TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 7

Hand-out time:	Monday, November 11, 2013, at 12:00
Hand-in deadline:	Friday, November 22, 2013, at 12:00

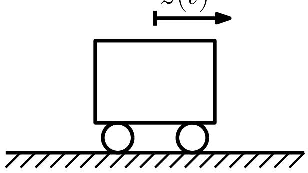
The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Linear quadratic regulation and minimum-energy estimation

Consider a cart for which the equation of motion is given by

$$\ddot{z}(t) - 2\dot{z}(t) = 2u(t) + d(t),$$

where $z(t)$ is the distance of the cart, $u(t)$ is an input force and $d(t)$ is a disturbance force. We are interested in controlling the velocity $\dot{z}(t)$ of the cart.



a) Write the equation of motion in the following form

$$\dot{x}(t) = Ax(t) + Bu(t) + \bar{B}d(t), \tag{1}$$

with state $x(t) = \dot{z}(t)$.

b) Is the system (1) controllable? Motivate your answer.

For the moment, we assume that $d(t) = 0$. We will design a feedback controller of the form

$$u(t) = -Kx(t). \tag{2}$$

c) For which values of K is the closed-loop system (1)-(2) marginally stable? Motivate your answer.

d) For which values of K is the closed-loop system (1)-(2) exponentially stable? Motivate your answer.

We select the gain K such that the control input (2) minimizes the cost function

$$J_{LQR} = \int_0^\infty \left[Qx^2(t) + Ru^2(t)\right] dt$$

of the linear quadratic regulation problem, with constants $Q, R > 0$. The gain K is given by

$$K = \frac{BP}{R},$$

where P is the positive definite solution of the Riccati equation

$$-\frac{B^2}{R}P^2 + 2AP + Q = 0.$$

e) Compute K as a function of Q and R.

f) From your answers of d) and e), determine for which values of $Q > 0$ and $R > 0$ the closed-loop system (1)-(2) is exponentially stable. Motivate your answer.

g) Let $Q = 6$. Determine the value for $R > 0$ such that $K = 3$.

We assume that the state $x(t)$ is unknown and can only be measured. The difference between the measured state $y(t)$ and the actual state $x(t)$ is given by $n(t)$, i.e.

$$n(t) = y(t) - x(t). \tag{3}$$

Note that $n(t)$ can be considered as measurement noise. Instead of the actual state $x(t)$, the controller in (2) uses the measured state $y(t)$, which leads to

$$u(t) = -Ky(t). \tag{4}$$

h) Write the equation (3) as

$$y(t) = Cx(t) + n(t), \tag{5}$$

and determine the constant C.

The system (1), (4) and (5) is depicted in Fig. 1.

i) Determine the transfer function from the disturbance $d(t)$ to the state $x(t)$ of the system (1), (4) and (5) as a function of K, i.e. determine $\frac{x(s)}{d(s)}$ as a function of K.

j) Determine the transfer function from the disturbance $n(t)$ to the state $x(t)$ of the system (1), (4) and (5) as a function of K, i.e. determine $\frac{x(s)}{n(s)}$ as a function of K.

We assume that $d(t)$ is a low-frequency signal, and that $n(t)$ is a high-frequency signal.

k) Calculate $\lim_{s\to 0} \frac{x(s)}{d(s)}$ as a function of K.

l) How should the value of K be chosen such that the effect of the disturbance $d(t)$ on the state $x(t)$ is small? Motivate your answer.

m) Calculate $\lim_{s\to\infty} \frac{x(s)}{n(s)}$ as a function of K.

n) How should the value of K be chosen such that the effect of the disturbance $n(t)$ on the state $x(t)$ is small? Motivate your answer.

In the remaining part of the exercise, we set $K = 3$. To balance the effect of the disturbances $d(t)$ and $n(t)$, we will design a state estimator of the following form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \tag{6}$$

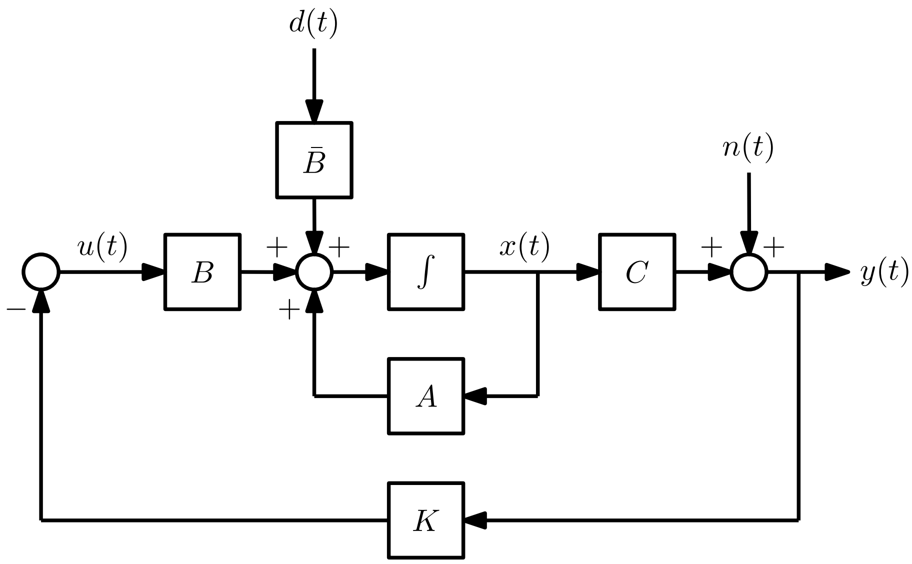


Fig. 1: System (1), (4) and (5).

with estimator gain L. We will use the estimated state $\hat{x}(t)$ as an input for the feedback controller in (2), which leads to

$$u(t) = -K\hat{x}(t). \tag{7}$$

o) Similar to Fig. 1, draw a block diagram of the system with controller and state estimator, which is given by the equations (1), (5), (6) and (7).

p) We define the estimation error $e(t) = x(t) - \hat{x}(t)$. Show that the system (1), (5), (6) and (7) can be written in the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d(t) \\ n(t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} d(t) \\ n(t) \end{bmatrix}.$$

q) For which values of L is the system (1), (5), (6) and (7) BIBO stable (bounded-input bounded-output stable)? Motivate your answer.

We select the gain L such that the state estimator (6) minimizes the cost function

$$J_{MEE} = \int_{-\infty}^{t} \left[S(y(t) - C\hat{x}(t))^2 + T d^2(t) \right] dt$$

of the minimum-energy estimation problem, with constants $S, T > 0$. The gain L is given by

$$L = WCS,$$

where W is the positive definite solution of the Riccati equation

$$-C^2SW^2 + 2AW + \frac{\bar{B}^2}{T} = 0.$$

r) Compute L as a function of S and T.

s) From your answers of q) and r), determine for which values of $S > 0$ and $T > 0$ the system (1), (5), (6) and (7) is BIBO stable. Motivate your answer.

t) Determine the transfer function from the disturbance $d(t)$ to the state $x(t)$ of the system (1), (5), (6) and (7) as a function of L.

u) Determine the transfer function from the disturbance $n(t)$ to the state $x(t)$ of the system (1), (5), (6) and (7) as a function of L.

Again, we assume that $d(t)$ is a low-frequency signal, and that $n(t)$ is a high-frequency signal.

v) How should the value of L be chosen such that the effect of the disturbance $d(t)$ on the state $x(t)$ is small? Motivate your answer.

w) How should the value of L be chosen such that the effect of the disturbance $n(t)$ on the state $x(t)$ is small? Motivate your answer.

We compare the system with output feedback in (1), (4) and (5) with the system with state-estimated feedback in (1), (5), (6) and (7).

x) Using $K = 3$ and $L = 12$, is the effect of the low-frequency disturbance $d(t)$ on the state $x(t)$ smaller or larger with output feedback than with state-estimated feedback? Motivate your answer.

y) Using $K = 3$ and $L = 12$, is the effect of the high-frequency disturbance $n(t)$ on the state $x(t)$ smaller or larger with output feedback than with state-estimated feedback? Motivate your answer.