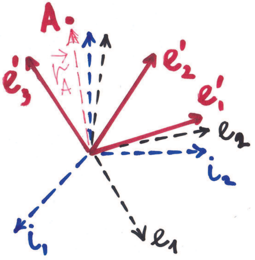
&= \sum_{k=1}^{3}\dot{\Omega}_k \vec{i}_k + \sum_{k=1}^{3}\dot{\omega}_k \vec{e}_k + \vec{\omega}^{\,e} \times \left[\sum_{k=1}^{3}\omega_k \vec{e}_k\right]
\end{aligned}
$$

Complex motion of a rigid body





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$$\begin{aligned}\vec{\varepsilon}^{\,abs} &= \frac{d}{dt}\vec{\omega}^{\,abs} = \frac{d}{dt}\left[\vec{\omega}^{\,e} + \vec{\omega}^{\,e'}\right] = \frac{d}{dt}\left[\sum_{k=1}^{3}\Omega_k \vec{i}_k + \sum_{k=1}^{3}\omega_k \vec{e}_k\right] \\ &= \sum_{k=1}^{3}\dot{\Omega}_k \vec{i}_k + \sum_{k=1}^{3}\dot{\omega}_k \vec{e}_k + \vec{\omega}^{\,e} \times \vec{\omega}^{\,e'}\end{aligned}$$

Modeling and Control of Robots

Lecture 4: Kinematics of a Rigid Body (cont'd).

Anton Shiriaev

January 19, 2020

Learning outcomes: Rotation matrices and their properties. Elementary rotations. Composition of rotations. Parametrization of rotation matrices: Euler angles.

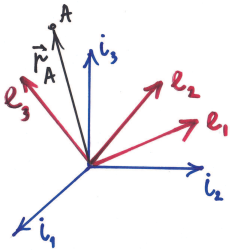
Outline

1. Rotation matrices and their properties
2. Elementary Rotations in $\mathbb{R}^3$
3. Composition of Rotations
4. Parametrization of Rotations: Euler Angles

Rotation matrices and their properties

Rotation matrix

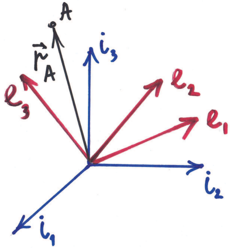
Given world and body frames having the same origin, coordinates of any point of the rigid body (such as a point A) can be written in either of frames



$$\vec{r}^{\,i}_A = \begin{bmatrix} x^i_A \\ y^i_A \\ z^i_A \end{bmatrix}, \quad \vec{r}^{\,e}_A = \begin{bmatrix} x^e_A \\ y^e_A \\ z^e_A \end{bmatrix}$$

The 3×3 matrix $\mathbf{R}$ created by three basis vectors is known as a rotation

$$\mathbf{R} := [\vec{e_1},\ \vec{e_2},\ \vec{e_1}] = \left[\begin{array}{c|c|c} \vec{i_1} \cdot \vec{e_1} & \vec{i_1} \cdot \vec{e_2} & \vec{i_1} \cdot \vec{e_3} \\ \vec{i_2} \cdot \vec{e_1} & \vec{i_2} \cdot \vec{e_2} & \vec{i_2} \cdot \vec{e_3} \\ \vec{i_3} \cdot \vec{e_1} & \vec{i_3} \cdot \vec{e_2} & \vec{i_3} \cdot \vec{e_3} \end{array}\right]$$



Given world and body frames having the same origin, coordinates of any point of the rigid body (such as a point A) can be written in either of frames

$$\vec{r}^{\,i}_A = \begin{bmatrix} x^i_A \\ y^i_A \\ z^i_A \end{bmatrix}, \quad \vec{r}^{\,e}_A = \begin{bmatrix} x^e_A \\ y^e_A \\ z^e_A \end{bmatrix}$$

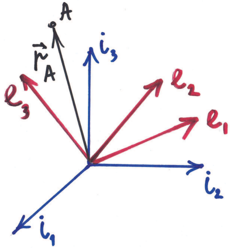
The vectors $\vec{r}^{\,i}_A$ and $\vec{r}^{\,e}_A$ are related by the equation

$$\vec{r}^{\,i}_A = x^e_A \cdot \vec{e_1} + y^e_A \cdot \vec{e_2} + z^e_A \cdot \vec{e_3}, \quad \vec{e_k} = \begin{bmatrix} \vec{i_1} \cdot \vec{e_k} \\ \vec{i_2} \cdot \vec{e_k} \\ \vec{i_3} \cdot \vec{e_k} \end{bmatrix}$$

The 3×3 matrix $\mathbf{R}$ created by three basis vectors is known as a rotation

$$\mathbf{R} := [\vec{e_1}, \vec{e_2}, \vec{e_1}] = \left[\begin{array}{c|c|c} \vec{i_1} \cdot \vec{e_1} & \vec{i_1} \cdot \vec{e_2} & \vec{i_1} \cdot \vec{e_3} \\ \vec{i_2} \cdot \vec{e_1} & \vec{i_2} \cdot \vec{e_2} & \vec{i_2} \cdot \vec{e_3} \\ \vec{i_3} \cdot \vec{e_1} & \vec{i_3} \cdot \vec{e_2} & \vec{i_3} \cdot \vec{e_3} \end{array}\right]$$

Rotation matrix



Given world and body frames having the same origin, coordinates of any point of the rigid body (such as a point A) can be written in either of frames

$$\vec{r}^{\,i}_A = \begin{bmatrix} x^i_A \\ y^i_A \\ z^i_A \end{bmatrix}, \quad \vec{r}^{\,e}_A = \begin{bmatrix} x^e_A \\ y^e_A \\ z^e_A \end{bmatrix}$$

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The 3×3 matrix $\mathbf{R}$ created by three basis vectors is known as a rotation

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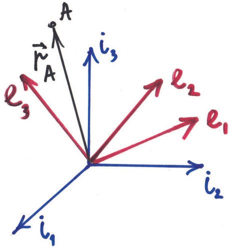

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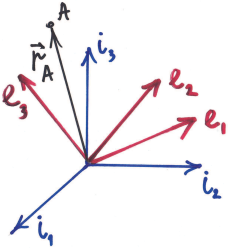

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Rotation matrix: properties

The rows of a rotation $\mathbf{R}$ provide coordinates of vectors $\vec{i_k}$ in the e-basis

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Rotation matrix: properties

The rows of a rotation $\mathbf{R}$ provide coordinates of vectors $\vec{i}_k$ in the e-basis

$$\mathbf{R} = \begin{bmatrix} \vec{i}_1 \cdot \vec{e}_1 & \vec{i}_1 \cdot \vec{e}_2 & \vec{i}_1 \cdot \vec{e}_3 \\ \vec{i}_2 \cdot \vec{e}_1 & \vec{i}_2 \cdot \vec{e}_2 & \vec{i}_2 \cdot \vec{e}_3 \\ \vec{i}_3 \cdot \vec{e}_1 & \vec{i}_3 \cdot \vec{e}_2 & \vec{i}_3 \cdot \vec{e}_3 \end{bmatrix} = \left[\begin{array}{ccc} \vec{i}_1 \cdot \vec{e}_1 & \vec{i}_1 \cdot \vec{e}_2 & \vec{i}_1 \cdot \vec{e}_3 \\ \hdashline \vec{i}_2 \cdot \vec{e}_1 & \vec{i}_2 \cdot \vec{e}_2 & \vec{i}_2 \cdot \vec{e}_3 \\ \hdashline \vec{i}_3 \cdot \vec{e}_1 & \vec{i}_3 \cdot \vec{e}_2 & \vec{i}_3 \cdot \vec{e}_3 \end{array}\right]$$

Since i-and e-frames are made of orthogonal vectors of size one, then

$$\mathbf{R}\,\mathbf{R}^T = I_3 \quad \Rightarrow \quad \boxed{\mathbf{R}^{-1} = \mathbf{R}^T}$$

Determinant of a rotation matrix $\mathbf{R}$ is equal to one. Indeed

$$1 = \det I_1 = \det\left(\mathbf{R}\,\mathbf{R}^T\right) = \left(\det \mathbf{R}\right)^2 \quad \Rightarrow \quad \det \mathbf{R} = \pm 1$$

At least one of eigenvalues λ_k of a rotation matrix $\mathbf{R}$ is equal to 1.

$$\boxed{\lambda \in \mathbb{C} \text{ is an eigenvalue of } \mathbf{R}} \quad \Leftrightarrow \quad \boxed{\det\left(\lambda \cdot I_3 - \mathbf{R}\right) = 0}$$

$$\det\left(I_3 - \mathbf{R}\right) \quad = \quad \det\left(\mathbf{R}\,\mathbf{R}^T - \mathbf{R}\right) = \det\left(\mathbf{R}\,\left[\mathbf{R}^T - I_3\right]\right)$$

Rotation matrix: properties

The rows of a rotation $\mathbf{R}$ provide coordinates of vectors $\vec{i_k}$ in the e-basis

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Elementary Rotations in $\mathbb{R}^3$

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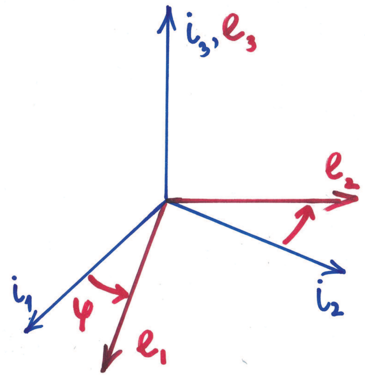
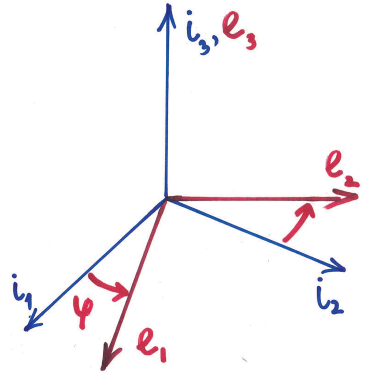
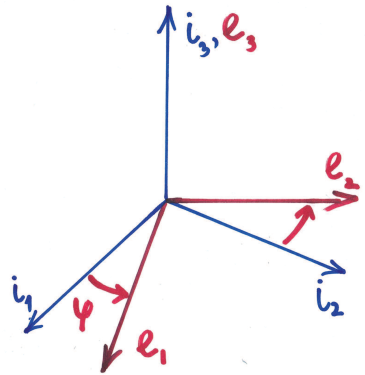


e-frame is rotated with respect to i-frame about z_0-axis by angle ϕ



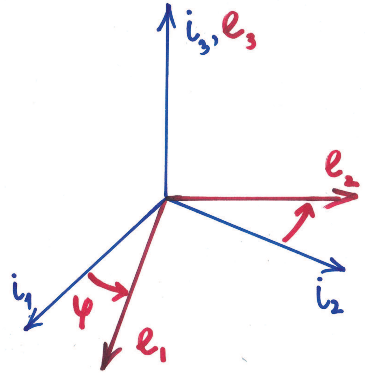
$$
\begin{array}{lcl|lcl|lcl}
x_1^0(\phi)x_0 & = & \cos\phi & y_1^0(\phi)x_0 & = & -\sin\phi & z_1^0(\phi)x_0 & = & 0 \\
x_1^0(\phi)y_0 & = & \sin\phi & y_1^0(\phi)y_0 & = & \cos\phi & z_1^0(\phi)y_0 & = & 0 \\
x_1^0(\phi)z_0 & = & 0 & y_1^0(\phi)z_0 & = & 0 & z_1^0(\phi)z_0 & = & 1
\end{array}
$$

Elementary Rotations in $\mathbb{R}^3$



$$R_1^0(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left\{ =: R_{z,\phi} \right\}$$

Elementary Rotations in $\mathbb{R}^3$



This elementary rotation around z-axis satisfies the properties

$$R_{z,0} = I_3, \qquad R_{z,\theta} R_{z,\phi} = R_{z,\theta+\phi}, \qquad [R_{z,\phi}]^{-1} = R_{z,-\phi}$$

Elementary Rotations in $\mathbb{R}^3$

Then, the expression for an elementary rotation around z-axis is

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, one can derive matrices representing elementary rotations

$$R_{x,\theta}, \quad R_{y,\theta}$$

Elementary Rotations in $\mathbb{R}^3$

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Similarly, one can derive matrices representing elementary rotations

$$R_{x,\theta}, \quad R_{y,\theta}$$

They are

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \qquad R_{y,\theta} = \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix}$$

Then, the expression for an elementary rotation around z-axis is

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Composition of Rotations

Rotation with Respect to Current Axis

Given three frames:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad (o_2, x_2, y_2, z_2),$$

any point p in the space will have three representations:

$$p^0 = [u_0, v_0, w_0]^T, \quad p^1 = [u_1, v_1, w_1]^T, \quad p^2 = [u_2, v_2, w_2]^T$$

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We know that

$$p^0 = R_1^0\, p^1, \quad p^1 = R_2^1\, p^2, \quad p^0 = R_2^0\, p^2$$

How are the rotation matrices R_1^0, R_2^1 and R_2^0 related?

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We can compute p^0 in two different ways

$$p^0 = R_1^0\, p^1 = R_1^0\, R_2^1\, p^2, \qquad p^0 = R_2^0\, p^2$$

Rotation with Respect to Current Axis

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$$p^0 = R_1^0\, p^1 = R_1^0\, R_2^1\, p^2, \qquad p^0 = R_2^0\, p^2$$

The combined rotation will be: $R_2^0 = R_1^0\, R_2^1$

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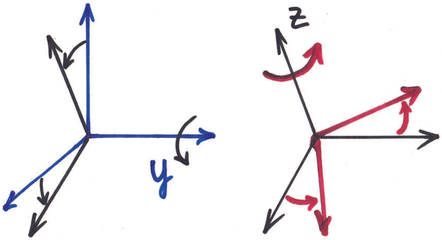
Example: Rotation w.r.t. Current Axis



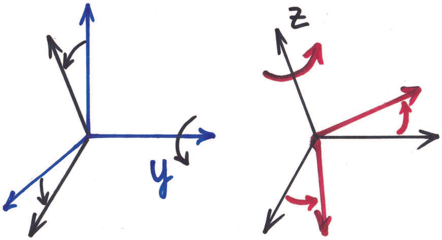
Composition of rotations about current axes

Suppose we rotate

- first the frame by angle ϕ around current y-axis,
- then rotate by angle θ around the current z-axis.

Find the combined rotation.

Example: Rotation w.r.t. Current Axis

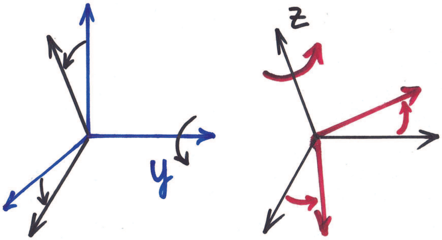


Composition of rotations about current axes

The rotations around y- and z-axis are basic rotations

$$R_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \qquad R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Rotation w.r.t. Current Axis



Composition of rotations about current axes

Therefore the overall rotation is

$$R = R_{y,\phi} R_{z,\theta} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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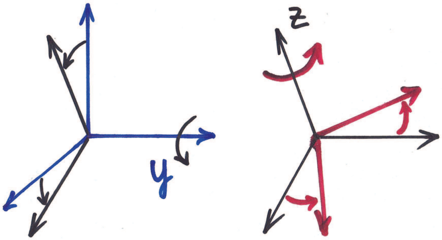

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Therefore the overall rotation is

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Example: Rotation w.r.t. Current Axis

Observation: Rotations do not commute

$$R_{y,\phi} R_{z,\theta} \neq R_{z,\theta} R_{y,\phi}$$

So that the order of rotations is **important!!!**

Indeed

$$R_{y,\phi} R_{z,\theta} = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}$$

and

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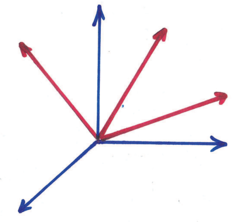
Parametrization of Rotations: Euler Angles

Parametrization of Rotation

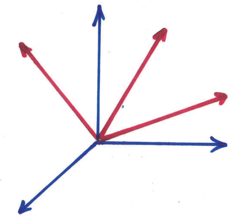


The problem settings assume that there are $\vec{i}$-and $\vec{e}$-frames and we need to find

- a set of parameters $(p_1, p_2, \ldots)$
- computational procedure

that allows representing one frame with respect to another

Parametrization of Rotation



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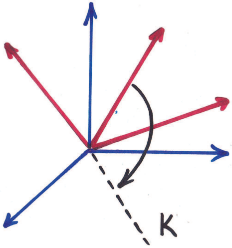
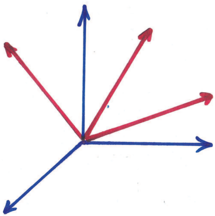


K is a line of nodes

Parametrization of Rotation



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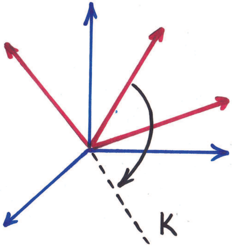
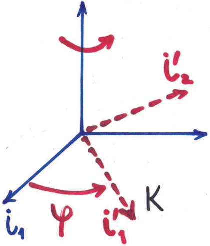
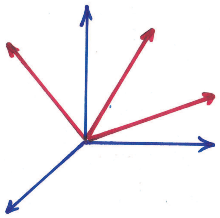


K is a line of nodes



φ is a precession angle

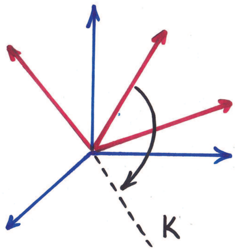
Parametrization of Rotation



The problem settings assume that there are $\vec{i}$-and $\vec{e}$-frames and we need to find

- a set of parameters $(p_1, p_2, \ldots)$
- computational procedure

that allows representing one frame with respect to another



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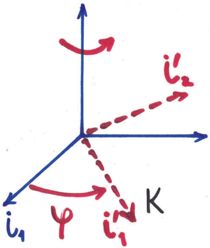

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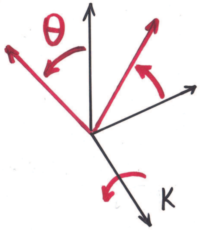
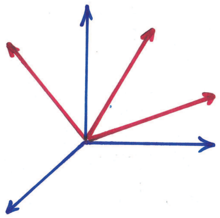


θ is a nutation angle



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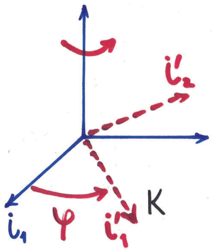

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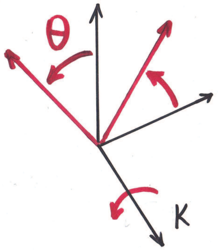

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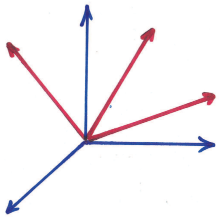

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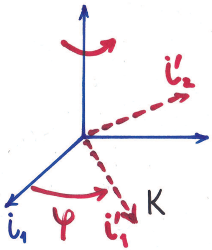

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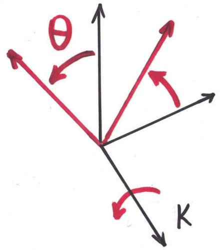
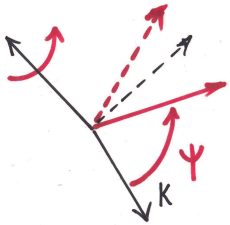
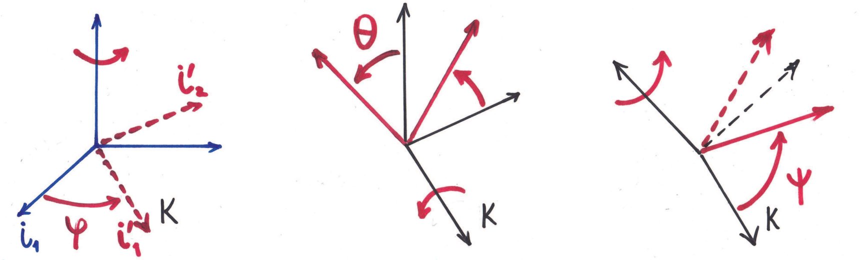


θ is a nutation angle



ψ is an intrinsic rotation

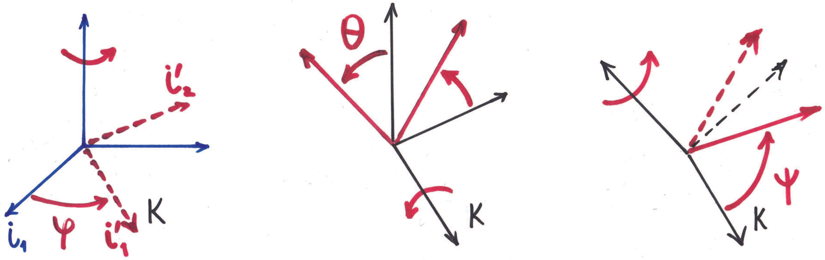
Parametrization of Rotation



Three consecutive rotations will result in the combined rotation

$$R = R(\varphi, \theta, \psi)$$

Parametrization of Rotation



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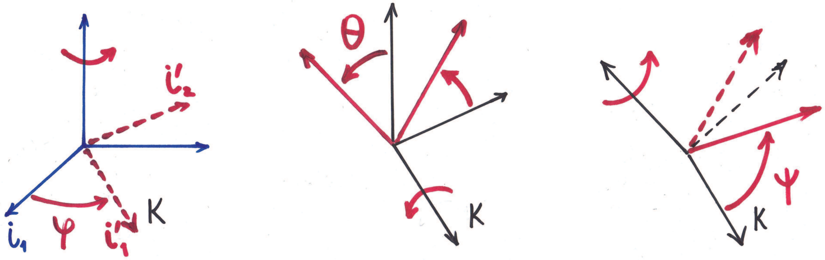
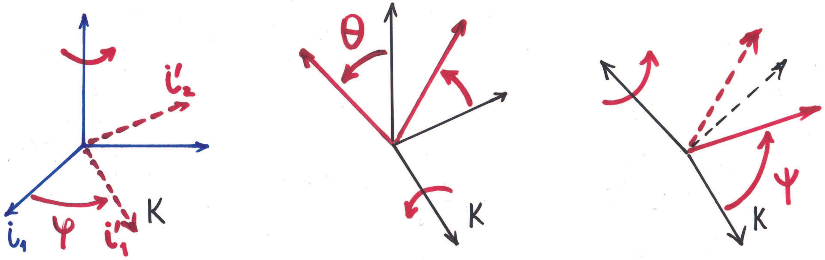
Parametrization of Rotation



Three consecutive rotations will result in the combined rotation

$$R \quad = \quad R(\varphi, \theta, \psi) = R_{z,\varphi} \cdot R_{x,\theta} \cdot R_{z,\psi}$$

Parametrization of Rotation



Three consecutive rotations will result in the combined rotation

$$
\begin{aligned}
R &= R(\varphi,\theta,\psi) = R_{z,\varphi}\cdot R_{x,\theta}\cdot R_{z,\psi} \\
&= \begin{bmatrix} c_\varphi & -s_\varphi & 0 \\ s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} \cdot \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{ZXZ}
\end{aligned}
$$

Parametrization of Rotation

Depending on choice/sequence of elementary rotations, we have various representations for the same rotation matrix

$$R_{XYX}, \quad R_{XYZ}, \quad R_{XZX}, \quad R_{XZY}, \quad R_{YXY}, \quad R_{YXZ}, \quad \ldots$$

$$R_{ZYZ} := R_{z,\phi} \cdot R_{y,\theta} \cdot R_{z,\psi}$$

Parametrization of Rotation

Depending on choice/sequence of elementary rotations, we have various representations for the same rotation matrix

$$R_{XYX}, \quad R_{XYZ}, \quad R_{XZX}, \quad R_{XZY}, \quad R_{YXY}, \quad R_{YXZ}, \quad \ldots$$

$$R_{ZYZ} := \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \cdot \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Parametrization of Rotation

Depending on choice/sequence of elementary rotations, we have various representations for the same rotation matrix

$$R_{XYX}, \quad R_{XYZ}, \quad R_{XZX}, \quad R_{XZY}, \quad R_{YXY}, \quad R_{YXZ}, \quad \ldots$$

$$R_{ZYZ} := \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Determining Euler Angles for a given Rotation Matrix

Given the rotation matrix $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

How to find angles ϕ, θ, ψ?

$$R = R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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Elements marked in red help to solve the problem

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \xrightarrow{?} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\begin{aligned} {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 &= 1 \\ {r_{31}}^2 + {r_{32}}^2 + {r_{33}}^2 &= 1 \end{aligned}$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \overset{?}{\rightarrow} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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Case 1: $r_{33} = \pm 1$

$$r_{13} = r_{23} = r_{31} = r_{32} = 0$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \xrightarrow{?} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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$$r_{33} = 1 \quad \Rightarrow \quad \cos\theta = 1, \quad \sin\theta = 0$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{?}{\rightarrow} \begin{bmatrix} \cos(\phi+\psi) & -\sin(\phi+\psi) & 0 \\ \sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} {r_{13}}^2 + {r_{23}}^2 + {r_{33}}^2 &= 1 \\ {r_{31}}^2 + {r_{32}}^2 + {r_{33}}^2 &= 1 \end{aligned}$$

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$$r_{33} = 1 \quad \Rightarrow \quad \cos\theta = 1, \quad \sin\theta = 0$$

$$\theta = 0, \quad \phi + \psi = \mathrm{atan2}\,(Y, X) = \mathrm{atan2}\,(r_{21}, r_{11})$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \overset{?}{\rightarrow} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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Case 2: ${r_{33}}^2 < 1$

$${r_{13}}^2 + {r_{23}}^2 \neq 0, \quad \left\{ {r_{31}}^2 + {r_{32}}^2 \neq 0 \right\}$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \stackrel{?}{\rightarrow} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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$${r_{13}}^2 + {r_{23}}^2 \neq 0, \quad \left\{ {r_{31}}^2 + {r_{32}}^2 \neq 0 \right\}$$

$$r_{33} = \cos\theta, \quad \left\{ (\sin\theta)^2 + (\cos\theta)^2 = 1 \right\}$$

Determining Euler Angles for a given Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \overset{?}{\rightarrow} \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

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$$ {r_{13}}^2 + {r_{23}}^2 \neq 0, \quad \left\{ {r_{31}}^2 + {r_{32}}^2 \neq 0 \right\}$$

$$r_{33} = \cos\theta, \quad \sin\theta = \pm\sqrt{1 - {r_{33}}^2}$$

$$\theta = \mathrm{atan2}(\pm\sqrt{1 - {r_{33}}^2}, r_{33})$$

$$\phi = \mathrm{atan2}(\pm r_{23}, \pm r_{13}), \quad \psi = \mathrm{atan2}(\pm r_{32}, \mp r_{31})$$

Modeling and Control of Robots

Lecture 5: Kinematics of a Rigid Body (cont'd).

Anton Shiriaev

January 25, 2020

Learning outcomes: Skew symmetric matrices, Euler and Poisson equations.

Outline

1. On time evolution of body frame
   - Euler kinematic equations
   - Poisson equations

On time evolution of body frame

Problem formulation

Suppose we are given

- a location of the origin $\vec{R}_0(\cdot)$ at time t
- an orientation $\mathbf{R}(\cdot) = \left[\vec{e}_1(\cdot), \vec{e}_2(\cdot), \vec{e}_3(\cdot)\right]$ of the body frame at time t
- a vector of velocity $\frac{d}{dt}\vec{R}_0(\cdot)$ of the origin at time t
- a vector of angular velocity $\vec{\omega}(\cdot)$ of the body frame at time t

$$\vec{R}_0(t) \mapsto$$

$$\mathbf{R}(t) = \left[\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right] \mapsto$$

Problem formulation

Suppose we are given

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- a vector of angular velocity $\vec{\omega}(\cdot)$ of the body frame at time t

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

How to compute/approximate the status of the body frame at
$t \mapsto (t + dt)$?

$$\vec{R}_0(t) \;\mapsto$$

$$\mathbf{R}(t) = \left[\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right] \;\mapsto$$

Problem formulation

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$$\Downarrow \quad \Downarrow \quad \Downarrow$$

How to compute/approximate the status of the body frame at
$t \mapsto (t + dt)$?

$$\vec{R}_0(t) \;\mapsto\; \boxed{\vec{R}_0(t+dt) = \vec{R}_0(t) + \frac{d}{dt}\vec{R}_0(t) \cdot dt}$$

$$\mathbf{R}(t) = \left[\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right] \;\mapsto$$

Problem formulation

Suppose we are given

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$$\Downarrow \quad \Downarrow \quad \Downarrow$$

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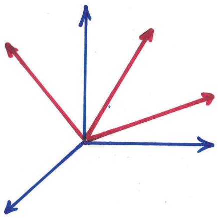
$$\vec{R}_0(t) \ \mapsto \ \boxed{\vec{R}_0(t + dt) = \vec{R}_0(t) + \frac{d}{dt}\vec{R}_0(t) \cdot dt}$$

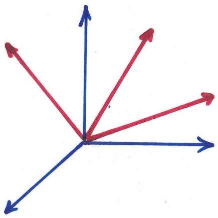
$$\mathbf{R}(t) = \left[\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)\right] \ \mapsto \ \boxed{\mathbf{R}(t + dt) = ?????}$$

Parametrization of Rotations by Euler Angles



Given $\vec{i}$-and $\vec{e}$-frames, three Euler angles are used to represent an orientation of one frame with respect to another

Parametrization of Rotations by Euler Angles



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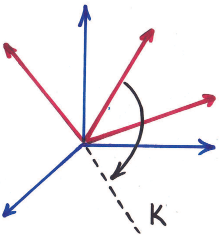

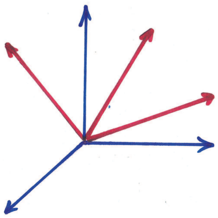

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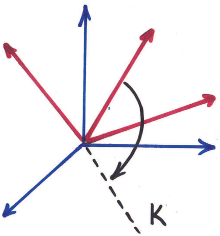
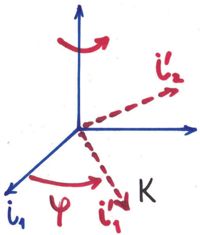
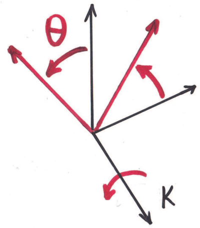
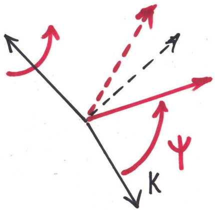




φ is a precession angle



θ is a nutation angle



ψ is an intrinsic rotation

Parametrization of Rotations by Euler Angles

$$\big[\varphi(t), \theta(t), \psi(t)\big]$$

$$\varphi(t+dt) \approx \varphi(t) + \quad , \qquad \theta(t+dt) \approx \theta(t) + \quad ,$$

$$\psi(t+dt) \approx \psi(t) +$$

How to define $\frac{d}{dt}\varphi(t)$, $\frac{d}{dt}\theta(t)$ and $\frac{d}{dt}\psi(t)$?

Parametrization of Rotations by Euler Angles

$$
\begin{array}{c}
\left[\varphi(t), \theta(t), \psi(t)\right] \\
\downarrow \\
\mathbf{R}(t) \\
\parallel \\
R_{z,\varphi(t)} \cdot R_{x,\theta(t)} \cdot R_{z,\psi(t)}
\end{array}
$$

$$\varphi(t+dt) \approx \varphi(t) + \qquad, \quad \theta(t+dt) \approx \theta(t) + \qquad,$$
$$\psi(t+dt) \approx \psi(t) + \qquad$$

How to define $\frac{d}{dt}\varphi(t)$, $\frac{d}{dt}\theta(t)$ and $\frac{d}{dt}\psi(t)$?

Parametrization of Rotations by Euler Angles

$$
\begin{array}{ccc}
\big[\varphi(t), \theta(t), \psi(t)\big] & \longmapsto & \big[\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}\big] = \big[\varphi(t + {\color{red}dt}), \theta(t + {\color{red}dt}), \psi(t + {\color{red}dt})\big] \\
\downarrow & & \\
\mathbf{R}(t) & & \\
\parallel & & \\
R_{z,\varphi(t)} \cdot R_{x,\theta(t)} \cdot R_{z,\psi(t)} & &
\end{array}
$$

$$\varphi(t + dt) \approx \varphi(t) + \qquad\qquad, \qquad \theta(t + dt) \approx \theta(t) + \qquad\qquad,$$

$$\psi(t + dt) \approx \psi(t) + \qquad\qquad$$

How to define $\frac{d}{dt}\varphi(t)$, $\frac{d}{dt}\theta(t)$ and $\frac{d}{dt}\psi(t)$?

Parametrization of Rotations by Euler Angles

$$
\begin{array}{ccc}
\big[\varphi(t), \theta(t), \psi(t)\big] & \longmapsto & [\tilde{\varphi}, \tilde{\theta}, \tilde{\psi}] = \big[\varphi(t + dt), \theta(t + dt), \psi(t + dt)\big] \\
\downarrow & & \downarrow \\
\mathbf{R}(t) & & \mathbf{R}(t + dt) \\
\| & & \| \\
R_{z,\varphi(t)} \cdot R_{x,\theta(t)} \cdot R_{z,\psi(t)} & & R_{z,\tilde{\varphi}} \cdot R_{x,\tilde{\theta}} \cdot R_{z,\tilde{\psi}}
\end{array}
$$

$$\varphi(t + dt) \approx \varphi(t) + \qquad , \quad \theta(t + dt) \approx \theta(t) + \qquad ,$$

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\mathbf{R}(t) & & \mathbf{R}(t+dt) \\
\| & & \| \\
R_{z,\varphi(t)} \cdot R_{x,\theta(t)} \cdot R_{z,\psi(t)} & & R_{z,\tilde{\varphi}} \cdot R_{x,\tilde{\theta}} \cdot R_{z,\tilde{\psi}}
\end{array}
$$

$$
\varphi(t+dt) \approx \varphi(t) + \frac{d}{dt}\varphi(t)dt, \quad \theta(t+dt) \approx \theta(t) + \frac{d}{dt}\theta(t)dt,
$$

$$
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\mathbf{R}(t) & & \mathbf{R}(t+dt) \\
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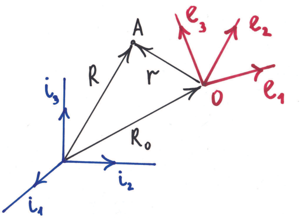
\end{array}
$$

$$
\varphi(t+dt) \approx \varphi(t) + \boxed{\frac{d}{dt}\varphi(t)dt}, \quad \theta(t+dt) \approx \theta(t) + \boxed{\frac{d}{dt}\theta(t)dt},
$$

$$
\psi(t+dt) \approx \psi(t) + \boxed{\frac{d}{dt}\psi(t)dt}
$$

How to define $\frac{d}{dt}\varphi(t)$, $\frac{d}{dt}\theta(t)$ and $\frac{d}{dt}\psi(t)$?

Angular Velocity



The vector of angular velocity is defined earlier as a function that allows defining velocity of any point A of the rigid body expressed in the i-frame as

$$\boxed{\frac{d}{dt}\vec{R}_A(t) = \frac{d}{dt}\vec{R}_0(t) + \vec{\omega}(t) \times \vec{r}(t)}$$

where coordinates of that point A of the rigid body written in the i-frame are

$$\vec{R}_A(t) = \vec{R}_0(t) + \vec{r}(t)$$

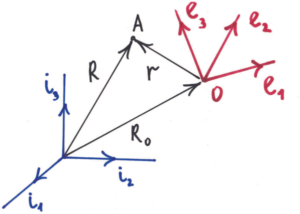
Hence, if the angular velocity is decomposed into the sum

$$\vec{\omega}(t) = \vec{\omega}_1(t) + \cdots + \vec{\omega}_N(t),$$

then

$$\frac{d}{dt}\vec{r}(t) = \vec{\omega}_1(t) \times \vec{r}(t) + \cdots + \vec{\omega}_N(t) \times \vec{r}(t)$$

Angular Velocity



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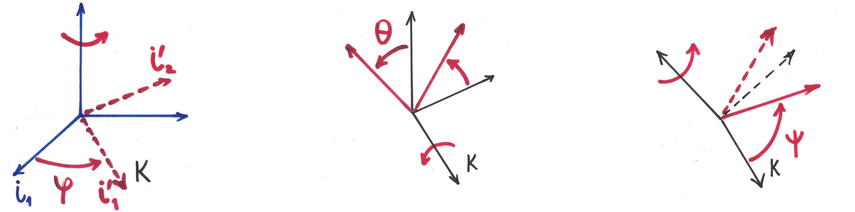
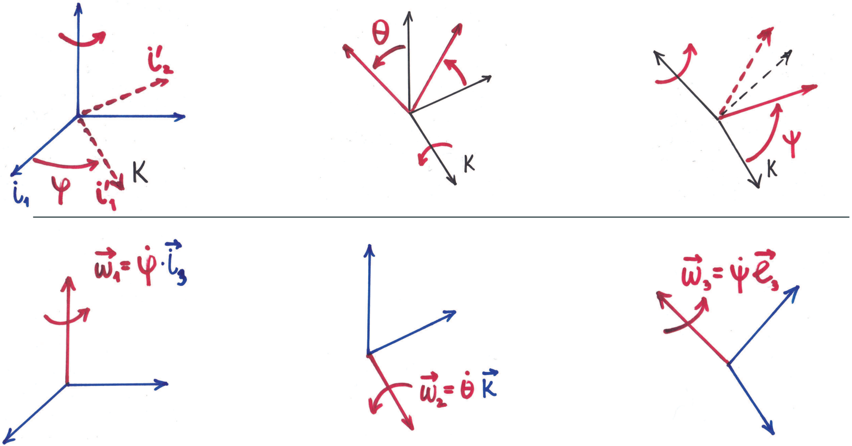
Angular Velocity



Therefore, the vector of angular velocity can be written as



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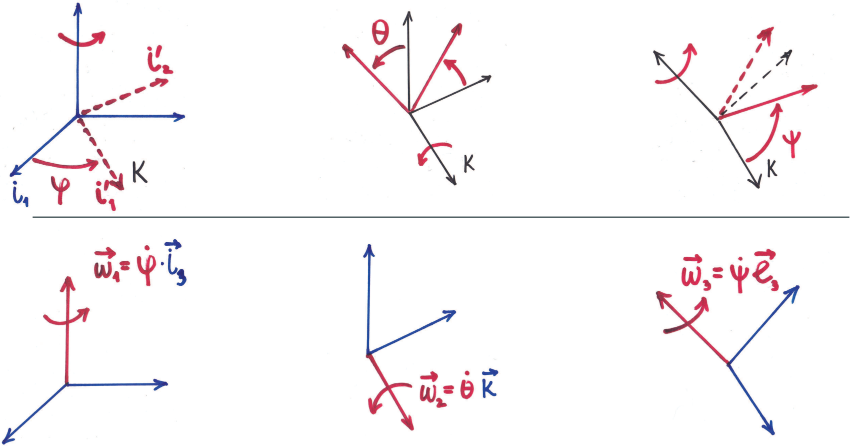
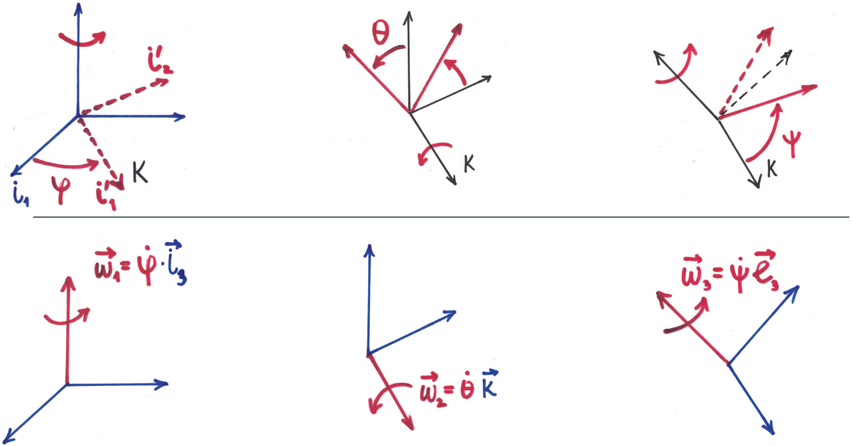
Angular Velocity



Therefore, the vector of angular velocity can be written as

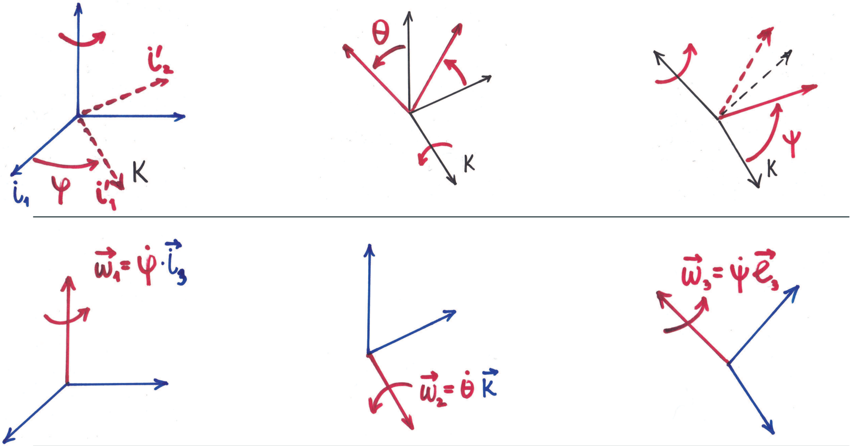
$$\vec{\omega} = \dot{\varphi} \cdot \vec{i}_3 + \dot{\theta} \cdot \vec{K} + \dot{\psi} \cdot \vec{e}_3$$

Angular Velocity



Therefore, the vector of angular velocity can be written as

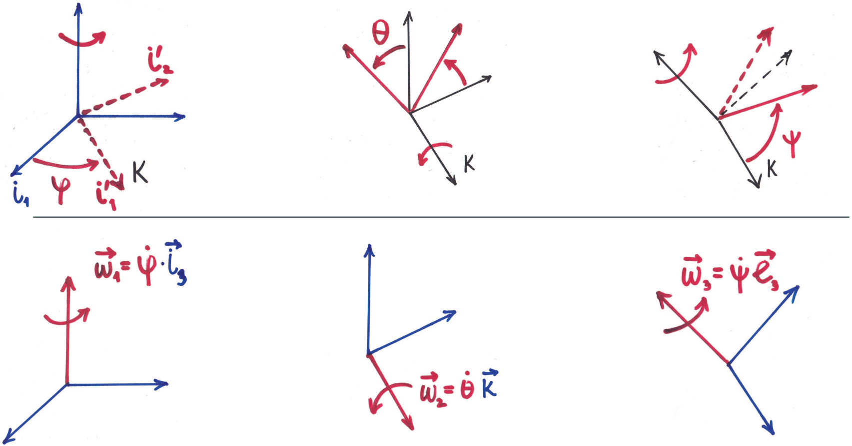
$$\boxed{\vec{\omega} = \dot{\varphi} \cdot \vec{i_3} + \dot{\theta} \cdot \vec{K} + \dot{\psi} \cdot \vec{e_3}} \quad = \quad \omega_1^i \cdot \vec{i_1} + \omega_2^i \cdot \vec{i_2} + \omega_3^i \cdot \vec{i_3}$$



Therefore, the vector of angular velocity can be written as

$$\boxed{\vec{\omega} = \dot{\varphi} \cdot \vec{i}_3 + \dot{\theta} \cdot \vec{K} + \dot{\psi} \cdot \vec{e}_3} \quad = \quad \omega_1^i \cdot \vec{i}_1 + \omega_2^i \cdot \vec{i}_2 + \omega_3^i \cdot \vec{i}_3$$
$$= \quad \omega_1^e \cdot \vec{e}_1 + \omega_2^e \cdot \vec{e}_2 + \omega_3^e \cdot \vec{e}_3$$

Angular Velocity

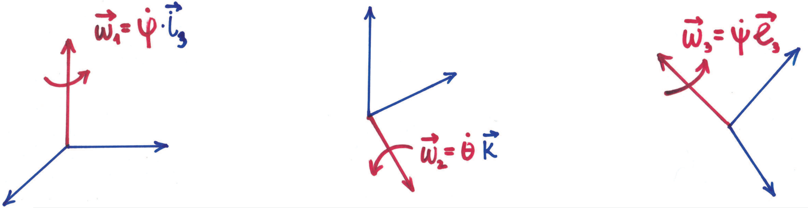


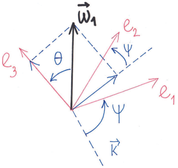
Therefore, the vector of angular velocity can be written as

$$\boxed{\vec{\omega} = \dot{\varphi} \cdot \vec{i}_3 + \dot{\theta} \cdot \vec{K} + \dot{\psi} \cdot \vec{e}_3} \quad = \quad \omega_1^i \cdot \vec{i}_1 + \omega_2^i \cdot \vec{i}_2 + \omega_3^i \cdot \vec{i}_3$$

$$= \quad \boxed{p \cdot \vec{e}_1 + q \cdot \vec{e}_2 + r \cdot \vec{e}_3}$$

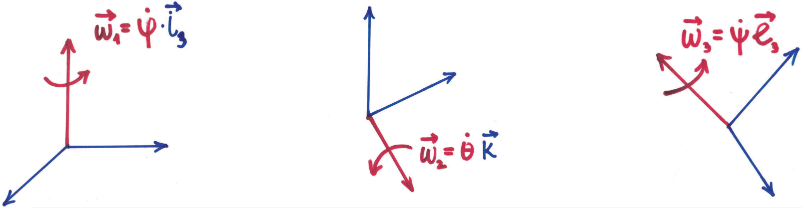
Euler kinematic equations

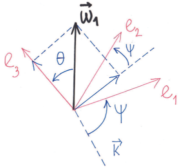




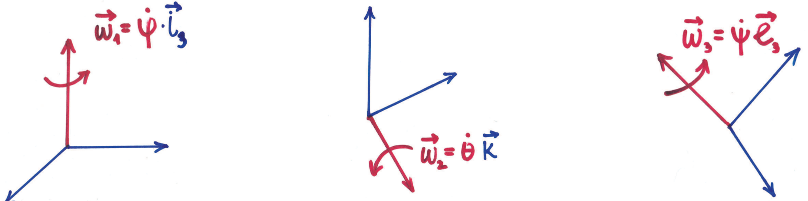
Therefore, the vector of angular velocity can be written as

$$
\begin{aligned}
p &= \dot{\varphi} \cdot \sin\theta \cdot \cos\psi - \dot{\theta} \cdot \sin\psi \\
q &= \dot{\varphi} \cdot \sin\theta \cdot \sin\psi + \dot{\theta} \cdot \cos\psi \\
r &= \dot{\varphi} \cdot \cos\theta + \dot{\psi}
\end{aligned}
$$



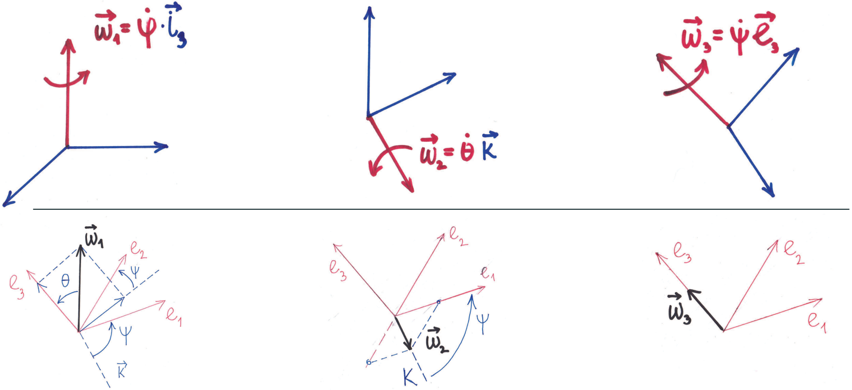


$$\vec{\omega}_1 \quad = \quad \dot{\varphi} \cdot \vec{i}_3 = \dot{\varphi} \cdot \left[\cos\theta \cdot \vec{e}_3 + \sin\theta \cdot \cos\psi \cdot \vec{e}_2 + \sin\theta \cdot \sin\psi \cdot \vec{e}_1 \right]$$



$$
\begin{aligned}
\vec{\omega}_1 &= \dot{\varphi} \cdot \vec{i}_3 = \dot{\varphi} \cdot \left[\cos\theta \cdot \vec{e}_3 + \sin\theta \cdot \cos\psi \cdot \vec{e}_2 + \sin\theta \cdot \sin\psi \cdot \vec{e}_1 \right] \\
\vec{\omega}_2 &= \dot{\theta} \cdot \vec{K} = \dot{\theta} \cdot \left[\cos\psi \cdot \vec{e}_1 - \sin\psi \cdot \vec{e}_2 \right]
\end{aligned}
$$

Euler kinematic equations

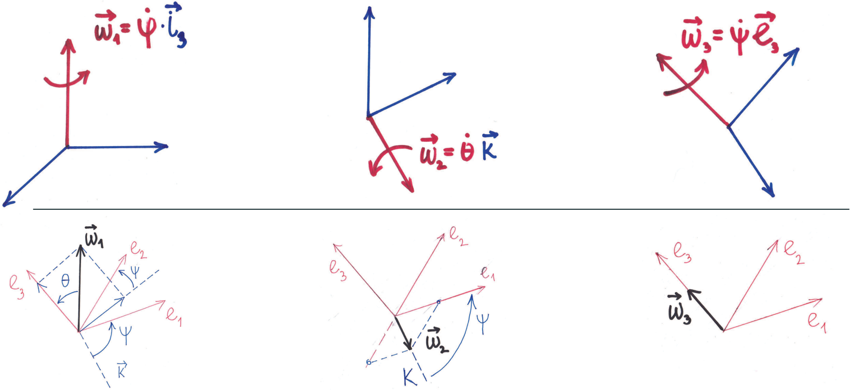


$$\vec{\omega}_1 = \dot{\varphi} \cdot \vec{i}_3 = \dot{\varphi} \cdot \left[\cos\theta \cdot \vec{e}_3 + \sin\theta \cdot \cos\psi \cdot \vec{e}_2 + \sin\theta \cdot \sin\psi \cdot \vec{e}_1\right]$$

$$\vec{\omega}_2 = \dot{\theta} \cdot \vec{K} = \dot{\theta} \cdot \left[\cos\psi \cdot \vec{e}_1 - \sin\psi \cdot \vec{e}_2\right]$$

$$\vec{\omega}_3 = \dot{\psi} \cdot \vec{e}_3$$

Euler kinematic equations



Therefore, the vector of angular velocity can be written as

$$
\begin{aligned}
\vec{\omega} &= p \cdot \vec{e}_1 + q \cdot \vec{e}_2 + r \cdot \vec{e}_3 \\
&= \left[\dot{\varphi} \cdot \sin\theta \cdot \cos\psi - \dot{\theta} \cdot \sin\psi\right] \cdot \vec{e}_1 \\
&\qquad + \left[\dot{\varphi} \cdot \sin\theta \cdot \sin\psi + \dot{\theta} \cdot \cos\psi\right] \cdot \vec{e}_2 + \left[\dot{\varphi} \cdot \cos\theta + \dot{\psi}\right] \cdot \vec{e}_3
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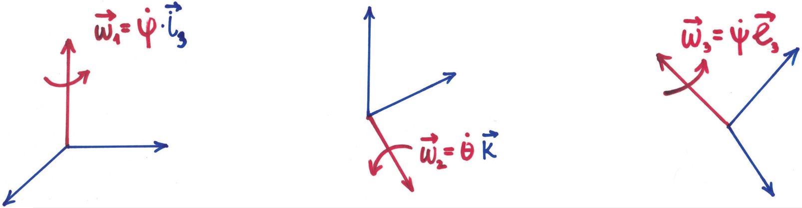

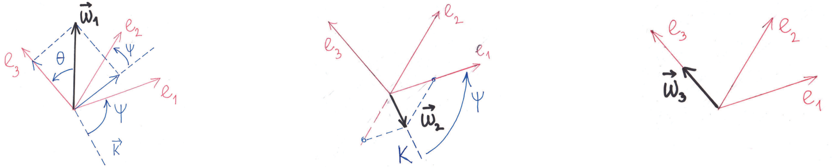

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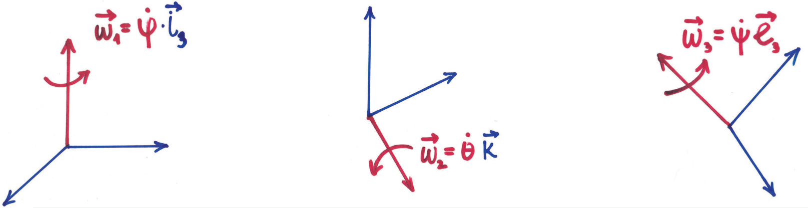

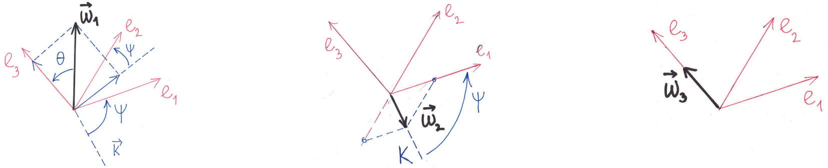

Therefore, the vector of angular velocity can be written as

$$\left.\begin{array}{rcl} p & = & \dot{\varphi}\cdot\sin\theta\cdot\cos\psi - \dot{\theta}\cdot\sin\psi \\ q & = & \dot{\varphi}\cdot\sin\theta\cdot\sin\psi + \dot{\theta}\cdot\cos\psi \\ r & = & \dot{\varphi}\cdot\cos\theta + \dot{\psi} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcl} \dot{\varphi} & = & f_1(\varphi,\theta,\psi,p,q,r) \\ \dot{\theta} & = & f_2(\varphi,\theta,\psi,p,q,r) \\ \dot{\psi} & = & f_3(\varphi,\theta,\psi,p,q,r) \end{array}\right.$$

Poisson equations

To derive singularity-free equations, consider properties of rotation matrices

For any θ the matrix $R(t)$ is a rotation so that

$$R(t)R^T(t) = R^T(t)R(t) = I_3$$

Therefore

$$\frac{d}{dt}\Big[R(t)R^T(t)\Big] = \underbrace{\frac{d}{dt}\Big[R(t)\Big]\, R^T(t)} + \underbrace{R(t)\frac{d}{dt}\Big[R^T(t)\Big]} = \frac{d}{dt}I_3 = 0$$

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$$\Rightarrow \qquad \boxed{\frac{d}{dt}\Big[R(t)\Big] = S(t)R(t), \quad S(\cdot) \text{ is skew-symmetric}}$$

Skew Symmetric Matrices

Definition

A $n \times n$ matrix S is skew symmetric and denoted $S \in so(n)$ if

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The matrix components of $S \in so(n)$ must obey

$$s_{ij} + s_{ji} = 0, \quad i = 1, \ldots, n,\ j = 1, \ldots, n$$

$$\Rightarrow \quad \left\{ \boxed{s_{ii} = 0} \quad \text{and} \quad \boxed{s_{ij} = -s_{ji} \quad \forall\, i \neq j} \right\}$$

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For example, if $n = 3$, then any $S \in so(3)$ has the form

$$S = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

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For example, if $n = 3$, then any $S \in so(3)$ has the form

$$S = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix} = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Properties

Given $\vec{a} = [a_x; a_y; a_z]$ and $S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$ the following properties hold:

1. $S(\alpha\vec{a} + \beta\vec{c}) = \alpha S\left(\vec{a}\right) + \beta S\left(\vec{c}\right)$

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$$S(\vec{a})\,\vec{p} \quad = \quad \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} p_z a_y - p_y a_z \\ p_x a_z - p_z a_x \\ p_y a_x - p_x a_y \end{bmatrix}$$

Skew Symmetric Matrices

Properties

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3. $RS(\vec{a})\,R^T = S\left(R\vec{a}\right)$ for any $R \in \mathcal{SO}(3)$, $\vec{a} \in \mathbb{R}^3$

$$\begin{aligned} R\left[S(\vec{a})\,R^T\vec{p}\right] &= R\,[\vec{a} \times R^T\vec{p}] = R[\vec{a} \times \vec{b}] \\ &= R\vec{a} \times R\vec{b} = R\vec{a} \times R\left(R^T\vec{p}\right) = R\vec{a} \times \vec{p} = S(R\vec{a})\,\vec{p} \end{aligned}$$

Skew Symmetric Matrices

Properties

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1. $S(\alpha\vec{a} + \beta\vec{c}) = \alpha S\left(\vec{a}\right) + \beta S\left(\vec{c}\right)$
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3. $RS(\vec{a})\,R^T = S\left(R\vec{a}\right)$ for any $R \in \mathcal{SO}(3)$, $\vec{a} \in \mathbb{R}^3$
4. $x^T S x = 0$ for any $S \in so(3)$, $x \in \mathbb{R}^3$

Example 4.2: Derivative of Rotation Matrix

If $R(t) = R_{z,\theta(t)}$, $\theta(t) = t$, that is the basic rotation around the axis z, then

$$S = \frac{d}{d\theta}\left[R_{z,\theta}\right] R_{z,\theta}^T = \frac{d}{d\theta}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

Example 4.2: Derivative of Rotation Matrix

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$$\begin{aligned} S &= \frac{d}{d\theta}\Big[R_{z,\theta}\Big]\, R_{z,\theta}^T = \frac{d}{d\theta}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

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&= \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = S\left(\vec{a}\right)
\end{aligned}
$$

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&= S(\vec{k}), \qquad \vec{k} = [0,\ 0,\ 1]^T
\end{aligned}
$$

Poisson equations

The differential equation

$$\frac{d}{dt}\Big[R(t)\Big] = S(t)R(t), \quad S(\cdot) \text{ is defined by } \omega(\cdot)$$

is singularity-free!

However, it is a rear situation that it can be solve analytically!

Rotation of a rigid body is very different from translation!

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Modeling and Control of Robots

Lecture 6: Kinematics of a Rigid Body (cont'd).

Anton Shiriaev
January 26, 2020

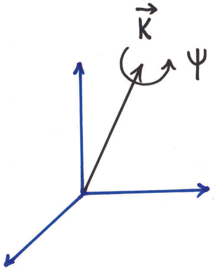
Learning outcomes: Alternative parametrization of rotation matrices using angle/axis representation and unit quaternions. Homogeneous transformation. Terminology

Outline

1. Alternative parametrization of rotations
   - Angle/Axis representation
   - Unit Quaternions
2. Homogeneous Transformations
3. Kinematic Chains
   - Assumptions and Terminology

Alternative parametrization of rotations

Angle/Axis Representation of Rotations



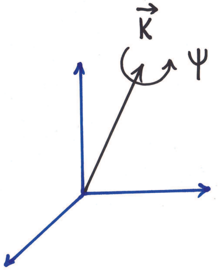
Euler: Any rotation matrix $\mathbf{R}$ can be expressed by a single rotation on angle ψ about a suitable axis defined by a vector $\vec{k} = [k_x;\ k_y;\ k_z]$ of unit length, denoted as $\boxed{\mathbf{R}_{\vec{k},\psi}}$

New interpretation: it acts on vectors of the same frame

How to compute the corresponding vector $\vec{k}$ and angle ψ

for a given rotation matrix $\mathbf{R}$?

Angle/Axis Representation of Rotations



Euler: Any rotation matrix **R** can be expressed by a single rotation on angle ψ about a suitable axis defined by a vector $\vec{k} = [k_x;\ k_y;\ k_z]$ of unit length, denoted as $\mathbf{R}_{\vec{k},\psi}$

It used to be that we interpreted **R** as a mapping of vectors $\vec{r}^{\,e} \mapsto \vec{r}^{\,i}\ (:= \mathbf{R}\vec{r}^{\,e})$ given in different frames!

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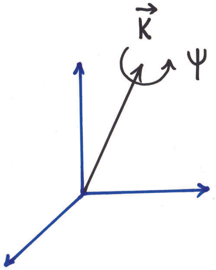
How to compute the corresponding vector $\vec{k}$ and angle ψ for a given rotation matrix **R**?

Angle/Axis Representation of Rotations



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Given $\vec{r} \in \mathbb{R}^3$, $\mathbf{R}$ transforms it into new vector $\vec{r}\,' = \mathbf{R}\vec{r}$ of the same frame

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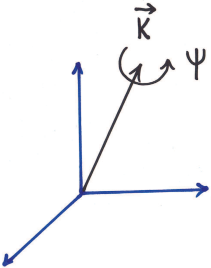
Angle/Axis Representation of Rotations



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---

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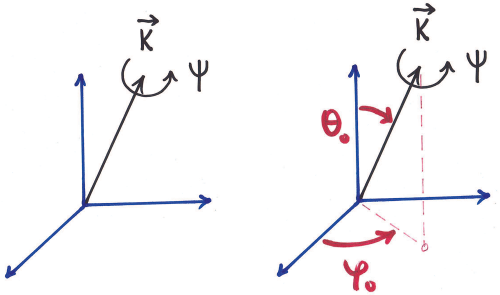
Angle/Axis Representation of Rotations

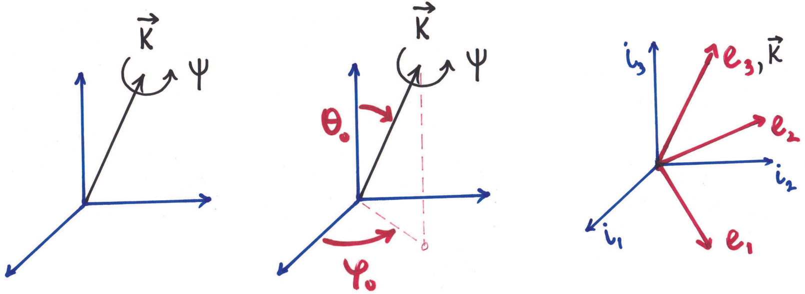


For a given $\vec{k}$ and ψ, the rotation $\mathbf{R}_{\vec{k},\psi}$ is associated with new one

$$\tilde{R} = R_{z,\phi_0} \cdot R_{y,\theta_0} \cdot R_{z,\psi}$$

Therefore, $\mathbf{R}_{\vec{k},\psi} = \left[R_{z,\phi_0} \cdot R_{y,\theta_0}\right] \cdot R_{z,\psi} \cdot \left[R_{z,\phi_0} \cdot R_{y,\theta_0}\right]^{-1}$

Angle/Axis Representation of Rotations



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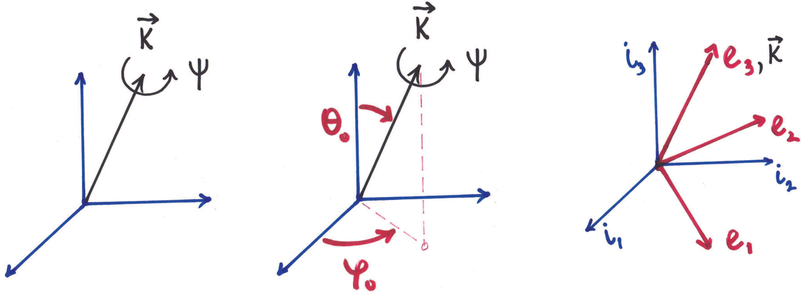

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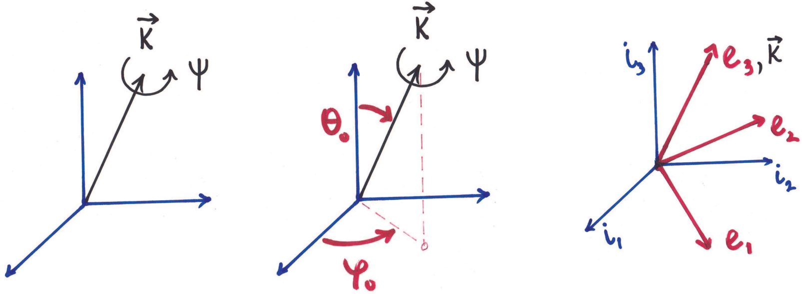
- $[R_{z,\phi_0} \cdot R_{y,\theta_0}]^{-1}$ maps any vector of i-frame into a vector of e-frame

Angle/Axis Representation of Rotations



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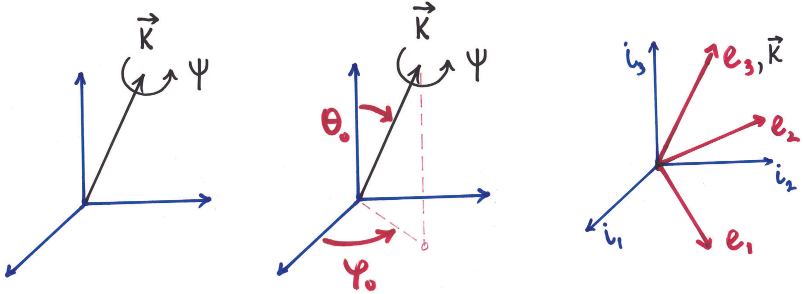
- Such vector of e-frame is rotated by $R_{z,\psi}$

Angle/Axis Representation of Rotations



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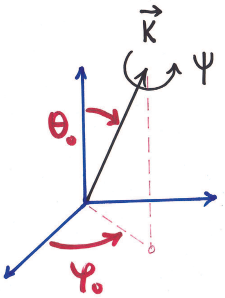
- The rotated vector is mapped back to i-frame by $\left[R_{z,\phi_0} \cdot R_{y,\theta_0}\right]$

Angle/Axis Representation of Rotations



Given the formula

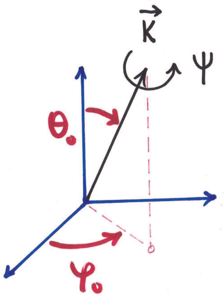
$$\begin{aligned}\mathbf{R}_{\vec{k},\psi} &= [R_{z,\phi_0} \cdot R_{y,\theta_0}] \cdot R_{z,\psi} \cdot [R_{z,\phi_0} \cdot R_{y,\theta_0}]^{-1} \\ &= \mathbf{R}\end{aligned}$$

we start searching for $\vec{k}$, ψ for $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

Consider the trace of the rotation matrix $\mathbf{R}$

$$\mathrm{tr}(\mathbf{R}) = r_{11} + r_{22} + r_{33}$$

Angle/Axis Representation of Rotations



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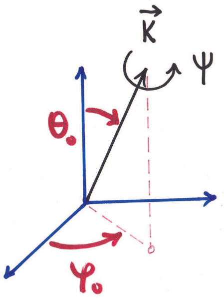

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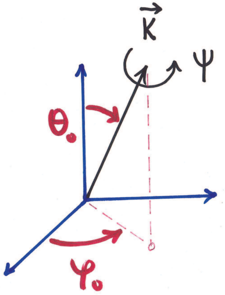
$$\mathrm{tr}\big(\mathbf{R}\big) \;=\; r_{11} + r_{22} + r_{33} = \mathrm{tr}\Big([R_{z,\phi_0} \cdot R_{y,\theta_0}] \cdot R_{z,\psi} \cdot [R_{z,\phi_0} \cdot R_{y,\theta_0}]^{-1} \Big)$$

Angle/Axis Representation of Rotations



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$$\begin{aligned} \mathbf{R}_{\vec{k},\psi} &= [R_{z,\phi_0} \cdot R_{y,\theta_0}] \cdot R_{z,\psi} \cdot [R_{z,\phi_0} \cdot R_{y,\theta_0}]^{-1} \\ &= \mathbf{R} \end{aligned}$$

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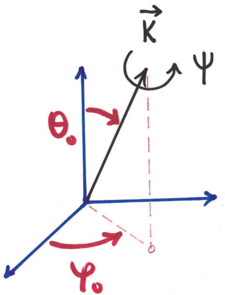
Consider the trace of the rotation matrix $\mathbf{R}$

$$\mathrm{tr}\big(\mathbf{R}\big) \quad = \quad r_{11} + r_{22} + r_{33} = \mathrm{tr}\Big([R_{z,\phi_0} \cdot R_{y,\theta_0}] \cdot R_{z,\psi} \cdot [R_{z,\phi_0} \cdot R_{y,\theta_0}]^{-1} \Big)$$

$$\boxed{\mathrm{tr}(A \cdot B) = \mathrm{tr}(B \cdot A)}$$

Angle/Axis Representation of Rotations



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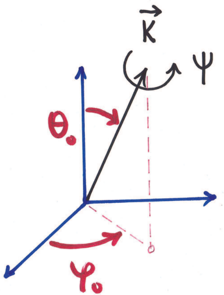
Angle/Axis Representation of Rotations



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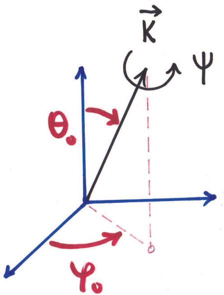
Angle/Axis Representation of Rotations



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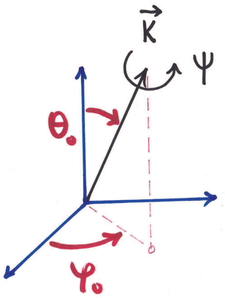
Angle/Axis Representation of Rotations



Given the formula

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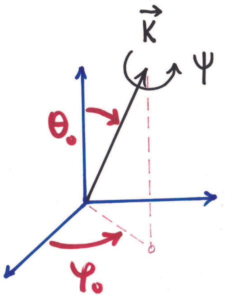
Angle/Axis Representation of Rotations



Given the formula

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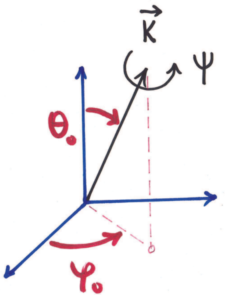
Angle/Axis Representation of Rotations



Given the formula

$$
\begin{aligned}
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&= \mathbf{R}
\end{aligned}
$$

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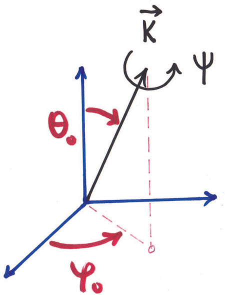
Consider a matrix $S := [\mathbf{R} - \mathbf{R}^T]$. It is skew symmetric:

$$S = -S^T$$

Angle/Axis Representation of Rotations



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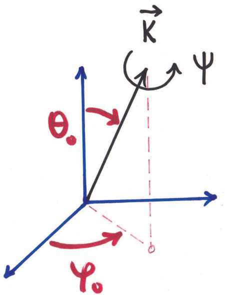

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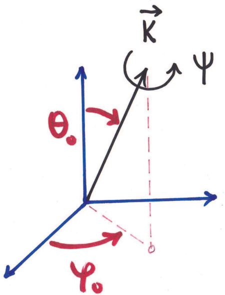

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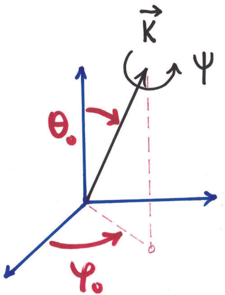

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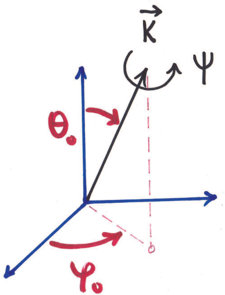

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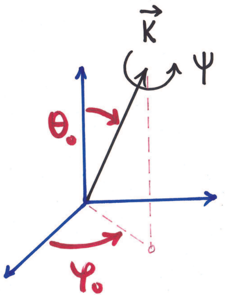

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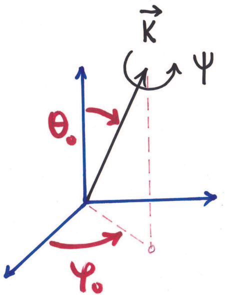

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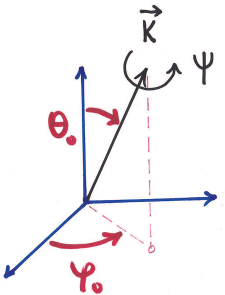

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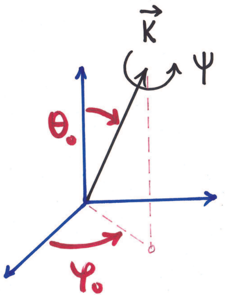

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Unit Quaternions

Quaternions are generally written as a scalar plus a vector

$$Q = \lambda_0 + \vec{\lambda} = \lambda_0 + \left[i\lambda_1 + j\lambda_2 + k\lambda_3\right], \qquad Q = \left[\lambda_0,\, \vec{\lambda}\right]$$

and serve an extension of complex numbers: $Q_1 + Q_2$, $Q_1 \circ Q_2 \mapsto Q_3$.

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It is done by computing a conjugate to quaternion with zero scalar part:

- Consider $\vec{r} \in \mathbb{R}^3$ and a quaternion $Q_r = \left[0, \, \vec{r}\right]$
- Compute the conjugate element to Q_r by the rule

$$Q_r \;\mapsto\; Q_{\vec{k},psi} \circ Q_r \circ \bar{Q}_{\vec{k},psi} =: Q_{r'}$$

- It turns out that $\vec{r}\,' = \mathbf{R}_{\vec{k},\psi}\vec{r}$

Homogeneous Transformations

Rigid Motions

A rigid motion is an ordered pair (R, d), where $R \in \mathcal{SO}(3)$ and $d \in \mathbb{R}^3$. The group of all rigid motions is known as Special Euclidean Group denoted by $\mathcal{SE}(3)$.

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If there are 3 frames corresponding to 2 rigid motions

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \end{aligned}$$

then the overall motion is

$$\boxed{p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0}$$

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HT is a convenient way to write the coordinate transformation

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Homogeneous Transformation

Given a rigid motion $(R, d) \in \mathcal{SE}(3)$, the 4×4-matrix

$$H = \begin{bmatrix} R & d \\ 0_{1\times 3} & 1 \end{bmatrix}$$

is called homogeneous transformation associated with (R, d).

To use HTs for coordinate transformations, we need to extend the vectors p^0 and p^1 by one coordinate. Namely

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \qquad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

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Kinematic Chains

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Basic Assumptions and Terminology:

- A robot manipulator is composed of a set of links connected together by joints.
- Joints can be either
  - revolute joint (a rotation by an angle about fixed axis)
  - prismatic joint (a displacement along a single axis)
  - more complicated joints (of 2 or 3 degrees of freedom) are represented as combinations of the simplest ones.
- Each joint connects two links. A robot manipulator with n joints will have $(n+1)$ links.
- We number joints from 1 to n, and links from 0 to n. So that joint i connects links $(i-1)$ and i.
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Basic Assumptions and Terminology:

- The link 0 is fixed. We call it a base.

- With the i^{th} joint, we associate a joint variable

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

- For each link we attach rigidly the coordinate frame, $(o_i x_i y_i z_i)$ for the link i.

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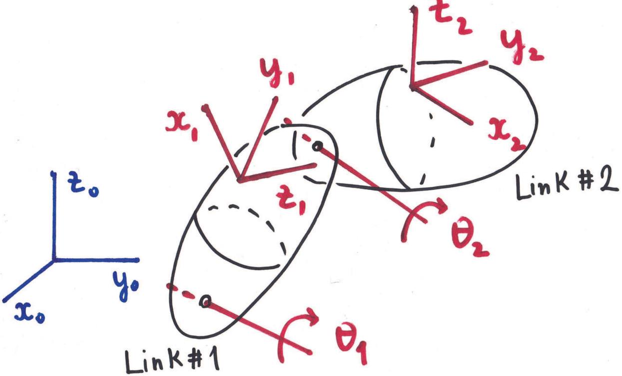

Coordinate frames attached to a open-chain manipulator

Kinematic Chains – Transformation Matrix

Basic Assumptions and Terminology:

- Suppose A_i is a homogeneous transformation that gives

  position and **orientation**

  of the frame $(o_i x_i y_i z_i)$ w.r.t. the frame $(o_{i-1} x_{i-1} y_{i-1} z_{i-1})$.

- The matrix A_i is changing as the robot configuration changes.

- The HT A_i is a function of a scalar variable: $A_i = A_i(q_i)$.

- Homogeneous transformation that expresses the position and orientation of $(o_j x_j y_j z_j)$ w.r.t. $(o_i x_i y_i z_i)$

  is called a transformation matrix.

Kinematic Chains – Transformation Matrix

Basic Assumptions and Terminology:

- Suppose A_i is a homogeneous transformation that gives

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- Homogeneous transformation that expresses the position and orientation of $(o_j x_j y_j z_j)$ w.r.t. $(o_i x_i y_i z_i)$

$$T_j^i = \begin{cases} A_{i+1} A_{i+2} \cdots A_{j-1} A_j & \text{if } i < j \\ I & \text{if } i = j \\ T_j^i = (T_i^j)^{-1} & \text{if } i > j \end{cases}$$

  is called a transformation matrix.

Modeling and Control of Robots

Lecture 7: Kinematics of an Open Chain Manipulator

Anton Shiriaev
February 1, 2020

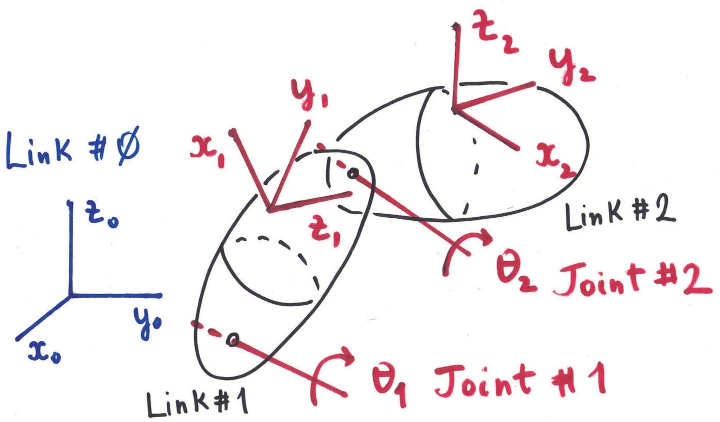
Learning outcomes: DH-parameters and convention for defining a pose of an open kinematic chain manipulator. Examples. Forward Kinematics for Open Chain Manipulators.

Outline

1. DH-parameters and convention
2. Examples
3. Forward Kinematics for Open Chain Manipulators

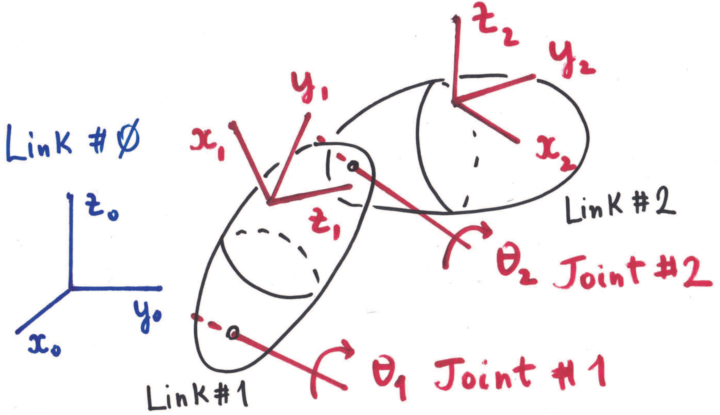
DH-parameters and convention

Open Kinematic Chain Manipulator



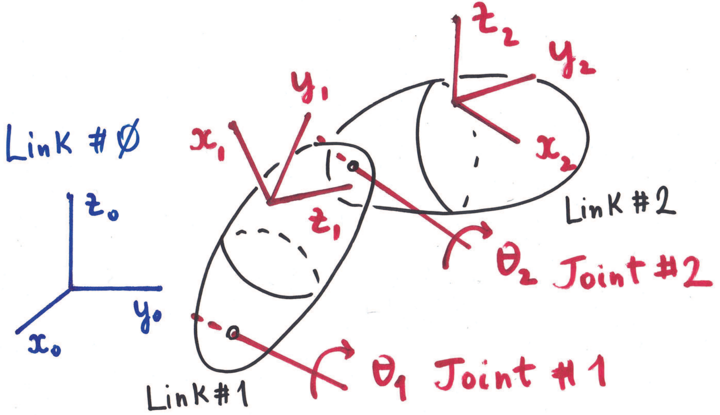
Generic choices of the world frame (Link # 0) and
frames attached to Links # 1, 2, . . . of a manipulator

Open Kinematic Chain Manipulator



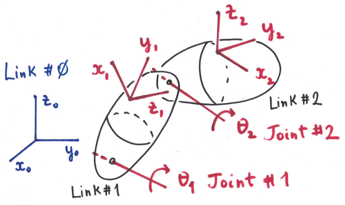
Each of homogenous transforms between frames H_1^0, H_2^1, H_3^2, ...
will depend on **6** parameters!

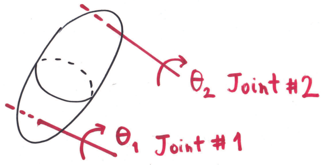
Open Kinematic Chain Manipulator



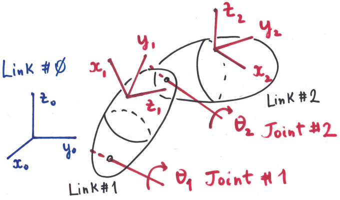
Each of homogenous transforms between frames H_1^0, H_2^1, H_3^2, ...
will depend on **6** parameters! Can it be **5**? **4**? **3**? ...

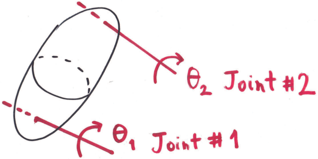
Recursive Procedure for Assigning Frames





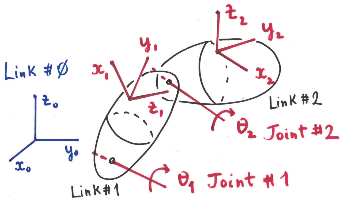
Recursive Procedure for Assigning Frames

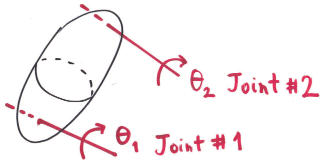




We can choose wisely the world frame (O_0, x_0, y_0, z_0) – for the link $\mathbf{0}$

Recursive Procedure for Assigning Frames



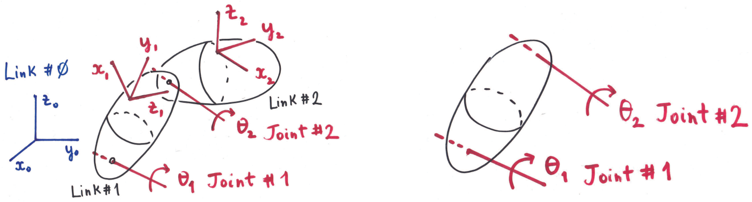


We can choose wisely the world frame $\left(O_0, x_0, y_0, z_0\right)$ – for the link $\mathbf{0}$

$$\Downarrow$$

choose wisely the frame $\left(O_1, x_1, y_1, z_1\right)$ – for the link $\mathbf{1}$

Recursive Procedure for Assigning Frames



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$$\Downarrow$$

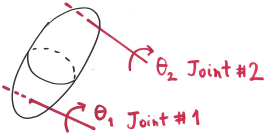
choose wisely the frame (O_1, x_1, y_1, z_1) – for the link **1**

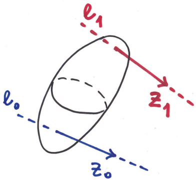
$$\Downarrow$$

choose wisely the frame (O_2, x_2, y_2, z_2) – for the link **2**

$$\Downarrow$$

$$\dots$$

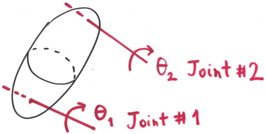


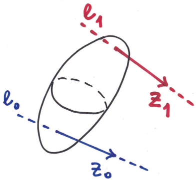


Given two lines ℓ_0 and ℓ_1 in $\mathbb{R}^3$, can we find another line that simultaneously

- do intersect ℓ_0 and ℓ_1
- is orthogonal to ℓ_0 and ℓ_1?

Recursive Procedure for Assigning Frames





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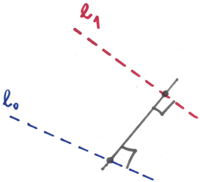
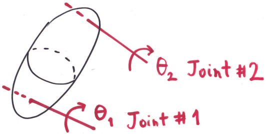
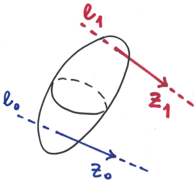


Recursive Procedure for Assigning Frames





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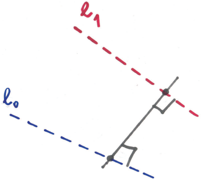
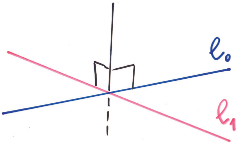
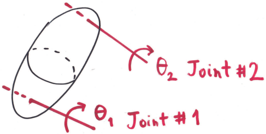
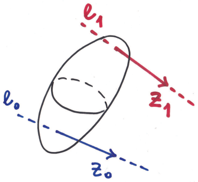


In case the lines ℓ_0 and ℓ_1 **intersect** then the line – that passes through their common point and orthogonal to the plane created as a span of ℓ_0 and ℓ_1 – does the job !!







Given two lines ℓ_0 and ℓ_1 in $\mathbb{R}^3$, can we find another line that simultaneously

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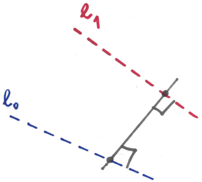
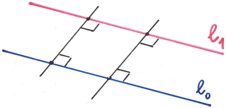
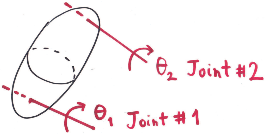
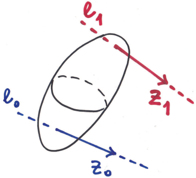


In case the lines ℓ_0 and ℓ_1 **are parallel** then there are infinitely many lines that do the job !!







Given two lines ℓ_0 and ℓ_1 in $\mathbb{R}^3$, can we find another line that simultaneously

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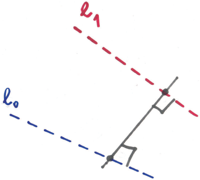
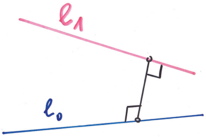
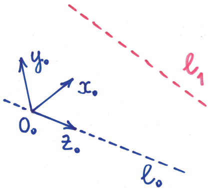


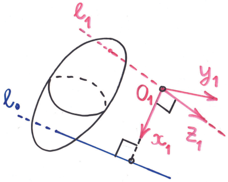
In case the lines ℓ_0 and ℓ_1 **are not parallel** and **do not intersect** then there is a point on each of lines which are closest to other line. The line passes through these points does the job !!



Recursive Procedure for Assigning Frames

How many parameters we need for moving the 0-frame into the 1-frame?





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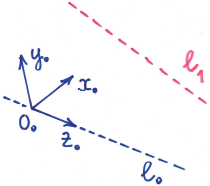

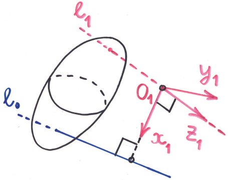

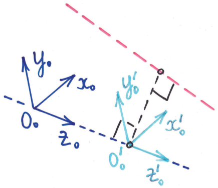

Translation along $\vec{z}_0$
on $\overrightarrow{O_o O_o'}$

Recursive Procedure for Assigning Frames

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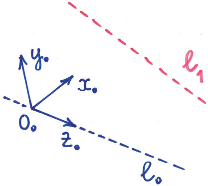
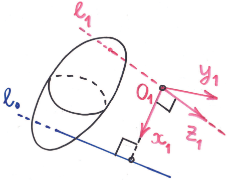
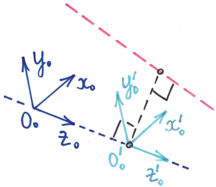
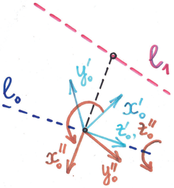






Translation along $\vec{z}_0$ on $\overrightarrow{O_o O_o'}$

$\mapsto$



Rotation about $\vec{z}_0$ to match the perpendicular by $\vec{x}_0$

Recursive Procedure for Assigning Frames

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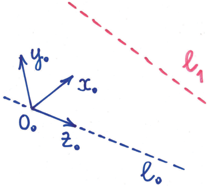
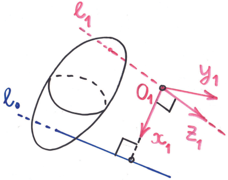
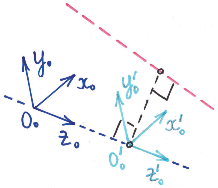
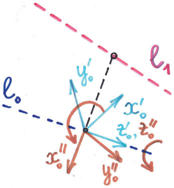






Translation along $\vec{z}_0$ on $\overrightarrow{O_o O_o'}$

$\mapsto$



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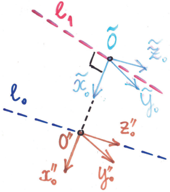


Translation along $\vec{x}_1$ on $\overrightarrow{O''\tilde{O}}$

Recursive Procedure for Assigning Frames

How many parameters we need for moving the 0-frame into the 1-frame?

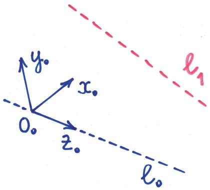
The HT A_1^0 becomes equal to a product

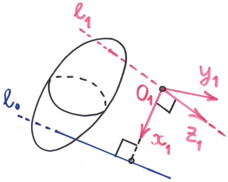
$$A_1^0 = [\mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d}] \cdot [\mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}]$$

$$= \left[\begin{array}{ccc|c} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Recursive Procedure for Assigning Frames

How many parameters we need for moving the 0-frame into the 1-frame?





The parameters of the four basic transformations are known as

- a: link length
- α: link twist
- d: link offset
- θ: link angle

Assigning Frames to Meet DH Convention

Given a robot manipulator with

- n revolute and/or prismatic joints
- $(n+1)$ links.

The task is to define

coordinate frames for each link so that HTs

between frames can be written following the DH convention.

The procedure for assigning $(n+1)$ frames to $(n+1)$ links

- is iterative by defining frame i using frame $i-1$;
- is generic although the assignment of frames is not unique.

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Procedure for DH Frame Assignment – Step 1

Step 1 – Choice of z-axes:

- Choose z_0-axis along the actuation line of the 1^{st}-link;
- Choose z_1-axis along the actuation line of the 2^{nd}-link;
- ...
- Choose $z_{(n-1)}$-axis along the actuation line of the n^{th}-link.

Procedure for DH Frame Assignment – Step 1

Step 1 – Choice of z-axes:

- Choose z_0-axis along the actuation line of the 1^{st}-link;
- Choose z_1-axis along the actuation line of the 2^{nd}-link;
- ...
- Choose $z_{(n-1)}$-axis along the actuation line of the n^{th}-link.

We need to complete the task and assign:

- point on each of z_i-axis that will be the origin of the i^{th}-frame
- x_i-axis for each frame so that two DH-conditions hold

  DH1: The axis x_i is perpendicular to the axis z_{i-1}
  DH2: The axis x_i intersects the axis z_{i-1}

- y_i-axis for each frame

Procedure for DH Frame Assignment – Step 2

Step 2 – Choice of x-axes and origins o:

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame.
- For the i^{th}-frame, the z_i axis is already fixed.
- To meet conditions DH1 and DH2

  the x_i-axis must intersect z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$

- 3 cases for assigning the new origin o_i and the x_i-axis:

  1.

     $\Rightarrow$ one common perpendicular line exists between both vectors

  2.

     $\Rightarrow$ infinitely many common perpendicular lines to choose from

  3.

     $\Rightarrow$ normal vector of the plane spanned by z_i and z_{i-1}

Procedure for DH Frame Assignment – Step 2

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Procedure for DH Frame Assignment – Step 2

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  the x_i-axis must intersect z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$
- 3 cases for assigning the new origin o_i and the x_i-axis:

  1. z_i and z_{i-1} are not coplanar:

     $\Rightarrow$ one common perpendicular line exists between both vectors
  2.

     $\Rightarrow$ infinitely many common perpendicular lines to choose from
  3.

     $\Rightarrow$ normal vector of the plane spanned by z_i and z_{i-1}

Procedure for DH Frame Assignment – Step 2

Step 2 – Choice of x-axes and origins o:

- Suppose that we have chosen the $(i-1)^{th}$-frame and need to proceed with the i^{th}-frame.
- For the i^{th}-frame, the z_i axis is already fixed.
- To meet conditions DH1 and DH2
  the x_i-axis must intersect z_{i-1} and $x_i \perp z_{i-1}$ and $x_i \perp z_i$
- 3 cases for assigning the new origin o_i and the x_i-axis:
  1. z_i and z_{i-1} are not coplanar:
     $\Rightarrow$ one common perpendicular line exists between both vectors
  2. z_i and z_{i-1} are parallel:
     $\Rightarrow$ infinitely many common perpendicular lines to choose from
  3.
     $\Rightarrow$ normal vector of the plane spanned by z_i and z_{i-1}

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     $\Rightarrow$ infinitely many common perpendicular lines to choose from
  3. z_i and z_{i-1} intersect:

     $\Rightarrow$ normal vector of the plane spanned by z_i and z_{i-1}

Procedure for DH Frame Assignment – Step 3

Step 3 – Choice of y-axes:

If we have already chosen the vectors z_i, x_i and the point o_i for the i^{th}-frame, y_i can be assigned by

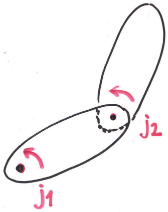
$$\text{cross-product operation:} \quad \vec{y_i} = \vec{z_i} \times \vec{x_i}$$

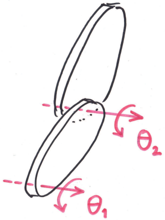
For most robots z_{n-1} and z_n coincide so the final transformation is

- translation by d_n along z_{n-1}-axis
- rotation by θ_n about z_n-axis

Examples

Example: DH-frames for Planar Two-link Robot





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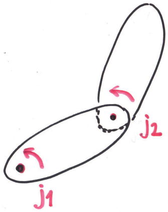

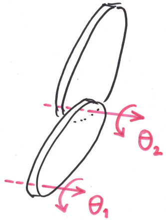

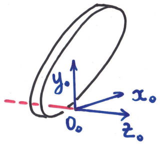

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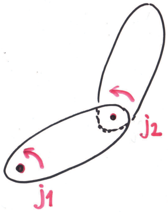
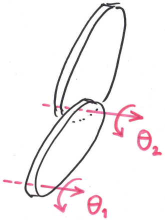
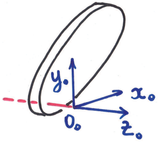
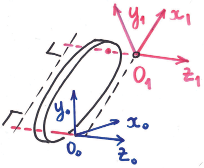






$\mapsto$



θ_1 is angle between $\vec{x}_0$ and $\vec{x}_1$

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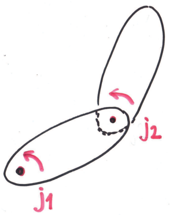

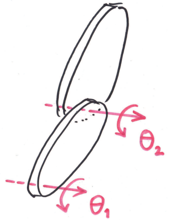

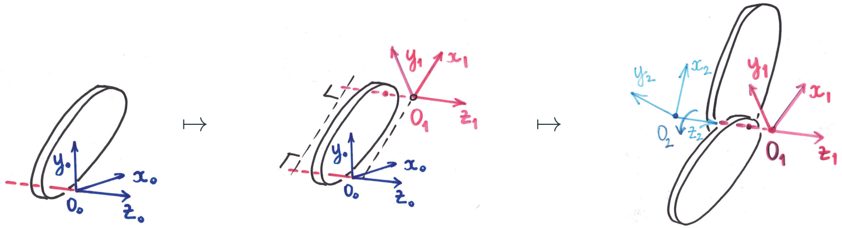

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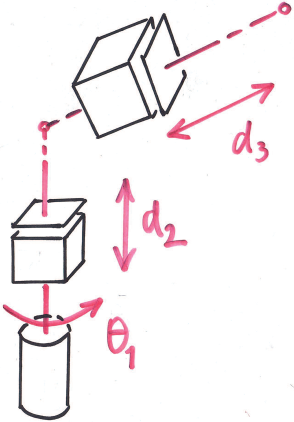
Example: DH-frames for RPP Robot



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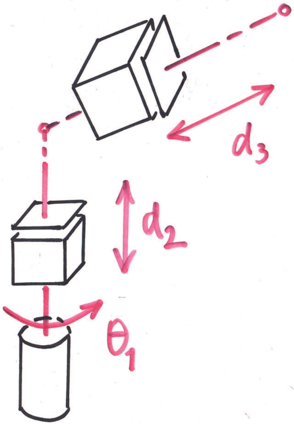

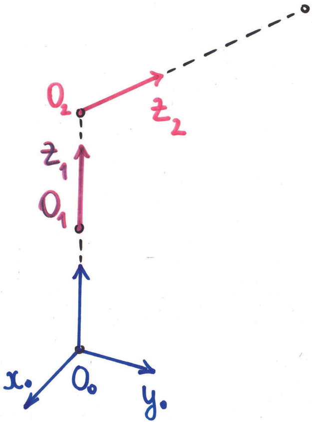

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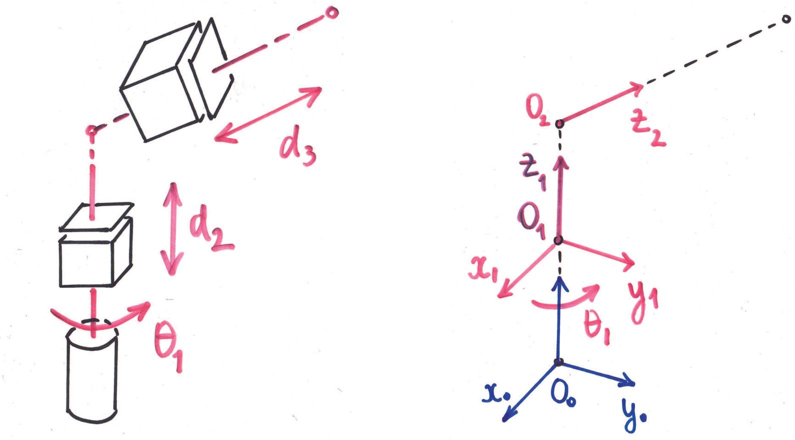

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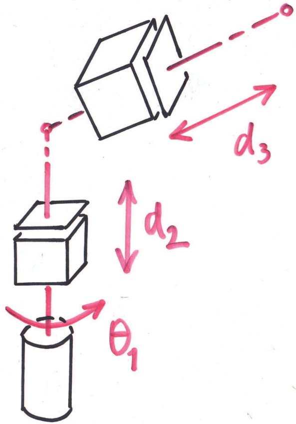

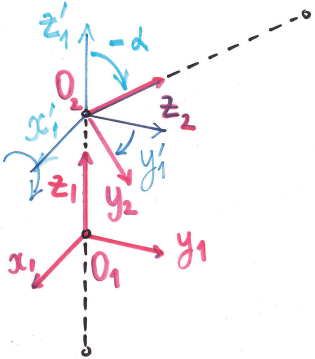

Forward Kinematics

Forward Kinematics for Open Chain Manipulators

Suppose a position and an orientation of the end-effector (tool) frame with respect to the base (world) frame are denoted as

$$o_n^0, \qquad R_n^0.$$

They can be expressed as a homogeneous transform dependent on joint variables $q_1, \ldots, q_n$

$$T_n^0 = A_1(q_1) \cdot A_2(q_2) \cdots A_{n-1}(q_{n-1}) \cdot A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

with

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Given the status of joint variables $q_1, \ldots, q_n$, the formula allows computing the pose the robot's end-effector in the world frame

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$$\Rightarrow \quad T_j^i = A_{i+1} A_{i+2} \cdots A_{j-1} A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}$$

with

$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}, \quad o_j^i = o_{j-1}^i + R_{j-1}^i o_{j-1}^j$$

Modeling and Control of Robots

Lecture 8: Kinematics of an Open Chain Manipulator

Anton Shiriaev

February 2, 2021

Learning outcomes: Inverse Kinematics for Open Chain Manipulators. Kinematic Decoupling. Manipulator Jacobian

Outline

Forward Kinematics

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Given the status of joint variables $q_1, \ldots, q_n$, the formula allows computing where the robot's end-effector will be and its orientation

Inverse Kinematics

Problem Formulation

Given a 4×4 matrix of homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in \mathcal{SE}(3)$$

The task is to find a solution of the equation

$$\boxed{T_n^0\left(q_1, \ldots, q_n\right) = A_1(q_1) A_2(q_2) \cdots A_n(q_n) = H}$$

Why is this task important?

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Why is this task important?

Robotic manipulations (tool behaviors) are typically described in the task frame but our control variables live in the joint space!

We have 12 equations to solve w.r.t. n variables $q_1, \ldots, q_n$

Requirements for Solutions

It is often advantageous to find a solution of

$$T_n^0\left(q_1,\ldots,q_n\right) = H = \left[\begin{array}{ccc|c} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

in analytical form

$$q_k = f_k\left(h_{11},\, h_{12},\, \ldots,\, h_{34}\right), \quad k = 1,\, \ldots,\, n$$

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For a closed-loop system the signals $q_k = q_k^\star(t)$ are time references to follow, so they must be computed as fast as possible.

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If there are several solutions, then on-line numerical procedures could find one that we do not want

$\Rightarrow$ Reference signals might have a jump!

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in analytical form

$$q_k = f_k\left(h_{11},\, h_{12},\, \ldots,\, h_{34}\right), \quad k = 1,\, \ldots,\, n$$

Various constraints (such as joint variables' limits, static/moving obstacles in the task space, ...) demand to test/analyze candidates $q_k = q_k^{\star}(t)$ for references in advance

Kinematic Decoupling

Given $R \in \mathcal{SO}(3)$ and $o \in \mathbb{R}^3$

- In general, solving inverse kinematics is quite difficult
- For open chain robot manipulators
  - with at least 6 joints, and
  - with the last 3 joint axes intersecting in one point (spherical wrist, wrist center)

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  - inverse position kinematics
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Angular Velocity of Moving Frame

Angular Velocity

If $R(t) \in \mathcal{SO}(3)$ is time-varying, then its time-derivative is

$$\frac{d}{dt}R(t) = S(t)R(t) = S(\omega(t))R(t), \quad S(\cdot) \in so(3)$$

The vector $\omega(t)$ will be the

angular velocity

of the rotating frame w.r.t. to the fixed frame at time t.

Consider a point p rigidly attached to a moving frame, then

$$p^0(t) = R_1^0(t)\, p^1$$

Differentiating this expression we obtain

$$\begin{aligned}\frac{d}{dt}\left[p^0(t)\right] &= \frac{d}{dt}\left[R_1^0(t)\right] p^1 = S(\omega(t))R_1^0(t)p^1 = \omega(t) \times R_1^0(t)p^1 \\ &= \omega(t) \times p^0(t)\end{aligned}$$

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Addition of Angular Velocities

An angular velocity is a free vector

$$\omega_{i,j}^{k}(t)$$

that corresponds to $\left\{ \frac{d}{dt} R_j^i(t) \right\}$ expressed in coordinate frame k.

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Taking time derivative of the left and right sides we have

$$\frac{d}{dt}R_2^0(t) \quad = \quad \frac{d}{dt}\left[R_1^0(t)\right]R_2^1(t) + R_1^0(t)\frac{d}{dt}\left[R_2^1(t)\right]$$

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&= \left[S(\omega_{0,1}^0(t)) R_1^0(t)\right] R_2^1(t) + R_1^0(t) \left[S(\omega_{1,2}^1(t)) R_2^1(t)\right]
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&= S(\omega_{0,1}^0(t))\underbrace{R_1^0(t)R_2^1(t)}_{=R_2^0(t)} + R_1^0(t)\left[S(\omega_{1,2}^1(t))R_2^1(t)\right]
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&= S(\omega_{0,1}^0(t))R_2^0(t) + \underbrace{R_1^0(t)S(\omega_{1,2}^1(t))R_1^0(t)^T}_{=S(R_1^0(t)\omega_{1,2}^1(t))}R_1^0(t)R_2^1(t)
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$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

Taking time derivative of the left and right sides we have

$$\begin{aligned}
\frac{d}{dt}R_2^0(t) &= \frac{d}{dt}\left[R_1^0(t)\right]R_2^1(t) + R_1^0(t)\frac{d}{dt}\left[R_2^1(t)\right] \\
&= S(\omega_{0,2}^0(t))R_2^0 \\
&= S(\omega_{0,1}^0(t))R_2^0(t) + S(R_1^0(t)\omega_{1,2}^1(t))\underbrace{R_1^0(t)R_2^1(t)}_{=R_2^0(t)}
\end{aligned}$$

Addition of Angular Velocities

An angular velocity is a free vector

$$\omega_{i,j}^{k}(t)$$

that corresponds to $\left\{\frac{d}{dt}R_j^i(t)\right\}$ expressed in coordinate frame k.

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&= S(\omega_{0,2}^0(t))R_2^0 \\
&= \Big[S(\omega_{0,1}^0(t)) + S(R_1^0(t)\omega_{1,2}^1(t))\Big]R_2^0(t)
\end{aligned}$$

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\end{aligned}$$

$$\Rightarrow \qquad \boxed{S(\omega_{0,2}^0(t)) = S(\omega_{0,1}^0(t)) + S(R_1^0(t)\omega_{1,2}^1(t))}$$

Addition of Angular Velocities (cont'd)

The relation

$$S(\omega^0_{0,2}(t)) = S(\omega^0_{0,1}(t)) + S(R^0_1(t)\omega^1_{1,2}(t))$$

together with the property $S(a) + S(c) = S(a + c)$ imply that the angular velocity can be computed as

$$\boxed{\omega^0_{0,2}(t) = \omega^0_{0,1}(t) + R^0_1(t)\omega^1_{1,2}(t) = \omega^0_{0,1}(t) + \omega^0_{1,2}(t)}$$

Given n-moving frames with the same origins as for the fixed one

$$R^0_n(t) = R^0_1(t)R^1_2(t)\cdots R^{n-1}_n(t) \;\Rightarrow\; \frac{d}{dt}R^0_n(t) = S(\omega^0_{0,n}(t))R^0_n(t)$$

$$\begin{aligned}\omega^0_{0,n}(t) &= \omega^0_{0,1}(t) + \omega^0_{1,2}(t) + \omega^0_{2,3}(t) + \cdots + \omega^0_{n-1,n}(t)\\ &= \omega^0_{0,1} + R^0_1\omega^1_{1,2} + R^0_2\omega^2_{2,3} + \cdots + R^0_{n-1}\omega^{n-1}_{n-1,n}\end{aligned}$$

Addition of Angular Velocities (cont'd)

The relation

$$S(\omega_{0,2}^0(t)) = S(\omega_{0,1}^0(t)) + S(R_1^0(t)\omega_{1,2}^1(t))$$

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\end{aligned}$$

Manipulator Jacobian

Concept of a Manipulator Jacobian

Given an n-link manipulator with joint variables $q_1, \ldots, q_n$

- Let $T_n^0(q)$ be the homogeneous transformation between the end-effector frame and base frame

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}, \quad q = \Big[q_1, \ldots, q_n\Big]^T$$

  so that $\forall\, p$ with coordinates p^n its coordinates in the base frame are

$$p^0 = R_n^0\, p^n + o_n^0$$

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  so that $\forall\, p$ with coordinates p^n its coordinates in the base frame are

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- As the robot moves joint variables become functions of time

$$t \;\to\; q(t) = \Big[q_1(t), \ldots, q_n(t)\Big]^T$$

  so that

$$\boxed{p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))}$$

Concept of a Manipulator Jacobian

Given the formula for the coordinates of p

$$\boxed{p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))}\,,$$

how to compute its velocity $\frac{d}{dt} p^0(t)$?

By the chain rule

$$\begin{aligned}
\frac{d}{dt} p^0(t) &= \frac{d}{dt}\left[R_n^0(q(t))\, p^n\right] + \frac{d}{dt}\left[o_n^0(q(t))\right] \\
&= \frac{\partial}{\partial q}\left[R_n^0(q)\, p^n\right]\Big|_{q=q(t)} \frac{d}{dt} q(t) + \frac{\partial}{\partial q}\left[o_n^0(q)\right]\Big|_{q=q(t)} \frac{d}{dt} q(t) \\
&= \quad \frac{d}{dt} q(t)
\end{aligned}$$

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&= [\cdots]\, \tfrac{d}{dt} q(t)
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&= \boxed{[\cdots]}\, \frac{d}{dt}q(t)
\end{aligned}$$

At the same time, the velocity of the point $p(\cdot)$ is given by

$$\frac{d}{dt}p^0(t) = \frac{d}{dt}\Big[R_n^0(q(t))\Big]\, p^n + \frac{d}{dt}\left[o_n^0(q(t))\right]$$

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&= [\cdots]\, \frac{d}{dt}q(t)
\end{aligned}$$

At the same time, the velocity of the point $p(\cdot)$ is given by

$$\begin{aligned}
\frac{d}{dt}p^0(t) &= S(\omega_{0,n}^0(t))R_n^0(q(t))p^n + v_n^0(t) \\
&= \omega_{0,n}^0(t) \times \left[p^0(t) - o_n^0(q(t))\right] + v_n^0(t)
\end{aligned}$$

Conclusions:

- to compute velocity of any point in the end-effector frame, it is necessary to know
  - an angular velocity $\omega_{0,n}^{0}(t)$ of the end-effector frame;
  - a linear velocity $v_n^0(t)$ of the the end-effector frame origin.

Concept of a Manipulator Jacobian

Conclusions:

- to compute velocity of any point in the end-effector frame, it is necessary to know
  - an angular velocity $\omega_{0,n}^0(t)$ of the end-effector frame;
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- the Euler formula for computing the velocity of a point

$$\frac{d}{dt} p^0(t) \quad = \quad \boxed{\omega_{0,n}^0(t) \times \left[p^0(t) - o_n^0(q(t))\right] + v_n^0(t)}$$

provides an explicit form for realizing the differential relation

$$\frac{d}{dt} p^0(t) \quad = \quad \boxed{[\cdots]} \, \frac{d}{dt} q(t)$$

defined by velocities of joint variables of the manipulator

Concept of a Manipulator Jacobian

Given

- a n-link manipulator with joint variables $q_1, \ldots, q_n$
- its particular motion $q(t) = \left[q_1(t), \ldots, q_n(t) \right]^T$

What do the vector functions $\omega_{0,n}^0(t)$ and $v_n^0(t)$ depend on?

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We will search them in the form

$$v^0_n(t) = J_v(q(t))\,\frac{d}{dt}q(t) \qquad \omega^0_{0,n}(t) = J_\omega(q(t))\,\frac{d}{dt}q(t)$$

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The $6 \times n$-matrix function $J(\cdot)$ defined by

$$\xi(t) = \begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = J(q(t))\, \tfrac{d}{dt} q(t) = \begin{bmatrix} J_v(q(t)) \\ J_\omega(q(t)) \end{bmatrix} \tfrac{d}{dt} q(t)$$

is the manipulator Jacobian; $\xi(t)$ is a vector of body velocities.

Manipulator Jacobian: Computing Angular Velocity

Given n-moving frames with the same origins as the fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \tfrac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \Rightarrow\; \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If ($\rho_i = 1$), then

- axis of rotation coincides with z_i
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$, where $\vec{k} = [0, 0, 1]^T$

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If i^{th}-joint is revolute ($\rho_i = 1$), then

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Therefore

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Manipulator Jacobian: Computing Angular Velocity

Given n-moving frames with the same origins as the fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \tfrac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

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Given n-moving frames with the same origins as the fixed one

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$$\omega_{0,n}^0(t) = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0 = \left[\rho_1 z_0^0,\; \rho_2 z_1^0,\; \ldots,\; \rho_n z_{n-1}^0\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

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Therefore

$$\omega_{0,n}^0(t) \;=\; \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0 = \underbrace{\left[\rho_1 z_0^0,\; \rho_2 z_1^0,\; \ldots,\; \rho_n z_{n-1}^0\right]}_{= J_\omega} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Manipulator Jacobian: Computing Linear Velocity of a Frame

The linear velocity $v_n^0(t)$ of the end-effector is the time-derivative of $o_n^0(t)$ and $v_n^0(t) \equiv 0$ if $\dot{q} \equiv 0$.

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Therefore there are functions $J_{v_1}(q(t)), \ldots, J_{v_n}(q(t))$ such that

$$v_n^0(t) \quad = \quad \tfrac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n$$

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Therefore there are functions $J_{v_1}(q(t))$, ..., $J_{v_n}(q(t))$ such that

$$\begin{aligned} v_n^0(t) &= \tfrac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n \\ &= \left[\frac{\partial}{\partial q_1} o_n^0(t)\right]\dot{q}_1 + \left[\frac{\partial}{\partial q_2} o_n^0(t)\right]\dot{q}_2 + \cdots + \left[\frac{\partial}{\partial q_n} o_n^0(t)\right]\dot{q}_n \end{aligned}$$

Manipulator Jacobian: Computing Linear Velocity of a Frame

For a motion of the end effector due to displacing a prismatic joint i along the $\vec{z}_{i-1}$-axis with velocity $\frac{d}{dt} d_i$, when all other joints are kept fixed, we have

$$\tfrac{d}{dt} o_n^0(t) = \tfrac{d}{dt} d_i(t)\, R_{i-1}^0 \vec{k} = \tfrac{d}{dt} d_i(t)\, z_{i-1} = J_{v_i} \dot{d}_i$$

Manipulator Jacobian: Computing Linear Velocity of a Frame

For a motion of the end effector due to activating a revolute joint i and rotating about the $\vec{z}_{i-1}$-axis with angular velocity $\frac{d}{dt}\theta_i \vec{z}_{i-1}$ when all other joints are kept fixed, we have

$$\frac{d}{dt} o_n^0(t) = R_{i-1}^0 \left[\omega_{i-1,n}^{i-1} \times o_n^{i-1}\right] = \left[\dot{\theta}_i z_{i-1}\right] \times [o_n - o_{i-1}] = J_{v_i} \dot{\theta}_i$$

Manipulator Jacobian: Computing Linear Velocity of a Frame

The linear velocity $v_n^0(t)$ of the end-effector is the time-derivative of $o_n^0(t)$ and $v_n^0(t) \equiv 0$ if $\dot{q} \equiv 0$.

Therefore there are functions $J_{v_1}(q(t)), \ldots, J_{v_n}(q(t))$ such that

$$
\begin{aligned}
v_n^0(t) &= \tfrac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n \\
&= \left[\frac{\partial}{\partial q_1} o_n^0(t)\right]\dot{q}_1 + \left[\frac{\partial}{\partial q_2} o_n^0(t)\right]\dot{q}_2 + \cdots + \left[\frac{\partial}{\partial q_n} o_n^0(t)\right]\dot{q}_n
\end{aligned}
$$

$$
J_{v_i} = \begin{cases} z_{i-1}^0 \times \left[o_n^0 - o_{i-1}^0\right] & \text{for revolute joint} \\ z_{i-1}^0 & \text{for prismatic joint} \end{cases}
$$

Recipe

Each column of the Jacobian corresponds to a particular joint i of the manipulator and takes the following form:

$$J_i = \begin{cases} \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix} & \text{if joint } i \text{ is revolute} \\ \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} & \text{if joint } i \text{ is prismatic} \end{cases}$$

Analytical Manipulator Jacobian

Analytical Manipulator Jacobian

Given a homogeneous transform to a robot end effector

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}, \quad q = \Big[q_1, \ldots, q_n\Big]$$

A minimal representation for the end-effector orientation can be

$$R_n^0(q) = R(\alpha) = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

with $\alpha = [\phi, \theta, \psi]^T$ being the Euler angles
(e.g. ZXZ-parametrization).

Analytical Manipulator Jacobian

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If the robot moves $q = q(t)$, then $\omega_{0,n}^0(t)$ is defined by

$$\tfrac{d}{dt} R_n^0(q(t)) = S(\omega_{0,n}^0(t)) R_n^0(q(t))$$

Analytical Manipulator Jacobian

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If the robot moves $q = q(t)$, then $\omega_{0,n}^0(t)$ is defined by

$$\tfrac{d}{dt} R(\alpha(t)) = S(\omega_{0,n}^0(t)) R(\alpha(t))$$

Angular velocity is related to the end-effector pose parameters by

$$\boxed{\omega_{0,n}^0(t) = B(\alpha(t)) \tfrac{d}{dt} \alpha(t)}$$

Analytical Manipulatior Jacobian (2)

The analytical Jacobian relates joint velocities to the time derivative of pose parameters of the end effector

$$X = \begin{bmatrix} o_n^0(q) \\ \alpha(q) \end{bmatrix} \quad \Rightarrow \quad \dot{X} = \begin{bmatrix} v_n^0 \\ \dot{\alpha} \end{bmatrix} = J_a(q)\ \dot{q}$$

Both manipulator and analytical Jacobian are closely related

$$\begin{aligned} \begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \qquad \dot{q}(t) \quad &= \begin{bmatrix} v_n^0(t) \\ B(\alpha(t))\dot{\alpha}(t) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & B(\alpha(t)) \end{bmatrix} \begin{bmatrix} v_n^0(t) \\ \dot{\alpha}(t) \end{bmatrix} \\ &= \qquad \dot{q}(t) \end{aligned}$$

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Both manipulator and analytical Jacobian are closely related

$$\begin{aligned}
\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = J(q(t))\,\dot{q}(t) \quad &= \quad \begin{bmatrix} v_n^0(t) \\ B(\alpha(t))\dot{\alpha}(t) \end{bmatrix} \\
&= \quad \begin{bmatrix} I & 0 \\ 0 & B(\alpha(t)) \end{bmatrix} \begin{bmatrix} v_n^0(t) \\ \dot{\alpha}(t) \end{bmatrix} \\
&= \quad \begin{bmatrix} I & 0 \\ 0 & B(\alpha(t)) \end{bmatrix} J_a(q(t))\,\dot{q}(t)
\end{aligned}$$

Modeling and Control of Robots

Lecture 9: Steps in Calibration of Kinematics of an Open Chain Manipulator

Anton Shiriaev

February 9, 2021

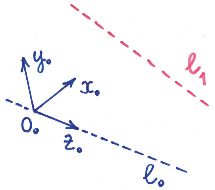
Learning outcomes: Calibration of DH-parameters of open-chain manipulator kinematics

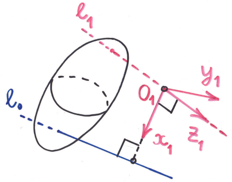
Outline

1. Modified DH-parameters
2. KUKA LWR: m-DH-parameters Calibration
3. Summary of Kinematics

Modified DH-parameters

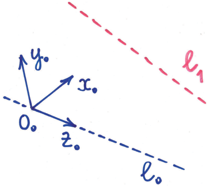
How many parameters we need for moving the 0-frame into the 1-frame?

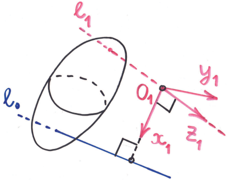


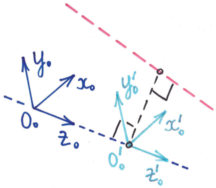


DH-parameters for Assigning Frames

How many parameters we need for moving the 0-frame into the 1-frame?







Translation along $\vec{z}_0$ on $\overrightarrow{O_o O_o'}$

DH-parameters for Assigning Frames

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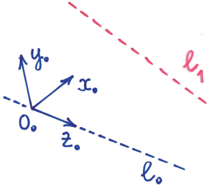
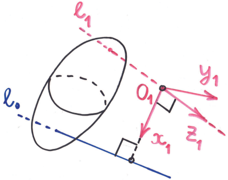
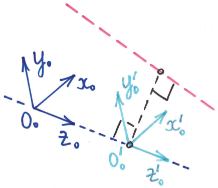
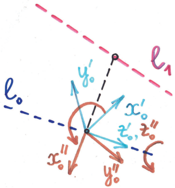






Translation along $\vec{z}_0$ on $\overrightarrow{O_o O_o'}$

$\mapsto$



Rotation about $\vec{z}_0$ to match the perpendicular by $\vec{x}_0$

DH-parameters for Assigning Frames

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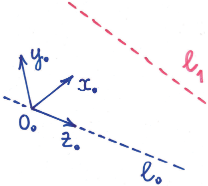
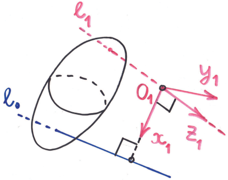
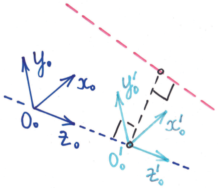
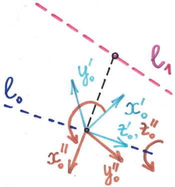


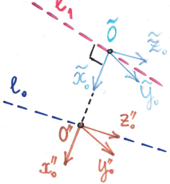




$\mapsto$



$\mapsto$



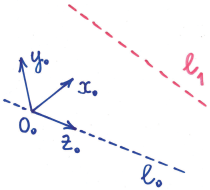
Translation along $\vec{z}_0$ on $\overrightarrow{O_o O_o'}$

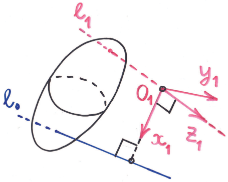
Rotation about $\vec{z}_0$ to match the perpendicular by $\vec{x}_0$

Translation along $\vec{x}_1$ on $\overrightarrow{O'' \tilde{O}}$

DH-parameters for Assigning Frames

How many parameters we need for moving the 0-frame into the 1-frame?





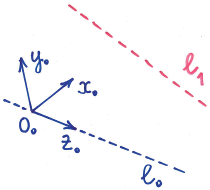
The HT A_1^0 becomes equal to a product

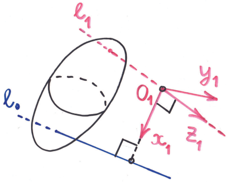
$$A_1^0 = [\mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d}] \cdot [\mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}]$$

$$= \left[\begin{array}{ccc|c} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

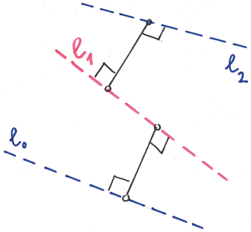
Modified DH-parameters for Assigning Frames

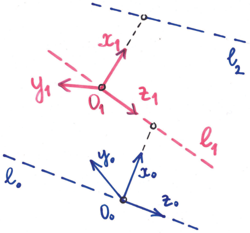
Original DH-frames are





The set of modified DH-parameters defines different frames for links





Modified DH-parameters for Assigning Frames

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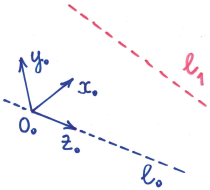
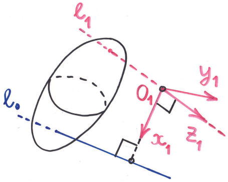
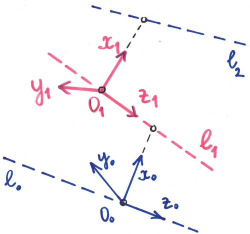




The set of modified DH-parameters defines different frames for links

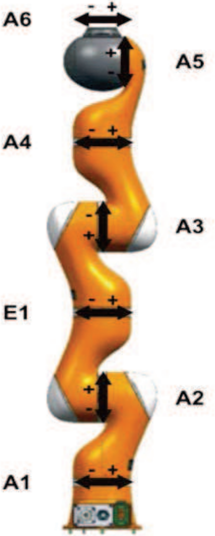
$$A_1^0 = [\mathrm{Trans}_{x,d} \cdot \mathrm{Rot}_{x,\alpha}] \cdot$$

$$[\mathrm{Trans}_{z,r} \cdot \mathrm{Rot}_{z,\theta}]$$



KUKA LWR: m-DH-parameters Calibration

Calibration of m-DH-parameters for KUKA LWR



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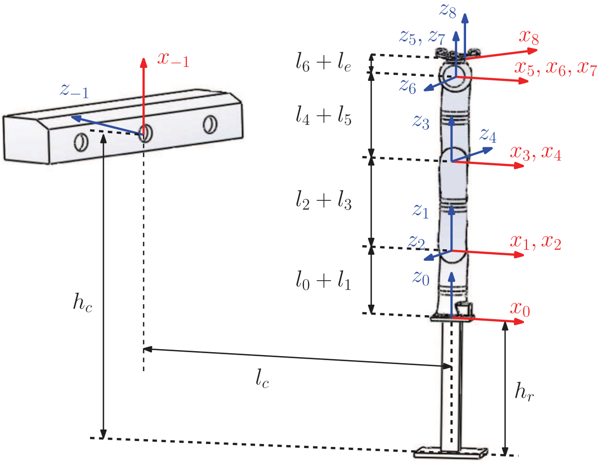
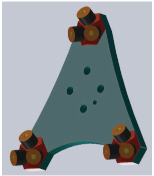
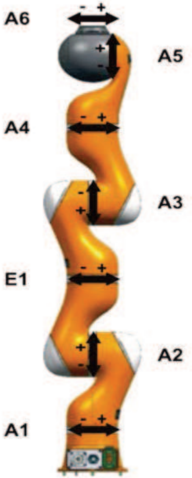




https://www.youtube.com/watch?v=PzcKRwr76kk

Table 1: Nominal mDH parameters

Frame	α_i, [rad]	d_i, [m]	θ_i, [rad]	r_i, [m]
0	0	0	1.55	−4.3
1	1.5	0.1	$q_1 - 0.79$	−0.14
2	$\frac{\pi}{2}$	0	q_2	0
3	$-\frac{\pi}{2}$	0	q_3	0.4
4	$-\frac{\pi}{2}$	0	q_4	0
5	$\frac{\pi}{2}$	0	q_5	0.39
6	$\frac{\pi}{2}$	0	q_6	0
7	$-\frac{\pi}{2}$	0	q_7	0
8	0	0.0015	0.79	0.093



$$A_i^{i-1} = [\mathrm{Trans}_{x,d} \cdot \mathrm{Rot}_{x,\alpha}] \cdot [\mathrm{Trans}_{z,r} \cdot \mathrm{Rot}_{z,\theta}]$$

Calibration of m-DH-parameters for KUKA LWR

Given the vector of joint variables

$$\vec{q} = \big[q_1,\, q_2,\, \ldots,\, q_7\big],$$

the pose of the end-effector y can be

- measured by an external sensor $\rightsquigarrow$ y_m

The calibration task is to minimize the error

$$\triangle y = y_m - y_c(\hat{\Phi}) \;\mapsto\; 0$$

by updating an estimate $\hat{\Phi}$ for Φ.

Calibration of m-DH-parameters for KUKA LWR

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- computed y_c as a function of our estimate of DH-parameters

$$\Phi = \Big[\, \{\alpha_{-1},\, d_{-1},\, \theta_{-1},\, r_{-1}\}\,,\, \{\alpha_0,\, d_0,\, \theta_0,\, r_0\}\,,\, \ldots,\, \{\alpha_8,\, d_8,\, \theta_8,\, r_8\} \Big]$$

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Nonlinear constrained optimization and nontrivial experiment design!!!

Comments on design of experiment

- We cannot do infinitely many experiments!

Calibration of m-DH-parameters for KUKA LWR

Comments on design of experiment

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- We need to choose a number of robot configurations $\vec{q}^1, \ldots, \vec{q}^M$ that
  - is reasonably small and
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$$\left\{\vec{q}^1, \ldots, \vec{q}^M\right\} \mapsto \vec{q}^{M+1}$$

that allows to improve the current estimate of DH-parameters in a best possible way? Which of DH-parameters are identifiable?

Calibration of m-DH-parameters for KUKA LWR

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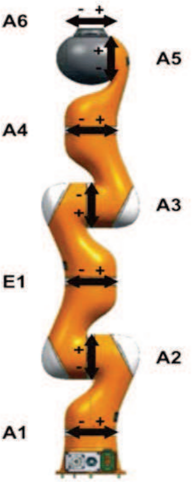
- How to recast the task as another optimization problem?

Table 2: Calibrated mDH parameters

Frame	α_i, [rad]	d_i, [m]	θ_i, [rad]	r_i, [m]
0	0	0	1.54925	-4.40661
1	1.52726	0.06978	$q_1 - 0.7994$	-0.13754
2	1.57092	-0.00026	$q_2 - 0.0042$	$3.9 \cdot 10^{-5}$
3	-1.57019	-0.00034	$q_3 - 0.0098$	0.39997
4	-1.57066	0.00104	$q_4 - 0.0065$	$2.6 \cdot 10^{-4}$
5	1.57069	-0.00095	$q_5 + 0.0011$	0.39082
6	1.57144	0.00037	$q_6 - 0.0016$	$1.6 \cdot 10^{-4}$
7	-1.57116	-0.00059	q_7	0
8	0.00053	-0.00021	0.7859	0.09109



$$A_i^{i-1} = [\mathrm{Trans}_{x,d} \cdot \mathrm{Rot}_{x,\alpha}] \cdot [\mathrm{Trans}_{z,r} \cdot \mathrm{Rot}_{z,\theta}]$$

Calibration of m-DH-parameters for KUKA LWR

Table 4: Forward kinematics errors (volumetric)

Parameter set	**Position, [mm]**		**Orientation, [deg]**	
	max	average	max	average
Nominal	9.6806	5.9019	0.8058	0.5636
Calibrated	1.7592	0.7965	0.2268	0.1042

Conclusions:

- It might be challenging to calibrate the robot kinematics globally

The kinematic model should be upgraded!

Calibration of m-DH-parameters for KUKA LWR

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Conclusions:

- It might be challenging to calibrate the robot kinematics globally
- The robot has limited ability for performing accurately point-to-point motions and, therefore, maneuvers

The kinematic model should be upgraded!

Calibration of m-DH-parameters for KUKA LWR

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Conclusions:

- It might be challenging to calibrate the robot kinematics globally
- The robot has limited ability for performing accurately point-to-point motions and, therefore, maneuvers
- The robot has limited ability for performing accurately contact operations that can be programmed in advance

The kinematic model should be upgraded!

Calibration of m-DH-parameters for KUKA LWR

Table 4: Forward kinematics errors (volumetric)

Parameter set	Position, [mm]		Orientation, [deg]	
	max	average	max	average
Nominal	9.6806	5.9019	0.8058	0.5636
Calibrated	1.7592	0.7965	0.2268	0.1042

Conclusions:

- It might be challenging to calibrate the robot kinematics globally
- The robot has limited ability for performing accurately point-to-point motions and, therefore, maneuvers
- The robot has limited ability for performing accurately contact operations that can be programmed in advance

The kinematic model should be upgraded!

Summary of Kinematics

Topics/questions covered in Kinematics part

- Formulas for velocity and acceleration of a point in curvilinear coordinates

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- Parametrization of rotation matrices using angle/axis representation and unit quaternions. Homogeneous transformations
- DH-parameters and convention for defining a pose of an open kinematic chain manipulator, forward and inverse kinematics for open chain manipulators, manipulator Jacobian

Complimentary reading materials for Kinematics part

- Compendium of the similar class by Dr. Freidovich
- Chapters 1 and 14 of "Handbook of robotics"

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Modeling and Control of Robots

Lecture 1: Control. Introduction to State Space Models and Linear Feedback Controller Designs based on Optimization

Anton Shiriaev
February 8, 2021

Learning outcomes: State space models for linear control systems. Controllability. Observability. Observer-based output feedback controllers and separation principle. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for LTI systems. Linear quadratic regulator (LQR) and algebraic Riccati equation (ARE).

1. State Space Models for Linear Control Systems
   - Controllability of a LTI Control System
   - Stability and Stabilizability of LTI Systems
   - Observability of LTI Control Systems
   - Detectability of a LTI System
2. Observers and Observer-Based Feedback Controllers
   - Luenberger Observers
3. $\mathcal{H}_2$ and $\mathcal{H}_\infty$-Norms of Linear Stable Systems
   - Linear Quadratic Regulator and Minimization of $\mathcal{H}_2$-norm
   - LQR and algebraic Riccati equation

State Space Models for LTI Systems

LTI Control Systems

It is a dynamical system described by the differential equations

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Here

- $x(t) \in R^n$ is called the state of the system;
- $x(t_0)$ is called the initial condition;
- $u(t) \in R^m$ is called the system input;
- $y(t) \in R^p$ is the system output;

- If $m = 1$, then the system is called single input
- If $p = 1$, then the system is called single output
- If both conditions hold $m = p = 1$, then it is a SISO system

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LTI Control Systems

Laplace transforms of the input, state and output signals of the dynamical system

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

result in an alternative representation

$$Y(s) = G(s)U(s) = \boxed{\left[C\left(sI - A\right)^{-1}B + D\right]}\,U(s)$$

We will use the following notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C\left(sI - A\right)^{-1}B + D$$

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Here

$$Y(s) = \int\limits_{0}^{+\infty} y(t)e^{-st}dt, \quad U(s) = \int\limits_{0}^{+\infty} u(t)e^{-st}dt$$

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Controllability of a LTI control system

The dynamical system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

or the pair (A, B) is called controllable (R. Kalman), if for

- any initial condition $x(t_0) = x_0$
- any time interval $[t_0, t_1]$
- any final state x_1

there is a control input $u(\cdot)$ defined on $[t_0, t_1]$ such that the solution $x(\cdot)$ of the system originated from $x(t_0) = x_0$ at $t = t_0$ arrives to the chosen final state $x(t_1) = x_1$ at $t = t_1$.

Otherwise, the system or the pair (A, B) is called uncontrollable.

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Criteria for Controllability

The following statements are equivalent:

- the pair (A, B) is controllable;
- the matrix $W_c(t) := \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$ is positive definite for any $t > 0$;

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- The eigenvalues of the matrix $A + BF$ can be freely assigned by a choice of the matrix gain F.

Stability and Stabilizability of LTI systems

The dynamical system

$$\frac{d}{dt}x(t) = Ax(t)$$

is called asymptotically stable if all the eigenvalues of the matrix A have negative real parts, i.e. $\boxed{\text{if } \det(\lambda I_n - A) = 0 \Rightarrow \mathrm{Re}(\lambda) < 0}$

The dynamical system or the pair (A, B)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

is called stabilizable, if there is a linear state feedback controller

$$u = Fx$$

such that the closed loop system

$$\frac{d}{dt}x(t) = Ax(t) + BFx(t) = \Big(A + BF\Big)\, x(t)$$

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The following statement are equivalent:

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- For any $\mathrm{Re}\,\lambda \geq 0$ and $x \in C^n$ are such that

$$x^* A = \lambda x^*,$$

  the inequality $x^* B \neq 0$ holds.

Observability of LTI Control Systems

The LTI system or the pair (C, A)

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

is called observable, if for

- any time interval $[t_0, t_1]$ with $t_1 > t_0$;
- any control input $u(t)$ and the corresponding output $y(t)$ recorded on the time interval $[t_0, t_1]$

the initial value x_0 can be determined from the pair $[y(\cdot), u(\cdot)]$.

Otherwise, the system or the pair (C, A) are called unobservable.

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Criteria for Observability

The next statements are equivalent to the observability of (C, A):

- the matrix $W_o(t) := \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ is positive definite for any $t > 0$;

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- The matrix $\begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix}$ has full-column rank for $\forall \lambda \in C$;
- The eigenvalues of the matrix $A + LC$ can be freely assigned by a choice of the observer gain L;
- The pair (A^T, C^T) is controllable.

Detectability of a LTI System

The dynamical system

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

or the pair (C, A) is called detectable, if there exists a matrix L such that the system

$$\frac{d}{dt}x(t) = Ax(t) + LCx(t) = (A + LC)\,x(t)$$

is asymptotically stable.

Observer-Based Feedback Controllers

Problem Formulation (State Estimation):

Given a dynamical system

$$
\begin{aligned}
\tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

and given an ability to measure and record time-signals $y(\cdot)$, $u(\cdot)$,

- When (under what conditions) is it possible to reconstruct (asymptotically estimate) the state vector $x(\cdot)$ at time t?

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Given a dynamical system

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- When (under what conditions) is it possible to reconstruct (asymptotically estimate) the state vector $x(\cdot)$ at time t?
- How to reconstruct (asymptotically estimate) the state vector $x(\cdot)$ at time t of the system exploiting just instant records of $y(\cdot)$ and $u(\cdot)$ at time t?

Solution (State Estimation):

Given a dynamical system

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

then an observer exists if and only if (C, A) is detectable.

If the pair (C, A) is detectable, then a state $\hat{x}(\cdot)$ of the observer (due to David Luenberger)

$$\tfrac{d}{dt}\hat{x}(t) = Aq(t) + Bu(t) + L\Big(C\hat{x}(t) + Du(t) - y(t)\Big)$$

converges exponentially to the state of the system $x(\cdot)$ provided that a matrix L is chosen to make $(A + LC)$ asymptotically stable.

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Problem Formulation (Output Feedback Controller Design):

Given a LTI system

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

the following assignment are important:

- How to synthesize a output feedback controller to stabilize the origin $x = 0$ of the LTI system?

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the following assignment are important:

- How to synthesize a output feedback controller to stabilize the origin $x = 0$ of the LTI system?
- How to analyze the closed loop system dynamics?

Solution (Output Feedback Controller Design):

Given a LTI control system

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

if the pair (A, B) is stabilizable, i.e. $\exists\, F$: $A + BF$ is stable , then

the state feedback controller $u = Fx$ stabilizes the system

The output controller is designed based on the Luenberger observer

$$\tfrac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L\Big(C\hat{x}(t) + Du(t) - y(t)\Big)$$

$$u = F\hat{x}(t)$$

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$$\begin{aligned} \frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + L\Big(C\hat{x}(t) + Du(t) - y(t)\Big) \\ &= \Big(A + LC\Big)\hat{x}(t) + \Big(B + LD\Big)u(t) - Ly \end{aligned}$$

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\tfrac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + L\Big(C\hat{x}(t) + Du(t) - y(t)\Big) \\
&= \Big(A + LC\Big)\,\hat{x}(t) + \Big(B + LD\Big)\,u(t) - Ly \\
&= \Big(A + LC + BF + LDF\Big)\,\hat{x}(t) - Ly \\
u &= F\hat{x}(t)
\end{aligned}
$$

Analysis of the Closed Loop System:

The closed loop system dynamics can be re-written as

$$
\begin{aligned}
\tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \\
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\end{aligned}
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The closed loop system dynamics can be re-written as

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\begin{aligned}
\tfrac{d}{dt}x(t) &= Ax(t) + BF\hat{x}(t) \\
\tfrac{d}{dt}\hat{x}(t) &= \Big(A + LC + BF + LDF\Big)\,\hat{x}(t) - L(Cx(t) + DF\hat{x}(t))
\end{aligned}
$$

Consider the error variable $e := x - \hat{x}$, then its dynamics are

$$
\begin{aligned}
\tfrac{d}{dt}e(t) &= \tfrac{d}{dt}x(t) - \tfrac{d}{dt}\hat{x}(t) \\
&= \Big[Ax(t) + BF\hat{x}(t)\Big] - \Big[\Big(A + LC + BF\Big)\,\hat{x}(t) - LCx(t)\Big] \\
&= (A + LC)\,(x(t) - \hat{x}(t)) = (A + LC)\,e(t)
\end{aligned}
$$

$$
\tfrac{d}{dt}\hat{x}(t) = \Big(A + BF\Big)\,\hat{x}(t) - LC(x(t) - \hat{x}(t))
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The closed loop system dynamics can be re-written as

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\frac{d}{dt}e(t) &= \frac{d}{dt}x(t) - \frac{d}{dt}\hat{x}(t) \\
&= \Big[Ax(t) + BF\hat{x}(t)\Big] - \Big[\Big(A + LC + BF\Big)\,\hat{x}(t) - LCx(t)\Big] \\
&= (A + LC)\,(x(t) - \hat{x}(t)) = \boxed{(A + LC)\,e(t)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\hat{x}(t) &= \Big(A + BF\Big)\,\hat{x}(t) - LC(x(t) - \hat{x}(t)) \\
&= \boxed{\Big(A + BF\Big)\,\hat{x}(t) - LCe(t)}
\end{aligned}
$$

Analysis of the Closed Loop System:

The closed loop system dynamics can be re-written as

$$
\begin{aligned}
\tfrac{d}{dt}x(t) &= Ax(t) + BF\hat{x}(t) \\
\tfrac{d}{dt}\hat{x}(t) &= \Big(A + LC + BF + LDF\Big)\,\hat{x}(t) - L(Cx(t) + DF\hat{x}(t))
\end{aligned}
$$

Consider the error variable $e := x - \hat{x}$, then its dynamics are

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Analysis of the Closed Loop System (cont'd):

The closed loop system dynamics in coordinates $[e; \hat{x}]$ are

$$\begin{aligned} \tfrac{d}{dt} e(t) &= (A + LC)\, e(t) \\ \tfrac{d}{dt} \hat{x}(t) &= \Big(A + BF\Big)\, \hat{x}(t) - LCe(t) \end{aligned}$$

In the matrix form, the dynamics are

$$\frac{d}{dt}\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} (A + LC) & 0 \\ -LC & \Big(A + BF\Big) \end{bmatrix} \begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix}$$

When is this system stable?

When the matrices on the diagonal

$$(A + LC) \quad \text{and} \quad \Big(A + BF\Big)$$

are asymptotically stable!

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Summary: Observer-Based Output Feedback Controller

Given a LTI control system

$$
\begin{aligned}
\tfrac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

The output feedback controller for LTI system is then

$$
\begin{aligned}
\tfrac{d}{dt}\hat{x}(t) &= \Big(A + LC + BF + LDF\Big)\,\hat{x}(t) - Ly, \\
u &= F\hat{x}(t)
\end{aligned}
$$

The transfer function of the output controller is

$$
\begin{aligned}
U(s) &= K(s)Y(s) = \left[\begin{array}{c|c} A + LC + BF + LDF & -L \\ \hline F & 0 \end{array}\right] Y(s) \\
&= \Big[-F\Big(sI - [A + LC + BF + LDF]\Big)^{-1} L\Big]\, Y(s)
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Suppose that (A, B) is stabilizable and (C, A) is detectable

$$\exists\, F,\ \exists\, L: \ \Big(A + BF\Big),\ (A + LC) \ \text{are stable}$$

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$\mathcal{H}_2$ and $\mathcal{H}_\infty$-Norms of Linear Stable Systems

$\mathcal{H}_2$ and $\mathcal{H}_\infty$-Norms of Linear Stable Systems

Consider a linear stable system

$$\frac{d}{dt}X(t) \;=\; AX(t) + Bw(t), \quad z(t) = CX(t) + Dw(t)$$

with

- $X(t) \in R^n$ is a state of the system;
- $w(t) \in R^m$ is a disturbance;
- $z(t) \in R^p$ is a system's output to measure its properties

---

The tasks are

- To compute outputs of the system for some class of disturbances $w(\cdot)$;

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The tasks are

- To compute outputs of the system for some class of disturbances $w(\cdot)$;
- To use some outputs for evaluating a size or norm of the linear stable system.

Example:

Consider the system

$$\ddot{x} = -\dot{x} + u + w, \quad u \text{ is control variable, } w \text{ is disturbance}$$

To stabilize the position $x_e = 0$, one suggests a PD-controller

$$u = -K_p \cdot x - K_d \cdot \dot{x}$$

The closed loop system is then

$$\ddot{x} = -K_p \cdot x - (K_d + 1) \cdot \dot{x} + w$$

The state-space model of the system is

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K_p & -(K_d+1) \end{bmatrix}}_{=A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=B} w, \quad z = \underbrace{[1, 0]}_{=C} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

How to access the quality of the feedback design?

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How to access the quality of the feedback design?

$\mathcal{H}_2$ and $\mathcal{H}_\infty$-Norms of Linear Stable Systems

Two classes of disturbances are often considered:

- $w(\cdot)$ is an impulse function, i.e.

$$w(t) = \delta(t - t_0)$$

$\mathcal{H}_2$ and $\mathcal{H}_\infty$-Norms of Linear Stable Systems

Two classes of disturbances are often considered:

- $w(\cdot)$ is an impulse function, i.e.

$$w(t) = \delta(t - t_0)$$

- $w(\cdot)$ is a linear combination of sin's and cos's

$$\begin{aligned} w(t) &= a_1 \sin(\omega_1 t) + \cdots + a_N \sin(\omega_N t) + \\ &\qquad + b_1 \cos(\omega_1 t) + \cdots + b_N \cos(\omega_N t) \end{aligned}$$

Both cases lead to two different norms to measure a size of the linear system.

$\mathcal{H}_2$-Norm of Linear Stable Systems

An output of a stable linear SISO system

$$\dot{X}(t) = AX(t) + Bw(t), \quad z(t) = CX(t), \quad X(0) = 0$$

to an impulse function $w(\cdot)$ applied at $t = 0$ is

$$z(t) = \begin{cases} Ce^{At}B, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

The size of this system can be measured in terms of energy of the output $z(t)$, i.e.

$$\|G(s)\|_2 := \sqrt{\int_0^{+\infty} z^2(t)dt}$$

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$$\dot{X}(t) = AX(t) + Bw(t), \quad z(t) = CX(t), \quad X(0) = 0$$

to a function

$$w(t) = \sin(\omega t)$$

in steady-state is equal to

$$z_{ss}(t) = |G(j\omega)| \cdot \sin\Big(\omega t + \arg G(j\omega)\Big)$$

with $G(s) = C(sI - A)^{-1}B$.

The norm of the system can be measured in terms of the maximal amplification over all frequencies

$$|G(s)|_\infty = \sup_{\omega \in R^1} \Big| G(j\omega) \Big|$$

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Linear Quadratic Regulator and Minimization of $\mathcal{H}_2$-norm

Consider a plant

$$\frac{d}{dt}x = Ax + B_1 w + B_2 u \quad \Big| \quad x(0) = 0 \quad \Big| \quad \begin{aligned} z &= C_1 x + D_1 u \\ y &= C_2 x \end{aligned}$$

The task is

To synthesize a linear controller $U(s) = C(s)Y(s)$ that minimizes the $\mathcal{H}_2$-norm of the transfer function of the closed loop system from $w(\cdot) \to z(\cdot)$

- If w is a scalar, then $w(t) = \delta(t)$;
- If $w = [w_1; \ldots; w_m]$ is a vector, then its components are

$$w_i(t) = \alpha_i \delta(t), \quad \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2 = 1$$

Here $\delta(t)$ is the delta function applied at $t = 0$.

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Let us assume that $B_1 = I_n$ and w is a vector of dimension n

---

Since w is a vector, $w = [w_1;\, w_2;\, \dots;\, w_n]$, with components

$$w_i = \alpha_i \delta(t), \quad \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1,$$

then one can rewrite the system dynamics as

$$\frac{d}{dt}x = Ax + B_2 u \quad \Big| \quad x(0) = a \quad \Big| \quad \begin{array}{rcl} z & = & C_1 x + D_1 u \\ y & = & C_2 x \end{array}$$

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with the initial conditions moved from the origin to the point

$$a = [\alpha_1;\, \alpha_2;\, \ldots;\, \alpha_n].$$

Linear Quadratic Regulator and Minimization of $\mathcal{H}_2$-norm

Consider a plant

$$\frac{d}{dt}x = Ax + B_2 u \quad \Big| \quad x(0) = a \quad \Big| \quad \begin{array}{rcl} z & = & C_1 x + D_1 u \\ y & = & C_2 x \end{array}$$

and $a = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ with $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1$

Let us assume that $C_2 = I_n$, i.e. all states are measured To summarize, given a plant

$$\frac{d}{dt}x = Ax + B_2 u \quad \Big| \quad x(0) = a \quad \Big| \quad \begin{array}{rcl} z & = & C_1 x + D_1 u \\ y & = & x \end{array}$$

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---

the $\mathcal{H}_2$-norm of the closed loop system (for any controller) is

$$\int_0^\infty \|z(t)\|^2 \, dt = \int_0^\infty \|C_1 x + D_1 u\|^2 \, dt$$

$$= \int_0^\infty \{x^T C_1^T C_1 x + 2x^T C_1^T D_1 u + u^T D_1^T D_1 u\} \, dt$$

Linear Quadratic Regulator and Minimization of $\mathcal{H}_2$-norm

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and $a = [\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_n]^T$ with $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1$,

---

the $\mathcal{H}_2$-norm of the closed loop system (for any controller) is

$$\int_0^\infty \|z(t)\|^2\, dt = \int_0^\infty \|C_1 x + D_1 u\|^2\, dt$$

$$= \int_0^\infty \{x^T C_1^T C_1 x + 2x^T C_1^T D_1 u + u^T D_1^T D_1 u\}\, dt$$

$$= \int_0^\infty \{x^T Q x + 2x^T g u + u^T R u\}\, dt$$

$\Uparrow$

One needs to find a controller that minimizes this integral

Linear Quadratic Regulator and Minimization of $\mathcal{H}_2$-norm

To summarize, given a plant

$$\frac{d}{dt}x = Ax + B_2 u \quad \Bigg| \quad x(0) = a \quad \Bigg| \quad \begin{array}{rcl} z & = & C_1 x + D_1 u \\ y & = & x \end{array}$$

and $a = [\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_n]^T$ with $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1$,

---

the $\mathcal{H}_2$-norm of the closed loop system (for any controller) is

$$\begin{aligned}
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&= \int_0^\infty \{x^T C_1^T C_1 x + 2x^T C_1^T D_1 u + u^T D_1^T D_1 u\}\, dt \\
&= \int_0^\infty \{x^T Q x + 2x^T g u + u^T R u\}\, dt
\end{aligned}$$

$$\Uparrow$$

One needs to find a controller that minimizes this integral

Example

Consider a plant

$$\frac{d}{dt}x = u, \quad x(0) = 1, \quad \begin{cases} z &= C_1 x + D_1 u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= x \end{cases}$$

The $\mathcal{H}_2$ norm of the closed loop system (for any controller) will be

$$\int_0^\infty \|z(t)\|^2 \, dt = \int_0^\infty \|C_1 x + D_1 u\|^2 \, dt$$

$$= \int_0^\infty \{x^T C_1^T C_1 x + 2x^T C_1^T D_1 u + u^T D_1^T D_1 u\} \, dt$$

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Example (cont'd)

Consider an optimization problem

$$\Phi\left(u(\cdot)\right) = \int_0^{\infty} \left\{x(t)^2 + u(t)^2\right\} dt \to \min$$

where $\dot{x} = u$, $x(0) = 1 \quad \Rightarrow \quad x(t) = x(0) + \int_0^t u(\tau) d\tau$

Extreme cases:

- $u \equiv 0 \;\Rightarrow\; \begin{cases} \int_0^{\infty} |u(\tau)|^2 d\tau = 0 \\ x(t) = x(0) = 1, \quad \int_0^{\infty} x(\tau)^2 d\tau = +\infty \end{cases}$

Example (cont'd)

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- $u = \begin{cases} -1/\varepsilon, & 0 < t < \varepsilon \\ 0, & t \geq \varepsilon \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} x(\varepsilon) = x(0) + \int_0^\varepsilon (-1/\varepsilon)\, d\tau = 1 - 1 = 0 \\ \int_0^\infty |u(\tau)|^2 d\tau = \int_0^\varepsilon |-1/\varepsilon|^2\, d\tau = 1/\varepsilon \end{cases}$$

Example (cont'd)

Let $u(\cdot)$ be any signal on $[0, +\infty)$ such that

$$\Phi\left(u(\cdot)\right) = \int_0^{+\infty} \left\{x(\tau)^2 + u(\tau)^2\right\} d\tau < +\infty$$

Then the solution of the system $\dot{x} = u$, $x(0) = 1$, satisfies the properties:

- $x(t) \to 0 \quad \text{as} \quad t \to +\infty$

Example (cont'd)

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$$\Phi\left(u(\cdot)\right) = \int_0^{+\infty} \left\{x(\tau)^2 + u(\tau)^2\right\} d\tau < +\infty$$

Then the solution of the system $\dot{x} = u$, $x(0) = 1$, satisfies the properties:

- $x(t) \to 0 \quad \text{as} \quad t \to +\infty$
- For any constant $p \in \mathbb{R}^1$ the value of the integral

$$\int\limits_0^{+\infty} 2 \cdot x(\tau) \cdot p \cdot u(\tau) d\tau = \int\limits_0^{+\infty} 2 \cdot x(\tau) \cdot p \cdot \dot{x}(\tau) d\tau = \int\limits_0^{+\infty} d\left(p \cdot x(\tau)^2\right)$$

$$= \quad p \cdot x(+\infty)^2 - p \cdot x(0)^2 = p \cdot 0^2 - p \cdot 1^2 = -p$$

is independent of the control input $u(\cdot)$!

Example (cont'd)

Based on this calculation, one can rewrite the performance index as

$$\Phi\left(u(\cdot)\right) = \int_0^\infty \left\{x(\tau)^2 + u(\tau)^2\right\} d\tau$$

$$= \int_0^\infty \left\{x(\tau)^2 + u(\tau)^2\right\} d\tau + \left(p + \int_0^\infty 2 \cdot x(\tau) \cdot p \cdot u(\tau) d\tau\right)$$

We will be interested to search for a constant p such that

$$u^2 + x^2 + 2 \cdot x \cdot p \cdot u \equiv (u - k \cdot x)^2, \quad \forall\, x, \forall\, u \in \mathbb{R}^1$$

If such constant p is found, then the performance index is

$$\Phi\left(u(\cdot)\right) = p + \int_0^\infty \left\{x(\tau)^2 + u(\tau)^2 + 2 \cdot x(\tau) \cdot p \cdot u(\tau)\right\} d\tau$$

$$= \boxed{p + \int_0^\infty \left(u(\tau) - k \cdot x(\tau)\right)^2 d\tau} \geq p$$

Example (cont'd)

Based on this calculation, one can rewrite the performance index as

$$\Phi\Big(u(\cdot)\Big) = \int_0^\infty \Big\{x(\tau)^2 + u(\tau)^2\Big\}\, d\tau$$
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Example (cont'd)

The algebraic equation

$$x^2 + u^2 + 2 \cdot x \cdot p \cdot u \equiv (u - k \cdot x)^2, \quad \forall\, x, \forall\, u \in \mathbb{R}$$

is, in fact, three equations

$$\boxed{x^2 = k^2 \cdot x^2}, \quad \boxed{2 \cdot x \cdot p \cdot u = -2 \cdot x \cdot k \cdot u}, \quad \boxed{u^2 = u^2} \ \forall\, x, \forall\, u \in \mathbb{R}$$

imposed on the factors of the corresponding monomials

$$1 = k^2, \qquad p = -k, \qquad 1 = 1.$$

They have two solutions

$$p_1 = -1, \qquad p_2 = 1$$

that indicate two candidates for an optimal control strategy

$$u = k \cdot x = (-p_1) \cdot x = \boxed{1 \cdot x}, \qquad u = k \cdot x = (-p_2) \cdot x = \boxed{-1 \cdot x}$$

Example (cont'd)

The algebraic equation

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$$u = k \cdot x = (-p_1) \cdot x = \boxed{1 \cdot x}, \qquad u = k \cdot x = (-p_2) \cdot x = \boxed{-1 \cdot x}$$

Example (cont'd)

If $p = -1$, then $k = 1$ and the solution of the system

$$\dot{x} = u = k \cdot x = x$$

is unbounded! Such control signal cannot be optimal!

If $p = 1$, then $k = -1$ and the solution of the system

$$\dot{x} = u = k \cdot x = -x$$

is exponentially converges to zero!

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

The case $p = 1$ corresponds to the optimal control strategy

$$\boxed{u_{opt}(t) = -1 \cdot x(t)}$$

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The case $p = 1$ corresponds to the optimal control strategy

$$\boxed{u_{opt}(t) = -1 \cdot x(t)}$$

LQR and algebraic Riccati equation

Consider the general problem to find a control input $u(\cdot)$ for a LTI control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = a, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

that minimizes the quadratic performance index

$$\Phi\left(u(\cdot)\right) = \int_0^\infty \sigma\left(x(\tau), u(\tau)\right) d\tau = \int_0^\infty \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^T \begin{bmatrix} Q & g \\ g^T & R \end{bmatrix} \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} d\tau$$

$$= \int_0^\infty \left\{ x(\tau)^T Q x(\tau) + 2x(\tau)^T g u(\tau) + u(\tau)^T R u(\tau) \right\} d\tau$$

LQR and algebraic Riccati equation

In the example with $\dot{x} = u$, we searched for a constant p such that

$$\underbrace{x^2 + u^2}_{:=\sigma_{exm}(x,u)} + 2 \cdot x \cdot p \cdot u = (u - k \cdot x)^2, \quad \forall\, x,\, u \in \mathbb{R}$$

It allowed rewriting the performance index as

$$\Phi_{exm} = \int_0^\infty \sigma_{exm}\Big(x(\tau), u(\tau)\Big)\, d\tau = p x(0)^2 + \int_0^\infty \Big(u(\tau) - k \cdot x(\tau)\Big)^2 d\tau$$

For general settings $\dot{x} = Ax + Bu$, and we search for $n \times n$-matrix P such that the following identity holds for any $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\sigma(x, u) + 2x^T P \Big(Ax + Bu\Big) \quad \equiv \quad (u - kx)^T R (u - kx)$$

LQR and algebraic Riccati equation

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$$\begin{aligned} \sigma(x, u) + 2x^T P \left(Ax + Bu\right) &\equiv (u - kx)^T R (u - kx) \\ &= |u - kx|_R^2 \end{aligned}$$

$$\Phi = \int_0^\infty \sigma\left(x(\tau), u(\tau)\right) d\tau = x(0)^T P x(0) + \int_0^\infty |u(\tau) - kx(\tau)|_R^2 \, d\tau$$

LQR and algebraic Riccati equation

As for the example, the equality of quadratic forms

$$\sigma(x,u) + 2x^{T} P \left(Ax + Bu\right) \quad = \quad (u - kx)^{T} R (u - kx)$$

gives two nontrivial equations with respect to P and k

$$\begin{aligned} x^{T}\left(Q + PA + A^{T}P\right)x &= x^{T}\left(k^{T}Rk\right)x \\ 2x^{T}\left(g + PB\right)u &= -2x^{T}\left(k^{T}R\right)u \end{aligned}$$

In the matrix form these equations are

$$Q + PA + A^{T}P = k^{T}Rk, \quad g + PB = -k^{T}R$$

$$\Downarrow$$

$$\begin{aligned} \boxed{Q + PA + A^{T}P} &= \left[-(g+PB)R^{-1}\right] R \left[-(g+PB)R^{-1}\right]^{T} \\ &= \boxed{(g+PB)R^{-1}(g+PB)^{T}} \end{aligned}$$

They can be rewritten as one quadratic equation w.r.t. P if $\det R \neq 0$.

LQR and algebraic Riccati equation

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LQR and algebraic Riccati equation

The algebraic Riccati equation

$$Q + PA + A^T P \quad = \quad (g + PB)R^{-1}(g + PB)^T$$

- is matrix and nonlinear;
- might have several solutions $P_1, \ldots, P_N$;

LQR and algebraic Riccati equation

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$$Q + PA + A^T P \;=\; (g + PB)R^{-1}(g + PB)^T$$

- is matrix and nonlinear;
- might have several solutions $P_1, \ldots, P_N$;
- we are looking for the particular solution P such that

$u = kx$
is stabilizing feedback

$\Leftrightarrow$

$\Big(A + Bk\Big) = \Big(A - BR^{-1}(g^T + B^T P)\Big)$
is asymptotically stable

LQR and algebraic Riccati equation

The algebraic Riccati equation

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- Such solution P is called stabilizing.

LQR and algebraic Riccati equation

The algebraic Riccati equation

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- Such solution P is called stabilizing.
- Check the Matlab command: `are(...)`

Modeling and Control of Robots

Lecture 11: Dynamics of a System of Points

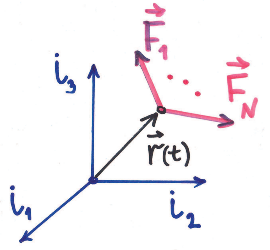
Anton Shiriaev
February 15, 2021

Learning outcomes: Dynamics of a point; linear momentum, angular momentum, kinetic energy a system of points and their rates of change.

Outline

Dynamics of a System of Point Masses

Dynamics of a Point

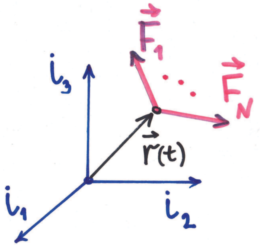


According to the 2nd Newton law

$$m \cdot \frac{d^2}{dt^2}\vec{r} = \vec{F}_1 + \cdots + \vec{F}_N$$

where m is the mass of the point; $\vec{F}_1, \ldots, \vec{F}_N$ are all the forces acting on the point

Dynamics of a Point



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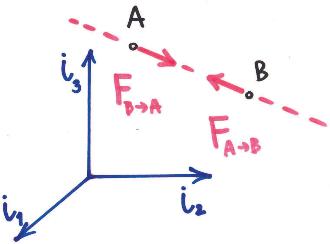
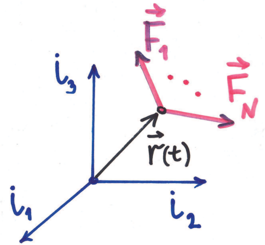


According to the 3rd Newton law, the forces $\vec{F}_{B\to A}$ and $\vec{F}_{A\to B}$ are

- acting the same line in $\mathbb{R}^3$



According to the 2nd Newton law

$$m \cdot \frac{d^2}{dt^2}\vec{r} = \vec{F}_1 + \cdots + \vec{F}_N$$

where m is the mass of the point; $\vec{F}_1$, . . . , $\vec{F}_N$ are all the forces acting on the point

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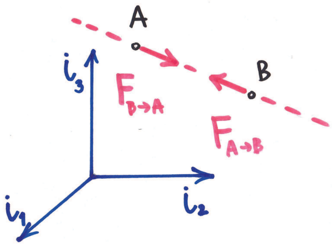
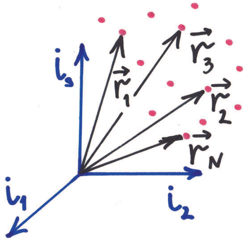


According to the 3rd Newton law, the forces $\vec{F}_{B\to A}$ and $\vec{F}_{A\to B}$ are

- acting the same line in $\mathbb{R}^3$
- equal in magnitude and opposite in sign:

$$\vec{F}_{B\to A} = -\vec{F}_{A\to B}$$



The dynamics of a system of points $\left\{\vec{r}_i\right\}$ are then

$$
\begin{array}{rcl}
m_1 \cdot \frac{d^2}{dt^2}\vec{r}_1 & = & \sum \vec{F}_1^{in} + \sum \vec{F}_1^{ex} \\
 & \dots & \\
m_i \cdot \frac{d^2}{dt^2}\vec{r}_i & = & \sum \vec{F}_i^{in} + \sum \vec{F}_i^{ex} \\
 & \dots &
\end{array}
$$

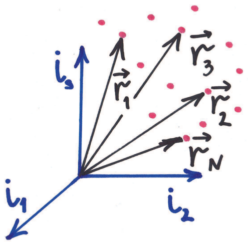
Consider a point C – known as a center of mass, – which coordinates are

$$\vec{r}_C(t) := \frac{\sum_i m_i \cdot \vec{r}_i(t)}{\sum_i m_i}$$

$$\Rightarrow \qquad M \cdot \frac{d^2}{dt^2}\vec{r}_C = \sum_i m_i \cdot \frac{d^2}{dt^2}\vec{r}_i(t)$$

The vector $\vec{P}(t) := M \cdot \frac{d}{dt}\vec{r}_C(t)$ is a linear momentum:

Dynamics of a System of Point Masses: Linear Momentum



The dynamics of a system of points $\{\vec{r}_i\}$ are then

$$
\begin{aligned}
m_1 \cdot \tfrac{d^2}{dt^2}\vec{r}_1 &= \sum \vec{F}_1^{in} + \sum \vec{F}_1^{ex} \\
&\dots \\
m_i \cdot \tfrac{d^2}{dt^2}\vec{r}_i &= \sum \vec{F}_i^{in} + \sum \vec{F}_i^{ex} \\
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How to use this huge set of equations?

Consider a point C – known as a center of mass, – which coordinates are

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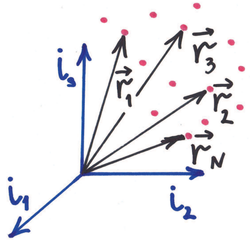

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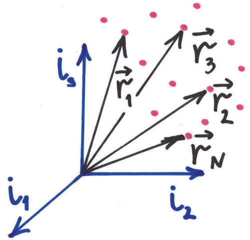

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$$\vec{r}_C(t) := \frac{\sum_i m_i \cdot \vec{r}_i(t)}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i(t)}{M}$$

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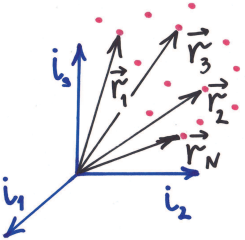
Dynamics of a System of Point Masses: Linear Momentum



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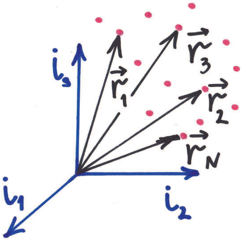

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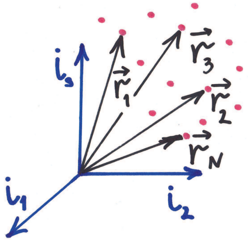
Dynamics of a System of Point Masses: Linear Momentum



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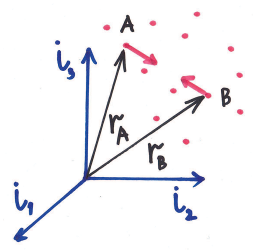
Dynamics of a System of Point Masses: Angular Momentum



Pre-multiply each of the equations by $\vec{r}_i \times$ and sum all

$$
\begin{array}{rcl}
\vec{r}_1 \times \left[m_1 \cdot \frac{d^2}{dt^2} \vec{r}_1 \right] & = & \vec{r}_1 \times \left[\sum \vec{F}_1^{in} + \sum \vec{F}_1^{ex} \right] \\
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\vec{r}_i \times \left[m_i \cdot \frac{d^2}{dt^2} \vec{r}_i \right] & = & \vec{r}_i \times \left[\sum \vec{F}_i^{in} + \sum \vec{F}_i^{ex} \right] \\
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\end{array}
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The vector function

$$\vec{L}(t) := \sum_i \left[\vec{r}_i(t) \times m_i \cdot \frac{d}{dt} \vec{r}_i(t) \right]$$

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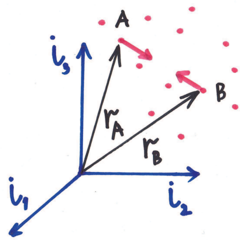
Dynamics of a System of Point Masses: Angular Momentum



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$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

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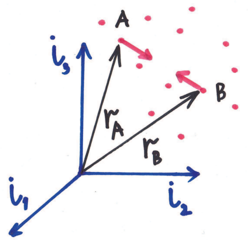

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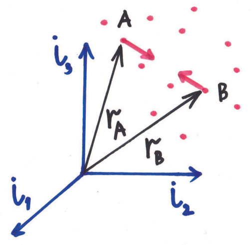
Dynamics of a System of Point Masses: Angular Momentum



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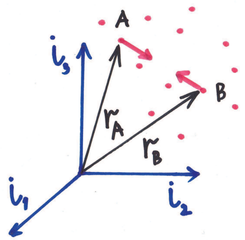
Dynamics of a System of Point Masses: Angular Momentum



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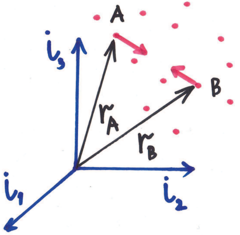

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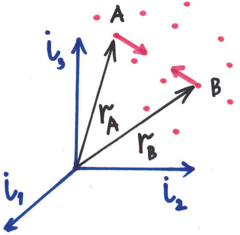

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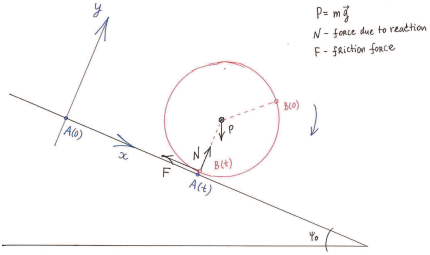
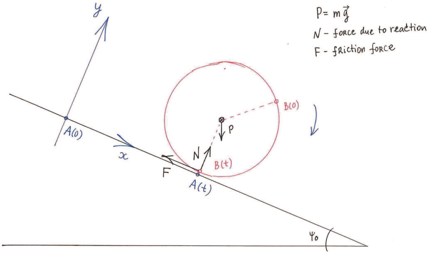
Example: disc rolling on inclined line



How to derive equations of motion?

Example: disc rolling on inclined line



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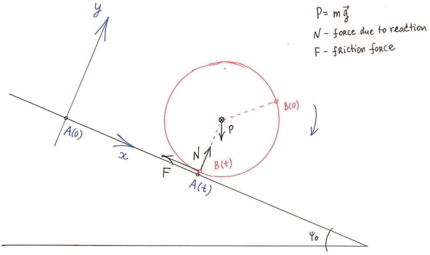

How to derive equations of motion?

Equations of motion of rigid body are

- Dynamics of centre of mass: $\frac{d}{dt}\vec{P} = \frac{d}{dt}\left(m\vec{v}_C\right) = \sum_k \vec{F}_k^{ex}$

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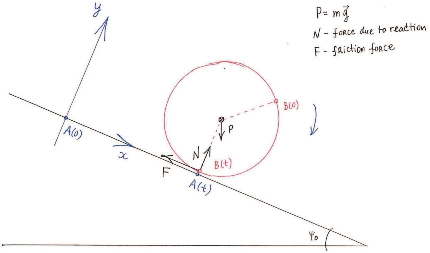

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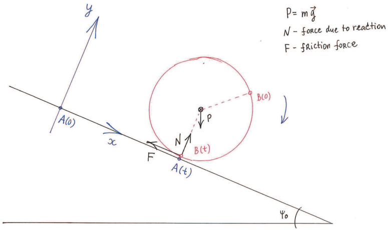
Equations of motion of rigid body are

- Dynamics of centre of mass: $\frac{d}{dt}\vec{P} = \frac{d}{dt}\left(m\vec{v}_C\right) = \sum_k \vec{F}_k^{ex}$
- Angular momentum's rate of change: $\frac{d}{dt}\vec{L} = \sum_k \vec{M}_k^{ex}$

Example: disc rolling on inclined line



The dynamics of center of mass

$$m \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ \ddot{z}_C \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt} \vec{L}^O \quad := \quad \sum_i \frac{d}{dt} \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i$$

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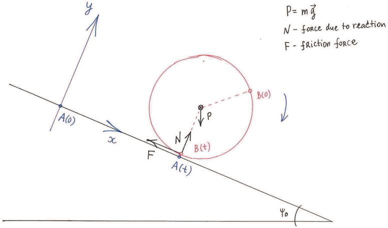

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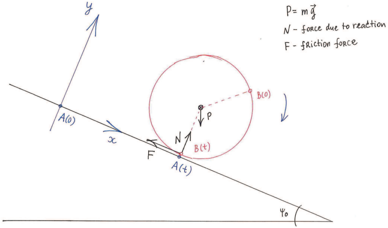

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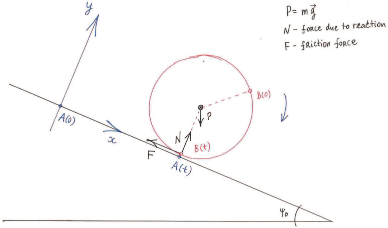
The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt} \vec{L}^O \quad := \quad \sum_i \frac{d}{dt} \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i$$

Example: disc rolling on inclined line



The dynamics of center of mass

$$m \begin{bmatrix} \ddot{x}_C \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} F_x \\ 0 \\ 0 \end{bmatrix}$$

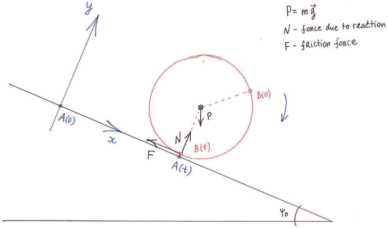
Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt}\vec{L}^O \quad := \quad \sum_i \frac{d}{dt}\vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i$$

Example: disc rolling on inclined line



The dynamics of center of mass

$$
\begin{aligned}
m \cdot \ddot{x}_C &= m \cdot g \cdot \sin\psi_0 - F \\
\ddot{y}_C &= 0 \\
0 &= N - m \cdot g \cdot \cos\psi_0
\end{aligned}
$$

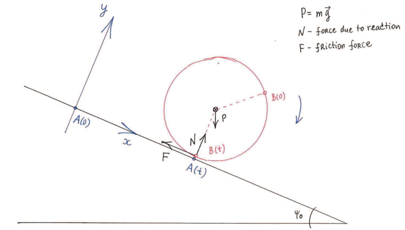
Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O := \sum_i \vec{R}_i \times m_i \tfrac{d}{dt}\vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

$$\tfrac{d}{dt}\vec{L}^O := \sum_i \tfrac{d}{dt}\vec{R}_i \times m_i \tfrac{d}{dt}\vec{R}_i + \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2}\vec{R}_i$$

Example: disc rolling on inclined line



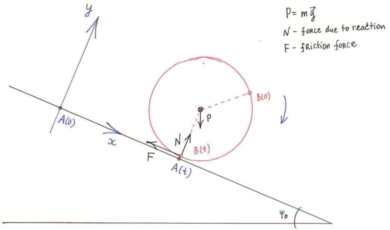
The dynamics of angular momentum $\vec{L}(\cdot)$ depends on a choice of the origin it is computed about!

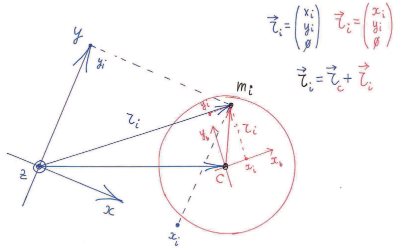
Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt}\vec{L}^O \quad := \quad \sum_i \frac{d}{dt}\vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i$$





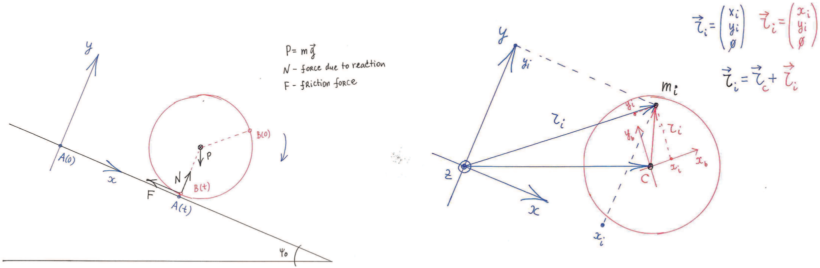
Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i = \sum_i \left(\vec{R}_C + \vec{r}_i\right) \times m_i \frac{d}{dt}\left(\vec{R}_C + \vec{r}_i\right)$$

The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt}\vec{L}^O \quad := \quad \sum_i \frac{d}{dt}\vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i$$

Example: disc rolling on inclined line



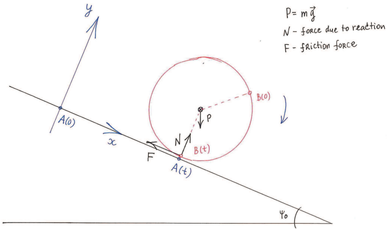
Relative to the origin of world frame $O = A(0)$, angular momentum is

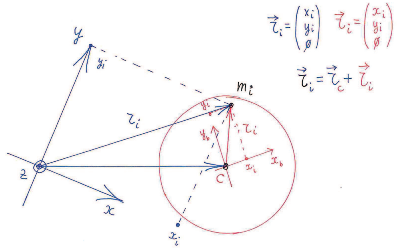
$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i = \sum_i \left(\vec{R}_C + \color{red}{\vec{r}_i} \right) \times m_i \tfrac{d}{dt} \left(\vec{R}_C + \color{red}{\vec{r}_i} \right)$$

The rate of change of $\vec{L}^O$ is then

$$\tfrac{d}{dt} \vec{L}^O \quad := \quad \sum_i \tfrac{d}{dt} \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2} \vec{R}_i$$

Example: disc rolling on inclined line



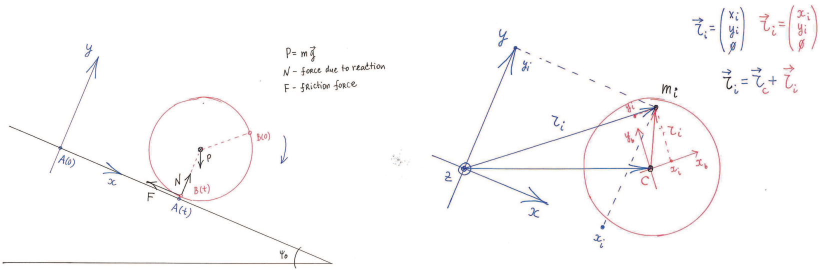


Relative to the origin of world frame $O = A(0)$, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i = \sum_i \left(\vec{R}_C + \vec{r}_i \right) \times m_i \tfrac{d}{dt} \left(\vec{R}_C + \vec{r}_i \right)$$

The rate of change of $\vec{L}^O$ is then

$$\begin{aligned} \tfrac{d}{dt} \vec{L}^O \quad &:= \quad \sum_i \tfrac{d}{dt} \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2} \vec{R}_i \\ &= \quad \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2} \vec{R}_i = \sum_i \vec{R}_i \times \vec{F}_i \end{aligned}$$



Relative to the origin of world frame $O = A(0)$, angular momentum is

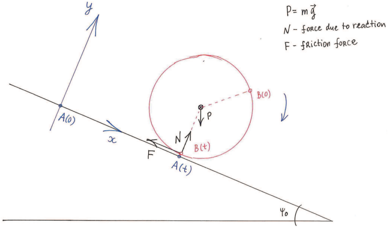
$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i = \sum_i \left(\vec{R}_C + \vec{r}_i \right) \times m_i \tfrac{d}{dt} \left(\vec{R}_C + \vec{r}_i \right)$$

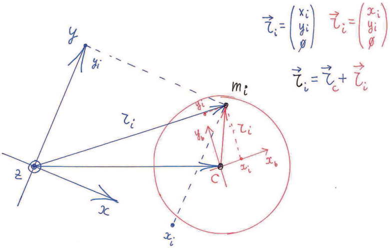
The rate of change of $\vec{L}^O$ is then

$$\tfrac{d}{dt} \vec{L}^O \quad := \quad \sum_i \tfrac{d}{dt} \vec{R}_i \times m_i \tfrac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2} \vec{R}_i$$

$$= \quad \sum_i \vec{R}_i \times m_i \tfrac{d^2}{dt^2} \vec{R}_i = \sum_i \vec{R}_i \times \vec{F}_i = \boxed{\sum_k \vec{R}_k \times \vec{F}_k^{ex}}$$

Example: disc rolling on inclined line



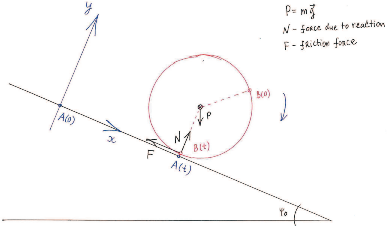


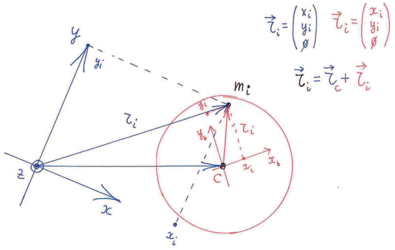
Relative to the origin of the inertia frame, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \tfrac{d}{dt}\vec{R}_i = \sum_i \left(\vec{R}_C + \color{red}{\vec{r}_i}\right) \times m_i \tfrac{d}{dt}\left(\vec{R}_C + \color{red}{\vec{r}_i}\right)$$

Relative to the centre of mass, angular momentum is

$$\vec{L}^C \quad := \quad \sum_i \color{red}{\vec{r}_i} \times m_i \tfrac{d}{dt}\vec{R}_i = \sum_i \color{red}{\vec{r}_i} \times m_i \tfrac{d}{dt}\left(\vec{R}_C + \color{red}{\vec{r}_i}\right)$$





Relative to the origin of the inertia frame, angular momentum is

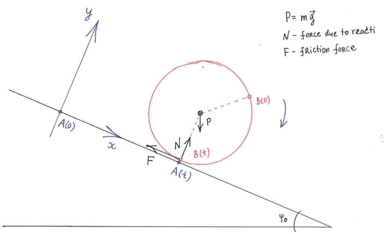
$$\vec{L}^{O} \quad := \quad \sum_i \vec{R}_i \times m_i \tfrac{d}{dt}\vec{R}_i = \sum_i \left(\vec{R}_C + \vec{r}_i\right) \times m_i \tfrac{d}{dt}\left(\vec{R}_C + \vec{r}_i\right)$$

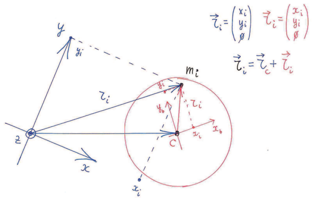
Relative to the centre of mass, angular momentum is

$$\vec{L}^{C} \quad := \quad \sum_i \vec{r}_i \times m_i \tfrac{d}{dt}\vec{R}_i = \sum_i \vec{r}_i \times m_i \tfrac{d}{dt}\left(\vec{R}_C + \vec{r}_i\right)$$

$$\boxed{\tfrac{d}{dt}\vec{L}^{C}} \quad = \quad \sum_i \tfrac{d}{dt}\vec{r}_i \times m_i \tfrac{d}{dt}\vec{R}_i + \sum_i \vec{r}_i \times m_i \tfrac{d^2}{dt^2}\vec{R}_i = \boxed{\sum_k \vec{r}_k \times \vec{F}_k^{ex}}$$

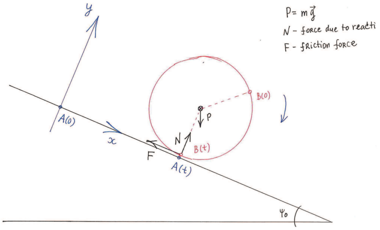
Example: disc rolling on inclined line

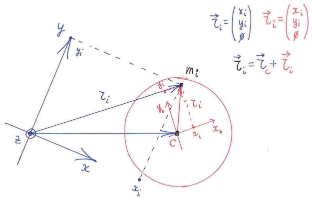




$$\vec{r}_i^{\,O} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=R_{z,\theta}} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix}, \qquad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

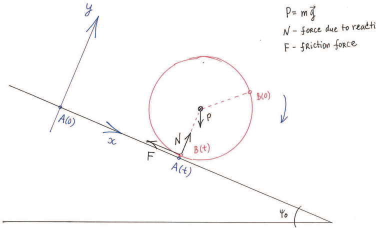
Example: disc rolling on inclined line

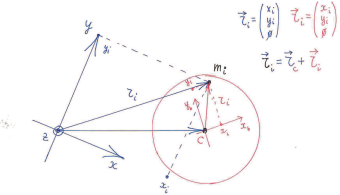




$$\color{red}{\vec{r}_i}^O = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \color{red}{x_i} \\ \color{red}{y_i} \\ \color{red}{0} \end{bmatrix} = R_{z,\theta} \color{red}{\vec{r}_i}^D, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\color{red}{\vec{r}_i}^O$$

$$\vec{L}^C \;=\; \textstyle\sum_i \color{red}{\vec{r}_i}^O \times m_i \vec{v}_i$$

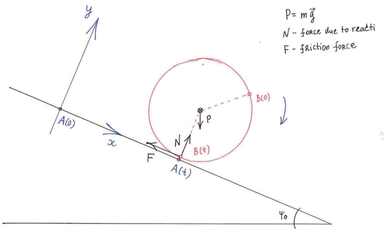


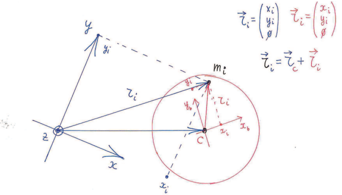


$$\color{red}{\vec{r}_i}^{O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \color{red}{x_i} \\ \color{red}{y_i} \\ \color{red}{0} \end{bmatrix} = R_{z,\theta} \color{red}{\vec{r}_i}^{D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\color{red}{\vec{r}_i}^{O}$$

$$\vec{L}^C = \sum_i \color{red}{\vec{r}_i}^{O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\color{red}{\vec{r}_i}^{D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\color{red}{\vec{r}_i}^{D} \right] \right)$$

Example: disc rolling on inclined line

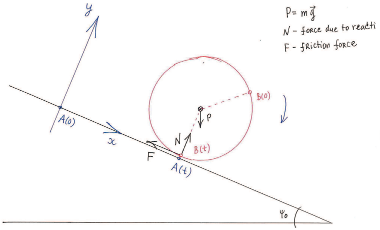


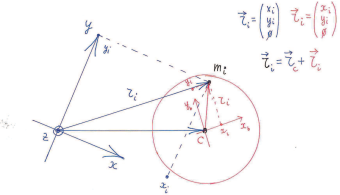


$$\textcolor{red}{\vec{r}_i}^{O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \textcolor{red}{x_i} \\ \textcolor{red}{y_i} \\ \textcolor{red}{0} \end{bmatrix} = R_{z,\theta} \textcolor{red}{\vec{r}_i}^{D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\textcolor{red}{\vec{r}_i}^{O}$$

$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \textcolor{red}{\vec{r}_i}^{O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\textcolor{red}{\vec{r}_i}^{D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\textcolor{red}{\vec{r}_i}^{D} \right] \right) \\ &= \textstyle\sum_i \textcolor{red}{\vec{r}_i}^{O} \times m_i \tfrac{d}{dt}\vec{R}_C + \sum_i R_{z,\theta}\textcolor{red}{\vec{r}_i}^{D} \times m_i \tfrac{d}{dt}\left[R_{z,\theta}\textcolor{red}{\vec{r}_i}^{D} \right] \end{aligned}$$

Example: disc rolling on inclined line





$$\vec{r_i}^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r_i}^{\,D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\vec{r_i}^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \vec{r_i}^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r_i}^{\,D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\vec{r_i}^{\,D}\right]\right) \\ &= \boxed{\left(\textstyle\sum_i m_i \vec{r_i}^{\,O}\right)} \times \tfrac{d}{dt}\vec{R}_C + \textstyle\sum_i R_{z,\theta}\vec{r_i}^{\,D} \times m_i \tfrac{d}{dt}\left[R_{z,\theta}\vec{r_i}^{\,D}\right] \end{aligned}$$

Example: disc rolling on inclined line

$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\vec{r}_i^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D} \right] \right) \\ &= R_{z,\theta} \left(\textstyle\sum_i m_i \vec{r}_i^{\,D} \times \boxed{ (R_{z,\theta})^{-1} \tfrac{d}{d\theta}\left[R_{z,\theta} \right] \vec{r}_i^{\,D} } \right) \tfrac{d}{dt}\theta \end{aligned}$$

Example: disc rolling on inclined line

$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \qquad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r}_i^{\,O} \times m_i\vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i\left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D}\right]\right) \\ &= R_{z,\theta}\left(\sum_i m_i\vec{r}_i^{\,D} \times \boxed{(R_{z,\theta})^{-1}\frac{d}{d\theta}\left[R_{z,\theta}\right]\vec{r}_i^{\,D}}\right)\frac{d}{dt}\theta \end{aligned}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = \begin{bmatrix} -y_i \\ x_i \\ 0 \end{bmatrix}$$

Example: disc rolling on inclined line

$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D}\right]\right) \\ &= R_{z,\theta}\left(\sum_i m_i \vec{r}_i^{\,D} \times \boxed{\left(R_{z,\theta}\right)^{-1} \frac{d}{d\theta}\left[R_{z,\theta}\right]\vec{r}_i^{\,D}}\right)\frac{d}{dt}\theta \end{aligned}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = \begin{bmatrix} -y_i \\ x_i \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} \times \begin{bmatrix} -y_i \\ x_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ {x_i}^2 + {y_i}^2 \end{bmatrix}$$

Example: disc rolling on inclined line

$$\vec{r}_i{}^O = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i{}^D, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\vec{r}_i{}^O$$

$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \vec{r}_i{}^O \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i{}^D \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\vec{r}_i{}^D \right] \right) \\ &= \left[0;\ 0;\ \textstyle\sum_i m_i \left(x_i{}^2 + y_i{}^2 \right) \tfrac{d}{dt}\theta \right] \end{aligned}$$

Example: disc rolling on inclined line

$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\vec{r}_i^{\,O}$$

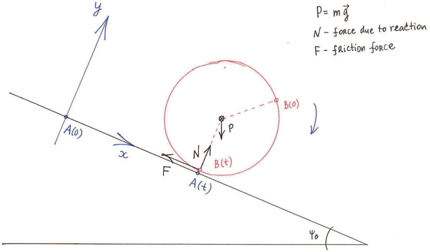
$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D}\right]\right) \\ &= \left[0;\ 0;\ J_z \tfrac{d}{dt}\theta \right] \end{aligned}$$

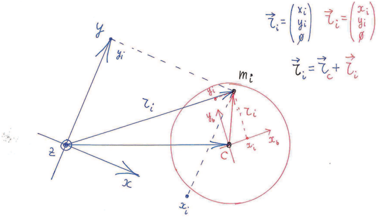
Example: disc rolling on inclined line

$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\vec{r}_i^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \textstyle\sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\tfrac{d}{dt}\vec{R}_C + \tfrac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D} \right] \right) \\ &= \left[0;\ 0;\ \boxed{J_z \tfrac{d}{dt}\theta} \right], \qquad \text{for a disc: } J_z = \tfrac{1}{2}(\text{Mass}) * (\text{Radius})^2 \end{aligned}$$

Example: disc rolling on inclined line





The rate of change of angular momentum $\vec{L}^C$ is then

$$\frac{d}{dt}\begin{bmatrix} 0 \\ 0 \\ J_z\dot{\theta} \end{bmatrix} = \vec{r}_P{}^O \times \vec{P} + \vec{r}_N{}^O \times \vec{N} + \vec{r}_F{}^O \times \vec{F}$$

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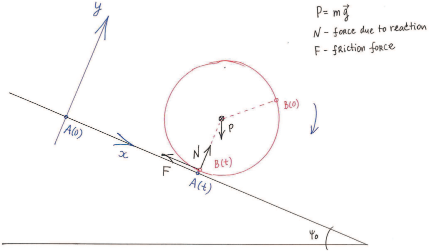

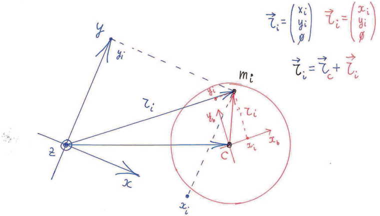

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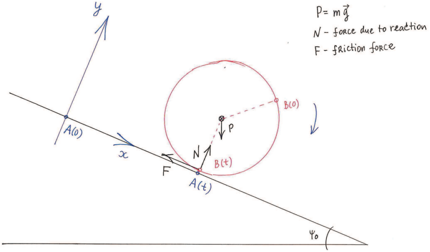

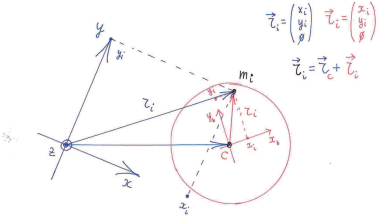

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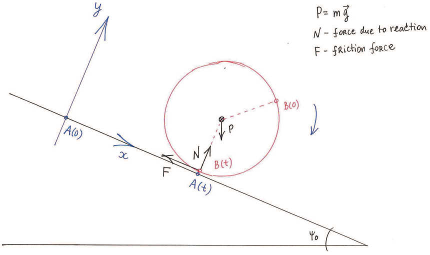

How many unknown variables?

The dynamics of the system are

$$
\begin{aligned}
m \cdot \ddot{x}_C &= m \cdot g \cdot \sin \psi_0 - F \\
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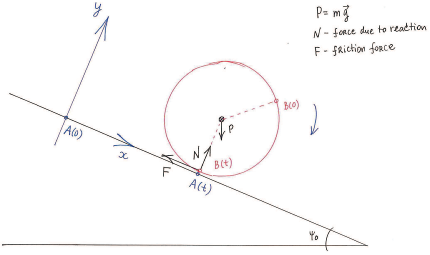

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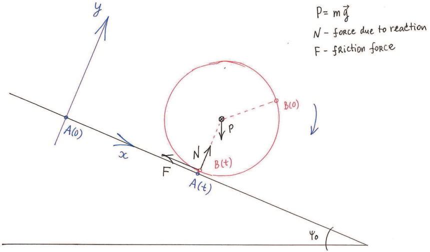

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How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

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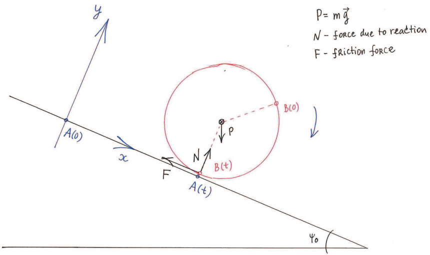

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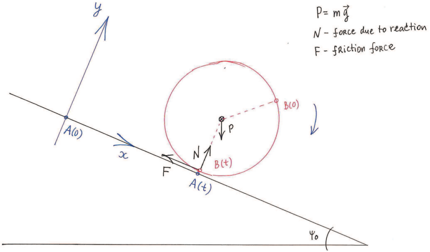
How many nontrivial equations?

Example: disc rolling on inclined line



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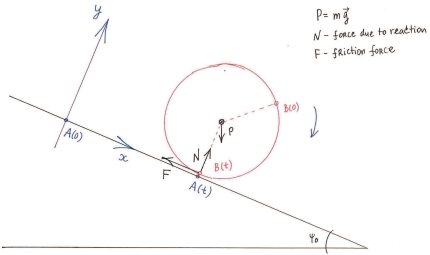
How many nontrivial equations? **Two!**

Example: disc rolling on inclined line



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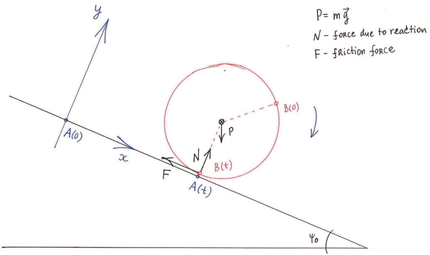
We need one more equation for the system!

Example: disc rolling on inclined line



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If the disc slides for some time interval $\Rightarrow$ $\boxed{F(t) = \mu \cdot N}$

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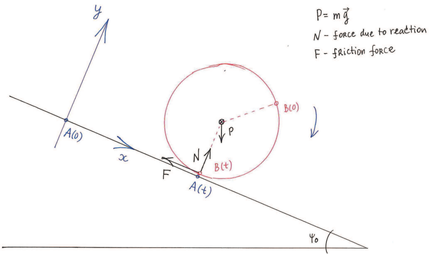

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If the disc slides for some time interval $\Rightarrow$ $\boxed{F(t) = \mu \cdot N}$

If the disc rolls without sliding for some interval

$$\Rightarrow \quad \boxed{x_C(t) - x_C(0) = -R_d \cdot (\theta(t) - \theta(0))}$$

Angular Momentum's Properties

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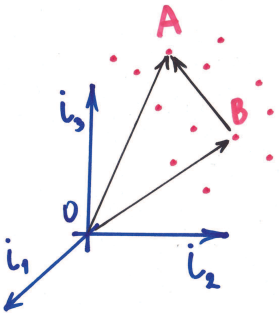
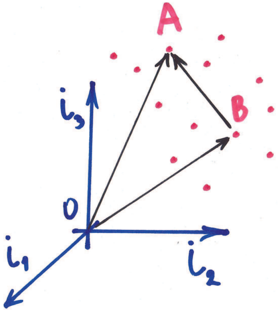


We can compute the vector of angular momentum $\vec{L}$ of a system about any point, how it will change if we shift such a point to new location?

By definition

$$= \sum_i \vec{r}_{Ai} \times m_i \frac{d}{dt} \vec{r}_{Oi}$$

Dynamic invariants:



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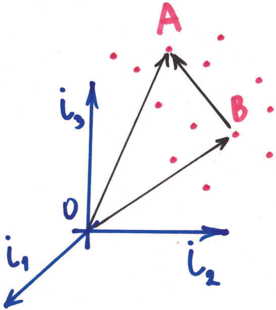

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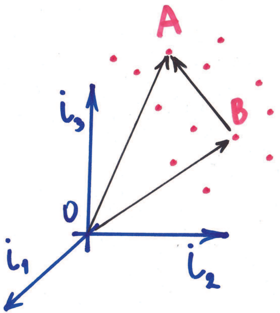

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Dynamic invariants: $\boxed{\vec{L}_A \cdot \overrightarrow{AB} \equiv \vec{L}_B \cdot \overrightarrow{AB}}$, $\boxed{\vec{L}_A \cdot \vec{P} \equiv \vec{L}_B \cdot \vec{P}}$

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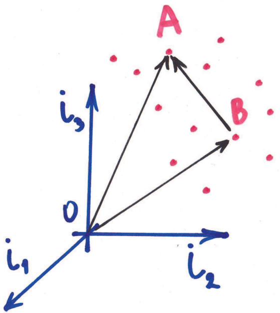

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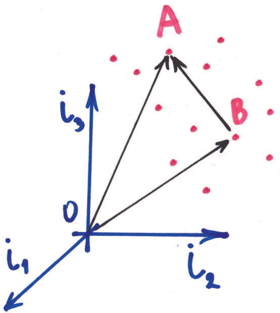

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Kinetic Energy

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By definition the kinetic energy of the system of points $\{\vec{r}_i\}$ is

$$T = \frac{1}{2}\sum_i m_i \left[\frac{d}{dt}\vec{r}_{Oi}\right]^2$$

Since any point coordinates can be written as $\vec{r}_{Oi} = \vec{r}_C + \vec{r}_{Ci}$, then

$$T = \frac{1}{2}\sum_i m_i \left[\frac{d}{dt}\vec{r}_C + \frac{d}{dt}\vec{r}_{Ci}\right]^2$$

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$$T = \frac{1}{2}\sum_i m_i \left[\frac{d}{dt}\vec{r}_{Oi}\right]^2$$

Since any point coordinates can be written as $\vec{r}_{Oi} = \vec{r}_C + \vec{r}_{Ci}$, then

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \left[\frac{d}{dt}\vec{r}_C + \frac{d}{dt}\vec{r}_{Ci}\right]^2 \\ &= \frac{1}{2}\sum_i m_i \Big[\left[\frac{d}{dt}\vec{r}_C\right]^2 + 2\left(\frac{d}{dt}\vec{r}_C\right)\cdot\left(\frac{d}{dt}\vec{r}_{Ci}\right) + \left[\frac{d}{dt}\vec{r}_{Ci}\right]^2\Big] \\ &= \frac{1}{2}M\left[\frac{d}{dt}\vec{r}_C\right]^2 + \left(\frac{d}{dt}\vec{r}_C\right)\cdot\Big(\sum_i m_i \frac{d}{dt}\vec{r}_{Ci}\Big) + \sum_i m_i \left[\frac{d}{dt}\vec{r}_{Ci}\right]^2 \end{aligned}$$

$$\vec{r}_C(t) = \frac{\sum_i m_i \vec{r}_{Oi}(t)}{\sum_i m_i} \quad \Rightarrow \quad \boxed{\sum_i m_i \vec{r}_{Ci}(t)} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt}\left\{\sum_i m_i \vec{r}_{Ci}(t)\right\}} = 0$$

$$T = \frac{1}{2}M\left[\frac{d}{dt}\vec{r}_C\right]^2 + \sum_i m_i \left[\frac{d}{dt}\vec{r}_{Ci}\right]^2, \qquad M = \sum_i m_i$$

Kinetic Energy

By definition the kinetic energy of the system of points $\{\vec{r}_i\}$ is

$$T = \frac{1}{2}\sum_i m_i \left[\frac{d}{dt}\vec{r}_{Oi}\right]^2$$

then its time derivative along a motion of the system is

$$\frac{d}{dt}T \quad = \quad \frac{1}{2}\frac{d}{dt}\left\{\sum_i m_i \left[\vec{v}_i \cdot \vec{v}_i\right]\right\}$$

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then its time derivative along a motion of the system is

$$\begin{aligned}\frac{d}{dt}T &= \frac{1}{2}\frac{d}{dt}\left\{\sum_i m_i\left[\vec{v}_i\cdot\vec{v}_i\right]\right\} = \sum_i \vec{v}_i\cdot\frac{d}{dt}\left\{m_i\vec{v}_i\right\} \\ &= \sum_i \vec{v}_i\cdot\left\{\sum F_i^{in} + \sum F_i^{ex}\right\}\end{aligned}$$

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$$\begin{aligned}
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&= \sum_i \vec{v}_i\cdot\left\{\sum F_i^{in} + \sum F_i^{ex}\right\} \\
&= \Big\{\text{Elementary work done by INTERNAL forces}\Big\} \\
&\qquad + \Big\{\text{Elementary work done by EXTERNAL forces}\Big\}
\end{aligned}$$

Modeling and Control of Robots

Lecture 13: Dynamics of Rigid Bodies

Anton Shiriaev

February 22, 2021

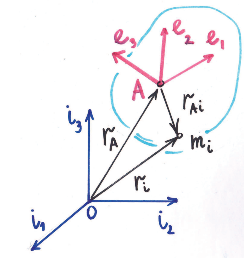
Learning outcomes: Euler equations of motion of a rigid body and Euler-Lagrange equations for a mechanical system subject to holonomic constraints

Outline

1. Euler equations of motion of a rigid body
2. Euler-Lagrange equations of motion of mechanical system

Euler equations of motion

Equations of Motion of a Rigid Body



Any motion of rigid body is consistent with Eqns.

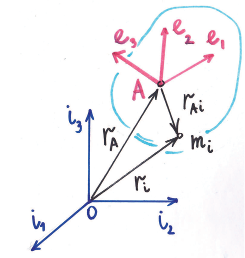
$$\tfrac{d}{dt}\big(M\vec{v}_C\big) = \textstyle\sum_i \vec{F}_i^{ex}, \;\; \tfrac{d}{dt}\big(\vec{L}_A\big) = \textstyle\sum_i \vec{r}_{Ai} \times \vec{F}_i^{ex} =: \vec{M}_A$$

where M is the total mass of the rigid body; $\vec{F}_i^{ex}$ are external forces applied the rigid body; $\vec{v}_C$ is the velocity of its center of mass; $\vec{L}_A$ is the angular momentum computed about the point A

According to (p. 4, Lecture 12), $\vec{L}_A$ can be computed as

$$\vec{L}_A = \big(M\cdot\vec{r}_{AC}\big)\times\vec{v}_A + J_A\vec{\omega}, \quad J_A(t) = \textstyle\sum_i m_i\cdot\Big(\big|\vec{r}_{Ai}(t)\big|^2\cdot I_3 - \vec{r}_{Ai}(t)\,\vec{r}_{Ai}(t)^T\Big)$$

Equations of Motion of a Rigid Body



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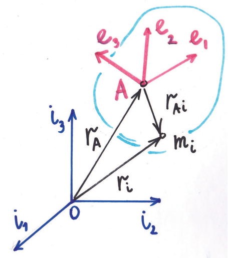
where $\vec{r}_{AC}$ is the vector connecting A and the center of mass; $\vec{v}_A$ is the velocity of A; and $J_A(t)$ is its inertia tensor about the point A.

Equations of Motion of a Rigid Body



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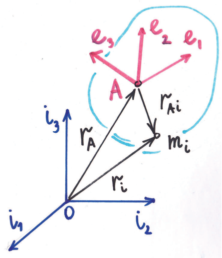
The inertia tensor $J_A(\cdot)$, if expressed the e-frame, is constant

$$\boxed{J_A^i(t) = R(t)\, J_A^e\, R(t)^T} \qquad J_A^e = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{12} & I_{22} & -I_{23} \\ -I_{13} & -I_{23} & I_{33} \end{bmatrix}$$

Equations of Motion of a Rigid Body



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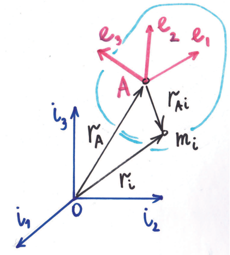
Euler Equations of Motion of a Rigid Body with a Fixed Point

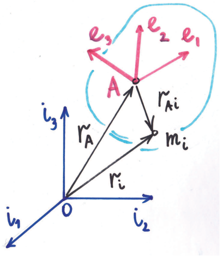


Suppose the point A of the rigid body is fixed.

The equation becomes rewritten as

$$\frac{d}{dt}\left(\left[R(t)J_A^e R(t)^T\right] R(t)\vec{\omega}^e\right) = \frac{d}{dt}\left(R(t)J_A^e\vec{\omega}^e\right) = R(t)\vec{M}_A^e(t)$$

Euler Equations of Motion of a Rigid Body with a Fixed Point



Suppose the point A of the rigid body is fixed.

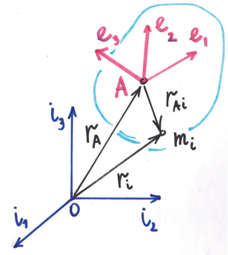
$$\Downarrow$$

Then the angular momentum about A is equal to

$$\vec{L}_A \quad = \quad \left(M \cdot \vec{r}_{AC}\right) \times \vec{v}_A + J_A \vec{\omega}$$

The equation becomes rewritten as

$$\frac{d}{dt}\left(\left[R(t) J_A^e R(t)^T\right] R(t) \vec{\omega}^e\right) \quad = \quad \frac{d}{dt}\left(R(t) J_A^e \vec{\omega}^e\right) = R(t) \vec{M}_A^e(t)$$



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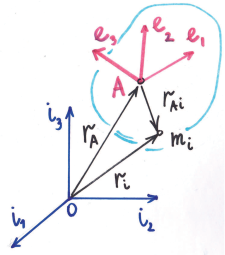

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The rotation between the i-and e-frames defines

$$\vec{\omega}^i = R\vec{\omega}^e, \quad \vec{M}_A^i = R\vec{M}_A^e, \quad J_A^i = R J_A^e R^T$$

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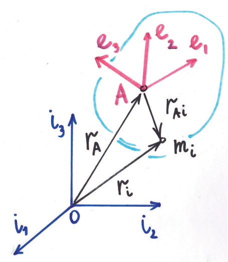

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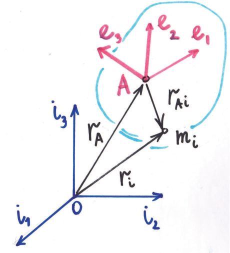

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The equation $\boxed{\frac{d}{dt}(\vec{L}_A^i) = \vec{M}_A^i}$ becomes rewritten as

$$\begin{aligned} \frac{d}{dt}\left(\left[R(t) J_A^e R(t)^T\right] R(t)\vec{\omega}^e\right) &= \frac{d}{dt}\left(R(t) J_A^e \vec{\omega}^e\right) = R(t)\vec{M}_A^e(t) \\ &= \frac{d}{dt}\left[R(t)\right] \boxed{J_A^e \vec{\omega}^e} + R(t) \boxed{\frac{d}{dt}\left[J_A^e \vec{\omega}^e\right]} \end{aligned}$$

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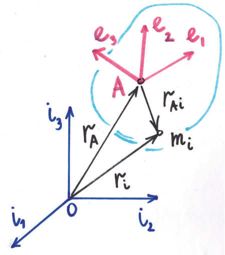

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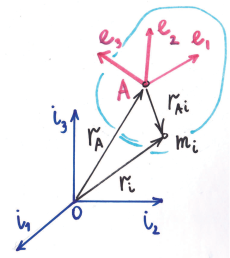
$$= \frac{d}{dt}\left[R(t)\right] \boxed{J_A^e \vec{\omega}^e} + R(t) \boxed{\frac{d}{dt}\left[J_A^e \vec{\omega}^e\right]}$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} I_1 \cdot p \\ I_2 \cdot q \\ I_3 \cdot r \end{bmatrix}$$

Euler Equations of Motion of a Rigid Body with a Fixed Point



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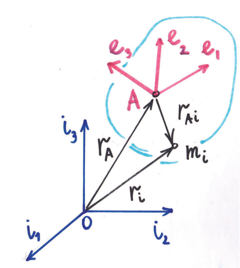

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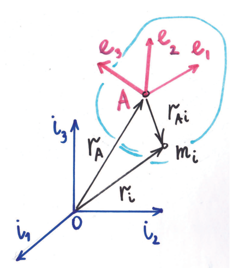
$$= \frac{d}{dt}\left[R(t)\right] J_A^e \vec{\omega}^e + R(t) \frac{d}{dt}\left[J_A^e \vec{\omega}^e\right]$$

$$\Downarrow$$

Euler Equations of Motion of a Rigid Body with a Fixed Point



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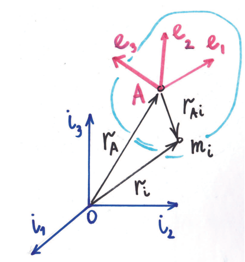
&= \frac{d}{dt}\left[R(t)\right]\boxed{J_A^e\vec{\omega}^e} + R(t)\boxed{\frac{d}{dt}\left[J_A^e\vec{\omega}^e\right]} \\
&\Downarrow \\
J_A^e\frac{d}{dt}\left[\vec{\omega}^e\right] + R(t)^T\frac{d}{dt}\left[R(t)\right]J_A^e\vec{\omega}^e &= \vec{M}_A^e(t)
\end{aligned}$$

Euler Equations of Motion of a Rigid Body with a Fixed Point



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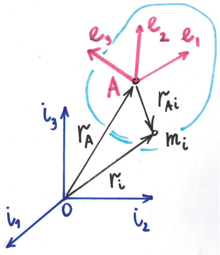
$$= \frac{d}{dt}\left[R(t)\right] \boxed{J_A^e \vec{\omega}^e} + R(t) \boxed{\frac{d}{dt}\left[J_A^e \vec{\omega}^e\right]}$$

$$\Downarrow$$

$$J_A^e \frac{d}{dt}\left[\vec{\omega}^e\right] + \underbrace{R(t)^T \frac{d}{dt}\left[R(t)\right]}_{= S(\vec{\omega}^e)} J_A^e \vec{\omega}^e \;=\; \vec{M}_A^e(t)$$

see PSS No.4



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&= \quad \frac{d}{dt}\left[R(t)\right] \boxed{J_A^e\vec{\omega}^e} + R(t)\,\boxed{\frac{d}{dt}\left[J_A^e\vec{\omega}^e\right]} \\
&\;\;\Downarrow \\
J_A^e\frac{d}{dt}\left[\vec{\omega}^e\right] + \vec{\omega}^e \times J_A^e\vec{\omega}^e \quad &= \quad \vec{M}_A^e(t)
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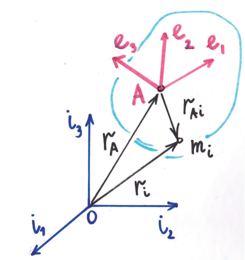
Euler Equations of Motion of a Rigid Body with a Fixed Point



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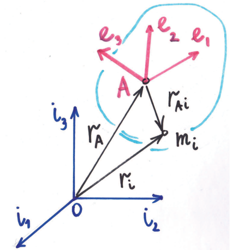

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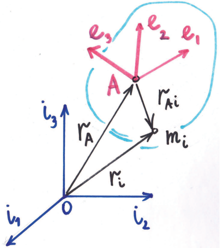

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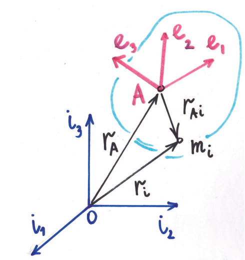

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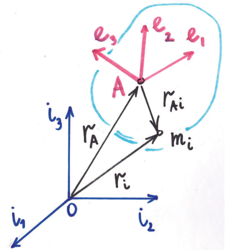

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Lecture No.5, p.8

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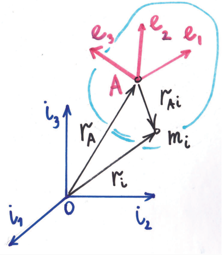

If the body has no fixed points, what arguments are to use?

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Its angular momentum about center of mass C is

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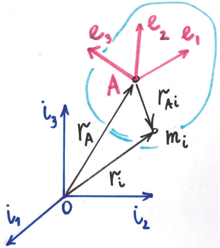

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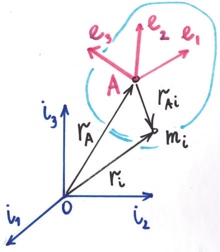

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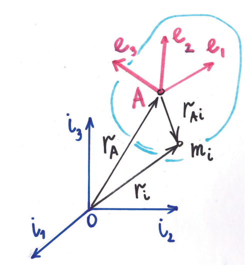

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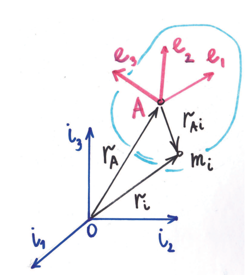

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Euler-Lagrange equations

Euler-Lagrange equations of motion of mechanical system

Computational Procedure

For a system with n DOF we shall

1. introduce a set of generalized coordinates $(q_1, \ldots, q_n)$;
2. formulate kinetic energy $\mathcal{T}(q, \dot{q})$ and potential energy $\mathcal{P}(q)$;
3. compute the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{P}$ and derive the Euler-Lagrange equation as

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \tau_k \ , \qquad k = 1, \ldots, n$$

This procedure is useful for complex systems such as multi-link robots. Each DOF is affected by external (generalized) forces τ_k.

Example: point-mass dynamics in excessive coordinates

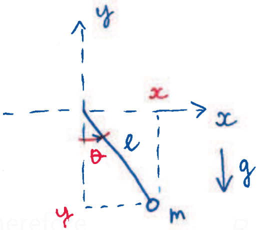
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f\left(x(t), y(t)\right) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x;\ R_y]$ is the reaction force due to the constraint.



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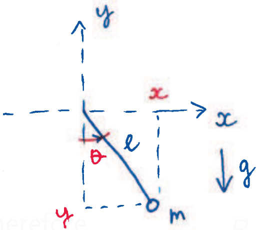

How to find the reaction force
$R = R(x, y, \dot{x}, \dot{y})$**?**

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The constraint $f(\cdot) \equiv 0$ implies that

$$\tfrac{d}{dt} f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall\, t.$$

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For defining λ consider the 2^{nd} derivative of the constraint $f(\cdot) \equiv 0$

$$\begin{aligned} 0 \equiv \tfrac{d^2}{dt^2} f &= 2\dot{x}^2 + 2x \cdot \ddot{x} + 2\dot{y}^2 + 2y \cdot \ddot{y} \\ &= 2\dot{x}^2 + 2x \cdot (\tfrac{1}{m}\lambda \cdot x) + 2\dot{y}^2 + 2y \cdot (\tfrac{1}{m}\lambda \cdot y - g) \\ &= 2\dot{x}^2 + 2\dot{y}^2 + 2\tfrac{1}{m}\lambda \cdot (x^2 + y^2) - 2y \cdot g \end{aligned}$$

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The Lagrangian multiplier is then equal to

$$\lambda = \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right)$$

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Therefore $$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

The point mass dynamics in excessive coordinates (x, y) are

$$\begin{aligned} m \cdot \ddot{x} &= \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x \\ m \cdot \ddot{y} &= \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g \end{aligned}$$

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f\left(x(t), y(t)\right) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin\theta(t), \quad y(t) = -l \cdot \cos\theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = T - \Pi &= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - m \cdot g \cdot y \\ &= \tfrac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos\theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} &= m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin\theta\right), \end{aligned}$$

which is the equation of the mathematical pendulum.

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which is the equation of the mathematical pendulum.

Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

$$\begin{aligned} m \cdot \ddot{x} &= \tfrac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x \\ m \cdot \ddot{y} &= \tfrac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g \end{aligned} \tag{1}$$

and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0 \tag{2}$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

Example: point-mass dynamics

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Modeling and Control of Robots

Lecture 14: Euler-Lagrange Equations

Anton Shiriaev

February 23, 2021

Learning outcomes: Virtual displacements and Euler-Lagrange equations for a mechanical system subject to holonomic constraints; dynamics of open-chain robot manipulator

Outline

1. On concept of a virtual displacement of mechanical system

2. Euler-Lagrange equations of motion of mechanical system

3. Dynamics of Open-Chain Manipulator

Euler-Lagrange equations of motion of mechanical system

Computational Procedure

For a system with n DOF we shall

1. introduce a set of generalized coordinates $(q_1, \ldots, q_n)$;
2. formulate kinetic energy $\mathcal{K}(q, \dot{q})$ and potential energy $\mathcal{P}(q)$;
3. compute the Lagrangian $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and derive the Euler-Lagrange equation as

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \tau_k \ , \qquad k = 1, \ldots, n$$

This procedure is useful for complex systems such as multi-link robots. Each DOF is affected by external (generalized) forces τ_k.

Acceleration of a point mass in curvilinear coordinates in $\mathbb{R}^3$

To represent the point's acceleration in new frame, compute the products

$$\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2, \qquad \left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3$$

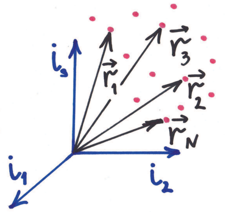
The computations done in Lecture 1 resulted in

$$\begin{aligned}
\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_1 &= \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_1}\right) - \frac{\partial K}{\partial q_1}} \\
\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_2 &= \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_2}\right) - \frac{\partial K}{\partial q_2}} \\
\left(\tfrac{d^2}{dt^2}\vec{r}\right)\cdot\vec{e}_3 &= \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_3}\right) - \frac{\partial K}{\partial q_3}}
\end{aligned}$$

where the scalar function $K(\cdot)$ is defined as

$$K = \tfrac{1}{2}\cdot\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \tfrac{1}{2}\cdot\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

On concept of a virtual displacement

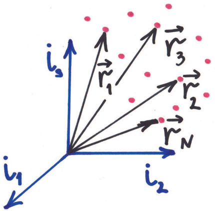


A constraint imposed on k particles with coordinates $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k \in \mathbb{R}^3$ is called holonomic, if it is of the form

$$\boxed{g_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k) = 0}, \quad i = 1, 2, \ldots, l$$

Differentiating $g_i(\cdot)$ with respect to time, we obtain a new constraint

$$\frac{d}{dt} g_i = \frac{\partial g_i}{\partial r_1} \frac{d}{dt} r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \frac{d}{dt} r_k = 0 \Rightarrow$$



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For example, given two particles joined by massless rigid rod of length l, then the corresponding holonomic constraint is

$$\vec{r}_1, \vec{r}_2 \in \mathbb{R}^3 : \ \left\|\vec{r}_1 - \vec{r}_2\right\|^2 = \left(\vec{r}_1 - \vec{r}_2\right)^T \left(\vec{r}_1 - \vec{r}_2\right) = l^2$$

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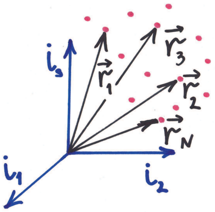
Constraints and constrained forces



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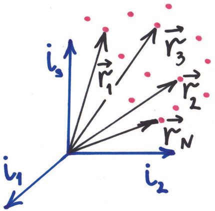
Presence of a constraint implies that there exists a force

so that the constraint holds $\Rightarrow$ It is referred to as constraint force.

Differentiating $g_i(\cdot)$ with respect to time, we obtain a new constraint

Constraints and constrained forces



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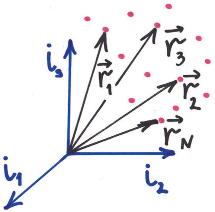

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The constraint of the form

$$\omega_1(r_1, \ldots r_k) dr_1 + \cdots + \omega_k(r_1, \ldots r_k) dr_k = 0$$

is called non-holonomic if it cannot be integrated back.

Concept of Generalized Coordinates

If a system of k particles is subject to holonomic constraints then

- it is possible to express their coordinates as functions of fewer than $3k$ variables

$$\left.\begin{array}{rcl} r_1 & = & r_1(q_1, \ldots, q_n) \\ & \vdots & \\ r_k & = & r_k(q_1, \ldots, q_n) \end{array}\right\} \quad \text{such that} \quad \left\{\begin{array}{rcl} g_1(r_1, \ldots, r_k) & = & 0 \\ & \vdots & \\ g_l(r_1, \ldots, r_k) & = & 0 \end{array}\right.$$

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- This smallest number n is called number of degrees of freedom.

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- The smallest set of such independent variables is called generalized coordinates.
- This smallest number n is called number of degrees of freedom.
- Even for a system with an infinite number of particles one might have a finite number of degrees of freedom.

Concept of Virtual Displacement

Given a system of k-particles and a holonomic constraint

$$g_i(r_1,\, r_2,\, \ldots,\, r_k) = 0, \qquad i = 1,\, 2,\, \ldots,\, l$$

or alternatively

$$\frac{\partial g_i}{\partial r_1}\, dr_1 + \cdots + \frac{\partial g_i}{\partial r_k}\, dr_k = 0, \qquad i = 1,\, 2,\, \ldots,\, l$$

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A set of infinitesimal displacements is called virtual displacements

$$\delta r_1, \ \delta r_2, \ \ldots, \ \delta r_k$$

if it is consistent with the constraint, i.e.

$$\frac{\partial g_i}{\partial r_1} \delta r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \delta r_k = 0, \qquad i = 1, 2, \ldots, l$$

Consider a system of k-particles and suppose that

- it has a holonomic constraint such that constraint forces f_i^c act on some of the particles;

The total sum of all forces applied to the i^{th}-particle is zero

$$\sum_i \left(f_i^c + f_i^e\right) = 0$$

Then the work done by forces applied to i^{th}-particle is zero, i.e.

$$= \sum_i \left(f_i^c + f_i^e\right) \delta r_i = \underbrace{\sum_i f_i^c \delta r_i}_{=0} +$$

Consider a system of k-particles and suppose that

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Principle of Virtual Work

Consider a system of k-particles and suppose that

- it has a holonomic constraint such that constraint forces f_i^c act on some of the particles;
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- the system is in equilibrium (at rest).

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$\Rightarrow\Rightarrow$ The work done by external forces at equilibrium is zero $\Leftarrow\Leftarrow$

D'Alembert's Principle

Consider a system of k-particles and suppose that

- there are holonomic constraints, i.e. constrained forces f_i^c act on some of particles;
- there are external forces f_i^e acting on particles;

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D'Alembert's principle transforms a work of a moving rigid body into a work of an equivalent static system by adding inertial forces

$$-\sum_i f_i^c \delta r_i = \boxed{0 = \sum_i \left(f_i^e - \frac{d}{dt}\left[m_i \dot{r}_i \right] \right) \delta r_i}$$

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However, the virtual constraints δr_i are not independent! We must

- rewrite $\sum_i f_i^e \delta r_i$ as function of generalized coordinates
- rewrite $\sum_i \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i$ as function of generalized coordinates

Euler-Lagrange equations

Right-hand side of the main equation of dynamics

Virtual displacements are computed in terms of generalized coordinates as

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j, \quad i = 1, \ldots, k$$

Then the external forces are

$$\begin{aligned} \sum_{i=1}^{k} f_i^e \delta r_i &= \sum_{i=1}^{k} f_i^e \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} f_i^e \frac{\partial r_i}{\partial q_j} \right) \delta q_j \\ &= \sum_{j=1}^{n} \psi_j \delta q_j \end{aligned}$$

The functions ψ_j are called generalized forces.

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The functions ψ_j are called generalized forces.

Right-hand side of the main equation of dynamics

The inertial forces can be rewritten as

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i \quad = \quad \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

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&= \sum_{j=1}^{n} \sum_{i=1}^{k} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j
\end{aligned}$$

Right-hand side of the main equation of dynamics

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$$\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right] = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]$$

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$$\begin{aligned}\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i &= \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{k} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j\end{aligned}$$

$$\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right] = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]$$

$$\Rightarrow \quad \sum_{i=1}^{k} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] \right\}$$

Right-hand side of the main equation of dynamics

The inertial forces can be rewritten as

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i \;=\; \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\} \right] \delta q_j$$

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$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\} \right] \delta q_j$$

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \quad \Rightarrow \quad \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

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$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] = \sum_{l=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial}{\partial q_j} \left[\sum_{l=1}^{n} \frac{\partial r_i}{\partial q_l} \dot{q}_l \right] = \frac{\partial v_i}{\partial q_j}$$

Right-hand side of the main equation of dynamics

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$$\begin{aligned}
\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i &= \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) \\
&= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right] \right\} \right] \delta q_j \\
&= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i v_i \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i v_i \frac{\partial v_i}{\partial q_j} \right\} \right] \delta q_j \\
&= \sum_{j=1}^{n} \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j
\end{aligned}$$

where $\mathcal{K} = \sum_{i=1}^{k} \frac{1}{2} m_i \left|v_i\right|^2$ is the kinetic energy of each particle.

Right-hand side of the main equation of dynamics

Elementary work done by all forces in the system along any virtual displacement is $0 = \sum_i \left(f_i^e - \frac{d}{dt}\left[m_i \dot{r}_i\right]\right) \delta r_i$, where

$$\sum_{i=1}^{k} f_i^e \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

$$\sum_{i=1}^{k} \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j}\right] \delta q_j$$

$$\sum_{j=1}^{n} \left\{\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j\right\} \delta q_j = 0$$

Right-hand side of the main equation of dynamics

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With independent virtual displacements δq_j we obtain equations

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} = \psi_j, \quad j = 1, \ldots n$$

Right-hand side of the main equation of dynamics

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$$\begin{aligned} \sum_{i=1}^{k} f_i^e \delta r_i &= \sum_{j=1}^{n} \psi_j \delta q_j \\ \sum_{i=1}^{k} \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i &= \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j}\right] \delta q_j \end{aligned}$$

$$\sum_{j=1}^{n} \left\{\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j\right\} \delta q_j = 0$$

Grouping the generalized forces ψ_j (potential/others) results in

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, \quad \psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j, \quad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

Dynamics of Open-Chain Manipulator

Equations of Motion of Open-Chain Manipulator

Kinetic Energy for an Open Chain Manipulator

In computing the kinetic energy of the robot

$$\mathcal{K} = \sum \frac{1}{2} m_i \left| v_{c,i} \right|^2 + \frac{1}{2} \omega_i^T \mathcal{I}_i \omega_i$$

we need to express the body velocities $\left\{ v_{c,i};\, \omega_i \right\}$ of each link

a function of generalized coordinates q and generalized velocities $\dot{q}$

These relations are given by the Jacobian of the manipulator

$$v_{c,i} = J_{v_i}(q)\dot{q}, \quad \omega_i = J_{\omega_i}(q)\dot{q}$$

The final form of kinetic energy is

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$$\mathcal{K} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_{Ci} R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

Potential Energy for an Open Chain Manipulator

The potential energy of the i^{th}-link is

$$\mathcal{P}_i = m_i g^T r_{c,i}$$

where $r_{c,i}$ is the position of its center of mass

The total potential energy of the robot is then

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The total potential energy of the robot is then

$$\boxed{\mathcal{P} = \sum_{i=1}^{n} \mathcal{P}_i = \sum_{i=1}^{n} m_i g^T r_{c,i}}$$

Equations of Motion of Open-Chain Manipulator

- The kinetic energy is given as

$$\begin{aligned} \mathcal{K} &= \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_{Ci} R_i(q)^T J_{\omega_i}(q) \right] \dot{q} \\ &= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{k,j} d_{kj}(q)\dot{q}_k \dot{q}_j \end{aligned}$$

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k$$

Equations of Motion of Open-Chain Manipulator

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&= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{k,j} d_{kj}(q)\dot{q}_k\dot{q}_j
\end{aligned}
$$

Properties of Inertia Matrix $D(q)$

Observe the following properties of the inertia matrix for any manipulator

1. $D(q) \in \mathbb{R}^{n\times n}$ is square
2. $D(q) = D(q)^T$ is symmetric by construction
3. $\dot{q}^T D(q)\dot{q} > 0$ is positive definite since kinetic energy is positive or identical zero iff $\dot{q} = 0$.

Equations of Motion of Open-Chain Manipulator

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\end{aligned}$$

- If generalized forces are potential, then $\psi_k = -\frac{\partial \mathcal{P}}{\partial q_k} + \tau_k$

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k$$

Equations of Motion of Open-Chain Manipulator

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- We can introduce a scalar function $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and write the equation of motion in compact form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$$

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial (\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k$$

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\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k
$$

$$
\frac{d}{dt}\frac{\partial (\mathcal{K} - \mathcal{P})}{\partial \dot{q}_k} - \frac{\partial (\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k
$$

Equations of Motion of Open-Chain Manipulator

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Equations of Motion of Open-Chain Manipulator

The equations of motion have the particular structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k, \qquad k = 1, \ldots, n$$

The first term is computed as

$$\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q}\right] = \sum_{j=1}^{n} d_{kj}\dot{q}_j$$

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and

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$$= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\right)\dot{q}_j$$

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$$= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

The second term is computed as

$$
\begin{aligned}
\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}
\end{aligned}
$$

Equations of Motion of Open-Chain Manipulator

The second term is computed as

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P}$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}$$

The equations of motion are finally computed as

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j+\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j+\frac{\partial}{\partial q_k}\mathcal{P}=\tau_k$$

Equations of Motion of Open-Chain Manipulator

The second term is computed as

$$\begin{aligned}\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q} - \frac{\partial}{\partial q_k}\mathcal{P} \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial}{\partial q_k}\mathcal{P}\end{aligned}$$

The equations of motion are finally computed as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k\,, \quad k = 1,\ldots,n$$

with Christoffel symbols $\{c_{ijk}\}$, gradient of the potential energy

$$c_{ijk}(q) = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right), \quad g_k(q) = \frac{\partial}{\partial q_k}\mathcal{P}$$

Equations of Motion of Open-Chain Manipulator

The second term is computed as

$$\begin{aligned}\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P} \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}\end{aligned}$$

The equations of motion are finally computed as

$$\boxed{\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j+\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j+g_k(q)=\tau_k}\,, \quad k=1,\ldots,n$$

given in the vector format by

$$\boxed{D(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=\tau}$$

with Coriolis/centrifugal forces corr.t. $C(q)$, $c_{kj}=\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$

Modeling and Control of Robots

Lecture 15: Properties of Euler-Lagrange Equations

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Learning outcomes: Dynamics of Euler-Lagrange systems and change of generalized coordinates;

Outline

1. Properties of Euler-Lagrange equations

2. Properties of E-L equations with a natural Lagrangian

Properties of Euler-Lagrange equations

Change of coordinates and Euler-Lagrange equations

Procedure for a holonomic n-DOF mechanical system

1. introduce a set of generalized coordinates $q = (q_1;\ \ldots;\ q_n)$;
2. introduce the Lagrangian $\mathcal{L} = \mathcal{L}(q, \dot q, t)$ and write the system dynamics as a set of the Euler-Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q, \dot q, t)}{\partial \dot q_k}\right] - \frac{\partial \mathcal{L}(q, \dot q, t)}{\partial q_k} = 0, \qquad k = 1, \ldots, n \tag{1}$$

The dynamics of the mechanical system (1) can be often rewritten as

the first order ODE:

Suppose we introduce new generalized coordinates $\tilde q$

Change of coordinates and Euler-Lagrange equations

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the first order ODE: $\boxed{\frac{d}{dt}X = F(X, t), \quad X = \big(q;\, \dot{q}\big)}$

Suppose we introduce new generalized coordinates $\tilde{q}$

Change of coordinates and Euler-Lagrange equations

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$q = q\big(\tilde{q}\big)$

Change of coordinates and Euler-Lagrange equations

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Suppose we introduce new generalized coordinates $\tilde{q}$

$$q = q\big(\tilde{q}\big) \quad \Rightarrow \quad \dot{q} = \left[\frac{\partial q}{\partial \tilde{q}}\right]\dot{\tilde{q}}$$

Change of coordinates and Euler-Lagrange equations

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$$q = q\big(\tilde{q}\big) \quad \Rightarrow \quad \dot{q} = \left[\frac{\partial q}{\partial \tilde{q}}\right]\dot{\tilde{q}} \quad \Rightarrow \quad \boxed{\frac{d}{dt}\tilde{X} = \tilde{F}(\tilde{X}, t), \quad \tilde{X} = \big(\tilde{q};\ \dot{\tilde{q}}\big)}$$

Change of coordinates and Euler-Lagrange equations

Procedure for a holonomic n-DOF mechanical system

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$$q = q\big(\tilde{q}\big) \quad \Rightarrow \quad \dot{q} = \left[\frac{\partial q}{\partial \tilde{q}}\right] \dot{\tilde{q}} \quad \Rightarrow \quad \boxed{\frac{d}{dt} \tilde{X} = \tilde{F}(\tilde{X}, t), \quad \tilde{X} = \big(\tilde{q};\, \dot{\tilde{q}}\big)}$$

Do we need to use Eqn. (1) for deriving the dynamics in new coordinates?

Change of coordinates and Euler-Lagrange equations

Procedure for a holonomic n-DOF mechanical system

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Suppose we introduce new generalized coordinates $\tilde{q}$

$$q = q(\tilde{q}) \quad \Rightarrow \quad \dot{q} = \left[\frac{\partial q}{\partial \tilde{q}}\right]\dot{\tilde{q}} \quad \Rightarrow \quad \boxed{\frac{d}{dt}\tilde{X} = \tilde{F}(\tilde{X}, t), \quad \tilde{X} = (\tilde{q};\, \dot{\tilde{q}})}$$

Not necessary! We can start the procedure using new coordinates $\tilde{q}$!

Lagrangian and Euler-Lagrange equations

Given a Lagrangian $\mathcal{L}_a(\cdot)$ and the Euler-Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_a(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}_a(q,\dot{q},t)}{\partial q_k} = 0, \qquad k = 1,\ldots,n \tag{2}$$

can we introduce another Lagrangian $\mathcal{L}_b(\cdot)$ such that the equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_b(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}_b(q,\dot{q},t)}{\partial q_k} = 0, \qquad k = 1,\ldots,n \tag{3}$$

literally coincide with Eqns. (2)???

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}_1}\right] - \frac{\partial \mathcal{L}_c}{\partial q_1} =$$

Lagrangian and Euler-Lagrange equations

Given a Lagrangian $\mathcal{L}_a(\cdot)$ and the Euler-Lagrange equations

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Case 1: If $\mathcal{L}_b(q,\dot{q},t) = c \cdot \mathcal{L}_a(q,\dot{q},t)$, $c \neq 0$, then this is true!!!

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}_1}\right] - \frac{\partial \mathcal{L}_c}{\partial q_1} =$$

Lagrangian and Euler-Lagrange equations

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Case 2: Given a function $f(\cdot)$, new Lagrangian $\mathcal{L}_b(\cdot)$ can be defined as

$$\mathcal{L}_b(\cdot) = \boxed{\mathcal{L}_a(\cdot) + \tfrac{d}{dt}\big[f(q,t)\big]}$$

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Lagrangian and Euler-Lagrange equations

Given a Lagrangian $\mathcal{L}_a(\cdot)$ and the Euler-Lagrange equations

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Lagrangian and Euler-Lagrange equations

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$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_c}{\partial \dot{q}_1}\right] - \frac{\partial \mathcal{L}_c}{\partial q_1} = \frac{d}{dt}\left[\frac{\partial f}{\partial q_1}\right] - \frac{\partial}{\partial q_1}\left[\frac{\partial f}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f}{\partial q_n}\dot{q}_n + \frac{\partial f}{\partial t}\right]$$

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$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}_c}{\partial \dot q_1}\right] - \frac{\partial \mathcal{L}_c}{\partial q_1} = \frac{d}{dt}\left[\frac{\partial f}{\partial q_1}\right] - \frac{\partial}{\partial q_1}\left[\frac{\partial f}{\partial q_1}\dot q_1 + \cdots + \frac{\partial f}{\partial q_n}\dot q_n + \frac{\partial f}{\partial t}\right] \boxed{= 0}$$

Lagrangian and Euler-Lagrange equations

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literally coincide with Eqns. (4)???

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$$\mathcal{L}_a = \tfrac{1}{2}\dot{\theta}^2 - \cos\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} + \sin\theta = 0}$$

Lagrangian and Euler-Lagrange equations

Given a Lagrangian $\mathcal{L}_a(\cdot)$ and the Euler-Lagrange equations

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$$\mathcal{L}_a = \tfrac{1}{2}\dot{\theta}^2 - \cos\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} + \sin\theta = 0} \qquad\qquad \mathcal{L}_b = \tfrac{1}{2}\dot{\theta}^2 - \cos\theta + \underbrace{d\cdot\dot{\theta}}_{\frac{d}{dt}\left[d\cdot\theta\right]}$$

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$$\mathcal{L}_a = \tfrac{1}{2}\dot{\theta}^2 - \cos\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} + \sin\theta = 0} \quad \Leftarrow \quad \mathcal{L}_b = \tfrac{1}{2}\dot{\theta}^2 - \cos\theta + \underbrace{d\cdot\dot{\theta}}_{\frac{d}{dt}\left[d\cdot\theta\right]}$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_k}\right] - \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{6}$$

consider the function

$$H(q,\dot q,t) := \dot q_1 \cdot \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_1} + \cdots + \dot q_n \cdot \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_n} - \mathcal{L}(q,\dot q,t)$$

Its time derivative along a solution of the E-L system (6) is

$$\frac{d}{dt}H = \left\{ \ddot q_1 \cdot \frac{\partial \mathcal{L}}{\partial \dot q_1} + \dot q_1 \cdot \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q_1}\right]\right\} + \cdots + \left\{ \ddot q_n \cdot \frac{\partial \mathcal{L}}{\partial \dot q_n} + \dot q_n \cdot \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q_n}\right]\right\}$$

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Its time derivative along a solution of the E-L system (6) is

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Generalized energy of Euler-Lagrange equations

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Its time derivative along a solution of the E-L system (6) is

$$\begin{aligned} \tfrac{d}{dt}H = & \Big\{\ddot{q}_1 \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_1} + \dot{q}_1 \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right]\Big\} + \cdots + \Big\{\ddot{q}_n \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_n} + \dot{q}_n \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_n}\right]\Big\} \\ & - \Big\{\tfrac{\partial}{\partial q_1}\mathcal{L}\tfrac{d}{dt}q_1 + \cdots + \tfrac{\partial}{\partial q_n}\mathcal{L}\tfrac{d}{dt}q_n \\ & \qquad + \tfrac{\partial}{\partial \dot{q}_1}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_1 + \cdots + \tfrac{\partial}{\partial \dot{q}_n}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_n + \tfrac{\partial}{\partial t}\mathcal{L}\Big\} \end{aligned}$$

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$$H(q,\dot{q},t) := \dot{q}_1 \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_1} + \cdots + \dot{q}_n \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_n} - \mathcal{L}(q,\dot{q},t)$$

Its time derivative along a solution of the E-L system (6) is

$$\begin{aligned}\tfrac{d}{dt}H = &\left\{\ddot{q}_1 \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_1} + \dot{q}_1 \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right]\right\} + \cdots + \left\{\ddot{q}_n \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_n} + \dot{q}_n \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_n}\right]\right\} \\ &- \left\{\tfrac{\partial}{\partial q_1}\mathcal{L}\tfrac{d}{dt}q_1 + \cdots + \tfrac{\partial}{\partial q_n}\mathcal{L}\tfrac{d}{dt}q_n \right. \\ &\qquad \left. + \tfrac{\partial}{\partial \dot{q}_1}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_1 + \cdots + \tfrac{\partial}{\partial \dot{q}_n}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_n + \tfrac{\partial}{\partial t}\mathcal{L}\right\}\end{aligned}$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_k}\right] - \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{6}$$

consider the function

$$H(q,\dot q,t) := \dot q_1 \cdot \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_1} + \cdots + \dot q_n \cdot \frac{\partial \mathcal{L}(q,\dot q,t)}{\partial \dot q_n} - \mathcal{L}(q,\dot q,t)$$

Its time derivative along a solution of the E-L system (6) is

$$\begin{aligned}
\tfrac{d}{dt}H = &\Big\{\ddot q_1 \cdot \tfrac{\partial \mathcal{L}}{\partial \dot q_1} + \dot q_1 \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot q_1}\right]\Big\} + \cdots + \Big\{\ddot q_n \cdot \tfrac{\partial \mathcal{L}}{\partial \dot q_n} + \dot q_n \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot q_n}\right]\Big\} \\
&- \Big\{\tfrac{\partial}{\partial q_1}\mathcal{L}\tfrac{d}{dt}q_1 + \cdots + \tfrac{\partial}{\partial q_n}\mathcal{L}\tfrac{d}{dt}q_n \\
&\qquad + \tfrac{\partial}{\partial \dot q_1}\mathcal{L}\cdot\tfrac{d^2}{dt^2}q_1 + \cdots + \tfrac{\partial}{\partial \dot q_n}\mathcal{L}\cdot\tfrac{d^2}{dt^2}q_n + \tfrac{\partial}{\partial t}\mathcal{L}\Big\} \\
= \quad & \dot q_1 \cdot \Big\{\tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot q_1}\right] - \tfrac{\partial}{\partial q_1}\mathcal{L}\Big\} + \cdots + \dot q_n \cdot \Big\{\tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot q_n}\right] - \tfrac{\partial}{\partial q_n}\mathcal{L}\Big\} - \tfrac{\partial}{\partial t}\mathcal{L}
\end{aligned}$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{6}$$

consider the function

$$H(q,\dot{q},t) := \dot{q}_1 \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_1} + \cdots + \dot{q}_n \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_n} - \mathcal{L}(q,\dot{q},t)$$

Its time derivative along a solution of the E-L system (6) is

$$\begin{aligned}
\tfrac{d}{dt}H = &\Big\{\ddot{q}_1 \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_1} + \dot{q}_1 \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right]\Big\} + \cdots + \Big\{\ddot{q}_n \cdot \tfrac{\partial \mathcal{L}}{\partial \dot{q}_n} + \dot{q}_n \cdot \tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_n}\right]\Big\} \\
&- \Big\{\tfrac{\partial}{\partial q_1}\mathcal{L}\tfrac{d}{dt}q_1 + \cdots + \tfrac{\partial}{\partial q_n}\mathcal{L}\tfrac{d}{dt}q_n \\
&\qquad + \tfrac{\partial}{\partial \dot{q}_1}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_1 + \cdots + \tfrac{\partial}{\partial \dot{q}_n}\mathcal{L} \cdot \tfrac{d^2}{dt^2}q_n + \tfrac{\partial}{\partial t}\mathcal{L}\Big\} \\
= \quad & \dot{q}_1 \cdot \Big\{\tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right] - \tfrac{\partial}{\partial q_1}\mathcal{L}\Big\} + \cdots + \dot{q}_n \cdot \Big\{\tfrac{d}{dt}\left[\tfrac{\partial \mathcal{L}}{\partial \dot{q}_n}\right] - \tfrac{\partial}{\partial q_n}\mathcal{L}\Big\} - \tfrac{\partial}{\partial t}\mathcal{L} \\
= \quad & \dot{q}_1 \cdot \tau_1 + \cdots + \dot{q}_n \cdot \tau_n - \tfrac{\partial}{\partial t}\mathcal{L}
\end{aligned}$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{7}$$

the function is known as a **generalized energy**

$$H(q,\dot{q},t) := \dot{q}_1 \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_1} + \cdots + \dot{q}_n \cdot \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_n} - \mathcal{L}(q,\dot{q},t)$$

Indeed, if $\boxed{\frac{\partial}{\partial t}\mathcal{L} \equiv 0}$ and external forces are absent $\boxed{\tau_k \equiv 0,\ \forall\, k}$, then the time derivative of $H(\cdot)$ along a solution of the E-L system (7) is equal to

$$\tfrac{d}{dt}H(\cdot) \quad = \quad \dot{q}_1 \cdot \tau_1 + \cdots + \dot{q}_n \cdot \tau_n - \tfrac{\partial}{\partial t}\mathcal{L} = \boxed{0}$$

What is the generalized energy $H(\cdot)$ if the Lagrangian is $\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - \mathcal{P}(q)$?

$$H(q,\dot{q}) = \dot{q}^T \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \left\{\tfrac{1}{2}\dot{q}^T D(q)\dot{q} - \mathcal{P}(q)\right\}$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{7}$$

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Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q_k} = \tau_k, \qquad k = 1,\ldots,n \tag{7}$$

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Generalized energy of Euler-Lagrange equations

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$$H(q,\dot q) = \dot q^{T}\left[\frac{\partial \mathcal{L}}{\partial \dot q}\right] - \left\{\tfrac{1}{2}\dot q^{T} D(q)\dot q - \mathcal{P}(q)\right\} = \dot q^{T}\Big[D(q)\dot q\Big] - \tfrac{1}{2}\dot q^{T} D(q)\dot q + \mathcal{P}(q)$$

Generalized energy of Euler-Lagrange equations

Given a set of coordinates, a Lagrangian $\mathcal{L}(\cdot)$, and the E-L equations

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E-L equations with a natural Lagrangian

Euler-Lagrange equations with a natural Lagrangian

A system with Lagrangian $\mathcal{L}(q, \dot{q})$ is called **natural** if

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} - \mathcal{P}(q)$$

Any open chain manipulator with the kinetic and potential energies

$$\mathcal{K} = \sum \tfrac{1}{2} m_i \left|v_{c,i}\right|^2 + \tfrac{1}{2}\omega_i^T \mathcal{I}_i \omega_i, \quad \mathcal{P} = \sum_{i=1}^{n} m_i g^T r_{c,i}$$

is obviously natural :)

The Euler-Lagrange equations can be rewritten as (Lecture 14)

$$k = 1, \ldots, n$$

with $\{c_{ijk}\}$ being the Christoffel symbols and $\{g_k(q)\}$ coefficients of gradient of the potential energy defined by

$$c_{ijk}(q) = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right), \quad g_k(q) = \frac{\partial}{\partial q_k}\mathcal{P}$$

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The Euler-Lagrange equations can be rewritten as (Lecture 14)

$$\boxed{\sum_{j=1}^{n} d_{kj}(q)\ddot q_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot q_i \dot q_j + g_k(q) = \tau_k} \quad k = 1,\ldots,n$$

with $\left\{c_{ijk}\right\}$ being the Christoffel symbols and $\left\{g_k(q)\right\}$ coefficients of gradient of the potential energy defined by

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A system with Lagrangian $\mathcal{L}(q, \dot{q})$ is called **natural** if

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is obviously natural :)

The equations of motion

$$\boxed{\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k}, \quad k = 1, \ldots, n$$

can be rewritten in the vector format as

$$\boxed{D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau}$$

where elements of the matrix function $C(\cdot)$ are $c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$

Skew Symmetry of $\frac{d}{dt}\left[D(q)\right] - 2C(q,\dot{q})$

To check that $N(\cdot)$ is skew symmetric

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q,\dot{q}) \ , \quad \boxed{N^T = -N}$$

consider its $(k,j)^{th}$-element

$$n_{kj} = \tfrac{d}{dt} d_{kj} - 2c_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i$$

Skew Symmetry of $\frac{d}{dt}\big[D(q)\big] - 2C(q,\dot q)$

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$$\begin{aligned} n_{kj} \ = \ \tfrac{d}{dt} d_{kj} - 2c_{kj} \quad &= \quad \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot q_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot q_i \\ &= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot q_i \end{aligned}$$

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n_{kj} \ = \ \tfrac{d}{dt} d_{kj} - 2c_{kj} \quad &= \quad \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i
\end{aligned}$$

Skew Symmetry of $\frac{d}{dt}\left[D(q)\right] - 2C(q,\dot q)$

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$$
\begin{aligned}
n_{kj} = \tfrac{d}{dt} d_{kj} - 2c_{kj} \quad &= \quad \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot q_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot q_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot q_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i
\end{aligned}
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Skew Symmetry of $\frac{d}{dt}\left[D(q)\right] - 2C(q,\dot q)$

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&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot q_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i \\
&= \quad \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot q_i
\end{aligned}$$

$$\boxed{\Rightarrow \qquad n_{kj} = -n_{jk}}$$

Passivity Relation

Given a mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau \quad \Leftrightarrow \quad D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

with

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Its (generalized) energy is given by

$$H(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \;\Leftrightarrow\; \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau}$$

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Its (generalized) energy is given by

$$H(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

Let us recompute $\frac{d}{dt}H$

Passivity Relation (2)

Differentiating $H(\cdot)$ along a solution of the system, we have

$$\begin{aligned}\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\ &= \tfrac{1}{2}\ddot{q}^T D(q)\dot{q} + \tfrac{1}{2}\dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}\end{aligned}$$

Passivity Relation (2)

Differentiating $H(\cdot)$ along a solution of the system, we have

$$\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \tfrac{1}{2}\ddot{q}^T D(q)\dot{q} + \tfrac{1}{2}\dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}$$

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Differentiating $H(\cdot)$ along a solution of the system, we have

$$\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \tfrac{1}{2}\ddot{q}^T D(q)\dot{q} + \tfrac{1}{2}\dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - G(q)\right] + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}$$

Here we used: $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$

Passivity Relation (2)

Differentiating $H(\cdot)$ along a solution of the system, we have

$$\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \tfrac{1}{2}\ddot{q}^T D(q)\dot{q} + \tfrac{1}{2}\dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
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&= \dot{q}^T\tau + \dot{q}^T\left(\tfrac{1}{2}\tfrac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)\dot{q} + \dot{q}^T\left(\frac{\partial \mathcal{P}}{\partial q} - G(q)\right)
\end{aligned}$$

Passivity Relation (2)

Differentiating $H(\cdot)$ along a solution of the system, we have

$$
\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
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\end{aligned}
$$

Passivity Relation (2)

Differentiating $H(\cdot)$ along a solution of the system, we have

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\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
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&= \dot{q}^T D(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \tfrac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
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\end{aligned}
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Differentiating $H(\cdot)$ along a solution of the system, we have

$$
\begin{aligned}
\tfrac{d}{dt}H &= \tfrac{d}{dt}\left[\tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
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&= \dot{q}^T\tau
\end{aligned}
$$

Passivity Relation (3)

The differential relation

$$\boxed{\frac{d}{dt}H = \dot{q}^T\tau}$$

can be integrated so that

$$\begin{aligned}\int_0^T \frac{d}{dt}H(q(t),\dot{q}(t))dt &= H(q(T),\dot{q}(T)) - H(q(0),\dot{q}(0)) \\ &= \int_0^T \dot{q}(t)^T\tau(t)dt\end{aligned}$$

The amount of energy dissipated by the system has a lower bound.

These relations are called

- passivity (dissipativity) relation
- passivity (dissipativity) relation in the integral form

Passivity Relation (3)

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$$\int_0^T \dot{q}(t)\tau(t)dt = H(q(T),\dot{q}(T)) - H(q(0),\dot{q}(0)) \geq -H(q(0),\dot{q}(0))$$

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These relations are called

- passivity (dissipativity) relation
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TTK4195: Modeling and Control of Robots

Lecture 12: Multivariable Control of Robot Manipulators

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Outline

Improved Dynamical Model

Given a mechanical system with n-degrees of freedom $q = \left[q_1, \ldots, q_n\right]^T$

$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau} \quad , \quad \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

Improved Dynamical Model

Given a mechanical system with n-degrees of freedom $q = \left[q_1, \ldots, q_n\right]^T$

$$\boxed{D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau} \quad , \quad \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

Suppose that each degree of freedom k is controlled by a geared DC-motor

$$\boxed{J_{m,k}\ddot{\theta}_{m,k} + B_{m,k}\dot{\theta}_{m,k} = \frac{K_{m,k}}{R_k} V_k - \frac{1}{r_k}\tau_k} \quad , \quad k = 1, \ldots, n$$

where

- $\theta_{m,k}$ is the k^{th} motor angle;
- r_k is the k^{th} gear ratio;
- $J_{m,k}$, $B_{m,k}$, $K_{m,k}$, R_k are parameters the k^{th} DC-motor.

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Given a mechanical system with n-degrees of freedom $q = \left[q_1, \ldots, q_n\right]^T$

$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau} \ , \quad \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

Suppose that each degree of freedom k is controlled by a geared DC-motor

$$\boxed{J_{m,k}\ddot{\theta}_{m,k} + B_{m,k}\dot{\theta}_{m,k} = \frac{K_{m,k}}{R_k} V_k - \frac{1}{r_k}\tau_k} \ , \quad k = 1, \ldots, n$$

and that the motor and the link angles are related by

$$\theta_{m,k} = r_k q_k, \quad k = 1, \ldots, n$$

Improved Dynamical Model

Given a mechanical system with n-degrees of freedom $q = \left[q_1, \ldots, q_n\right]^T$

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$$\boxed{J_{m,k}\ddot{\theta}_{m,k} + B_{m,k}\dot{\theta}_{m,k} = \frac{K_{m,k}}{R_k} V_k - \frac{1}{r_k}\tau_k}\ , \quad k = 1, \ldots, n$$

and that the motor and the link angles are related by

$$\theta_{m,k} = r_k q_k, \quad k = 1, \ldots, n$$

Then the actuator equations are

$$\boxed{r_k^2 J_{m,k}\ddot{q}_k + r_k^2 B_{m,k}\dot{q}_k = r_k \frac{K_{m,k}}{R_k} V_k - \tau_k, \quad k = 1, \ldots, n}$$

Improved Dynamical Model (2)

The dynamical systems

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau\ , \quad \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

$$r_k^2 J_{m,k}\ddot{q}_k + r_k^2 B_{m,k}\dot{q}_k = r_k \frac{K_{m,k}}{R_k} V_k - \tau_k, \quad k = 1,\ \ldots,\ n$$

can be combined, if we exclude τ!

Improved Dynamical Model (2)

The dynamical systems

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau\ , \quad \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

$$r_k^2 J_{m,k}\ddot{q}_k + r_k^2 B_{m,k}\dot{q}_k = r_k \frac{K_{m,k}}{R_k} V_k - \tau_k, \quad k = 1,\ \ldots,\ n$$

can be combined, if we exclude τ!

Indeed, it is

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u}$$

where

- $\boxed{M(q) = D(q) + J}$ with $J = \mathrm{diag}\left(r_1^2 J_{m,1}, \ldots, r_n^2 J_{m,n}\right)$
- $B = \left[r_1^2 B_{m,1}, r_2^2 B_{m,2}, \ldots, r_n^2 B_{m,n}\right]^T$ being friction coefficients
- $u = [u_1, u_2, \ldots, u_n]^T$ with $u_k = r_k \frac{K_{m,k}}{R_k} V_k,\ \ k = 1, \ldots, n$

Outline

PD-Controller Design

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

We are interested

- to design a controller that stabilizes a particular configuration of the robot: $q = q_d$
- to analyze the closed-loop system.

PD-Controller Design

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We are interested

- to design a controller that stabilizes a particular configuration of the robot: $q = q_d$
- to analyze the closed-loop system.

Assume that $B = 0$ and $G(q) = 0$.

PD-Controller Design

Given a mechanical system

$$\boldsymbol{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u}$$

We are interested

- to design a controller that stabilizes a particular configuration of the robot: $\boldsymbol{q = q_d}$
- to analyze the closed-loop system.

Assume that $\boldsymbol{B = 0}$ and $\boldsymbol{G(q) = 0}$.

The first control law to analyze is

$$\boldsymbol{u = -K_p\,(q - q_d) - K_d\dot{q}}$$

with $\boldsymbol{K_p}$ and $\boldsymbol{K_d}$ being diagonal matrices with positive elements.

PD-Controller Design (2)

To analyze the behavior of the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p\,(q - q_d) - K_d\dot{q}$$

consider a scalar function, the Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\,(q - q_d)^T\, K_p\,(q - q_d)$$

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$$V(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + \tfrac{1}{2}\,(q - q_d)^T\,K_p\,(q - q_d)$$

Its time-derivative along solutions of the closed-loop system is

$$\tfrac{d}{dt}V \;\;=\;\; \dot{q}^T M(q)\ddot{q} + \dot{q}^T \tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d)$$

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$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d) \\
&= \dot{q}^T\left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d)
\end{aligned}$$

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To analyze the behavior of the closed-loop system

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\tfrac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d) \\
&= \dot{q}^T\left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d) \\
&= \dot{q}^T\left[u + K_p\,(q - q_d)\right] + \dot{q}^T\left\{\tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right] - C(q,\dot{q})\right\}\dot{q}
\end{aligned}$$

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To analyze the behavior of the closed-loop system

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$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}[M(q)]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}[M(q)]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u + K_p(q - q_d)\right] + \underbrace{\dot{q}^T \left\{\frac{d}{dt}\frac{1}{2}[D(q) + J] - C(q,\dot{q})\right\}\dot{q}}_{=0}
\end{aligned}$$

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Its time-derivative along solutions of the closed-loop system is

$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d) \\
&= \dot{q}^T\left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \tfrac{d}{dt}\tfrac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p\,(q - q_d) \\
&= \dot{q}^T\left[u + K_p\,(q - q_d)\right] \\
&= \dot{q}^T\left[-K_p\,(q - q_d) - K_d\dot{q} + K_p\,(q - q_d)\right] = -\dot{q}^T K_d\dot{q}
\end{aligned}$$

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To analyze the behavior of the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p\,(q - q_d) - K_d\dot{q}$$

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$$V(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + \tfrac{1}{2}\,(q - q_d)^T\, K_p\,(q - q_d)$$

Its time-derivative along solutions of the closed-loop system is

$$\boxed{\frac{d}{dt}V = -\dot{q}^T K_d \dot{q} \le 0}$$

Therefore

- V is positive definite, $V(q,\dot{q}) = 0 \Rightarrow \{q = q_d, \dot{q} = 0\}$
- $V(q(t),\dot{q}(t))$ is monotonically decreasing!

$\Rightarrow$ $\boxed{\exists \lim_{t\to+\infty} V(q(t),\dot{q}(t)) = V_\infty}$ and $\boxed{\exists \lim_{t\to+\infty} \dot{q}(t) = \dot{q}_\infty(t) = 0}$

PD-Controller Design (3)

To analyze the behavior of the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p\,(q - q_d) - K_d\dot{q}$$

consider a scalar function, the Lyapunov function candidate

$$V(q,\dot{q}) = \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + \tfrac{1}{2}\,(q - q_d)^T\,K_p\,(q - q_d)$$

If we substitute this limit trajectory into the dynamics, we obtain

$$M(q_\infty)\underbrace{\ddot{q}_\infty}_{=0} + C(q_\infty,\dot{q}_\infty)\underbrace{\dot{q}_\infty}_{=0} = -K_p\,(q_\infty - q_d) - K_d\underbrace{\dot{q}_\infty}_{=0}$$

that is

$$0 = -K_p\,(q_\infty - q_d)$$

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that is

$$0 = -K_p\left(q_\infty - q_d\right)$$

$$K_p = \operatorname{diag}\left(K_{p1}, K_{p2}, \ldots, K_{pn}\right) > 0 \;\Rightarrow\; \boxed{q_\infty = q_d}$$

Outline

PD-Controller Design with Gravity Compensation

Given a mechanical system

$$\boldsymbol{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u}$$

How to modify the controller

$$\boldsymbol{u = -K_p\,(q - q_d) - K_d\dot{q}}$$

if $\boldsymbol{B \neq 0}$ and $\boldsymbol{G(q) \neq 0}$?

PD-Controller Design with Gravity Compensation

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How to modify the controller

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if $B \neq 0$ and $G(q) \neq 0$?

The modified control law

$$u = -K_p\,(q - q_d) - K_d\dot{q} + G(q)$$

is stabilizing, if K_p and K_d are diagonal matrices such that

$$K_p > 0 \qquad K_d - B > 0$$

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Feedback Linearization

We consider a class of nonlinear systems of the form

$$\boxed{\dot{x} = f(x) + g(x)u}$$

What is the explicit state equation for a mechanical system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + G(q) = u$$

which in our case is fully actuated?

Feedback Linearization

We consider a class of nonlinear systems of the form

$$\boxed{\dot{x} = f(x) + g(x)u}\,, \quad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^{2n}\,, \quad u \in \mathbb{R}^{n}$$

What is the explicit state equation for a mechanical system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + G(q) = u$$

which in our case is fully actuated?

The state equation is

$$\boxed{\dot{x} = \begin{bmatrix} \dot{q} \\ M(q)^{-1}\left[u - C(q, \dot{q})\dot{q} - B\dot{q} - G(q)\right] \end{bmatrix}}$$

Feedback Linearization

We consider the mechanical system with states $x = [q; \dot{q}] \in \mathbb{R}^{2n}$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ M(q)^{-1}\left[u - C(q,\dot{q})\dot{q} - B\dot{q} - G(q)\right] \end{bmatrix}$$

Does there exist a state feedback control

$$u = \alpha(x) + \beta(x) v$$

and a change of variable $z = T(x)$ such that the closed-loop system

$$\dot{z} = A_z z + B_z v$$

is linear and the pair (A_z, B_z) is stabilizable?

Feedback Linearization (2)

We consider the mechanical system with states $\boldsymbol{x} = [\boldsymbol{q}; \dot{\boldsymbol{q}}] \in \mathbb{R}^{2n}$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1} \left[\boldsymbol{u} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{B}\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})\right] \end{bmatrix}$$

Lets choose $\boldsymbol{z} = \boldsymbol{x} = [\boldsymbol{q}; \dot{\boldsymbol{q}}]$ and the control transformation

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q})$$

What is the dynamics of the closed-loop system?

Feedback Linearization (2)

We consider the mechanical system with states $x = [q; \dot{q}] \in \mathbb{R}^{2n}$

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Lets choose $z = x = [q; \dot{q}]$ and the control transformation

$$u = M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

What is the dynamics of the closed-loop system?

$$\ddot{q} = v \quad \Rightarrow \quad \boxed{\dot{x} = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} x + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} v}, \quad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

- $\ddot{q} = v$ is known as double integrator system
- $u = M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$ is called inverse dynamics control that makes the close-loop system linear and decoupled.

Outline

Inverse Dynamics + PD Control

Given a mechanical system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= M(q)a_q + C(q, \dot{q})\dot{q} + B\dot{q} + G(q) \\ a_q &= \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \end{aligned}$$

Inverse Dynamics + PD Control

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Then the closed loop system is

$$\ddot{q} = a_q = \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big)$$

Inverse Dynamics + PD Control

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$$\ddot{q} = a_q = \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big)$$

It can be rewritten in error variables as

$$\boxed{\ddot{e} + K_d\dot{e} + K_p e = 0}\ , \quad e = q_d(t) - q$$

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$$

Inverse Dynamics + PD Control (2)

Trajectory Tracking – inverse dynamics controller

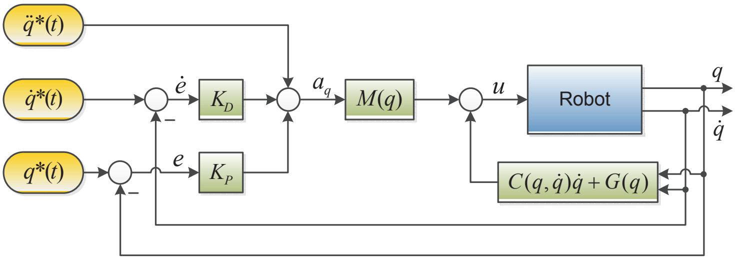

Figure: Schematic of linearizing and decoupling control law $\boldsymbol{u}$ plus PD tracking controller; here friction coefficient $\boldsymbol{B}$ is neglected for simplicity.

We could for instance choose $\boldsymbol{K_p} = \mathbf{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\boldsymbol{K_d} = \mathbf{diag}(2\omega_1, \ldots, 2\omega_n)$ to shape the tracking performance.

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Task Space Representation of Inverse Dynamics

Given a mechanical system and the linearizing feedback control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

$$u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

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$$u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

Let $X = \begin{bmatrix} o_n^0(q) \\ \alpha(q) \end{bmatrix} \in \mathbb{R}^6$ be the end-effector pose using a minimal representation of $SO(3)$ so that we get the relations

$$\begin{aligned} \dot{X} &= J_a(q)\ \dot{q} \\ \ddot{X} &= \dot{J}_a(q,\dot{q})\ \dot{q} + J_a(q)\ \ddot{q} \end{aligned}$$

Task Space Representation of Inverse Dynamics

Given a mechanical system and the linearizing feedback control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

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Now modify a_q for a linearization and decoupling in task space:

$$a_X = \dot{J}_a(q,\dot{q})\ \dot{q} + J_a(q)\ a_q \quad \Leftrightarrow \quad a_q = J_a^{-1}(q)\left[a_X - \dot{J}_a(q,\dot{q})\ \dot{q}\right]$$

resulting in the double integrator system $\ddot{X} = a_X$ for which we can also design tracking controller. Use pseudoinverse if Jacobian is not square.

Outline

Robust and Adaptive Motion Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, the controller

$$\begin{aligned} u &= \boxed{M(q)v + C(q,\dot{q})\dot{q} + G(q)} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

cannot be safely implemented if the model parameters are uncertain!

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cannot be safely implemented if the model parameters are uncertain!

The approximation might be used

$$\begin{aligned} u &= \boxed{[M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

Robust and Adaptive Motion Control (2)

Parameter uncertainties and unmodelled effects of the robot dynamics give rise for two control approaches:

$$\boldsymbol{u} = \boxed{[\boldsymbol{M}(\boldsymbol{q}) + \triangle \boldsymbol{M}]\, \boldsymbol{v} + [\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \triangle \boldsymbol{C}]\, \dot{\boldsymbol{q}} + [\boldsymbol{G}(\boldsymbol{q}) + \triangle \boldsymbol{G}]}$$

$$\boldsymbol{v} = \ddot{\boldsymbol{q}}_{\boldsymbol{d}}(\boldsymbol{t}) - \boldsymbol{K_p}(\boldsymbol{q} - \boldsymbol{q_d}(\boldsymbol{t})) - \boldsymbol{K_d}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_{\boldsymbol{d}}(\boldsymbol{t}))$$

- Robust Control:
  Design $\boldsymbol{K_p}$, $\boldsymbol{K_d}$ and $\boldsymbol{q_d}(\boldsymbol{t})$ such that the error signal

$$\boldsymbol{e}(\boldsymbol{t}) = \boldsymbol{q}(\boldsymbol{t}) - \boldsymbol{q_d}(\boldsymbol{t}) \approx \boldsymbol{0} \qquad \forall\, \{\triangle \boldsymbol{M}, \triangle \boldsymbol{C}, \triangle \boldsymbol{G}\} \in \boldsymbol{\mathcal{W}}$$

Robust and Adaptive Motion Control (2)

Parameter uncertainties and unmodelled effects of the robot dynamics give rise for two control approaches:

$$u = [M(q) + \triangle M]\, v + [C(q,\dot q) + \triangle C]\,\dot q + [G(q) + \triangle G]$$

$$v = \ddot q_d(t) - K_p(q - q_d(t)) - K_d(\dot q - \dot q_d(t))$$

- Robust Control:
  Design K_p, K_d and $q_d(t)$ such that the error signal

  $$e(t) = q(t) - q_d(t) \approx 0 \qquad \forall\, \{\triangle M, \triangle C, \triangle G\} \in \mathcal{W}$$

- Adaptive Control: Improve estimates for

  $$M(q), \quad C(q,\dot q), \quad G(q)$$

  in the course of regulating the system.

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Robust Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$

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$$\boxed{\ddot{q} = v + \eta(q, \dot{q}, v)}$$

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Robust Feedback Linearization (2)

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w$$

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$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w$$

It can be rewritten as

$$\ddot{q} = v + \eta(q, \dot{q}, v)$$

or

$$(\ddot{q} - \ddot{q}_d(t)) + K_d\,(\dot{q} - \dot{q}_d(t)) + K_p\,(q - q_d(t)) = w + \eta(q, \dot{q}, v)$$

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or

$$\frac{d}{dt} e = \begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix} e + \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} \left[w + \eta(t, e, w) \right], \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

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$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [G(q) + \triangle G]$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))\ \boxed{+w}$$

It can be rewritten as

$$\ddot{q} = v + \eta(q, \dot{q}, v)$$

or

$$(\ddot{q} - \ddot{q}_d(t)) + K_d\,(\dot{q} - \dot{q}_d(t)) + K_p\,(q - q_d(t)) = w + \eta(q, \dot{q}, v)$$

or

$$\frac{d}{dt}e = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix}}_{A} e + \underbrace{\begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix}}_{B} [w + \eta(t, e, w)]\ , \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\ , \quad \alpha < 1$$

The condition $\alpha = \left\|M^{-1}(q)\triangle M - I\right\| < 1$ determines how close our estimate $\triangle M$ must be the true inertia matrix.

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \ , \quad \alpha < 1$$

Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

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$$\|\eta(t, e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\ , \quad \alpha < 1$$

Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^T Pe$, then

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P\frac{d}{dt}e \\ &= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(t, e, w)\right] \end{aligned}$$

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\;, \quad \alpha < 1$$

Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

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Consider a Lyapunov function candidate as $V = e^T Pe$, then

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P\frac{d}{dt}e \\ &= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(t, e, w)\right] \\ &\leq 0 \quad \leftarrow \end{aligned}$$

How to achieve this by choosing w?

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \le \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\ , \quad \alpha < 1$$

Let us look at the second term

$$\frac{d}{dt}V \quad = \quad -e^T Q e + 2\underbrace{e^T P B}_{z^T}\left[w + \eta(t, e, w)\right]$$

when w has the form

$$w = -\rho(t, e)\frac{z}{\sqrt{z^T z}}\,, \qquad z = B^T P e\,, \qquad \rho(t, e) \text{ is a function to choose}$$

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t,e,w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t,e,w)\| \le \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\;, \quad \alpha < 1$$

Let us look at the second term

$$\frac{d}{dt}V \;=\; -e^TQe + 2\underbrace{e^TPB}_{z^T}\left[w + \eta(t,e,w)\right]$$

when w has the form

$$w = -\rho(t,e)\frac{z}{\sqrt{z^Tz}}\,, \qquad z = B^TPe, \qquad \rho(t,e) \text{ is a function to choose}$$

$$z^T\left(-\rho\frac{z}{\sqrt{z^Tz}} + \eta\right) \le -\rho\|z\| + |z^T\eta|$$

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t,e,w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t,e,w)\| \le \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\;, \quad \alpha < 1$$

Let us look at the second term

$$\frac{d}{dt}V \;=\; -e^T Q e + 2\underbrace{e^T P B}_{z^T}\left[w + \eta(t,e,w)\right]$$

when w has the form

$$w = -\rho(t,e)\frac{z}{\sqrt{z^T z}}\,, \qquad z = B^T P e, \qquad \rho(t,e) \text{ is a function to choose}$$

$$z^T\left(-\rho\frac{z}{\sqrt{z^T z}} + \eta\right) \le -\rho\|z\| + \|z\|\;\|\eta\| = \|z\|\left(-\rho + \|\eta\|\right)$$

Robust Feedback Linearization (4)

To sum up, we search for a scalar function $\rho(t, e)$ such that

and

$$\boxed{(-\rho + \|\eta\|) \leq 0} \;\Leftrightarrow\; \boxed{\|\eta\| \leq \rho}$$

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

with

$$w = -\rho(t, e) \frac{z}{\sqrt{z^T z}}$$

Robust Feedback Linearization (4)

To sum up, we search for a scalar function $\rho(t, e)$ such that

and

$$(-\rho + \|\eta\|) \leq 0 \quad \Leftrightarrow \quad \|\eta\| \leq \rho$$

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

with

$$w = -\rho(t, e) \frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha \left\| \rho(t, e) \frac{z}{\sqrt{z^T z}} \right\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \leq \rho(t, e)$$

Robust Feedback Linearization (4)

To sum up, we search for a scalar function $\rho(t, e)$ such that

and

$$\boxed{(-\rho + \|\eta\|) \leq 0} \;\Leftrightarrow\; \boxed{\|\eta\| \leq \rho}$$

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

with

$$w = -\rho(t, e) \frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha\rho(t, e) + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \leq \rho(t, e)$$

Robust Feedback Linearization (4)

To sum up, we search for a scalar function $\rho(t, e)$ such that

$$\boxed{(-\rho + \|\eta\|) \leq 0} \quad \Leftrightarrow \quad \boxed{\|\eta\| \leq \rho}$$

and

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

with

$$w = -\rho(t, e) \frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha \rho(t, e) + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \leq \rho(t, e)$$

$$\boxed{\rho(t, e) \geq \frac{1}{1 - \alpha} \left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \right]}$$

Robust Feedback Linearization – Final Controller

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [G(q) + \triangle G]$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w$$

$$w = \begin{cases} -\rho(t,e)\frac{z}{\sqrt{z^T z}}\,, & \text{if } z = B^T P e \neq 0 \\ 0\,, & \text{if } z = B^T P e = 0 \end{cases}$$

where $\rho(t,e)$ is any function that satisfies the inequality

$$\rho(t,e) \geq \frac{1}{1-\alpha}\left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t)\right]$$

with constants α, γ_1,γ_2 and $\gamma_3(t)$ obtained from the inequality

$$\|\eta(\cdot)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

and

$$\eta(q,\dot{q},v) = M^{-1}[\triangle M v + \triangle C \dot{q} + \triangle G]\,, \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

Lecture 13: Multivariable Control of Robot Manipulators (cont'd)

- Robust and Adaptive Motion Control
  - Example: Robust Motion Control
  - Adaptive Motion Control Based on Feedback Linearization
- Passivity-Based Motion Control

TTK4195: Modeling and Control of Robots

Lecture 17: Multivariable Control of Robot Manipulators

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Spring 2021

Outline

1. Trajectory Tracking
   - Feedback Linearization
   - Joint Space Inverse Dynamics
   - Task Space Inverse Dynamics

2. Robust and Adaptive Motion Control
   - Robust Control Based on Feedback Linearization
   - Adaptive Control Based on Feedback Linearization

Outline

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   - Robust Control Based on Feedback Linearization
   - Adaptive Control Based on Feedback Linearization

Feedback Linearization

We consider a class of nonlinear systems of the form

$$\boxed{\dot{x} = f(x) + g(x)u}$$

What is the explicit state equation for a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

which in our case is fully actuated?

Feedback Linearization

We consider a class of nonlinear systems of the form

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}}\,, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} \in \mathbb{R}^{2n}\,, \quad \boldsymbol{u} \in \mathbb{R}^{n}$$

What is the explicit state equation for a mechanical system

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{u}$$

which in our case is fully actuated?

The state equation is

$$\boxed{\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1}\left[\boldsymbol{u} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{B}\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})\right] \end{bmatrix}}$$

Feedback Linearization

We consider the mechanical system with states $\boldsymbol{x} = [\boldsymbol{q}; \dot{\boldsymbol{q}}] \in \mathbb{R}^{2n}$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1} \left[\color{red}{\boldsymbol{u}} \color{black} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{B}\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \right] \end{bmatrix}$$

Does there exist a state feedback control

$$\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\beta}(\boldsymbol{x})\color{red}{\boldsymbol{v}}$$

and a change of variable $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x})$ such that the closed-loop system

$$\dot{\boldsymbol{z}} = \boldsymbol{A}_z \boldsymbol{z} + \boldsymbol{B}_z \color{red}{\boldsymbol{v}}$$

is linear and the pair $(\boldsymbol{A}_z, \boldsymbol{B}_z)$ is stabilizable?

Feedback Linearization (2)

We consider the mechanical system with states $\boldsymbol{x} = [\boldsymbol{q}; \dot{\boldsymbol{q}}] \in \mathbb{R}^{\boldsymbol{2n}}$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})^{-1} [\boldsymbol{u} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{B}\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})] \end{bmatrix}$$

Lets choose $\boldsymbol{z} = \boldsymbol{x} = [\boldsymbol{q}; \dot{\boldsymbol{q}}]$ and the control transformation

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q})$$

What is the dynamics of the closed-loop system?

Feedback Linearization (2)

We consider the mechanical system with states $x = \left[q; \dot{q}\right] \in \mathbb{R}^{2n}$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ M(q)^{-1}\left[u - C(q,\dot{q})\dot{q} - B\dot{q} - G(q)\right] \end{bmatrix}$$

Lets choose $z = x = \left[q; \dot{q}\right]$ and the control transformation

$$u = M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

What is the dynamics of the closed-loop system?

$$\ddot{q} = v \quad \Rightarrow \quad \boxed{\dot{x} = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} x + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} v}, \quad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

- $\ddot{q} = v$ is known as double integrator system
- $u = M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$ is called inverse dynamics control that makes the close-loop system linear and decoupled.

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   - Feedback Linearization
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   - Robust Control Based on Feedback Linearization
   - Adaptive Control Based on Feedback Linearization

Inverse Dynamics + PD Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) \\ a_q &= \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \end{aligned}$$

Inverse Dynamics + PD Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) \\ a_q &= \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \end{aligned}$$

Then the closed loop system is

$$\ddot{q} = a_q = \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big)$$

Inverse Dynamics + PD Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) \\ a_q &= \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \end{aligned}$$

Then the closed loop system is

$$\ddot{q} = a_q = \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big)$$

It can be rewritten in error variables as

$$\boxed{\ddot{e} + K_d\dot{e} + K_p e = 0}\ , \quad e = q_d(t) - q$$

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$$

Inverse Dynamics + PD Control (2)

Trajectory Tracking – inverse dynamics controller

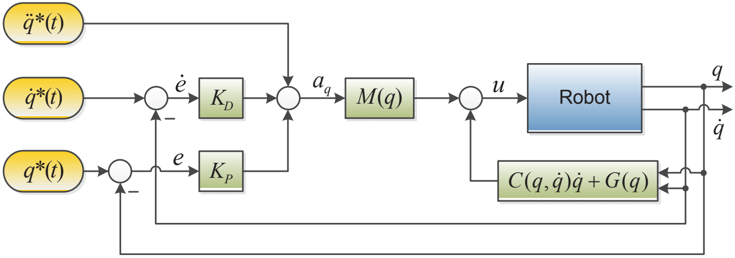

Figure: Schematic of linearizing and decoupling control law $\boldsymbol{u}$ plus PD tracking controller; here friction coefficient $\boldsymbol{B}$ is neglected for simplicity.

We could for instance choose $\boldsymbol{K_p} = \mathbf{diag}(\boldsymbol{\omega_1^2}, \ldots, \boldsymbol{\omega_n^2})$ and $\boldsymbol{K_d} = \mathbf{diag}(\mathbf{2}\boldsymbol{\omega_1}, \ldots, \mathbf{2}\boldsymbol{\omega_n})$ to shape the tracking performance.

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Task Space Representation of Inverse Dynamics

Given a mechanical system and the linearizing feedback control

$$\boldsymbol{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u}$$

$$\boldsymbol{u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)}$$

Task Space Representation of Inverse Dynamics

Given a mechanical system and the linearizing feedback control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

$$u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

Let $X = \begin{bmatrix} o_n^0(q) \\ \alpha(q) \end{bmatrix} \in \mathbb{R}^6$ be the end-effector pose using a minimal representation of $SO(3)$ so that we get the relations

$$\begin{aligned} \dot{X} &= J_a(q)\ \dot{q} \\ \ddot{X} &= \dot{J}_a(q,\dot{q})\ \dot{q} + J_a(q)\ \ddot{q} \end{aligned}$$

Task Space Representation of Inverse Dynamics

Given a mechanical system and the linearizing feedback control

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + G(q) = u$$

$$u = M(q)a_q + C(q,\dot{q})\dot{q} + B\dot{q} + G(q)$$

Let $X = \begin{bmatrix} o_n^0(q) \\ \alpha(q) \end{bmatrix} \in \mathbb{R}^6$ be the end-effector pose using a minimal representation of $SO(3)$ so that we get the relations

$$\begin{aligned} \dot{X} &= J_a(q)\ \dot{q} \\ \ddot{X} &= \dot{J}_a(q,\dot{q})\ \dot{q} + J_a(q)\ \ddot{q} \end{aligned}$$

Now modify a_q for a linearization and decoupling in task space:

$$a_X = \dot{J}_a(q,\dot{q})\ \dot{q} + J_a(q)\ a_q \quad \Leftrightarrow \quad \boxed{a_q = J_a^{-1}(q)\left[a_X - \dot{J}_a(q,\dot{q})\ \dot{q}\right]}$$

resulting in the double integrator system $\boxed{\ddot{X} = a_X}$ for which we can also design tracking controller. Use pseudoinverse if Jacobian is not square.

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Robust and Adaptive Motion Control

Given a mechanical system

$$\boldsymbol{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u}$$

and the desired trajectory $\boldsymbol{q_d = q_d(t)}$, the controller

$$\begin{aligned} \boldsymbol{u} &= \boxed{\boldsymbol{M(q)v + C(q,\dot{q})\dot{q} + G(q)}} \\ \boldsymbol{v} &= \boldsymbol{\ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))} \end{aligned}$$

cannot be safely implemented if the model parameters are uncertain!

Robust and Adaptive Motion Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

and the desired trajectory $q_d = q_d(t)$, the controller

$$\begin{aligned} u &= \boxed{M(q)v + C(q,\dot{q})\dot{q} + G(q)} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

cannot be safely implemented if the model parameters are uncertain!

The approximation might be used

$$\begin{aligned} u &= \boxed{[M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

Robust and Adaptive Motion Control (2)

Parameter uncertainties and unmodelled effects of the robot dynamics give rise for two control approaches:

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$
$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

- Robust Control:
  Design K_p, K_d and $q_d(t)$ such that the error signal

$$e(t) = q(t) - q_d(t) \approx 0 \qquad \forall\, \{\triangle M, \triangle C, \triangle G\} \in \mathcal{W}$$

Robust and Adaptive Motion Control (2)

Parameter uncertainties and unmodelled effects of the robot dynamics give rise for two control approaches:

$$u = \boxed{[M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]}$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

- Robust Control:
  Design K_p, K_d and $q_d(t)$ such that the error signal

  $$e(t) = q(t) - q_d(t) \approx 0 \qquad \forall\, \{\triangle M, \triangle C, \triangle G\} \in \mathcal{W}$$

- Adaptive Control: Improve estimates for

  $$M(q), \quad C(q,\dot{q}), \quad G(q)$$

  in the course of regulating the system.

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Robust Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

Robust Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

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$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$

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$$\begin{aligned} M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q) &= \\ &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G] \end{aligned}$$

Robust Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

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Given a trajectory $q = q_d(t)$, consider the closed loop system

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$$\ddot{q} = v + \eta(q, \dot{q}, v)$$

or

$$(\ddot{q} - \ddot{q}_d(t)) + K_d\,(\dot{q} - \dot{q}_d(t)) + K_p\,(q - q_d(t)) = w + \eta(q, \dot{q}, v)$$

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or

$$\frac{d}{dt}e = \begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix} e + \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} \left[w + \eta(t, e, w)\right], \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

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Robust Feedback Linearization (3)

To continue with design of $\boldsymbol{w}$ for the system

$$\frac{d}{dt}\boldsymbol{e} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{B}\left[\boldsymbol{w} + \boldsymbol{\eta}(t, \boldsymbol{e}, \boldsymbol{w})\right]$$

we need to impose some assumptions on $\boldsymbol{\eta}(\cdot)$, namely

$$\|\boldsymbol{\eta}(t, \boldsymbol{e}, \boldsymbol{w})\| \leq \alpha \|\boldsymbol{w}\| + \gamma_1 \|\boldsymbol{e}\| + \gamma_2 \|\boldsymbol{e}\|^2 + \gamma_3(t) \ , \quad \alpha < 1$$

The condition $\alpha = \left\|\boldsymbol{M}^{-1}(q)\triangle\boldsymbol{M} - \boldsymbol{I}\right\| < 1$ determines how close our estimate $\boldsymbol{M} + \triangle\boldsymbol{M}$ should be to the true inertia matrix $\boldsymbol{M}$.

Robust Feedback Linearization (3)

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \leq \alpha\|w\| + \gamma_1\|e\| + \gamma_2\|e\|^2 + \gamma_3(t)\ , \quad \alpha < 1$$

Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

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Consider a Lyapunov function candidate as $V = e^T P e$, then

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P\frac{d}{dt}e \\ &= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(t, e, w)\right]\end{aligned}$$

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&= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(t, e, w)\right] \\
&\le 0 \quad \leftarrow \quad \text{How to achieve this by choosing } w\text{?}
\end{aligned}$$

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$$\frac{d}{dt}e = Ae + B\left[w + \eta(t, e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(t, e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t)\ , \quad \alpha < 1$$

Let us look at the second term

$$\frac{d}{dt}V \;=\; -e^T Q e + 2\underbrace{e^T P B}_{z^T}\left[w + \eta(t, e, w)\right]$$

when w has the form

$$w = -\rho(t, e)\frac{z}{\sqrt{z^T z}}\,, \qquad z = B^T P e, \qquad \rho(t, e) \text{ is a function to choose}$$

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$$z^T\left(-\rho\frac{z}{\sqrt{z^T z}} + \eta\right) \le -\rho\|z\| + |z^T\eta|$$

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$$z^T\left(-\rho\frac{z}{\sqrt{z^Tz}} + \eta\right) \leq -\rho\|z\| + \|z\|\;\|\eta\| = \|z\|\left(-\rho + \|\eta\|\right)$$

Robust Feedback Linearization (4)

To sum up, we search for a scalar function $\rho(t, e)$ such that

$$\boxed{(-\rho + \|\eta\|) \leq 0} \;\Leftrightarrow\; \boxed{\|\eta\| \leq \rho}$$

and

$$\|\eta(t, e, w)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

with

$$w = -\rho(t, e) \frac{z}{\sqrt{z^T z}}$$

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$$w = -\rho(t, e)\frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha\left\|\rho(t, e)\frac{z}{\sqrt{z^T z}}\right\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t) \le \rho(t, e)$$

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These two inequalities imply the next one

$$\alpha \rho(t, e) + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \leq \rho(t, e)$$

$$\boxed{\rho(t, e) \geq \frac{1}{1 - \alpha} \left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t) \right]}$$

Robust Feedback Linearization – Final Controller

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [G(q) + \triangle G]$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w$$

$$w = \begin{cases} -\rho(t,e)\frac{z}{\sqrt{z^T z}}\ , & \text{if } z = B^T P e \neq 0 \\ 0\ , & \text{if } z = B^T P e = 0 \end{cases}$$

where $\rho(t,e)$ is any function that satisfies the inequality

$$\rho(t,e) \geq \frac{1}{1-\alpha}\left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t)\right]$$

with constants α, γ_1,γ_2 and $\gamma_3(t)$ obtained from the inequality

$$\|\eta(\cdot)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

and

$$\eta(q,\dot{q},v) = M^{-1}[\triangle M v + \triangle C \dot{q} + \triangle G]\,, \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

Chattering Phenomenon of Robust Controller

Suppose the control variable w is scalar such that

$$w = \begin{cases} -\rho(e)\frac{z}{\sqrt{z^T z}}\ , & \text{if } z = B^T P e \neq 0 \\ 0\ , & \text{if } z = B^T P e = 0 \end{cases}$$

becomes

$$w = \begin{cases} -\rho(e) \cdot \text{sign}(z)\ , & \text{if } z = B^T P e \neq 0 \\ 0\ , & \text{if } z = B^T P e = 0 \end{cases}$$

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This means that

whenever $z(t)$ is changing its sign

$\Downarrow$

the control signal switches rapidly with non-zero increment!

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One may implement a continuous approximation of the discontinuous control as

$$w = \begin{cases} -\rho(e) \cdot \text{sign}(z) \,, & \text{if } z = B^T P e \,, \quad |z| > \delta \\ -\frac{1}{\delta}\rho(e) \cdot z \,, & \text{if } z = B^T P e \,, \quad |z| \leq \delta \end{cases}$$

Re-Design of Robust Controller (1-DOF case)

The closed loop system can be rewritten as

$$\frac{d}{dt}e = Ae + B\left[w(e) + \eta(e, w(e))\right], \quad w(\cdot) \in \mathbb{R}^1, \quad \eta(\cdot) \in \mathbb{R}^1$$

where we have assumed that

$$\|\eta(e, w(e))\| \leq \rho(e)$$

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Here

$$A = \begin{bmatrix} 0 & 1 \\ -K_p & -K_d \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

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$$A^T P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^T P e$, then

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P \frac{d}{dt}e \\ &= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(e, w)\right] \end{aligned}$$

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$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T P e + e^T P \frac{d}{dt}e \\ &= -e^T Q e + 2e^T PB\left[w + \eta(e, w)\right] \end{aligned}$$

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where we have assumed that

$$\|\eta(e, w(e))\| \leq \rho(e)$$

If

$$w(e) = \begin{cases} -\rho(e) \cdot \text{sign}(z), & \text{if } z = B^T P e, \ |z| > \delta \\ -\frac{1}{\delta}\rho(e) \cdot z, & \text{if } z = B^T P e, \ |z| \leq \delta \end{cases}$$

then for those e that satisfy $\|B^T P e\| > \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &< 2z\left[-\rho(e) \cdot \text{sign}(z) + \eta(e, w)\right] < -2|z|\left[\rho(e) - \frac{\eta(e, w)}{\text{sign}(z)}\right] \end{aligned}$$

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then for those e that satisfy $\|B^T P e\| \leq \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &= -e^T Q e + 2z\left[-\frac{1}{\delta}\rho(e) \cdot z + \eta(e, w)\right] \end{aligned}$$

Re-Design of Robust Controller (1-DOF case)

The closed loop system can be rewritten as

$$\frac{d}{dt}e = Ae + B\left[w(e) + \eta(e, w(e))\right], \quad w(\cdot) \in \mathbb{R}^1, \quad \eta(\cdot) \in \mathbb{R}^1$$

where we have assumed that

$$\|\eta(e, w(e))\| \leq \rho(e)$$

If

$$w(e) = \begin{cases} -\rho(e) \cdot \mathrm{sign}(z), & \text{if } z = B^T P e, \ |z| > \delta \\ -\dfrac{1}{\delta}\rho(e) \cdot z, & \text{if } z = B^T P e, \ |z| \leq \delta \end{cases}$$

then for those e that satisfy $\|B^T P e\| \leq \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &= -e^T Q e + 2\Big[-\frac{1}{\delta}\rho(e) \cdot |z|^2 + \underbrace{z \cdot \eta(e, w)}_{\leq |z| \cdot |\eta|}\Big] \end{aligned}$$

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then for those e that satisfy $\|B^T Pe\| \le \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Qe + 2e^T PB\left[w + \eta(e, w)\right] \\ &\le -e^T Qe + 2\left[-\frac{1}{\delta}\rho(e) \cdot |z|^2 + \rho(e)|z|\right] \end{aligned}$$

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$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &\leq -e^T Q e + 2\rho(e) \cdot \left[-\frac{1}{\delta} \cdot |z|^2 + |z|\right] \end{aligned}$$

What is maximum?

Re-Design of Robust Controller (1-DOF case)

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Re-Design of Robust Controller (1-DOF case)

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$$\begin{aligned} \frac{d}{dt}V &= -e^T Qe + 2e^T PB\left[w + \eta(e, w)\right] \\ &\leq -e^T Qe + 2\rho(e) \cdot \frac{\delta}{4} = -e^T Qe + \rho(e) \cdot \frac{\delta}{2} \end{aligned}$$

Re-Design of Robust Controller (1-DOF case)

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then for those e that satisfy $\|B^T P e\| \leq \delta$ we have

$$\frac{d}{dt}V \quad \leq \quad -e^T Q e + \rho(e) \cdot \frac{\delta}{2} \leq 0$$

$$\text{whenever} \quad e^T Q e > \frac{\delta}{2}\rho(e)$$

Re-Design of Robust Controller (1-DOF case)

To conclude:

- To avoid chattering, the next modification is suggested

$$\boldsymbol{w(e)} = \begin{cases} -\boldsymbol{\rho(e)} \cdot \mathrm{sign}(\boldsymbol{z}), & \text{if } \boldsymbol{z = B^T P e}, \ |\boldsymbol{z}| > \boldsymbol{\delta} \\ -\dfrac{\boldsymbol{1}}{\boldsymbol{\delta}}\boldsymbol{\rho(e)} \cdot \boldsymbol{z}, & \text{if } \boldsymbol{z = B^T P e}, \ |\boldsymbol{z}| \le \boldsymbol{\delta} \end{cases}$$

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- With such modification we cannot prove asymptotic stability of the closed loop system :(

Re-Design of Robust Controller (1-DOF case)

To conclude:

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- With such modification we cannot prove asymptotic stability of the closed loop system :(
- However, we were able to show that the value of Lyapunov function will be discreasing everywhere, except some vicinity of the origin of error dynamics.
- The size of this set can be modified and depends on $\boldsymbol{\delta}$.
- Such property is called ultimate boundedness .

Example: 1-DOF Mass Without Gravity and Friction

Consider a target reference $q_d(t)$ for the linear 1-DOF system

$$(M + \triangle M)\,\ddot{q} = u\,, \quad M = 1\,, \quad \triangle M = \varepsilon$$

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The control transformations $u \to v \to w$ are as follows

$$\begin{aligned} u &= M v \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w \end{aligned}$$

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Let $K_p = K_d = 1$, then the transformed system is

$$\begin{aligned} (1+\varepsilon)\,\ddot{q} &= v \\ (\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) &= w - \varepsilon\ddot{q} \\ (\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) &= w + \eta(v) \end{aligned}$$

where

$$\eta(v) = -\frac{\varepsilon}{1+\varepsilon}\left[\ddot{q}_d(t) - (q - q_d(t)) - (\dot{q} - \dot{q}_d(t)) + w\right]$$

Example: 1-DOF Mass Without Gravity and Friction (2)

To proceed with robust design we need to

- rewrite the system

$$(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = w + \eta(\cdot)$$

  into state-space form

$$\frac{d}{dt}e = Ae + B\left[w + \eta(\cdot)\right], \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

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- solve the Lyapunov equation

$$A^T P + PA = -Q \,, \quad \text{say for} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- solve the Lyapunov equation
$$A^T P + PA = -Q\,, \quad \text{say for} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- find a range of ε so that the function
$$\eta(t, e, w) = -\frac{\varepsilon}{1+\varepsilon}\left[\ddot{q}_d(t) - e(t) - \dot{e}(t) + w\right]$$
satisfies the bound
$$\|\eta(t, e, w)\| \le \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3(t), \quad \alpha < 1$$

Example: 1-DOF Mass Without Gravity and Friction (3)

Lets try the robust controller for the reference $q_d(t) = \sin(t)$.

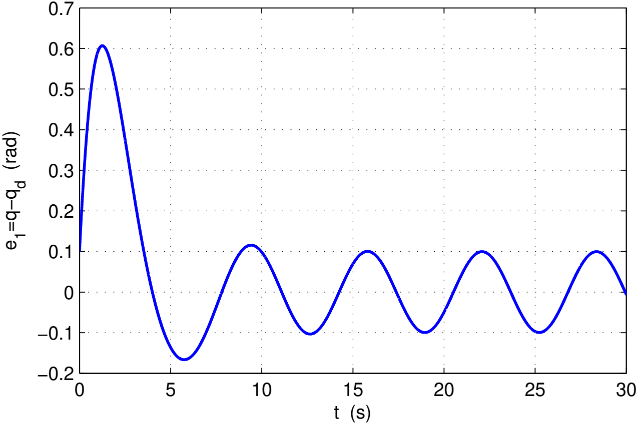

Figure: Error signal for nominal feedback linearization
$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$

Example: 1-DOF Mass Without Gravity and Friction (3)

Lets try the robust controller for the reference $q_d(t) = \sin(t)$.

Suppose that the following choice of ρ holds for our uncertainty

$$\|\eta(t, e, w)\| \leq \rho(t, e) = \frac{\varepsilon}{1+\varepsilon}\big(|\ddot{q}_d(t)| + \|e\|\big) \ , \quad \varepsilon = 0.1$$

We want to compare the two robust feedback linearizing controllers

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w$$

1. $$w = \begin{cases} -\rho(e) \cdot \mathrm{sign}(z) \ , & \text{if } z = B^T P e \neq 0 \\ 0 \ , & \text{if } z = B^T P e = 0 \end{cases}$$

2. $$w = \begin{cases} -\rho(e) \cdot \mathrm{sign}(z) \ , & \text{if } z = B^T P e \ , \quad |z| > \delta \\ -\dfrac{1}{\delta}\rho(e) \cdot z \ , & \text{if } z = B^T P e \ , \quad |z| \leq \delta \end{cases} \qquad \delta = 0.01$$

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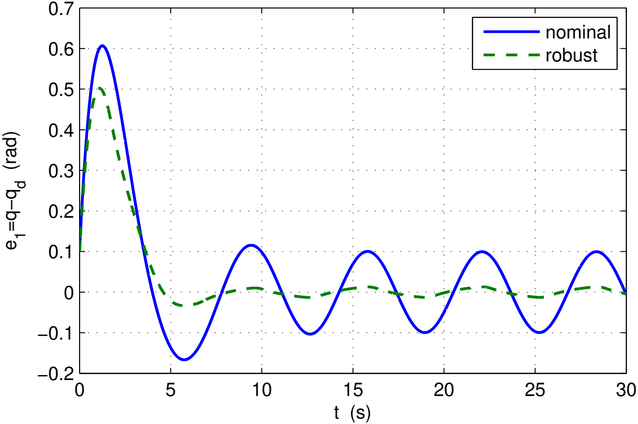

Figure: Error signal for robust feedback linearization with discontinuous $\boldsymbol{w}$.

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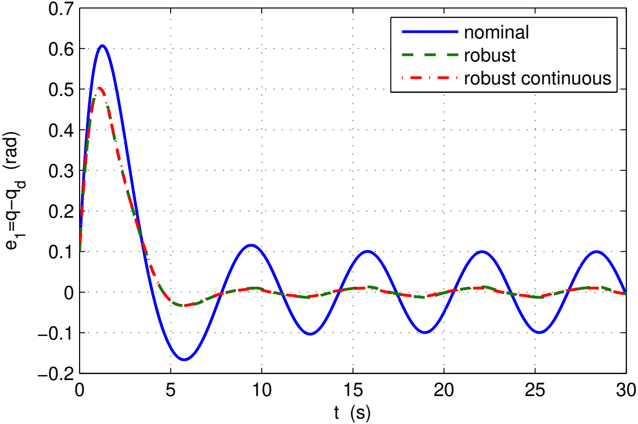

Figure: Error signal for robust feedback linearization with continuous w.

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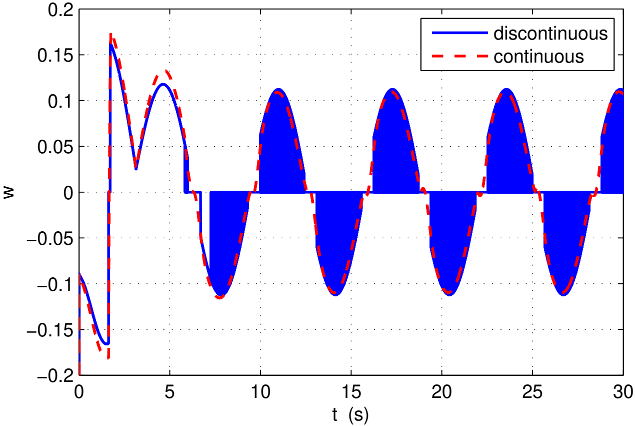

Figure: Comparison of control signal w.

Outline

1. Trajectory Tracking
   - Feedback Linearization
   - Joint Space Inverse Dynamics
   - Task Space Inverse Dynamics

2. Robust and Adaptive Motion Control
   - Robust Control Based on Feedback Linearization
   - Adaptive Control Based on Feedback Linearization

Adaptive Feedback Linearization

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

and the desired trajectory $q = q_d(t)$, introduce the controller

$$\begin{aligned} u &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w \end{aligned}$$

What is the difference with the robust design?

- In robust design, the coefficients of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{G}(\cdot)$ were fixed.
- Now, they are variables to tune and $w = 0$!

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What is the difference with the robust design?

- In robust design, the coefficients of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{G}(\cdot)$ were fixed.
- Now, they are variables to tune and $w = 0$!

Let us find the dynamical equations for updating values of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{G}(\cdot)$ provided that we measure $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$.

Adaptive Feedback Linearization (2)

Given a trajectory $q = q_d(t)$, consider the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q)$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

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$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \\ Y(q,\dot{q},\ddot{q})\theta &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \end{aligned}$$

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Given a trajectory $q = q_d(t)$, consider the closed-loop system

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$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \\ Y(q,\dot{q},\ddot{q})\theta + \hat{M}(q)\ddot{q} &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) + \hat{M}(q)\ddot{q} \end{aligned}$$

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Given a trajectory $q = q_d(t)$, consider the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q)$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

$$\begin{aligned}
M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \\
Y(q,\dot{q},\ddot{q})\theta + \hat{M}(q)\ddot{q} &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) + \hat{M}(q)\ddot{q} \\
\ddot{q} &= v + \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta} - \theta\right]
\end{aligned}$$

where

- θ is the vector of true parameters of the model
- $\hat{\theta}$ is the vector of estimates
- $Y(\cdot)$ is the regressor evaluated for $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$.

Adaptive Feedback Linearization (2)

Given a trajectory $q = q_d(t)$, consider the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$
$$u = \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q)$$
$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \\ Y(q,\dot{q},\ddot{q})\theta + \hat{M}(q)\ddot{q} &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) + \hat{M}(q)\ddot{q} \\ \ddot{q} &= v + \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta} - \theta\right] \end{aligned}$$

The closed-loop dynamics is then given as

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta} - \theta\right]$$

Adaptive Feedback Linearization (3)

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q}) \left[\hat{\theta} - \theta\right]$$

has variables we can change! How to do this update?

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has variables we can change! How to do this update?

Let us rewrite the system into state space form

$$\frac{d}{dt} e = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{= A} e + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{= B} \Phi \left[\hat{\theta} - \theta\right], \quad e = \begin{bmatrix} q - q_d(t) \\ \dot{q} - \dot{q}_d(t) \end{bmatrix}$$

Adaptive Feedback Linearization (3)

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \boxed{\hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q})} \left[\hat{\theta} - \theta\right]$$

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$$\frac{d}{dt} e = Ae + B\Phi \left[\hat{\theta} - \theta\right]$$

Solve the Lyapunov equation

$$A^T P + PA = -Q < 0, \quad P > 0$$

and consider the Lyapunov function candidate

$$V = e^T P e + \frac{1}{2} \left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right], \quad \Gamma > 0$$

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has variables we can change! How to do this update?

Let us rewrite the system into state space form

$$\frac{d}{dt} e = Ae + B\Phi \left[\hat{\theta} - \theta\right]$$

Then

$$\begin{aligned}
\frac{d}{dt} V &= \frac{d}{dt}\left\{e^T P e\right\} + \frac{d}{dt}\left\{\frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right]\right\} \\
&= \left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right]^T Pe + e^T P \left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right] + \\
&\qquad + \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - \color{red}{0}\right]^T \Gamma \left[\hat{\theta} - \theta\right] + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\frac{d}{dt}\hat{\theta} - \color{red}{0}\right]
\end{aligned}$$

Adaptive Feedback Linearization (3)

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \boxed{\hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q})} \left[\hat{\theta} - \theta\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into state space form

$$\frac{d}{dt} e = Ae + B\Phi \left[\hat{\theta} - \theta\right]$$

Then

$$\begin{aligned} \frac{d}{dt} V &= \left[Ae + B\Phi \left[\hat{\theta} - \theta\right]\right]^T Pe + e^T P \left[Ae + B\Phi \left[\hat{\theta} - \theta\right]\right] + \\ &\quad + \frac{1}{2} \left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]^T \Gamma \left[\hat{\theta} - \theta\right] + \frac{1}{2} \left[\hat{\theta} - \theta\right]^T \Gamma \left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right] \\ &= e^T \left[A^T P + PA\right] e + \left[\hat{\theta} - \theta\right]^T \left\{\Phi^T B^T Pe + \Gamma \frac{d}{dt}\hat{\theta}\right\} \end{aligned}$$

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \boxed{\hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q})} \left[\theta - \hat{\theta}\right]$$

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Adaptive Feedback Linearization (4)

With the proposed update law for $\hat{\boldsymbol{\theta}}$ the closed-loop system is

$$\frac{d}{dt}\boldsymbol{e} = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{B}\boldsymbol{\Phi}\left[\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right], \quad \frac{d}{dt}\hat{\boldsymbol{\theta}} = -\boldsymbol{\Gamma}^{-1}\boldsymbol{\Phi}^{T}\boldsymbol{B}^{T}\boldsymbol{P}\boldsymbol{e}$$

Adaptive Feedback Linearization (4)

With the proposed update law for $\hat{\theta}$ the closed-loop system is

$$\frac{d}{dt}e = Ae + B\Phi\left[\theta - \hat{\theta}\right], \quad \frac{d}{dt}\hat{\theta} = -\Gamma^{-1}\Phi^T B^T P e$$

The inequality

$$\frac{d}{dt}V = \frac{d}{dt}\left[e^T P e + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right]\right] = -e^T Q e$$

implies that (thanks to Barbalat lemma)

$e(t) \to 0$ as $t \to +\infty$ and $\left[\hat{\theta}(t) - \theta\right]$ is bounded

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$e(t) \to 0$ as $t \to +\infty$ and $\left[\hat{\theta}(t) - \theta\right]$ is bounded

Recall that

- we have to measure $\ddot{q}$ for computing the regressor;
- the matrix $\hat{M}(q)$ at each time moment should be invertible.

TTK4195: Modeling and Control of Robots

Lecture 18: Multivariable Control of Robot Manipulators

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Spring 2021

Outline

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1. Passivity-Based Motion Control
   - Robust Control
   - Adaptive Control

Passivity-Based Motion Control

Given a trajectory $q = q_d(t)$, consider the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = M(q)a + C(q,\dot{q})b + G(q) - Kr\ , \quad K = \mathrm{diag}(k_1, \ldots, k_n)$$

where variables a, b and r must be chosen.

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Substituting the control law, we obtain

$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr = 0$$

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Let us impose some relations between the signals

$$\frac{d}{dt}r = \ddot{q} - a\ , \qquad r = \dot{q} - b$$

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Let us impose some relations between the signals and the reference

$$\frac{d}{dt}r = \ddot{q} - a \ , \qquad r = \dot{q} - b \ , \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

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Then the closed loop dynamics is

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = 0$$

Passivity-Based Motion Control (2)

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda (q - q_d(t))$$

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\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda (q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T \frac{d}{dt}[M(q)]\, r + \\
&\qquad\qquad +2(q - q_d(t))^T K\Lambda \frac{d}{dt}(q - q_d(t))
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&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t)) \\
&= r^T\,\boxed{\left[-C(q,\dot{q})r - Kr\right]} + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right] r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t))
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&= -r^T K r + r^T\left[\frac{1}{2}\frac{d}{dt}M(q) - C(q,\dot{q})\right] r + \\
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where

$$r = (\dot{q} - \dot{q}_d(t)) + \Lambda (q - q_d(t))$$

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&= -\left(\frac{d}{dt}\tilde{q} + \Lambda\tilde{q}\right)^T K \left(\frac{d}{dt}\tilde{q} + \Lambda\tilde{q}\right) + 2\tilde{q}^T K\Lambda \frac{d}{dt}\tilde{q}
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&= -\left[\frac{d}{dt}\tilde{q}\right]^T K \frac{d}{dt}\tilde{q} - \tilde{q}^T \Lambda^T K \Lambda \tilde{q} < 0
\end{aligned}$$

Outline

1. Passivity-Based Motion Control
    - Robust Control
    - Adaptive Control

Passivity-Based Robust Control

Given a trajectory $q = q_d(t)$, consider the closed-loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

$$u = \hat{M}(q)a + \hat{C}(q,\dot{q})b + \hat{G}(q) - Kr \;, \quad K = \mathrm{diag}(k_1, \ldots, k_n)$$

with the variables a, b, r defined by

$$\frac{d}{dt}r = \ddot{q} - a \;, \qquad r = \dot{q} - b \;, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

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The dynamics can be rewritten in regressor form

$$M(q)\left[\ddot{q} \qquad\right] + C(q,\dot{q})\left[\dot{q} \qquad\right] + Kr =$$

$$= \left[\hat{M}(q) \qquad\qquad\right]a + \left[\hat{C}(q,\dot{q}) \qquad\qquad\right]b + \left[\hat{G}(q) \qquad\qquad\right]$$

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Given a trajectory $q = q_d(t)$, consider the closed-loop system

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$$= \left[\hat{M}(q) - M(q)\right]a + \left[\hat{C}(q,\dot{q}) - C(q,\dot{q})\right]b + \left[\hat{G}(q) - G(q)\right]$$

$$= Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

Passivity-Based Robust Control (2)

Lets introduce an additional control variable to the closed-loop dynamics

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{r}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{r} + \boldsymbol{K}\boldsymbol{r} &= \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{a}, \boldsymbol{b}) \left[\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right] \\ \hat{\boldsymbol{\theta}} &= \boldsymbol{\theta}_0 + \boldsymbol{w} \end{aligned}$$

with nominal parameter vector $\boldsymbol{\theta}_0$ so that the uncertainty is given by

$$\triangle\boldsymbol{\theta} = \boldsymbol{\theta}_0 - \boldsymbol{\theta} = \mathbf{const.}$$

Passivity-Based Robust Control (2)

Lets introduce an additional control variable to the closed-loop dynamics

$$\boldsymbol{M(q)\dot{r} + C(q,\dot{q})r + Kr \quad = \quad Y(q,\dot{q},a,b)\left[w + \triangle\theta\right]}$$

with nominal parameter vector $\boldsymbol{\theta_0}$ and uncertainty
$\boldsymbol{\triangle\theta = \theta_0 - \theta = \text{const.}}$

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$$M(q)\dot{r} + C(q,\dot{q})r + Kr \quad = \quad Y(q,\dot{q},a,b)\left[w + \triangle\theta\right]$$

with nominal parameter vector θ_0 and uncertainty
$\triangle\theta = \theta_0 - \theta = \mathrm{const.}$

With bounded uncertainty

$$\|\triangle\theta\| = \|\theta - \theta_0\| \leq \rho\ , \quad \rho \geq 0$$

the tracking error is uniformly ultimate bounded by the control law

$$w = \begin{cases} -\rho\dfrac{z}{\sqrt{z^T z}}\ , & \text{if } z = Y(q,\dot{q},a,b)^T\ r\ , \quad |z| > \delta \\ -\dfrac{\rho}{\delta}z\ , & \text{if } z = Y(q,\dot{q},a,b)^T\ r\ , \quad |z| \leq \delta \end{cases}$$

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1. Passivity-Based Motion Control
   - Robust Control
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Passivity-Based Adaptive Control

Given a trajectory $q = q_d(t)$, consider the closed-loop system

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$$u = \hat{M}(q)a + \hat{C}(q,\dot{q})b + \hat{G}(q) - Kr\ , \quad K = \mathrm{diag}(k_1, \ldots, k_n)$$

with the variables a, b, r defined by

$$\frac{d}{dt}r = \ddot{q} - a\ , \qquad r = \dot{q} - b\ , \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

Passivity-Based Adaptive Control

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The dynamics can be rewritten in regressor form

$$M(q)\left[\ddot{q} \qquad \right] + C(q,\dot{q})\left[\dot{q} \qquad \right] + Kr =$$

$$= \left[\hat{M}(q) \qquad \qquad \right] a + \left[\hat{C}(q,\dot{q}) \qquad \qquad \right] b + \left[\hat{G}(q) \qquad \qquad \right]$$

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$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr =$$

$$= \left[\hat{M}(q) - M(q)\right] a + \left[\hat{C}(q,\dot{q}) - C(q,\dot{q})\right] b + \left[\hat{G}(q) - G(q)\right]$$

$$= Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

Passivity-Based Adaptive Control (2)

To find the update law for $\hat{\theta}$-variable of the system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = Y(q,\dot{q},a,b)\left[\hat{\theta}-\theta\right]$$

we will use the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t)) + \frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma\left[\hat{\theta}-\theta\right]$$

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Its time-derivative along a solution of the system is

$$\begin{aligned}\frac{d}{dt}V \;\; &= \;\; -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda^T K \Lambda \tilde{q} + \\ &\qquad + \left[\hat{\theta}-\theta\right]^T \left\{\Gamma \frac{d}{dt}\hat{\theta} + Y(q,\dot{q},a,b)^T r\right\}\end{aligned}$$

Passivity-Based Adaptive Control (2)

To find the update law for $\hat{\theta}$-variable of the system

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Its time-derivative along a solution of the system is

$$\begin{aligned}\tfrac{d}{dt}V \;&=\; -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda^T K \Lambda \tilde{q} + \\ &\qquad + \left[\hat{\theta} - \theta\right]^T \underbrace{\left\{\Gamma \tfrac{d}{dt}\hat{\theta} + Y(q,\dot{q},a,b)^T r\right\}}_{!=0} \le 0\end{aligned}$$

$$\tfrac{d}{dt}\hat{\theta} = -\Gamma^{-1} Y(q,\dot{q},a,b)^T r$$

Modeling and Control of Robots

Lecture 19: Underactuated Mechanical Systems (Part I)

Anton Shiriaev
March 15, 2021

Learning outcomes: Examples of successful feedback linearization for underactuated systems; Concepts of moving Ponicare sections, transverse coordinates and transverse linearization for a solution of a nonlinear system; Andronov-Vitt theorem.

Outline

1. FL for Inertia Wheel Pendulum

Feedback Linearization (FL) for Mechanical Systems

FL for a controlled mechanical system with n-degrees of freedom

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q,\dot{q})u, \quad q \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

is the search of

- a change of coordinates $x := \left[q; \dot{q}\right] \mapsto z$
- a feedback transformation $u = \alpha(x) + \beta(x)v$

such that the dynamics become linear and stabilizable

$$\dot{z} = A_z z + B_z v, \qquad \left(A_z, B_z\right) \text{ stabilizable}$$

It is indeed possible provided that $m = n$ and $\operatorname{rank} B(q,\dot{q}) = m$!!!

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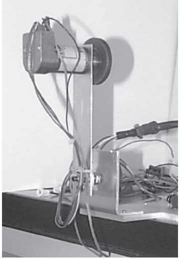
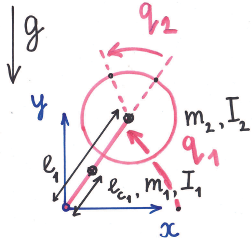
What if the system has fewer control inputs than DoF ($m < n$)?

FL for Inertia Wheel Pendulum

Dynamics of Inertia Wheel Pendulum





Physical parameters of the system:

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\begin{aligned}
\ell_1 &= 0.125\ [m]\\
\ell_{c_1} &= 0.063\ [m]\\
m_1 &= 0.02\ [kg]\\
m_2 &= 0.3\ [kg]\\
I_1 &= 4.7\cdot 10^{-5}\ [kg\cdot m^2]\\
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\end{aligned}
$$

The kinetic energy of the system is

$$
\begin{aligned}
\mathcal{K} &= \left[\tfrac{1}{2}\,m_1 v_{c_1}^2 + \tfrac{1}{2}\,J_1\omega_1^2\right] + \left[\tfrac{1}{2}\,m_2 v_{c_2}^2 + \tfrac{1}{2}\,J_2\omega_2^2\right]\\
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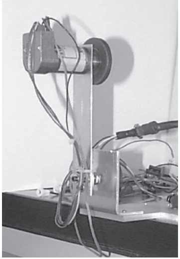

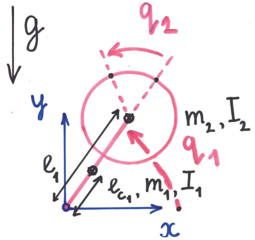

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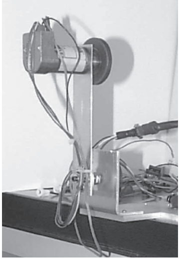
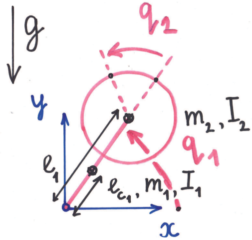
Dynamics of Inertia Wheel Pendulum





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$$\begin{cases} x_{c_1} = \ell_{c_1} \cdot \cos\left(q_1\right) \\ y_{c_1} = \ell_{c_1} \cdot \sin\left(q_1\right) \end{cases}$$

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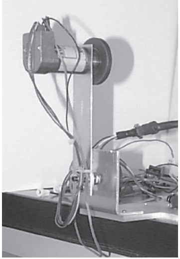

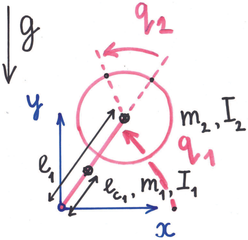

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$$
\left\{\begin{aligned} x_{c_1} &= \ell_{c_1}\cdot\cos\left(q_1\right)\\ y_{c_1} &= \ell_{c_1}\cdot\sin\left(q_1\right)\end{aligned}\right. \quad\Rightarrow\quad \left\{\begin{aligned} \dot{x}_{c_1} &= -\ell_{c_1}\cdot\sin\left(q_1\right)\cdot\dot{q}_1\\ \dot{y}_{c_1} &= \ell_{c_1}\cdot\cos\left(q_1\right)\cdot\dot{q}_1\end{aligned}\right.
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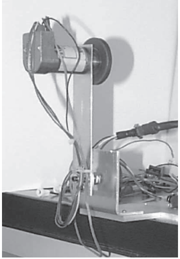

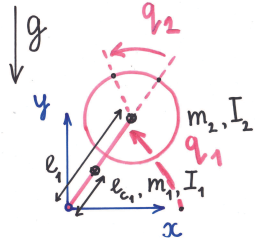

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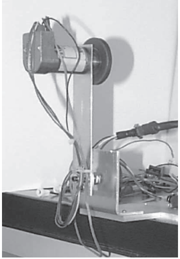

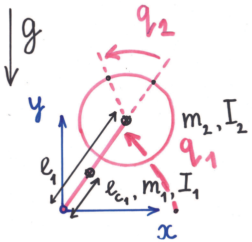

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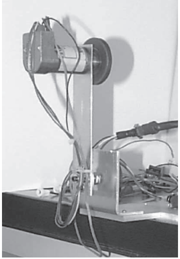

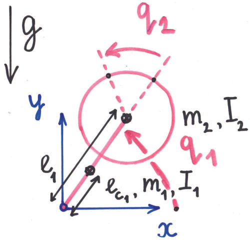

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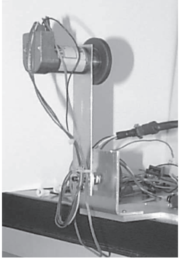

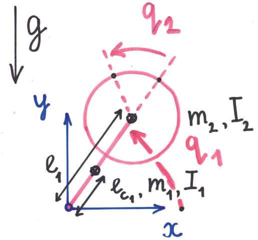

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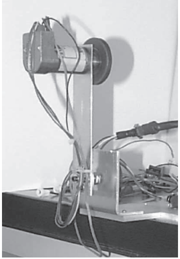

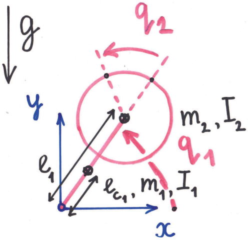

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&= \tfrac{1}{2} \cdot \left[m_1 \ell_{c_1}^2 + J_1 + m_2 \ell_{c_2}^2 + J_2\right] \cdot \dot{q}_1^2 + J_2 \cdot \dot{q}_1 \cdot \dot{q}_2 + \tfrac{1}{2} \cdot J_2 \cdot \dot{q}_2^2
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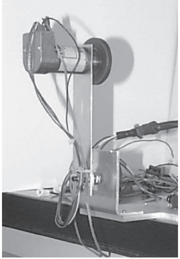
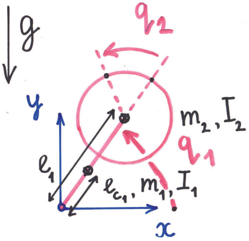
Dynamics of Inertia Wheel Pendulum





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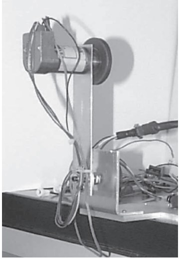

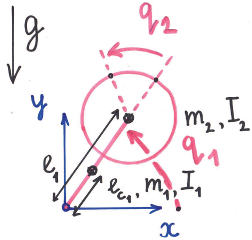

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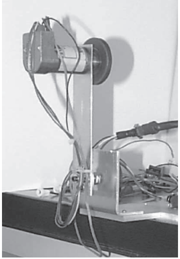
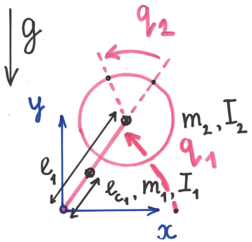
The potential energy of the system is

$$\mathscr{P} \quad = \quad m_1 \cdot g \cdot y_{c_1} + m_2 \cdot g \cdot y_{c_2}$$

Dynamics of Inertia Wheel Pendulum





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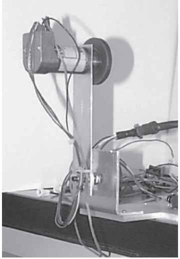
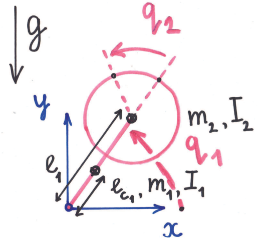
I_1 &= 4.7 \cdot 10^{-5}\ [kg \cdot m^2]\\
I_2 &= 3.2 \cdot 10^{-5}\ [kg \cdot m^2]
\end{aligned}
$$

The potential energy of the system is

$$
\begin{aligned}
\mathscr{P} &= m_1 \cdot g \cdot y_{c_1} + m_2 \cdot g \cdot y_{c_2}\\
&= m_1 \cdot g \cdot \ell_{c_1} \cdot \sin q_1 + m_2 \cdot g \cdot \ell_1 \cdot \sin q_1
\end{aligned}
$$

Dynamics of Inertia Wheel Pendulum





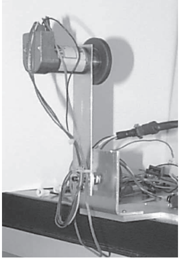
Physical parameters of the system:

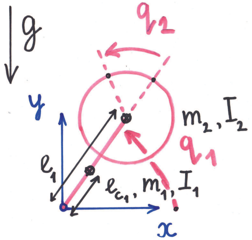
ℓ_1	$=$	$0.125\ [m]$
ℓ_{c_1}	$=$	$0.063\ [m]$
m_1	$=$	$0.02\ [kg]$
m_2	$=$	$0.3\ [kg]$
I_1	$=$	$4.7 \cdot 10^{-5}\ [kg \cdot m^2]$
I_2	$=$	$3.2 \cdot 10^{-5}\ [kg \cdot m^2]$

The potential energy of the system is

$$\begin{aligned}\mathscr{P} &= m_1 \cdot g \cdot y_{c_1} + m_2 \cdot g \cdot y_{c_2} \\ &= m_1 \cdot g \cdot \ell_{c_1} \cdot \sin q_1 + m_2 \cdot g \cdot \ell_1 \cdot \sin q_1 \\ &= \gamma \cdot \sin q_1\end{aligned}$$

Dynamics of Inertia Wheel Pendulum





Physical parameters of the system:

$$\begin{aligned} \ell_1 &= 0.125\ [m] \\ \ell_{c_1} &= 0.063\ [m] \\ m_1 &= 0.02\ [kg] \\ m_2 &= 0.3\ [kg] \\ I_1 &= 4.7 \cdot 10^{-5}\ [kg \cdot m^2] \\ I_2 &= 3.2 \cdot 10^{-5}\ [kg \cdot m^2] \end{aligned}$$

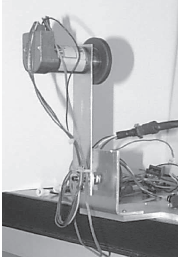
The kinetic and potential energies of the system

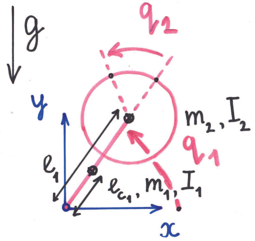
$$\mathcal{K} = \tfrac{1}{2} \cdot d_1 \cdot \dot{q}_1^2 + d_2 \cdot \dot{q}_1 \cdot \dot{q}_2 + \tfrac{1}{2} \cdot d_2 \cdot \dot{q}_2^2, \quad \mathcal{P} = \gamma \cdot \sin q_1$$

the dynamics of the IWP are

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}(\mathcal{K} - \mathcal{P})\right] - \frac{\partial}{\partial q_1}(\mathcal{K} - \mathcal{P}) &= 0 \\ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_2}(\mathcal{K} - \mathcal{P})\right] - \frac{\partial}{\partial q_2}(\mathcal{K} - \mathcal{P}) &= u \end{aligned}$$

Dynamics of Inertia Wheel Pendulum





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The kinetic and potential energies of the system

$$\mathcal{K} = \frac{1}{2} \cdot d_1 \cdot \dot{q}_1^2 + d_2 \cdot \dot{q}_1 \cdot \dot{q}_2 + \frac{1}{2} \cdot d_2 \cdot \dot{q}_2^2, \quad \mathcal{P} = \gamma \cdot \sin q_1$$

the dynamics of the IWP are

$$\begin{cases} d_1 \cdot \ddot{q}_1 + d_2 \cdot \ddot{q}_2 + \gamma \cdot \cos q_1 = 0 \\ d_2 \cdot \ddot{q}_1 + d_2 \cdot \ddot{q}_2 = u \end{cases}$$

Feedback Linearization for Inertia Wheel Pendulum

Is the IWP dynamics of the IWP feedback linearizable?

$$
\begin{cases}
d_1 \ddot{q}_1 + d_2 \ddot{q}_2 + \gamma \cos q_1 & = & 0 \\
d_2 \ddot{q}_1 + d_2 \ddot{q}_2 & = & u
\end{cases}
$$

Given a, $b \in \mathbb{R}^1$, consider the scalar function $z_1 := a \cdot q_1 + b \cdot q_2$

Its time derivative is $z_2 := \frac{d}{dt} z_1 = a \cdot \dot{q}_1 + b \cdot \dot{q}_2$

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Feedback Linearization for Inertia Wheel Pendulum

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$$z_3 = -\frac{a\cdot\gamma}{d_1-d_2}\cos q_1 - \frac{a\cdot u}{d_1-d_2} + \frac{b\cdot\gamma}{d_1-d_2}\cos q_1 + \frac{b\cdot d_1\cdot u}{(d_1-d_2)d_2}$$

Feedback Linearization for Inertia Wheel Pendulum

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Feedback Linearization for Inertia Wheel Pendulum

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Feedback Linearization for Inertia Wheel Pendulum

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$$\begin{array}{rcl} z_3 & = & -\dfrac{a\cdot\gamma}{d_1 - d_2}\cos q_1 - \dfrac{a\cdot u}{d_1 - d_2} + \dfrac{b\cdot\gamma}{d_1 - d_2}\cos q_1 + \dfrac{b\cdot d_1\cdot u}{(d_1 - d_2)\, d_2} \\ & = & (\cdots) + \left[\dfrac{b\cdot d_1}{(d_1 - d_2)\, d_2} - \dfrac{a}{d_1 - d_2}\right]\cdot u \\ & = & L\cdot\cos q_1 + \left[\, 0 \,\right]\cdot u \qquad \text{if } b = d_2,\ a = d_1 \end{array}$$

Feedback Linearization for Inertia Wheel Pendulum

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Its 3rd time derivative is $z_4 := \frac{d}{dt}z_3 = -L\cdot\sin(q_1)\cdot\dot{q}_1$

Its 4th time derivative is $\frac{d}{dt}z_4 = \varepsilon(\cdot) + \delta(\cdot)\cdot u$

Feedback Linearization for Inertia Wheel Pendulum

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Feedback Linearization for Inertia Wheel Pendulum

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Feedback Linearization for Inertia Wheel Pendulum

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If $a = d_1$, $b = d_2$ and $z_1 := a\cdot q_1 + b\cdot q_2$, then the dynamics is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v$$

Is this linear system stabilizable? controllable?

Feedback Linearization for Inertia Wheel Pendulum

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Modeling and Control of Robots

Lecture 19: Underactuated Mechanical Systems (Part II)

Anton Shiriaev
March 15, 2021

Learning outcomes: Examples of successful feedback linearization for under-actuated systems; Concepts of moving Ponicare sections, transverse coordinates and transverse linearization for a solution of a nonlinear system; Andronov-Vitt theorem.

Outline

1. Transverse dynamics and transverse coordinates
   - Moving Poincare sections
   - Andronov-Vitt theorem
   - Challenges in orbital feedback stabilization
   - Generic choice of transverse coordinates

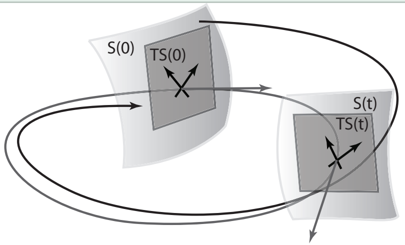
Transverse dynamics

Dynamics in a vicinity of a cycle

Given a T-periodic solution $x^*(\cdot)$ of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties introduce a family of transverse sections $\{S(\tau)\}$, $\tau \in [0, T]$, which union can recreate a tubing vicinity of the cycle.



Dynamics in a vicinity of a cycle

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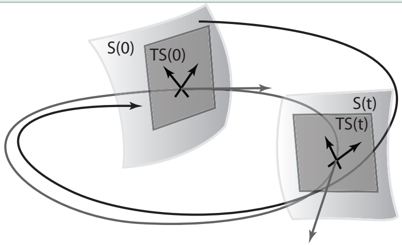
$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties introduce a family of transverse sections $\{S(\tau)\}$, $\tau \in [0, T]$, which union can recreate a tubing vicinity of the cycle.



Defining hypersurfaces $\{S(\tau)\}_{\tau\in[0,T]}$ implies a change of coordinates:

$$\mathbb{R}^{2n} \ni x(\tau) \mapsto \left[\theta(\tau) \in \mathbb{R}^1;\, x_\perp \in S(\tau)\right], \qquad \tau \in [0, T]$$

such that in new coordinates the periodic solution $x^*(\cdot)$ is of the form

$$\theta = \theta^*(t), \quad x_\perp^*(t) \equiv \mathbf{0}, \qquad \forall\, t$$

Asymptotic orbital stability of T-periodic solution $x^*(\cdot)$ means that $x_\perp(t) \to 0$ as $t \to \infty$

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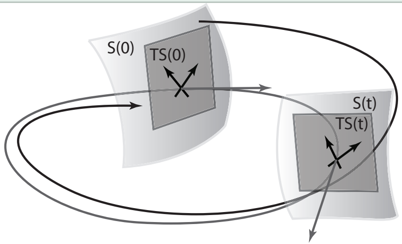

The linearization of the full dynamics of the system in a vicinity of $x^*(\cdot)$

$$\dot{z} = \left[\frac{\partial}{\partial x} f(x)\right]\Big|_{x=x^*(t)} z = A(t)z, \quad z \in \mathbb{R}^{2n}$$

- cannot be asymptotically stable since $\theta(t) \not\to 0$ as $t \to +\infty$

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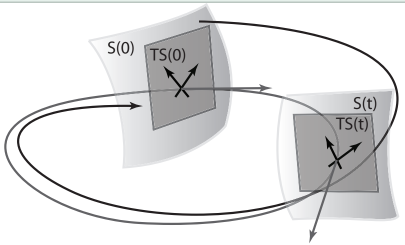

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- has a linear invariant subspace of co-dimension 1, which vectors approximate time evolution of transverse coordinates $x_\perp(\cdot)$.

Andronov-Vitt theorem (1930)

Given a T-periodic solution

$$x^*(t) = x^*(t+T) \quad \forall\, t$$

of the system

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$$\tfrac{d}{dt}\Phi(t) = A(t)\Phi(t), \qquad \Phi(0) = I_{2n}$$

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One of eigenvalues of the monodromy matrix $\Phi(T)$ equals to 1.

If the amplitudes of $(2n-1)$ eigenvalues of the matrix $\Phi(T)$ are less than 1, then the solution $x^*(\cdot)$ is asymptotically orbitally stable.

Challenges in orbital feedback stabilization

Given a T-periodic solution $x^*(\cdot)$ of the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^{2n}, \ u \in \mathbb{R}^m,$$

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Suppose it exists, then such orbitally stabilizing feedback controller

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will satisfy the interpolation condition

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For a smooth $U(\cdot)$, it can be written as (check Hadamard lemma)

$$\boxed{u = U(x) = K(x)x_\perp(x), \qquad x_\perp \in \mathbb{R}^{2n-1}}$$

where $x_\perp(\cdot)$ are transverse coordinates defined for the motion $x^*(\cdot)$

Challenges in orbital feedback stabilization

The system dynamics in a vicinity of the solution $x^*(\cdot)$ can be written in new coordinates

$$\tilde{x} = [\theta;\, x_\perp], \quad \theta \in \mathbb{R}^1,\ x_\perp \in \mathbb{R}^{2n-1}$$

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$$\tfrac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u.$$

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With a feedback controller candidate

$$u = K(\tilde{x})x_\perp$$

the closed loop system becomes

$$\tfrac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})K(\tilde{x})x_\perp.$$

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$$\dot{z} = \left[\frac{\partial}{\partial \tilde{x}}\tilde{f}(\tilde{x})\right]\Bigg|_{\tilde{x}=\tilde{x}^*(t)} z + \left[\frac{\partial}{\partial \tilde{x}}\left\{\tilde{g}(\tilde{x})K(\tilde{x})x_\perp)\right\}\right]\Bigg|_{\tilde{x}=\tilde{x}^*(t)} z$$

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$$\dot{z} = A(t)z + B(t)v, \quad v = K\left(\tilde{x}^*(t)\right)\delta x_\perp, \quad z \in \mathbb{R}^{2n},\ \delta x_\perp \in \mathbb{R}^{2n-1}$$

Challenges in orbital feedback stabilization

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By Andronov-Vitt theorem the linear feedback controller with the gain $k(\cdot)$ is not stabilizing the origin of this system

Challenges in orbital feedback stabilization

Orbital stabilization of the solution $x^*(\cdot)$ the system **by linearization**

$$x \in \mathbb{R}^{2n}: \quad \frac{d}{dt}x = f(x) + g(x)u, \quad [x^*(\cdot);\ u^*(\cdot)]$$

requires stabilization of **linearization** of transverse coordinates' dynamics

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---

This means that with change of coordinates

$$\boxed{x \mapsto \tilde{x} = [\theta \in \mathbb{R}^1;\ x_\perp \in \mathbb{R}^{2n-1}]}$$

it requires stabilizing an **invariant subspace** of the linearization

$$\dot{z} = A(t)z + B(t)v, \quad v = k(t)\delta x_\perp, \quad z \in \mathbb{R}^{2n},\ \delta x_\perp \in \mathbb{R}^{2n-1},\ v \in \mathbb{R}^n$$

of the system defined by variations of transverse coordinates $x_\perp(\cdot)$:

$$\boxed{\delta x_{1\perp} \equiv 0, \quad \delta x_{2\perp} \equiv 0, \quad \ldots, \quad \delta x_{(2n-1)\perp} \equiv 0}$$

Generic choice of transverse coordinates

Given a T-periodic pair $[\, x^*(\cdot), u^*(\cdot) \,]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\; u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

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such that the $(2n \times 2n)$-matrix

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Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Modeling and Control of Robots

Lecture 20: Underactuated Mechanical Systems (Part III)

Anton Shiriaev
March 16, 2021

Learning outcomes: Zhukovsky stability of a solution of nonlinear system. Procedures for computing transverse linearization and Leonov lemma. Examples of energy based controller design for a pendulum and transverse coordinates for a forced cart-pendulum motion

Outline

1. Transverse dynamics and transverse coordinates
   - Moving Poincare sections
   - Generic choice of transverse coordinates
2. Zhukovsky stability and linearization
   - Definition
   - Leonov lemma
   - Example: Energy-based control of a pendulum
3. Example: a pendulum on a cart

Transverse dynamics

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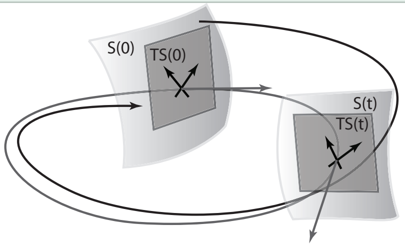
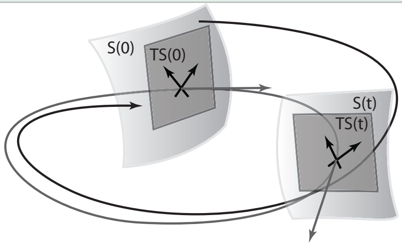


Dynamics in a vicinity of a cycle



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Generic choice of transverse coordinates

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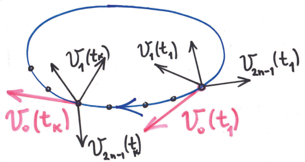

such that the next $(2n \times 2n)$-matrix is of rank $2n$

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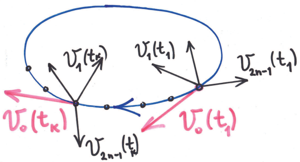

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Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Zhukovsky stability and linearization

Zhukovsky stability

Consider a nonlinear dynamic system, one of its solutions

$$\dot{x} = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n},\ t \in [0, +\infty)$$

and a set of homeomorphisms (re-parametrizations of time)

$$Hom := \{\tau(\cdot) |\, \tau : [0, +\infty) \to [0, +\infty),\, \tau(0) = 0\}$$

Zhukovsky stability

Consider a nonlinear dynamic system, one of its solutions

$$\dot{x} = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n}, \ t \in [0, +\infty)$$

and a set of homeomorphisms (re-parametrizations of time)

$$Hom := \{\tau(\cdot) |\, \tau : [0, +\infty) \to [0, +\infty),\, \tau(0) = 0\}$$

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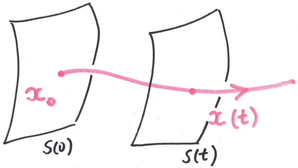

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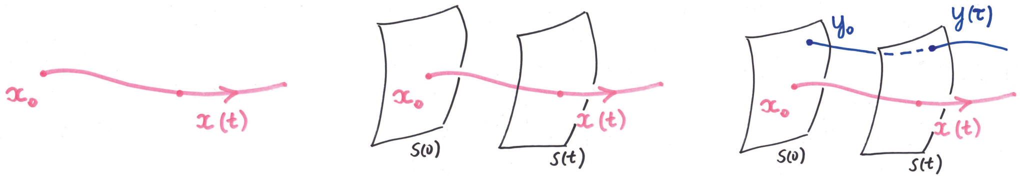

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Definition

The solution $x^0(t) = x(t, x_0)$ of the system is referred to as

- **stable**, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y_0 \in \mathbb{R}^n$ from the set $|x_0 - y_0| < \delta$ there exists $\tau(\cdot) \in Hom$ such that

$$|x(t, x_0) - x(\tau(t), y_0)| < \varepsilon \quad \forall \, t \geq 0$$

- **asymptotically stable**, if, in addition, for some $\delta_0 > 0$ and any y_0 from the set $\{y_0 : |x_0 - y_0| < \delta_0\}$ there exists $\tau(\cdot) \in Hom$ such that

$$\lim_{t \to +\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

Lyapunov, Zhukovsky and Poincare stability

Consider a nonlinear dynamic system and one of its solutions

$$\frac{d}{dt}x = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n}, \ t \in [0, +\infty)$$

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$$\Downarrow$$

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$$\xcancel{\Uparrow}\ \xcancel{\Uparrow}\ \xcancel{\Uparrow}$$

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Linearization

Linearization of a nonlinear dynamic system around its solution

$$\tfrac{d}{dt}x = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n},\ t \in [0, +\infty)$$

is the method for approximating the error by solving a linear system

$$x(t, y_0) - x(t, x_0) \approx w(t) \quad \Leftarrow \tfrac{d}{dt}w = A(t)w,\ w(0) = y_0 - x_0$$

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Here the matrix function $A(\cdot)$ is determined as a Jacobi matrix

$$\boxed{A(t) = \frac{\partial}{\partial x} f(x)\Big|_{x=x^0(t)}}$$

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How to compute the first order approximation for the error

$$x(\tau(t), y_0) - x(t, x_0) \approx v(t) \qquad \boxed{\Leftarrow \ \frac{d}{dt}v = \mathcal{A}(t)v, \ \ v(0) = y_0 - x_0?}$$

Leonov lemma

Given a nominal $x(t, x_0)$ and a perturbed $x(t, y_0)$ solutions of the system

$$\frac{d}{dt}x = f(x), \quad x \in \mathbb{R}^{2n},$$

suppose there is a homeomorphism $\tau = \tau(t) \in Hom$ defined by the Eqn

$$\boxed{[x(\tau, y_0) - x(t, x_0)]^T f(x(t, x_0)) \equiv 0, \quad \forall\, t \geq 0}$$

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Consider the difference between these solutions at times t and $t \to \tau(t)$

$$z(t) := x(\tau(t), y_0) - x(t, x_0), \quad z \in \mathbb{R}^{2n}$$

and re-write its time derivative as

$$\frac{d}{dt}z(t) \quad = \quad \frac{d}{d\tau}x(\tau(t), y_0)\, \frac{d}{dt}\tau(t) - \frac{d}{dt}x(t, x_0)$$

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$$\Rightarrow \quad \boxed{\mathcal{A}(t) = \left\{ \frac{\partial f(x)}{\partial x} - \frac{f(x)f(x)^T}{|f(x)|^2}\left(\frac{\partial f(x)}{\partial x} + \left[\frac{\partial f(x)}{\partial x}\right]^T \right) \right\}\Bigg|_{x=x(t,x_0)}}$$

Leonov lemma. Remarks

As seen, the matrix function $\mathcal{A}(\cdot)$ in the formula

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The vector $z(\cdot)$ has the same number of elements as the state $x(\cdot) \in \mathbb{R}^{2n}$

$$z(\cdot) = [z_1(\cdot);\, z_2(\cdot);\, \ldots;\, z_{2n}(\cdot)]\,,$$

and at each time moment satisfies the Eqn

$$z(t)^{\top} f(x(t, x_0)) = 0, \quad \forall t \geq 0.$$

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$$z(t)^T f(x(t, x_0)) = 0, \quad \forall t \geq 0.$$

$\Rightarrow$ $z(\cdot)$ is an excessive set of transverse coordinates for $x(t, x_0)$

New auxiliary linear system!

Given a nominal $x(t, x_0)$ solution of the system

$$\tfrac{d}{dt}x = f(x), \quad x \in \mathbb{R}^{2n},$$

one can explore its (local) properties by analyzing linear systems:

$$\boxed{\tfrac{d}{dt}w = A(t)w} \quad \text{with} \quad A(t) := \tfrac{\partial}{\partial x}f(x)\Big|_{x=x^0(t)}$$

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Asymptotic stability of the origin of the linearized system **might not imply** asymptotic (Lyapunov or Zhukovsky) stability of the solution of the nonlinear system!

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where the matrix function $\mathcal{A}(t)$ is defined in Leonov lemma.

Then the scalar function $\rho(\cdot)$ defined as a scalar product

$$\rho(t) := v(t)^T f(x^0(t))$$

is a constant on solutions of the linear system, i.e.

$$\rho(t) = v(t)^T f(x^0(t)) \equiv v(0)^T f(x^0(0)), \qquad \forall\, t \geq 0.$$

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Hence, if v_0 is chosen orthogonal to $f(x_0)$, then $v(t)^T f(x^0(t)) \equiv 0$, $\forall\, t$!

Example: Energy-based control of a pendulum

Consider the dynamics of a pendulum

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u, \quad a > 0,\ c > 0$$

augmented with the closed loop feedback controller

$$\boxed{u = -\Phi\left(\dot{\theta} \cdot \left[E(\theta, \dot{\theta}) - E_0\right]\right),} \quad E(\theta, \dot{\theta}) = \tfrac{a}{2}\,\dot{\theta}^2 + c \cdot (1 - \cos\theta).$$

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Here, E_0 is a non-negative constant, the scalar function $\phi(\cdot)$ is such that

$$\boxed{s \cdot \phi(s) > 0, \quad \forall\, s \neq 0}$$

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If $\theta_0(t)$ is a motion of the pendulum such that

$$E(\theta_0(t), \dot{\theta}_0(t)) = E_0 \quad \Rightarrow \quad u = 0.$$

Therefore, $\theta_0(t)$ is one of solutions of the closed loop system. **Let us investigate its stability**

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$$\tfrac{d}{dt}\Big[E(\theta, \dot{\theta}) - E_0\Big]^2 \;=\; 2 \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big] \cdot \tfrac{d}{dt} E(\theta, \dot{\theta})$$

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Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u = -\phi(\dot{\theta}) \cdot \left[E(\theta, \dot{\theta}) - E_0\right]$$

with $\{x_1 = \theta;\ x_2 = \dot{\theta}\}$ is written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{c}{a}\sin x_1 - \frac{\phi(x_2)}{a}\left[\frac{a}{2}x_2^2 + c(1-\cos x_1) - E_0\right] \end{bmatrix}$$

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The linear system defined for the solution $x_1^0(t) = \theta_0(t)$, $x_2^0(t) = \dot{\theta}_0(t)$ is

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{A}(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with

$$\mathcal{A}(t) = \left\{ \frac{\partial f(x)}{\partial x} - \frac{f(x)f(x)^T}{|f(x)|^2}\left(\frac{\partial f(x)}{\partial x} + \left[\frac{\partial f(x)}{\partial x}\right]^T\right)\right\}\Bigg|_{x=x^0(t)}$$

Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u = -\phi(\dot{\theta}) \cdot \left[E(\theta, \dot{\theta}) - E_0\right]$$

with $\{x_1 = \theta;\ x_2 = \dot{\theta}\}$ is written as

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The linear system defined for the solution $x_1^0(t) = \theta_0(t)$, $x_2^0(t) = \dot{\theta}_0(t)$ is

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Solutions $v^{\perp}(\cdot)$ approximating transverse dynamics are defined by

$$0 \equiv v^{\perp}(t)^T f(x^0(t))$$

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$$0 \equiv v_1^{\perp}(t) f_1(x^0(t)) + v_2^{\perp}(t) f_2(x^0(t))$$

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Example (Cont'd)

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Solutions $v^{\perp}(\cdot)$ approximating trans ... ned by

How to find $\lambda(\cdot)$?

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$$\tfrac{d}{dt}[\lambda] \cdot \left(v_0^{\perp}\right)^T v_0^{\perp} + \lambda \cdot \left(v_0^{\perp}\right)^T \tfrac{d}{dt}\left[v_0^{\perp}\right] = \lambda \cdot \left(v_0^{\perp}\right)^T \mathcal{A}(t) v_0^{\perp}$$

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$$\tfrac{d}{dt}\lambda = \boxed{\left[\ldots\right]} \cdot \lambda$$

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$$\lambda(T) = \exp\left[-\int_0^T \dot{\theta}_0(t) \cdot \phi(\dot{\theta}_0(t))dt\right] \cdot \lambda(0)$$

Example: a pendulum on a cart

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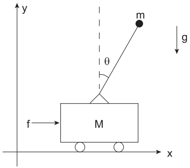
Consider a pendulum (a point of a mass m at the distance ℓ from the suspension point) attached to a cart of a mass M, for which one can apply a force f along the horizontal



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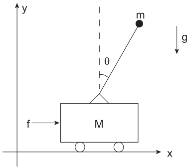

When $M = m = 1$ $[kg]$ and $\ell = 1$ $[m]$ the dynamics of the system written in coordinates (x, θ) are

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= f \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

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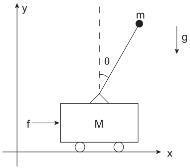

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I have derived the dynamics using the Euler-Lagrange equations :)

Transverse coordinates for a forced motion

Suppose we have the forced motion $\left[x^*(\cdot), \theta^*(\cdot)\right]$ of the system

$$
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2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= f \\
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that admits the following *"nested"* representation:

- $x^*(\cdot)$ defined by $\theta^*(\cdot)$ through the geometric relation

$$
\boxed{x^*(t) = \phi\left(\theta^*(t)\right)}
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$$\boxed{x^*(t) = \phi\left(\theta^*(t)\right)}$$

- $\theta^*(\cdot)$ defined as a solution of the reduced dynamics

$$\boxed{\left[1 + \cos\theta \cdot \phi'(\theta)\right] \ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0}$$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot{\theta}_0]$ such that the solution $\theta(t) = \theta\big(t, \theta_0, \dot{\theta}_0\big)$ of the system

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\, \ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

creates the nominal behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot{\theta}_0\right)\right).$$

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The functions of $[\theta, x, \dot{\theta}, \dot{x}]$ that define three transverse coordinates for the motion are

$$x_{1\perp} \quad := y = x - \phi(\theta)$$

$$x_{2\perp} \quad := \dot{y} = \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} \quad := I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

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$$\alpha(s) = 1 + \cos s \cdot \phi'(s), \quad \beta(s) = \cos s \cdot \phi''(s), \quad \gamma(s) = -g \cdot \sin s$$

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&\quad\;\; = \dot{\theta}^2 - h^2\left(\theta, \theta_0, \dot{\theta}_0\right)
\end{aligned}
$$