TTK4115 Linear System Theory
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Solution to homework assignment 3

Problem 1: Controllability tests

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -6 \\ 2 & 2 & -10 & -10 \end{bmatrix}.$$

b) Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 2 = n$, we conclude that the system is controllable.

c) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

d) The Popov-Belevitch-Hautus test for controllability states that the given system is controllable if and only if for all $\lambda \in \mathbb{C}$ the condition

$$\text{rank} \begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix} = n = 2.$$

holds. Note that the matrix $\mathbf{A} - \lambda\mathbf{I}$ has rank $n = 2$ for all $\lambda \in \mathbb{C}$, except when λ is an eigenvalue of $\mathbf{A}$. This implies that the above condition automatically holds for all $\lambda \in \mathbb{C}$ that are not eigenvalues of $\mathbf{A}$. Hence, we only need to check if the above condition holds for eigenvalues of $\mathbf{A}$. For $\lambda = \lambda_1$, we have

$$\text{rank} \begin{bmatrix} \mathbf{A} - \lambda_1\mathbf{I} & \mathbf{B} \end{bmatrix} = \text{rank} \begin{bmatrix} 3 & -3 & 0 & 0 \\ 4 & -4 & 2 & 2 \end{bmatrix} = 2.$$

Similarly, for $\lambda = \lambda_2$, we have

$$\text{rank} \begin{bmatrix} \mathbf{A} - \lambda_2\mathbf{I} & \mathbf{B} \end{bmatrix} = \text{rank} \begin{bmatrix} 4 & -3 & 0 & 0 \\ 4 & -3 & 2 & 2 \end{bmatrix} = 2.$$

Therefore, for $\lambda \in \mathbb{C}$ the above condition holds, which implies that the system is controllable.

e) For the Lyapunov test, it is required that the matrix $\mathbf{A}$ is a stability matrix (also called a Hurwitz matrix), which implies that the eigenvalues of $\mathbf{A}$ should have strictly negative real parts.

f) From c), we know that the eigenvalues of $\mathbf{A}$ are given by $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence, the eigenvalues of $\mathbf{A}$ have strictly negative real parts. This implies that the condition on $\mathbf{A}$ to apply the Lyapunov test is satisfied.

g) To find the matrix $\mathbf{W}$, we solve the Lyapunov equation

$$\mathbf{AW} + \mathbf{WA}^T = -\mathbf{BB}^T.$$

Note that $\mathbf{W}$ is a symmetric matrix, i.e. $\mathbf{W} = \mathbf{W}^T$. Let $\mathbf{W}$ be given by

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix},$$

where w_1, w_2 and w_3 are constant that are yet to be determined. Substituting the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{W}$ in the Lyapunov equation, we obtain

$$\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} 2w_1 - 3w_2 & 2w_2 - 3w_3 \\ 4w_1 - 5w_2 & 4w_2 - 5w_3 \end{bmatrix} + \begin{bmatrix} 2w_1 - 3w_2 & 4w_1 - 5w_2 \\ 2w_2 - 3w_3 & 4w_2 - 5w_3 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}.$$

From this, we obtain the equations

$$\begin{aligned} 4w_1 - 6w_2 &= 0, \\ 4w_1 - 3w_2 - 3w_3 &= 0, \\ 8w_2 - 10w_3 &= -8, \end{aligned}$$

which can be written in the following form:

$$\begin{bmatrix} 4 & -6 & 0 \\ 4 & -3 & -3 \\ 0 & 8 & -10 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}.$$

Solving for w_1, w_2 and w_3 yields $w_1 = 6$, $w_2 = 4$ and $w_3 = 4$. Hence, we obtain the matrix

$$\mathbf{W} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}.$$

h) The system is controllable if the matrix $\mathbf{W}$ is positive definite. The matrix $\mathbf{W}$ is positive definite if and only if all its leading principle minors are positive. The leading principle minors of $\mathbf{W}$ are

$$w_1 = 6 \qquad \text{and} \qquad \det(\mathbf{W}) = \begin{vmatrix} 6 & 4 \\ 4 & 4 \end{vmatrix} = 8.$$

Because all leading principle minors of $\mathbf{W}$ are positive, we conclude that the system is controllable.

Problem 2: Controllable decompositions and stabilizability

a) From the Popov-Belevitch-Hautus test for controllability, it follows that if system (1) is controllable, the condition

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\end{bmatrix} = n$$

is satisfied for all $\lambda \in \mathbb{C}$. This also implies that the condition

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda^*\mathbf{I} & \mathbf{B}\end{bmatrix} = n$$

is satisfied for all eigenvalues λ^* of $\mathbf{A}$. Hence, all eigenvalues of $\mathbf{A}$ are controllable eigenvalues. Therefore, system (1) has n controllable eigenvalues and no uncontrollable eigenvalues if system (1) is controllable.

b) From the the Popov-Belevitch-Hautus test for stabilizability, it follows that if system (1) is stabilizable, the condition

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\end{bmatrix} = n$$

is satisfied for all $\lambda \in \mathbb{C}$ for which $\mathbb{R}[\lambda] \geq 0$ (the real part of λ is larger than or equal to zero). If the eigenvalue λ^* of $\mathbf{A}$ is an uncontrollable eigenvalue, then

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda^*\mathbf{I} & \mathbf{B}\end{bmatrix} < n.$$

This implies that system (1) is only stabilizable if the uncontrollable eigenvalue λ^* has a real part smaller than zero (in that case it is not covered by the rank condition of the Popov-Belevitch-Hautus test for stabilizability). Hence, if system (1) is stabilizable, all uncontrollable eigenvalues need to have a strictly negative real part.

c) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \begin{vmatrix}-4-\lambda & -4 & -10\\ 0 & -2-\lambda & 5\\ 0 & 0 & 3-\lambda\end{vmatrix} = (-4-\lambda)(-2-\lambda)(3-\lambda)\\ &= -(\lambda+4)(\lambda+2)(\lambda-3) = 0.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -4$, $\lambda_2 = -2$ and $\lambda_3 = 3$.

d) For the eigenvalues $\lambda_1 = -4$, $\lambda_2 = -2$ and $\lambda_3 = 3$, we subsequently have

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda_1\mathbf{I} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & -4 & -10 & 4\\ 0 & 2 & 5 & -2\\ 0 & 0 & 7 & -1\end{bmatrix} = 2 < n,$$

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda_2\mathbf{I} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-2 & -4 & -10 & 4\\ 0 & 0 & 5 & -2\\ 0 & 0 & 5 & -1\end{bmatrix} = 3 = n$$

and

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda_3\mathbf{I} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-7 & -4 & -10 & 4\\ 0 & -5 & 5 & -2\\ 0 & 0 & 0 & -1\end{bmatrix} = 3 = n.$$

Hence, we obtain that $\lambda_2 = -2$ and $\lambda_3 = 3$ are controllable eigenvalues, while $\lambda_1 = -4$ is an uncontrollable eigenvalue.

e) Because not all eigenvalues of $\mathbf{A}$ are controllable ($\lambda_1 = -4$ is uncontrollable), from a) it follows that system (2) is not controllable.

f) Because all uncontrollable eigenvalues have strictly negative real parts ($\lambda_1 = -4$ is the only uncontrollable eigenvalue), from b) we conclude that system (2) is stabilizable.

g) The eigenvectors $\mathbf{v}_i$ can be obtained from the kernel of the matrix $(\mathbf{A} - \lambda_i\mathbf{I})$ for $i = 1, 2, 3$:

$$\begin{aligned}\ker(\mathbf{A}-\lambda_1\mathbf{I}) &= \ker\left(\begin{bmatrix}0 & -4 & -10\\ 0 & 2 & 5\\ 0 & 0 & 7\end{bmatrix}\right)\\ &= \ker\left(\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) \qquad \Longrightarrow \quad \mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\ker(\mathbf{A}-\lambda_2\mathbf{I}) &= \ker\left(\begin{bmatrix}-2 & -4 & -10\\ 0 & 0 & 5\\ 0 & 0 & 5\end{bmatrix}\right)\\ &= \ker\left(\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right) \qquad \Longrightarrow \quad \mathbf{v}_2 = \begin{bmatrix}2\\-1\\0\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\ker(\mathbf{A}-\lambda_3\mathbf{I}) &= \ker\left(\begin{bmatrix}-7 & -4 & -10\\ 0 & -5 & 5\\ 0 & 0 & 0\end{bmatrix}\right)\\ &= \ker\left(\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\right) \qquad \Longrightarrow \quad \mathbf{v}_3 = \begin{bmatrix}2\\-1\\-1\end{bmatrix}.\end{aligned}$$

h) Because $\mathbf{q}_1$ and $\mathbf{q}_2$ are eigenvectors that correspond to controllable eigenvalues, we select $\mathbf{q}_1 = \mathbf{v}_2$ and $\mathbf{q}_2 = \mathbf{v}_3$, which correspond to the controllable eigenvalues $\lambda_2 = -2$ and $\lambda_3 = 3$, respectively. The eigenvector $\mathbf{q}_3$ that corresponds to the uncontrollable eigenvalue is given by $\mathbf{q}_3 = \mathbf{v}_1$, which corresponds to the uncontrollable eigenvalue $\lambda_1 = -4$. Hence, we obtain the matrix

$$\mathbf{Q} = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix} = \begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1\end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

i) Use the similarity transform $\mathbf{x}(t) = \mathbf{Q}\hat{\mathbf{x}}(t)$, system (2) is transformed to

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t).$$

with matrices

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and

$$\hat{\mathbf{B}} = \mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The solution for $\hat{\mathbf{B}}$ is not unique and depends on how the eigenvalues are chosen. The third element of $\hat{\mathbf{B}}$, however, is always zero.

j) From i), it is easy to see that the matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ can be written as

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}_c & \hat{\mathbf{A}}_{12} \\ \mathbf{0} & \hat{\mathbf{A}}_u \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \hat{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix}.$$

with

$$\hat{\mathbf{A}}_c = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \hat{\mathbf{A}}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\hat{\mathbf{A}}_u = \begin{bmatrix} -4 \end{bmatrix}, \qquad \hat{\mathbf{B}}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 3: State feedback

a) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B}\end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -4 & -6 \end{bmatrix}.$$

b) Because the controllability matrix has full row rank, i.e. $\text{rank}(\mathcal{C}) = 2 = n$, we conclude that the system is controllable.

c) Combining the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$ and $u(t) = -\mathbf{K}\mathbf{x}(t)$, we obtain

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t),$$

with

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}.$$

d) The characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\begin{aligned}\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix}1 & 0\\ -2 & 0\end{bmatrix} - \begin{bmatrix}3\\ -4\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) &= \begin{vmatrix}1 - 3k_1 - \lambda & -3k_2\\ -2 + 4k_1 & 4k_2 - \lambda\end{vmatrix}\\ &= (1 - 3k_1 - \lambda)(4k_2 - \lambda) + 3k_2(-2 + 4k_1)\\ &= \lambda^2 + (-1 + 3k_1 - 4k_2)\lambda - 2k_2.\end{aligned}$$

e) The characteristic polynomial of $\bar{\mathbf{A}}$ should be equal to

$$\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) = (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda) = (-2 - \lambda)(-4 - \lambda) = \lambda^2 + 6\lambda + 8 = 0.$$

Comparing this to the characteristic polynomial obtained in d), we obtain the equations

$$-1 + 3k_1 - 4k_2 = 6 \qquad \text{and} \qquad -2k_2 = 8.$$

Hence, we obtain the values

$$k_1 = -3 \qquad \text{and} \qquad k_2 = -4.$$

f) Using the values $k_1 = -3$ and $k_2 = -4$, we get

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix}1 & 0\\ -2 & 0\end{bmatrix} - \begin{bmatrix}3\\ -4\end{bmatrix}\begin{bmatrix}-3 & -4\end{bmatrix} = \begin{bmatrix}10 & 12\\ -14 & -16\end{bmatrix}.$$

The eigenvalues of $\bar{\mathbf{A}}$ can be computed as follows:

$$\begin{aligned}\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) = \begin{vmatrix}10 - \lambda & 12\\ -14 & -16 - \lambda\end{vmatrix} &= (10 - \lambda)(-16 - \lambda) + 168\\ &= \lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0.\end{aligned}$$

Hence, the eigenvalues of $\bar{\mathbf{A}}$ are given by $\lambda_1 = \bar{\lambda}_1 = -2$ and $\lambda_2 = \bar{\lambda}_2 = -4$.