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TTK4135 – Lecture 10
Model Predictive Control

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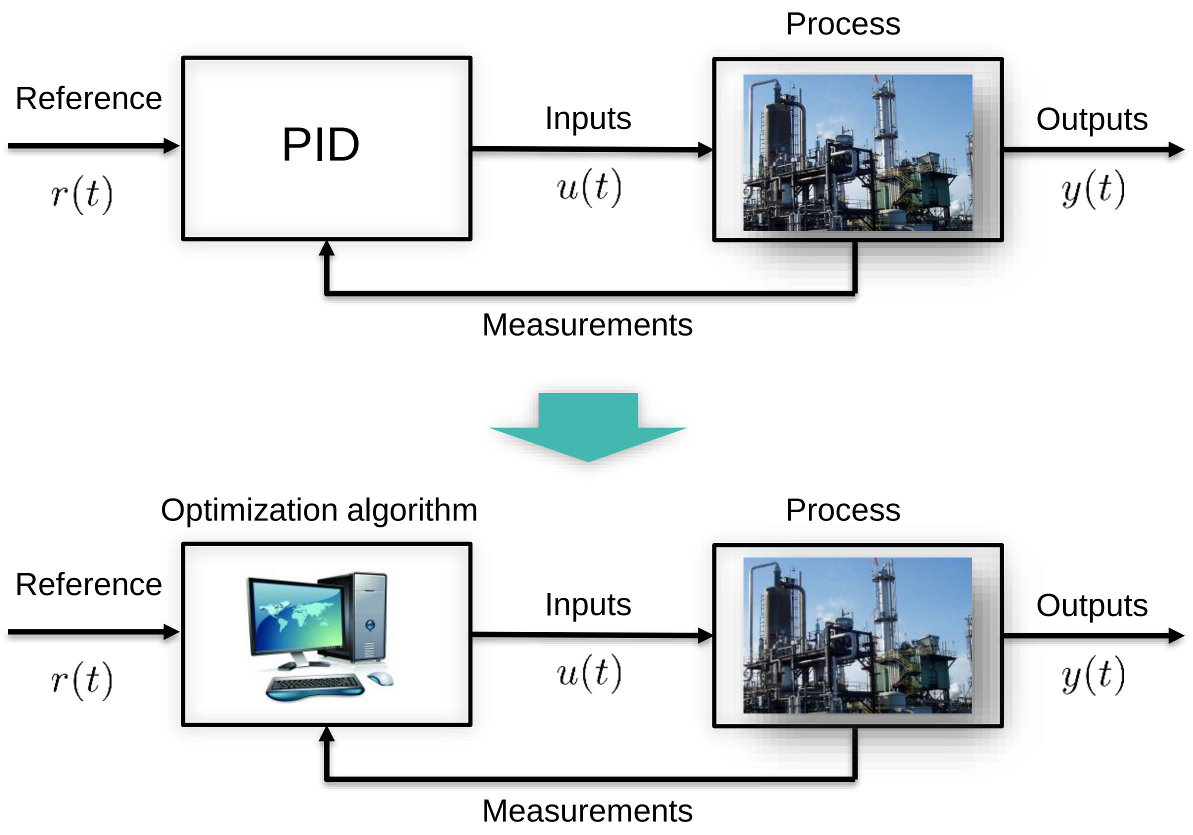
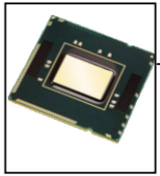
Model Predictive control – control based on optimization



A model of the process is used to compute the control signals (inputs) that optimize predicted future process behavior

MPC: Applications


Computer control — ns

µs — Power systems



Traction control — ms

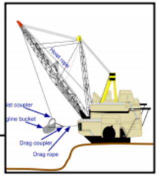
Seconds — Buildings



Refineries — Minutes

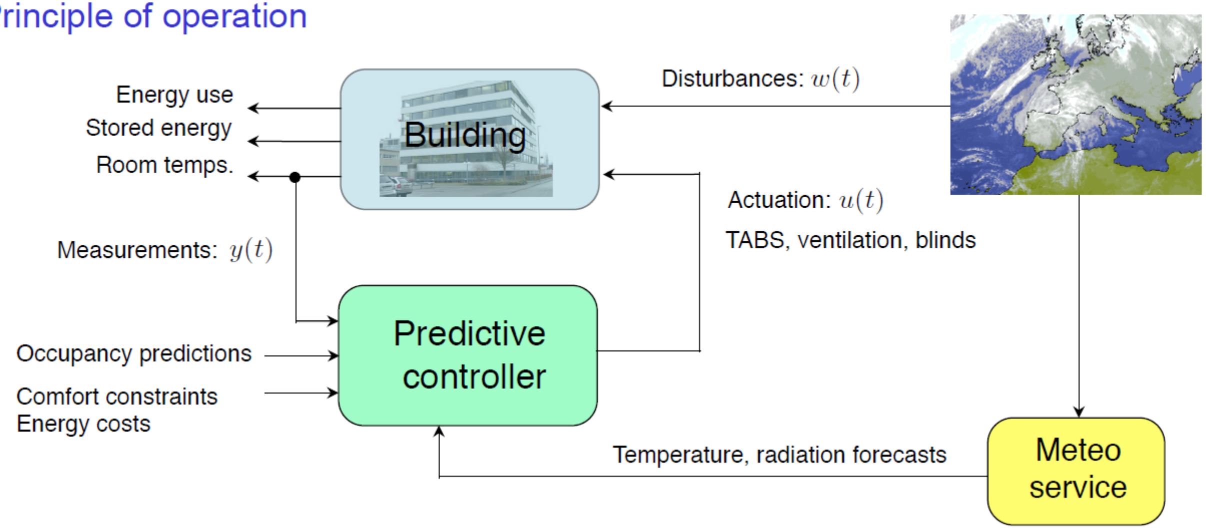
Hours — Nurse rostering


Train scheduling — Days

Weeks — Production planning


Model predictive control (MPC)

Principle of operation



$$\text{Predicted Cost} = \underset{u(t)}{\text{minimize}} \ \text{Expected} \left(\sum_{t}^{t+N} \text{energy cost(t)} \right)$$ ← Minimize the predicted energy cost

$$\text{subject to} \quad u(t) \in \mathcal{U}$$ ← Actuation within limits

$$x(t) \in \mathcal{X}$$ ← Predicted temperatures within limits

$$x(t+1) = f(x(t), u(t), w(t))$$ ← Predicted dynamics of the building

From ETH

Open-loop optimization with linear state-space model

QP

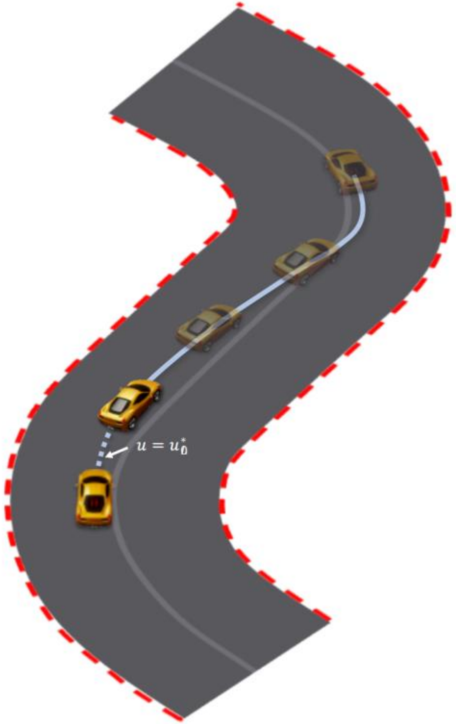
$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + d_{x,t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t + d_{u,t}u_t + \frac{1}{2}\Delta u_t^\top S \Delta u_t$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, \quad t = \{0,\dots,N-1\}\\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1,\dots,N\}\\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0,\dots,N-1\}\\
-\Delta u^{\text{high}} &\leq \Delta u_t \leq \Delta u^{\text{high}}, \quad t = \{0,\dots,N-1\}
\end{aligned}$$

where

$$\begin{aligned}
&x_0 \text{ and } u_{-1} \text{ is given}\\
\Delta u_t &:= u_t - u_{t-1}\\
z^\top &:= (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top)\\
n &= N \cdot (n_x + n_u)\\
Q_t &\succeq 0 \quad t = \{1,\dots,N\}\\
R_t &\succeq 0 \quad t = \{0,\dots,N-1\}
\end{aligned}$$



Open-loop dynamic optimization problem as QP

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q x_{t+1} + \frac{1}{2}u_t^\top R u_t$$

subject to

$$\begin{aligned}
x_{t+1} &= Ax_t + Bu_t, \quad t = \{0,\dots,N-1\} \\
x^{\text{low}} &\leq x_t \leq x^{\text{high}}, \quad t = \{1,\dots,N\} \\
u^{\text{low}} &\leq u_t \leq u^{\text{high}}, \quad t = \{0,\dots,N-1\}
\end{aligned}$$

where

$$\begin{aligned}
&x_0 \text{ is given} \\
&z^\top := (u_0^\top, x_1^\top, \dots, u_{N-1}^\top, x_N^\top) \\
&Q \succeq 0, \quad R \succ 0
\end{aligned}$$

Model predictive control principle

Optimization algorithm

$$\min_{z} \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t$$

$$\begin{aligned} \text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1 \\ & x^{\text{low}} \le x_t \le x^{\text{high}}, \quad t = 1, \ldots, N \\ & u^{\text{low}} \le u_t \le u^{\text{high}}, \quad t = 0, \ldots, N-1 \\ & x_0 = x(t) \end{aligned}$$

Reference

Inputs

$u(t) = u_0^*$

Process

Outputs

$y(t)$

Plant state $x(t)$

At each sample time:

- Measure or estimate current state $x(t)$
- Find optimal input sequence $U^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$
- Implement only the first element of sequence: $u(t) = u_0^*$

Model predictive control principle

Optimization algorithm

$$
\begin{aligned}
\min_{z} \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \ldots, N-1 \\
& x^{\text{low}} \le x_t \le x^{\text{high}}, \quad t = 1, \ldots, N \\
& u^{\text{low}} \le u_t \le u^{\text{high}}, \quad t = 0, \ldots, N-1 \\
& x_0 = x(t)
\end{aligned}
$$

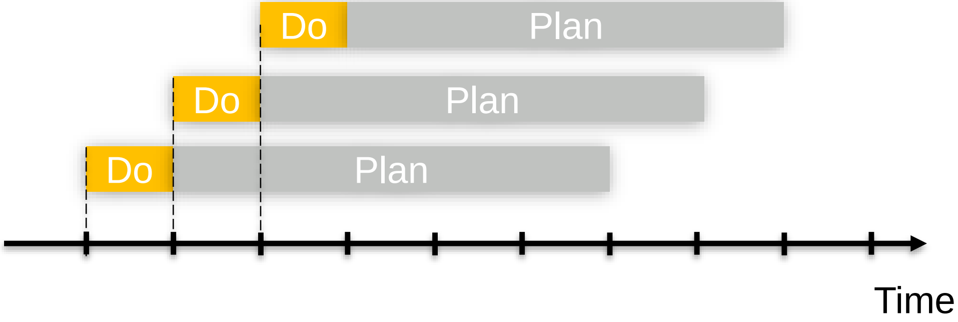
Process

Reference

Inputs $u(t) = u_0^*$

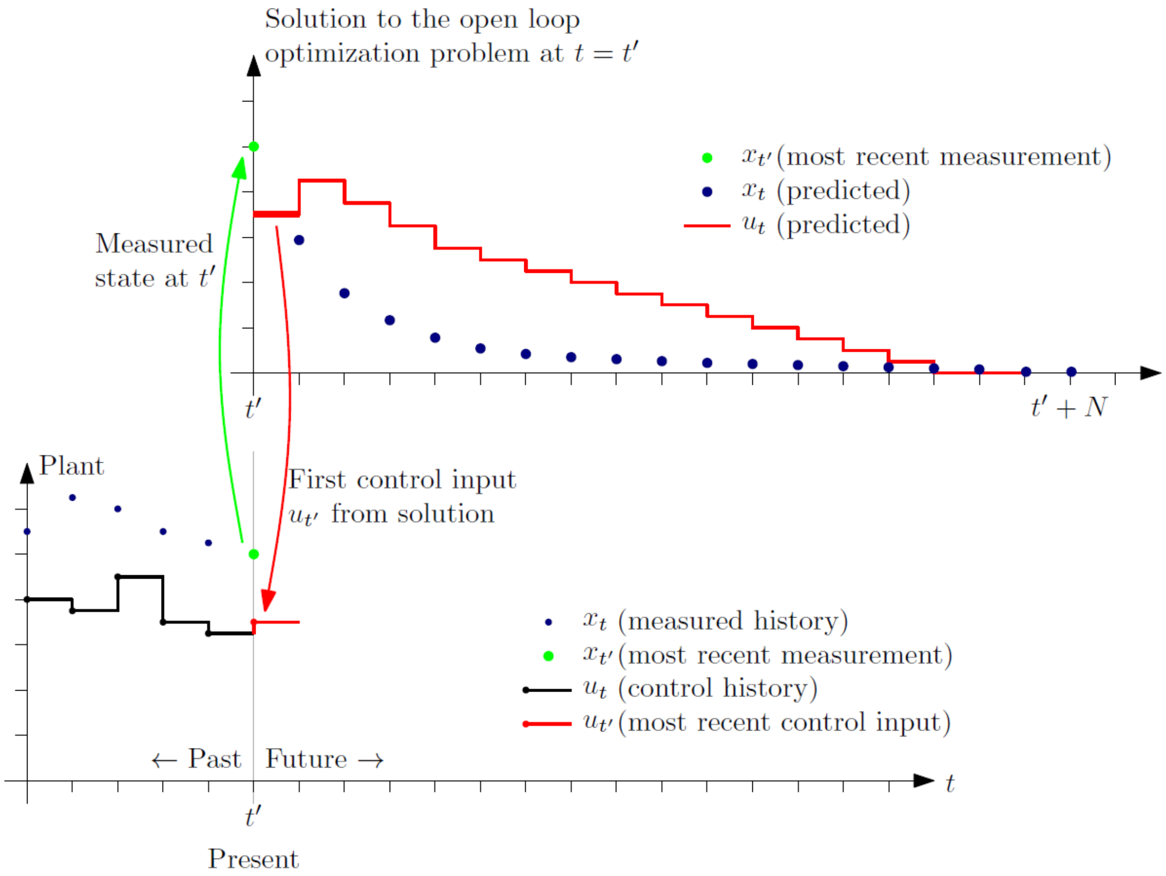
Outputs $y(t)$

Plant state $x(t)$



Why? This introduces feedback!

Model predictive control principle



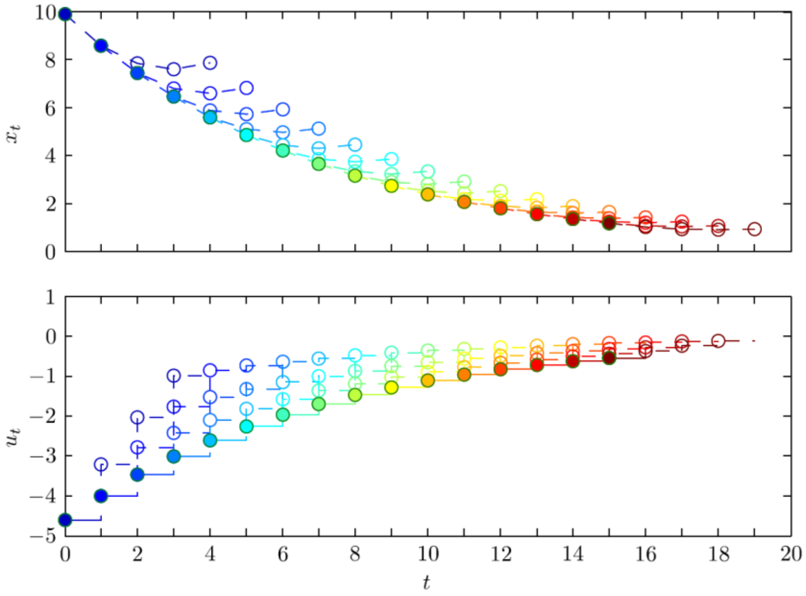
(MPC animation)

Feasibility (or: is there a solution to the QP?)

Open-loop vs closed-loop trajectories

$$\min \sum_{t=0}^{4} x_{t+1}^2 + 4\,u_t^2$$

$$\text{s.t.} \qquad x_{t+1} = 1.2x_t + 0.5u_t, \quad t = 0, \dots, 4$$



- Closed-loop trajectories different from open-loop (optimized) trajectories!
- It is the closed-loop trajectories that must analyzed for feasibility and stability.

Example: Is MPC alway stable?

Design MPC for $x_{t+1} = 1.2x_t + u_t$, no constraints, *N* = *2*

Example: Is MPC alway stable?

Design MPC for $x_{t+1} = 1.2x_t + u_t$, no constraints, *N = 2*

MPC optimality implies stability?

$$\min \sum_{t=0}^{1} x_{t+1}^2 + r\, u_t^2$$

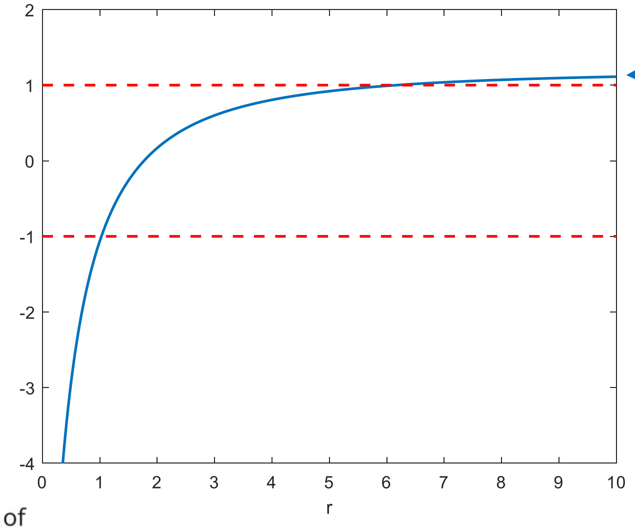
$$\text{s.t.} \qquad x_{t+1} = 1.2x_t + u_t, \quad t = 0, 1$$

MPC solution

$$u_t = -\frac{1.2 + 2.64r}{1 + 3.2r + r^2} x_t$$

MPC closed loop

$$x_{t+1} = \underbrace{\left(1.2 - \frac{1.2 + 2.64r}{1 + 3.2r + r^2}\right)}\, x_t$$



MPC and stability

Nominal vs robust stability

- "Nominal stability": Stability when optimization model = plant model
  - No "model-plant mismatch", no disturbances
- "Robust stability": Stability when optimization model ≠ plant model
  - "Model-plant mismatch" and/or disturbances (more difficult to analyze)

Requirements for nominal stability:

- Stabilizability ((A,B) stabilizable)
- Detectability ((A,D) detectable)
  - D is a matrix such that $Q = D^T D$ (that is, "D is matrix square root of Q")
  - Detectability: No modes can grow to infinity without being "visible" through Q

- But more is needed to guarantee stability…

How to achieve nominal stability?

$$
\begin{aligned}
\min_{z} \quad & \sum_{t=0}^{N-1} \frac{1}{2} x_{t+1}^\top Q x_{t+1} + \frac{1}{2} u_t^\top R u_t \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\
& x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\
& u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1
\end{aligned}
$$

- Choose prediction horizon equal to infinity ($N = \infty$)
  - Usually not possible
- For given N, design Q and R such that MPC is stable (cf. example)
  - Difficult in general!
- Change the optimization problem such that
  - The new problem gives a finite upper bound of infinite horizon problem cost
  - The constraints is guaranteed to hold after the prediction horizon

$$
\begin{aligned}
\min_{z} \quad & \sum_{t=0}^{N-1} \left(\frac{1}{2} x_t^\top Q x_t + \frac{1}{2} u_t^\top R u_t \right) + \frac{1}{2} x_N^\top P x_N \\
\text{s.t.} \quad & x_{t+1} = A x_t + B u_t, \quad t = 0, 1, \dots, N-1 \\
& x^{\text{low}} \leq x_t \leq x^{\text{high}}, \quad t = 1, \dots, N \\
& u^{\text{low}} \leq u_t \leq u^{\text{high}}, \quad t = 0, \dots, N-1 \\
& x_N \in \mathcal{S}
\end{aligned}
$$

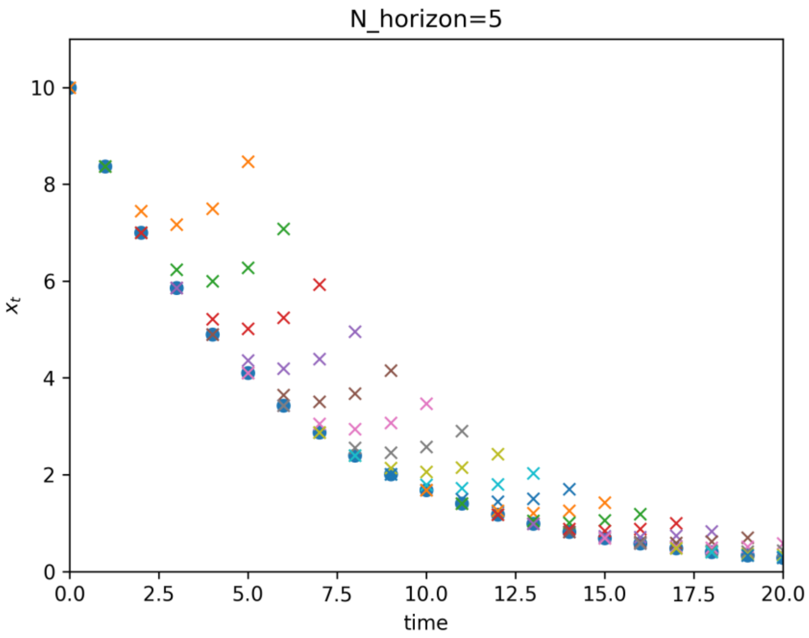
Terminal cost

Terminal constraint

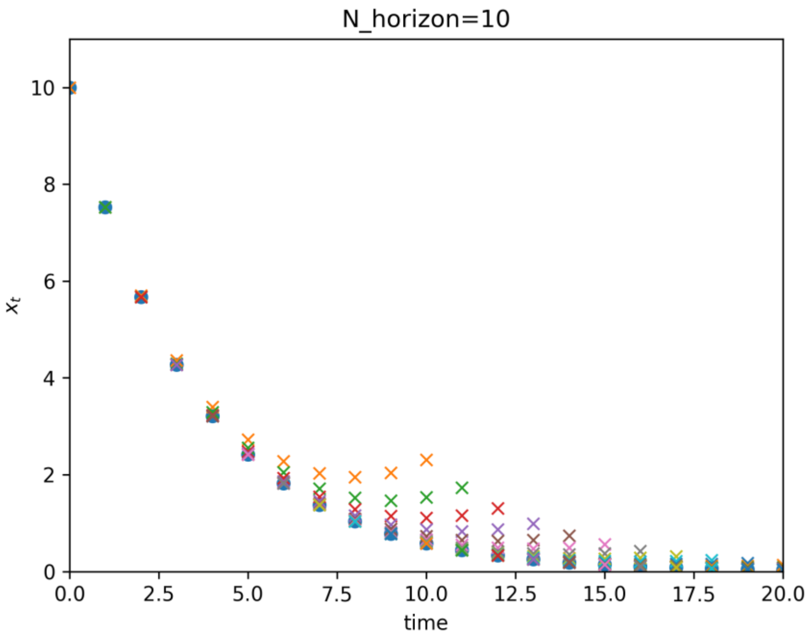
  - Fairly straightforward to do in theory, but not always done

- Typically, in practice: Choose N “large”
  - Stability guaranteed for N large enough, but difficult/conservative to compute this limit
  - So what is “large enough” in practice? Rule of thumb: longer than dominating dynamics

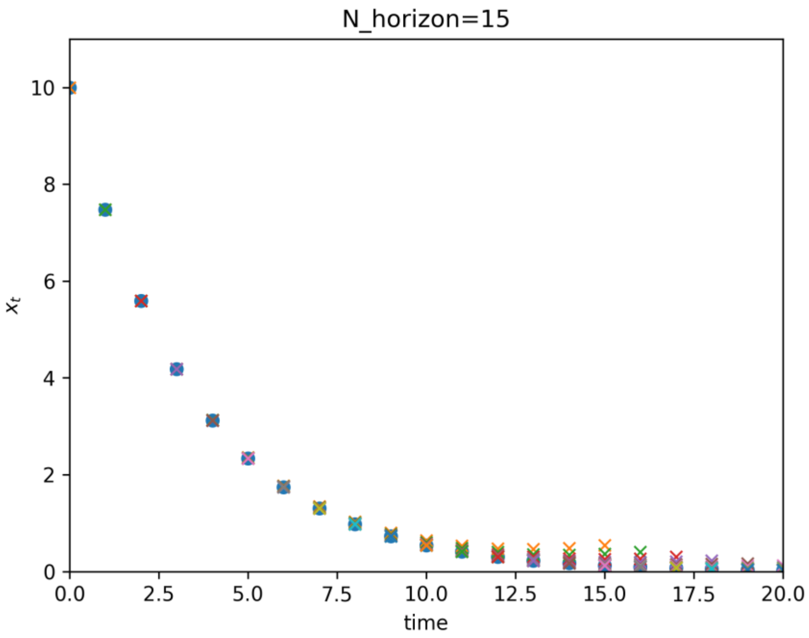
Open-Loop vs Closed-Loop: N = 5



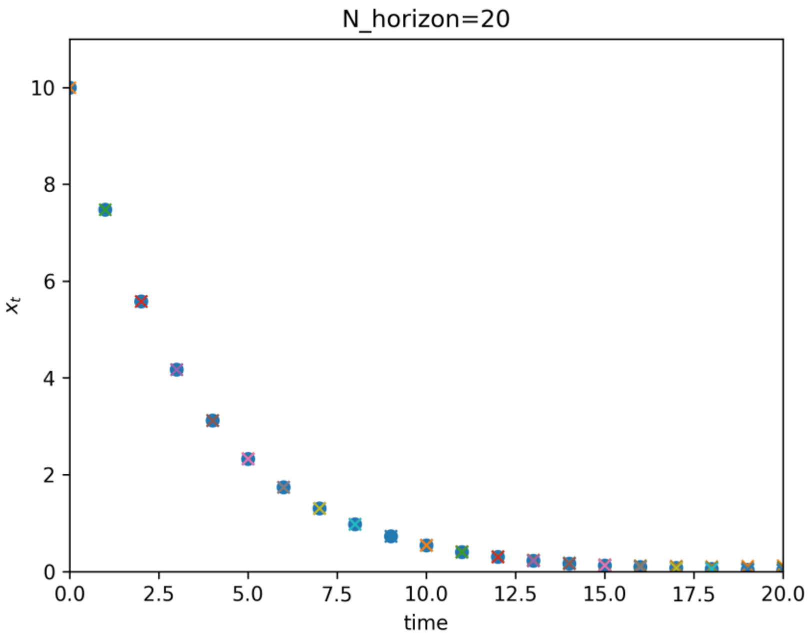
Open-Loop vs Closed-Loop: N = 10



Open-Loop vs Closed-Loop: N = 15



Open-Loop vs Closed-Loop: N = 20



Note: This assumes no model-plant mismatch!

Why MPC over PID control?

Advantages of MPC:

- MPC handles constraints in a transparent way
  - Physical constraints (actuator limits), performance constraints, safety limits, …
- MPC is by design multivariable (MIMO)
- MPC gives “optimal” performance (but what is the optimal objective?)

Disadvantage with MPC

- Online complexity
- Requires models! Increased commisioning cost?
- Difficult to maintain?

“Squeeze and shift”

How MPC (or improved/advanced control in general) improves profitability