NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Design of Digital Filters: IIR

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 10.3.3 IIR filter design by the bilinear transformation
  - 10.3.4 Characteristics of commonly used analog filters

- A compressed overview of topics treated in the lecture, see "Design av digitale filtre" on Blackboard

*Level of detail is defined by lectures and problem sets

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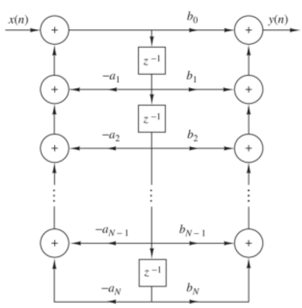
Contents and learning outcomes

- IIR filter
- Bilinear transformation
- Examples

IIR filters



- Moving and recursive averages
- Filter has both poles and zeros

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

- IIR filters designed, $\{a_k\}$ and $\{b_k\}$, by specifying poles and zeros in the z-plane
- In general IIR filters can, for a given filter order, satisfy a tighter specification than FIR filters (lower computational complexity)

IIR filters...

- In contrast to FIR filter design, IIR filters are typically designed by utilizing known analog filter design
  - Take an analog design and transform it to the digital domain
  - Nice thing: closed-form solutions exist
  - How to transform the analog solutions to discrete-time?
- Three ways of describing an analog filter
  - System function: $H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{n=0}^{N} \alpha_k s^k}$
  - Impulse response: $H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt$
  - Differential equations: $\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$
- We will be using the system function

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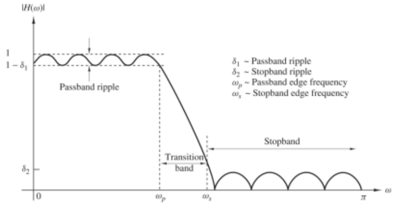
IIR filters...



- IIR filter design quite different from the FIR design
  - Step 1: State filter specs of digital filter, $\{\omega_p, \omega_s, \delta_1, \delta_2\}$
  - Step 2: Map the specs to analog domain, $\omega_p \rightarrow \Omega_p$, $\omega_s \rightarrow \Omega_s$
  - Step 3: Design an analog filter (for resistors, capacitors, and inductors) using the Laplace transform $H(s)$
  - Step 4: The design in digital domain is obtained using mapping $s = f(z)$, or $H(z) = H(s)|_{s=f(z)}$

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Transformation between s- and z-planes

- We need to transform an analog design into a digital design
  - How to go from the s-plane to the z-plane?

$$H(z) = H_a(s)|_{s=f(z)}$$

- Demands on the mapping?
  - Stable analog filters need to be mapped to stable digital filters

$$Re\{s\} < 0 \Rightarrow |z| < 1$$

  - Imaginary axis in s-plane mapped to unit circle in z-plane

$$Re\{s\} = 0 \Rightarrow |z| = 1 \Leftrightarrow j\Omega \rightarrow e^{j\omega}$$

- The bilinear tranformation satisfies these conditions
- Alternative method is to sample analog impulse respose

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Impulse invariance method

- Sample analog impulse response

$$h[n] = h_a(t)|_{t=nT} \overset{\mathcal{F}}{\leftrightarrow} H(\Omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

- Frequency mapping: $\omega = \Omega T$
  - Simple and linear
  - Suffers from potential aliasing $\Rightarrow$ not useful for highpass
- Transfer function:

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k} \rightarrow H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}$$

- No explicit mapping $s = f(z)$, but mapping of poles

$$s_k = p_k \rightarrow z_k = e^{p_k T}$$

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Impulse invariance method...

- Procedure for transforming $H_a(s)$ to $H(z)$:
  1. Find poles of $H_a(s)$, p_k
  2. Express $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}$
  3. Finally, $H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}$

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Bilinear transformation...

- The bilinear transform, a conformal mapping, provides an explicit mapping between s-plane and z-plane

$$s = \frac{2}{T}\frac{z-1}{z+1}, \text{ or } z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$$

- Setting $s = \sigma + j\Omega$ and $z = e^{j\omega}$, we get the frequency mapping

$$\omega = 2\arctan\frac{\Omega T}{2} \text{ or } \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

- The discrete-time filter's system function is given by

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

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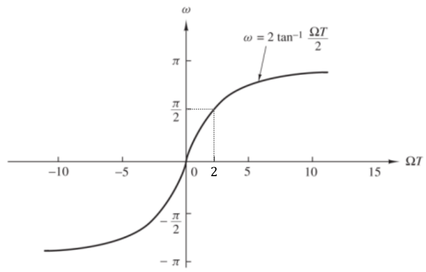
Bilinear transformation...

- Example: Fill in the table using $z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$ and $\omega = 2\arctan\frac{\Omega T}{2}$

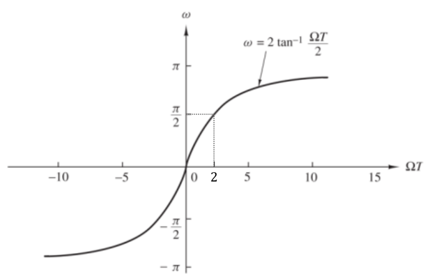
s	z	ω
0		
∞		
$\frac{2j}{T}$		
$-\frac{2}{T}$		



Bilinear transformation...

- Example: Fill in the table using $z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$ and $\omega = 2\arctan\frac{\Omega T}{2}$

s	z	ω
0	1	0
∞	-1	π
$\frac{2j}{T}$	j	$\pi/2$
$-\frac{2}{T}$	0	N/A



Bilinear transformation...

- Substitute $s = \sigma + j\Omega$ in $|z| = \frac{\left|\frac{2}{T}+s\right|}{\left|\frac{2}{T}-s\right|}$, look at $\sigma < 0, \sigma = 0, \sigma > 0$
  - For $\sigma < 0 \Rightarrow |z| = \left|\frac{\frac{2}{T}+\sigma+j\Omega}{\frac{2}{T}-\sigma-j\Omega}\right| < 1$
  - For $\sigma = 0 \Rightarrow |z| = \left|\frac{\frac{2}{T}+j\Omega}{\frac{2}{T}-j\Omega}\right| = 1$
  - For $\sigma > 0 \Rightarrow |z| = \left|\frac{\frac{2}{T}+\sigma+j\Omega}{\frac{2}{T}-\sigma-j\Omega}\right| > 1$
- Entire left half-plane maps into the inside of unit circle
- Imaginary axis maps onto the unit circle

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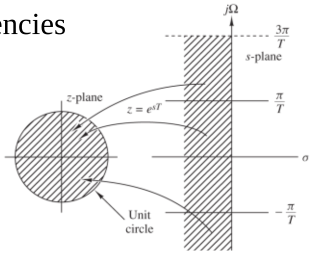
Bilinear transformation...

- The mapping satisfies the conditions for stability and mapping of the $j\Omega$-axis to the unit circle
- Reversible mapping of frequency axis, i.e.,

$$\Omega \in (-\infty, \infty) \leftrightarrow \omega \in (-\pi, \pi]$$

- Nonlinear relation between analog and digital frequencies
  - Need to pre-warp digital frequencies
- Magnitude levels unaffected
- No aliasing
  - Can design all filter types



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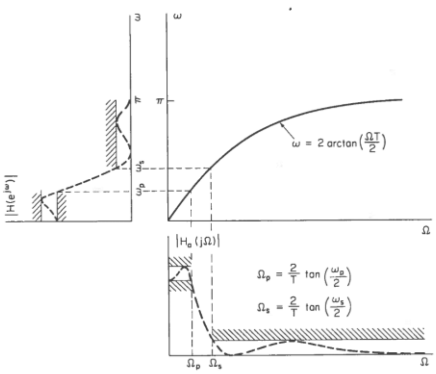
Bilinear transformation...

- Transformation of $H_a(s)$ to $H(z)$:

$$H(z) = H_a(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$



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Bilinear transformation...

- Example: Transform $H_a(s) = \frac{s+1}{s^2+5s+6}$ into a digital filter using the bilinear transformation. You may choose $T = 1$

$$H(z) = H_a(s)\big|_{s=2\frac{1-z^{-1}}{1+z^{-1}}} = \frac{2\frac{1-z^{-1}}{1+z^{-1}}+1}{\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2+5\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)+6}$$

$$= \frac{2\frac{1-z^{-1}}{1+z^{-1}}+1}{\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2+5\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)+6} = \frac{3+2z^{-1}-z^{-2}}{20+4z^{-1}}$$

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Example: Bandpass filter

- Design three digital IIR bandpass filters with resonance frequencies

$$\omega_{r_1} = \frac{\pi}{4}, \omega_{r_2} = \frac{\pi}{2}, \text{ and } \omega_{r_3} = \frac{3\pi}{4}$$

by converting the analog filter with system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

using the the bilinear transformation

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Example: Bandpass filter...

- Poles of analog filter

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{s+0.1}{(s-[-0.1+j3])(s-[-0.1-j3])} = \frac{s+0.1}{(s-p_r)(s-p_r^*)}$$

reveals analog resonance frequency $\Omega_r = 3$ rad/s

- Use frequency relation between Ω and ω_{r_i} to obtain T_i

$$\Omega = \frac{2}{T_i}\tan\frac{\omega_i}{2} \Rightarrow T_i = \frac{2}{\Omega}\tan\frac{\omega_i}{2}$$

$$T_1 = \frac{2}{3}\tan\frac{\pi}{8}, T_2 = \frac{2}{3}, \text{ and } T_3 = \frac{2}{3}\tan\frac{\pi}{8}$$

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Example: Bandpass filter...

- For each T_i apply the bilinear transform:

$$H_i(z) = H_a(s)|_{s=\frac{2}{T_i}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\left(\frac{2}{T_i}\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(\frac{2}{T_i}\frac{1-z^{-1}}{1+z^{-1}}+0.1\right)^2+9}$$

$$= \frac{(2T_i+0.1T_i^2)+0.2T_i^2z^{-1}+(0.1T_i^2-2T_i)z^{-2}}{(4+0.4T+9.01T_i^2)+(18.02T_i^2-8)z^{-1}+(4-0.4T_i+9.01T_i^2)z^{-2}}$$

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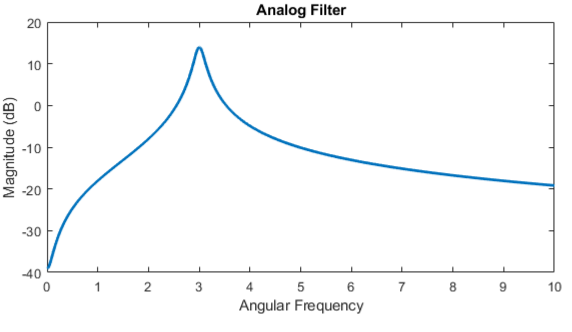
Example: Bandpass filter...



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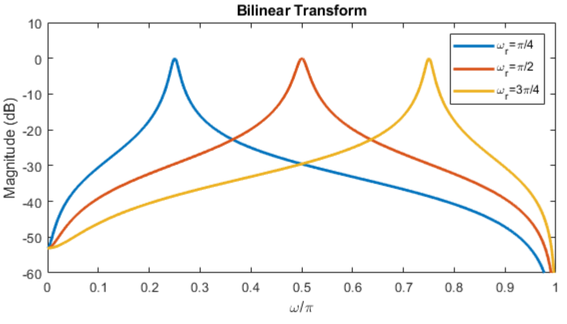
Example: Bandpass filter...



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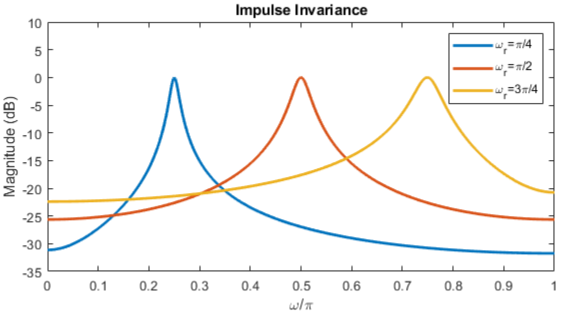
Example: Bandpass filter...



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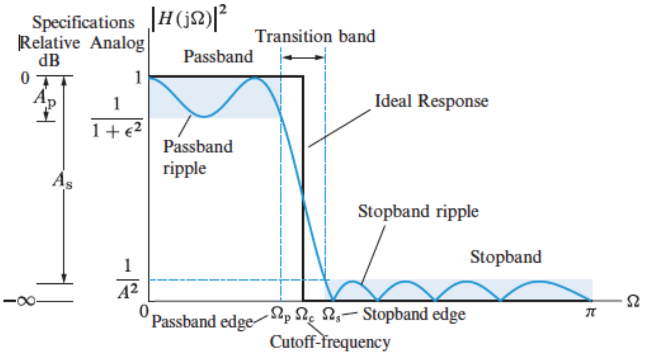
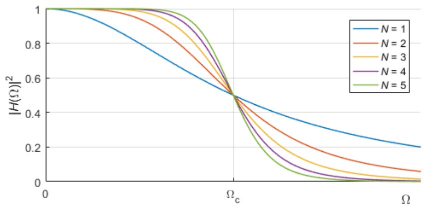
Analog filter specifications



Three classes of IIR filters

- Butterworth filters
  - In Matlab: `butter`
  - No ripples (oscillations) in $|H(\omega)|$, maximally flat
  - Smoothest transition from passband to stopband
- Chebyshev filters (two types)
  - `cheby1` and `cheby2` commands in Matlab
  - Ripples in either passband or stopband
- Elliptic filters
  - `ellip` in Matlab
  - Ripples in both passband and stopband
  - Sharpest transition from passband to stopband for a given order

Butterworth filter



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\epsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

- N poles on a circle with radius Ω_c in the s-plane
- Notice: $|H(0)|^2 = 1$, $|H(\Omega_c)|^2 = 0.5$ for all N
  $|H(\Omega)|^2$ monotonically decreasing
- Choose filter order depending on flatness of passband and how rapid decay in stopband

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Butterworth filter...

- How to find $H(s)$:

$$|H(\Omega)|^2 = H(\Omega)H^*(\Omega) = H(s)H(-s)|_{s=j\Omega} = \left.\frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}\right|_{s=j\Omega}$$

- Poles can be found from

$$1 + \left(-p_k^2/\Omega_c^2\right)^N = 0 \Rightarrow p_k = \Omega_c e^{\frac{j\pi}{2}} e^{\frac{j(2k+1)\pi}{2N}}, k = 0, \ldots, 2N-1$$

- Poles in $H(s)$: p_k in the left half-plane $k = 0, \ldots, N-1$

$$H(s) = \frac{1}{(s-p_0)(s-p_1)\ldots(s-p_{N-1})}$$

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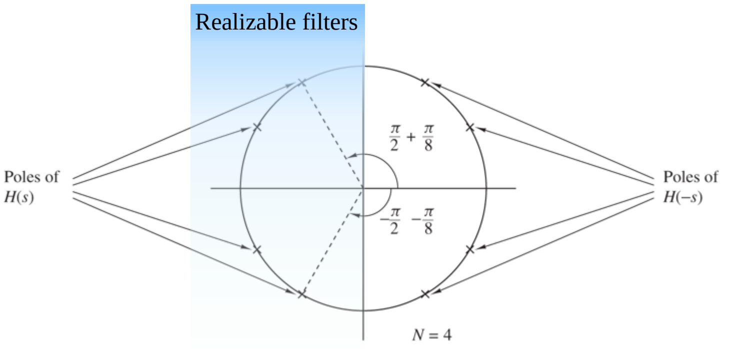
Butterworth filter...



$$p_k = \Omega_C e^{\frac{j\pi}{2}} e^{\frac{j(2k+1)\pi}{2N}}, k = 0, \dots, N-1$$

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Butterworth filter...

- Example: Design a Butterworth filter, order $N = 2$, with half-power (digital) frequency $\omega_c = \frac{\pi}{4}$
- In analog frequency domain, $\omega_c = \frac{\pi}{4}$ corresponds to

$$\Omega = \frac{2}{T}\tan\frac{\pi}{8} = \frac{2(\sqrt{2}-1)}{T}$$

- Poles and system function:

$$p_0 = \Omega_C e^{\frac{j\pi}{2}} e^{\frac{j\pi}{4}} = \frac{2(\sqrt{2}-1)}{T} e^{\frac{j3\pi}{4}}, p_1 = \Omega_C e^{\frac{j\pi}{2}} e^{\frac{j3\pi}{4}} = \frac{2(\sqrt{2}-1)}{T} e^{\frac{-j3\pi}{4}}$$

$$H(z) = \frac{1}{(s-p_0)(s-p_1)}\bigg|_{\frac{\frac{2}{T}(1-z^{-1})}{1+z^{-1}}} = \frac{T^2}{4} \cdot \frac{1+2z^{-1}+z^{-2}}{(6-3\sqrt{2}) - 4(\sqrt{2}-1)z^{-1} + (2-\sqrt{2})z^{-2}}$$

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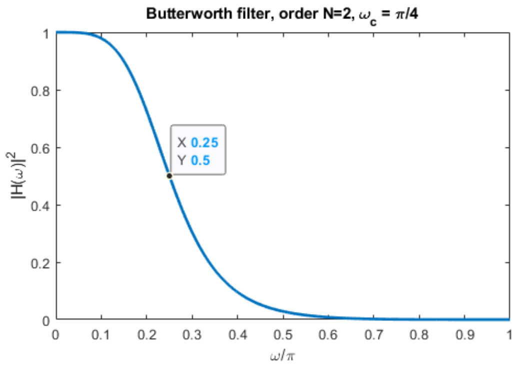
Butterworth filter...




```
[B,A] = butter(2,1/4);
[H,w]=freqz(B,A,(0:pi/500:pi));
plot(w/pi,(abs(H)).^2/max(abs(H)).^2)
```

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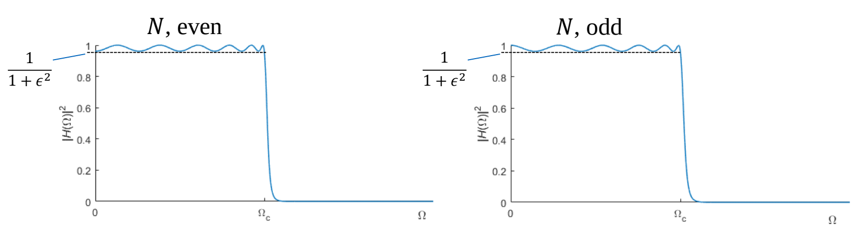
Chebyshev I



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)},\ (T_N\ (x)\ N\text{th-order Chebyshev pol.})$$

- Parameter ϵ decides ripple in passband
- Poles lying on an ellipse in the s-plane

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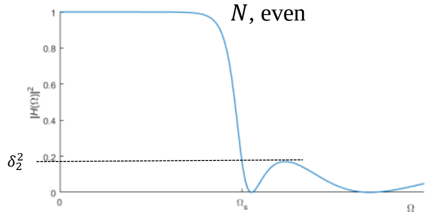
Chebyshev II



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[T_N^2(\Omega_s/\Omega_p)/T_N^2(\Omega_s/\Omega)\right]}$$

- Parameter ϵ decides ripple in stopband
- Poles on an ellipse in s-plane
- Zeros on the imaginary axis ($j\Omega$-axis)

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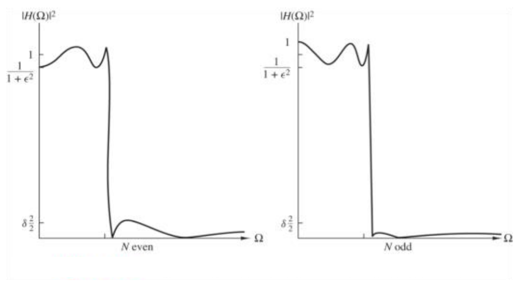
Elliptic filter



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 U_N^2(\Omega/\Omega_c)}$$ ($U_N\ (x)$ Nth-order Jacobi elliptic function)

- Parameter ϵ decides ripple in passband
- Sharpest transition from passband to stopband among discussed filters
- Zeros on the imaginary axis ($j\Omega$-axis)

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Summary

- Today we discussed:
  - IIR filter design
- Next:
  - Wiener filters

Matlab: fdatool

- Type `fdatool` at Matlab command prompt: