Introduction to Balance Equations

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Topic

- In this lecture we will see how some of the basic equations describing the motion of fluids are built
- We will just look at the basic principles & maths involved there
- The goal is that if you encounter that type of modelling in the future, you will hopefully recognise some ideas we will discuss here

Control Volume & Material Volume

"Volumes" are arbitrary shapes in the 3D space

- Material Volume: encloses a specific set of "particles". Think of it as a "balloon" that contains a specific amount of gas. Used to "keep track" of a specific amount of fluid

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We need to describe how volumes move, and evolve, and how their content evolves

Material Derivative

Keeps track of how a certain quantity is evolving in time, as seen from a moving particle

- Imagine particle is moving in 3D, and quantity (e.g. temperature, pressure, etc.) is changing in time
- What changes does the particle see?

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Material derivative over scalar fields:

- Scalar field $\phi(\mathbf{x}, t)$ (e.g. pressure, temperature)
- Particle velocity $\mathbf{v}(t) \in \mathbb{R}^3$
- Material derivative:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \mathbf{x}} \mathbf{v}$$

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Note that $\frac{D}{Dt}$ works as a "total derivative" based *specifically* on the "particles velocity" $\mathbf{v}$

Material derivative over vector fields:

- Vector field $\mathbf{u}(\mathbf{x}, t)$ (e.g. magnetism, stress)
- Material derivative:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{v}$$

Divergence Theorem

- Fundamental theorem of calculus
- Relates integral over a surface ∂V to integral over the volume V contained inside the surface

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Theorem

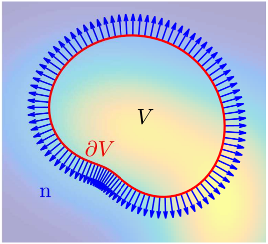
For fields:

$$\int_{\partial V} \phi \mathbf{n} \, \mathrm{d}A$$

where (for 3D)

$$\phi \in \mathbb{R}, \quad \mathbf{n} \in \mathbb{R}^3,$$

and $\mathbf{n}$ is the normal (of norm 1) to the surface ∂V at each of its points



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For fields:

$$\int_{\partial V} \phi \mathbf{n} \, \mathrm{d}A = \int_V \nabla \phi \, \mathrm{d}V$$

where (for 3D)

$$\phi \in \mathbb{R}, \quad \mathbf{n} \in \mathbb{R}^3, \quad \nabla \phi = \begin{bmatrix} \partial \phi / \partial x_1 \\ \partial \phi / \partial x_2 \\ \partial \phi / \partial x_3 \end{bmatrix} \in \mathbb{R}^3$$

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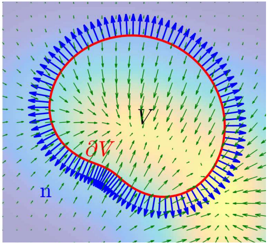

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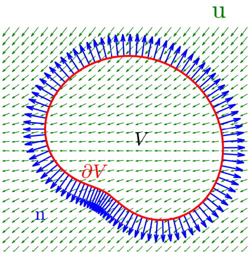

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$$\mathbf{u} \in \mathbb{R}^3, \quad \mathbf{n} \in \mathbb{R}^3, \quad \nabla \circ \mathbf{u} = \frac{\partial \mathbf{u}_1}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{u}_2}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{u}_3}{\partial \mathbf{x}_3} \in \mathbb{R}$$

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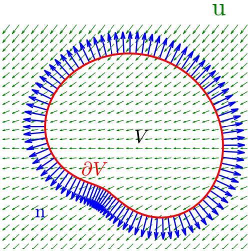

Dilation

- Volume (m^3) contained inside ∂V at a given time t:

$$\mathrm{Vol}(t) = \int_{V(t)} \mathrm{d}V \in \mathbb{R},$$

- $V(t)$ can change shape, dilate, etc.
- Surface $\partial V(t)$ as some "velocity" $\mathbf{v}(x, t)$ for each of its points
- How does $\mathrm{Vol}(t)$ change with time?

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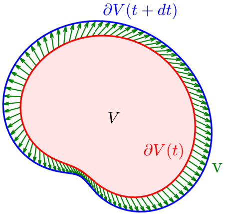

where

- $\mathbf{n}\cdot\mathbf{v}$ provides (at any of its points) the velocity of the surface $\partial V(t)$ *in the direction perpendicular to the surface*
- $\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}A$ is an infinitesimal volume added to $V(t)$ (per time unit) due to the surface moving

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$$\frac{\mathrm{DVol}(t)}{\mathrm{D}t} = \int_{\partial V(t)} \mathbf{n} \cdot \mathbf{v}\, \mathrm{d}A \underbrace{=}_{\text{Div. Theo.}} \int_{V(t)} \nabla \circ \mathbf{v}\, \mathrm{d}V$$

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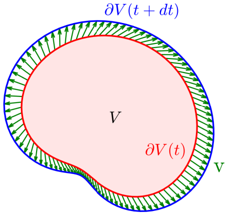

Reynolds transport theorem

- Consider arbitrary field $\phi(x, t) \in \mathbb{R}$, and volume $V(t)$
- How does $\int_{V(t)} \phi \, \mathrm{d}V$ change in time? I.e. what is $\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \phi \, \mathrm{d}V$?
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If V is a control volume (arbitrary):

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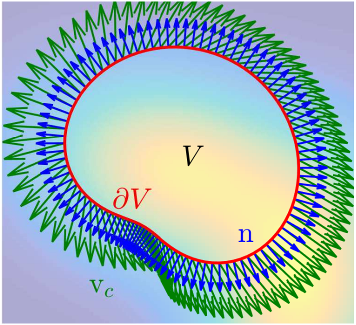
where $\mathbf{v}_\mathrm{c} \in \mathbb{R}^3$ is the velocity of the surface $\partial V(t)$



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If V is a material control volume (moves with the particles):

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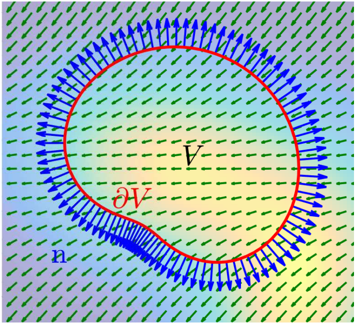
where $\mathbf{v} \in \mathbb{R}^3$ is the velocity of the particles



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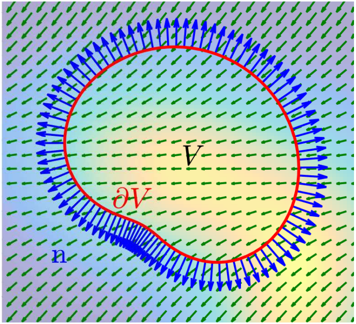

Definition:

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{V(t)} \phi \, \mathrm{d}V = \int_{V(t)} \frac{\partial \phi}{\partial t} \, \mathrm{d}V + \int_{\partial V(t)} \phi \, \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A \qquad \text{for any volume...}$$

Reynolds transport theorem (cont')

Remarks

- If $V(t)$ is a material volume, then it follows that:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V(t)} \phi \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi \,\mathrm{d}V$$

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$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\int_{V(t)} \phi \,\mathrm{d}V &= \int_{V(t)} \left(\frac{\partial \phi}{\partial t} + \nabla \circ (\phi\,\mathbf{v})\right)\mathrm{d}V \\ &= \int_{V(t)} \left(\frac{\mathrm{D}\phi}{\mathrm{D}t} + \phi\,\nabla \circ \mathbf{v}\right)\mathrm{d}V\end{aligned}$$

  where the last equality holds using the definition of $\frac{\mathrm{D}}{\mathrm{D}t}$ and the chain rule

Reynolds transport theorem (cont')

- For modelling, we usually want to work with arbitrary (often fixed) control volume, where particles can enter and leave
- Laws usually hold on material control volumes (following the particles), because that's where conservation principle apply. E.g. the mass contained in a material control volume is constant.
- How to connect an arbitrary control volume to a material control volume?

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Use:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_{\mathrm{c}}(t)}\phi\,\mathrm{d}V = \int_{V_{\mathrm{c}}(t)}\frac{\partial\phi}{\partial t}\,\mathrm{d}V + \int_{\partial V_{\mathrm{c}}(t)}\phi\,\mathbf{v}_{\mathrm{c}}\cdot\mathbf{n}\,\mathrm{d}A \quad \text{(Reynolds for arbitrary control volume)}$$

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Assume $V_\mathrm{c}(t) = V(t)$ *at a specific time* t (not all times), then a substraction results in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\mathrm{c}(t)} \phi \,\mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\mathrm{c}(t)} \phi \,\mathrm{d}V - \int_{\partial V_\mathrm{c}(t)} \phi \,(\mathbf{v}-\mathbf{v}_\mathrm{c})\cdot \mathbf{n}\,\mathrm{d}A$$

Reynolds transport theorem (cont')

If V_{c} is fixed, then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_{\mathrm{c}}(t)} \phi \,\mathrm{d}V = \int_{V_{\mathrm{c}}(t)} \frac{\partial \phi}{\partial t}\,\mathrm{d}V \qquad \text{and} \qquad \mathbf{v}_{\mathrm{c}} = 0$$

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We then have:

$$\boxed{\int_{V_{\mathrm{c}}(t)}\frac{\partial\phi}{\partial t}\,\mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_{V_{\mathrm{c}}(t)}\phi\,\mathrm{d}V - \int_{\partial V_{\mathrm{c}}(t)}\phi\,\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A}$$

These relationships are useful to build the balance laws

Mass balance

- The first balance law we will look at is the mass balance, describing how the mass in the fluid evolves
- We consider

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  which is constant if V is a material control volume (contains a fixed amount of particles)
- Then we have the "principle of conservation":

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{D}}{\mathrm{D}t} \int_V \rho \, \mathrm{d}V = 0$$

Mass balance (cont')

- The Reynolds transport theorem then entails that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V} \rho \,\mathrm{d}V = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \circ (\rho\, \mathbf{v})\right) \mathrm{d}V$$
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$$= \int_{V(t)} \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \, \nabla \circ \mathbf{v} \right) \mathrm{d}V = 0$$

- Because the relationships above hold for any choice of volume $V(t)$, the integrands in the two last integrals must be zero, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla \circ (\rho \, \mathbf{v}) = 0, \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \, \nabla \circ \mathbf{v} = 0$$

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must hold.

- We can recall that by definition of the material derivative:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{v}^\top \nabla \rho$$

where $\frac{\mathrm{D}\rho}{\mathrm{D}t}$ describes the local change in ρ in a material element (element moving with the particles), while $\frac{\partial \rho}{\partial t}$ describes the change in ρ in a "spatial" element (i.e. an arbitrary element fixed in space) due to the particles moving

Mass balance (cont')

An additional integral relationship holds for the mass balance over a fixed control volume. Recall that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\mathrm{c}(t)} \phi \,\mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\mathrm{c}(t)} \phi \,\mathrm{d}V - \int_{\partial V_\mathrm{c}(t)} \phi \,(\mathbf{v} - \mathbf{v}_\mathrm{c}) \cdot \mathbf{n} \,\mathrm{d}A$$

holds for any volume and any ϕ.

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holds for any volume and any ϕ. For V_{c} fixed, $\mathbf{v}_{\mathrm{c}} = 0$ such that for $\phi = \rho$

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Hence

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Hence

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This relationship is useful to relate the changes of mass within a fixed volume V_c to the flow through the surface ∂V_c of that volume.

Continuity equation

Apply Reynolds transportation theorem to $\phi = \rho\psi$ for an arbitrary field ψ:

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\psi \mathrm{d}V &\overset{\text{Reynolds}}{=} \int_V \left(\frac{\mathrm{D}\rho\psi}{\mathrm{D}t} + \rho\psi\nabla\circ\mathbf{v}\right)\mathrm{d}V \\
&\overset{\text{calculus}}{=} \int_V \left(\rho\frac{\mathrm{D}\psi}{\mathrm{D}t} + \psi\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\psi\nabla\circ\mathbf{v}\right)\mathrm{d}V \\
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&\overset{\text{mass balance}}{=} \int_V \rho\frac{\mathrm{D}\psi}{\mathrm{D}t}\mathrm{d}V
\end{aligned}$$

Continuity equation

Apply Reynolds transportation theorem to $\phi = \rho\psi$ for an arbitrary field ψ:

$$
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Hence we have:

$$
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Momentum Balance

Newton's Law element volume $\mathrm{d}V$:

- Mass: $\mathrm{d}m = \rho \mathrm{d}V$
- Velocity $\mathbf{v}$
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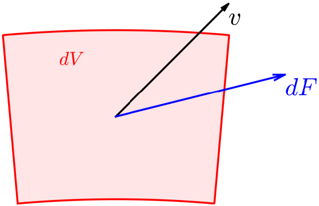

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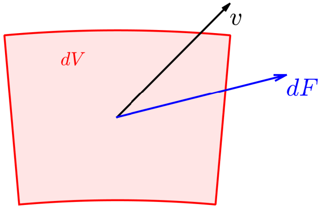

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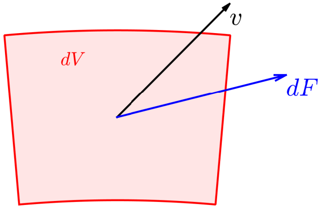

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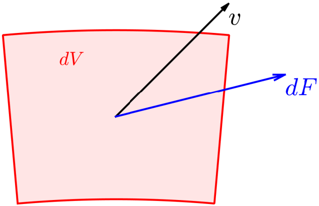

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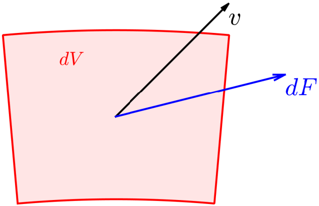

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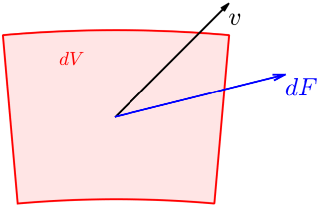

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- Here $\mathbf{F}$ shall be understood as the resultant of all the forces acting on the volume $V(t)$, which includes the forces acting on the surface of the volume (resulting from the pressure on the surface) as well as the "external forces" acting on the particles inside the volume (e.g. from an electric field acting on ionized particles)

Momentum Balance (cont')

$\mathbf{F}$ shall be understood as the resultant of all the forces acting on the volume $V(t)$, i.e.

$$\mathbf{F} = \int_{V(t)} \mathbf{f}\rho \mathrm{d}V - \int_{\partial V(t)} p\mathbf{n}\mathrm{d}A$$

where

- $\mathbf{f}$ is a vector representing a "force per mass unit" acting on an element of volume $\mathrm{d}V$. Its unit is N/kg.
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- It follows that

$$\boxed{\mathbf{F} = \int_{V(t)} \left[\rho\mathbf{f} - \nabla p\right] \mathrm{d}V}$$

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We collected the following relationships:

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- This last relationship is valid for any material control volume $V(t)$, hence:

$$\boxed{\rho(\mathbf{x}, t) \frac{\mathrm{D}\mathbf{v}(\mathbf{x}, t)}{\mathrm{D}t} = \rho \mathbf{f} - \nabla p}$$

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- A material volume has a fixed set of particles
- Particles are the only "energy carriers" (kinetic and potential energy)
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$$\mathrm{d}E = \underbrace{\left(u + \frac{1}{2}\mathbf{v}^\top\mathbf{v} + \phi\right)}_{:=e}\rho\mathrm{d}V$$

where

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The force $\mathbf{f}$ on the particles is often deriving from a potential energy, i.e. $\mathbf{f} = -\nabla\phi$

Energy Balance (cont')

- Energy in volume $V(t)$:

$$E = \int_{V(t)} \rho e \mathrm{d}V$$

- Time variations:

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\int_{\partial V} p\mathbf{v} \cdot \mathbf{n}\,\mathrm{d}A - \int_{\partial V} \mathbf{j}_Q \cdot \mathbf{n}\,\mathrm{d}A$$

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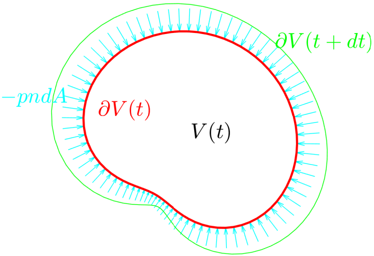
where $-p\mathbf{v}\cdot\mathbf{n}\,\mathbf{d}A$ is the power transferred to volume $V(t)$ through the element of surface $\mathrm{d}A$ due to the surface velocity $\mathbf{v}$. The scalar product $\mathbf{v}\cdot\mathbf{n}$ is the surface velocity in the direction orthogonal to the surface, and $-p\mathbf{n}\mathbf{d}A$ is the force acting on the element of surface $\mathrm{d}A$ in the orthogonal direction.



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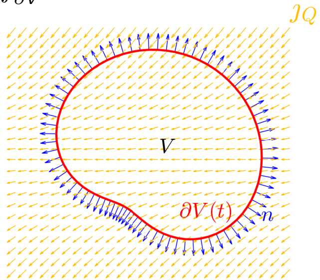
where $\mathbf{j}_Q$ is a vector representing the heat flux in $\mathrm{J/m^2}$. The scalar product $\mathbf{j}_Q \cdot \mathbf{n}$ represents the heat flux in the direction orthogonal to the surface ∂V.



Energy Balance (cont')

Divergence theorem entails that:

$$\int_{\partial V} p\mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A = \int_V \nabla \circ (p\mathbf{v}) \, \mathrm{d}V, \qquad \int_{\partial V} \mathbf{j}_Q \cdot \mathbf{n} \, \mathrm{d}A = \int_V \nabla \circ \mathbf{j}_Q \, \mathrm{d}V$$

And continuity equation entails that:

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Hence

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This last equality is valid for any material volume V, hence:

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