NTNU – Trondheim
Norwegian University of
Science and Technology

TTT4120 Digital Signal Processing
Fall 2019

Lecture: The Sampling Theorem

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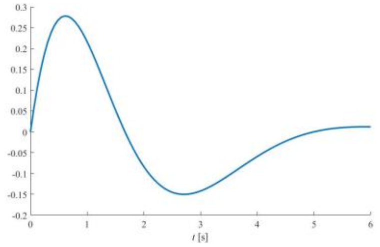
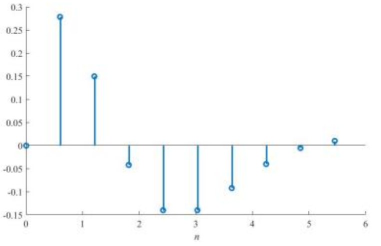
Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 1.4.2 The sampling theorem
  - 1.4.6 Digital to analog conversion
  - 6.1 Ideal sampling and reconstruction of continuous-time signals

*Level of detail is defined by lectures and problem sets

Preliminary questions



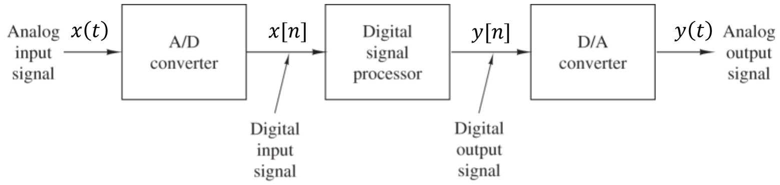


- How fast must we sample the continuous-time signal (left) without losing information?
- What continuous-time signal corresponds to the discrete-time signal (right)?

Contents and learning outcomes

- Sampling of sinusoids and aliasing (partially covered Lect.1)
- Sampling theorem:
  – Ideal reconstruction of continuous-time signals
- Wagon wheel effect

Periodic sampling



- Sampling – Processing – Reconstruction
- A signal is read (sampled) at a regular interval

$$x[n] = x(t_n) = x(nT) = x\left(\frac{n}{F_s}\right)$$

- Sampling interval $T = \frac{1}{F_s}$, F_s being the sampling frequency

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Periodic sampling…

- Examples of sampling rate standards:
  – CD audio: $F_s = 44.1$ kHz
  – TV frame rate: $F_s = 100, 200, 400$ fr/s

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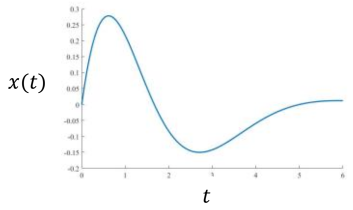
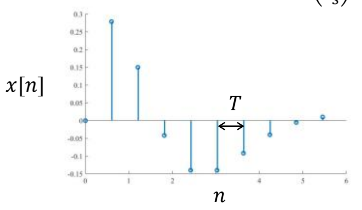
Periodic sampling…

$$x[n] = x(t_n) = x(nT) = x\left(\frac{n}{F_s}\right)$$





- Under what conditions is $x[n]$ a good representation of $x(t)$?
  - Appropriate choice of T or F_s
- Under what conditions can $x(t)$ be recovered from $x[n]$?
  - Interpolation formula is needed
- Conditions are provided by the sampling theorem

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Sampling of sinusoids and aliasing

- Why considering sinusoidal signals?
- Many practical signals can be represented by the Fourier transform (or Fourier series)

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$
$$= \int_{-\infty}^{\infty} X(F)(\cos 2\pi Ft + j \sin 2\pi Ft) dF$$

- The concepts of sampling a single sinusoidal signal carry over to the case of more complicated signals.

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Sampling of sinusoids and aliasing…

- Consider the continuous-time signal

$$x(t) = \cos \Omega t = \cos 2\pi F t$$

with angular frequency Ω [rad/s], or frequency F [Hz]

- Periodic sampling at regular time intervals $t_n = nT = 1/F_s$

$$x[n] \equiv x(t_n) = \cos 2\pi F n T = \cos 2\pi \frac{F}{F_s} n = \cos \underbrace{2\pi f}_{\omega} n$$

- Spectrum of digital signal is periodic with period $\omega = 2\pi$ (or $f = 1$), where $f = 1/2$ represents the highest frequency

$$\therefore\ f = \frac{F}{F_s} \leq \frac{1}{2}$$

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Sampling of sinusoids and aliasing…

- Example 1: Sample signal $x(t) = \cos 2\pi 400 t$ at F_s=1000

$$\begin{aligned} x[n] &= \cos 2\pi 400 n T \\ &= \cos 2\pi \tfrac{400}{1000} n = \cos 2\pi (0.4 + k)\, n \end{aligned}$$

- Spectrum of sampled signal $X(f)$ obtained directly from

$$x[n] = \tfrac{1}{2}\left(e^{j2\pi(0.4+k)n} + e^{-j2\pi(0.4+k)n}\right)$$

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Sampling of sinusoids and aliasing…

- Example 2: Sample signal $x(t) = \cos 2\pi 400t + \cos 2\pi 800t$ at F_s=1000

$$x[n] = \cos 2\pi 400nT + \cos 2\pi 800nT$$
$$= \cos 2\pi(0.4 + k)n + \cos 2\pi(\underbrace{0.8}_{1-0.2} + k)\, n$$
$$= \cos 2\pi(0.4 + k)n + \cos 2\pi(-0.2 + k)\, n$$

- Spectrum of sampled signal $X(f)$ obtained directly from:

$$x[n] = \frac{1}{2}(e^{j2\pi 0.4n} + e^{-j2\pi 0.4n} + e^{j2\pi 0.2n} + e^{-j2\pi 0.2n})$$

- Distortion: highest analog frequency (800 Hz) appears as low-frequency component in digital spectrum (200 Hz)

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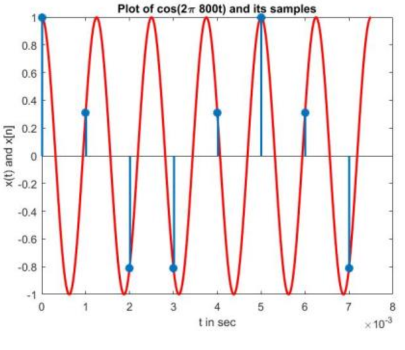
Sampling of sinusoids and aliasing…

- Example 2 (cont.): Simultaneous plot of $\cos 2\pi 800t$ and its samples when $F_s = 1000$



- Samples appear to be from $\cos 2\pi 200t$

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Sampling of sinusoids and aliasing…

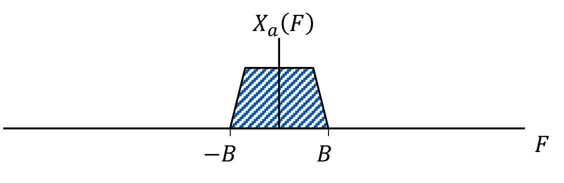
- To avoid aliasing (folding of high frequency components around $f = 1/2$, the following condition must be satisfied (Lecture 1)

$$\frac{F}{F_s} \leq \frac{1}{2}, \forall F \Rightarrow F_s \geq 2F_{\max}$$

- We shall see that any bandlimited continuous-time signal can be reconstructed if sampled above the Nyquist rate $2F_{\max}$

Ideal reconstruction of continuous-time signals

- Bandlimited signal:



A signal is bandlimited if there exists a finite frequency B (or Ω_B) such that $X_a(F)$ (or $X_a(\Omega)$) is zero for $F > B$ (or $\Omega > \Omega_B$). The frequency $B = \Omega_B/2\pi$ is called the signal bandwidth in Hz.

Ideal reconstruction of continuous-time …

- Sampling Theorem:

A bandlimited analog signal $x_a(t)$ can be reconstructed from its sample values $x[n] = x_a(nT)$ if the signal is sampled at rate

$$F_s = \frac{1}{T} \geq 2F_{\max} = 2B,$$

where $F_{max} = B$ is the highest frequency contained in $x_a(t)$. Otherwise aliasing would result in $x[n]$.

- Sampling rate $F_{\mathrm{N}} = 2F_{\max}$ is called the Nyquist rate
- Highest analog frequency represented in $x[n]$ is $\frac{F_s}{2}$

Ideal reconstruction of continuous-time …

- Example: What is the Nyquist rate for the following signals?

$$x_1(t) = \cos 2\pi 400t + \cos 2\pi 800t$$

$$x_2(t) = \cos 100\pi t + 3\cos 200\pi t$$

$$x_3(t) = \cos 150\pi t + 10\sin(600\pi t + \theta)$$

- Can the signals be sampled at rate F_{N} without problems?

Ideal reconstruction of continuous-time …

- Continuous-time signal: $x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF$

  $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$

- Discrete-time signal: $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df$

  $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$

- Relationship between f and F

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df = x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi FnT} dF$$

$$= \sum_{k=-\infty}^{\infty} \int_{(k-1)\, F_s/2}^{(k+1)\, F_s/2} X_a(F) e^{j2\pi FnT} dF$$

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Ideal reconstruction of continuous-time …

- Make use following relations

$$f = \frac{F}{F_s},\ e^{j2\pi FnT} = e^{\frac{j2\pi F}{F_s}n} = e^{\frac{j2\pi n}{F_s}(F-kF_s)}$$

- Then, we can manipulate the former expression into

$$\frac{1}{F_s}\int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X(F) e^{\frac{j2\pi nF}{Fs}} dF = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \sum_{k=-\infty}^{\infty} X_a(F-kF_s)\, e^{\frac{j2\pi F}{Fs}n} dF$$

- Relation between sampled and analog spectra

$$\boxed{\begin{array}{c} X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F-kF_s), \text{ or} \\ X(f) = F_s \sum_{k=-\infty}^{\infty} X_a([f-k]F_s) \end{array}}$$

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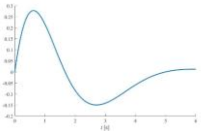
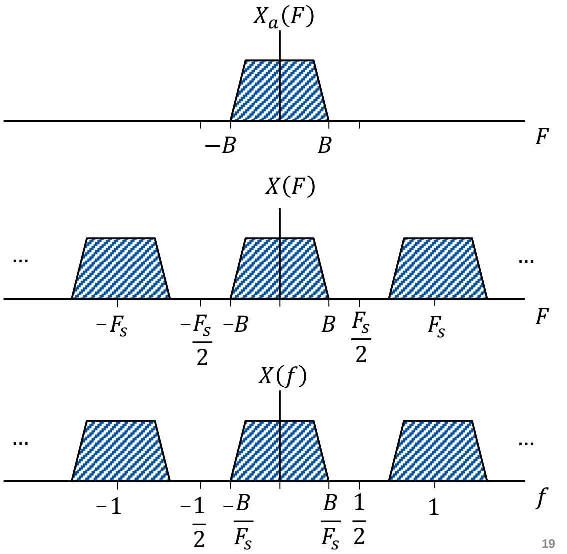
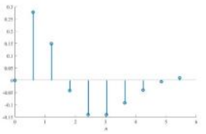
Ideal reconstruction of continuous-time …

- No aliasing







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Ideal reconstruction of continuous-time …

- If discrete-time signal $x[n]$ has no aliasing in spectrum $X(F)$

$$X_a(F) = \begin{cases} \frac{1}{F_s} X(F), & |F| \leq \frac{F_s}{2} \\ 0, & |F| > \frac{F_s}{2} \end{cases}$$

- Analog signal can be reconstructed from samples $x[n]$

$$x_a(t) = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X_a(f) e^{j2\pi Ft} dF = \frac{1}{F_s} \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X(F) e^{j2\pi Ft} dF = \cdots$$

$$\boxed{= \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\pi}{T}[t-nT]\right)}{\frac{\pi}{T}[t-nT]} = \sum_{n=-\infty}^{\infty} x[n] g[t-nT]}$$

- Interpolation function is a *sinc* function

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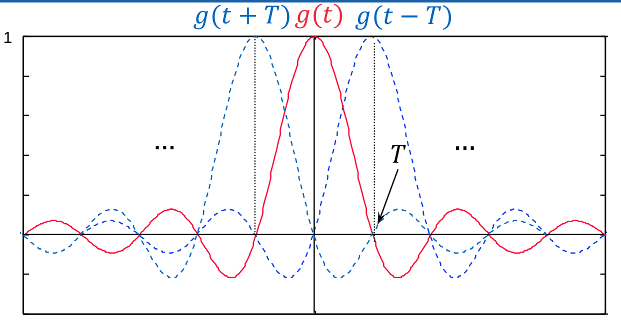
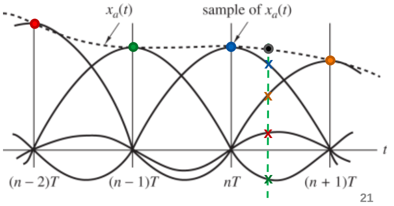
Ideal reconstruction of continuous-time …



$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\pi}{T}[t-nT]\right)}{\frac{\pi}{T}[t-nT]}$$



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Ideal reconstruction of continuous-time …

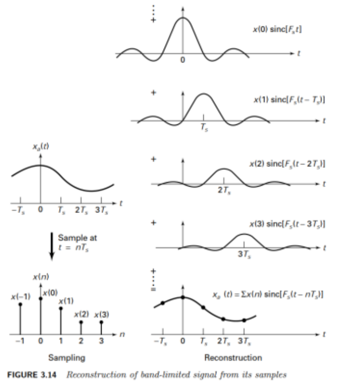

FIGURE 3.14 *Reconstruction of band-limited signal from its samples*

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]g[t-nT]$$

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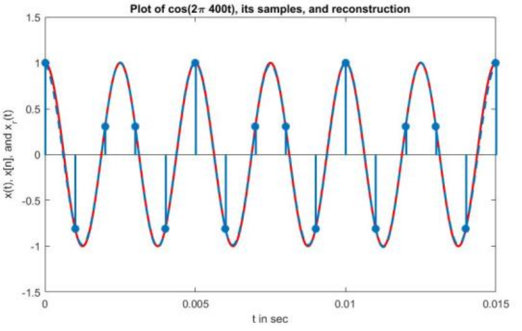
Ideal reconstruction of continuous-time …

- Revisiting Example 1: Simultaneous plot of $\cos 2\pi 400t$, its samples, and reconstruction when $F_s = 1000$



- Perfect reconstruction is possible

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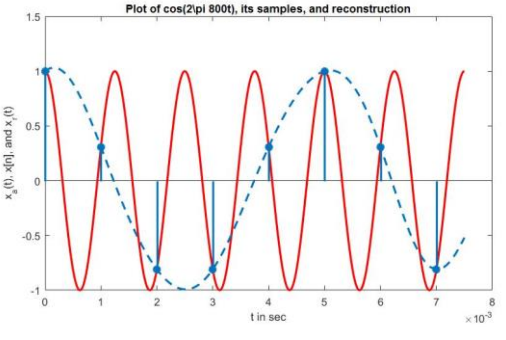
Ideal reconstruction of continuous-time …

- Revisiting Example 2: Simultaneous plot of $\cos 2\pi 800t$, its samples, and reconstruction when $F_s = 1000$



- Reconstruction of folded signal component $\cos 2\pi 200t$

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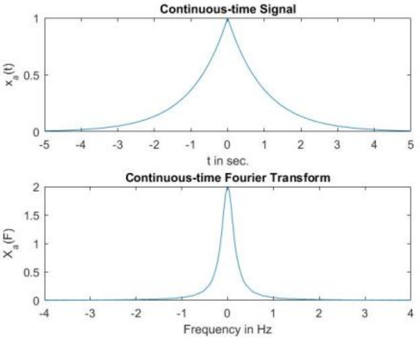
Ideal reconstruction of continuous-time …

- Example 3: Sample $x_a(t) = e^{-|t|}$ at rates $F_{s_1} = 5$ and $F_{s_2} = 1$.



$$X(F) = \frac{2}{1 + (2\pi F)^2}$$

- How about spectra $X(F)$ and $X(f)$ for the two sampling rates? Sketch and draw conclusions about the reconstructed signals?

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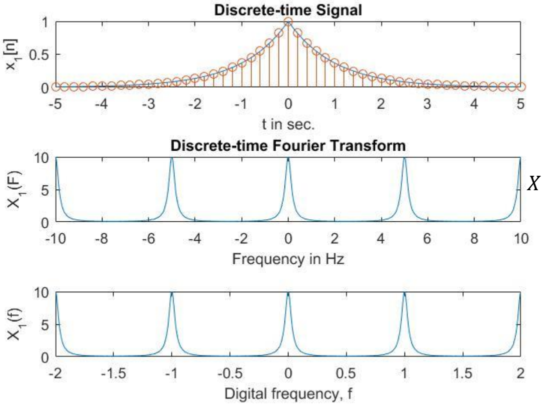
Ideal reconstruction of continuous-time …

- Example 3 (cont.): $x_1[n] = e^{-|n|T_1} = e^{-\frac{|n|}{F_{s_1}}}, F_{s_1} = 5$



$$X\left(\frac{F}{F_s}\right) = \frac{1 - e^{\frac{-2}{F_s}}}{\left(1 - 2e^{-\frac{n}{F_s}}\cos\frac{2\pi F}{F_s} + e^{\frac{-2}{F_s}}\right)}$$

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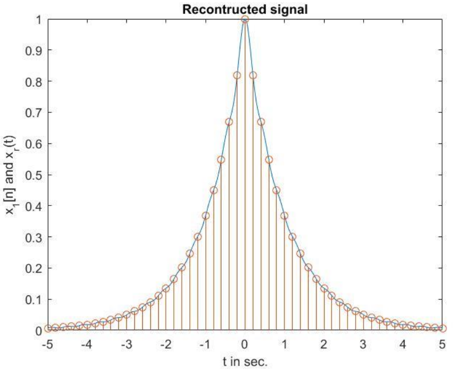
Ideal reconstruction of continuous-time …

- Example 3 (cont.): $x_1[n] = e^{-nT_1} = e^{-\frac{n}{F_{S1}}}, F_{s_1} = 5$



- Excellent reconstruction.

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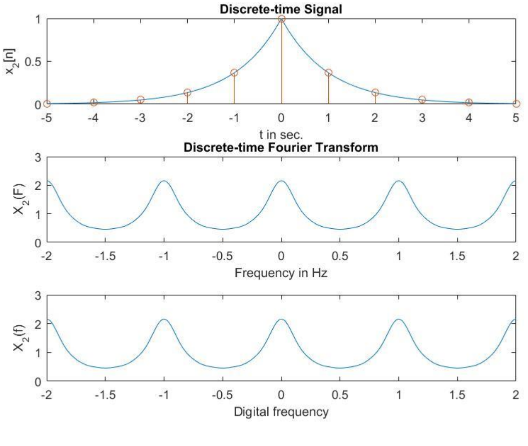
Ideal reconstruction of continuous-time …

- Example 3 (cont.): $x_2[n] = e^{-nT_2} = e^{-\frac{n}{F_{S2}}}, F_{s_2} = 1$



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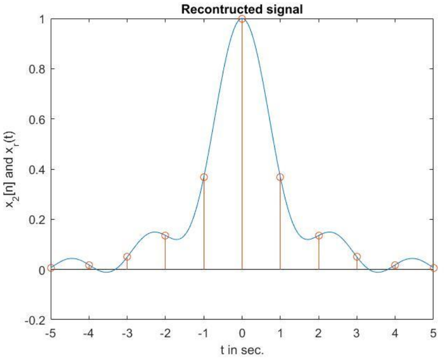
Ideal reconstruction of continuous-time …

- Example 3 (cont.): $x_2[n] = e^{-nT_2} = e^{-\frac{n}{F_{s2}}}, F_{s_2} = 1$



- Reconstructed signal quite different from actual one (aliasing).

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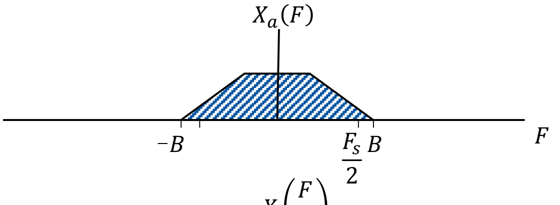
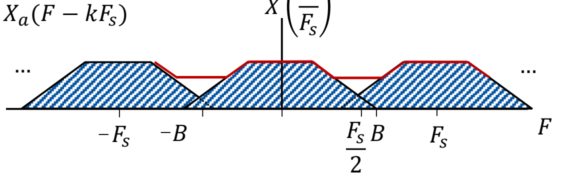
Ideal reconstruction of continuous-time …

- Aliasing



$$X(F/F_s) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$



- Interpolation will produce $\hat{x}_a(t)$ corresponding to aliased spectrum
- Prefilter $x_a(t)$ to limit bandwidth before sampling

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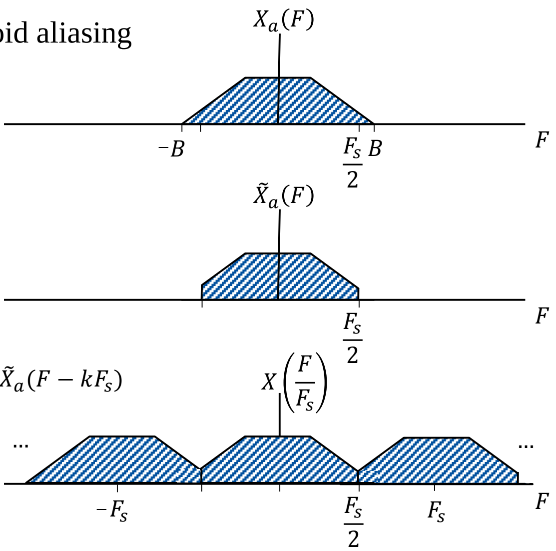
Ideal reconstruction of continuous-time …

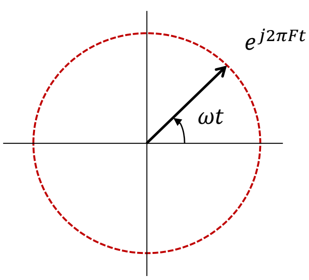
- Prefilter to avoid aliasing



$$X(F/F_s) = F_s \sum_{k=-\infty}^{\infty} \tilde{X}_a(F - kF_s)$$

Example: Wagon wheel effect

- Illusion of a wheel spinning in wrong direction
- Imagine phasor rotating at angular speed $\omega = 2\pi F$ rad/sec

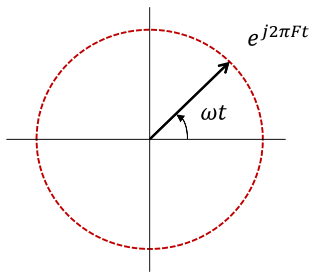


- Starting at $t = 0$, take a snapshot every T seconds, i.e., $nT = \frac{n}{F_s}$
- Find values of T such that the sampled phasor appears to rotate in clockwise direction rather than counter-clockwise?

Example: Wagon wheel effect…

- Demo on ItsLearning:

    `sampling_rotating_phasor.m`



Matlab

```
F = 1; Fs = 5; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);

F = 1; Fs = 4; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);

F = 1; Fs = 1.3; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);
```


Summary

Today:

- Sampling of analog and aliasing
- Sampling theorem
- Ideal reconstruction of analog signals

Next:

- Sampling in frequency domain: Discrete Fourier Transform