TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Solution to homework assignment 4

Problem 1: Optimal set-point control (computer exercise)

a) Using the equations of motion, we obtain

$$
\begin{aligned}
\dot{x} &= \dot{x}, \\
\ddot{x} &= -\frac{l_x + x}{r_1(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) \\
&\quad + \frac{l_x - x}{r_2(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right), \\
\dot{y} &= \dot{y}, \\
\ddot{y} &= -\frac{m_c g}{\left(m_c + \frac{m_w l_y^2}{8r^2}\right)} + \frac{l_y - y}{r_1(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) \\
&\quad + \frac{l_y - y}{r_2(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\dot{x} &= f_1(x,\dot{x},y,\dot{y},F_1,F_2), \\
\ddot{x} &= f_2(x,\dot{x},y,\dot{y},F_1,F_2), \\
\dot{y} &= f_3(x,\dot{x},y,\dot{y},F_1,F_2), \\
\ddot{y} &= f_4(x,\dot{x},y,\dot{y},F_1,F_2),
\end{aligned}
$$

with

$$
\begin{aligned}
f_1(x,\dot{x},y,\dot{y},F_1,F_2) &= \dot{x},\\
f_2(x,\dot{x},y,\dot{y},F_1,F_2) &= -\frac{l_x + x}{r_1(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right)\\
&\quad + \frac{l_x - x}{r_2(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right),\\
f_3(x,\dot{x},y,\dot{y},F_1,F_2) &= \dot{y},\\
f_4(x,\dot{x},y,\dot{y},F_1,F_2) &= -\frac{m_c g}{\left(m_c + \frac{m_w l_y^2}{8r^2}\right)} + \frac{l_y - y}{r_1(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right)\\
&\quad + \frac{l_y - y}{r_2(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right).
\end{aligned}
$$

b) For $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$, we obtain

$$
\begin{aligned}
f_1(0,0,0,0,0,0) &= 0,\\
f_2(0,0,0,0,0,0) &= 0,\\
f_3(0,0,0,0,0,0) &= 0,\\
f_4(0,0,0,0,0,0) &= -\frac{m_c g}{m_c + \frac{m_w l_y^2}{8r^2}} + \frac{m_w g l_y}{2r\left(m_c + \frac{m_w l_y^2}{8r^2}\right)},
\end{aligned}
$$

where we used that $r_1(0,0) = r_2(0,0) = r$. The functions f_1, f_2, f_3 and f_4 are zero if

$$
m_w = \frac{2m_c r}{l_y} = \frac{2m_c\sqrt{l_x^2 + l_y^2}}{l_y} = \frac{2\cdot 3\sqrt{16^2 + 12^2}}{12} = \frac{2\cdot 3\cdot 20}{12} = 10.
$$

c) Linearizing the equations of motion, we obtain

$$
\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u},
$$

with

$$
\mathbf{z} = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} \qquad \text{and} \qquad \mathbf{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},
$$

and matrices

$$
\mathbf{A} = \left.\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \dot{x}} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \dot{y}}\\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \dot{x}} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \dot{y}}\\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \dot{x}} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \dot{y}}\\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial \dot{x}} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \dot{y}}
\end{bmatrix}\right|_{\substack{x=\dot{x}=y=\dot{y}=0\\ F_1=F_2=0}}
\qquad \text{and} \qquad
\mathbf{B} = \left.\begin{bmatrix}
\frac{\partial f_1}{\partial F_1} & \frac{\partial f_1}{\partial F_2}\\
\frac{\partial f_2}{\partial F_1} & \frac{\partial f_2}{\partial F_2}\\
\frac{\partial f_3}{\partial F_1} & \frac{\partial f_3}{\partial F_2}\\
\frac{\partial f_4}{\partial F_1} & \frac{\partial f_4}{\partial F_2}
\end{bmatrix}\right|_{\substack{x=\dot{x}=y=\dot{y}=0\\ F_1=F_2=0}}.
$$

The corresponding derivatives for $\mathbf{A}$ are given by

$$
\begin{aligned}
&\frac{\partial f_1}{\partial x} = 0, \quad \frac{\partial f_1}{\partial \dot{x}} = 1, \quad \frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_1}{\partial \dot{y}} = 0, \\
&\qquad\qquad \frac{\partial f_2}{\partial \dot{x}} = 0, \qquad\qquad\qquad \frac{\partial f_2}{\partial \dot{y}} = 0, \\
&\frac{\partial f_3}{\partial x} = 0, \quad \frac{\partial f_3}{\partial \dot{x}} = 0, \quad \frac{\partial f_3}{\partial y} = 0, \quad \frac{\partial f_3}{\partial \dot{y}} = 1, \\
&\qquad\qquad \frac{\partial f_4}{\partial \dot{x}} = 1, \qquad\qquad\qquad \frac{\partial f_4}{\partial \dot{y}} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial f_2}{\partial x} &= -\frac{1}{r_1(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) - \frac{1}{r_2(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right) \\
&\quad + \frac{(l_x + x)^2}{r_1^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) + \frac{(l_x - x)^2}{r_2^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right) \\
&= -\frac{(l_y - y)^2}{r_1^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) - \frac{(l_y - y)^2}{r_2^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right), \\
\frac{\partial f_2}{\partial y} &= -\frac{(l_x + x)(l_y - y)}{r_1^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) + \frac{(l_x - x)(l_y - y)}{r_2^3(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right), \\
\frac{\partial f_4}{\partial x} &= -\frac{(l_x + x)(l_y - y)}{r_1^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) + \frac{(l_x - x)(l_y - y)}{r_2^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right), \\
\frac{\partial f_4}{\partial y} &= -\frac{1}{r_1(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) - \frac{1}{r_2(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right) \\
&\quad + \frac{(l_y - y)^2}{r_1^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) + \frac{(l_y - y)^2}{r_2^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right) \\
&= -\frac{(l_x + x)^2}{r_1^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_1\right) - \frac{(l_x - x)^2}{r_2^3(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}\left(\frac{m_w g}{4} + F_2\right).
\end{aligned}
$$

Substituting $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$, we obtain

$$
\begin{aligned}
\frac{\partial f_2}{\partial x} &= -\frac{m_w g l_y^2}{2r^3\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}, & \frac{\partial f_2}{\partial y} &= 0, \\
\frac{\partial f_4}{\partial x} &= 0, & \frac{\partial f_4}{\partial y} &= -\frac{m_w g l_x^2}{2r^3\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}.
\end{aligned}
$$

Therefore, we obtain

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{m_w g l_y^2}{2r^3\left(m_c + \frac{m_w l_x^2}{8r^2}\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{m_w g l_x^2}{2r^3\left(m_c + \frac{m_w l_y^2}{8r^2}\right)} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.2323 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.4550 & 0 \end{bmatrix}.$$

The corresponding derivatives for $\mathbf{B}$ are given by

$$\frac{\partial f_1}{\partial F_1} = 0, \qquad \frac{\partial f_1}{\partial F_2} = 0,$$
$$\frac{\partial f_3}{\partial F_1} = 0, \qquad \frac{\partial f_3}{\partial F_2} = 0,$$

and

$$\frac{\partial f_2}{\partial F_1} = -\frac{l_x + x}{r_1(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}, \qquad \frac{\partial f_2}{\partial F_2} = \frac{l_x - x}{r_2(x,y)\left(m_c + \frac{m_w l_x^2}{8r^2}\right)},$$
$$\frac{\partial f_4}{\partial F_1} = \frac{l_y - y}{r_1(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}, \qquad \frac{\partial f_4}{\partial F_2} = \frac{l_y - y}{r_2(x,y)\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}.$$

Substituting $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$, we obtain

$$\frac{\partial f_2}{\partial F_1} = -\frac{l_x}{r\left(m_c + \frac{m_w l_x^2}{8r^2}\right)}, \qquad \frac{\partial f_2}{\partial F_2} = \frac{l_x}{r\left(m_c + \frac{m_w l_x^2}{8r^2}\right)},$$
$$\frac{\partial f_4}{\partial F_1} = \frac{l_y}{r\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}, \qquad \frac{\partial f_4}{\partial F_2} = \frac{l_y}{r\left(m_c + \frac{m_w l_y^2}{8r^2}\right)}.$$

Therefore, we obtain

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -\frac{l_x}{r\left(m_c + \frac{m_w l_x^2}{8r^2}\right)} & \frac{l_x}{r\left(m_c + \frac{m_w l_x^2}{8r^2}\right)} \\ 0 & 0 \\ \frac{l_y}{r\left(m_c + \frac{m_w l_y^2}{8r^2}\right)} & \frac{l_y}{r\left(m_c + \frac{m_w l_y^2}{8r^2}\right)} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 \\ -0.2105 & 0.2105 \\ 0 & 0 \\ 0.1739 & 0.1739 \end{bmatrix}.$$

d) The controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix}$$
$$\approx \begin{bmatrix} 0 & 0 & -0.2105 & 0.2105 & 0 & 0 & 0.0489 & -0.0489 \\ -0.2105 & 0.2105 & 0 & 0 & 0.0489 & -0.0489 & 0 & 0 \\ 0 & 0 & 0.1739 & 0.1739 & 0 & 0 & -0.0791 & -0.0791 \\ 0.1739 & 0.1739 & 0 & 0 & -0.0791 & -0.0791 & 0 & 0 \end{bmatrix}.$$

Because the controllability matrix has full row rank, i.e. rank($\mathcal{C}$) $= 4 = n$, we conclude that the system is controllable.

e) Because

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} = \mathbf{Cz},$$

we obtain that

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

f) The observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.2323 & 0 & 0 & 0 \\ 0 & 0 & -0.4550 & 0 \\ 0 & -0.2323 & 0 & 0 \\ 0 & 0 & 0 & -0.4550 \end{bmatrix}.$$

Because the observability matrix has full column rank, i.e. rank($\mathcal{O}$) $= 4 = n$, we conclude from that the system is observable.

g) Because

$$\mathbf{z}_c = \begin{bmatrix} x_c \\ 0 \\ y_c \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \mathbf{Er}_c,$$

we obtain that

$$\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

h) From the equation $\mathbf{0} = \mathbf{Az}_c + \mathbf{Bu}_c$, it follows that

$$\mathbf{z}_c = -\mathbf{A}^{-1}\mathbf{Bu}_c.$$

Substituting this in $\mathbf{r}_c = \mathbf{Cz}_c$ gives

$$\mathbf{r}_c = -\mathbf{CA}^{-1}\mathbf{Bu}_c.$$

Therefore, we obtain

$$\mathbf{u}_c = \mathbf{Fr}_c,$$

with

$$\mathbf{F} = -\left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \approx \begin{bmatrix} -0.5518 & 1.3080 \\ 0.5518 & 1.3080 \end{bmatrix}.$$

i) Differentiating both sides of the equation $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{z}_c$ with respect to time gives

$$\begin{aligned} \dot{\tilde{\mathbf{z}}} &= \dot{\mathbf{z}} - \underbrace{\dot{\mathbf{z}}_c}_{=0} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} = \mathbf{A}(\tilde{\mathbf{z}} + \mathbf{z}_c) + \mathbf{B}(\tilde{\mathbf{u}} + \mathbf{u}_c) \\ &= \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}} + \underbrace{\mathbf{A}\mathbf{z}_c + \mathbf{B}\mathbf{u}_c}_{=0} = \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}}. \end{aligned}$$

Moreover, we have

$$\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{r}_c = \mathbf{C}\mathbf{z} - \mathbf{C}\mathbf{z}_c = \mathbf{C}(\mathbf{z} - \mathbf{z}_c) = \mathbf{C}\tilde{\mathbf{z}}.$$

Therefore, we obtain

$$\begin{aligned} \dot{\tilde{\mathbf{z}}} &= \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}}, \\ \tilde{\mathbf{r}} &= \mathbf{C}\tilde{\mathbf{z}}. \end{aligned}$$

j) The positive definite solution $\mathbf{W}$ of the Riccati equation

$$\mathbf{A}^T\mathbf{W} + \mathbf{W}\mathbf{A} + \mathbf{C}^T\mathbf{Q}\mathbf{C} - \mathbf{W}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{W} = \mathbf{0}$$

is given by

$$\mathbf{W} \approx \begin{bmatrix} 15.3189 & 9.3056 & 0 & 0 \\ 9.3056 & 14.4899 & 0 & 0 \\ 0 & 0 & 16.1400 & 8.4459 \\ 0 & 0 & 8.4459 & 16.7105 \end{bmatrix}.$$

Substituting this in the equation $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{W}$ gives

$$\mathbf{K} \approx \begin{bmatrix} -1.9591 & -3.0505 & 1.4688 & 2.9062 \\ 1.9591 & 3.0505 & 1.4688 & 2.9062 \end{bmatrix}.$$

Note that $\mathbf{K}$ can be obtained by using the MATLAB function lqr. The function lqr assumes that the cost function for the linear quadratic regulator is a function of the state and the input of the system. We can write the cost function

$$J_{LQR} = \int_0^\infty \tilde{\mathbf{r}}^T(t)\mathbf{Q}\tilde{\mathbf{r}}(t) + \tilde{\mathbf{u}}^T(t)\mathbf{R}\tilde{\mathbf{u}}(t)dt$$

as a function of the state and the input of the error system using the equation $\tilde{\mathbf{r}} = \mathbf{C}\tilde{\mathbf{z}}$, which leads to

$$J_{LQR} = \int_0^\infty \tilde{\mathbf{z}}^T(t)\mathbf{C}^T\mathbf{Q}\mathbf{C}\tilde{\mathbf{z}}(t) + \tilde{\mathbf{u}}^T(t)\mathbf{R}\tilde{\mathbf{u}}(t)dt.$$

Hence, we have that the weighting matrices are given by $\mathbf{C}^T\mathbf{Q}\mathbf{C}$ and $\mathbf{R}$. The matrix $\mathbf{K}$ can be obtained from the command $\mathsf{K} = \mathsf{lqr}(\mathsf{A}, \mathsf{B}, \mathsf{C}.' * \mathsf{Q} * \mathsf{C}, \mathsf{R})$. Alternatively, the MATLAB function lqry can be used.

k) The control input $\mathbf{u}$ is computed as follows:

$$\begin{aligned}\mathbf{u} &= \tilde{\mathbf{u}} + \mathbf{u}_c = -\mathbf{K}\tilde{\mathbf{z}} + \mathbf{F}\mathbf{r}_c = -\mathbf{K}(\mathbf{z} - \mathbf{z}_c) + \mathbf{F}\mathbf{r}_c \\ &= -\mathbf{K}(\mathbf{z} - \mathbf{E}\mathbf{r}_c) + \mathbf{F}\mathbf{r}_c = -\mathbf{K}\mathbf{z} + (\mathbf{K}\mathbf{E} + \mathbf{F})\mathbf{r}_c.\end{aligned}$$

Therefore, we have

$$\mathbf{u} = -\mathbf{M}_1\mathbf{z} + \mathbf{M}_2\mathbf{r}_c,$$

with

$$\mathbf{M}_1 = \mathbf{K} \approx \begin{bmatrix} -1.9591 & -3.0505 & 1.4688 & 2.9062 \\ 1.9591 & 3.0505 & 1.4688 & 2.9062 \end{bmatrix}$$

and

$$\mathbf{M}_2 = \mathbf{K}\mathbf{E} + \mathbf{F} \approx \begin{bmatrix} -2.5109 & 2.7768 \\ 2.5109 & 2.7768 \end{bmatrix}.$$

l) The extended error system is obtained from the equations

$$\begin{aligned}\dot{\underline{\tilde{\mathbf{z}}}} &= \begin{bmatrix} \dot{\tilde{\mathbf{z}}} \\ \dot{\tilde{\mathbf{R}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}} \\ \tilde{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}} \\ \mathbf{C}\tilde{\mathbf{z}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{R}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \underline{\tilde{\mathbf{z}}} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{u}}\end{aligned}$$

and

$$\underline{\tilde{\mathbf{r}}} = \begin{bmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{R}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}\tilde{\mathbf{z}} \\ \tilde{\mathbf{R}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{R}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \underline{\tilde{\mathbf{z}}},$$

where $\mathbf{I}$ is the identity matrix. Therefore, we have

$$\begin{aligned}\dot{\underline{\tilde{\mathbf{z}}}} &= \underline{\mathbf{A}}\underline{\tilde{\mathbf{z}}} + \underline{\mathbf{B}}\tilde{\mathbf{u}}, \\ \underline{\tilde{\mathbf{r}}} &= \underline{\mathbf{C}}\underline{\tilde{\mathbf{z}}},\end{aligned}$$

with

$$\underline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \qquad \underline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

m) The positive definite solution $\underline{\mathbf{W}}$ of the Riccati equation

$$\underline{\mathbf{A}}^T\underline{\mathbf{W}} + \underline{\mathbf{W}}\underline{\mathbf{A}} + \underline{\mathbf{C}}^T\underline{\mathbf{Q}}\underline{\mathbf{C}} - \underline{\mathbf{W}}\underline{\mathbf{B}}\underline{\mathbf{R}}^{-1}\underline{\mathbf{B}}^T\underline{\mathbf{W}} = \mathbf{0}$$

is given by

$$\underline{\mathbf{W}} \approx \begin{bmatrix} 21.5453 & 13.4854 & 0 & 0 & 5.1933 & 0 \\ 13.4854 & 17.4432 & 0 & 0 & 3.3588 & 0 \\ 0 & 0 & 21.8262 & 13.0697 & 0 & 5.1127 \\ 0 & 0 & 13.0697 & 20.7875 & 0 & 4.0659 \\ 5.1933 & 3.3588 & 0 & 0 & 4.7954 & 0 \\ 0 & 0 & 5.1127 & 4.0659 & 0 & 5.0643 \end{bmatrix}.$$

Substituting this in the equation $\underline{\mathbf{K}} = \underline{\mathbf{R}}^{-1}\underline{\mathbf{B}}^T\underline{\mathbf{W}}$ gives

$$\underline{\mathbf{K}} \approx \begin{bmatrix} -2.8390 & -3.6722 & 2.2730 & 3.6152 & -0.7071 & 0.7071 \\ 2.8390 & 3.6722 & 2.2730 & 3.6152 & 0.7071 & 0.7071 \end{bmatrix}.$$

n) For ease of notion, we introduce the matrices

$$\underline{\mathbf{K}}_1 \approx \begin{bmatrix} -2.8390 & -3.6722 & 2.2730 & 3.6152 \\ 2.8390 & 3.6722 & 2.2730 & 3.6152 \end{bmatrix}$$

and

$$\underline{\mathbf{K}}_2 \approx \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix},$$

such that $\underline{\mathbf{K}} = \begin{bmatrix} \underline{\mathbf{K}}_1 & \underline{\mathbf{K}}_2 \end{bmatrix}$. The control input $\mathbf{u}$ with integral effect is given by

$$\begin{aligned} \mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_c &= -\underline{\mathbf{K}}\tilde{\underline{\mathbf{z}}} + \mathbf{F}\mathbf{r}_c = -\underline{\mathbf{K}}_1\tilde{\mathbf{z}} - \underline{\mathbf{K}}_2\tilde{\mathbf{R}} + \mathbf{F}\mathbf{r}_c \\ &= -\underline{\mathbf{K}}_1(\mathbf{z} - \mathbf{z}_c) - \underline{\mathbf{K}}_2 \int_0^t \tilde{\mathbf{r}}(\tau) d\tau + \mathbf{F}\mathbf{r}_c \\ &= -\underline{\mathbf{K}}_1(\mathbf{z} - \mathbf{E}\mathbf{r}_c) - \underline{\mathbf{K}}_2 \int_0^t \mathbf{r}(\tau) - \mathbf{r}_c d\tau + \mathbf{F}\mathbf{r}_c \\ &= -\underline{\mathbf{K}}_1\mathbf{z} + (\underline{\mathbf{K}}_1\mathbf{E} + \mathbf{F})\mathbf{r}_c - \underline{\mathbf{K}}_2 \int_0^t \mathbf{C}\mathbf{z}(\tau) - \mathbf{r}_c d\tau. \end{aligned}$$

Therefore, the control input $\mathbf{u}$ can be written as

$$\mathbf{u} = -\underline{\mathbf{M}}_1\mathbf{z} + \underline{\mathbf{M}}_2\mathbf{r}_c - \underline{\mathbf{M}}_3 \int_0^t \mathbf{C}\mathbf{z}(\tau) - \mathbf{r}_c d\tau,$$

with

$$\underline{\mathbf{M}}_1 = \underline{\mathbf{K}}_1 \approx \begin{bmatrix} -2.8390 & -3.6722 & 2.2730 & 3.6152 \\ 2.8390 & 3.6722 & 2.2730 & 3.6152 \end{bmatrix},$$

$$\underline{\mathbf{M}}_2 = \underline{\mathbf{K}}_1\mathbf{E} + \mathbf{F} \approx \begin{bmatrix} -3.3908 & 3.5810 \\ 3.3908 & 3.5810 \end{bmatrix}$$

and

$$\underline{\mathbf{M}}_3 = \underline{\mathbf{K}}_2 \approx \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}.$$

o) The plots of x and y as a function of time are given in Fig. 1 and Fig. 2, respectively. The corresponding Simulink model is depicted in Fig. 3.

p) Looking at the plots in Fig. 1 and Fig. 2, we see that without integral effect the solutions for x and y have a static deviation with respect to their corresponding set-point values x_c and y_c (the set-point variables x and y do not converge to the set-point values x_c and y_c, but settle at a distance from x_c and y_c). The plots clearly show that with integral effect, the static deviation is removed. However, a possible downside to adding an integral effect is that we can have a larger overshoot (as shown in Fig. 1) due to integrator windup when the set-point variable has not converged yet.

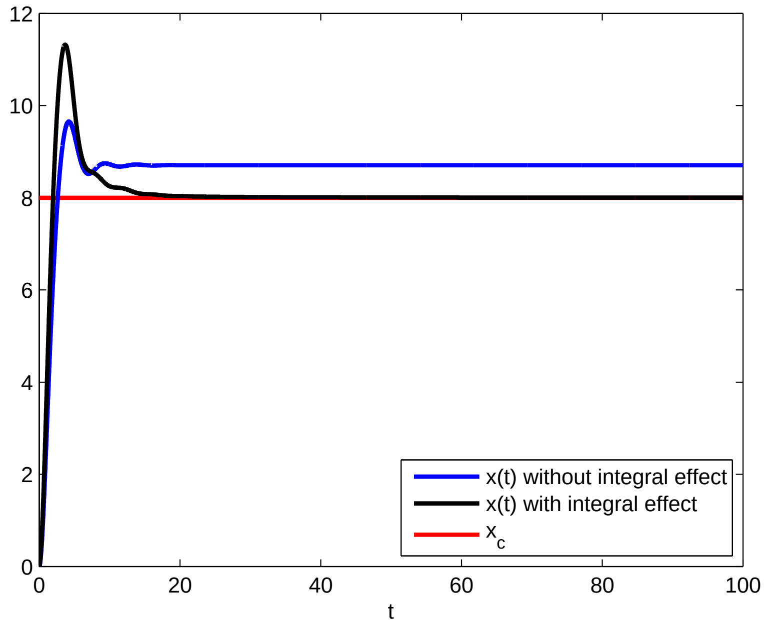

Fig. 1: Set-point variable x.

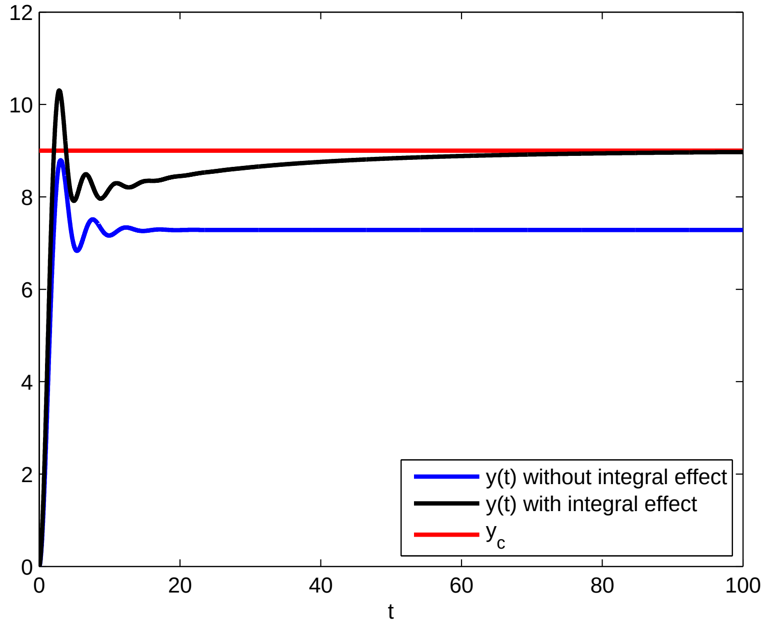

Fig. 2: Set-point variable y.

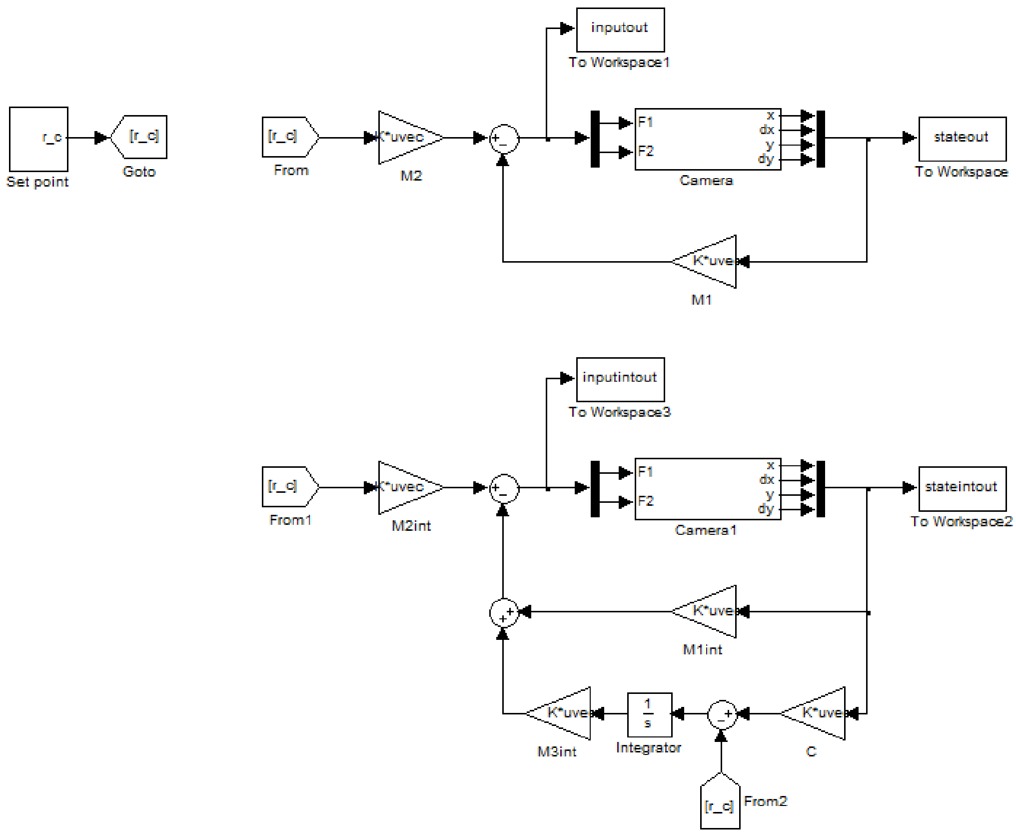

Fig. 3: Simulink model.

q) To get a faster response, the values on the diagonal of $\mathbf{Q}$ need to be increased. (Alternatively, the values on the diagonal of $\mathbf{R}$ can be decreased.) Simulation results for different values of $\mathbf{Q}$ (while $\mathbf{R}$ is kept constant) are shown in Fig. 4 and Fig. 5. From this results that larger values on the diagonal of $\mathbf{Q}$ increase the convergence speed. (Note that if the values on the diagonal of $\mathbf{Q}$ are increase, the corresponding control input is increased as well.)

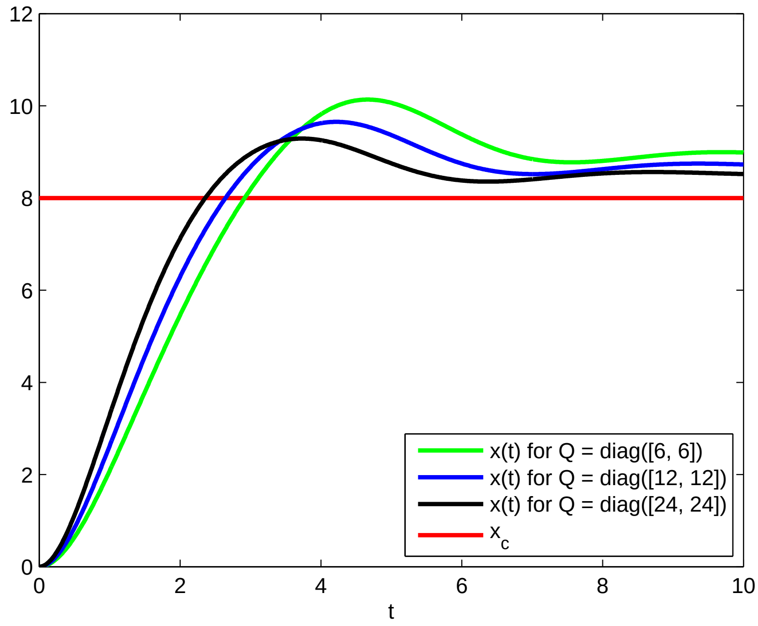

Fig. 4: Set-point variable x for different values of Q.

Problem 2: Output-feedback controllers

a) Substituting the output equation $y(t) = \mathbf{C}\mathbf{x}(t) + v$ in the equation for the controller $u(t) = -k_p y(t)$, we immediately obtain

$$u(t) = -k_p \mathbf{C}\mathbf{x}(t) - k_p v.$$

Hence, we have

$$u(t) = -\mathbf{K}_p \mathbf{x}(t) + q_p,$$

with $\mathbf{K}_p = k_p \mathbf{C}$ and $q_p = -k_p v$.

b) Substituting the control lay $u(t) = -\mathbf{K}_p \mathbf{x}(t) + q_p$ in the state equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{w}$, we obtain the closed-loop system dynamics

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K}_p)\mathbf{x}(t) + \mathbf{B}q_p + \mathbf{w}.$$

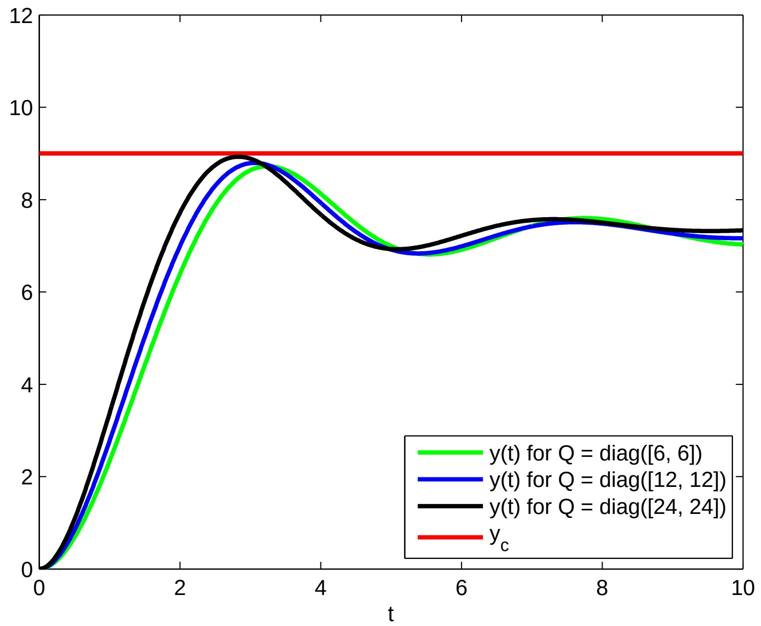

Fig. 5: Set-point variable y for different values of Q.

The stability of the closed-loop system is only determined by the system matrix $\mathbf{A} - \mathbf{B}\mathbf{K}_p$. If the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ all have a negative real part, the system is asymptotically stable. Note, however, that due to the constant disturbances, the state $\mathbf{x}(t)$ and output $y(t)$ do not necessarily converge to the origin even if the system is asymptotically stable.

The eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$, which is given by

$$\det(\mathbf{A} - \mathbf{B}\mathbf{K}_p - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 2 \\ -1-2k_p & -2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 4k_p - 6.$$

The eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ are equal to the roots the characteristic polynomial of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$. Hence, we obtain the eigenvalues

$$\lambda_{1,2} = 1 \pm \sqrt{7 - 4k_p}.$$

From this, we conclude that there exists no value of k_p such that both eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ have a negative real part. Therefore, there exists no value of k_p such that the closed-loop system is asymptotically stable.

c) Using that the disturbance v is constant, we obtain from $y(t) = \mathbf{C}\mathbf{x}(t) + v$ that

$$\dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t).$$

Substituting $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{w}$ yields

$$\dot{y}(t) = \mathbf{C}\mathbf{A}\mathbf{x}(t) + \mathbf{C}\mathbf{B}u(t) + \mathbf{C}\mathbf{w}.$$

When we substitute this and $y(t) = \mathbf{C}\mathbf{x}(t) + v$ in the control law $u(t) = -k_p y(t) - k_d \dot{y}(t)$, we get

$$\begin{aligned} u(t) &= -k_p\mathbf{C}\mathbf{x}(t) - k_p v - k_d\mathbf{C}\mathbf{A}\mathbf{x}(t) - k_d\mathbf{C}\mathbf{B}u(t) - k_d\mathbf{C}\mathbf{w} \\ &= -(k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A})\mathbf{x}(t) - k_d\mathbf{C}\mathbf{B}u(t) - k_p v - k_d\mathbf{C}\mathbf{w}. \end{aligned}$$

It follows that

$$(1 + k_d\mathbf{C}\mathbf{B})u(t) = -(k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A})\mathbf{x}(t) - k_p v - k_d\mathbf{C}\mathbf{w},$$

which leads to

$$u(t) = -\frac{k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A}}{1 + k_d\mathbf{C}\mathbf{B}}\mathbf{x}(t) - \frac{k_p v + k_d\mathbf{C}\mathbf{w}}{1 + k_d\mathbf{C}\mathbf{B}}.$$

Hence, we obtain

$$u(t) = -\mathbf{K}_{pd}\mathbf{x}(t) + q_{pd},$$

with

$$\mathbf{K}_{pd} = \frac{k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A}}{1 + k_d\mathbf{C}\mathbf{B}} \qquad \text{and} \qquad q_{pd} = -\frac{k_p v + k_d\mathbf{C}\mathbf{w}}{1 + k_d\mathbf{C}\mathbf{B}}.$$

d) Substituting the values of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the expression for $\mathbf{K}_{pd}$, we have

$$\mathbf{K}_{pd} = \frac{k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A}}{1 + k_d\mathbf{C}\mathbf{B}} = \begin{bmatrix} k_p + 4k_d & 2k_d \end{bmatrix}.$$

The eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_{pd}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{B}\mathbf{K}_{pd}$, which is given by

$$\det(\mathbf{A} - \mathbf{B}\mathbf{K}_{pd} - \lambda\mathbf{I}) = \begin{vmatrix} 4 - \lambda & 2 \\ -1 - 2k_p - 8k_d & -2 - 4k_d - \lambda \end{vmatrix} = \lambda^2 + (4k_d - 2)\lambda + 4k_p - 6.$$

If the eigenvalues $\lambda_{1,2}$ are equal to $\lambda_{1,2} = -1 \pm i$, then the characteristic polynomial is given by

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda) = (-1 + i - \lambda)(-1 - i - \lambda) = \lambda^2 + 2\lambda + 2.$$

Hence, we obtain the equations

$$4k_d - 2 = 2 \qquad \text{and} \qquad 4k_p - 6 = 2.$$

Solving for k_p and k_d yields $k_p = 2$ and $k_d = 1$.

e) Substituting the control law $u(t) = -\mathbf{K}_{pd}\mathbf{x}(t) + q_{pd}$ in the state-space equation, we obtain

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K}_{pd})\mathbf{x}(t) + \mathbf{B}q_{pd} + \mathbf{w},$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + v.$$

Because k_p and k_d are chosen such that the poles of the closed-loop system are equal to $-1 \pm i$, the system is asymptotically stable. For constant disturbances, this implies that the state $\mathbf{x}(t)$ will converge to an equilibrium point as t goes to infinity. Therefore, we have $\lim_{t\to\infty} \dot{\mathbf{x}}(t) = 0$. From the state-space equation, for $\dot{\mathbf{x}}(t) = 0$, the state $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = -(\mathbf{A} - \mathbf{B}\mathbf{K}_{pd})^{-1}(\mathbf{B}q_{pd} + \mathbf{w}).$$

Note that $\mathbf{A} - \mathbf{B}\mathbf{K}_{pd}$ is invertible because none of the poles of the closed-loop system, and therefore none of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_{pd}$, are equal to zero. Substituting $k_p = 2$ and $k_d = 1$ and the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in the expressions for $\mathbf{K}_{pd}$ and q_{pd}, we obtain

$$\mathbf{K}_{pd} = \frac{k_p\mathbf{C} + k_d\mathbf{C}\mathbf{A}}{1 + k_d\mathbf{C}\mathbf{B}} = \begin{bmatrix} 6 & 2 \end{bmatrix} \qquad \text{and} \qquad q_{pd} = -\frac{k_p v + k_d\mathbf{C}\mathbf{w}}{1 + k_d\mathbf{C}\mathbf{B}} = -4.$$

Subsequently, it follows that

$$\begin{aligned} \mathbf{x}(t) &= -(\mathbf{A} - \mathbf{B}\mathbf{K}_{pd})^{-1}(\mathbf{B}q_{pd} + \mathbf{w}) \\ &= -\begin{bmatrix} 4 & 2 \\ -13 & -6 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -\begin{bmatrix} -3 & -1 \\ 6.5 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -12 \\ 25 \end{bmatrix}. \end{aligned}$$

The corresponding steady-state output is given by

$$y(t) = \mathbf{C}\mathbf{x}(t) + v = -12 + 3 = -9.$$