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TTK4135 – Lecture 9
Linear Quadratic Control

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Outline

- Recap: Open-loop linear dynamic optimization problem
- Recap: Three ways of solving this
  - Two batch methods (-> QPs)
  - One recursive method

- Today: Linear Quadratic Control (= “The recursive method”)
  - Finite horizon
  - Infinite horizon

Reference: F&H Ch. 4.3-4.4

Last time: Dynamic open-loop optimization (with linear state-space model)

$$\min_{z \in \mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q_{t+1} x_{t+1} + d_{x,t+1} x_{t+1} + \frac{1}{2} u_t^\top R_t u_t + d_{u,t} u_t + \frac{1}{2} \Delta u_t^\top S \Delta u_t$$

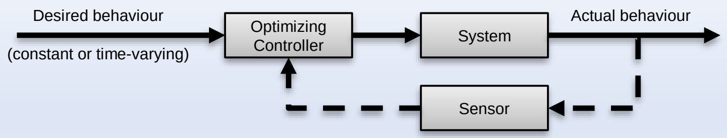
subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0, \dots, N-1\} \\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1, \dots, N\} \\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
-\Delta u^{\text{high}} &\leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0, \dots, N-1\} \\
Q_t &\succeq 0, \quad t = \{1, \dots, N\} \\
R_t &\succ 0, \quad t = \{0, \dots, N-1\}
\end{aligned}$$

where

$$\begin{aligned}
x_0 & \text{ and } u_{-1} \text{ is given} \\
\Delta u_t &:= u_t - u_{t-1} \\
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u)
\end{aligned}$$

Open-loop vs closed-loop/feedback:

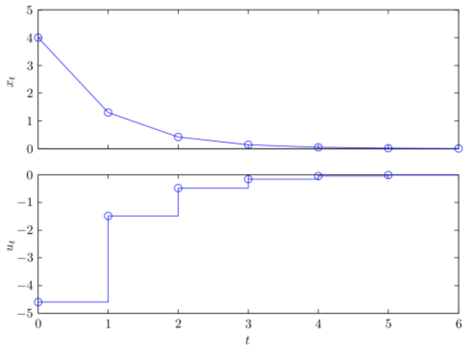


The significance of weigths

$$\min \sum_{t=0}^{5} q\, x_{t+1}^2 + r\, u_t^2$$

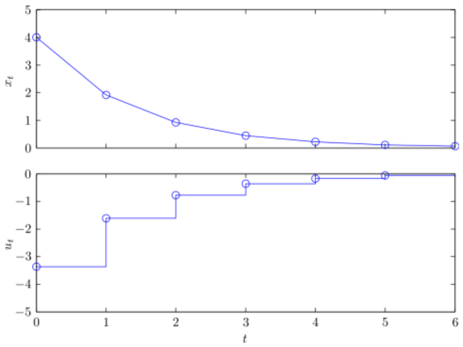
$$\text{s.t.} \qquad x_{t+1} = 0.9x_t + 0.5u_t, \quad t = 0, \ldots, 5$$

$q = 5,\ r = 1$



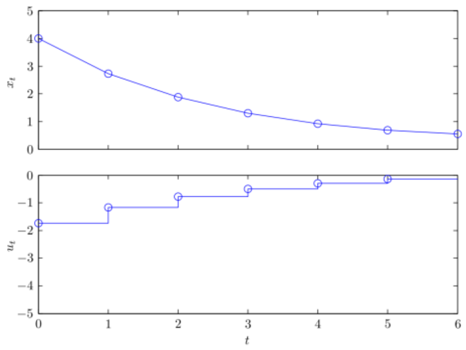
$$\sum_{t=0}^{N-1} x_{t+1}^2 = 1.9, \qquad \sum_{t=0}^{N-1} u_t^2 = 23.6$$

$q = 2,\ r = 1$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 4.8, \qquad \sum_{t=0}^{N-1} u_t^2 = 14.7$$

$q = 1,\ r = 2$



$$\sum_{t=0}^{N-1} x_{t+1}^2 = 14.3, \qquad \sum_{t=0}^{N-1} u_t^2 = 5.3$$

Linear quadratic control: Dynamic optimization without constraints

$$\min_{z} \sum_{t=0}^{N-1} x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$

$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

Three approaches for solution:

- Batch approach v1, "full space" – solve as QP
- Batch approach v2, "reduced space" – solve as QP
- Recursive approach – solve as linear state feedback

Also work with input- and state constraints!

Only work without constraints!

Linear Quadratic Control
Batch approach v1, "Full space" QP

- Formulate with model as equality constraints, all inputs and states as optimization variables

$$
\begin{aligned}
\min_z \; & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.} \; & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1 \\
& z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top
\end{aligned}
$$

$$
\left\{
\begin{aligned}
\min_z \; & \frac{1}{2} z^\top \begin{pmatrix} R & & & & \\ & Q & & & \\ & & R & & \\ & & & \ddots & \\ & & & & Q \end{pmatrix} z \\
\text{s.t.} \; & \begin{pmatrix} -B & I & & & & & \\ & -A & -B & I & & & \\ & & & -A & -B & I & \\ & & & & \ddots & \ddots & \\ & & & & -A & -B & I \end{pmatrix} z = \begin{pmatrix} A x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
& z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top
\end{aligned}
\right.
$$

Linear Quadratic Control
Batch approach v2, "Reduced space" QP

$$\min_z \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$
$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

- Use model to eliminate states as variables
  - Future states as function of inputs and initial state

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{pmatrix} x_0 + \begin{pmatrix} B & & & & \\ AB & B & & & \\ A^2 & AB & B & & \\ \vdots & \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \dots & B \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = S^x x_0 + S^u U$$

  - Insert into objective (no constraints!)

$$\min_U \frac{1}{2} (S^x x_0 + S^u U)^\top \mathbf{Q} (S^x x_0 + S^u U) + \frac{1}{2} U^\top \mathbf{R} U$$

$$\mathbf{Q} = \begin{pmatrix} Q & & \\ & Q & \\ & & \ddots \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R & & \\ & R & \\ & & \ddots \end{pmatrix}$$

  - Solution found by setting gradient equal to zero:

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -\left((S^u)^\top \mathbf{Q} S^u + \mathbf{R}\right)^{-1} (S^u)^\top \mathbf{Q} S^x x_0 = -F x_0$$

Linear Quadratic Control
Recursive approach

$$\min_{z} \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$
$$z = (u_0, x_1, u_1, \ldots, u_{N-1}, x_N)^\top$$

- By writing up the KKT-conditions, we can show (we will do this today) that the solution can be formulated as:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned}
K_t &= R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, && t = 0, \ldots, N-1 \\
P_t &= Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, && t = 0, \ldots, N-1 \\
P_N &= Q
\end{aligned}$$

Comments to the three solution approaches

- All give same numerical solution
  - If problem is strictly convex (Q psd, R pd), solution is unique
- The batch approaches give an open-loop solution, the recursive approach give a closed-loop (feedback) solution

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = -Fx_0 \qquad \text{vs} \qquad u_t = -K_t x_t$$

- Constraints on inputs and states:
  - Easy for batch approaches (both becomes convex QPs)
  - Difficult for the recursive approach

- How to to add feedback (and thereby robustness) to batch approaches?
  - Model predictive control! (Next time)

Today: The recursive solution (LQ control)

KKT Conditions (Thm 12.1)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} c_i(x) = 0, & i \in \mathcal{E}, \\ c_i(x) \geq 0, & i \in \mathcal{I}. \end{array}$$

Lagrangian: $$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

KKT-conditions (First-order necessary conditions): If x^* is a local solution and LICQ holds, then there exist λ^* such that

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && \text{(stationarity)} \\ c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, && \\ c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(primal feasibility)} \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(dual feasibility)} \\ \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. && \text{(complementarity condition/ complementary slackness)} \end{aligned}$$

LQR:

$$\min_{z} \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1$$
$$z = (u_0, x_1, u_1, \dots, u_{N-1}, x_N)^\top$$

KKT:

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\
c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\
c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\
\lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\
\lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}.
\end{aligned}$$

Second-order conditions

Theorem 12.6 (Second-Order Sufficient Conditions).

Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions (12.34) are satisfied. Suppose also that

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\, w \neq 0. \tag{12.65}$$

Then x^* is a strict local solution for (12.1).

- Critical directions:

$$w \in \mathcal{C}(x^*, \lambda^*) \;\Leftrightarrow\; \begin{cases} \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{E}, \\ \nabla c_i(x^*)^T w = 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0, \\ \nabla c_i(x^*)^T w \geq 0, & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0. \end{cases} \tag{12.53}$$

- The critical directions are the “allowed” directions where it is not clear from KKT-conditions whether the objective will decrease or increase

Thm 16.4: For convex QP, KKT is sufficient

- From N&W, p. 464:

KKT conditions

> For convex QP, when G is positive semidefinite, the conditions (16.37) are in fact sufficient for x^* to be a global solution, as we now prove.
>
> **Theorem 16.4.**
>
> *If x^* satisfies the conditions (16.37) for some λ_i^*, $i \in \mathcal{A}(x^*)$, and G is positive semidefinite, then x^* is a global solution of (16.1).*

- That is: Since the solution of the Riccati equation implies the KKT conditions are fulfilled, Thm 16.4 means that the Riccati equation gives the global solution

  – Side-remark: It is, in fact, the *unique* global solution. If G is positive definite (implied by Q positive definite), this follows from the proof of Thm 16.4. If Q positive semidefinite, further arguments are necessary (for instance using Thm 12.6 as in the note).

- Finite horizon LQ controller

$$\begin{aligned}
\min_{z\in\mathbb{R}^n} f(z) &= \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t \\
\text{subject to} \quad x_{t+1} &= A_t x_t + B_t u_t, \quad t = 0,\dots,N-1 \\
x_0 &= \text{given} \\
Q_t &\succeq 0 \quad t = 1,\dots,N \\
R_t &\succ 0 \quad t = 0,\dots,N-1
\end{aligned}$$

where

$$\begin{aligned}
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
n &= N \cdot (n_x + n_u)
\end{aligned}$$

- State feedback solution

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned}
K_t &= R_t^{-1}B_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0,\dots,N-1 \\
P_t &= Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, && t = 0,\dots,N-1 \\
P_N &= Q_N
\end{aligned}$$

(discrete) Riccati equation

Linear quadratic control (finite horizon)

- The optimal solution to LQ control is a linear, time-varying state feedback:

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned} K_t &= R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

- Note that the gain matrix K_t is independent of the states, and can therefore be computed in advance (knowing A_t, B_t, Q_t, R_t)

- The matrix (difference) equation

$$\begin{aligned} P_t &= Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad && t = 0, \dots, N-1 \\ P_N &= Q_N \end{aligned}$$

is called the (discrete) *Riccati equation*

- Note that the “boundary condition” is given at the end of the horizon, and the P_t -matrices must be found iterating backwards in time

Example

$$u_t = -K_t x_t$$

where the feedback gain matrix is derived by

$$\begin{aligned}
K_t &= R^{-1}B^\top P_{t+1}(I + BR^{-1}B^\top P_{t+1})^{-1}A, && t = 0, \ldots, N-1 \\
P_t &= Q + A^\top P_{t+1}(I + BR^{-1}B^\top P_{t+1})^{-1}A, && t = 0, \ldots, N-1 \\
P_N &= Q
\end{aligned}$$

Example

$$\min \sum_{t=0}^{10} \frac{1}{2}x_{t+1}^2 + \frac{1}{2}r\ u_t^2$$

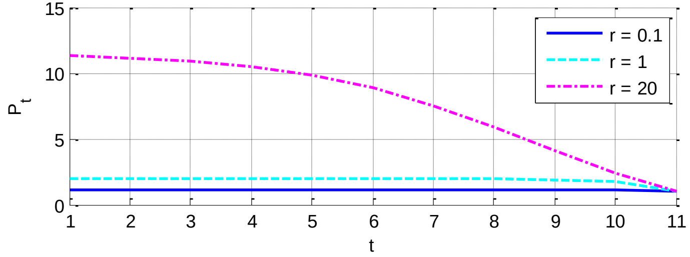
$$\text{s.t.} \qquad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1, \dots, 10$$

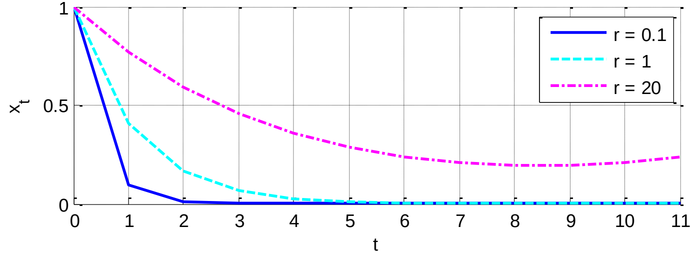
LQ solution

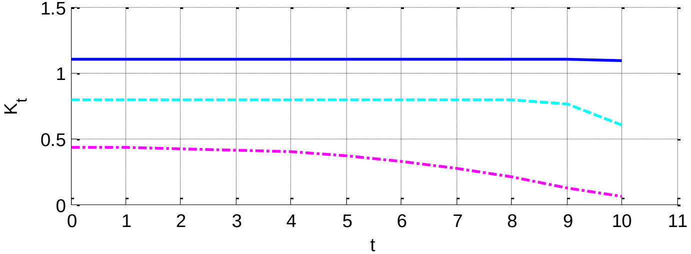
$$P_t = 1 + \frac{1.44rP_{t+1}}{P_{t+1} + r}, \quad t = 10, \dots, 1$$

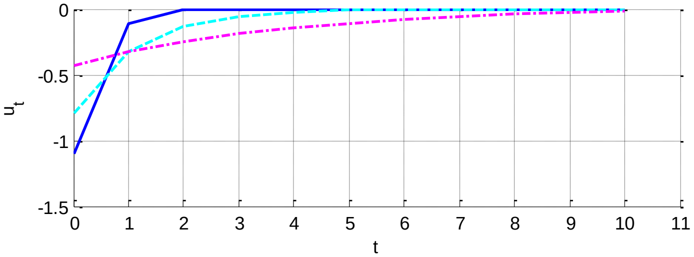
$$P_{11} = 1$$

$$K_t = 1.2\frac{P_{t+1}}{P_{t+1} + r}, \quad t = 0, \dots, 10$$





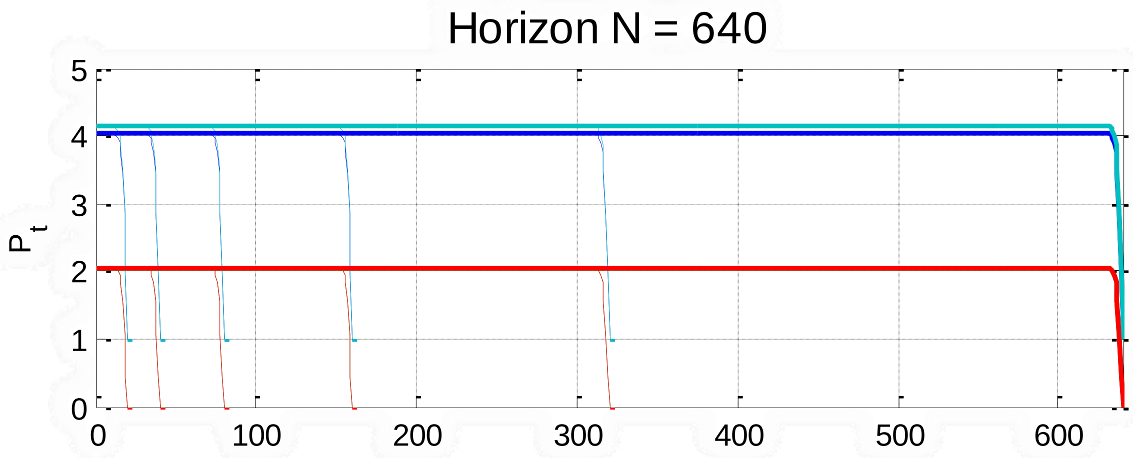




Increasing LQ horizon

$$\min \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1$$

$$A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.125 \\ 0.5 \end{pmatrix}, \quad Q = I, \quad R = 1.$$



Infinite horizon LQ control

$$\min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$
$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots$$

Fact: Steady-state ($P_{t+1} = P_t$) backwards-in-time solution of Riccati equation is infinite horizon solution

$$u_t = -K_t x_t$$

where

$$K_t = R^{-1} B^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad t = 0, \dots, N-1$$
$$P_t = Q + A^\top P_{t+1} (I + B R^{-1} B^\top P_{t+1})^{-1} A, \qquad t = 0, \dots, N-1$$
$$P_N = Q$$

$\Rightarrow$

$$u_t = -K x_t$$

where

$$K = R^{-1} B^\top P (I + B R^{-1} B^\top P)^{-1} A$$
$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A$$

Infinite horizon LQ control

Theorem: The solution (when one exists) to

$$\begin{aligned} \min \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\ \text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots \end{aligned}$$

is given by

$$u_t = -K x_t$$

where

$$\begin{aligned} K &= R^{-1} B^\top P (I + B R^{-1} B^\top P)^{-1} A \\ P &= Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A \end{aligned}$$

(Discrete-time Algebraic Riccati Equation, DARE)

Two central questions:

- When does a solution exist?
- When is the closed-loop stable?

Controllability vs stabilizability
Observability vs detectability

- Stabilizable: All unstable modes are controllable
  (that is: all uncontrollable modes are stable)

- Detectability: All unstable modes are observable
  (that is: all unobservable modes are stable)

- Controllability implies stabilizability
- Observability implies detectability

Riccati equations

- Discrete-time Riccati equation in the note (and lecture)

$$P_t = Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \quad P_N = Q_N$$

- However, another, equivalent, form is found in other sources:

$$P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (R_t + B_t^\top P_{t+1} B_t)^{-1} B_t^\top P_{t+1} A_t, \quad P_N = Q_N$$

- The latter is more numerically stable due to “enforced symmetry”

- The trick used to get the different formulas is the “Matrix Inversion Lemma” (a very useful Lemma in control theory, optimization, …)

- Discrete-time Algebraic Riccati equation (DARE) in the note (and lecture)

$$P = Q + A^\top P (I + B R^{-1} B^\top P)^{-1} A$$

- Equivalent form (e.g. Matlab)

$$P = Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A$$

  – Note: This is a quadratic equation with two solutions. The one we want is the positive definite solution (the “stabilizing” solution).

```
>> help dare
dare  Solve discrete-time algebraic Riccati equations.

   [X,L,G] = dare(A,B,Q,R,S,E) computes the unique
           stabilizing solution X of the discrete-time
           algebraic Riccati equation
```