TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 1

Hand-out time:	Monday, August 19, 2013, at 12:00
Hand-in deadline:	Friday, August 30, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: State-space equation, transfer function and impulse response

Consider the system described by the differential equation

$$\ddot{y}(t) + 2\dot{y}(t) = \dot{u}(t) + 4u(t).$$

a) For this system, derive a state-space equation of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) - u(t) \end{bmatrix}$.

b) Assume zero initial conditions. Use $\hat{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$ to find the transfer function $\hat{G}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ for the system.

c) To check your answer of the previous question, compute $\hat{G}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ by applying the Laplace transform to the differential equation while assuming zero initial conditions.

d) Note that the transfer function $\hat{G}(s)$ can be written as $\hat{G}(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2}$, where α_1 and α_2 are constants. Use this to find the impulse response $G(t)$ of the system by taking the inverse Laplace transform of $\hat{G}(s)$, i.e. $G(t) = \mathcal{L}^{-1}[\hat{G}(s)]$.

Problem 2: Realizations

a) Give conditions under which a transfer matrix is realizable.

Consider the following transfer matrix:

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix}.$$

b) Use the conditions in a) to show that the transfer matrix $\hat{\mathbf{G}}(s))$ is realizable.

c) Show that the transfer matrix $\hat{\mathbf{G}}(s)$ can be written as $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, where $\hat{\mathbf{G}}_{sp}(s)$ is a strictly proper transfer matrix and $\mathbf{D}$ is a constant matrix.

d) Find the least common denominator

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

for the transfer functions of the transfer matrix $\hat{\mathbf{G}}(s)$, where n is the degree of the denominator and $\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ are constants. Moreover, write $\hat{\mathbf{G}}_{sp}(s)$ in the following form:

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}\left[\mathbf{N}_1 s^{n-1} + \mathbf{N}_2 s^{n-2} + \cdots + \mathbf{N}_{n-1}s + \mathbf{N}_n\right],$$

where $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_r$ are constant matrices.

e) Find a realization of $\hat{\mathbf{G}}(s)$ using the set of equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\alpha_1\mathbf{I} & -\alpha_2\mathbf{I} & \cdots & -\alpha_{r-1}\mathbf{I} & -\alpha_r\mathbf{I} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \cdots & \mathbf{N}_{r-1} & \mathbf{N}_r \end{bmatrix}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $\mathbf{I}$ is the identity matrix.

Problem 3: Similarity transforms and equivalent state-space equations

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t), \end{aligned} \tag{1}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \text{and} \qquad D = 2.$$

Consider the coordinate transformation

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}, \tag{2}$$

with

$$\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}.$$

a) Using the coordinate transformation (2), the system (1) can be written in the following form:

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}u(t), \\ y(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{D}u(t). \end{aligned} \tag{3}$$

Determine the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ and $\bar{D}$.

b) Are the systems (1) and (3) algebraically equivalent? Motivate your answer.

c) Are the systems (1) and (3) zero-state equivalent? Motivate your answer.

Consider the system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}u(t), \end{aligned} \tag{4}$$

with

$$\tilde{A} = -1, \qquad \tilde{B} = 2, \qquad \tilde{C} = 3 \qquad \text{and} \qquad \tilde{D} = 2.$$

d) Are the systems (1) and (4) algebraically equivalent? Motivate your answer.

e) Are the systems (1) and (4) zero-state equivalent? Motivate your answer.

Problem 4: Solutions of state-space equations

Consider the following state-space equation:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t), \end{aligned}$$

with state $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 3 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 1 \end{bmatrix}.$$

a) Compute $e^{\mathbf{A}t}$ by taking the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$, i.e. compute $e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$.

b) Let $u(t) = 1$ for all t. Compute $y(t)$ as a function of the initial conditions $\mathbf{x}(0)$.

c) Suppose that $y(1) = y(2) = 4$. Calculate $\mathbf{x}(0)$ assuming that $u(t) = 1$ for all t.