Modeling using Lagrange

Sébastien Gros

ESS101, 2017

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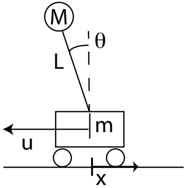

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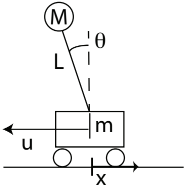
- Note that $\mathbf{q} \neq$ states !! Often states are $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$



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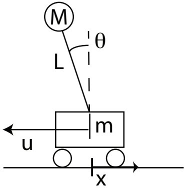
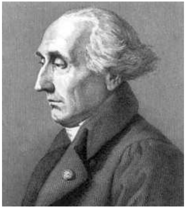


Lagrange (1788) function:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \underbrace{T(\mathbf{q}, \dot{\mathbf{q}})}_{\text{kinetic energy}} - \underbrace{V(\mathbf{q})}_{\text{potential energy}}$$



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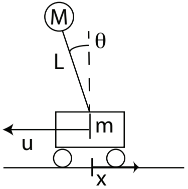

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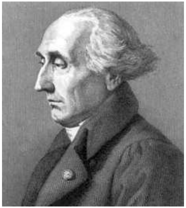
then the integral action:

$$\mathcal{I} = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})\, \mathrm{d}t$$

is minimised by the systems (free) trajectory.



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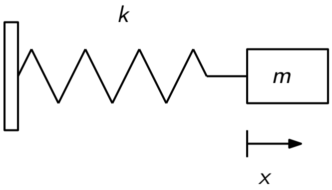

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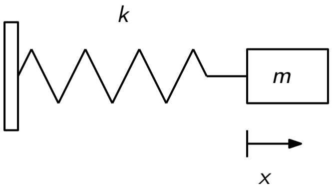

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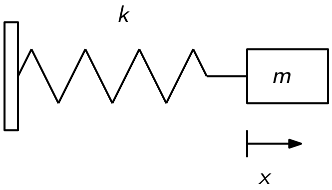

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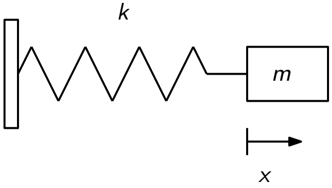

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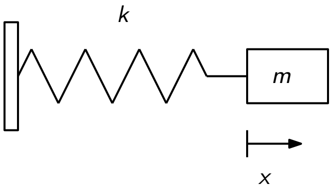

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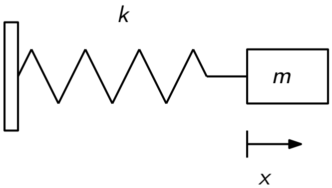

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From *variational calculus*, the free trajectories satisfy (Euler-Lagrange equation):

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0$$

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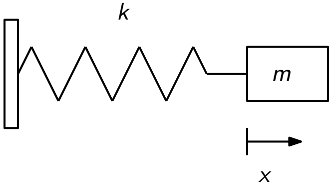

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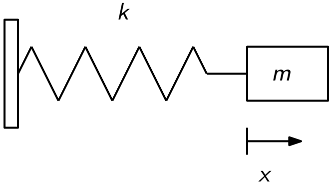

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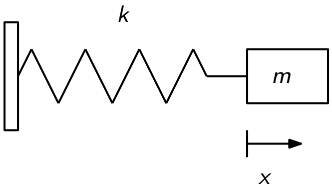

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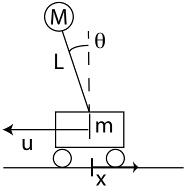
Yield the ODE:

$$m\ddot{x} + kx = 0$$

Example - Pendulum on a Cart

Generalized coordinates $\mathbf{q} = \begin{bmatrix} x \\ \theta \end{bmatrix}$



Kinetic energy: $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\theta}^2 + LM\dot{\theta}\dot{x}\cos\theta$

Potential energy: $V(\mathbf{q}) = MgL\cos\theta$

Lagrange function: $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$

From *variational calculus*, the free trajectories satisfy:

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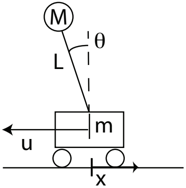

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$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = \begin{bmatrix} (M + m)\dot{x} + ML\dot{\theta}\cos(\theta) \\ ML^2\dot{\theta} + ML\dot{x}\cos(\theta) \end{bmatrix}, \quad \frac{\partial \mathcal{L}}{\partial \mathbf{q}}^\top = \begin{bmatrix} 0 \\ MgL\sin(\theta) - ML\dot{\theta}\dot{x}\sin(\theta) \end{bmatrix}$$

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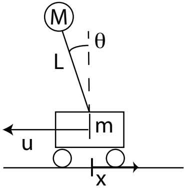

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Useful tip: the whole procedure can be easily coded in a Computer Algebra System.

Structure of the Lagrange Equation

Most often the kinetic energy reads as:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top W(\mathbf{q})\dot{\mathbf{q}}$$

for some matrix $W(\mathbf{q})$ (symmetric positive-definite). While the potential energy V is a function of $\mathbf{q}$ only. One can then observe that:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = \frac{\partial T}{\partial \dot{\mathbf{q}}}^\top = W(\mathbf{q})\dot{\mathbf{q}}, \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)\dot{\mathbf{q}} = W(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left[W(\mathbf{q})\dot{\mathbf{q}}\right]\dot{\mathbf{q}} \tag{1b}$$

such that the Lagrange equation yields

$$\underbrace{W(\mathbf{q})\ddot{\mathbf{q}}}_{\equiv\text{“}m\cdot a\text{”}} = \underbrace{-\frac{\partial V}{\partial \mathbf{q}}^\top}_{\text{forces from potentials}} + \underbrace{\frac{\partial T}{\partial \mathbf{q}}^\top - \frac{\partial}{\partial \mathbf{q}}\left[W(\mathbf{q})\dot{\mathbf{q}}\right]\dot{\mathbf{q}}}_{\text{quadratic in } \dot{\mathbf{q}}}$$

Note that the

- complexity of this equation will mostly be due to the last two terms
- these two terms disappear if matrix $W(\mathbf{q})$ is constant (not depending of $\mathbf{q}$)

External Forces

Consider a system described by the **generalized coordinates** $\mathbf{q}$ with:

$$\text{Kinetic energy:} \quad T(\mathbf{q}, \dot{\mathbf{q}})$$
$$\text{Potential energy:} \quad V(\mathbf{q})$$

Define the **Lagrange function**: $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T - V$.

Then the free dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0$$

... and the *forced* dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}$$

where $\mathbf{Q}$ are the **generalized forces**, defined such that the **virtual work** condition:

$$\underbrace{\delta W}_{work} = \langle \mathbf{Q}, \delta \mathbf{q} \rangle$$

is satisfied for all compatible displacement $\delta \mathbf{q}$.

How to use that ?!? Suppose force $\mathbf{F}_i$ applied at point $\mathbf{p}_i(\mathbf{q})$ in the system

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Define the **Lagrange function**: $\mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}\right) = T - V$.

Then the free dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0$$

... and the *forced* dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}$$

where $\mathbf{Q}$ are the **generalized forces**, defined such that the **virtual work** condition:

$$\underbrace{\delta W}_{\textit{work}} = \langle \mathbf{Q},\, \delta \mathbf{q} \rangle$$

is satisfied for all compatible displacement $\delta \mathbf{q}$.

How to use that ?!? Suppose force $\mathbf{F}_i$ applied at point $\mathbf{p}_i\left(\mathbf{q}\right)$ in the system

$$\delta W = \sum_i \mathbf{F}_i^\top \frac{\partial \mathbf{p}_i}{\partial \mathbf{q}} \delta \mathbf{q} = \mathbf{Q}^\top \delta \mathbf{q}, \ \forall \delta \mathbf{q} \qquad \text{then} \qquad \mathbf{Q} = \sum_i \mathbf{F}_i^\top \frac{\partial \mathbf{p}_i}{\partial \mathbf{q}}$$

Constrained Lagrange Mechanics

Consider a system described by the **generalized coordinates** $\mathbf{q}$ with:

$$
\begin{aligned}
\text{Kinetic energy:} \quad & T(\mathbf{q}, \dot{\mathbf{q}}) \\
\text{Potential energy:} \quad & V(\mathbf{q}) \\
\text{Constraints:} \quad & \mathbf{C}(\mathbf{q}) = 0
\end{aligned}
$$

Define the **Lagrange function**:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{C}(\mathbf{q})$$

where $\mathbf{z}$ is a set of "helper variables" (true name is *Lagrange multipliers*) that have the size of our constraint function.

Then the dynamics are given by:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} &= \mathbf{Q} \\
\mathbf{C}(\mathbf{q}) &= 0
\end{aligned}
$$

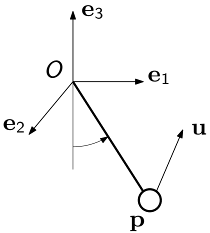
The constraints enter the dynamics via:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \frac{\partial T}{\partial \mathbf{q}} - \frac{\partial V}{\partial \mathbf{q}} - \mathbf{z}^\top \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

The "force" keeping the system on $\mathbf{C}(\mathbf{q}) = 0$ is in the space spanned by $\nabla_{\mathbf{q}} \mathbf{C}_i$

3D pendulum in Lagrange Mechanics

Generalized coordinates: $\mathbf{q} \equiv \mathbf{p}$, and:



$$\text{Kinetic energy:} \quad T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}}$$

$$\text{Potential energy:} \quad V(\mathbf{q}) = mg\mathbf{e}_3^\top \mathbf{p}$$

$$\text{Constraints:} \quad \mathbf{C}(\mathbf{q}) = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right)$$

3D pendulum in Lagrange Mechanics

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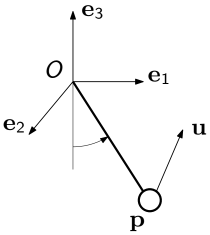

Lagrange function: $\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{e}_3^\top \mathbf{p} - \frac{1}{2} z \left(\mathbf{p}^\top \mathbf{p} - L^2\right)$ yields:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{p}} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\ddot{\mathbf{p}} \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = -mg\mathbf{e}_3 - z\mathbf{p}$$

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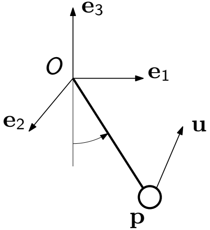

Lagrange function: $\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{e}_3^\top \mathbf{p} - \frac{1}{2} z \left(\mathbf{p}^\top \mathbf{p} - L^2\right)$ yields:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{p}} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\ddot{\mathbf{p}} \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = -mg\mathbf{e}_3 - z\mathbf{p}$$

Using $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{u}$ the dynamics read as

$$m\ddot{\mathbf{p}} + mg\mathbf{e}_3 + z\mathbf{p} = \mathbf{u}$$
$$\frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right) = 0$$

Delta robot in Lagrange Mechanics

L: length "long" arms
l: length "small" arms
d: distance center-motors

- Position of the nacelle $\mathbf{p} \in \mathbb{R}^3$.

Delta robot in Lagrange Mechanics

- Position of the nacelle $\mathbf{p} \in \mathbb{R}^3$.
- Position of the rods end point:

$$\mathbf{p}_k^{\mathrm{R}} = \begin{bmatrix} \cos\gamma_k & -\sin\gamma_k & 0 \\ \sin\gamma_k & \cos\gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d + l\cos\alpha_k \\ 0 \\ -l\sin\alpha_k \end{bmatrix}$$

where $\gamma_{1,2,3} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

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Kinetic energy: $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} + \frac{1}{2} \sum_{k=1}^{3} J \dot{\alpha}_k^2$

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Potential energy: $$V(\mathbf{q}) = mg\mathbf{p}_3 + \frac{1}{2} \sum_{k=1}^{3} Mgl \sin\alpha_k$$

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where $\gamma_{1,2,3} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

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Potential energy: $V(\mathbf{q}) = mg\mathbf{p}_3 + \frac{1}{2} \sum_{k=1}^{3} Mgl \sin\alpha_k$

Constraints: $\mathbf{C}_k(\mathbf{q}) = \left\| \mathbf{p} - \mathbf{p}_k^{\mathrm{R}} \right\|^2 - L^2, \quad k = 1, 2, 3$

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$$\text{Kinetic energy:} \quad T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} + \frac{1}{2} \sum_{k=1}^{3} J \dot{\alpha}_k^2$$

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Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} + \sum_{k=1}^{3} \left[\frac{1}{2} J \dot{\alpha}_k^2 - mg\mathbf{p}_3 - \frac{1}{2} Mgl\sin\alpha_k + \mathbf{z}_k \left(\left\| \mathbf{p} - \mathbf{p}_k^{\mathrm{R}} \right\|^2 - L^2 \right) \right]$$

Structure with constraints

Most often the kinetic energy reads as:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top W(\mathbf{q})\dot{\mathbf{q}}$$

for some matrix $W(\mathbf{q})$ (symmetric positive-definite), while the potential energy V and the constraints $\mathbf{C}$ are functions of $\mathbf{q}$ only. One can then observe that the Lagrange equation yields:

$$\underbrace{W(\mathbf{q})\ddot{\mathbf{q}}}_{\equiv\text{“}m\cdot a\text{”}} = \underbrace{-\frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z}}_{\text{forces from constraints}} \quad \underbrace{-\frac{\partial V}{\partial \mathbf{q}}^\top}_{\text{forces from potentials}} \quad + \underbrace{\frac{\partial T}{\partial \mathbf{q}}^\top - \frac{\partial}{\partial \mathbf{q}}\left[W(\mathbf{q})\dot{\mathbf{q}}\right]\dot{\mathbf{q}}}_{\text{quadratic in } \dot{\mathbf{q}}}$$

Note that

- if the constraints are quadratic, then $\frac{\partial \mathbf{C}}{\partial \mathbf{q}}$ is linear
- the same remarks on complexity hold as in the unconstrained case

Solving the constrained Lagrange equation?

The dynamics are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}$$
$$\mathbf{C}\left(\mathbf{q}\right) = 0$$

but they don't define $\mathbf{z}$ properly... Indeed, these equations also read as:

$$W(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{Q} - \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z} - \frac{\partial V}{\partial \mathbf{q}}^\top + \frac{\partial T}{\partial \mathbf{q}}^\top - \frac{\partial}{\partial \mathbf{q}}\left[W(\mathbf{q})\dot{\mathbf{q}}\right]\dot{\mathbf{q}} \tag{2a}$$
$$\mathbf{C}\left(\mathbf{q}\right) = 0 \tag{2b}$$

and one can observe that

- $\ddot{\mathbf{q}} \in \mathbb{R}^{n_q}$ is defined by (2a) (n_q equations linear in $\ddot{\mathbf{q}}$), as a function of $\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}$
- Equation (2b) informs us on conditions that $\mathbf{q}$ ought to satisfy, but it does not help us computing $\mathbf{z}$ as such

This difficulty can be construed as follows. $\mathbf{z}$ "adjusts" the forces in the system and therefore influences the "accelerations" in the system. Since $\mathbf{C}\left(\mathbf{q}\right)$ is a function of the "positions" $\mathbf{q}$, it is not directly influenced by $\mathbf{z}$, but indirectly via the influence of $\mathbf{z}$ on $\ddot{\mathbf{q}}$. To tackle this problem, we need to make this influence appear explicitly...

Model transformation

Consider the time derivatives of $\mathbf{C}$:

$$\dot{\mathbf{C}}\left(\mathbf{q},\dot{\mathbf{q}}\right) = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}\dot{\mathbf{q}} \tag{3a}$$

$$\ddot{\mathbf{C}}\left(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}\right) = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right]\dot{\mathbf{q}} \tag{3b}$$

((3b) is similar to (1b))

We observe that $\ddot{\mathbf{C}}$ is function of $\ddot{\mathbf{q}}$ which is itself function of $\mathbf{z}$. Now we have an explicit influence of $\mathbf{z}$ on the constraints (via $\ddot{\mathbf{C}}$)!!

Note that we want $\mathbf{C}\left(\mathbf{q}\right) = 0$ for all time t, hence:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{C} = 0$$

must hold for all k and t. In particular

$$\ddot{\mathbf{C}}\left(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}\right) = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}\ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}\left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right]\dot{\mathbf{q}} = 0 \tag{4}$$

must hold at all time.

Model transformation

Using (2a) and (4), we observe that the dynamics satisfy:

$$
\begin{aligned}
W(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z} &= \mathbf{Q} - \frac{\partial V}{\partial \mathbf{q}}^\top + \frac{\partial T}{\partial \mathbf{q}}^\top - \frac{\partial}{\partial \mathbf{q}}\left[W(\mathbf{q})\dot{\mathbf{q}}\right]\dot{\mathbf{q}} && \text{(5a)} \\
\frac{\partial \mathbf{C}}{\partial \mathbf{q}}\ddot{\mathbf{q}} &= -\frac{\partial}{\partial \mathbf{q}}\left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}}\dot{\mathbf{q}}\right]\dot{\mathbf{q}} && \text{(5b)}
\end{aligned}
$$

Equation (5a) is the same as (2a). Equation (5b) does not provide $\mathbf{z}$ as such but it does provide information on $\ddot{\mathbf{q}}$, which can be used in conjunction with (5a) to resolve $\mathbf{z}$... See next slide.

Model transformation (cont')

It is useful here to rewrite (5a)-(5b) in a matrix form, to make their structure visible:

$$\underbrace{\begin{bmatrix} W(\mathbf{q}) & \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \\ \frac{\partial \mathbf{C}}{\partial \mathbf{q}} & 0 \end{bmatrix}}_{:=\mathcal{M}(q)} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial V}{\partial \mathbf{q}}^\top + \frac{\partial T}{\partial \mathbf{q}}^\top - \frac{\partial}{\partial \mathbf{q}} \left[W(\mathbf{q}) \dot{\mathbf{q}} \right] \dot{\mathbf{q}} \\ -\frac{\partial}{\partial \mathbf{q}} \left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right] \dot{\mathbf{q}} \end{bmatrix} \tag{6}$$

If $\mathcal{M}(\mathbf{q})$ is full rank (i.e. invertible), then (6) delivers $\ddot{\mathbf{q}}$, $\mathbf{z}$ as functions of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\mathbf{Q}$. Because $W(\mathbf{q})$ is positive definite, matrix $\mathcal{M}(\mathbf{q})$ is full rank if $\frac{\partial \mathbf{C}}{\partial \mathbf{q}}$ is full row rank, i.e. if the constraints are not "degenerate". This would happen if e.g. the system is "over-constrained", i.e. hyperstatic, or if $\frac{\partial \mathbf{C}}{\partial \mathbf{q}}$ has a row of zeros.

Similarly to previous remarks, if $W(\mathbf{q})$ is constant, then (6) reduces to:

$$\begin{bmatrix} W & \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \\ \frac{\partial \mathbf{C}}{\partial \mathbf{q}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} - \frac{\partial V}{\partial \mathbf{q}}^\top \\ -\frac{\partial}{\partial \mathbf{q}} \left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right] \dot{\mathbf{q}} \end{bmatrix} \tag{7}$$

To make W constant, one needs to pick a "coordinate system" (i.e. the $\mathbf{q}$) such that the kinetic energy function is a "simple form" of the $\dot{\mathbf{q}}$. In general, choosing a cartesian coordinate system will do the trick!!

Some more remarks

For $W(\mathbf{q})$ constant:

$$\mathcal{M}(\mathbf{q}) \left[\begin{array}{c} \ddot{\mathbf{q}} \\ \mathbf{z} \end{array} \right] = \mathbf{c}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}\right)$$

where

$$\mathcal{M}(\mathbf{q}) = \left[\begin{array}{cc} W & \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^{\top} \\ \frac{\partial \mathbf{C}}{\partial \mathbf{q}} & 0 \end{array} \right]$$

$$\mathbf{c}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}\right) = \left[\begin{array}{c} \mathbf{Q} - \frac{\partial V}{\partial \mathbf{q}}^{\top} \\ -\frac{\partial}{\partial \mathbf{q}} \left[\frac{\partial \mathbf{C}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right] \dot{\mathbf{q}} \end{array} \right]$$

- In order to get an explicit model, one needs to form:

$$\left[\begin{array}{c} \ddot{\mathbf{q}} \\ \mathbf{z} \end{array} \right] = \mathcal{M}(\mathbf{q})^{-1} \mathbf{c}\left(\mathbf{q}, \dot{\mathbf{q}}\right)$$

  This stunt is not always advisable, as $\mathcal{M}(\mathbf{q})^{-1}$ is often very complex (especially if $\mathcal{M}(\mathbf{q})$ is large), even if $\mathcal{M}(\mathbf{q})$ is simple. More on that later...

- One can write the explicit state space form:

$$\dot{\mathbf{x}} = \left[\begin{array}{c} \dot{\mathbf{q}} \\ \mathcal{M}(\mathbf{q})^{-1} \mathbf{c}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}\right) \end{array} \right], \quad \mathbf{x} = \left[\begin{array}{c} \mathbf{q} \\ \dot{\mathbf{q}}, \mathbf{Q} \end{array} \right]$$

- The expressions involved in $\mathcal{M}(\mathbf{q})$, $\mathbf{c}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{Q}\right)$ (or in (6)) can be readily computed by a Computer Algebra System (Matlab symbolic toolbox, Maple, Mathematica, ect...)

Consistency conditions

We have built (6) (and (7)) by imposing $\ddot{\mathbf{c}} = 0$ at all time instead of $\mathbf{c} = 0$. This allowed us to determine the unknown variables $\mathbf{z}$.

However, we need to note here that $\ddot{\mathbf{c}} = 0$ holding at all time implies:

$$\mathbf{c}\left(\mathbf{q}(t)\right) = \mathbf{c}(\mathbf{q}(0)) + \dot{\mathbf{c}}(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \cdot t$$

(regardless of the trajectory $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$ followed by the model), hence

The constraints $\mathbf{c}\left(\mathbf{q}(t)\right) = 0$ hold at all time if and only if:

$$\mathbf{c}(\mathbf{q}(0)) = 0 \tag{8a}$$

$$\dot{\mathbf{c}}(\mathbf{q}(0), \dot{\mathbf{q}}(0)) = 0 \tag{8b}$$

Equations (8a)-(8b) impose *consistency conditions* on the initial conditions:

$$\mathbf{x}(0) = \left[\begin{array}{c} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{array}\right]$$

in order to be <u>admissible</u> in the model. A failure to satisfy (8a)-(8b) will typically deliver trajectories $\mathbf{x}(t)$ that are mathematically well defined (the simulation can usually be run without any problem) but physically meaningless, as the constraints $\mathbf{c}\left(\mathbf{q}(t)\right)$ will be violated.