NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2019

Lecture: The Discrete Fourier Transform

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 7.1.1 Frequency-domain sampling and reconstruction of discrete-time signals
  - 7.1.2 The discrete Fourier transform (DFT)
  - 7.2 Properties of the DFT

*Level of detail is defined by lectures and problem sets

Preliminary questions

- To perform frequency analysis of sequence $x[n]$ we need to convert it into its frequency-domain representation
- In our toolkit we find the discrete-time Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Is this a convenient representation?

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Contents and learning outcomes

- Frequency-domain sampling and reconstruction
- Discrete Fourier Transform (DFT)
- Properties of the DFT

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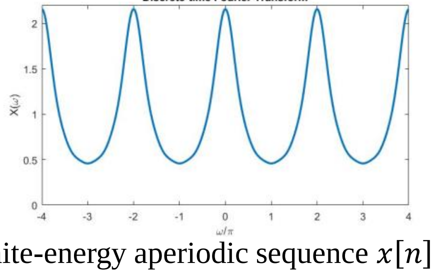
Motivation: Discrete Fourier transform

- Discrete Fourier transform (DFT) and inverse DFT (IDFT)
  - linear filtering of long sequences
  - frequency (spectrum) analysis
  - power spectrum estimation

- Efficient implementation using fast Fourier transform (FFT)

Frequency-domain sampling

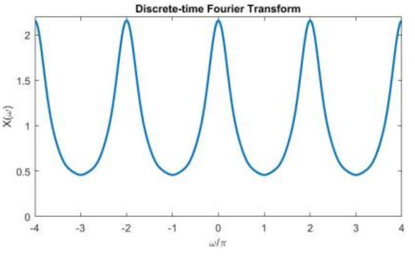


- Consider finite-energy aperiodic sequence $x[n]$ with DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

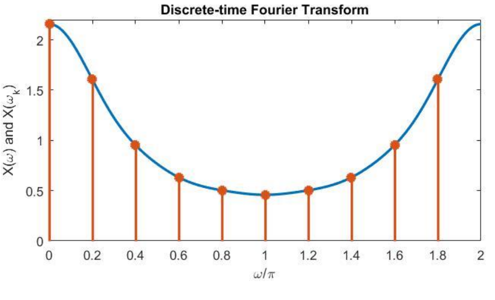
- Spectrum $X(\omega)$ is continuous but 2π-periodic
- Sample spectrum periodically in frequency
  - Benefits of performing such sampling?
  - Is sampled spectrum anymore related to $x[n]$?

Frequency-domain sampling…



- Discussion:
  - Sampling *continuous-time* signal versus sampling *continuous-frequency* signal
  - Periodicity in transform-domain

Frequency-domain sampling…

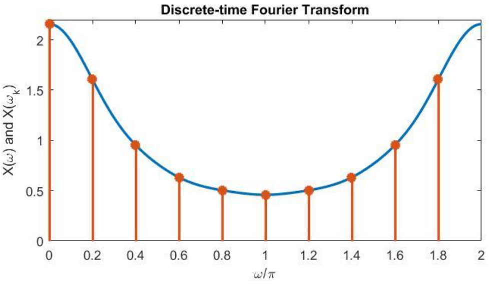


- In interval $0 \leq \omega \leq 2\pi$, take N equidistant samples,

$$X(\omega_k) = X(\omega)|_{\omega=\omega_k},$$

$$\omega_k = \frac{2\pi k}{N}, k = 0, \ldots, N-1$$

Frequency-domain sampling…



- DTFT $X(\omega)$ evaluated at ω_k

$$X(\omega_k) = \sum_{n=-\infty}^{\infty} x[n] e^{-\frac{j2\pi k}{N} n},\ k = 0, \dots, N-1$$

- Make use of identity $e^{-\frac{j2\pi k}{N} n} = e^{-\frac{j2\pi k}{N}(n+N)}$

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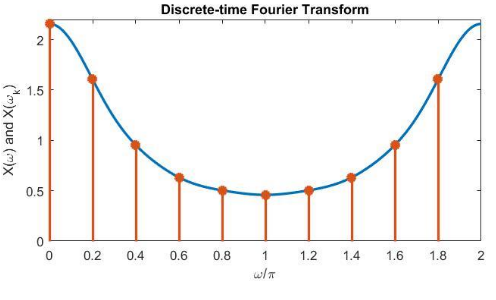
Frequency-domain sampling…



- DTFT $X(\omega)$ evaluated at ω_k

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-\frac{j2\pi k}{N} n}$$

$$= \cdots + \sum_{n=-N}^{-1} x[n] e^{-\frac{j2\pi k}{N} n}$$

$$+ \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi k}{N} n}$$

$$+ \sum_{n=N}^{2N-1} x[n] e^{-\frac{j2\pi k}{N} n} + \cdots$$

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Frequency-domain sampling…

- DTFT $X(\omega)$ evaluated at ω_k

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN]\, e^{-\frac{j2\pi k}{N}n}$$
$$= \sum_{n=0}^{N-1} x_p[n] e^{-\frac{j2\pi k}{N}n}$$

- Periodic extension of $x[n]$

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$$

- Example 1: Given $x[n] = \delta[n] + 0.5\delta[n-1]$, sketch $x_p[n]$ for $N = 1$ and $N = 3$ and comment on the results

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Frequency-domain sampling…

- Clearly $x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$ is periodic with period N
- Express as a discrete-time Fourier series $\Longrightarrow$

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi k}{N}n},\ n = 0, \dots, N-1$$
$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{\frac{-j2\pi k}{N}n} = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$$

- Let us take stock and see where we stand

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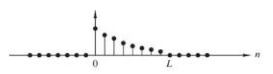
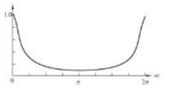
Frequency-domain sampling…

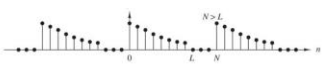


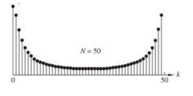


$$x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$$

$\Uparrow$? $\Downarrow$ Sample at $\omega_k = \frac{2\pi k}{N}$

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n - lN] \overset{DTFS}{\longleftrightarrow} X(\omega_k)$$





- When can $x[n]$ be recovered from $x_p[n]$?
  - Duration of sequence $x[n]$ versus period of $x_p[n]$?

Frequency-domain sampling…

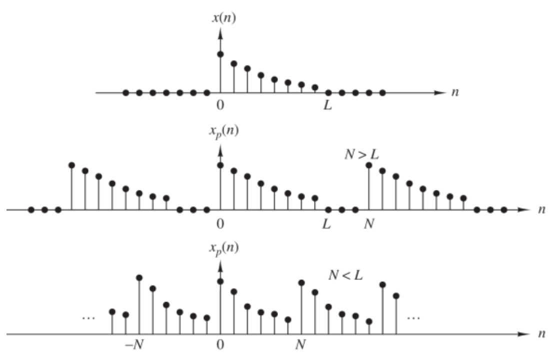
- Lesson learned:

The spectrum $X(\omega)$ of an aperiodic sequence $x[n]$ of finite duration L, can be recovered from samples $X(\omega_k)$, with $\omega_k = \frac{2\pi k}{N}$, if the number of samples $N \geq L$

- Procedure for closing the circle:
  1. Compute $x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi k}{N}n}$, $n = 0, \ldots, N-1$
  2. Set $x[n] = x_p[n]$ for $0 \leq n \leq N-1$, zero elsewhere
  3. Compute $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

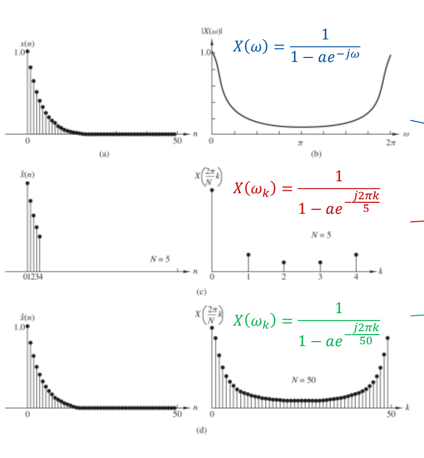
Frequency-domain sampling…

- Example 2: Which periodic extension of $x[n]$ can be used to recover spectrum $X(\omega)$?



Frequency-domain sampling…

- Example 3: Infinite duration sequences, reconstructed sequence will suffer from aliasing, $x[n] = a^n u[n]$, $|a| < 1$.



Matlab

```
B = [1]; A = [1 -0.9];
[X,w]=freqz(B,A,'whole');
figure, plot(w/pi,abs(X));
[x,n]=impz(B,A)
figure, stem(n,x),hold on
X1 = X(1:length(w)/5:end)
figure,
stem(ifft(X1),'r')
X2 = X(1:length(w)/50:end)
figure,
stem(ifft(X2),'g')
```

Discrete Fourier transform (DFT)

- Putting the bits and pieces together (remember)

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi k}{N}n}, n = 0, \dots, N-1$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n] e^{-\frac{j2\pi k}{N}n} = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$$

- For sequence $x[n]$ of length $L \leq N$, $x[n] = 0$, $L \leq n \leq N$

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi k}{N}n}, k = 0, \dots, N-1$$

$$\text{IDFT: } x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi k}{N}n}, n = 0, \dots, N-1$$

- Notation: $X(k) \equiv X(\omega_k)$, $X(k) = \text{DFT}_N\{x[n]\}$

Discrete Fourier transform (DFT)

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi k}{N}n}, k = 0, \dots, N-1$$

$$\text{IDFT: } x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi k}{N}n}, n = 0, \dots, N-1$$

- What happens when increasing $N > L$, L is kept fixed?
  - In frequency-domain?
  - In time-domain?
- Using $N > L$ samples for computing the DFT is commonly referred to as *zero padding* and improves resolution

Discrete Fourier transform (DFT)…

- Example 4: Plot N-point DFT of $x[n] = \sum_{l=0}^{3} \delta[n-l]$ for $N = 4$ and $N = 40$

$$X(\omega) = \sum_{n=0}^{L-1=3} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

$$X(k) = \sum_{n=0}^{L-1=3} e^{-\frac{j2\pi k}{N}n} = \frac{1-e^{-\frac{j2\pi k}{N}L}}{1-e^{-\frac{j2\pi k}{N}}} = \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

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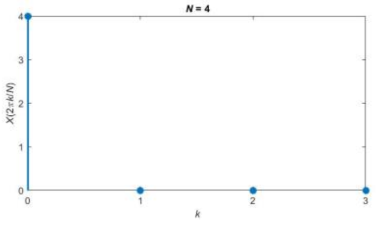
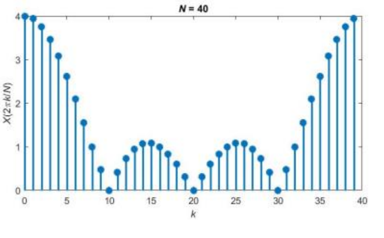
Discrete Fourier transform (DFT)…

- Example 4: Plot N-point DFT of $x[n] = \sum_{l=0}^{3} \delta[n-l]$ for $N = 4$ and $N = 40$






```
Matlab
L = 4; x = ones(1,L);
N = L*10; x_zp = [x,zeros(1,N-L)];
stem((0:N-1),abs(fft(x_zp,N)));
```

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Properties of the DFT

- Periodicity
- Linearity
- Time reversal
- Circular time shift
- Circular frequency shift
- Conjugation
- Circular convolution
- Multiplication of two sequences
- Parseval's theorem

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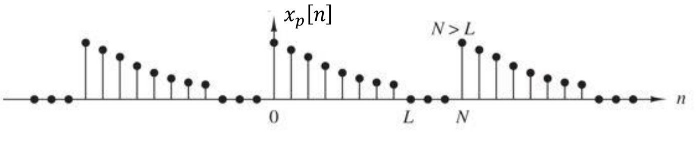
Properties of the DFT…

- Properties are similar to those of the DTFT
- Keep in mind is that operations on $X(k)$ in frequency domain corresponds to *operations on* $x_p[n]$ in time domain



$$x_p[n] = x[n, \text{modulo } N] = x([n])_N$$

$\Rightarrow$ Shifting $x_p[n]$ in time by for k units, $x_p[n-k]$, is identical to a circular shift of $x[n]$ in interval $0 \le n \le N-1$

$$x([n-k])_N \equiv x[n-k, \text{modulo } N]$$

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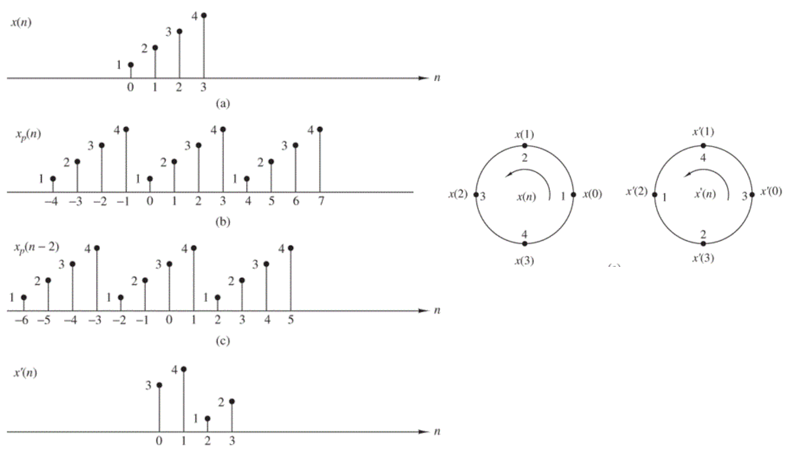
Properties of the DFT…

- Shifting:



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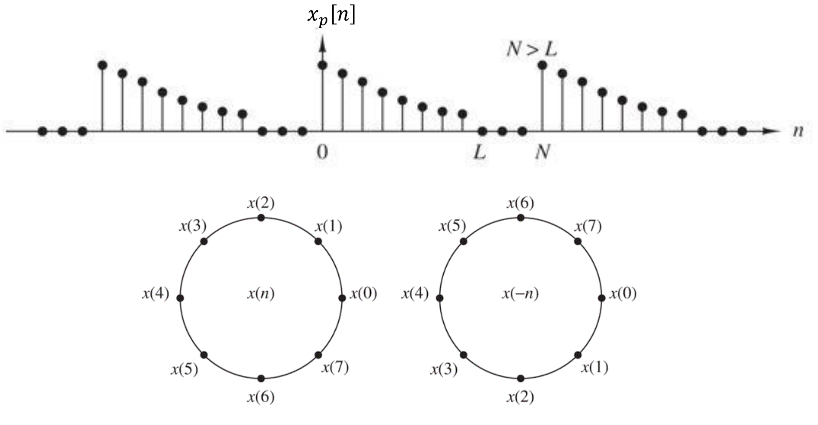
Properties of the DFT…

- Time-reversal:



$$x([-n])_N \equiv x[-n, \text{modulo } N] = x[N-n],\ 0 \le n \le N-1$$

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Properties of the DFT…

- Periodicity: $x[n] = x[n+N] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X(k) = X(k+N)$
- Linearity: $a_1 x_1[n] + a_2 x_2[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} a_1 X_1(k) + a_2 X_2(k)$
- Time reversal: $x[N-n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X(N-k)$
- Circular time shift: $x([n-l])_N \overset{\mathrm{DFT}_N}{\longleftrightarrow} X(k)e^{-j2\pi kl/N}$
- Circular frequency shift: $x[n]e^{j2\pi ln/N} \overset{\mathrm{DFT}_N}{\longleftrightarrow} X((k-l))_N$
- Conjugation: $x^*[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X^*(N-k)$
- Circular convolution: $x_1[n] \otimes_N x_2[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X_1(k)X_2(k)$
- Parseval's theorem: $\sum_{n=0}^{N-1} x[n]y^*[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} \frac{1}{N}\sum_{n=0}^{N-1} X(k)Y^*(k)$

Properties of the DFT…

- Circular convolution: $x_1[n] \otimes_N x_2[n] \overset{\mathrm{DFT}_N}{\longleftrightarrow} X_1(k)X_2(k)$

  $x_1[n] \otimes_N x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2([n-k])_N \, , n = 0,1, \ldots N-1$

- Linear convolution of causal sequences $x_1[n]$ and $x_2[n]$

$$x_1[n] * x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2[n-k]$$

- In general, $x_1[n] \otimes_N x_2[n] \neq x_1[n] * x_2[n]$

  $\Rightarrow$ important when applying the DFT to linear system analysis (next lecture)

Summary

Today:

- Frequency-domain sampling and reconstruction
- The DFT (discrete Fourier transform)
- Properties of the DFT

Next:

- Using DFT for filtering and frequency analysis