TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 6

Hand-out time:	Monday, October 28, 2013, at 12:00
Hand-in deadline:	Friday, November 8, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Kalman filter

Consider a plant where the relation between the input $i(t)$ and the output $o(t)$ is given by the transfer function

$$g(s) = \frac{o(s)}{i(s)} = \frac{10}{s+5}.$$

The input $i(t)$ consists of the control input $u(t)$ and an input disturbance $d(t)$ and is given by

$$i(t) = 2u(t) + d(t).$$

The disturbance $d(t)$ is the output of the following Wiener process:

$$d(t) = 5\int_0^t w(\tau)d\tau,$$

where $w(t)$ is Gaussian white noise. The output $o(t)$ of the plant is measured. The corresponding output measurement is given by

$$y(t) = o(t) + v(t),$$

where $v(t)$ is Gaussian white noise.

a) Draw a block diagram of the system.

b) Show that the system can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Hv(t), \end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$.

We use Euler discretization to discretize the obtained system. The corresponding discretized system is given by

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k, \\ y_k &= \mathbf{C}_d\mathbf{x}_k + H_d v_k, \end{aligned} \tag{1}$$

with matrices

$$\mathbf{A}_d = \mathbf{I} + \Delta t \mathbf{A}, \quad \mathbf{B}_d = \Delta t \mathbf{B}, \quad \mathbf{G}_d = \Delta t \mathbf{G}, \quad \mathbf{C}_d = \mathbf{C} \quad \text{and} \quad H_d = H,$$

where $\mathbf{I}$ is the identity matrix and $\Delta t = \frac{1}{5}$ denotes the sampling time. Note that the time index k of the discretized system corresponds to the time $t = k\Delta t$, i.e. $\mathbf{x}_k = \mathbf{x}(t)$ for $t = k\Delta t$.

c) Calculate the matrices $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{G}_d$, $\mathbf{C}_d$ and H_d.

d) Check if the obtained discrete-time system (1) is observable.

We will design a Kalman filter for the discrete-time system (1). We assume the Gaussian white noise processes w_k and v_k have a zero mean and are uncorrelated. Moreover, we assume that the variances of w_k and v_k are given by $Q = 3$ and $R = 1$, respectively. Summarizing, we assume that the following holds:

$$\begin{aligned}
E[w_k] &= 0, \quad \text{for all } k, \\
E[v_k] &= 0, \quad \text{for all } k, \\
E[w_k w_l] &= \begin{cases} Q = 3, & \text{if } k = l, \\ 0, & \text{otherwise}, \end{cases} \\
E[v_k v_l] &= \begin{cases} R = 1, & \text{if } k = l, \\ 0, & \text{otherwise}, \end{cases} \\
E[w_k v_l] &= 0, \quad \text{for all } k \text{ and } l.
\end{aligned}$$

Let the (a priori) state estimate be denoted by $\hat{\mathbf{x}}_k^-$. We define the estimation error to be

$$\mathbf{e}_k^- = \mathbf{x}_k - \hat{\mathbf{x}}_k^-.$$

The associated error covariance matrix is

$$\mathbf{P}_k^- = E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T].$$

We use the measurement y_k to update the state estimate $\hat{\mathbf{x}}_k^-$ according to the following update law:

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-), \tag{2}$$

where $\hat{\mathbf{x}}_k$ is the updated state estimate and $\mathbf{K}_k$ is the Kalman gain (yet to be determined). Now, we define the updated estimation error $\mathbf{e}_k$ and the associated updated error covariance matrix $\mathbf{P}_k$ as

$$\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$$

and

$$\mathbf{P}_k = E[\mathbf{e}_k \mathbf{e}_k^T] = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]. \tag{3}$$

e) Use the definition of $\mathbf{P}_k$ in (3) and the update law (2) to write $\mathbf{P}_k$ as a function of $\mathbf{P}_k^-$, $\mathbf{K}_k$, $\mathbf{C}_d$, H_d and R.

f) Use the equation

$$\frac{d(\text{trace } \mathbf{P}_k)}{d\mathbf{K}_k} = -2\mathbf{C}_d\mathbf{P}_k^- + 2(\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2)\mathbf{K}_k^T = \mathbf{0}^T$$

to find an expression for the Kalman gain $\mathbf{K}_k$ as a function of $\mathbf{P}_k^-$, $\mathbf{C}_d$, H_d and R.

To find the a priori state estimate $\hat{\mathbf{x}}_{k+1}^-$ for the state $\mathbf{x}_{k+1}$ at time index $k+1$, we use the state equation $\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k$. We set $\hat{\mathbf{x}}_{k+1}^- = E[\mathbf{x}_{k+1}] = E[\mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k]$.

g) Find an expression for $\hat{\mathbf{x}}_{k+1}^-$ as a function of $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{G}_d$, Q, $\mathbf{P}_k$, $\hat{\mathbf{x}}_k$ and u_k. Note that the best available estimate for $\mathbf{x}_k$ is $\hat{\mathbf{x}}_k$, i.e. $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k$, and that u_k is a known input.

h) Use the state equation $\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k$ and the expression for $\hat{\mathbf{x}}_{k+1}^-$ found in g) to find an expression for $\mathbf{P}_{k+1}^- = E[\mathbf{e}_{k+1}^-\mathbf{e}_{k+1}^{-T}] = E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T]$ as a function of $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{G}_d$, Q, $\mathbf{P}_k$, $\hat{\mathbf{x}}_k$ and u_k.

Consider the initial conditions

$$\hat{\mathbf{x}}_0^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_0^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the measured outputs and control inputs

$$\begin{aligned} y_0 &= 2, & u_0 &= 1, \\ y_1 &= -1, & u_1 &= -1, \\ y_2 &= -8. \end{aligned}$$

i) Use the obtained Kalman filter algorithm to calculate the state estimates $\hat{\mathbf{x}}_0$, $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$.

Problem 2: Extended Kalman filter

Consider the following discrete-time system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, u_k) + \mathbf{w}_k, \\ y_k &= h(\mathbf{x}_k) + v_k, \end{aligned} \tag{4}$$

where u_k is the input, y_k is the output, and where the state $\mathbf{x}_k$ and the functions $\mathbf{f}$ and h are given by

$$\mathbf{x}_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}_k, u_k) = \begin{bmatrix} -\frac{1}{2}x_{1,k}^2 - x_{2,k} \\ -x_{2,k} + u_k \end{bmatrix} \qquad \text{and} \qquad h(\mathbf{x}_k) = x_{1,k} + x_{2,k}^2.$$

The disturbances $\mathbf{w}_k$ and v_k are multivariate Gaussian noises. The corresponding covariance matrices are respectively given by

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R = 3.$$

We will design an extended Kalman filter for the system in (4). The initial conditions for the extended Kalman filter are given by an (a priori) state estimate $\hat{\mathbf{x}}_0^-$ and an (a priori) error covariance matrix $\mathbf{P}_0^-$, which are given by

$$\hat{\mathbf{x}}_0^- = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_0^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a) Compute the Kalman gain $\mathbf{K}_0$ from the equation

$$\mathbf{K}_0 = \mathbf{P}_0^- \mathbf{H}_0^T (\mathbf{H}_0 \mathbf{P}_0^- \mathbf{H}_0^T + R)^{-1},$$

where the matrix $\mathbf{H}_0$ is given by

$$\mathbf{H}_0 = \left. \frac{dh}{d\mathbf{x}_k} \right|_{\mathbf{x}_k = \hat{\mathbf{x}}_0^-}.$$

b) Let $y_0 = 17$. Compute the updated state estimate

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{x}}_0^- + \mathbf{K}_0 (y_0 - h(\hat{\mathbf{x}}_0^-)).$$

c) Now, similar to the Kalman filter, compute the updated error covariance matrix $\mathbf{P}_0$ from the equation

$$\mathbf{P}_0 = (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0) \mathbf{P}_0^- (\mathbf{I} - \mathbf{K}_0 \mathbf{H}_0)^T + \mathbf{K}_0 R \mathbf{K}_0^T,$$

where $\mathbf{I}$ is the identity matrix.

The projection ahead, from $k = 0$ to $k = 1$, is based on the formulas

$$\hat{\mathbf{x}}_1^- = \mathbf{f}(\hat{\mathbf{x}}_0, u_0) \qquad \text{and} \qquad \mathbf{P}_1^- = \mathbf{\Phi}_0 \mathbf{P}_0 \mathbf{\Phi}_0^T + \mathbf{Q},$$

where the matrix $\mathbf{\Phi}_0$ is given by

$$\mathbf{\Phi}_0 = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}_k} \right|_{\substack{\mathbf{x}_k = \hat{\mathbf{x}}_0 \\ u_k = u_0}}.$$

d) Let $u_0 = -5$. Show that $\hat{\mathbf{x}}_1^-$ and $\mathbf{P}_1^-$ are given by

$$\hat{\mathbf{x}}_1^- = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_1^- = \begin{bmatrix} 7 & 1 \\ 1 & 1.5 \end{bmatrix}.$$

In a more general form (not only for $k = 0$), the extended Kalman filter algorithm can be summarized as follows.

- *Compute the Kalman gain:*

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + R)^{-1}, \qquad \textit{with} \qquad \mathbf{H}_k = \left. \frac{dh}{d\mathbf{x}_k} \right|_{\mathbf{x}_k = \hat{\mathbf{x}}_k^-}.$$

- *Update the state estimate:*

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(y_k - h(\hat{\mathbf{x}}_k^-)).$$

- *Update the error covariance matrix:*

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k R \mathbf{K}_k^T.$$

- *Project ahead to $k+1$:*

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{f}(\hat{\mathbf{x}}_k, u_k) \qquad \textit{and} \qquad \mathbf{P}_{k+1}^- = \mathbf{\Phi}_k\mathbf{P}_k\mathbf{\Phi}_k^T + \mathbf{Q}, \qquad \textit{with} \qquad \mathbf{\Phi}_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}_k}\right|_{\substack{\mathbf{x}_k = \hat{\mathbf{x}}_k \\ u_k = u_k}}.$$

Consider the following outputs and input:

$$\begin{aligned} y_1 &= 3, \qquad & u_1 &= -1, \\ y_2 &= 2. \end{aligned}$$

e) Show that $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are given by

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{x}}_2 = \begin{bmatrix} -0.6 \\ -1.5 \end{bmatrix}.$$