TTK4115

Lecture 1

Introduction, LTI systems - representations and solutions

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This lecture

1. Motivation, Course goals
   - What will you learn?
   - 7 main topics
2. LTI systems
   - State space models
   - Transfer functions
   - Solutions
     - Convolution
   - More connections
3. Summary
4. Next time

Topic

1. Motivation, Course goals

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Learning objectives

- Basic *theory* of linear multi-input multi-output (MIMO) systems, important concepts like controllability, observability, stability
- Understanding state-space theory as an alternative to frequency domain theory
- Basic introduction to the theory of random systems and signals
- Knowledge about design of linear controllers and state estimators, including the Linear-Quadratic optimal controller and the Kalman Filter
- Skills in practical use of the theory through projects and labs (experimental and simulations using Matlab and Simulink)
- Get a solid basis for further studies in control engineering; Optimization and Control; Nonlinear Systems; Adaptive Control; Navigation and Vehicle Control; Advanced Process Control, Robotics; .....

Motivation

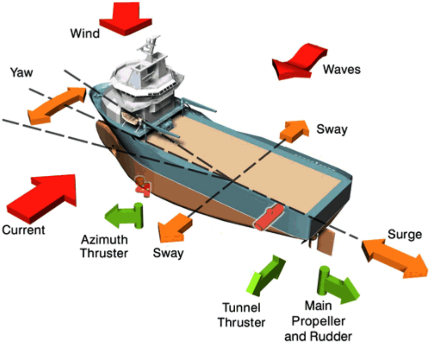
In the first course on control engineering, most of you have focused on frequency domain methods and SISO-systems.

Why state space theory?

- Mathematical description which easily handles many physical models
- Easier way to handle multiple-input multiple-output (MIMO) systems (in most cases)
- Generalizes to nonlinear systems
- Many advanced control design and estimation methods have been developed for systems on state space form

Example: Dynamic positioning system

Most of what you learn in this course is applied here!



Curriculum

1. Chi-Tsong Chen: Linear system theory and design. 4th International. New York: Oxford University Press, 2014. ISBN: 9780199964543
2. Robert G. Brown & Patrick Y. C. Hwang: Introduction to random signals and applied Kalman filtering : with MATLAB exercises. 4th. Hoboken, NJ: John Wiley & Sons, Inc., 2012. ISBN: 9780470609699

Style of lectures

- Due to the corona virus we can't have ordinary lectures in the auditorium
- Plan for this year (may be revised!):
  - One or more video lectures will be published for each week's topics
  - Full set of slides will be made available on Blackboard for each week
  - Weekly Zoom/Blackboard session for answering questions

Topic

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What will you learn? 7 main topics

1. Solution of state equations

Consider the *state equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

and the *output equation*

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).$$

Given the initial condition $\mathbf{x}(t_0)$ and the input $\mathbf{u}(t)$, $t_0 \leq t \leq t_1$. What is $\mathbf{x}(t_1)$ and $\mathbf{y}(t_1)$?

Why?

Basic knowledge. Understand the structure of the solution.

What will you learn? 7 main topics

2. Stability

What happens if we change the initial conditions $\mathbf{x}(0)$ of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ slightly?
Given the model $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$: if our input signals are bounded, will the outputs be bounded?

Why?

- Fundamental property of a dynamic system.
- Simpler to analyse stability than to solve the differential equations.
- Different formulations and techniques.
- Extensions to nonlinear systems.

What will you learn? 7 main topics

3. Canonical forms of state equations

By a change of coordinates, transform from

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

to

$$\begin{aligned}\dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t) \\ \mathbf{y}(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\mathbf{u}(t)\end{aligned}$$

where the new matrices $\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}$ have some desired properties, e.g. $\bar{\mathbf{A}}$ being diagonal, or makes the control design easier.

Why?

Basic knowledge. Understand your alternatives.

What will you learn? 7 main topics

4. Realizations

Given $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$. Find a state space model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

Why?

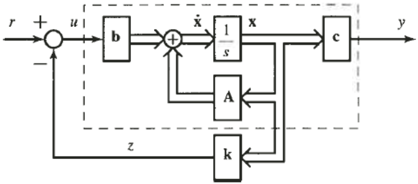
- Enables state space analysis.
- Basis for implementing filters, controllers and estimators in software and analog hardware.

What will you learn? 7 main topics

5. Controllability, State Feedback, Linear Quadratic Regulator (LQR)

We want to use the feedback $\mathbf{u}(t) = -\mathbf{k}\mathbf{x}(t)$. How do we choose $\mathbf{k}$ to get a desirable response? When can we control a system to a desired state?

Controllability: Do we have the right actuators to control the system to the desired set-point?



Why?

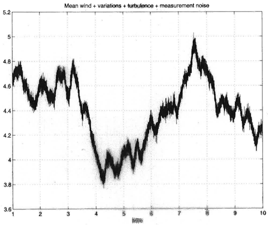
Control design approach for MIMO systems.

What will you learn? 7 main topics

6. Representing random signals and systems

How to describe a stochastic/random signal $u(t)$ in term of its statistical properties?

How to describe the response $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$ when $u(t)$ is a random input signal?



Why?

Analyse and minimize the effects of unknown measurement noise, disturbances and other uncertainty.

What will you learn? 7 main topics

7. Observability, state estimation, Kalman Filter

Assume the model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

Given $\mathbf{u}(t)$, $\mathbf{y}(t)$ for any $t_0 \leq t \leq t_1$. What is $\mathbf{x}(t_0)$? What is $\mathbf{x}(t_1)$?
Observability: Do we have the right measurements? What if our measurements are correlated with noise? What if estimated states are used for control feedback?

Why?

In real world applications we often cannot measure all states
Estimation allows us to compute what we cannot measure - combining sensors and model

Topic

Representations of linear systems

We will encounter three different representations of linear systems in this course. Our models will appear as:

State-space models

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned}$$

Transfer functions

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$$

Input-output descriptions

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$$

First week's lectures

In the first week's lectures we will see how these forms are connected.
Only lumped LTI systems are considered.

LTI: **L**inear **T**ime **I**nvariant

Topic

LTI systems overview

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}}$$

State-space model

Most of our models will be put on the state space form.

System states

Definition of state

The state $\mathbf{x}(t_0)$ of a system at a time t_0 is the information at t_0 that, together with the input $\mathbf{u}(t)$, for $t \geq t_0$, determines uniquely the output $\mathbf{y}(t)$ for all $t \geq t_0$.

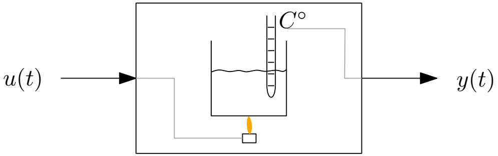
Explanation

- We do not need $\mathbf{u}(t)$ for $t < t_0$ to predict what will happen after t_0.
- We do not need $\mathbf{x}(t)$ for $t < t_0$ to predict what will happen after t_0.
- The state acts as a memory of these!

Concise statement

$$\left.\begin{array}{ll} \mathbf{x}(t_0) & \\ \mathbf{u}(t), & t > t_0 \end{array}\right\} \rightarrow \quad \mathbf{y}(t), \quad t \geq t_0$$

Water heater



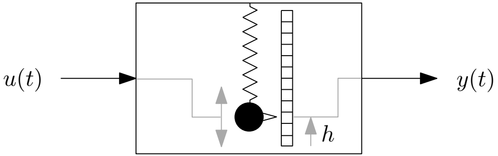
Question

What could be the state of the system?

Hints

If we know how hot the water is now at t_0....
... do we need to know the past temperatures to predict the future ones?

Pendulum



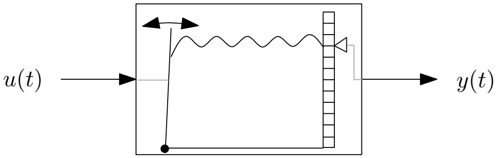
Question

What could be the states of the system? How many are there?

Hints

We know the position of the mass at t_0.
Since the mass may be moving vertically, we also need to know its speed at t_0.

Wave tank



Question

What could be the states of the system? How many are there?

Answer

The height of the water surface varies with time and along the horizontal axis.
This a **distributed** system, it has *infinitely*[1] many states!

[1] Systems with a *finite* number of states are called **lumped**

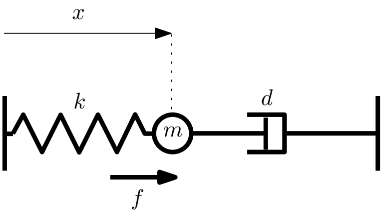
About states

- The choice of state variables is not unique
- The state does not have to be measurable
- The state does not have to be a physical quantity
- The state summarizes the history of the system

Example: Mass spring damper

Second order dynamics:

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = f(t)$$



Input/Output: force and displacement

$$y(t) \triangleq x(t), \quad u(t) \triangleq f(t)$$

States: displacement and velocity

$$x_1(t) \triangleq x(t), \quad x_2(t) \triangleq \dot{x}(t)$$

Example: Mass spring damper

Second order dynamics:

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = f(t)$$

Input/Output and states

$$y(t) \triangleq x(t), \quad u(t) \triangleq f(t)$$
$$x_1(t) \triangleq x(t), \quad x_2(t) \triangleq \dot{x}(t)$$

Array form

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Topic

LTI systems overview

$\mathcal{L}(\ldots)$

$$\boxed{\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] \\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}}$$

Laplace Transform

We can convert state space models (time domain) to the frequency domain with the **Laplace Transform**.

Transfer functions

Transfer functions model the system in the *frequency domain*.

State-space model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

Transform rule

$$\dot{\mathbf{x}}(t) \quad \Leftrightarrow \quad s\hat{\mathbf{x}}(s) - \mathbf{x}_0$$

Applying the rule

$$\begin{aligned}s\hat{\mathbf{x}}(s) - \mathbf{x}_0 &= \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s) \\ (s\mathbb{I} - \mathbf{A})\hat{\mathbf{x}}(s) &= \mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\end{aligned}$$

Laplace Transform of the state space model

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbb{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] \\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

Zero-state response, Zero-input response

Response

$$\hat{\mathbf{y}}(s) = \overbrace{\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{x}_0}^{\text{Zero-input}} + \overbrace{\left[\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\hat{\mathbf{u}}(s)}^{\text{Zero-state}}$$

Transfer matrix

In many cases the zero-input response is uninteresting because:

1. The zero-input response dies away exponentially fast if the system is stable.
2. The plant is initialized at $\mathbf{x}_0 = \mathbf{0}$.

Then the we may focus on the input/output response:

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s), \quad \hat{\mathbf{G}}(s) = \underbrace{\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}_{\text{transfer matrix}}$$

which is determined by the *transfer matrix* $\hat{\mathbf{G}}(s)$.

Example: Mass spring damper in the Laplace domain

Model

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Transform

Using:

$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1} \left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] + \mathbf{D}\hat{\mathbf{u}}(s)$$

leads to:

$$\hat{y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \hat{u}(s) \right)$$

Example: Mass spring damper in the Laplace domain

Laplace model:

$$\hat{y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \hat{u}(s) \right)$$

Inversion:

$$\left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} = \frac{1}{ms^2 + sd + k} \begin{bmatrix} d + ms & m \\ -k & ms \end{bmatrix}$$

Result

$$\hat{y}(s) = \underbrace{\frac{1}{ms^2 + sd + k} \begin{bmatrix} d + ms & m \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{Zero input response}} + \underbrace{\frac{1}{ms^2 + sd + k}}_{\hat{g}(s)} \hat{u}(s)$$

Topic

LTI systems overview

$\mathcal{L}(\dots)$

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]$$
$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

LTI sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Solutions in time

We may need to know the behavior of the system in time. This requires a solution of the differential state space model.

Solutions of LTI state equations

We want to solve:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

Initial value problem

We wish to find $\mathbf{y}(t)$ given $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{u}(t)$.

First order homogenous case

Let's try something simpler first:

$$\dot{x}(t) = ax(t), \quad y(t) = cx(t)$$

No input, *one* state.

Step 1: Premultiply and reformulate

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) \quad \Rightarrow \frac{d}{dt}[e^{-at}x(t)] = 0$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}[e^{-a\tau}x(\tau)]d\tau = e^{-at}x(t) - \underbrace{e^{-a0}x(0)}_{x(0)} = 0$$

Step 3: Recover solution

$$e^{-at}x(t) = x_0 \quad \Rightarrow x(t) = x_0 e^{at}$$

First order homogenous case

Insertion of solution:

$$\overbrace{\frac{d}{dt}[e^{at}x_0]}^{\dot{x}} = \overbrace{ae^{at}x_0}^{ax}$$

Check of the initial condition

$$\overbrace{e^{at}x_0\Big|_{t=0}}^{x(0)} = x_0$$

First order nonhomogenous case

Let's try something harder:

$$\dot{x}(t) = ax(t) + bu(t), \quad y(t) = cx(t) + du(t)$$

One input, *one* state.

Step 1: Premultiply and reformulate

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) + e^{-at}bu(t) \quad \Rightarrow \frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t)$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}[e^{-a\tau}x(\tau)]d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad \Rightarrow e^{-at}x(t) - \underbrace{e^{-a0}x(0)}_{x(0)} = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

Step 3: Recover solution

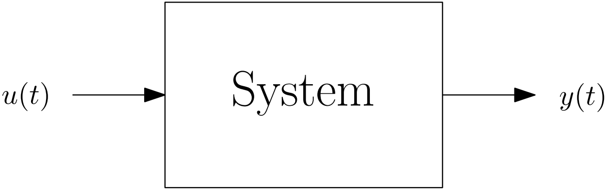
$$e^{-at}x(t) = x_0 + \int_0^t e^{-a\tau}bu(\tau)d\tau \quad \Rightarrow x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

Output response

$$y(t) = cx(t) + du(t) = \overbrace{ce^{at}x_0}^{\text{Free decay}} + c\overbrace{\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}^{\text{Convolution integral}} + du(t)$$

Convolution

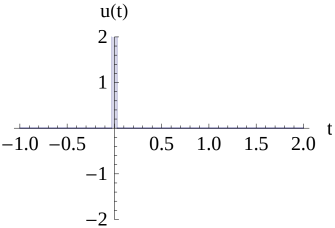
Consider now the SISO system:

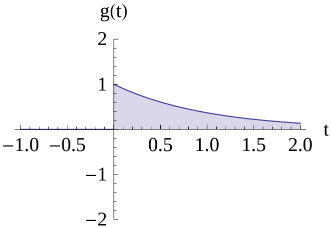


Convolution

Experiment 1

Jolt at t=0: $u(t) = 1 \times \delta(t)$





Impulse response

$y(t) = g(t)$ is the system's response to an impulse.

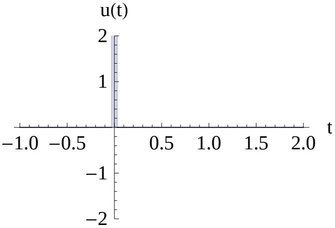
Causality

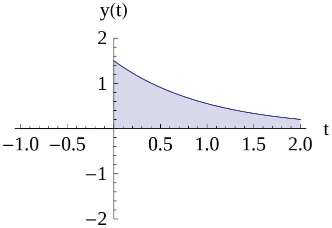
There is no response *before* the impulse.

Convolution

Experiment 2

Larger jolt: $u(t) = 1.5 \times \delta(t)$





Result

The response grows proportionally with the jolt magnitude.

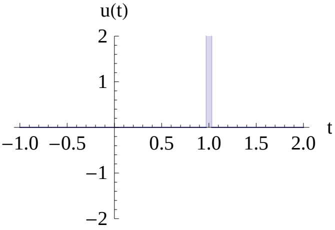
Homogeneity

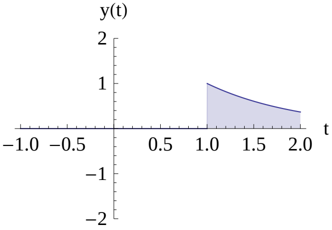
$$\alpha u(t) \to \alpha y(t)$$

Convolution

Experiment 3

Jolt is shifted to t=1: $u(t) = 1 \times \delta(t-1)$





Result

The response is the same, just shifted $y(t) = g(t-1)$

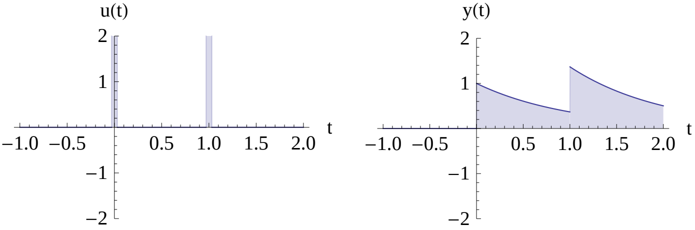
Time invariance

$$u(t-T) \rightarrow y(t-T)$$

Convolution

Experiment 4

Two jolts are superposed: $u(t) = 1 \times \delta(t) + 1 \times \delta(t-1)$



Result

The responses are superposed.

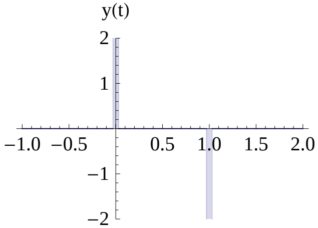
Additivity

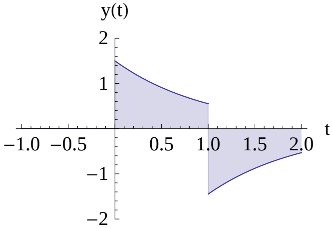
$$u_1(t) + u_2(t) \to y_1(t) + y_2(t)$$

Convolution

Experiment 5

Two jolts with different amplitudes are added: $u(t) = 1.5 \times \delta(t) - 2 \times \delta(t-1)$





Result

We see that the responses have been added proportionally.

Superposition

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Convolution

The idea

Convolution is simply:

- adding up all the impulse responses
- shifted to the time when the input happened: τ
- scaled proportionally with the input

In mathematics

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Output response

$$y(t) = cx(t) + du(t) = \overbrace{ce^{at}x_0}^{\text{Free decay}} + \overbrace{c\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}^{\text{Convolution integral}} + du(t)$$

The output response of an LTI system is described in terms of *convolution*.

Solution of an LTI system

Let's go for the whole thing:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

p inputs, n states, q outputs.

Step 1: Premultiply and reformulate

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Wait a minute!

What is $e^{-\mathbf{A}t}$?

The matrix exponential

A matrix analogue to the scalar exponential function
$e^{\mathbf{A}t}$ is the most important function of the matrix $\mathbf{A}$.

Analogy to e^{at}

The ordinary exponential function is defined by series expansion:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

Series expansion

The matrix exponential is defined analogously:

$$e^{\mathbf{A}t} = \mathbb{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$$

Computing $e^{\mathbf{A}t}$

We will learn how to compute the matrix exponential *exactly* next lesson.

Important properties of $e^{\mathbf{A}t}$

Properties

$$\begin{aligned} e^{\mathbf{0}} &= \mathbb{I} \\ e^{\mathbf{A}(t_1+t_2)} &= e^{\mathbf{A}t_1} e^{\mathbf{A}t_2} \\ \left[e^{\mathbf{A}t}\right]^{-1} &= e^{-\mathbf{A}t} \end{aligned}$$

Derivative of $e^{\mathbf{A}t}$

$$\begin{aligned} \frac{d}{dt}[e^{\mathbf{A}t}] = \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k\right] &= \sum_{k=1}^{\infty} \frac{t^{(k-1)}}{(k-1)!}\mathbf{A}^k \\ = \quad \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^{k+1} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \end{aligned}$$

Step 1: Premultiply and reformulate

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \quad \Rightarrow \frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}\left[e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right] d\tau = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad \Rightarrow e^{-\mathbf{A}t}\mathbf{x}(t) - \underbrace{e^{-\mathbf{A}0}\mathbf{x}(0)}_{\mathbf{x}(0)} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Step 3: Recover solution

$$e^{-\mathbf{A}t}\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad \Rightarrow \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Output response

$$\mathbf{y}(t) = \overbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}^{\text{Zero-input}} + \overbrace{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}^{\text{Zero-state}}$$

Impulse response matrix

$$\mathbf{G}(t-\tau) = \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B} + \mathbf{D}\delta(t-\tau)$$

Example: Mass spring damper solution

Definitions

We redefine the constants for simplicity:

$$k/m = \omega_0^2, \quad d = 0$$

Model

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Example: Mass spring damper solution

Response

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t)$$

F.y.i.

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos(t\omega_0) & \frac{\sin(t\omega_0)}{\omega_0} \\ -\omega_0\sin(t\omega_0) & \cos(t\omega_0) \end{bmatrix}$$

Response

$$y(t) = \underbrace{\begin{bmatrix} \cos(t\omega_0) & \frac{\sin(t\omega_0)}{\omega_0} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{Zero input response}} + \int_0^t \underbrace{\frac{\sin[(t-\tau)\omega_0]}{m\omega_0}}_{g(t-\tau)} u(\tau)d\tau$$

Topic

LTI systems overview

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

$\mathcal{L}(\dots)$

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

$\mathcal{L}^{-1}(\dots)$

LTI sol.

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

Inverse Laplace transform

The inverse Laplace transform gives us another route to the solution.

Frequency domain model

$$\mathbf{y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)\right] + \mathbf{D}\mathbf{u}(s)$$

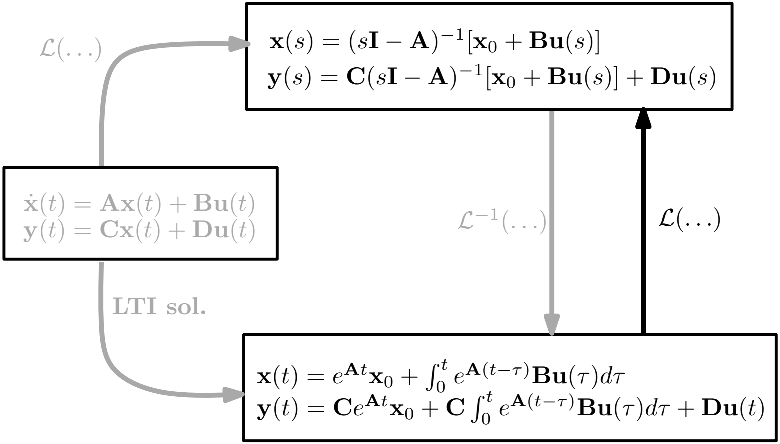
Important relations

$$\begin{aligned} \mathcal{L}^{-1}[(s\mathbb{I} - \mathbf{A})^{-1}] &= e^{\mathbf{A}t} \\ \mathcal{L}^{-1}[f(s)g(s)] &= \int_0^t f(t-\tau)g(\tau)d\tau \end{aligned}$$

Inverse Laplace Transform

$$\begin{aligned} \mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{y}(s)] &= \mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)\right] + \mathbf{D}\mathbf{u}(s)] \\ &= \underbrace{\mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0]}_{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0} + \underbrace{\mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)]}_{\int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau} + \underbrace{\mathcal{L}^{-1}[\mathbf{D}\mathbf{u}(s)]}_{\mathbf{D}\mathbf{u}(t)} \end{aligned}$$

LTI systems overview



Laplace transform

It is also possible to reverse the procedure.

Topic

LTI systems overview

$$
\begin{tikzcd}
 & \boxed{\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0+\mathbf{B}\hat{\mathbf{u}}(s)]\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0+\mathbf{B}\hat{\mathbf{u}}(s)]+\mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}} \arrow[dd, shift right=2, "\mathcal{L}^{-1}(\dots)"'] \\
\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)\end{aligned}} \arrow[ur, bend left, "\mathcal{L}(\dots)"] \arrow[dr, bend right, "\textbf{LTI sol.}"'] & \\
 & \boxed{\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ \mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0+\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau+\mathbf{D}\mathbf{u}(t)\end{aligned}} \arrow[uu, shift right=2, "\mathcal{L}(\dots)"']
\end{tikzcd}
$$

Lumped LTI systems

We have seen the different flavors of Lumped LTI systems and how to convert one representation to another.

Topic

Next time

Coming up next lesson

The matrix exponential

How do we calculate $e^{\mathbf{A}t}$?

Equivalent representations

Is it possible to find more convenient state equations that do the same job?

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Discretization

Useful for simulating systems

TTK4115

Lecture 2

Equivalent representations, useful forms, functions of square matrices

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

1. Equivalent Representations

2. Diagonalization

3. Recovering the Diagonal and Jordan Forms

4. Complex eigenvalues: Modal Form

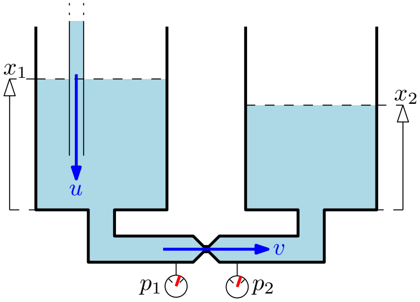
5. Physical significance of Eigenvalues/vectors

6. Functions of a Square Matrix

    Matrix Exponentials - Special Properties

Topic

Example: Tank system



Dynamics

Tank 1 balance:

$$S\dot{x}_1 = u - v = u - k\rho g[x_1 - x_2]$$

Tank 2 balance:

$$S\dot{x}_2 = v = k\rho g[x_1 - x_2]$$

Hydraulic model

The flow between the tanks is assumed proportional to the pressure differential over the constriction. The hydraulic head is $p = \rho g x$. Then:

$$v(t) = k[p_2(t) - p_1(t)] = k\rho g[x_1(t) - x_2(t)]$$

Output: Averaged tank level

$$y = \frac{1}{2}[x_1 + x_2]$$

ρ : Density of fluid
S : Tank cross-section
k : Constriction constant
g : Gravitational constant

Tank system state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad y(t) = \mathbf{c}\mathbf{x}(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Change of basis

Now define two *alternative* states:

$$\bar{x}_1(t) \triangleq \underbrace{\frac{1}{2}[x_1(t) + x_2(t)]}_{\text{Average}}, \quad \bar{x}_2(t) \triangleq \underbrace{[x_1(t) - x_2(t)]}_{\text{Difference}}$$

Transformation matrix

A transformation matrix **T** relates the two state vectors:

$$\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$$

System transformation

The system dynamics may be expressed in terms of the new states, if the transformation matrix is *invertible*:

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}, \quad \mathbf{x} = \mathbf{T}^{-1}\bar{\mathbf{x}}$$

Equivalence transformation

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{T}\mathbf{A}\mathbf{T}^{-1}}^{\bar{\mathbf{A}}}\bar{\mathbf{x}} + \overbrace{\mathbf{T}\mathbf{B}}^{\bar{\mathbf{B}}}\mathbf{u} \\ \mathbf{y} &= \underbrace{\mathbf{C}\mathbf{T}^{-1}}_{\bar{\mathbf{C}}}\bar{\mathbf{x}} + \underbrace{\mathbf{D}}_{\bar{\mathbf{D}}}\mathbf{u} \end{aligned}$$

Algebraic equivalence

If we can find an invertible matrix **T** that relate the two systems:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \quad \mathbf{y} = \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u} \end{aligned}$$

they are **algebraically equivalent**.

Tank example, cont.

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix}}^{\mathbf{T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}}^{\mathbf{T}^{-1}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Transform of **A**:

$$\bar{\mathbf{A}} \triangleq \mathbf{TAT}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix}$$

Transform of **B**:

$$\bar{\mathbf{B}} \triangleq \mathbf{TB} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix}$$

Transform of **C**:

$$\bar{\mathbf{C}} \triangleq \mathbf{CT}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Algebraic equivalence

Below are two **equivalent** tank models, that represent the **same dynamics**.

Representation 1 - **Original**

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}
$$

Representation 2 - **Transformed**

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}
\end{aligned}
$$

Change of state variables

Key points

- We converted the original state variables to a *linear combination* of an alternative set of state variables: $\mathbf{x} = \mathbf{T}^{-1}\bar{\mathbf{x}}$.
- A new *basis* $\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}$ is used to represent the system.
- The transformation $\bar{\mathbf{A}} = \mathbf{TAT}^{-1}$ is called an **equivalence/similarity** transformation.
- The choice of $\mathbf{T}$ is *not unique*. Some choices may be better than others, depending on the application.

Transfer function invariance

State equation

$$\dot{\bar{\mathbf{x}}} = \overbrace{\mathbf{TAT}^{-1}}^{\bar{\mathbf{A}}}\bar{\mathbf{x}} + \overbrace{\mathbf{TB}}^{\bar{\mathbf{B}}}\mathbf{u}, \quad \mathbf{y} = \overbrace{\mathbf{CT}^{-1}}^{\bar{\mathbf{C}}}\bar{\mathbf{x}} + \mathbf{Du}$$

Laplace transform of equivalent system

The transfer matrix is invariant under a similarity transformation:

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{C}}(s\mathbb{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \mathbf{D} = \mathbf{CT}^{-1}(s\mathbb{I} - \mathbf{TAT}^{-1})^{-1}\mathbf{TB} + \mathbf{D} = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Since poles and zeros are encoded in $\hat{\mathbf{G}}(s)$, these are invariant also.

Zero state equivalence

Tank example: representation 2

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$

Note that the two states are decoupled and cannot affect each other.

Tank example: representation 3

Remove the unmeasured state $x_2(t)$ to obtain a reduced order model:

$$\dot{\bar{x}}_1(t) = \frac{1}{2S} u(t), \quad y(t) = \bar{x}_1(t)$$

Zero-state equivalence

Claim: Representations 1-3 all have the same transfer function $g(s)$. They are **zero-state equivalent**.

Zero state equivalence

Transfer function for Representation 1/2[1]

$$g(s) = \mathbf{c}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s + \frac{2gk\rho}{S} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s+\frac{2gk\rho}{S}} \end{bmatrix} \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} = \frac{1}{2Ss}$$

Transfer function for Representation 3

$$g(s) = \frac{1}{2Ss}$$

[1] Recall that the transfer function is invariant to a similarity transformation.

Zero state equivalence

Zero-state equivalence

If the system:

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

has the same transfer function as the system:

$$\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}\}$$

they are *zero-state equivalent*.

Caution

- Algebraic equivalence $\Rightarrow$ Zero-state equivalence
- Zero-state equivalence $\not\Rightarrow$ Algebraic equivalence

Topic

Recall the definition of **eigenvalues** and **eigenvectors**:

$$\mathbf{A}\mathbf{q} = \lambda\mathbf{q}$$

These are very important in dynamics:

Consider the case where the state coincides with an eigenvector at some time:

$$\mathbf{x}(t) = \mathbf{q}\alpha(t), \quad t = 0$$

where $\alpha(t)$ scales the *constant*[a] eigenvector. Assume $\mathbf{u}(t) = \mathbf{0}, \; t > 0$, so that:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) \\ \Rightarrow \mathbf{q}\dot{\alpha}(t) &= \mathbf{A}\mathbf{q}\alpha(t) \\ \Rightarrow \mathbf{q}\dot{\alpha}(t) &= \lambda\mathbf{q}\alpha(t) \\ \Rightarrow \dot{\alpha}(t) &= \lambda\alpha(t) \end{aligned}$$

Solutions along an eigenvector stays along the eigenvector. Solving this problem is simple:

$$\alpha(t) = e^{\lambda t}\alpha(0) \quad \Rightarrow \mathbf{x}(t) = \mathbf{q}e^{\lambda t}\alpha(0)$$

..only the scalar factor changes in time.

[a] Eigenvectors are often normalized so that: $\mathbf{q}^T\mathbf{q} = 1$.

Diagonalization

Generalization

Consider next the case where $\mathbf{x}(t) \in \mathbb{R}^n$ coincides with a *linear combination* of n eigenvectors[a]:

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{q}_1\alpha_1(t) + \mathbf{q}_2\alpha_2(t) + \ldots + \mathbf{q}_n\alpha_n(t) \\
&= \overbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \ldots & \mathbf{q}_n \end{bmatrix}}^{\mathbf{Q}} \underbrace{\begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix}}_{\boldsymbol{\alpha}(t)} = \mathbf{Q}\boldsymbol{\alpha}(t)
\end{aligned}$$

[a]We have n linearly independent eigenvectors and n eigenvalues in cases where $\mathbf{A}$ is *semisimple*. This is most often the case.

Diagonalization: $\mathbf{x}(t) = \mathbf{Q}\boldsymbol{\alpha}(t)$

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) \\
\mathbf{Q}\dot{\boldsymbol{\alpha}}(t) &= \begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \dots & \mathbf{A}\mathbf{q}_n \end{bmatrix} \boldsymbol{\alpha}(t) \\
\mathbf{Q}\dot{\boldsymbol{\alpha}}(t) &= \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \dots & \lambda_n\mathbf{q}_n \end{bmatrix} \boldsymbol{\alpha}(t) \\
\mathbf{Q}\dot{\boldsymbol{\alpha}}(t) &= \overbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \end{bmatrix}}^{\mathbf{Q}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\boldsymbol{\Lambda}} \boldsymbol{\alpha}(t)
\end{aligned}
$$

Finally: $\mathbf{Q}\dot{\boldsymbol{\alpha}}(t) = \mathbf{Q}\boldsymbol{\Lambda}\boldsymbol{\alpha}(t)$

$$
\Rightarrow \begin{bmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \vdots \\ \dot{\alpha}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1\alpha_1(t) \\ \lambda_2\alpha_2(t) \\ \vdots \\ \lambda_n\alpha_n(t) \end{bmatrix}
$$

..this is far simpler than solving the system as is!

With: $\mathbf{Q}\dot{\alpha}(t) = \mathbf{Q}\mathbf{\Lambda}\alpha(t)$

$$\begin{bmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \vdots \\ \dot{\alpha}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1\alpha_1(t) \\ \lambda_2\alpha_2(t) \\ \vdots \\ \lambda_n\alpha_n(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}\alpha_1(0) \\ e^{\lambda_2 t}\alpha_2(0) \\ \vdots \\ e^{\lambda_n t}\alpha_n(0) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}}_{e^{\mathbf{\Lambda} t}} \begin{bmatrix} \alpha_1(0) \\ \alpha_2(0) \\ \vdots \\ \alpha_n(0) \end{bmatrix}$$

..we can solve large systems easily[a]:

$$\boldsymbol{\alpha}(t) = e^{\mathbf{\Lambda} t}\boldsymbol{\alpha}(0)$$

[a] This trick only works for *diagonal* matrices.

Finally with $\alpha(t) = e^{\Lambda t}\alpha(0)$:

Initial conditions are obtained as:

$$\mathbf{x}(0) = \mathbf{Q}\alpha(0) \quad \Rightarrow \mathbf{Q}^{-1}\mathbf{x}(0) = \alpha(0)$$

Hence:

$$\mathbf{x}(t) = \mathbf{Q}\alpha(t) = \mathbf{Q}e^{\Lambda t}\mathbf{Q}^{-1}\mathbf{x}(0)$$

Recall:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

which implies:

$$\mathbf{Q}e^{\Lambda t}\mathbf{Q}^{-1} \equiv e^{\mathbf{A}t}$$

Equivalence transform: $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$

The eigenvector matrix defines an equivalence transformation with $\mathbf{T} = \mathbf{Q}^{-1}$:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}}^{\boldsymbol{\Lambda}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{B}\mathbf{Q}^{-1}}^{\bar{\mathbf{B}}}\,\mathbf{u} \\ \mathbf{y} &= \underbrace{\mathbf{C}\mathbf{Q}}_{\bar{\mathbf{C}}}\,\bar{\mathbf{x}} + \underbrace{\mathbf{D}}_{\bar{\mathbf{D}}}\,\mathbf{u} \end{aligned}$$

Diagonalization

If the transform above is possible, the system has been **diagonalized**. ($\boldsymbol{\Lambda}$ has elements only on the main diagonal.)

Solutions

Compute:

$$\mathbf{y}(t) = \bar{\mathbf{C}} e^{\mathbf{\Lambda} t} \bar{\mathbf{x}}_0 + \bar{\mathbf{C}} \int_0^t e^{\mathbf{\Lambda}(t-\tau)} \bar{\mathbf{B}} \mathbf{u}(\tau) d\tau + \bar{\mathbf{D}}(t)$$

with:

$$e^{\mathbf{\Lambda} t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

Transfer functions

Compute

$$\mathbf{G}(s) = \bar{\mathbf{C}}(s\mathbb{I} - \mathbf{\Lambda})^{-1} \bar{\mathbf{B}} + \bar{\mathbf{D}}$$

with:

$$(s\mathbb{I} - \mathbf{\Lambda})^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{s-\lambda_n} \end{bmatrix}$$

The Jordan Form

The Jordan Form

- If there are repeated eigenvalues, some eigenvectors may also be repeated.
- Then $\mathbf{Q}$ will not have full rank, and no inverse exists.
- The Jordan form captures these cases, using *generalized eigenvectors*, to give an invertible $\mathbf{Q}$

Topic

Diagonal & Jordan forms

Eigenvalues & Eigenvectors

Definition:

$$\mathbf{Aq} = \lambda \mathbf{q} \quad \Rightarrow (\lambda \mathbb{I} - \mathbf{A})\mathbf{q} = \mathbf{0}$$

Idea

1. If $(\lambda \mathbb{I} - \mathbf{A})$ has full rank, only $\mathbf{q} = \mathbf{0}$ is possible[a]
2. The determinant of a matrix with full rank is never zero: $\det(\mathbf{M}) \neq 0$.
3. ..so we search for eigenvalues that make $|\lambda \mathbb{I} - \mathbf{A}| = 0$.
4. This is done by solving the characteristic polynomial $\Delta(\lambda) = |\lambda \mathbb{I} - \mathbf{A}| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$.
5. There are n solutions to the characteristic polynomial $\Delta(\lambda) = 0$, not necessarily distinct.
6. For each eigenvalue λ_i we identify the corresponding eigenvector $\mathbf{Aq}_i = \mathbf{q}_i \lambda_i$
7. ..by finding the *null space* of $(\lambda_i \mathbb{I} - \mathbf{A})$.

[a] This is known as the trivial solution.

Diagonal & Jordan forms

Eigenvalues & Eigenvectors

For λ_i, $i = 1 \dots n$, we have:

$$\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$$

with the associated eigenvectors $\mathbf{q}_i$.

We may also write: $\mathbf{AQ} = \mathbf{Q}\boldsymbol{\Lambda}$

$$\begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \cdots & \mathbf{A}\mathbf{q}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \cdots & \lambda_n\mathbf{q}_n \end{bmatrix}$$

$$\Rightarrow \mathbf{A} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \cdots & \lambda_n\mathbf{q}_n \end{bmatrix}$$

$$\Rightarrow \mathbf{A} \underbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix}}_{\mathbf{Q}} = \underbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\boldsymbol{\Lambda}}$$

Repeated eigenvalues

With:

$$\mathbf{AQ} = \mathbf{Q\Lambda}$$

Our aim is to transform the original system to the form:

$$\begin{aligned} \overbrace{\mathbf{Q}^{-1}\mathbf{Q}}^{\mathbb{I}}\,\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{Q}^{-1}\mathbf{Q}}^{\mathbb{I}}\,\mathbf{\Lambda}\bar{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{Bu} \\ \mathbf{y} &= \mathbf{CQ}\bar{\mathbf{x}} + \mathbf{Du} \end{aligned}$$

$\mathbf{Q}$ must be invertible to do this

This requires that $\mathbf{q}_1 \ldots \mathbf{q}_n$ are linearly independent. This is not always the case with repeated eigenvalues..

Repeated eigenvalues

Option 1

For a repeated eigenvalue λ_r: If $(\lambda_r \mathbb{I} - \mathbf{A})$ has *nullity*[a] larger than 1, we can find several linearly independent solutions to:

$$(\lambda_r \mathbb{I} - \mathbf{A})\mathbf{n} = \mathbf{0}$$

[a] $\text{null}(M) + \text{rank}(M) = n$

Distinct eigenvectors

If the nullity of $(\lambda_r \mathbb{I} - \mathbf{A})$ equals the number of repetitions of λ_r we can find distinct eigenvectors to form a full rank Q matrix:

$$\mathbf{A} \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix}$$

Repeated eigenvalues

Option 2

If the nullity of $(\lambda_r \mathbb{I} - \mathbf{A})$ is *less* than the repetitions of the eigenvalue, we don't have sufficient distinct eigenvectors.
In this case we can use a *Jordan block*:

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

Jordan Block, application

$$\mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix}$$

Modified equations

$$\Rightarrow \mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_1 \mathbf{v}_2 + 1\mathbf{v}_1 & \lambda_2 \mathbf{q}_2 & \lambda_3 \mathbf{q}_3 \end{bmatrix}$$

with linear combinations of eigenvectors:

$$\Rightarrow \mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad \mathbf{A}\mathbf{v}_2 = \lambda_1 \mathbf{v}_2 + 1\mathbf{v}_1$$

Example: 4 Repeated eigenvalues

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Nullity $(\lambda\mathbb{I} - \mathbf{A}) = 1$

Find the generalized eigenvectors

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \lambda\mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_1 &= 0 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^4\mathbf{v} &= 0 \\
\mathbf{A}\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_2 &= \mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^3\mathbf{v} &= \mathbf{v}_1 \\
\mathbf{A}\mathbf{v}_3 &= \lambda\mathbf{v}_3 + \mathbf{v}_2 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_3 &= \mathbf{v}_2 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^2\mathbf{v} &= \mathbf{v}_2 \\
\mathbf{A}\mathbf{v}_4 &= \lambda\mathbf{v}_4 + \mathbf{v}_3 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v} &= \mathbf{v}_3 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^1\mathbf{v} &= \mathbf{v}_3
\end{aligned}$$

All the eigenvectors are seen to issue from a chain generated by $\mathbf{v}$.

Procedure

1. Find the multiplicity of the repeated eigenvalue $\lambda_i : n_i$.
2. Find the nullity N of $(\lambda_i \mathbb{I} - \mathbf{A})$.
3. Generate N linearly independent eigenvectors, $\mathbf{v}_k$, $k = 1 \ldots N$, from the null-space of $(\lambda_i \mathbb{I} - \mathbf{A})$.
4. We are left with $n_i - N$ eigenvectors to find.
5. Use the generalized eigenvector scheme to generate the remaining vectors: $(\mathbf{A} - \lambda_i \mathbb{I})\mathbf{v}_{k,2} = \mathbf{v}_k$.
6. ..$(\mathbf{A} - \lambda_i \mathbb{I})\mathbf{v}_{k,3} = \mathbf{v}_{k,2}$
7. You can choose which of $\mathbf{v}_k$ to use.
8. Associate chains of these generated vectors with Jordan blocks.

Jordan form

With

- **Q** consisting of eigenvectors and generalized eigenvectors
- **J** the matrix with Jordan blocks for repeated eigenvalues and distinct eigenvalues along the diagonal

The system can be transformed like this:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{J}\bar{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{Q}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

Example

$$
\mathbf{J} = \begin{bmatrix}
\lambda_1 & & & & & & & & & & & \\
& \lambda_2 & 1 & 0 & 0 & & & & & & & \\
& & \lambda_2 & 1 & 0 & & & & & & & \\
& & & \lambda_2 & 1 & & & & & & & \\
& & & & \lambda_2 & & & & & & & \\
& & & & & \lambda_3 & 1 & 0 & & & & \\
& & & & & & \lambda_3 & 1 & & & & \\
& & & & & & & \lambda_3 & & & & \\
& & & & & & & & \lambda_3 & 1 & & \\
& & & & & & & & & \lambda_3 & & \\
& & & & & & & & & & \lambda_4 & 0 \\
& & & & & & & & & & & \lambda_4
\end{bmatrix}
$$

Topic

Modal Form

Modal Form

The modal form is useful when we have pairs of complex conjugated eigenvalues. It allows us to deal with only real numbers, as opposed to the Jordan form.

Modal form

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Characteristic equation

$$\begin{aligned} & \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right| \\ = \quad & \left(\lambda + \frac{d + \sqrt{d^2 - 4km}}{2m} \right) \left(\lambda + \frac{d - \sqrt{d^2 - 4km}}{2m} \right) \end{aligned}$$

Modal form

Example: Mass spring damper; $d = k = m = 1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Characteristic equation

$$\begin{aligned} & \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right| \\ = \quad & \left(\lambda + \frac{1 + \sqrt{1 - 4}}{2} \right) \left(\lambda + \frac{1 - \sqrt{1 - 4}}{2} \right) \\ = \quad & \left(\lambda + \frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \left(\lambda + \frac{1}{2} - \frac{i\sqrt{3}}{2} \right) \end{aligned}$$

Eigenvalues & Eigenvectors

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \quad \mathbf{Q} = \begin{bmatrix} \frac{1}{2}\left(-1 - i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) \\ 1 & 1 \end{bmatrix}$$

Similarity transform

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \mathbf{Q}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \mathbf{Q}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Result (with complex coefficients)

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Modal form

Do yet another similarity transform with

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

The modal form avoids imaginary numbers in the state equation.

Similarity transform to Modal Form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix} \mathbf{M} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \mathbf{M}^{-1} \begin{bmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix} \mathbf{M} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

The system on **modal** form

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{3}} \end{bmatrix} u \\ y &= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{aligned}$$

Modal form

General case

A diagonalized state equation with complex eigenvalues:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + i\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - i\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 + i\beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 - i\beta_2 \end{bmatrix}$$

Modal transform: $\mathbf{\Lambda}_m = \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & i & -i \end{bmatrix}$$

Modal form

General case

$$\mathbf{\Lambda}_m = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

Modal transform: $\mathbf{\Lambda}_m = \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & i & -i \end{bmatrix}$$

Canonical forms

Modal form

Modal form: $\Lambda_m = \mathbf{M}^{-1}\Lambda\mathbf{M}$

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

Matrix exponential:

$$e^{\Lambda_m t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t\alpha_1}\cos(t\beta_1) & e^{t\alpha_1}\sin(t\beta_1) & 0 & 0 & 0 \\ 0 & -e^{t\alpha_1}\sin(t\beta_1) & e^{t\alpha_1}\cos(t\beta_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{t\alpha_2}\cos(t\beta_2) & e^{t\alpha_2}\sin(t\beta_2) \\ 0 & 0 & 0 & 0 & -e^{t\alpha_2}\sin(t\beta_2) & e^{t\alpha_2}\cos(t\beta_2) \end{bmatrix}$$

Summary

Usage

- Distinct real eigenvalues: diagonal blocks: $\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{bmatrix}$
- Repeated real eigenvalues: Jordan blocks: $\begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$
- Complex eigenvalues: modal blocks: $\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$

Topic

Modal analysis

Definition: Modal analysis

The study of the dynamic properties of structures under vibrational excitation.

Applications

- Earthquake engineering
- Acoustics
- Aeroelasticity
- Fatigue analysis
- Architecture

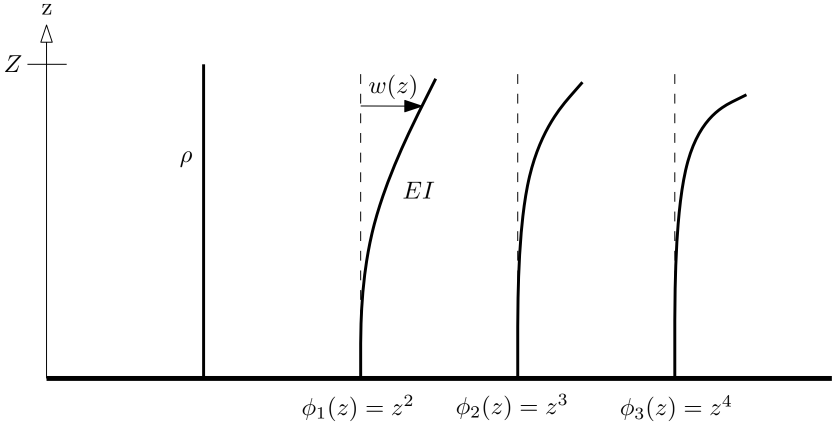
Methodology

Construct a dynamic model of the structure. Then find:

Eigenvalues: Complex part of eigenvalue corresponds to the *resonance frequency*.

Eigenvectors: These encode the *shape* of the resonant motion.

Example: Elastic rod



Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states x_i:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Kinetic energy

Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states x_i:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Kinetic energy

Let the mass-distribution be uniform with density ρ. Then the kinetic energy is a *quadratic form*:

$$\mathcal{K} = \frac{1}{2}\rho \int_0^Z \dot{w}^2(z) dz = \frac{1}{2}\rho \int_0^Z \left[\sum_{i=1}^{n} \phi_i(z)\dot{x}_i\right] \left[\sum_{j=1}^{n} \phi_j(z)\dot{x}_j\right] dz$$

$$= \frac{1}{2}\rho \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\left[\int_0^Z \phi_i(z)\phi_j(z) dz\right] \dot{x}_i \dot{x}_j\right) = \frac{1}{2}\dot{\mathbf{x}}^\top \mathbf{M}\dot{\mathbf{x}}, \quad M_{ij} = \rho \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_0^Z \phi_i(z)\phi_j(z) dz\right]$$

Potential energy

Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Potential energy

The potential energy is proportional to the specific elastic modulus EI and quadratic in beam *curvature*:

$$\mathcal{U} = \frac{1}{2} EI \int_0^Z w''^2(z) dz = \frac{1}{2} EI \int_0^Z \left[\sum_{i=1}^{n} \phi_i''(z) x_i\right] \left[\sum_{j=1}^{n} \phi_j''(z) x_j\right] dz$$

$$= \frac{1}{2} EI \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left[\int_0^Z \phi_i''(z)\phi_j''(z) dz\right] x_i x_j\right) = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{K}\mathbf{x}, \quad K_{ij} = EI \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\int_0^Z \phi_i''(z)\phi_j''(z) dz\right]$$

Equations of motion

Kinetic energy

$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{x}}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{x}}, \quad M_{ij} = \rho \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^Z \phi_i(z)\phi_j(z)dz\right]$$

Potential energy

$$\mathcal{U} = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{K}\mathbf{x}, \quad K_{ij} = EI \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^Z \phi_i''(z)\phi_j''(z)dz\right]$$

Lagrangian equations of motion, $\mathcal{L} = \mathcal{K} - \mathcal{U}$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{K}}{\partial \dot{\mathbf{x}}}\right] + \frac{\partial \mathcal{U}}{\partial \mathbf{x}} = \ddot{\mathbf{x}}^{\mathsf{T}}\mathbf{M} + \mathbf{x}^{\mathsf{T}}\mathbf{K} = \mathbf{0}^{\mathsf{T}}$$

State-space model

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

Eigenvalues/vectors

State-space model

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

Eigenvalues

$$|\lambda\mathbb{I} - \mathbf{A}| = \begin{vmatrix} \lambda\mathbb{I} & -\mathbb{I} \\ \mathbf{M}^{-1}\mathbf{K} & \lambda\mathbb{I} \end{vmatrix} = |\lambda^2\mathbb{I} + \mathbf{M}^{-1}\mathbf{K}| = |\lambda^2\mathbf{M} + \mathbf{K}| = 0$$

Imaginary eigenvalues result:

$$\lambda = 0 \pm j\omega \quad \Rightarrow |\mathbf{K} - \omega^2\mathbf{M}| = 0$$

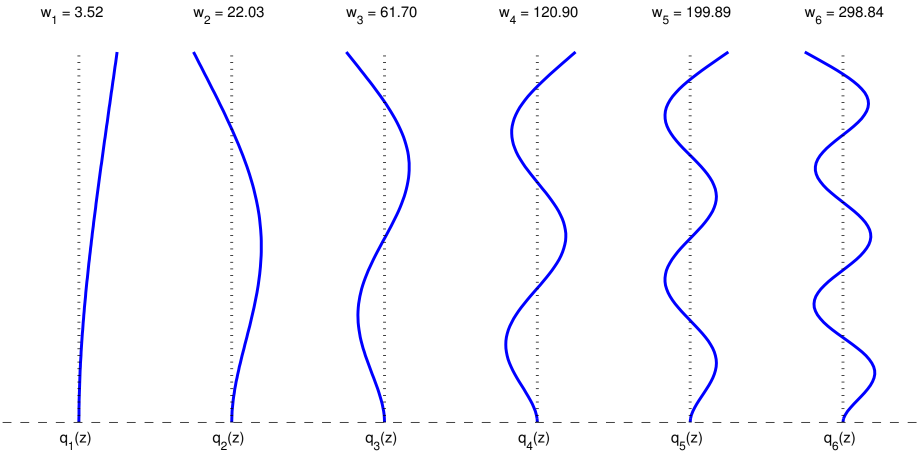
Eigenvectors

$$[\lambda\mathbb{I} - \mathbf{A}]\mathbf{q} = \begin{bmatrix} \lambda\mathbb{I} & -\mathbb{I} \\ \mathbf{M}^{-1}\mathbf{K} & \lambda\mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_{\dot{x}} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{q}_x - \mathbf{q}_{\dot{x}} \\ \mathbf{M}^{-1}\mathbf{K}\mathbf{q}_x + \lambda\mathbf{q}_{\dot{x}} \end{bmatrix}$$

Thus:

$$\mathbf{q}_{\dot{x}} = \lambda\mathbf{q}_x \quad \Rightarrow (\lambda^2\mathbb{I} + \mathbf{M}^{-1}\mathbf{K})\mathbf{q}_x = (\mathbf{K} - \omega^2\mathbf{M})\mathbf{q}_x = \mathbf{0}$$

Eigenvalues/vectors



Results: $\rho = 1,\ EI = 1,\ Z = 1$

The first six modeshapes[2] $m_i(z) = \sum_j^n (\phi_j(z) q_j^i)$ and frequencies ω_i are shown.

[2] q_j^i: i'th component of j'th eigenvector

Topic

Last lecture

State-space model

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Solution

$$\mathbf{y}(t) = \mathbf{C}e^{At}\mathbf{x_0} + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

This lecture

Computation and properties of the matrix exponential $e^{\mathbf{A}t}$

Last lecture

State-space model

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Laplace transform

$$\mathbf{y}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) + \mathbf{D}\mathbf{u}(s)$$

This lecture

Computation and properties of the matrix: $(s\mathbb{I} - \mathbf{A})^{-1}$

Properties of $e^{\mathbf{A}t}$

- $\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}$
- $e^{\mathbf{A}t} e^{\mathbf{A}\tau} = e^{\mathbf{A}(t+\tau)}$
- $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$
- $\mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$
- $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k$

Warning

$$e^{\mathbf{A}t} e^{\mathbf{B}t} \neq e^{(\mathbf{A}+\mathbf{B})t}$$

Note

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k$$

Functions of a square matrix

Computation of $e^{\mathbf{A}t}$

1. It is inconvenient to use an infinite series to compute $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$
2. There is a shortcut that allows a finite summation $e^{\mathbf{A}t} = \sum_{k=0}^{n-1} a_k(t)\mathbf{A}^k$
3. The *Cayley Hamilton Theorem* provides the recipe.

Cayley-Hamilton:
A matrix satisfies its own characteristic polynomial

$$\Delta(\lambda) = \det(\lambda\mathbb{I} - \mathbf{A}) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

so:

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbb{I} = \mathbf{0}$$

Why is $\Delta(\mathbf{A}) = \mathbf{0}$ important?

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1 \mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n \mathbb{I} = \mathbf{0}$$

$\mathbf{A}^n$

$$\mathbf{A}^n = -\alpha_1 \mathbf{A}^{n-1} - \cdots - \alpha_{n-1}\mathbf{A} - \alpha_n \mathbb{I}$$

$\mathbf{A}^n$

Can be written as a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$

$\mathbf{A}^{n+1}$

$$\overbrace{\mathbf{A}\mathbf{A}^n}^{\mathbf{A}^{n+1}} = -\alpha_1 \overbrace{\mathbf{A}\mathbf{A}^{n-1}}^{\mathbf{A}^n} - \cdots - \alpha_{n-1} \overbrace{\mathbf{A}\mathbf{A}}^{\mathbf{A}^2} - \alpha_n \overbrace{\mathbf{A}}^{\mathbf{A}} \mathbb{I}$$

$\mathbf{A}^{n+1}$

Can be written as a linear combination of $\{\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots \mathbf{A}\}$
..which is a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$

Functions of a square matrix

Why is $\Delta(\mathbf{A}) = \mathbf{0}$ important?

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1 \mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n \mathbb{I} = \mathbf{0}$$

Because:

It tells us that any polynomial function can be written as a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$!

Linear combination:

$$f(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1}$$

Functions of a square matrix

Linear combination:

$$h(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1} \mathbf{A}^{n-1}$$

Linear combination in terms of λ:

$$h(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1} \lambda^{n-1}$$

Functions of a square matrix

Procedure to compute $f(\mathbf{A})$

1. Given a function we wish to find: $f(\mathbf{A})$
2. Define the function of unknown coefficients $h(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$
3. For each eigenvalue, make an equation: $f(\lambda_i) = h(\lambda_i)$.
4. If the eigenvalue is repeated n_i times:
5. Use the derivatives $\left.\frac{d^l f(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i} = \left.\frac{d^l h(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i}$ for $l = 1 \ldots n_i - 1$ to generate additional equations.
6. Solve the n equations for $\beta_0 \ldots \beta_{n-1}$
7. Insert: $f(\mathbf{A}) = h(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1}$

Example: $\mathbf{A}^{10}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = \mathbf{A}^{10}$$

Computation

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$

$$h(\lambda) = \beta_0 + \beta_1 \lambda, \quad f(\lambda) = \lambda^{10}$$

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = 2^{10} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = 1^{10} \end{aligned}$$

Solution:

$$\beta_0 = -1022, \quad \beta_1 = 1023$$

Result

$$f(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} = -1022 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1023 \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1024 & 4092 \\ 0 & 1 \end{bmatrix}$$

Example: $e^{\mathbf{A}t}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = e^{\mathbf{A}t}$$

Computation

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$

$$h(\lambda) = \beta_0 + \beta_1\lambda, \quad f(\lambda) = e^{\lambda t}$$

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = e^{2t} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = e^{t} \end{aligned}$$

Solution:

$$\beta_0 = -e^t\left(-2 + e^t\right), \quad \beta_1 = e^t\left(-1 + e^t\right)$$

Result

$$f(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} = -e^t\left(-2 + e^t\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t\left(-1 + e^t\right)\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Example: $(s\mathbb{I} - A)^{-1}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = (s\mathbb{I} - A)^{-1}$$

Computation

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = \frac{1}{s-2} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = \frac{1}{s-1} \end{aligned}$$

Solution:

$$\beta_0 = \frac{-3+s}{2-3s+s^2}, \quad \beta_1 = \frac{1}{2-3s+s^2}$$

Result

$$f(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} = \frac{-3+s}{2-3s+s^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2-3s+s^2}\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Topic

Property 1:

$$f(\mathbf{PAP}^{-1}) = \mathbf{P}f(\mathbf{A})\mathbf{P}^{-1}$$

Recall that all matrix functions are linear combinations:

$$f(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} + \beta_2\mathbf{AA}\ldots + \beta_{n-1}\mathbf{A}^{n-1}$$

Insert similar matrix:

$$f(\mathbf{PAP}^{-1}) = \beta_0 \overbrace{\mathbb{I}}^{\mathbf{PP}^{-1}} + \beta_1\mathbf{PAP}^{-1} + \beta_2\mathbf{PA}\overbrace{\mathbf{P}^{-1}\mathbf{P}}^{\mathbb{I}}\mathbf{AP}^{-1} + \ldots$$

Clean up and rearrange:

$$f(\mathbf{PAP}^{-1}) = \beta_0\mathbf{PP}^{-1} + \beta_1\mathbf{PAP}^{-1} + \beta_2\mathbf{PA}^2\mathbf{P}^{-1} + \ldots$$

Q.E.D.:

$$f(\mathbf{PAP}^{-1}) = \mathbf{P}[\beta_0 + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 + \ldots]\mathbf{P}^{-1} = \mathbf{P}f(\mathbf{A})\mathbf{P}^{-1}$$

Property 2:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & a_3 & \\ & & & a_4 \end{bmatrix} \Rightarrow f(\mathbf{A}) = \begin{bmatrix} f(\mathbf{A}_1) & & & \\ & f(\mathbf{A}_2) & & \\ & & f(a_3) & \\ & & & f(a_4) \end{bmatrix}$$

Functions of a square matrix

Special cases

Diagonal matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \Rightarrow e^{\boldsymbol{\Lambda} t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix}$$

Functions of a square matrix

Special cases

Jordan Block

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \Rightarrow e^{\mathbf{J}t} = \begin{bmatrix} e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 & \frac{1}{3!}e^{t\lambda}t^3 \\ 0 & e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 \\ 0 & 0 & e^{t\lambda} & e^{t\lambda}t \\ 0 & 0 & 0 & e^{t\lambda} \end{bmatrix}$$

TTK4115

Lecture 3

Discretization, controllability, state feedback

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

Topic

Analog state space model

The continuous state space model is analog[1]:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

To simulate it *as is* would require an analog computer.

Discretization

Discretization is a necessary step for computer simulation. A recursive model is sought:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]\end{aligned}$$

Many methods are available for obtaining a discretized model. We examine the two most common methods: **Exact** and **Euler** discretization.

Approach

$$\text{TF} \xrightarrow{\text{Realize}} \text{CLTI} \xrightarrow{\text{Discretize}} \text{DLTI} \xrightarrow{\text{Recursion}} \text{Solution}$$

[1] Some systems are discrete by nature, such as financial systems or discrete filters. Most plants will however be continuous as they are based on a physical model.

LTI solution

The exact solution of the LTI system forms the theoretical basis of **exact** discretization.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Sampling

Time is discretized into intervals of duration T: $t = kT$. Sample index is denoted k.
The state solution from one sample to the next is:

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Here $\mathbf{x}[k] \triangleq \mathbf{x}(t)|_{t=kT}$ serves as an initial condition
The resulting solution is evaluated at $\mathbf{x}[k+1] \triangleq \mathbf{x}(t)|_{t=(k+1)T}$.

Piecewise constant input

The input is assumed to stay approximately constant between samples:

$$\mathbf{u}[k] \simeq \mathbf{u}(t), \quad kT \leq t < (k+1)T$$

Sampled model

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{Bu}[k]$$

Substitution rule

$$\int_{y(a)}^{y(b)} F(x)dx = \int_a^b F(y(x))\frac{dy}{dx}dx$$

Change of variable: $\alpha(\tau) \triangleq (k+1)T - \tau, \quad d\tau = -d\alpha$

Integration limits are simplified:

$$\tau_0 = kT \rightarrow \alpha_0 = T, \quad \tau_1 = (k+1)T \rightarrow \alpha_1 = 0$$

along with integrand:

$$e^{\mathbf{A}[(k+1)T-\tau]} \rightarrow e^{\mathbf{A}\alpha}$$

$$\mathbf{B}_d = \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{B} = \left(\int_0^T e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}$$

Discretization

Exactly discretized model

$$\begin{aligned}\mathbf{x}[k+1] &= \overbrace{e^{\mathbf{A}T}}^{\mathbf{A}_d}\mathbf{x}[k] + \overbrace{\left(\int_0^T e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}}^{\mathbf{B}_d}\mathbf{u}[k] \\ \mathbf{y}[k] &= \underbrace{\mathbf{C}}_{\mathbf{C}_d}\mathbf{x}[k] + \underbrace{\mathbf{D}}_{\mathbf{D}_d}\mathbf{u}[k]\end{aligned}$$

Discrete time system

This model is *exact* under the assumption:

$$\mathbf{u}[k] = \mathbf{u}(t), \quad kT \leq t \leq (k+1)T$$

It is recursive, and very efficient in implementation:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{u}[k]\end{aligned}$$

Discretization

Euler discretization

Euler's method proceeds via the definition of the derivative[2]:

$$\dot{\mathbf{x}}[k] \approx \frac{\mathbf{x}[k+1] - \mathbf{x}[k]}{T}$$

Thus:

$$\dot{\mathbf{x}}[k] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \quad \Rightarrow \mathbf{x}[k+1] = \mathbb{I}\mathbf{x}[k] + T\mathbf{A}\mathbf{x} + T\mathbf{B}\mathbf{u}$$

Stability

Euler's method may be unstable although the underlying plant is stable. This problem gets worse with larger timesteps. A mathematical criterion for first order systems may be stated as:

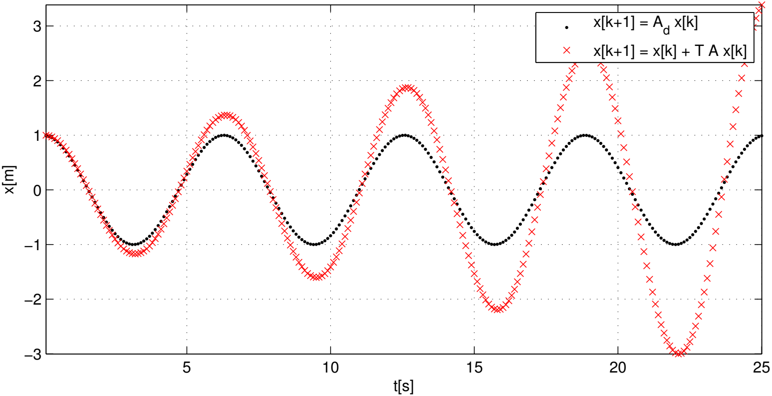
$$|1 + T\lambda| \leq 1$$

Insufficiently stable systems or large timesteps will result in a divergent solution.

[2] $x[k] = x(kT)$

Euler's Method vs Discretization

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{x}_e[k+1] &= \mathbf{x}_e[k] + T\mathbf{A}\mathbf{x}_e[k] + T\mathbf{B}\mathbf{u}[k]\end{aligned}$$



Topic

Example

Consider the single input **discrete** system:

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k] \quad \mathbf{A} \in \mathbb{R}^{4\times 4}$$

4 steps forward

Starting point is $\mathbf{x}_0$

$$\begin{aligned}
\mathbf{x}[1] &= \mathbf{A}\mathbf{x}[0] + \mathbf{b}u[0] \\
\mathbf{x}[2] &= \mathbf{A}\mathbf{x}[1] + \mathbf{b}u[1] \\
\mathbf{x}[3] &= \mathbf{A}\mathbf{x}[2] + \mathbf{b}u[2] \\
\mathbf{x}[4] &= \mathbf{A}\mathbf{x}[3] + \mathbf{b}u[3]
\end{aligned}$$

Last step may be written as:

$$\mathbf{x}[4] = \mathbf{A}^4\mathbf{x}[0] + \mathbf{A}^3\mathbf{b}u[0] + \mathbf{A}^2\mathbf{b}u[1] + \mathbf{A}\mathbf{b}u[2] + \mathbf{b}u[3]$$

The k'th step is linear in the initial condition and the sequence of inputs

$$\mathbf{x}[4] = \overbrace{\mathbf{A}^4\mathbf{x}[0]}^{\text{zir}} + \overbrace{\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} u[3] \\ u[2] \\ u[1] \\ u[0] \end{bmatrix}}^{\text{zsr}}$$

Key idea

Iff $\begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$ has *full rank*, we can choose $\mathbf{x}[4]$ as we like with our inputs.

Controllability matrix

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$

$$\mathcal{C} \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{(n-1)}\mathbf{b} \end{bmatrix}$$

the state can be placed anywhere with the right sequence of inputs.
-**This is controllability**.

Topic

1. Discretization

2. Controllability

Controllability Gramians

Eigenvector tests

Controllability in practice

3. State feedback

Pole placement

Controllability indices

Controllability Gramian

Given an LTI system:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Attempt to place the system at $\mathbf{x}_1$ at $t = t_1$:

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We clearly need the proper input signal $\mathbf{u}(t)$ to do this.

If we can find such an input, the system is controllable.

Controllability Gramian: definition

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)}d\tau$$

The existence of a nonsingular Controllability Gramian guarantees that a sufficient $\mathbf{u}(t)$ exists.

Place the system at $\mathbf{x}_1$ at $t = t_1$

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We need an input $\mathbf{u}(t)$ to do this.

Educated guess:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-t)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]$$

Result

$$\begin{aligned}
\mathbf{x}_1 &= e^{\mathbf{A}t_1}\mathbf{x}_0 - \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\left(\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]\right)d\tau \\
&= e^{\mathbf{A}t_1}\mathbf{x}_0 - \overbrace{\left(\int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}d\tau\right)}^{\mathbf{W}_c(t_1)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] \\
&= e^{\mathbf{A}t_1}\mathbf{x}_0 - \left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] = \underline{\mathbf{x}_1}
\end{aligned}$$

The gramian and controllability

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}(t-\tau)} d\tau$$

If $\mathbf{W}_c(t)$ is singular for t, a nonzero vector $\mathbf{v}$ must exist such that

$$\mathbf{v}^{\mathsf{T}}\mathbf{W}_c(t)\mathbf{v} = \int_0^t \overbrace{\mathbf{v}^{\mathsf{T}} e^{\mathbf{A}(t-\tau)}\mathbf{B}}^{\mathbf{0}} \overbrace{\mathbf{B}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}(t-\tau)}\mathbf{v}}^{\mathbf{0}} d\tau = \mathbf{0}$$

Say we want to move the system from initial value $e^{-\mathbf{A}t_1}\mathbf{v} \neq \mathbf{0}$ to $\mathbf{0}$

$$\mathbf{x}_1 = \mathbf{0} = \mathbf{v} + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Premultiplying by $\mathbf{v}^T$:

$$\mathbf{0} = \mathbf{v}^T\mathbf{v} + \int_0^{t_1} \overbrace{e^{\mathbf{v}^T\mathbf{A}(t_1-\tau)}\mathbf{B}}^{\mathbf{0}} \mathbf{u}(\tau)d\tau = ||\mathbf{v}||^2 + 0$$

Which contradicts $\mathbf{v} \neq \mathbf{0}$. So if the system is controllable, $\mathbf{W}_c(t)$ cannot be singular.

The gramian and the controllability matrix

Note:

- $e^{\mathbf{A}t}$ may be expressed as a linear combination of $\{\mathbb{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}\}$
- $e^{\mathbf{A}t}\mathbf{B}$ may be expressed as a linear combination of $\{\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\}$

If $\mathbf{W}_c(t)$ is invertible and $\mathcal{C}$ does not have full row rank:

There exists a nonzero $\mathbf{v}$ such that

$$\mathbf{v}^T\mathcal{C} = \mathbf{v}^T[\mathbf{B} \quad \mathbf{AB} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{0}$$

which implies that

$$\mathbf{v}^T\mathbf{A}^k\mathbf{B} = \mathbf{0} \qquad \text{for } k = 0, 1, 2, \ldots, n-1$$

Since $e^{\mathbf{A}t}\mathbf{B}$ can be expressed as a linear combination of $\{\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\}$:

$$\mathbf{v}^T e^{\mathbf{A}t}\mathbf{B} = \mathbf{0}$$

which contradicts the nonsingularity of $\mathbf{W}_c(t)$.

Which shows:

If $\mathcal{C}$ doesn't have full rank, $\mathbf{W}_c(t)$ cannot be invertible.

The gramian and the controllability matrix

If $\mathcal{C}$ has full row rank but $\mathbf{W}_c(t)$ is not invertible:

There exists a nonzero $\mathbf{v}$ such that

$$\mathbf{v}^T e^{\mathbf{A}t}\mathbf{B} = \mathbf{0} \qquad \text{for all } t$$

which implies that for $t = 0$, $\mathbf{v}^T\mathbf{B} = \mathbf{0}$.

Differentiating k times and setting $t = 0$, we get $\mathbf{v}^T\mathbf{A}^k\mathbf{B} = \mathbf{0}$, or:

$$\mathbf{v}^T[\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{v}^T\mathcal{C} = \mathbf{0}$$

which contradicts the hypothesis that $\mathcal{C}$ has full row rank.

Which shows:

If $\mathbf{W}_c(t)$ is not invertible, $\mathcal{C}$ cannot have full row rank.

Equivalent Statements on Controllability

Controllability Gramian

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

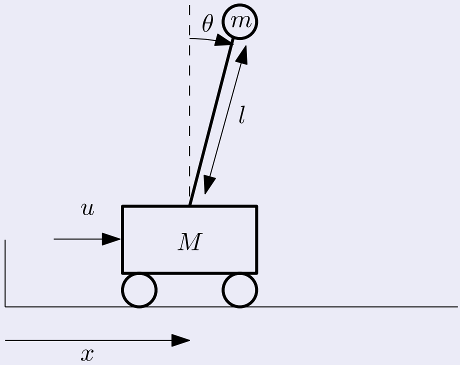
Controllability Matrix

$$\mathcal{C} = \overbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}^{np} \Big\} n$$

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$, the system is controllable.

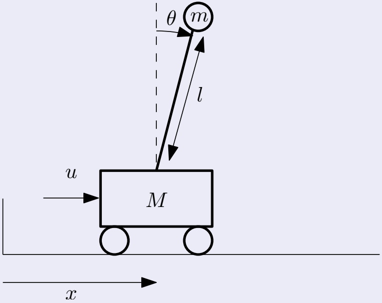
Controllability

Example



Controllability

Example



Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Controllability Matrix: Full row rank

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix}$$

Controllability Gramian: Invertible

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

Controllability Gramian

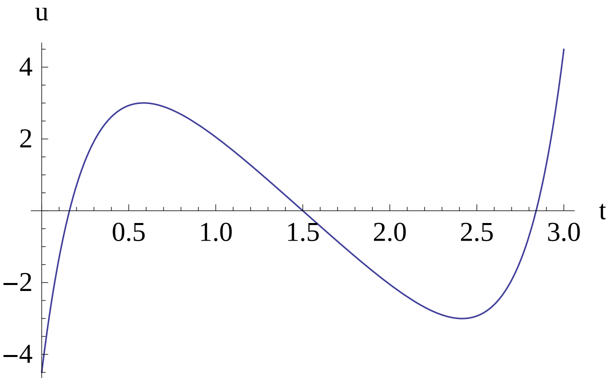
$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

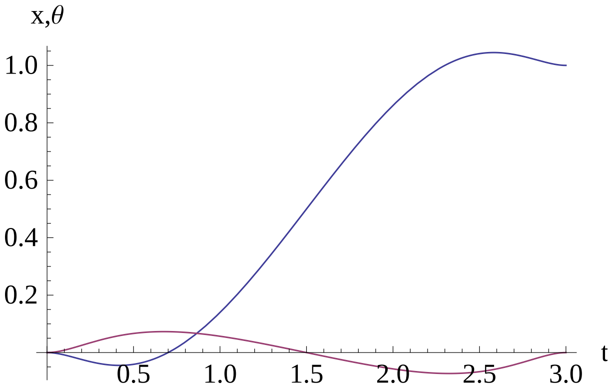


Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$



Topic

Eigenvector test

Basic idea

If the Controllability matrix has full rank, there is no vector $\mathbf{v}$ such that:

$$\mathcal{C}^\mathsf{T}\mathbf{v} = \begin{bmatrix} \mathbf{B}^\mathsf{T} \\ \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} \\ \dots \\ \mathbf{B}^\mathsf{T}(\mathbf{A}^\mathsf{T})^{(n-1)} \end{bmatrix} \mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \neq \mathbf{0}$$

Let $\mathbf{q}$ be an eigenvector of $\mathbf{A}^\mathsf{T}$: $\mathbf{A}^\mathsf{T}\mathbf{q} = \lambda\mathbf{q}$

$$\begin{bmatrix} \mathbf{B}^\mathsf{T} \\ \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} \\ \dots \\ \mathbf{B}^\mathsf{T}(\mathbf{A}^\mathsf{T})^{(n-1)} \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{B}^\mathsf{T} \\ \lambda\mathbf{B}^\mathsf{T} \\ \vdots \\ \lambda^{n-1}\mathbf{B}^\mathsf{T} \end{bmatrix} \mathbf{q}$$

Controllability and eigenvectors

Controllability is only possible if every eigenvector of $\mathbf{A}^\mathsf{T}$ is not in the null-space of $\mathbf{B}^\mathsf{T}$.

Eigenvector test

Controllability and eigenvectors

Controllability is only possible if every eigenvector of $\mathbf{A}^\mathsf{T}$ is not in the null-space of $\mathbf{B}^\mathsf{T}$.

Popov-Belevitch-Hautus - test

The PBH test gives an elegant test based on this insight.
An LTI system is controllable iff:

$$\text{rank}[\mathbf{A} - \lambda\mathbb{I} \;\; \mathbf{B}] = n, \quad \text{for all eigenvalues } \lambda \text{ of } \mathbf{A}$$

Topic

1. Discretization

2. Controllability

Controllability Gramians

Eigenvector tests

Controllability in practice

3. State feedback

Pole placement

Controllability indices

Controllable systems

Three important points

- If the pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable, so is $\{\mathbf{A} - \mathbf{BK}, \mathbf{B}\}$.
- If the system is controllable we can place the eigenvalues of the system exactly as desired.
- A controllable system can always be transformed to the controllable canonical form.

Controllability in practice

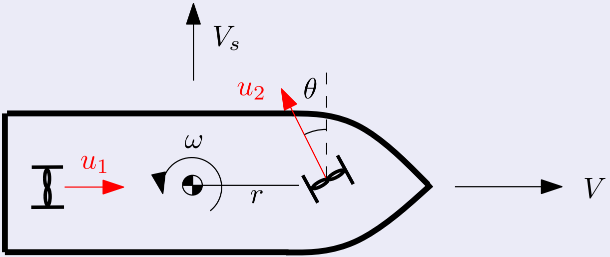
Caveats

Even if the controllability matrix has full rank, this does not mean that the system is easy to control in practice.

- The controller may require too large inputs.
- The closed loop response may be higly sensitive to modeling errors in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$.
- The closed loop eigenvalues may have been chosen unrealistically fast.
- Fast response requires powerful actuators and an accurate model.
- The system may be ”almost uncontrollable” in practice.

Controllable?

Dynamic positioning

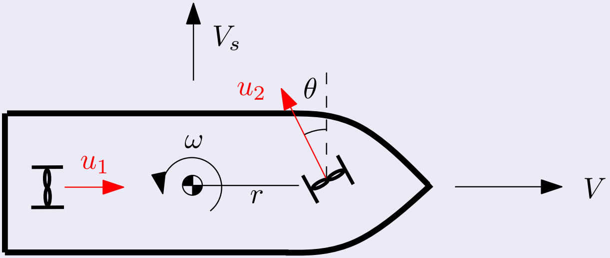


State equation

$$\begin{bmatrix} \dot{V} \\ \dot{V}_s \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & 0 & 0 \\ 0 & -\frac{d_s}{m} & 0 \\ 0 & 0 & -\frac{d_\omega}{J} \end{bmatrix} \begin{bmatrix} V \\ V_s \\ \omega \end{bmatrix} + \begin{bmatrix} 1/m & -\sin(\theta)/m \\ 0 & \cos(\theta)/m \\ 0 & \cos(\theta)r/J \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Controllable?

Dynamic positioning

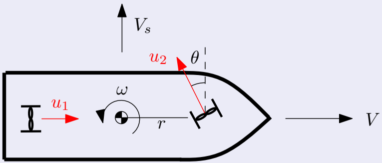


Controllability matrix

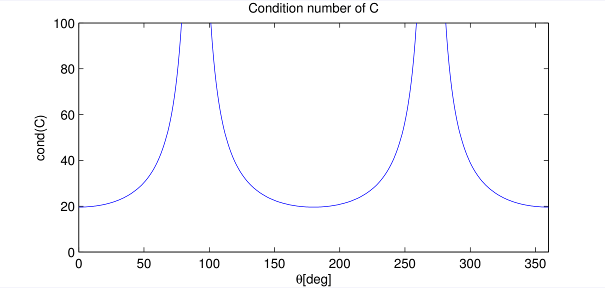
$$\mathcal{C} = \begin{bmatrix} \frac{1}{m} & -\frac{\sin(\theta)}{m} & -\frac{d}{m^2} & \frac{d\sin(\theta)}{m^2} & \frac{d^2}{m^3} & -\frac{d^2\sin(\theta)}{m^3} \\ 0 & \frac{\cos(\theta)}{m} & 0 & -\frac{\cos(\theta)d_s}{m^2} & 0 & \frac{\cos(\theta)d_s^2}{m^3} \\ 0 & \frac{r\cos(\theta)}{J} & 0 & -\frac{r\cos(\theta)d_\omega}{J^2} & 0 & \frac{r\cos(\theta)d_\omega^2}{J^3} \end{bmatrix}$$

Controllable?

Dynamic positioning



Condition number of $\mathcal{C}$

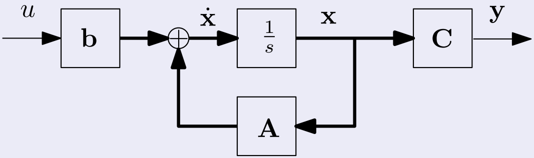


Topic

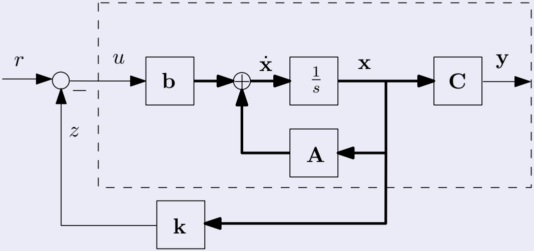
Topic

Open loop and feedback controlled systems

Open loop



State feedback: $u = r - \mathbf{kx}$



Closed loop dynamics

$$u = r - \mathbf{k}\mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r$$

State feedback

All states are available for feedback. Contrast this with *output feedback*.

First important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, there is a similarity transform $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ that will transform the system to controllable canonical form:

$$\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}^{\bar{\mathbf{A}}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{P}^{-1}\mathbf{b}}^{\bar{\mathbf{b}}}\,u \\
y &= \underbrace{\mathbf{c}\mathbf{P}}_{\bar{\mathbf{c}}}\,\bar{\mathbf{x}}
\end{aligned}$$

Controllable canonical form $n = 4$

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \bar{\mathbf{c}} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

$$g(s) = \bar{\mathbf{c}}(s\mathbb{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

First important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, there is a similarity transform $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ that will transform the system to controllable canonical form:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}^{\bar{\mathbf{A}}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{P}^{-1}\mathbf{b}}^{\bar{\mathbf{b}}}\,u \\ y &= \underbrace{\mathbf{c}\mathbf{P}}_{\bar{\mathbf{c}}}\,\bar{\mathbf{x}} \end{aligned}$$

Transformation

$$\mathbf{P} = \underbrace{\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\bar{\mathcal{C}}^{-1}}$$

Second important point

The pair $\{(\mathbf{A} - \mathbf{bk}), \mathbf{b}\}$ is only controllable if the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable.
- Feedback cannot produce or destroy controllability.

Reason[3]

Factorize closed loop controllability matrix to obtain:

$$
\begin{aligned}
\mathcal{C}_k &= \begin{bmatrix} \mathbf{b} & (\mathbf{A}-\mathbf{bk})\mathbf{b} & (\mathbf{A}-\mathbf{bk})^2\mathbf{b} & (\mathbf{A}-\mathbf{bk})^3\mathbf{b} \end{bmatrix} \\
&= \mathcal{C} \underbrace{\begin{bmatrix} 1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})^2\mathbf{b} \\ 0 & 1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b} \\ 0 & 0 & 1 & -\mathbf{kb} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Never singular}}
\end{aligned}
$$

---

[3]
$$\mathcal{C} = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$$

Third important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, we can place the eigenvalues exactly where we want!

Pole placement

Transform the plant to the controllable canonical form. Choose the **state feedback**:

$$u = \bar{\mathbf{k}}\bar{\mathbf{x}}, \quad \bar{\mathbf{k}} = \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \end{bmatrix}$$

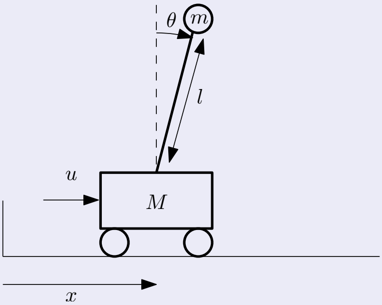
Closed loop dynamics

$$\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}} = \begin{bmatrix} -\alpha_1 - (\eta_1 - \alpha_1) & -\alpha_2 - (\eta_2 - \alpha_2) & -\alpha_3 - (\eta_3 - \alpha_3) & -\alpha_4 - (\eta_4 - \alpha_4) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Modified transfer function

$$\Rightarrow g_k(s) = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4}$$

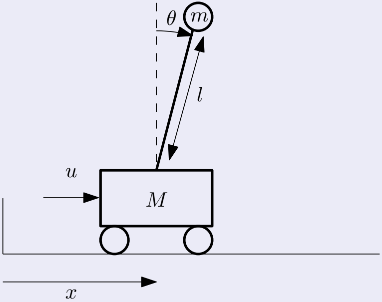
Example



Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Example



Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix} \quad \text{Full rank}$$

The system is controllable

It can be converted to controllable form!

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}| = s^4 - 15s^2 = s^2\left(-\sqrt{15} + s\right)\left(\sqrt{15} + s\right)$$

$$\alpha_2 = -15, \quad \alpha_{1,3,4} = 0$$

Transform

$$\mathbf{P} = \mathcal{C}\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -15 & 0 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}| = s^4 - 15s^2 = s^2\left(-\sqrt{15} + s\right)\left(\sqrt{15} + s\right)$$

$$\alpha_2 = -15, \quad \alpha_{1,3,4} = 0$$

Better characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}'| = s^2\left(\sqrt{15} + s\right)\left(\sqrt{15} + s\right) = s^4 + 2\sqrt{15}s^3 + 15s^2$$

$$\eta_1 = 2\sqrt{15}, \quad \eta_2 = 15, \quad \eta_{3,4} = 0$$

Pole placement: $u = \bar{\mathbf{k}}\bar{\mathbf{x}}$

$$\bar{\mathbf{k}} = \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{15} & 30 & 0 & 0 \end{bmatrix}$$

Transformed system

$$\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}} = \begin{bmatrix} -2\sqrt{15} & -15 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Reversion to original system $\bar{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{x}$

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} \quad \Rightarrow \mathbf{P}^{-1}\dot{\mathbf{x}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow \dot{\mathbf{x}} = \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}\bar{\mathbf{b}}\bar{\mathbf{k}}\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}$$

Transformed gain matrix

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & -60 & 0 & -4\sqrt{15} \end{bmatrix}$$

Original state equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ u \\ q \end{bmatrix}$$

Closed loop state equation $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 25 & 0 & 2\sqrt{15} \\ 0 & -15 & 0 & -2\sqrt{15} \end{bmatrix} \begin{bmatrix} x \\ \theta \\ u \\ q \end{bmatrix}$$

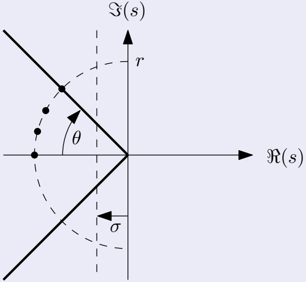
Pole placement

Procedure

1. Calculate the controllability matrix.
2. Find transform to controllable canonical form, $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$.
3. Choose gain matrix $\bar{\mathbf{k}}$ such that $u = -\bar{\mathbf{k}}\bar{\mathbf{x}}$ leads to the desired poles. Always possible.
4. Transform $\bar{\mathbf{k}}$ back to the original coordinates, $\mathbf{k} = \bar{\mathbf{k}}\mathbf{P}^{-1}$

Pole placement

Where to place poles



Considerations

- Close spacing of eigenvalues: sluggish response, large *u*.
- Large σ: fast response, large *u*.
- Large *r*: fast response, large *u*.
- Large θ: greater overshoot

Pole placement in MIMO systems

Controllable canonical form: p inputs

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{G}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} + \mathbf{D}$$

Pole placement in MIMO systems

Controllable canonical form: p inputs

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Pole placement

$$\bar{\mathbf{K}} = \begin{bmatrix} (\eta_1 - \alpha_1)\mathbb{I}_p & (\eta_2 - \alpha_2)\mathbb{I}_p & (\eta_3 - \alpha_3)\mathbb{I}_p & (\eta_4 - \alpha_4)\mathbb{I}_p \end{bmatrix}$$

$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4} + \mathbf{D}$$

MIMO control

Comments

- In matlab, use: ”K = place(A,B,eig)”
- Remember to always assign complex eigenvalues in *pairs*.
- MIMO pole placement alters the common denominator of $\mathbf{G}_{sp}(s)$.
- It may be difficult to tune the individual outputs.
- Only eigenvalues are considered, not input amplitudes.

Linear **Q**uadratic **R**egulator

The LQR allows more flexibility when specifying system performance. This is the preferred method for MIMO systems.

Topic

What's wrong with this **B**?

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer

Linear dependency in the columns. We disregard redundant inputs.

We have p inputs. Let:

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

Controllability matrix

$$\mathcal{C} = \left[\begin{array}{cccc|cccc|c|cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p & \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \cdots & \mathbf{A}\mathbf{b}_p & \ldots & \mathbf{A}^{n-1}\mathbf{b}_1 & \mathbf{A}^{n-1}\mathbf{b}_2 & \cdots & \mathbf{A}^{n-1}\mathbf{b}_p \end{array} \right]$$

Controllability index

The controllability indices of $\mathbf{b}_i$: μ_i, are the number of linearly independent columns associated with $\mathbf{b}_i$. If $\mathcal{C}$ has rank n, these indices sum to:

$$\mu_1 + \mu_2 + \cdots + \mu_p = n$$

The largest μ_i is the controllability index.

Multi-input controllability

Using these indices, we can show that it is sufficient to check the rank of:

$$\mathcal{C} = \left[\begin{array}{c|c|c|c} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-p}\mathbf{B} \end{array} \right]$$

Final notes

Property 1

Controllability is not affected by an equivalence transformation.

Property 2

Controllability is not affected by reordering the columns of $\mathbf{B}$.

TTK4115

Lecture 4

State feedback (continued), optimal control

Morten O. Alver (based on slides by Morten D. Pedersen

This lecture

1. Reference feed-forward

2. Integral effect

3. Optimal Control

4. Lunar lander

Topic

1. Reference feed-forward

2. Integral effect

3. Optimal Control

4. Lunar lander

Reference feed-forward

Aim: asymptotic convergence to reference

We want[1] $\mathbf{y}(t) \rightarrow \mathbf{r}$ as $t \rightarrow \infty$.

Implementation

$$\mathbf{u} = \underbrace{\mathbf{K}_r\mathbf{r}}_{\text{Reference feedforward}} - \underbrace{\mathbf{K}\mathbf{x}}_{\text{State Feedback}}$$

Equilibrium conditions

Assuming that feedback results in a stable equilibrium yields the steady-state condition:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{K}_r\mathbf{r} - \mathbf{K}\mathbf{x}) = \mathbf{0} \quad \Rightarrow (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_\infty = -\mathbf{B}\mathbf{K}_r\mathbf{r}_0 \quad \Rightarrow \mathbf{y}_\infty = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{K}_r\mathbf{r}_0$$

Finding $\mathbf{K}_r$

Inversion gives the correct feedforward gain:

$$\mathbf{K}_r = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Note that the number of references must be the same as the number of outputs.

[1] The elements in **y** are the variables we wish to control, not the measurement per sé. In state feedback *all* states are assumed known.

A common mistake

Suppose now that pure error feedback is used.:

$$\mathbf{u} = \mathbf{K}(\mathbf{r} - \mathbf{x})$$

This yields the system:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{r}$$

$$\Rightarrow \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)} = (s\mathbb{I} - [\mathbf{A} - \mathbf{B}\mathbf{K}])^{-1}\mathbf{B}\mathbf{K}$$

Assume that the reference is constant $\hat{\mathbf{r}}(s) = \mathbf{r}_0/s$. The final-value theorem yields (in general):

$$\mathbf{x}(\infty) = \lim_{s\to 0} \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)}\mathbf{r}_0 = -(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{K}\mathbf{r}_0 \neq \mathbf{r}_0$$

Correct approach

Suppose now that state feedback + reference feedforward is used instead:

$$\mathbf{u} = [(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}\mathbf{r} - \mathbf{Kx}$$

This yields the system:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}[(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}\mathbf{r}$$

$$\Rightarrow \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)} = (s\mathbb{I} - [\mathbf{A} - \mathbf{BK}])^{-1}\mathbf{B}[(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Assume that the reference is constant $\hat{\mathbf{r}}(s) = \mathbf{r}_0/s$. The final-value theorem yields:

$$\mathbf{x}(\infty) = \lim_{s\to 0} \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)}\mathbf{r}_0 = (\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}[(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}]^{-1}\mathbf{r}_0 = \mathbf{r}_0$$

Topic

Problem

What if we don't know our system perfectly? (We never do)
What about disturbances: **w**?

Solution

Use integral effect!

Integral effect

Plant[2]

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \overbrace{\mathbf{B}\mathbf{w}}^{\text{Disturbance}} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

Integrator state augmentation

$$\mathbf{x}_a = \int_0^t \mathbf{r}(\tau) - \mathbf{C}\mathbf{x}(\tau)\, d\tau$$

Augmented system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \mathbf{r}$$

[2] The disturbance **w** is assumed to act in a way that can be cancelled by the input.

Integral effect

Augmented system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \mathbf{r}$$

State feedback

As before state feedback + reference feedforward[3] is used:

$$\mathbf{u} = -\begin{bmatrix} \mathbf{K} & \mathbf{K}_a \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \mathbf{K}_r \mathbf{r}$$

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{K}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

[3] This is actually optional with integral effect.

Integral effect

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & -\mathbf{BK}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{BK}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

Steady state behavior - with feedforward

Provided that the feedback results in an asymptotically stable augmented system the output will tend towards a constant reference $\mathbf{r}_0$:

$$\dot{\mathbf{x}}_a = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{r}_0$$

The state vector exhibits the following limit:

$$\dot{\mathbf{x}} = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{K}_r\mathbf{r}_0 - \mathbf{w}_0]$$

If the feedforward is implemented as shown earlier ($\mathbf{K}_r = [\mathbf{C}(\mathbf{BK}-\mathbf{A})^{-1}\mathbf{B}]^{-1}$) we have:

$$\mathbf{Cx}_\infty - \mathbf{r}_0 = \mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}[\mathbf{w}_0 - \mathbf{K}_a\mathbf{x}_{a,\infty}] = \mathbf{0}$$

The integral action thus acts as a disturbance estimator/compensator:

$$\mathbf{K}_a\mathbf{x}_{a,\infty} = \mathbf{w}_0$$

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{K}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

Steady state behavior - without feedforward

Provided that the feedback results in an asymptotically stable augmented system the output will tend towards a constant reference $\mathbf{r}_0$:

$$\dot{\mathbf{x}}_a = \mathbf{0} \quad \Rightarrow \mathbf{C}\mathbf{x}_\infty = \mathbf{r}_0$$

The state vector exhibits the following limit:

$$\dot{\mathbf{x}} = \mathbf{0} \quad \Rightarrow \mathbf{C}\mathbf{x}_\infty = \mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{K})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{w}_0]$$

If the feedforward is *not implemented* we have:

$$\mathbf{C}\mathbf{x}_\infty - \mathbf{r}_0 = \mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{K})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{w}_0] - \mathbf{r}_0 = \mathbf{0}$$

The integral action now performs double duty and compensates for the disturbance as well as setting the correct bias[4]:

$$\mathbf{K}_a\mathbf{x}_{a,\infty} = -\mathbf{K}_r\mathbf{r}_0 + \mathbf{w}_0$$

[4] But only asymptotically..

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\tfrac{d}{m}, \quad \mathbf{B} = \tfrac{1}{m}, \quad \mathbf{C} = 1$$

State feedback

Control:

$$u = -kv + k_r r$$

Feedforward gain:

$$k_r = [\mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1} = d + k$$

Closed loop dynamics:

$$m\dot{v} = (d + k)(r - v) + w, \quad y = v$$

Stable equilibrium at $y = r$ if $w = 0$.

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\frac{d}{m}, \quad \mathbf{B} = \frac{1}{m}, \quad \mathbf{C} = 1$$

Augmented state space

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

State feedback

Control:

$$u = -kv - k_a x_a + k_r r$$

Closed loop dynamics:

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d+k}{m} & -\frac{k_a}{m} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{k_r}{m} \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\frac{d}{m}, \quad \mathbf{B} = \frac{1}{m}, \quad \mathbf{C} = 1$$

State feedback

Control:

$$u = -kv - k_a x_a + k_r r$$

Closed loop dynamics:

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d+k}{m} & -\frac{k_a}{m} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{k_r}{m} \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

Transfer functions

$$\hat{y}(s) = \frac{\overbrace{(d+k)s}^{\text{r.ff.}} - k_a}{ms^2 + (d+k)s - k_a}\hat{r}(s) + \frac{s}{ms^2 + (d+k)s - k_a}\hat{w}(s)$$

- Retaining reference feedforward may yield faster reference tracking.

Topic

1. Reference feed-forward

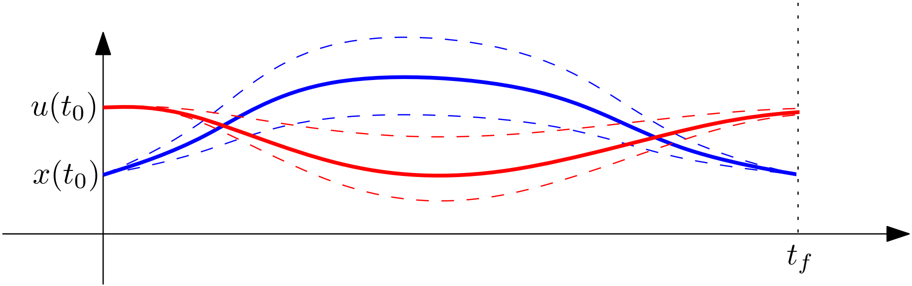
2. Integral effect

3. Optimal Control

4. Lunar lander

Optimal Control

$$J = \int_{t_0}^{t_f} x^T(\tau)Qx(\tau) + u(\tau)^T Ru(\tau)d\tau$$



Optimal control

Optimal control is concerned with finding an input $u(t)$ that minimizes a cost function over some interval of time, subject to system dynamics.

LQR

The **L**inear **Q**uadratic **R**egulator is a special case where the plant is **L**inear and the cost function is **Q**uadratic. The time interval is infinite.

Optimal Control

Finding a minimum

- Finding the control that minimizes the objective function is in general not trivial.
- Analytical solutions are in general very difficult to obtain.
- Discretizing the dynamics and finding the optimum numerically over a finite horizon is a viable method (MPC,NMPC).
- Linear models with a quadratic cost functions are a ”lucky” case.

Optimal Control

Finding a minimum

We want to find the minimum of:

$$J = \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt$$

for the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

The "trick" is to rewrite the cost functional on the special form:

$$\underline{J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt}$$

where:

- $H(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ is not affected by the input, directly or indirectly.
- $\Lambda(\mathbf{x}(t), \mathbf{u}(t))$ has an obvious minimum in terms of $\mathbf{u}(t)$.

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^{\infty} \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt$$

The functional[a] $H(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ above can be described as a *feedback invariant*. The functional takes the system inputs and states as arguments, but its value depends only on the initial condition $\mathbf{x}(0)$. It is not affected by the input, directly or indirectly.

[a] A functional takes functions as arguments and returns a scalar.

Feedback invariance

The functional:

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) \triangleq -\int_0^{\infty} \frac{d}{dt}\left[\mathbf{x}^{\mathsf{T}}(t)\mathbf{S}\mathbf{x}(t)\right] dt$$

happens to be feedback invariant as long as $\mathbf{x}(t) \to 0$ as $t \to \infty$:

$$-\int_0^{\infty} \frac{d}{dt}\left[\mathbf{x}^{\mathsf{T}}(t)\mathbf{S}\mathbf{x}(t)\right] dt = -\left[\mathbf{x}^{\mathsf{T}}(t)\mathbf{S}\mathbf{x}(t)\right]_0^{\infty} = \mathbf{x}^{\mathsf{T}}(0)\mathbf{S}\mathbf{x}(0) - \cancel{\mathbf{x}^{\mathsf{T}}(\infty)\mathbf{S}\mathbf{x}(\infty)}$$

The integrand can be rewritten as follows:

$$\frac{d}{dt}\left[\mathbf{x}^{\mathsf{T}}\mathbf{S}\mathbf{x}(t)\right] = \dot{\mathbf{x}}^{\mathsf{T}}\mathbf{S}\mathbf{x} + \mathbf{x}^{\mathsf{T}}\mathbf{S}\dot{\mathbf{x}} = \mathbf{x}^{\mathsf{T}}\mathbf{S}[\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}] + [\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}]^{\mathsf{T}}\mathbf{S}\mathbf{x}$$

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt \quad \left(= \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt \right)$$

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]^\mathsf{T}\mathbf{S}\mathbf{x}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]dt$$

Develop by adding and subtracting

$$\begin{aligned}
J &= H(\mathbf{x}, \mathbf{u}) + \left[\underline{\int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt} - H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) \right] \\
&= H(\mathbf{x}, \mathbf{u}) + \left[\int_0^\infty \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})^\mathsf{T}\mathbf{S}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{S}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})dt \right] \\
&= H(\mathbf{x}, \mathbf{u}) + \left[\int_0^\infty \underbrace{\mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C})\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}}_{\Lambda(\mathbf{x},\mathbf{u})} dt \right]
\end{aligned}$$

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt$$

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]^\mathsf{T}\mathbf{S}\mathbf{x}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]dt$$

$$\Lambda(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C})\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$$

Refine $\Lambda(\mathbf{x}, \mathbf{u})$ by ”completing the squares” (matrix version):

$$\mathbf{z}^T\mathbf{M}\mathbf{z} - 2\mathbf{b}^T\mathbf{z} = (\mathbf{z} - \mathbf{M}^{-1}\mathbf{b})^T\mathbf{M}(\mathbf{z} - \mathbf{M}^{-1}\mathbf{b}) - \mathbf{b}^T\mathbf{M}^{-1}\mathbf{b}$$

Thus we can rewrite $\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$ by choosing $\mathbf{u}$ as $\mathbf{z}$ and $-\mathbf{B}^T\mathbf{S}\mathbf{x}$ as $\mathbf{b}$:

$$\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x} = (\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x})^\mathsf{T}\mathbf{R}(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}) - \mathbf{x}^\mathsf{T}\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$$

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt$$

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{Ax}(t) + \mathbf{Bu}(t)]^\mathsf{T}\mathbf{Sx}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{Ax}(t) + \mathbf{Bu}(t)]dt$$

$$\Lambda(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{SA} + \mathbf{C}^\mathsf{T}\mathbf{QC} - \mathbf{SBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S})\mathbf{x} + (\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx})^\mathsf{T}\mathbf{R}(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx})$$

Minimization step 1:

Solve CARE:

$$\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{SA} + \mathbf{C}^\mathsf{T}\mathbf{QC} - \mathbf{SBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S} = 0$$

Minimization step 2:

Choose:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx}$$

The minimum is thus obtained:

$$\Lambda(\mathbf{x}(t), \mathbf{u}(t)) \equiv 0 \quad \Rightarrow J = \mathbf{x}^\mathsf{T}(0)\mathbf{Sx}(0)$$

LQR

Cost functional to be *minimized* w.r.t. $\mathbf{u}(t)$

$$J_{LQR} = \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt$$

with:

$$\mathbf{Q} > 0,\ \mathbf{Q} = \mathbf{Q}^\mathsf{T}, \quad \mathbf{R} > 0,\ \mathbf{R} = \mathbf{R}^\mathsf{T}$$

Output *energy*

$$\mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t)$$

Making this function smaller requires more input energy.

Control *energy*

$$\mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)$$

Making this function smaller requires less input energy which leads to higher output energy.

LQR

Minimal solution

The solution that minimizes the cost function for the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is a linear feedback:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$$

where:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{S}$$

leading to the closed loop system:

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{S}]\mathbf{x}(t)$$

LQR

Algebraic Riccati Equation

The matrix **S** is found by solving:

$$[\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A}] + \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S} = 0$$

where $\mathbf{S} > 0$ must be **positive definite**.

Example $\dot{x}(t) = -\lambda x(t) + u(t),\ y(t) = x(t)$

The (now scalar) p is found by solving:

$$-2\lambda p + q - \frac{p^2}{r} = 0$$

The solutions are:

$$p = -\lambda r \pm \sqrt{qr + \lambda^2 r^2} \quad : \text{Pick the } \underline{\text{positive}} \text{ solution}$$

The feedback matrix becomes:

$$u(t) = -kx(t) = -\frac{p}{r}x(t) = \left(\lambda - \sqrt{\frac{q}{r} + \lambda^2}\right) x(t)$$

LQR

Closed loop system $\dot{x}(t) = -\lambda x(t) - kx(t)$

State dynamics:

$$\begin{aligned}
\dot{x}(t) &= -\left(\lambda - \frac{p}{r}\right) x(t) \\
&= -\left(\lambda - [\lambda - \sqrt{\frac{q}{r} + \lambda^2}]\right) x(t) \\
&= -\left(\sqrt{\frac{q}{r} + \lambda^2}\right) x(t)
\end{aligned}$$

Input:

$$u(t) = -kx(t) = -\frac{p}{r}x(t) = \left(\lambda - \sqrt{\frac{q}{r} + \lambda^2}\right) x(t)$$

Note:

There is a direct tradeoff between q and r, ($\mathbf{Q}$ and $\mathbf{R}$ in the general case)

Tuning

Tuning is done by selecting the weights **Q** and **R**. We typically choose these as diagonal matrices.

Bryson's Rule (Rule of thumb)

$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2}$$

$$R_{jj} = \frac{1}{\text{maximum acceptable value of } u_j^2}$$

Matlab commands

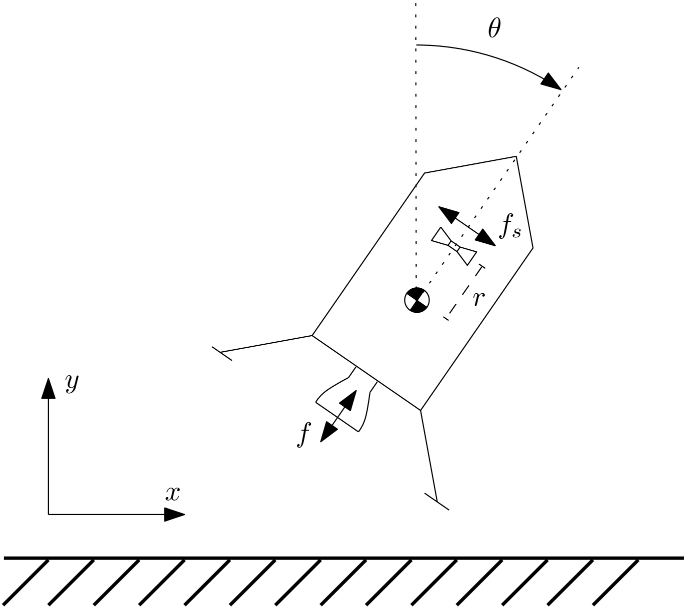
- "SYS = ss(A,B,C,D)"
- "[K,S,E] = lqr(SYS,Q,R,N)"
- "[K,S,E] = lqry(SYS,Q,R,N)"

Remarks

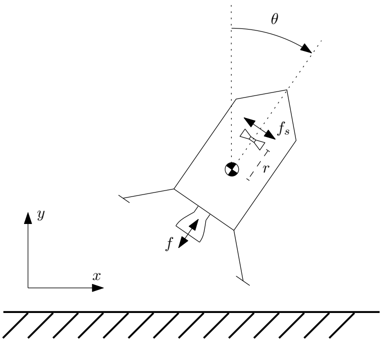
- $\{\mathbf{A}, \mathbf{B}\}$ must be controllable.
- If weighing the outputs, all unstable modes must show up in the output.

Topic

Lunar lander



Lunar lander



Nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} f \\ f_s \end{bmatrix} - \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

Lunar lander

Nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} f \\ f_s \end{bmatrix} - \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

Inputs

$$u_1 = f - mg \quad u_2 = f_s$$

Perturbed nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \sin(\theta) \\ \cos(\theta) - 1 \\ 0 \end{bmatrix}$$

Lunar lander

Perturbed nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \sin(\theta) \\ \cos(\theta) - 1 \\ 0 \end{bmatrix}$$

Linearized model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix}$$

Lunar lander

Linearized model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix}$$

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{j} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Lunar lander

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{j} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & \frac{gr}{j} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r}{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m} & 0 & 0 & 0 & \frac{gr}{j} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{r}{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Lunar lander

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{J} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Output

$$\mathbf{y} = \mathbf{C}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

Lunar lander

Simulink model

A Simulink model for the lunar lander with accompanying Matlab script is available on Blackboard.

Nonlinearities

The true system is not linear!

Actuator saturation

Thrusters are limited. In this example to:

$$0 \leq f \leq 2mg \qquad -0.1mg \leq f_s \leq 0.1mg$$

TTK4115

Lecture 5

Canonical forms, realizations, observability

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

1. Canonical Forms

2. Realizations

3. Observability

   Duality

Topic

Canonical Forms

Canonical Forms

- Using the change of basis $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$ we can change a system into infinitely many similar forms.
- Some of these forms are more useful than others.
- Some of these are called *canonical*.

Equivalence/Similarity transform

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\bar{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{T}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

Canonical Forms

Canonical Forms[1]

- Jordan Form
- Modal Form
- Companion form
- Controllable form
- Observable form

[1] This list is not exhaustive.

Canonical Forms

Jordan Form

The Jordan form is the most convenient to use when solving the system. We have seen that this form is very practical for finding solutions to LTI systems:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Diagonal matrix

$$\mathbf{A} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix}$$

Jordan Block

$$\mathbf{A} = \mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 & \frac{1}{3!}e^{t\lambda}t^3 \\ 0 & e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 \\ 0 & 0 & e^{t\lambda} & e^{t\lambda}t \\ 0 & 0 & 0 & e^{t\lambda} \end{bmatrix}$$

Topic

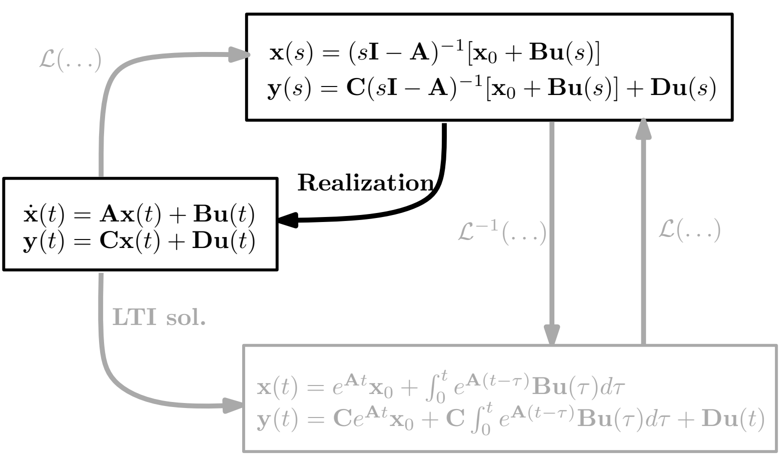
1. Canonical Forms

2. Realizations

3. Observability

Duality

LTI systems overview



Realizations

The final piece in the diagram

Realizations

Key points

Realization

- We have seen that a transformation $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$ can change the state equation..
- but the transfer function remains the same.
- When we realize, we start with a transfer function $\mathbf{H}(s)$..
- and generate a state-space $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$
- that yields $\mathbf{H}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Note

There are infinitely many state-spaces we could realize to!

Note

We usually go for a *canonical* form.

Proper

A transfer function must be proper to have a realization:

$$h(s) = \frac{n(s)}{d(s)} \quad \Rightarrow \deg d(s) \geq n(s)$$

$$|h_p(j\infty)| < \infty, \quad |h_{sp}(j\infty)| = 0$$

Rational

A transfer function must be rational to have a realization.

- The degrees of the numerator and denominator must be finite.
- All lumped LTI systems are rational.

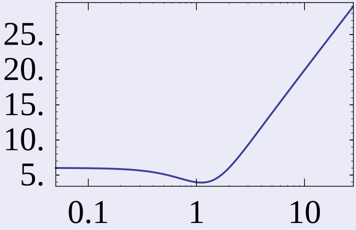
Proper transfer functions

We must have a proper transfer function for realization.

Example

$$h(s) = \frac{2 + 2s + s^2}{1 + s}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Signals are amplified

at infinite frequencies.. no device can do this

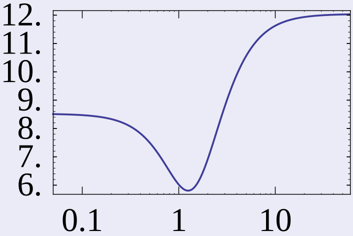
Proper transfer functions

Proper transfer functions behave nicely at high frequencies

Example

$$h(s) = \frac{4\left(2 + 2s + s^2\right)}{3 + 4s + s^2}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Answer

Yes, *but not strictly proper*. The transfer function is finite at infinite frequencies:

$$\lim_{\omega \to \infty} h(j\omega) = h_\infty \neq 0$$

Realizations

Conditions

Quiz

Are these transfer functions realizable?

- $g_1(s) = \frac{1}{s}$ (yes, strictly proper)
- $g_2(s) = s$ (no, not proper)
- $g_3(s) = \frac{1}{s+1}$ (yes, strictly proper)
- $g_4(s) = \frac{1}{s-1}$ (yes, strictly proper)
- $g_5(s) = \frac{s}{s+1}$ (yes, proper)
- $g_6(s) = e^{-\tau s}, \quad \tau > 0$ (no, not rational)
- $g_7(s) = \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}, \quad \tau > 0$ (yes, proper)

Improper/Proper/Strictly proper

Improper

$$H_{i.p.}(s) = k_P + k_D s + \frac{k_I}{s} = \frac{k_D s^2 + k_P s + k_I}{s} \quad \text{PID regulator}$$

Proper

$$H_p(s) = \frac{s}{Ts + 1} \quad \text{Band-limited differentiator}$$

Strictly proper

$$H_{s.p.}(s) = \frac{1}{s^2 m + ds + k} \quad \text{Mass-spring-damper}$$

Realizations
Strictly proper transfer functions

Decomposition

We decompose the proper transfer function as:

$$\mathbf{G}(s) = \overbrace{\mathbf{G}_{sp}(s)}^{\text{strictly proper}} + \overbrace{\mathbf{G}_{\infty}}^{\text{constant}}$$

Relation between system matrices and decomposed transfer function

$$\begin{aligned}\hat{\mathbf{y}}(s) &= \mathbf{G}_{sp}(s)\hat{\mathbf{u}}(s) + \mathbf{G}_{\infty}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

Matching

The crucial next step is to select a state-space model with unknown coefficients:

$$\Sigma_r : \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

that can represent our transfer-function.

Matching

We shall use the **Controllable Canonical Form** today. This is one of many choices.

Let's pick a nice **A** for the realization

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}$

$$(s\mathbb{I} - \mathbf{A})^{-1} = \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} \begin{bmatrix} s^3 & -s^2\alpha_2 - s\alpha_3 - \alpha_4 & -s^2\alpha_3 - s\alpha_4 & -s^2\alpha_4 \\ s^2 & s^3 + s^2\alpha_1 & -s\alpha_3 - \alpha_4 & -s\alpha_4 \\ s & s^2 + s\alpha_1 & s^3 + s^2\alpha_1 + s\alpha_2 & -\alpha_4 \\ 1 & s + \alpha_1 & s^2 + s\alpha_1 + \alpha_2 & s^3 + s^2\alpha_1 + s\alpha \end{bmatrix}$$

Let's pick a nice **B** for the realization too

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

What about $\mathbf{C}$?

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

Example: Mass spring damper transfer-function realization

$$\frac{x(s)}{f(s)} = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + sd + k} = \frac{1/m}{s^2 + s(d/m) + (k/m)}$$

Controllable canonical form, $n = 2$

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s\overbrace{n_1}^{0} + \overbrace{n_2}^{1/m}}{s^2 + s\underbrace{\alpha_1}_{d/m} + \underbrace{\alpha_2}_{k/m}}$$

Realization

The mass spring damper back on state-space form:

$$\mathbf{A} = \begin{bmatrix} -d/m & -k/m \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1/m \end{bmatrix}$$

Controllable canonical form: p inputs, q outputs

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$d(s)$

We have to find the common denominator of $\mathbf{G}_{sp}(s)$: $d(s) = s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find $\mathbf{G}_\infty$

$$\mathbf{D} = \mathbf{G}_\infty = \lim_{s\to\infty} \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Find $\mathbf{G}_{sp} = \mathbf{G}(s) - \mathbf{G}_\infty$

$$\mathbf{G}_{sp} = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{12}{1+2s} & \frac{3}{2+s} \\ \frac{1}{2+5s+2s^2} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find common denominator $d(s)$

$$\mathbf{G}_{sp} = \frac{1}{s^3 + (9/2)s^2 + 6s + 2} \begin{bmatrix} -6(2+s)^2 & 3(1+s/2)(1+2s) \\ 1+s/2 & (1/2+s)(1+s) \end{bmatrix}$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\begin{bmatrix} -24-24s & 3+\frac{15s}{2} \\ 1+\frac{s}{2} & \frac{1}{2}+\frac{3s}{2} \end{bmatrix} + s^2\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Find numerator matrices $\mathbf{N}_i$

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\overbrace{\begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}}^{\mathbf{N}_3} + s\overbrace{\begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}}^{\mathbf{N}_2} + s^2\overbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}^{\mathbf{N}_1}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left[\mathbf{N}_3 + s\mathbf{N}_2 + s^2\mathbf{N}_1\right] \quad \mathbf{G}_\infty = \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} \quad \mathbf{N}_2 = \begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \mathbf{N}_3 = \begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = s^3 + (9/2)s^2 + 6s + 2$$

Realize:

$$\mathbf{A} = \begin{bmatrix} -\alpha_1\mathbb{I}_p & -\alpha_2\mathbb{I}_p & -\alpha_3\mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \end{bmatrix}$$

Topic

1. Canonical Forms

2. Realizations

3. Observability

Duality

Observability

Observability: definition

The system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

is said to be **observable** if for any unknown initial state $\mathbf{x}(0)$ there exists a finite $t_1 > 0$ such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over $[0, t_1]$ suffices to determine uniquely the initial state $\mathbf{x}(0)$.
Otherwise, the system is **unobservable**.

Observability

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is observable.

Explanation

The solution to the state equation is:

$$\mathbf{y}(t) = \overbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)}^{\text{We don't know the in. conds.}} \overbrace{+\,\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}^{\text{We know the input.}}$$

Definition

The **unknown** part of the solution is:

$$\mathbf{y}_{\text{nat.}}(t) \triangleq \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

AKA the *natural response*.

Observability

1: Premultiply

$$e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(t) = e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

2: Integrate

$$\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}d\tau\,\mathbf{x}(0)$$

3: Recover the Gramian

$$\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \underbrace{\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}d\tau}_{\mathbf{W}_o(t_1)}\,\mathbf{x}(0)$$

4: Invert[2] and premultiply

$$\mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \mathbf{W}_o^{-1}(t_1)\mathbf{W}_o(t_1)\mathbf{x}(0) = \mathbb{I}\mathbf{x}(0)$$

[2] Only possible if the system is observable.

Observability

Only *one possible* $\mathbf{x}(0)$ for a given $\mathbf{y}_{\text{nat.}}(t)$

$$\mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \mathbf{x}(0)$$

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^{t} e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{C}e^{\mathbf{A}\tau}d\tau$$

is nonsingular for any $t > 0$, then the state equation is observable.
-**The initial conditions can be recovered from the output**

Observability

Observability Gramian

Otherwise the Gramian:

$$\exists\, \mathbf{v}: \quad \mathbf{v}^{\mathsf{T}}\mathbf{W}_o(t)\mathbf{v} = \int_0^t \mathbf{v}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{C}e^{\mathbf{A}\tau}\mathbf{v}\, d\tau = \int_0^t \left\|\mathbf{C}e^{\mathbf{A}\tau}\mathbf{v}\right\|^2 d\tau = 0$$

is **singular**.
-Then the state equation is **unobservable**.

$\mathbf{x}(0)$ cannot be identified uniquely given $\mathbf{y}_{\text{nat.}}(t)$

$$\int_0^{t_1} e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \mathbf{W}_o(t_1)[\mathbf{x}(0) + \mathbf{v}]$$

Topic

1. Canonical Forms

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Duality

Duality

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is **observable**.

Controllability Gramian

If the **Controllability Gramian**:

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is **controllable**.

Duality

System Gramians

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau \quad \mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

Imagine a system:

$$\dot{\mathbf{z}} = \mathbf{A}^\mathsf{T}\mathbf{z} + \mathbf{C}^\mathsf{T}\mathbf{u}$$

What is the controllability gramian?

$$\mathbf{W}_c^z(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

Duality

System Gramians

$$\underline{\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau} \quad \mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

Dual system:

$$\dot{\mathbf{z}} = \mathbf{A}^\mathsf{T}\mathbf{z} + \mathbf{C}^\mathsf{T}\mathbf{u}$$

Implication?

$$\underline{\mathbf{W}_c^z(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau}$$

Theorem of duality

The pair $\{\mathbf{A}, \mathbf{C}\}$ is observable if and only if
the pair $\{\mathbf{A}^\mathsf{T}, \mathbf{C}^\mathsf{T}\}$ is controllable.

Duality

The pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable iff:

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}\tau}\, d\tau$$

is nonsingular for all $t > 0$.

The **controllability matrix**:

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

has full *row rank*.

The pair $\{\mathbf{A}, \mathbf{C}\}$ is observable iff:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{C} e^{\mathbf{A}\tau}\, d\tau$$

is nonsingular for all $t > 0$.

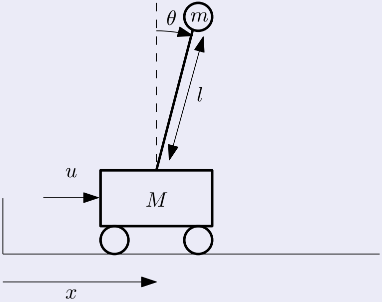
The **observability matrix**:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

has full *column rank*.

Observability

Example



Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Observability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Case I

The *angle* is measured:

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Observability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Case II

The *position* is measured:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

TTK4115

Lecture 6

State estimation, output feedback

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

1. State estimation

    Measurement equation inversion

    Open-loop estimator

    Closed-loop estimator

    Noise suppression: A simple example

    Band-limited differentiation: A simple example

2. Conveyor belt case study

3. Certainty Equivalence

4. Separation principle

Topic

1. State estimation

   Measurement equation inversion

   Open-loop estimator

   Closed-loop estimator

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State estimation

Problem

With the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Given $\mathbf{y}(t)$ and $\mathbf{u}(t)$.
What is $\mathbf{x}(t)$?

Applications

- State feedback without direct measurements
- Navigation, GPS, robotics, motion control
- Radar, sonar etc..
- Fault detection & diagnosis
- Disturbance estimation

State estimation

Notation

$$\begin{aligned} \mathbf{x} &: \text{Signal to be estimated} \\ \hat{\mathbf{x}} &: \text{Estimate} \\ \mathbf{y} &: \text{Output (no noise)} \\ \mathbf{y}_m &: \text{Measured output: } \mathbf{y}_m = \mathbf{y} + \mathbf{n} \\ \mathbf{n} &: \text{Noise} \end{aligned}$$

Estimation error

The estimation error is defined as:

$$\mathbf{e} \triangleq \mathbf{x} - \hat{\mathbf{x}}$$

The error is **dynamic**:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\hat{\mathbf{x}}}$$

Observer design is concerned with choosing the **estimate update law** $\dot{\hat{\mathbf{x}}}(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{u})$ in a clever way so that:

$$\mathbf{e} \to \mathbf{0}, \quad t \to \infty$$

Topic

1. State estimation

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    Open-loop estimator

    Closed-loop estimator

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    Band-limited differentiation: A simple example

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4. Separation principle

Method I: Measurement equation inversion

If

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

and **C** is square and invertible, we can reconstruct the states:

$$\mathbf{x}(t) = \mathbf{C}^{-1}\mathbf{y}(t)$$

If **C** is *not* square and invertible, but there are *more measurements than states*, we can use:

$$\mathbf{x}(t) = \left[\mathbf{C}^T\mathbf{C}\right]^{-1}\mathbf{C}^T\mathbf{y}(t)$$

which is a *least squares* estimator, minimizing:

$$\left[\mathbf{y}(t) - \mathbf{C}\mathbf{x}(t)\right]^T\left[\mathbf{y}(t) - \mathbf{C}\mathbf{x}(t)\right]$$

Topic

1. State estimation

Measurement equation inversion

Open-loop estimator

Closed-loop estimator

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Band-limited differentiation: A simple example

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Method II: Open-loop estimator

Idea

If we have a good model, we can use it to estimate the states:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

by simulating the model given the real input $\mathbf{u}(t)$.

Caveats

- Sensitive to modeling errors and disturbances
- Often slow convergence rate
- Does not work for unstable systems

Method II: Open-loop estimator

Plant:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x}
\end{aligned}
$$

Open loop estimator:

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \\
\hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}}
\end{aligned}
$$

Error dynamics

$$
\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}) = \mathbf{A}\mathbf{e}
$$

Note

$\mathbf{A}$ must be stable: $\mathbf{e} \to \mathbf{0}, \quad t \to \infty$

Topic

1. State estimation
   - Measurement equation inversion
   - Open-loop estimator
   - Closed-loop estimator
   - Noise suppression: A simple example
   - Band-limited differentiation: A simple example
2. Conveyor belt case study
3. Certainty Equivalence
4. Separation principle

Method III: Closed-loop estimator

Idea

Use an open-loop estimator in conjunction with a feedback from the measurement error:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})$$

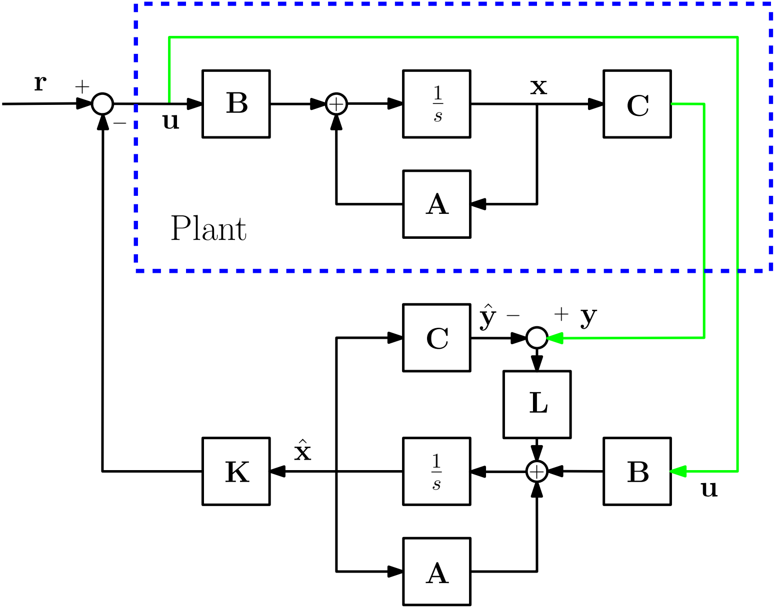
Key points

- The feedback corrects to a certain degree for uncertainties due to modeling errors and disturbances.
- Subject to certain conditions, $\mathbf{L}$ can be chosen such that $\hat{\mathbf{x}}(t) \to \mathbf{x}(t)$ with desired convergence rate.
- An asymptotically stable estimator can be achieved for an unstable system, due to stabilizing feedback.

Theorem

All eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be *assigned arbitrarily* by selecting a real constant matrix $\mathbf{L}$ if and only if $\{\mathbf{A}, \mathbf{C}\}$ are **observable**.

Method III: Closed-loop estimator



Method III: Closed-loop estimator

Plant:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

Closed loop estimator:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \overbrace{\mathbf{L}(\mathbf{y}_m - \hat{\mathbf{y}})}^{\text{Injection term}}, \quad \mathbf{L}(\mathbf{y}_m - \hat{\mathbf{y}}) = \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{L}\mathbf{n} = \mathbf{L}\mathbf{C}\mathbf{e} + \mathbf{L}\mathbf{n}\\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}}\end{aligned}$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} - \mathbf{L}\mathbf{n}$$

Note

$\mathbf{L}$ can be chosen so that $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$.
We can in fact choose the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ freely if the system is observable!

Topic

Noise suppression: A simple example

Plant

$$\dot{x} = -\frac{1}{T}x + \frac{1}{T}u, \quad y_m = x + n$$

Task: **Estimate x using an observer to suppress noise in the measurement.**

Closed loop estimator

Estimate update equation:

$$\dot{\hat{x}} = \underbrace{-\frac{1}{T}\hat{x} + \frac{1}{T}u}_{\text{Copy dynamics}} + \underbrace{l(y_m - \hat{y})}_{\text{Injection term}}$$

Error dynamics:

$$\dot{e} = \underbrace{\left(-\frac{1}{T} - l\right)}_{\mathbf{A-LC}} e - ln$$

Pole placement

Apply the *observer gain*:

$$l = -\frac{1}{T} + \frac{1}{T_d}$$

to place the pole at $\lambda_1 = -\frac{1}{T_d}$. A small time constant yields fast convergence.

Noise suppression: A simple example

Error dynamics

$$\dot{e} = -\frac{1}{T_d} e + \left(\frac{1}{T} - \frac{1}{T_d} \right) n$$

Error due to noise: Laplace model

- The observer low-passes the noise contribution in the estimate.
- Higher observer gain yields a higher cut-off frequency.

$$\frac{\hat{e}(s)}{\hat{n}(s)} = \frac{\left(\frac{1}{T} - \frac{1}{T_d} \right)}{\left(s + \frac{1}{T_d} \right)}$$

Compare with original situation:

$$\hat{x} = y_m \quad \Rightarrow \frac{\hat{e}(s)}{\hat{n}(s)} = -1$$

Topic

1. State estimation

   Measurement equation inversion

   Open-loop estimator

   Closed-loop estimator

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Band-limited differentiation: A simple example

Plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n$$

Task: **Estimate the derivative $\dot{x}_1 = x_2$ using an observer.**

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}}_{\text{Copy dynamics}} + \underbrace{\overbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}^{\mathbf{L}} \overbrace{(y_m - \hat{y})}^{\mathbf{Ce+n}}}_{\text{Injection term}}$$

Error dynamics:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} n$$

Band-limited differentiation: A simple example

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}}_{\text{Copy dynamics}} + \underbrace{\overbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}^{\mathbf{L}} \overbrace{(y_m - \hat{y})}^{\mathbf{Ce}+\mathbf{n}}}_{\text{Injection term}}$$

Error dynamics:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} n$$

Pole placement

Apply the *observer gain*:

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 2\omega_n \\ \omega_n^2 \end{bmatrix}$$

to place the poles at $\lambda_1 = \lambda_2 = -\omega_n$. A high natural frequency yields fast performance.

Band-limited differentiation: A simple example

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -2\omega_n & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 2\omega_n \\ \omega_n^2 \end{bmatrix} y_m$$

Band-limited differentiation: Laplace model

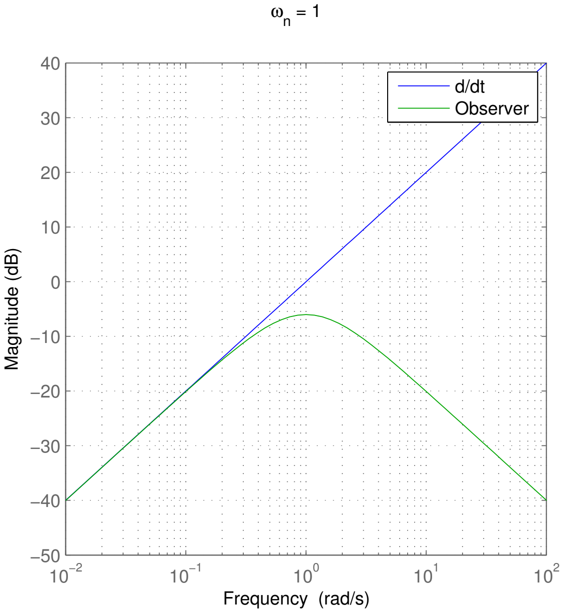
- The observer uses band-limited differentation.
- Higher observer gain yields a higher cut-off frequency.
- Warning: noise below cut-off is also differentiated.

$$\frac{\hat{\hat{x}}_2(s)}{\hat{y}_m(s)} = \frac{s\omega_n^2}{(s+\omega_n)^2}$$

Compare with original situation:

$$\frac{\hat{\hat{x}}_2(s)}{\hat{y}_m(s)} = s$$

Band-limited differentiation: A simple example



Topic

Example: Conveyor belt

Model

Transport belt model:

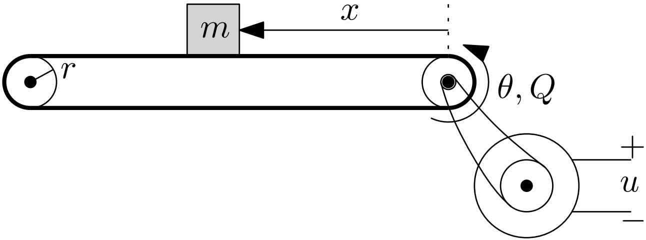
$$x = \theta r$$

Belt dynamics:

$$m\ddot{x}(t) + \frac{j}{r}\ddot{x}(t) + d\dot{x}(t) = \frac{1}{r}Q(t), \quad \text{Mass } m \text{ unknown}$$

Motor dynamics:

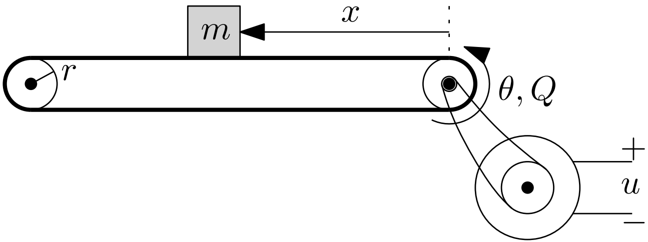
$$T\dot{Q}(t) + Q(t) = u(t)$$



Example: Conveyor belt

State-space dynamics

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & 0 & 0 \\ \frac{1}{j+mr} & -\frac{dr}{j+mr} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} \frac{1}{T} \\ 0 \\ 0 \end{bmatrix} u$$



Example: Conveyor belt

Parameters

$$T = 1, \quad j = 10, \quad r = 1, \quad d = 1, \quad \underbrace{m = 1}_{\text{unknown}}$$

Plant State-space dynamics - numerical

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{11} & -\frac{1}{11} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Eigenvalues:

$$\lambda = \left\{ 0, -1, -\frac{1}{11} \right\} \quad \text{Marginally stable}$$

Observer model: $m = 0$

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Three measurement models

Case 1: Encoder on roller wheel

$$y_m = x_m = \theta r + n_x \quad \Rightarrow y_m = \overbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}^{\mathbf{c}_1} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + n_x$$

Quantization noise on position signal: n_x.

Case 2: Add tachometer on roller shaft.

$$\mathbf{y}_m = \begin{bmatrix} \dot{x}_m \\ x_m \end{bmatrix} + \begin{bmatrix} n_{\dot{x}} \\ n_x \end{bmatrix} \quad \Rightarrow \mathbf{y}_m = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{C}_2} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} n_{\dot{x}} \\ n_x \end{bmatrix}$$

Quantization noise on speed signal: $n_{\dot{x}}$.

Case 3: Add strain transducer on motor shaft.

$$\mathbf{y}_m = \begin{bmatrix} Q \\ \dot{x}_m \\ x_m \end{bmatrix} + \begin{bmatrix} n_Q \\ n_{\dot{x}} \\ n_x \end{bmatrix} \quad \Rightarrow \mathbf{y}_m = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{C}_3} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} n_Q \\ n_{\dot{x}} \\ n_x \end{bmatrix}$$

Vibrations/electrical noise: n_Q.

All measurement models render the plant observable. Noise is assumed to be of *high frequency*.

Method I: Measurement equation inversion

Conveyor belt: **Case 3**

Only full state measurements may be used.

$$\mathbf{y}_m = \mathbf{C}_3\mathbf{x} + \mathbf{n}, \quad \mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Measurement equation inversion

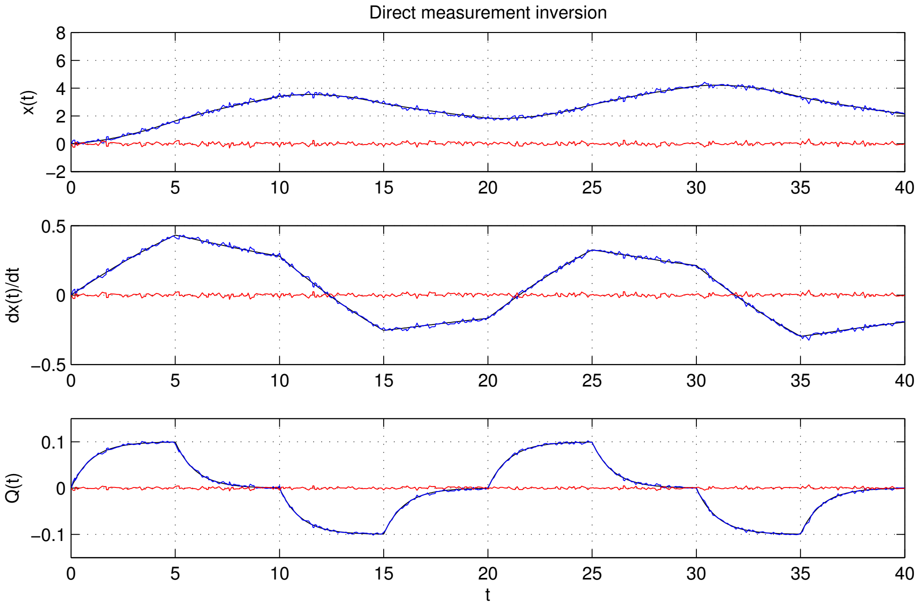
With $\mathbf{C}_3 = \mathbb{I}$:

$$\hat{\mathbf{x}}(t) = \mathbf{C}_3^{-1}\mathbf{y}_m(t) = \mathbf{x} + \mathbf{n}$$

The error reads as:

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} = -\mathbf{n}$$

Method I: Measurement equation inversion

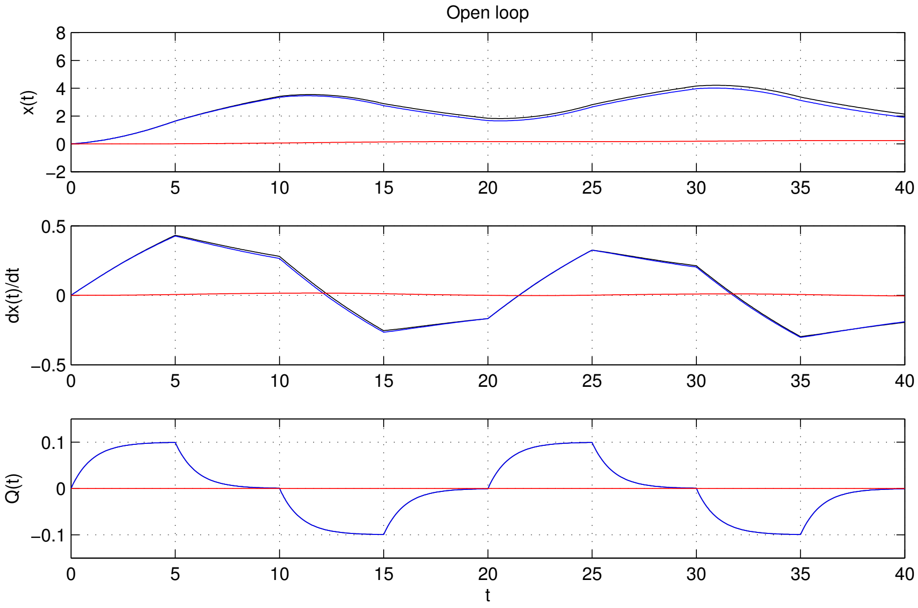


Method I: Measurement equation inversion

Comments

This method picks up the measurement noise by design: $\hat{\mathbf{x}} = \mathbf{y}_m$. With noisy sensors this may be problematic.

Method II: Open loop observer



Method II: Open loop observer

Comments

As expected the open-loop observer will diverge due to the model error in the simulator dynamics. On the other hand, the methodology is ”noise-proof”.

Method III: Closed loop observer

Pole-placement

Pole-placement for the observer gain is not a trivial exercise. Some general guidelines are:

- The error dynamics should be faster than the plant itself: $\times 2 - 20$.
- Noisy plants generally warrant a slower observer, i.e. placing poles with smaller negative real part. This to avoid amplification of noise.
- The Kalman filter is a closed loop observer. It takes the guesswork out of the tuning and is often the preferable approach. Pay attention in the last lecture weeks.

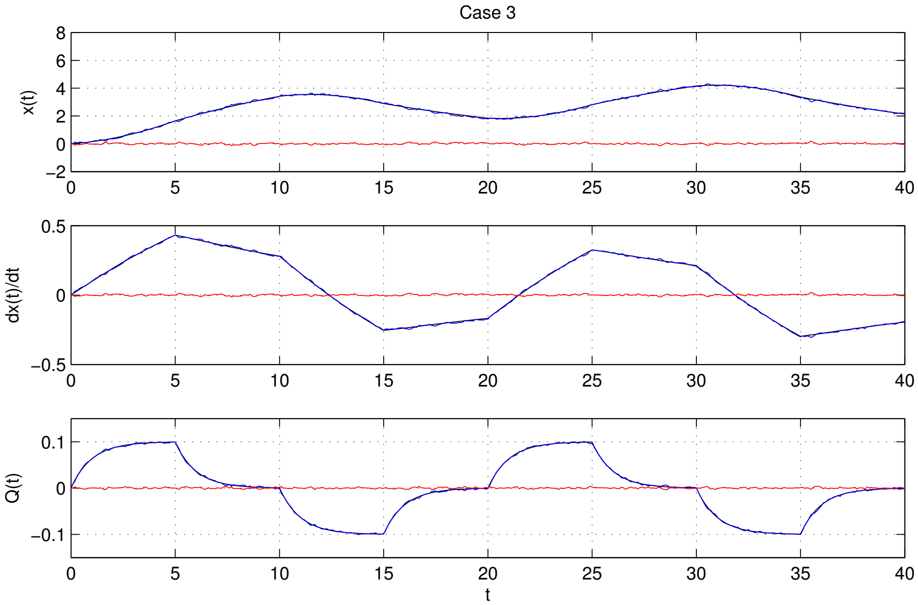
Placing poles for Cases 1,2,3

- Poles are initially placed along a circular arc of radius $R = 5$. The most negative pole in the plant is $\lambda_{\text{max}} = -1$. There is some noise in the plant so it is best to be conservative.
- The arc opens to an angle of 60°. This is to avoid bunching poles close together.

For the conveyor:

$$\lambda = -5\{e^{-30\cdot\frac{pi}{180}i},\ e^{0\cdot\frac{pi}{180}i},\ e^{+30\cdot\frac{pi}{180}i}\} = \{-4.33 + 2.5i,\ -5,\ -4.33 - 2.5i\}$$

Case 3: All states measured

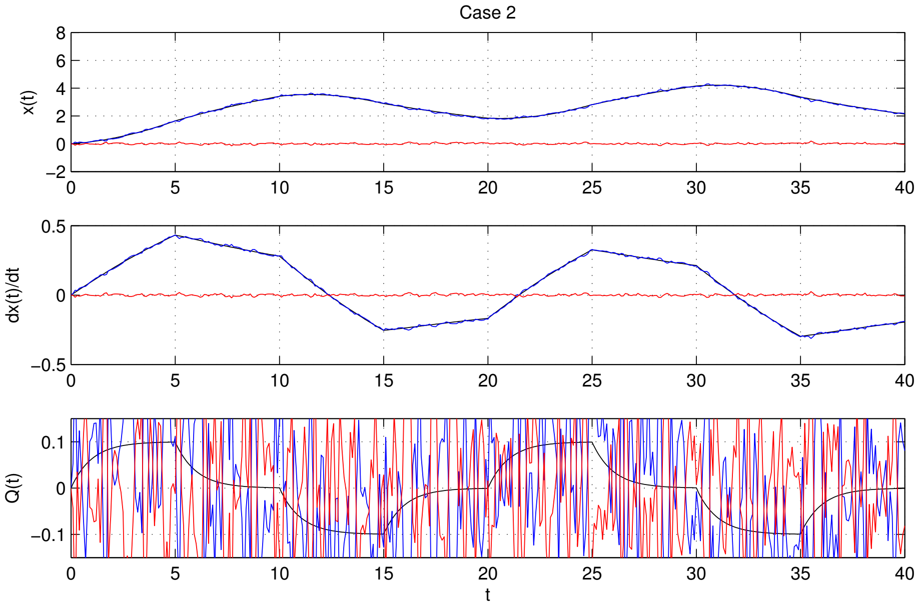


Case 3: All states measured

Comments

The observer filters out significant portions of the measurement noise. The convergence to the true states is due to the error feedback.

Case 2: Shaft torque not measured



Case 2: Shaft torque not measured

Comments

The lack of a torque measurement means *differentiation* must be employed in torque estimation, albeit not in a pure manner. This is seen in the belt dynamics:

$$\left(m + \frac{j}{r}\right)\ddot{x}(t) + d\dot{x}(t) = \frac{1}{r}Q(t)$$

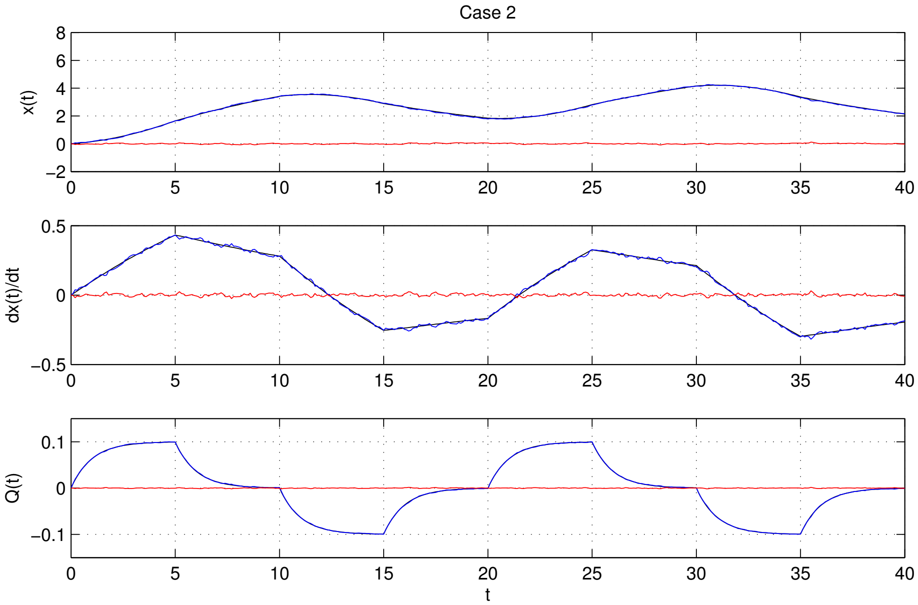
This suggests that the errors are due amplification of noise. The observer must be tuned so as to avoid this phenomena.

Better tuning

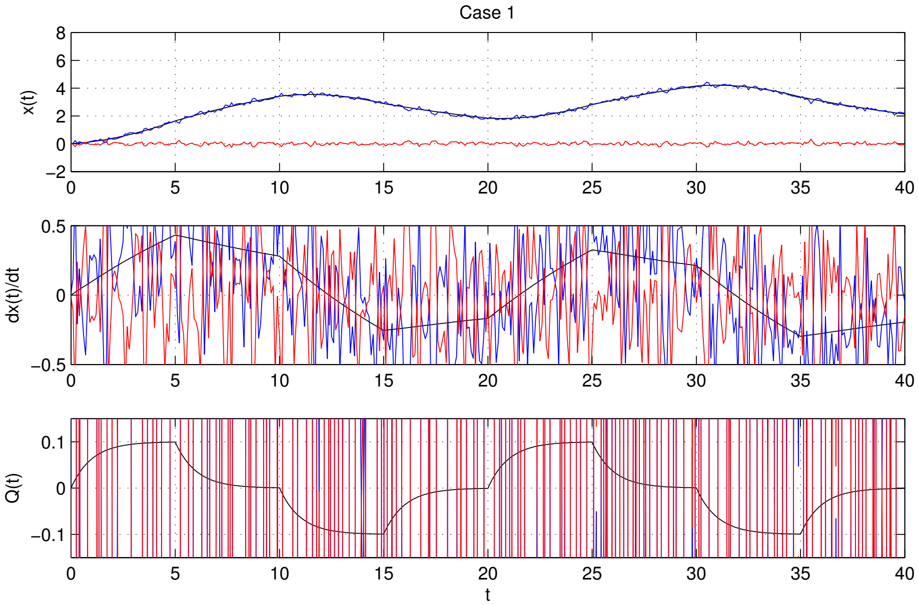
Moderating the eigenvalues and tuning yields a potentially better set of poles:

$$\lambda_{\text{tuned}} = \{-1,\ -2.3,\ -2.8\}$$

Case 2: Shaft torque not measured - Tuned



Case 1: Shaft torque and band velocity not measured



Case 1: Shaft torque and band velocity not measured

Comments

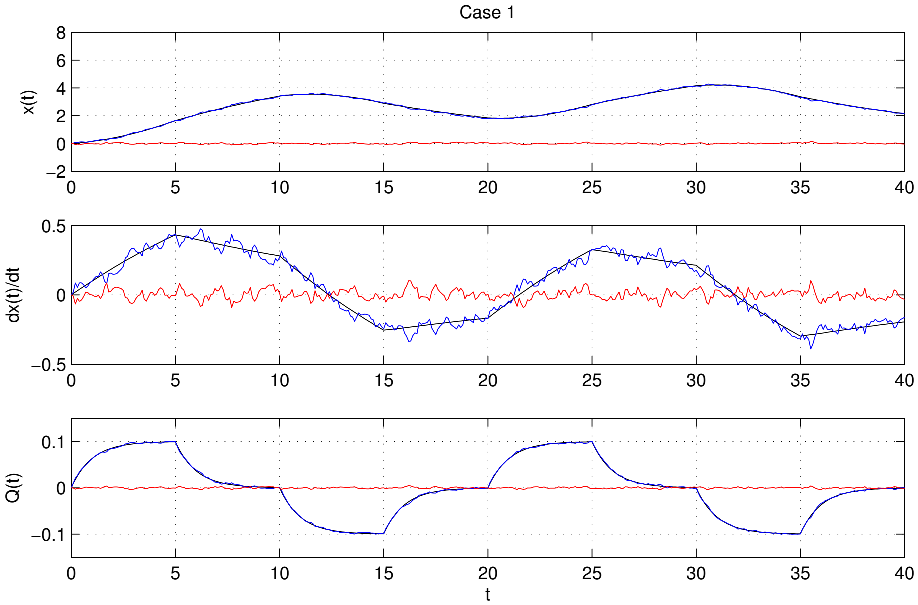
Here the noise amplification makes the observer worthless. A less aggressive tuning is necessary!

Better tuning

Moderating the eigenvalues and tuning yields a potentially better set of poles:

$$\lambda_{\text{tuned}} = \{-1,\ -1.5 + 1i,\ -1.5 - 1i\}$$

Case 1: Shaft torque and band velocity not measured - Tuned



Topic

Certainty Equivalence

If the full state is not available for feedback, one can use the estimate $\hat{\mathbf{x}}$, an approach known as *certainty equivalence*. The estimate is generated from the output $\mathbf{y}$, thus motivating the name *output feedback*.

Output feedback controller

The feedback no longer consists of a simple constant matrix $\mathbf{K}$, but is in fact a dynamic system of its own. This means that care must be exercised due to the added complexity.

In the loop

Plant :

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}$$

Estimator :

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}]$$

Output feedback :

$$\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} + \mathbf{K}_r\mathbf{r}, \quad \mathbf{K}_r = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Regulator system

Estimate update equation:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}[-\mathbf{K}\hat{\mathbf{x}} + \mathbf{K}_r\mathbf{r}] + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}] = (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_r\mathbf{r} + \mathbf{L}\mathbf{y}$$

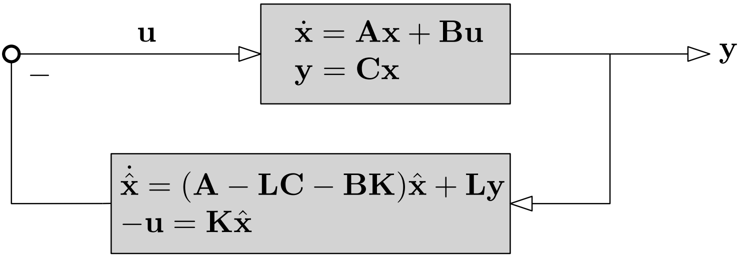
Laplace domain system[a]:

$$\hat{\hat{\mathbf{x}}}(s) = (s\mathbb{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}[\mathbf{B}\mathbf{K}_r\hat{\mathbf{r}}(s) + \mathbf{L}\hat{\mathbf{y}}(s)]$$

$$\Rightarrow \hat{\mathbf{u}}(s) = -\mathbf{K}(s\mathbb{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}[\mathbf{B}\mathbf{K}_r\hat{\mathbf{r}}(s) + \mathbf{L}\hat{\mathbf{y}}(s)] + \mathbf{K}_r\hat{\mathbf{r}}(s)$$

[a] $\hat{(\cdot)}$ denotes the Laplace Transform.

Block diagram: $\mathbf{r} \equiv \mathbf{0}$



Topic

Output feedback

Feedback from estimated states

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}} + \mathbf{r}) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} + \mathbf{B}\mathbf{r} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

Note

$$\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$$

Output feedback

Feedback from estimated states

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A}-\mathbf{BK})\mathbf{x}+\mathbf{BKe}+\mathbf{Br}\\ \mathbf{y} &= \mathbf{Cx}\end{aligned}$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A}-\mathbf{LC})\mathbf{e}$$

Closed loop system

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{bmatrix} &= \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{BK}\\ \mathbf{0} & \mathbf{A}-\mathbf{LC}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ \mathbf{0}\end{bmatrix}\mathbf{r}\\ \mathbf{y} &= \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}\end{aligned}$$

Output feedback

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Transfer function

$$\frac{\mathbf{y}(s)}{\mathbf{r}(s)} = \mathbf{G}(s) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} s\mathbb{I}-\mathbf{A}+\mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbb{I}-\mathbf{A}+\mathbf{LC} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}$$

$$= \mathbf{C}(s\mathbb{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}$$

Output feedback

Closed loop system

$$
\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} &= \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r} \\
\mathbf{y} &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}
\end{aligned}
$$

Separation principle

The input output response:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$$

and error dynamics:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e}$$

are separated!

Separation principle

The separation principle tells us that the overall loop is stable if the state-feedback and observer are individually stable.

Proof

The estimation error is denoted $\mathbf{e} \triangleq \mathbf{x} - \hat{\mathbf{x}}$, giving:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} \\ \dot{\mathbf{e}} &= \underbrace{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}_{\dot{\mathbf{x}}} - \underbrace{[\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{C}\mathbf{e}]}_{\dot{\hat{\mathbf{x}}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}\end{aligned}$$

The full system has this equation:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ 0 & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

and the characteristic polynomial shows that the eigenvalues are independent:

$$\begin{vmatrix} \lambda\mathbb{I} - (\mathbf{A} - \mathbf{B}\mathbf{K}) & -\mathbf{B}\mathbf{K} \\ 0 & \lambda\mathbb{I} - (\mathbf{A} - \mathbf{L}\mathbf{C}) \end{vmatrix} = |\lambda\mathbb{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})||\lambda\mathbb{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})|$$

TTK4115

Lecture 7

Stability

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

1. Internal Stability

    Lyapunov's Method

2. Input-Output stability

Topic

1. Internal Stability

    Lyapunov's Method

2. Input-Output stability

Internal Stability

Internal dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Stability in the sense of Lyapunov

Asymptotic stability : every finite initial state $\mathbf{x}_0$ produces a bounded response, and $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

Marginal stability : every finite initial state $\mathbf{x}_0$ produces a bounded response.

Eigenvalue Conditions

Marginal stability : All eigenvalues of $\mathbf{A}$ have **zero or negative** real parts. No Jordan blocks larger than 1×1 associated with zero eigenvalues[a].

Asymptotic stability : All eigenvalues of $\mathbf{A}$ have **negative real** parts.

Exponential stability : All eigenvalues of $\mathbf{A}$ have **negative real** parts. (Only for LTI systems).

Unstable : If none of the above conditions are met. One or more of the eigenvalues of $\mathbf{A}$ have **positive real** parts, or $\mathbf{A}$ has Jordan blocks larger than 1×1 associated with zero eigenvalues.

---

[a]

$$J = [0] : \text{ OK}, \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : \text{ NOT OK}$$

Exponential stability

Exponential stability

$$\|\mathbf{x}(t)\| = \|e^{\mathbf{A}t}\mathbf{x}_0\| \leq \|e^{\mathbf{A}t}\|\|\mathbf{x}_0\| \leq c^{-\lambda t}\|\mathbf{x}_0\|$$

Exponential stability

LTI system stability always implies *global exponential stability*. This is **not** the case for systems in general, and may be difficult to show.

Topic

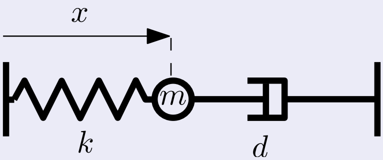
1. Internal Stability

    Lyapunov's Method

2. Input-Output stability

Lyapunov method

Example: Mass spring damper



This is the equation of motion of a mass spring damper system:

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

Energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

Energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\dot{\mathbf{x}}$$

$$\dot{E} = \frac{1}{2}\overbrace{\dot{\mathbf{x}}^T}^{\mathbf{x}^T\mathbf{A}^T}\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\overbrace{\dot{\mathbf{x}}}^{\mathbf{A}\mathbf{x}}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\mathbf{A}\mathbf{x} = \frac{1}{2}\mathbf{x}^T\left(\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A}\right)\mathbf{x}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T \left(\mathbf{A}^T\mathbf{M} + \mathbf{MA}\right)\mathbf{x} = -\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}, \quad \mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix}$$

Lyapunov method

Example: Mass spring damper

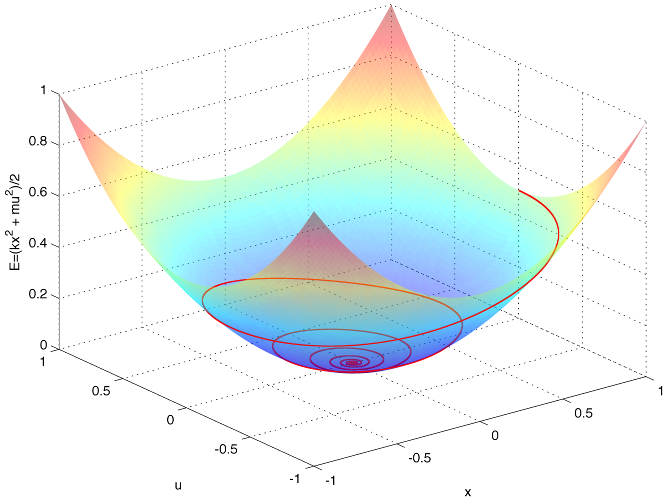
$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T \left(\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A}\right)\mathbf{x} = \overbrace{-\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}}^{-d\dot{x}^2 \leq 0}, \quad \mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix}$$

Lyapunov method



Lyapunov method

Lyapunov function

The Lyapunov function does not have to be the energy:

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{M}\mathbf{x}$$

In fact, it can be any function, usually called V that:

$$V(\mathbf{0}) = 0, \quad V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq 0$$

Lyapunov method

Lyapunov function: Example

Energy function:

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \mathbf{x}$$

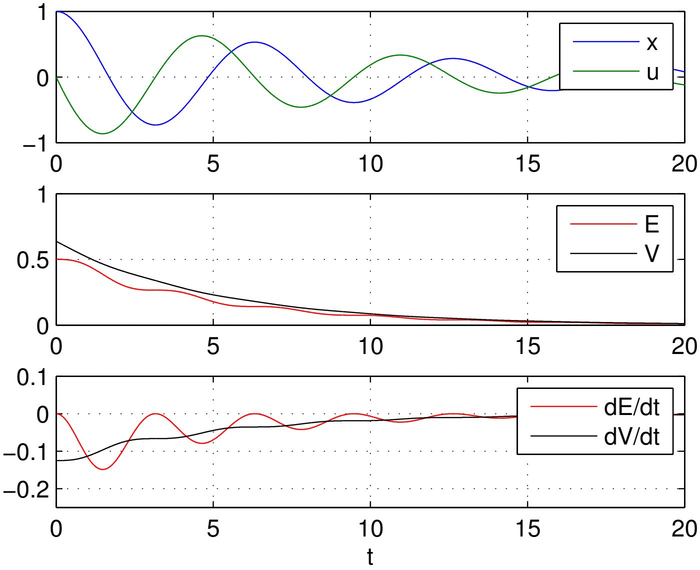
Energy "like" function

$$V(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} \frac{d^2+k(k+m)}{2dk} & \frac{m}{2k} \\ \frac{m}{2k} & \frac{m(k+m)}{2dk} \end{bmatrix} \mathbf{x}$$

Rate of change of "energy"

$$\begin{aligned} \dot{E} &= -d\dot{x}^2 = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix} \mathbf{x} \\ \dot{V} &= -\dot{x}^2 - x^2 = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \end{aligned}$$

Lyapunov method



Lyapunov method

Quadratic form

$$\frac{1}{2}\mathbf{x}^T\left(\mathbf{A}^T\mathbf{M}+\mathbf{M}\mathbf{A}\right)\mathbf{x}=-\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}$$

Remove states:

$$\cancel{\frac{1}{2}\mathbf{x}^T}\left(\mathbf{A}^T\mathbf{M}+\mathbf{M}\mathbf{A}\right)\cancel{\mathbf{x}}=-\cancel{\frac{1}{2}\mathbf{x}^T}\mathbf{N}\cancel{\mathbf{x}}$$

Lyapunov's equation

$$\mathbf{A}^T\mathbf{M}+\mathbf{M}\mathbf{A}=-\mathbf{N}$$

Lyapunov method

Lyapunov's equation

$$\underline{\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}}$$

Stability condition:

If for any given positive definite symmetric matrix **N**, the Lyapunov equation has a unique **symmetric positive definite** solution **M**, the system is **asymptotically stable**.

Positive definite

A matrix **P** is **positive definite** if:

$$\mathbf{x}^T\mathbf{P}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$$

But **semidefinite** if:

$$\mathbf{x}^T\mathbf{P}\mathbf{x} \geq 0$$

Positive definite

Positive definiteness

A symmetric $n \times n$ matrix **P** is positive definite if and only if:

- All its eigenvalues are positive
- Its leading principal minors are all positive
- There exists a nonsingular $n \times n$ matrix **L** so that $\mathbf{P} = \mathbf{L}^{\star}\mathbf{L}$.

Principal minors

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad \underbrace{\left\{ \begin{vmatrix} 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{vmatrix} \right\}}_{\text{Leading principal minors}}$$

Positive definite

Eigenvalue analysis

Symmetric matrices can be factored as:

$$\mathbf{M} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{M}$ and $\mathbf{Q}$ contains the eigenvectors.

Consequences

For an eigenvector $\mathbf{q}$ and corresponding eigenvalue λ:

$$\mathbf{q}^T\mathbf{M}\mathbf{q} = \lambda\mathbf{q}^T\mathbf{q} = \lambda|\mathbf{q}|^2$$

This reveals the following properties:

- $\lambda_{min}(\mathbf{M})|\mathbf{x}|^2 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq \lambda_{max}(\mathbf{M})|\mathbf{x}|^2$
- $\mathbf{M} > 0$ iff $\lambda_{min}(\mathbf{M}) > 0$

Summary

Equivalent conditions for the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

1. The system is asymptotically stable
2. The system is exponentially stable
3. All eigenvalues of **A** have strictly negative real parts.
4. For every symmetric positive definite matrix **Q**, there is a unique solution **P** to the following Lyapunov equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

   and **P** is symmetric and positive definite.

Topic

1. Internal Stability

Lyapunov's Method

2. Input-Output stability

Input-Output stability

LTI solution

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\text{Zero Input Resp.}} + \underbrace{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}_{\text{Zero State Resp.}}$$

Bounded Input

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Bounded Input Bounded Output stability

A system is said to be BIBO stable if every bounded input excites a bounded output for $\mathbf{x}(0) = \mathbf{0}$.

Asymptotic stability implies BIBO stability

Every pole of $\mathbf{G}(s)$ is an eigenvalue of $\mathbf{A}$.

Note

$\mathbf{G}(s)$ only tells us about BIBO stability.

Input-Output stability

SISO system

$$
\begin{aligned}
y(t) &= \int_0^t g(\tau)u(t-\tau)d\tau \\
\Rightarrow |y(t)| &= \left| \int_0^t g(\tau)u(t-\tau)d\tau \right| \\
\Rightarrow |y(t)| &\leq \int_0^t |g(\tau)||u(t-\tau)|d\tau \\
\Rightarrow |y(t)| &\leq \int_0^t |g(\tau)|u_m d\tau
\end{aligned}
$$

Bounded Input

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Input-Output stability

SISO system

$$
\begin{aligned}
|y(t)| &\leq \int_0^\infty |g(\tau)| u_m d\tau \\
|y(t)| &\leq M u_m \leq \infty
\end{aligned}
$$

Bounded Input

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Bounded Output

The output is bounded if:

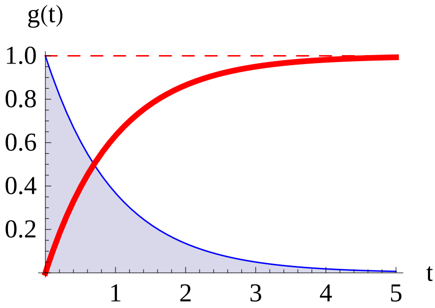
$$\int_0^\infty |g(\tau)| d\tau \leq M \leq \infty$$

SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 1

$$g(s) = \frac{1}{s+1} \quad \Rightarrow g(t) = e^{-t}$$

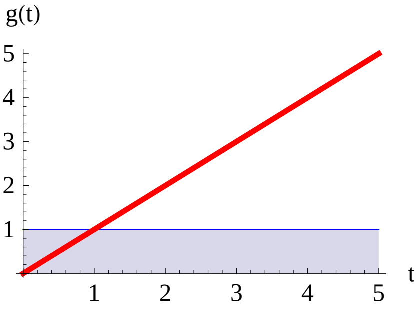


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 2

$$g(s) = \frac{1}{s} \quad \Rightarrow g(t) = 1$$

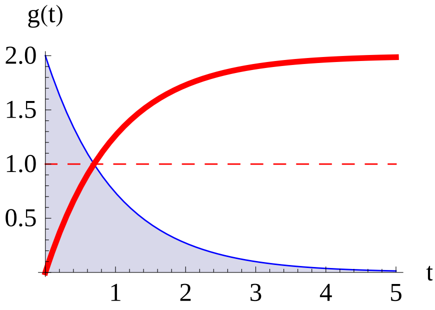


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 3

$$g(s) = \frac{-s+1}{s+1} \quad \Rightarrow g(t) = 2e^{-t} - \text{DiracDelta}[t]$$

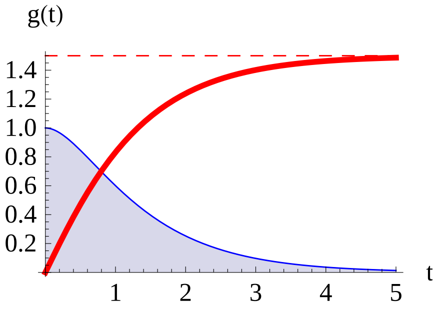


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 4

$$g(s) = \frac{s+3}{(s+1)(s+2)} \quad \Rightarrow g(t) = e^{-2t}\left(-1+2e^t\right)$$

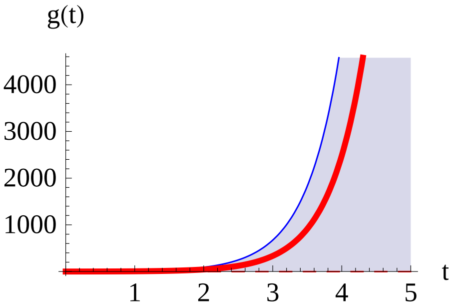


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 5

$$g(s) = \frac{s+3}{(s+1)(s-2)} \quad \Rightarrow g(t) = \frac{1}{3}e^{-t}\left(-2+5e^{3t}\right)$$

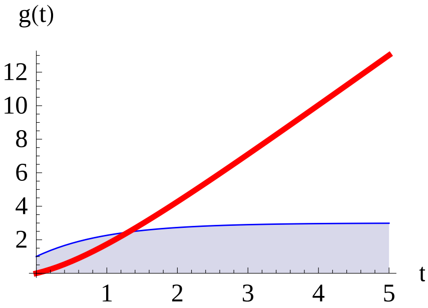


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 6

$$g(s) = \frac{s+3}{s(s+1)} \quad \Rightarrow g(t) = 3 - 2e^{-t}$$



Input-Output stability

If a system is BIBO stable:

- The output excited by $u(t) = A$ approches $\hat{g}(0)A$ as $t \to \infty$.
- The output excited by $u(t) = \sin(\omega t)$ approaches $|\hat{g}(j\omega)| \sin(\omega t + \angle\hat{g}(j\omega))$ as $t \to \infty$.

BIBO & Poles

A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part.

BIBO for MIMO

- A multivariable system is BIBO stable if every element of its impulse response matrix $\mathbf{G}(t)$: $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.
- A multivariable system is BIBO stable if and only of every pole of every element of its transfer matrix $\hat{\mathbf{G}}(s)$: $\hat{g}_{ij}(s)$ has a negative real part.

TTK4115

Lecture 8

Canonical decompositions & Minimal realizations

Morten O. Alver (based on slides by Morten D. Pedersen)

This lecture

1. Canonical decompositions
2. Minimal realizations
3. Next 4 weeks

Topic

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Zero state equivalence

Zero-state equivalence

If the system:

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

has the same transfer function as the system:

$$\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}\}$$

they are *zero-state equivalent*.

Canonical decomposition

- The above matrices may have different dimensions...
- ... but the transfer matrices are the same.
- Some information must be thrown away, that is not related to the transfer function!

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Are one of these states **uncontrollable**?

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \bar{\mathbf{b}} & \bar{\mathbf{A}}\bar{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \bar{b}_c & \bar{a}_c \bar{b}_c \\ 0 & 0 \end{bmatrix}$$

Canonical decompositions

Example:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
\end{aligned}
$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$
\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u(s)
$$

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \frac{1}{s-\bar{a}_c} & \frac{\bar{a}_{12}}{(s-\bar{a}_c)(s-\bar{a}_{nc})} \\ 0 & \frac{1}{s-\bar{a}_{nc}} \end{bmatrix} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u(s)$$

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \frac{\bar{b}_c}{s-\bar{a}_c} \\ 0 \end{bmatrix} u(s) = \frac{\bar{b}_c}{s-\bar{a}_c} \begin{bmatrix} \bar{c}_{11} \\ \bar{c}_{21} \end{bmatrix} u(s)$$

All information about the uncontrollable state is gone!

Canonical decompositions

General case: Controllability

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Notation

c: Controllable	$\bar{c}$: Uncontrollable
o: Observable	$\bar{o}$: Unobservable

Canonical decompositions

General case: Controllability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \bar{\mathbf{B}}_c & \bar{\mathbf{A}}_c\bar{\mathbf{B}}_c & \dots & \bar{\mathbf{A}}_c^{n-1}\bar{\mathbf{B}}_c \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}, \quad \rho(\mathcal{C}) = n_1 < n$$

Transform (theorem 6.6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$..

$$\mathbf{T} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{n_1} & \cdots & \mathbf{q}_n \end{bmatrix}$$

Use all n_1 linearly independent columns of $\mathcal{C}$, then fill in the rest so that $\mathbf{T}$ is invertible.

Canonical decompositions

General case: Controllability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transform (theorem 6.6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$..

$$\mathbf{T} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{n_1} & \cdots & \mathbf{q}_n \end{bmatrix}$$

Use all n_1 linearly independent columns of $\mathcal{C}$, then fill in the rest so that $\mathbf{T}$ is invertible.

Canonical decompositions

General case: Controllability

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Transfer matrix

$$
\begin{aligned}
\mathbf{G}(s) &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \left(s \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} - \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} + \mathbf{D} \\
&= \bar{\mathbf{C}}_c \left(s\mathbb{I} - \bar{\mathbf{A}}_c \right)^{-1} \bar{\mathbf{B}}_c + \mathbf{D}
\end{aligned}
$$

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Transform:

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \implies \mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Transformed system:

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{C}} = \mathbf{CT} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Canonical decompositions

Example:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u \\
y_1 &= \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
\end{aligned}
$$

Are one of these states **unobservable**?

Observability matrix:

$$
\mathcal{O} = \begin{bmatrix} \bar{c}_1 & 0 \\ \bar{c}_1 a_o & 0 \end{bmatrix}
$$

Canonical decompositions

Example:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u \\
y_1 &= \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
\end{aligned}
$$

$\bar{x}_2$ is **unobservable**.

Transfer matrix:

$$
y_1(s) = \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u(s)
$$

Canonical decompositions

Example:

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u \\ y_1 &= \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{aligned}$$

$\bar{x}_2$ is **unobservable**.

Transfer matrix:

$$y_1(s) = \frac{\bar{c}_1 \bar{b}_1}{s - \bar{a}_o} u(s)$$

No information about the unobservable state remains..

Canonical decompositions

General case: Observability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Notation

c: Controllable $\quad \bar{c}$: Uncontrollable
o: Observable $\quad \bar{o}$: Unobservable

General case: Observability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{Du}$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \\ \bar{\mathbf{C}}_o \bar{\mathbf{A}}_o & \mathbf{0} \end{bmatrix}, \quad \rho(\mathcal{O}) = n_2 < n$$

Transform (theorem 6.O6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT}$..

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{n_2} \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

Use all n_2 linearly independent **rows** of $\mathcal{O}$, then fill in the rest so that $\mathbf{T}^{-1}$ is invertible.

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Transform:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \implies \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed system:

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Canonical decompositions

General case: Observability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transfer matrix

$$\mathbf{G}(s) = \bar{\mathbf{C}}_o \left(s\mathbb{I} - \bar{\mathbf{A}}_o \right)^{-1} \bar{\mathbf{B}}_o + \mathbf{D}$$

Canonical decompositions

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Notation

c: Controllable $\quad$ $\bar{c}$: Uncontrollable

o: Observable $\quad$ $\bar{o}$: Unobservable

Kalman decomposition

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transfer matrix

$$\mathbf{y}(s) = \left[\bar{\mathbf{C}}_{co}(s\mathbb{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}\right]\mathbf{u}(s)$$

Kalman Decomposition Theorem (theorem 6.7)

Every state-space equation can be transformed into the form above.

Kalman decomposition

General case

$$\left[\begin{array}{c} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{array}\right] = \left[\begin{array}{cccc} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{array}\right] \left[\begin{array}{c} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{array}\right] + \left[\begin{array}{c} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{array}\right] \mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{cccc} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{array}\right] + \mathbf{D}\mathbf{u}$$

Transfer matrix

$$\mathbf{y}(s) = \left[\bar{\mathbf{C}}_{co}(s\mathbb{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}\right]\mathbf{u}(s)$$

The **same** transfer matrix

$$\mathbf{y}(s) = \left[\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{u}(s)$$

Kalman decomposition

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Zero state equivalent system

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_{co} &= \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u} \end{aligned}$$

Canonical decompositions

Implications

- Transfer matrices do not contain any information about the unobservable and uncontrollable parts of the system.
- This explains why transfer matrices may have lower order than the original system.
- Realizations of transfer functions can only produce the controllable and observable subsystem.
- We should consider the unobservable and uncontrollable subsystems also: are they stable?

Eigenvalues

Characteristic polynomial

$$
\begin{aligned}
\Delta(\lambda) &= \left|\lambda\mathbb{I} - \bar{\mathbf{A}}\right| = \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{co} & \mathbf{0} & -\bar{\mathbf{A}}_{13} & \mathbf{0} \\ -\bar{\mathbf{A}}_{21} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}} & -\bar{\mathbf{A}}_{23} & -\bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\bar{\mathbf{A}}_{43} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{vmatrix} \\
&= \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{co} & \mathbf{0} \\ -\bar{\mathbf{A}}_{21} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}} \end{vmatrix} \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ -\bar{\mathbf{A}}_{43} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{vmatrix} \\
&= \left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{co}\right| \underbrace{\left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}}\right| \left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o}\right| \left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}}\right|}_{\text{Not present in } \mathbf{G}(s)} = 0
\end{aligned}
$$

Note

The transfer matrix does not tell the full story. **Check the eigenvalues of the full state space model.**

Topic

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Minimal realizations

- We have seen that unobservable and uncontrollable subsystems are removed when going **to** the Laplace plane.
- There are infinitely many realizations of a proper rational transfer matrix $\mathbf{G}(s)$.
- By choosing a *minimal* realization, we do not create redundant unobservable and uncontrollable states.
- The resulting state space model will have the same dimensions as:

$$\begin{aligned}\dot{\bar{\mathbf{x}}}_{co} &= \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u}\end{aligned}$$

- which is *minimal*

Minimal realizations

Coprime fractions: SISO case

A state equation $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is a *minimal realization* of a proper rational function $\hat{g}(s)$ if and only if:

- The pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable.
- The pair $\{\mathbf{A}, \mathbf{c}\}$ is observable.
- $n = \dim(\mathbf{A}) = \deg(\hat{g}(s))$
- where $\hat{g}(s) = \frac{N(s)}{D(s)}$, and $N(s)$ and $D(s)$ do not have any common factors.
- I.e.: They are **coprime**, and $\frac{N(s)}{D(s)}$ is a **coprime fraction**.

Minimal realizations

Example

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{s^2 - 1}{4(s^3 - 1)} = \frac{(s-1)(1+s)}{4(s-1)\left(1+s+s^2\right)} = \underbrace{\frac{\cancel{(s-1)}(1+s)}{4\cancel{(s-1)}\left(1+s+s^2\right)}}_{\text{Coprime fraction}}$$

Minimal realizations

Coprimeness and minimal realizations

- If the transfer function is a coprime fraction, we only need to check the dimensions of $\mathbf{A}$ to verify whether the realized system is *minimal*: $\dim(\mathbf{A}) = \deg(\hat{g}(s))$
- This implies that the system is controllable and observable.
- If a fraction is coprime, every root of $D(s)$ is a root of $\hat{g}(s)$.
- The eigenvalues of the minimal realization are the poles of $\hat{g}(s)$.
- All minimal realizations are equivalent, and relate via an equivalence transform $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$.

Topic

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Upcoming subjects

Deterministic systems

The material we have covered in the first 8 weeks, from *Linear system theory and design* by Chi-Tsong Chen has focused *mostly* on deterministic systems.

- Mathematical models and model parameters have been assumed to be exact
- Model inputs $\mathbf{u}(t)$ have been assumed to be exact
- Measurements $\mathbf{y}(t)$ have mostly been assumed to be exact

... except in some cases where we have included model uncertainty, disturbances and measurement noise

Stochastic systems

In real systems, model uncertainty, disturbances and measurement noise are often important enough that they require proper treatment:

- Uncertainties must be modelled according to their statistical properties, and represented as *random processes*.
- Systems affected by stochastic disturbances or random input values are *stochastic systems*
- State estimation in stochastic systems needs to take their random properties into account (e.g. the *Kalman filter*)

These are the subjects of the next 4 weeks, and the material is covered by *Introduction to random signals and applied Kalman filtering* by Brown & Hwang.

Upcoming subjects

Coming subjects

- Characterization of random signals in terms of expectation, variance, autocorrelation, power spectrum, correlation/covariance
- Random processes: systems with random inputs or initial values
- Mean and (co)variances of random state-space systems
- Optimal estimation of random process: Kalman filter in continuous and discrete time
- Noise shaping for Kalman filter systems

TTK4115

Lecture 9

Random processes

Morten. O. Alver (based on material by Morten D. Pedersen/Tor Arne Johansen)

This lecture

1. Basic concepts of Random Processes, B&H 2.1-2.4
2. Autocorrelation functions, B&H 2.5
3. Spectral density functions, B&H 2.7-2.8
4. Common random processes, B&H 2.9-2.14

Topic

1. Basic concepts of Random Processes, B&H 2.1-2.4
2. Autocorrelation functions, B&H 2.5
3. Spectral density functions, B&H 2.7-2.8
4. Common random processes, B&H 2.9-2.14

Random Processes

Terminology

Deterministic process Completely predictable.
Random process Uncertain, not completely predictable[1]. *Can be characterized using statistical properties*.

NB!

random process = stochastic process = random signal = random system = stochastic system

[1] But not necessarily completely unpredictable

Distribution Functions

Probability *Distribution* Function

The probability **distribution** function[2] for a continuous random variable X is defined as

$$F_X(\theta) = P(X \leq \theta)$$

Properties

- $F_X(\theta) \to 0$, as $\theta \to -\infty$
- $F_X(\theta) \to 1$, as $\theta \to \infty$
- $F_X(\theta)$ is a nondecreasing function of θ

[2] No: sannsynlighetsfordelingsfunksjon

Distribution Functions

Probability *Density* Function

The probability **density** function[3] for a continuous random variable X is defined as

$$f_X(\theta) = \frac{d}{d\theta} F_X(\theta)$$

Properties

- $F_X(x) = \int_{-\infty}^{x} f_X(\theta) d\theta$
- $P(\theta_1 \leq X \leq \theta_2) = F_X(\theta_2) - F_X(\theta_1) = \int_{\theta_1}^{\theta_2} f_X(\theta) d\theta$
- $\int_{-\infty}^{\infty} f_X(\theta) d\theta = 1$
- $f_X(\theta) \geq 0$

[3] No: sannsynlighetstetthetsfunksjon

Expected value

Expected value

The expected value[4] for a continuous random variable X, written $E(X)$, $\bar{X}$, m_X or μ, is:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

The expected value of a function $g(X)$:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

[4] No: forventningsverdi

Moments

The k'th moment

The k'th moment of a continuous random variable X is defined by

$$E(X^k) = \int_{-\infty}^{\infty} x^k f_X(x)\, dx$$

The first moment is equal to the mean $E(X) = m_X$.

Central moments

The k'th central moment of a continuous random variable X is:

$$E[(X - m_X)^k] = \int_{-\infty}^{\infty} (x - m_X)^k f_x(x)\, dx$$

Properties:

- The **first** central moment is equal to zero.
- The **second** central moment is called the variance[5], $\mathrm{Var}\, X = E(X^2) - (E(X))^2$
- The square root of the variance is called standard deviation[6], denoted σ_X

[5] No: varians

[6] No: standardavvik

Moments

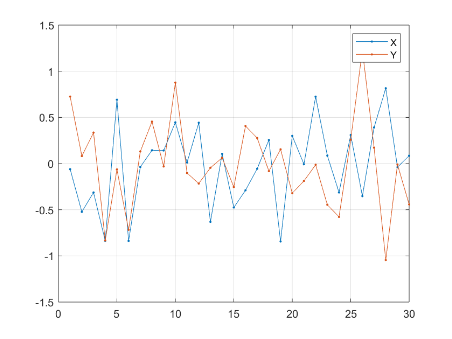
Correlation

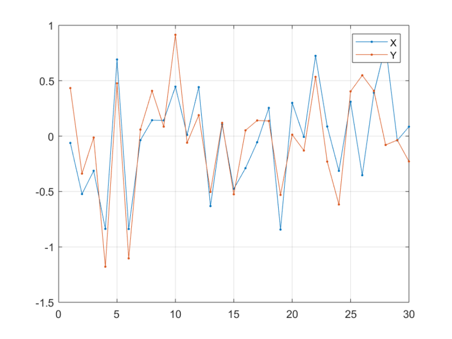
Independence

Independent random variables are said to be uncorrelated

$$E[XY] = E[X] \cdot E[Y]$$

Independent and correlated variables





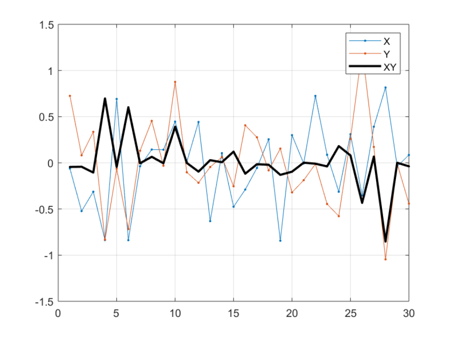
Correlation

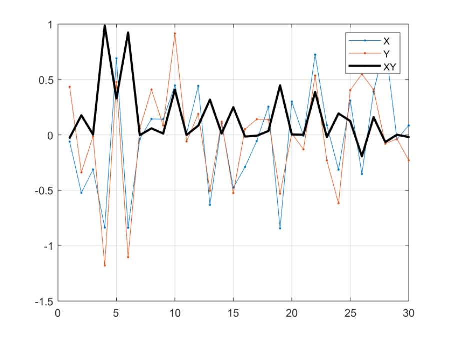
Independence

Independent random variables are said to be uncorrelated

$$E[XY] = E[X] \cdot E[Y]$$

Independent and correlated variables





Correlation

Independence

Independent random variables are said to be uncorrelated

$$E[XY] = E[X] \cdot E[Y]$$

Covariance

The covariance of X and Y is defined by

$$\mathrm{Cov}(X, Y) = E[(X - m_x)(Y - m_y)]$$

The *correlation coefficient* is defined by

$$\varrho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Topic

1. Basic concepts of Random Processes, B&H 2.1-2.4
2. Autocorrelation functions, B&H 2.5
3. Spectral density functions, B&H 2.7-2.8
4. Common random processes, B&H 2.9-2.14

Random Processes

Signals

Deterministic signal *Known* function of time, for example $x(t) = A\sin(\omega_0 t + \varphi)$.

Random signal *Random* function of time. At a given time t, $x(t)$ is the realization (or outcome) of a random variable $X(t)$.

Problem

Can we predict $X(t_2)$ if we know the outcome of $X(t_1)$? Are $X(t_1)$ and $X(t_2)$ correlated, for $t_1 \neq t_2$?

Tools

Random/Stochastic models characterize to what extent signal values at different time instants are correlated. The models assume the form of

- Correlation functions
- Spectral density functions
- Transfer functions
- State space models

Random Processes - Typical examples

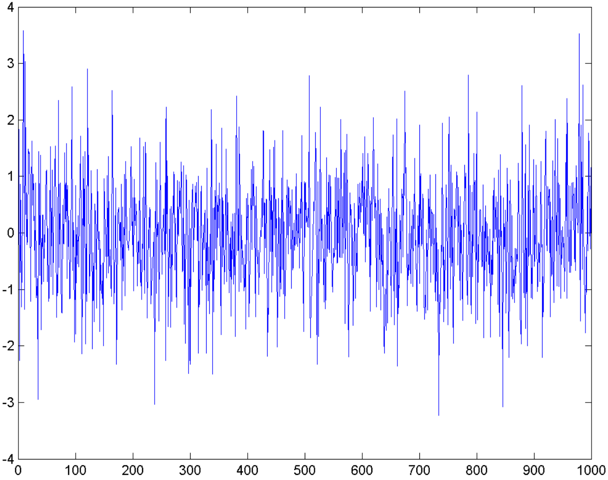


Figure: High-frequency measurement noise

Random Processes - Typical examples

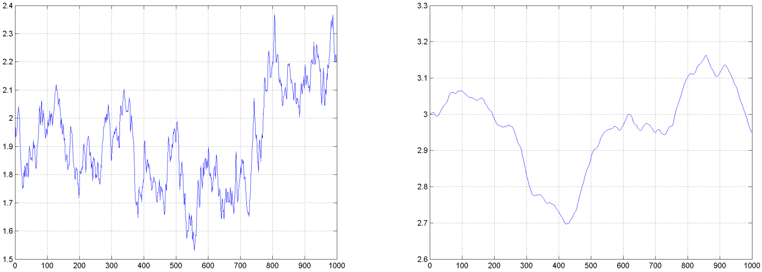


Figure: Process disturbance; e.g wind force, or variations in raw materials.

Random Processes - Typical examples

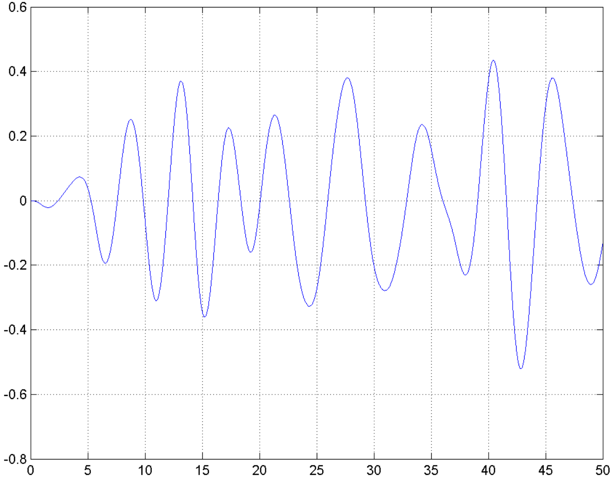


Figure: Irregular waves

Random Processes - Most general characterization

Joint probability density function

$$f_{X(t_1),X(t_2),X(t_3),\ldots}(x_1, x_2, x_3, \ldots)$$

Describes the probability that $X(t_1) = x_1$, $X(t_2) = x_2$, ..., simultaneously.

The Joint probability density function is not practical

Considers a large (possibly infinite) number of time instants $t_1, t_2, t_3, \ldots.$ and is generally too complicated for practical use.

Autocorrelation function

Autocorrelation/Autocovariance

The autocorrelation function for a random process $X(t)$ is defined by

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

The autocovariance function for a random process $X(t)$ is defined by

$$C_X(t_1, t_2) = E[(X(t_1) - m_x(t_1))(X(t_2) - m_x(t_2))]$$

where $m_x(t)$ is the mean of $X(t)$.

These quantities are equal if the process has zero mean.

Stationary random processes

The simultaneous joint probability density function can be shifted in time if the statistical properties of the process does not depend on time. If the process is stationary, then

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

reduces to

$$R_X(\tau) = E[X(t)X(t - \tau)] = E[X(t)X(t + \tau)]$$

Autocorrelation functions

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

Properties of autocorrelation functions for stationary processes

- $R_X(0)$ is the mean square value of $X(t)$.
- $R_X(0)$ is the variance of $X(t)$, if $X(t)$ has zero mean.
- R_X is symmetric; $R_X(\tau) = R_X(-\tau)$
- $|R_X(\tau)| \leq R_X(0)$ for all τ.
- No periodic components in $X(t)$ if and only if $\lim_{\tau\to\infty} R_X(\tau) = 0$. Why?

Realization of a stationary random process

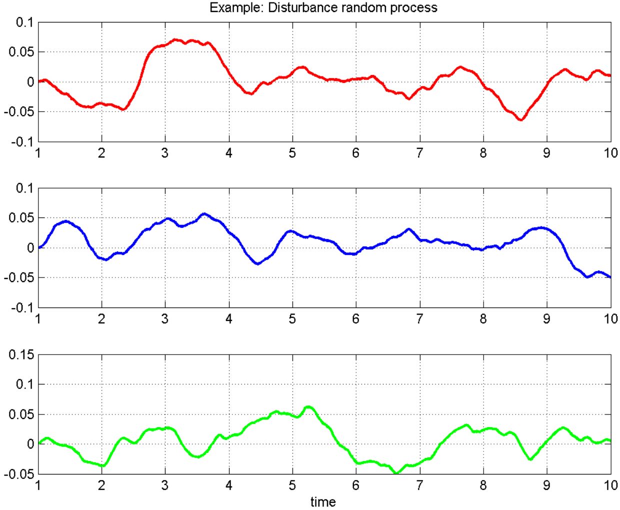
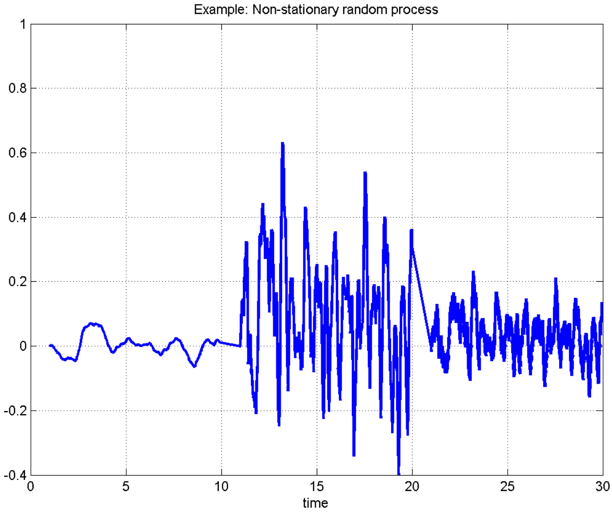


Figure: What happens if you run the same experiment many times? Same process - different realizations.

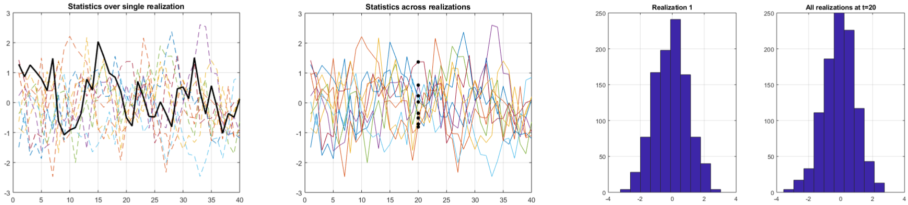
Realization of a non-stationary random process



Ergodicity

A random process is *ergodic* if averaging over (infinite) time can be used to compute expectations. Then, a single (infinite) realization is enough to capture all statistical properties.

$$R_X(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt$$



Topic

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Power spectral density function

Power spectral density

The power spectral density[7] is related to the autocorrelation function by the Fourier transform $\mathcal{F}$:

$$S_X(j\omega) = \mathcal{F}\left[R_X(\tau)\right] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

Autocorrelation function

The autocorrelation function can be found from the power spectral density function by the inverse Fourier transform $\mathcal{F}^{-1}$:

$$R_X(\tau) = \mathcal{F}^{-1}(S_X(j\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(j\omega) e^{j\omega\tau} d\omega$$

Hence, they are equivalent (same information)

[7] No: spektraltetthetsfunksjonen

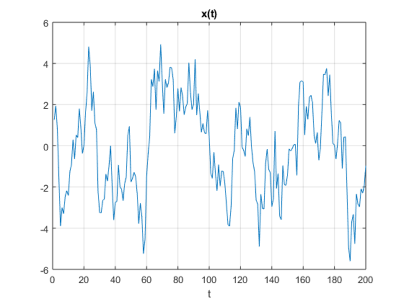
Power spectral density function

Example 2.9: Gauss-Markov process

Consider a random process $X(t)$ whose autocorrelation function is given by:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Example realization ($\sigma = 2$, $\beta = 0.2$)



Power spectral density function

Example 2.9: Gauss-Markov process

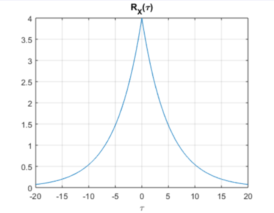
Consider a random process $X(t)$ whose autocorrelation function is given by:

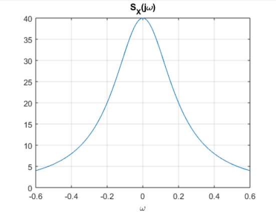
$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Spectral density function

$$S_X(j\omega) = \mathcal{F}[R_X(\tau)] = \frac{2\sigma^2\beta}{\omega^2 + \beta^2} = \frac{\sigma^2}{j\omega + \beta} + \frac{\sigma^2}{-j\omega + \beta}$$

Autocorrelation and spectral density ($\sigma = 2$, $\beta = 0.2$)





Power spectral density function

Example 2.9: Gauss-Markov process

Consider a random process $X(t)$ whose autocorrelation function is given by:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Estimating mean square value from spectral function

$$R_X(0) = E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)d\omega$$

For the example process $X(t)$:

$$E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sigma^2\beta}{\omega^2+\beta^2}d\omega = \frac{\sigma^2\beta}{\pi}\left[\frac{1}{\beta}\tan^{-1}\frac{\omega}{\beta}\right]_{-\infty}^{\infty} = \sigma^2$$

Periodogram and power spectral density

Estimating $S_X(j\omega)$ from data

Suppose $X_T(t)$ is a (finite) truncation of the random signal $X(t)$ at time T.
For a given realization $x_T(t)$ of $X_T(t)$ we define the periodogram

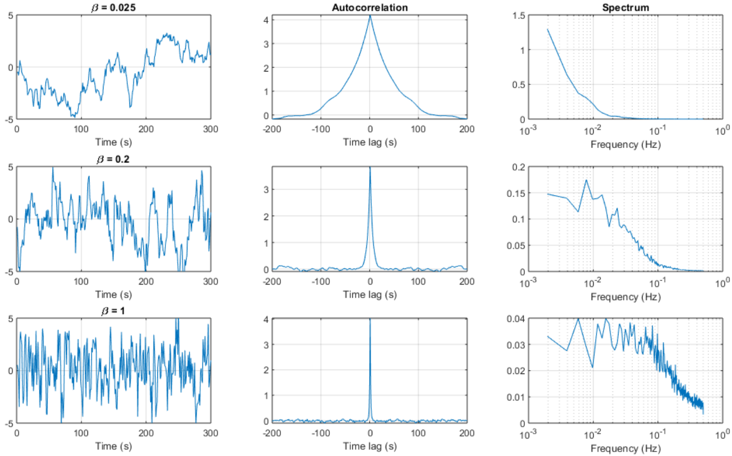
$$\frac{1}{T}\left|\mathcal{F}(x_T(t))\right|^2$$

For an ergodic process, we can show

$$E\left(\frac{1}{T}\left|\mathcal{F}(x_T(t))\right|^2\right) \to S_X(j\omega), \text{ as } T \to \infty$$

Practical estimation of $S_X(j\omega)$ from sampled data therefore often uses a combination of FFT (Fast Fourier Transform), window functions and averaging (see DSP course).

Example



β = 0.025
Autocorrelation
Spectrum
β = 0.2
β = 1
Time (s)
Time lag (s)
Frequency (Hz)

Topic

1. Basic concepts of Random Processes, B&H 2.1-2.4

2. Autocorrelation functions, B&H 2.5

3. Spectral density functions, B&H 2.7-2.8

4. Common random processes, B&H 2.9-2.14

Normal or Gaussian Random Variables

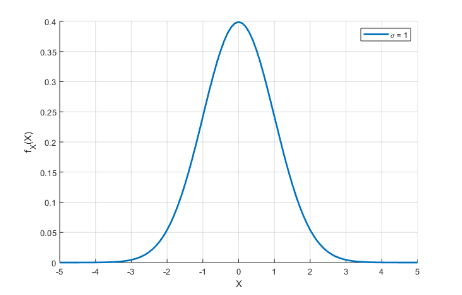
Gaussian distribution

The random variable X (scalar or vector) is called *Gaussian* or *normal* if its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - m_X)^2\right]$$

Gaussian distribution

Probability density functions of Gaussian distributions with mean 0:



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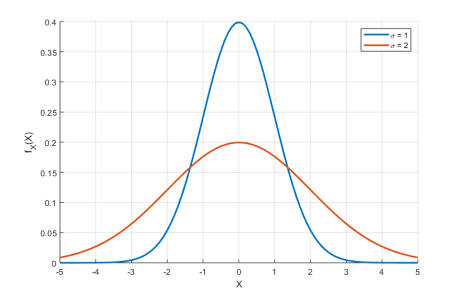

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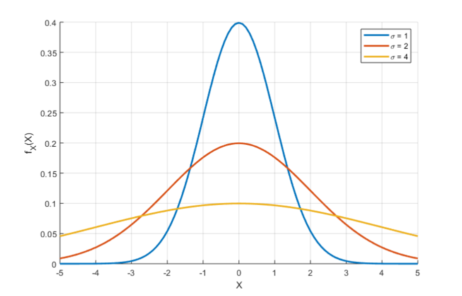


Normal or Gaussian Random Variables

Gaussian distribution

The random variable X (scalar or vector) is called *Gaussian* or *normal* if its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - m_X)^2\right]$$

Notation

The notation $X \sim N(m_x, \sigma^2)$ means that X is normally distributed with mean m_X and variance σ^2.

Multivariable Gaussian

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{C}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m}_X)\boldsymbol{C}^{-1}(\boldsymbol{x} - \boldsymbol{m}_X)\right]$$

The covariance matrix is defined by its elements

$$\boldsymbol{C}_{ij} = E(X_i - m_{X,i})(X_j - m_{X,j})$$

Gaussian distributions

- It is in many cases sufficient to know the means and covariances[8] of a random signal.
- If additional information is sought, a more explicit probability distribution must be found.
- The normal (Gaussian) distribution represents a wide variety of random processes found in nature and technology.
- The multivariable Gaussian probability distribution is completely specified by the mean vector and covariance matrix.

[8] Understood in the generalized sense including variances on the diagonal.

Gauss-Markov process

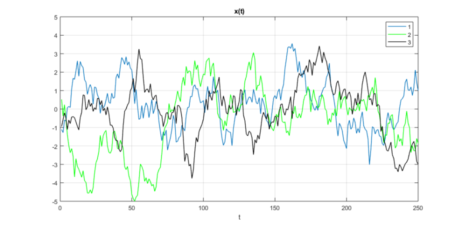
Gauss-Markov processes are Gaussian - i.e. their PDF is a normal distribution.
The process has an exponential autocorrelation function:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Simulating a Gauss-Markov process

To simulate the next value x_{t+1} of a Gauss-Markov process with parameters β and σ and time step dt:

$$x_{t+1} = f x_t + \sqrt{1 - f^2} \cdot N(0, \sigma^2)$$
$$f = e^{-\beta dt}$$



White noise

White noise

White noise[9] is a stationary random process having a constant spectral density function:

$$S_{wn}(j\omega) = A$$

Autocorrelation function

The corresponding autocorrelation function is

$$R_{wn}(\tau) = A\delta(\tau)$$

where $\delta(\cdot)$ is the *Dirac* function

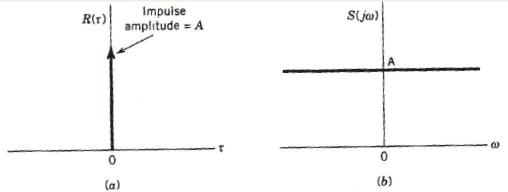
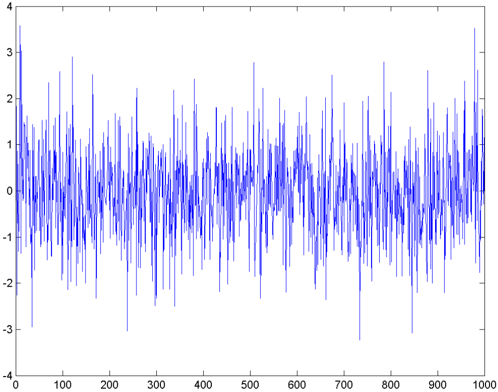


Figure 2.12 White noise. (*a*) Autocorrelation function. (*b*) Spectral density function.

[9] No: hvit støy

Example of white noise realization



White noise

Mathematical abstraction

- $X(t_1)$ and $X(t_2)$ are uncorrelated for all $t_1 \neq t_2$.
- Assuming zero-mean, the variance of the process is $R_X(0) = A\delta(0) \to \infty$. Must be interpreted and handled carefully.
- Assuming zero-mean, the power of the process is infinite (infinite area under the graph of $S_X(j\omega)$. Must be interpreted and handled carefully.

Useful because:

- Mathematically simple
- Many practical processes can be approximated close as white noise, or more generally *filtered* white noise.

Gaussian white noise

The white noise signal $\mathrm{n}(t)$ with variance $\mathcal{C}_n = \delta(0)q$ can be construed as having the probability distribution

$$f_{\mathrm{n}}(n) = \lim_{h\to\infty} \frac{1}{\sqrt{2\pi hq}} \mathrm{Exp}\left[-\frac{(n-m_n)^2}{2qh}\right]$$

Continuous time Gaussian white noise therefore takes on values in the interval $(-\infty, \infty)$ with equal probability.

Band-limited white noise

Definition

$$S_X(j\omega) = \begin{cases} A, & |\omega| \leq 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$

where W is the cut-off frequency.

Can show:

$$R_X(\tau) = 2WA\frac{\sin(2\pi W\tau)}{2\pi W\tau}$$

Properties

- "Almost uncorrelated" $X(t)$ and $X(t+\tau)$ for large $W\tau$.
- Uncorrelated for $\tau = 1/2W, 2/2W, 3/3W$, etc.
- Finite variance and finite power, hence practically more useful than white noise (e.g. in simulations)
- Ideal low-pass filtering of white noise (non-causal filter)

Wiener process ("random walk" or Brownian motion)

Integrated "white" noise

$$X(t) = \int_0^t U(\tau)d\tau$$

where $U(t)$ is normally distributed delta-correlated ("white") process, i.e. $U(t) \sim N(0, \sigma^2)$.

Properties

At time t, the Wiener process has the following probability density function:

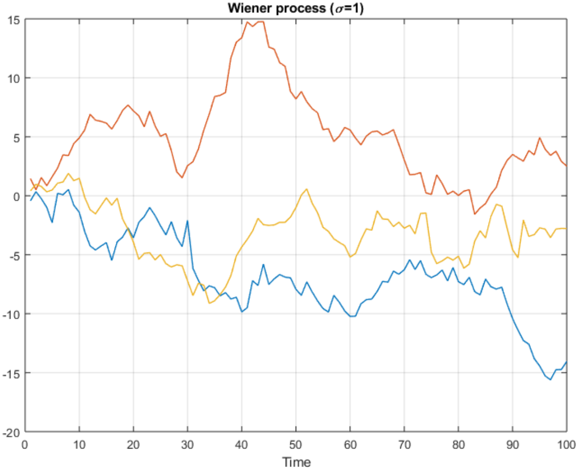
$$f_X(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$$

A normal distribution with $E[X(t)] = 0$ and $E[X^2(t)] = t$.

Applications

- Modelling of slowly random varying disturbances.
- Modelling of slowly random varying parameters and variables.
- Modelling of the accumulated effect of several independent random events, e.g. accumulated win/loss of a series of independent games.

Example of realizations of the same Wiener process



Summary

Key concepts

- Characterization of random processes:
  - Expectation, variance, moments, central moments
  - Probability distribution and probability density functions
  - Joint probability density function
  - Autocorrelation functions
  - Power spectral density functions
- Stationary random process: Statistical properties do not change with time.
- Ergodic random process: Statistical properties can be computed using time averaging from a single (infinite) time series.

Summary

Some common random processes

- Gauss-Markov process (exponential autocorrelation function)
- White noise (no correlations for $\tau \neq 0$)
- Band-limited white noise (finite variance and power)
- Wiener process (integrated white noise)

TTK4115

Lecture 10

Random state space systems

Morten. O. Alver (based on material by Morten D. Pedersen)

Topic

1. Random state space systems

2. Optimal estimation preview

Deterministic state-space model

Assume that $\mathbf{x}_0$ and $\mathbf{u}(t)$ are *known*. Then, the state space model given below is a deterministic process

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

It is in fact straightforward to compute the deterministic solution which is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau, \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Problem

What happens if

- $\mathbf{u}(t)$ is unknown and random? (Denoted $\mathbb{u}(t)$)
- $\mathbf{x}_0$ is unknown and random? (Denoted $\mathbb{x}_0$)

Then it follows that $\mathbf{y}(t)$ and $\mathbf{x}(t)$ must also be random and unknown! Here denoted by the symbols $\mathbb{y}(t)$ and $\mathbb{x}(t)$.

Uncertain state-space model

The state space model given below describes a *random* process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{x}(0) = \mathbb{x}_0$$
$$\mathbb{y} = \mathbf{C}\mathbb{x}$$

The *uncertain* solution follows from

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)$$

---

*Rather than attempting to find out what **will** happen, it is possible to find out what is **likely** to happen.*

What can one *expect* from $\mathbb{x}(t),\ \mathbb{x}_0,\ \mathbb{y}(t),\ \mathbb{u}(t)$?

The expectation operator[1] E can be used to identify a series of important quantities at each time t.

Mean : $m_x(t) = \mathrm{E}[\mathbb{x}(t)]$

Variance : $\mathrm{var}[\mathbb{x}(t)] = \mathrm{E}[(\mathbb{x}(t) - m_x(t))^2]$

Covariance : $\mathrm{cov}[\mathbb{x}_1(t), \mathbb{x}_2(t)] = \mathrm{E}[(\mathbb{x}_1(t) - m_{x_1}(t))(\mathbb{x}_2(t) - m_{x_2}(t))^\mathsf{T}]$

---

[1] A *linear* operator satisfying $\mathrm{E}[\mathbb{x} + c] = \mathrm{E}[\mathbb{x}] + c, \quad \mathrm{E}[\mathbb{x}_1 + \mathbb{x}_2] = \mathrm{E}[\mathbb{x}_1] + \mathrm{E}[\mathbb{x}_2], \quad \mathrm{E}[a\mathbb{x}] = a\mathrm{E}[\mathbb{x}]$.

Uncertain state-space model

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)$$

Mean of random process

Where to *expect* $\mathbb{x}(t)$ is found in the following manner

$$\begin{aligned}\mathbf{m_x}(t) \triangleq \mathrm{E}[\mathbb{x}(t)] = \mathrm{E}\left[e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)\right] \\ = e^{\mathbf{A}t}\mathrm{E}[\mathbb{x}_0] + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[\mathbb{u}(\tau)]\, d\tau, \quad \mathrm{E}[\mathbb{y}(t)] = \mathbf{C}\mathrm{E}[\mathbb{x}(t)]\end{aligned}$$

Model for the mean

Differentiating on both sides produces a simple model for the mean

$$\begin{aligned}\dot{\mathbf{m}}_\mathbf{x}(t) = \mathbf{A}\left[e^{\mathbf{A}t}\mathrm{E}[\mathbb{x}_0] + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{m_u}(\tau)\, d\tau\right] + \mathbf{B}\mathbf{m_u}(t) \\ = \mathbf{A}\mathbf{m_x}(t) + \mathbf{B}\mathbf{m_u}(t), \quad \mathbf{m_y}(t) = \mathbf{C}\mathbf{m_x}(t)\end{aligned}$$

Here, $\mathbf{m_u}(t) \triangleq \mathrm{E}[\mathbb{u}(t)]$ and $\mathbf{m}_{\mathbf{x}_0} \triangleq \mathrm{E}[\mathbb{x}_0]$, whilst $\mathbf{m_y}(t) \triangleq \mathrm{E}[\mathbb{y}(t)]$.

Uncertain state-space model

The state space model given below describes a *random* process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{x}(0) = \mathbb{x}_0$$
$$\mathbb{y} = \mathbf{C}\mathbb{x}$$

Mean of random process

Its **mean** (expected value) follows from using the expectancy operator on the preceding equation

$$\dot{\mathbf{m}}_{\mathbf{x}} = \mathbf{A}\mathbf{m}_{\mathbf{x}} + \mathbf{B}\mathbf{m}_{\mathbf{u}}, \quad \mathbf{m}_{\mathbf{x}}(0) = \mathbf{m}_{\mathbf{x}_0}$$
$$\mathbf{m}_{\mathbf{y}} = \mathbf{C}\mathbf{m}_{\mathbf{x}}$$

This result implies that deterministic models are found in the limit var$[\mathbb{x}] \to \mathbf{0}$.

Covariance & variance of random process

The *variance* provides a measure of the spread of the variable in question whereas the *covariance* measures the relation between two random variables. It is customary to collect this information in a *covariance matrix*.
A vector $\mathbb{x}(t) \in \mathbb{R}^n$ is thus equipped with the covariance matrix

$$\mathcal{C}_{\mathbf{x}}(t) \triangleq \mathrm{E}\begin{bmatrix} (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_n - m_{x_n}) \\ (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_n - m_{x_n}) \\ \dots & \dots & \ddots & \vdots \\ (\mathbb{x}_n - m_{x_n})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_n - m_{x_n})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_n - m_{x_n})(\mathbb{x}_n - m_{x_n}) \end{bmatrix}$$

A compact vectorial representation is given by

$$\mathcal{C}_{\mathbf{x}}(t) = \mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))^{\mathsf{T}}]$$

Note

- Variances are located on the *diagonal*.
- The covariance matrix is symmetric.

Uncertain state-space model about the *mean*

$$\begin{aligned}\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}} &= \mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}) \\ \mathbb{y} - \mathbf{m}_{\mathbf{y}} &= \mathbf{C}(\mathbb{x} - \mathbf{m}_{\mathbf{x}})\end{aligned}$$

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$$\mathcal{C}_{\mathbf{x}}(t) = \mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))^{\mathsf{T}}]$$

Computation

Direct differentiation yields a *covariance update equation*, viz.

$$\begin{aligned}\dot{\mathcal{C}}_{\mathbf{x}} &= \mathrm{E}[(\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}})(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}})^{\mathsf{T}}] \\ &= \mathrm{E}[(\mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}))(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}))^{\mathsf{T}}] \\ &= \mathbf{A}\mathcal{C}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m}_{\mathbf{u}})(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\mathbb{u} - \mathbf{m}_{\mathbf{u}})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}\end{aligned}$$

But, what is the covariance between $\mathbb{x}$ and $\mathbb{u}$?

Uncertain state-space model about the *mean*

$$\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t) = e^{\mathbf{A}t}(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0}) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))\, d\tau$$

Covariance computation

$$\begin{aligned}
&\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} \\
&= \mathrm{E}[e^{\mathbf{A}t}(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}] + \mathrm{E}\left[\left(\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))\, d\tau\right)(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\right] \\
&= e^{\mathbf{A}t}\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}\, d\tau
\end{aligned}$$

Covariance computation

$$
\begin{aligned}
&\mathrm{E}[(\mathbb{x}(t) - \mathbf{m_x}(t))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T} \\
= e^{\mathbf{A}t}&\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T}\, d\tau
\end{aligned}
$$

Causality

The input given in the interval $[0, t)$ cannot affect the initial conditions at $t = 0$ by having impacts *backwards* in time. Arguing from causality, one can assume

$$
\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}] = \mathbf{0}
$$

Autocovariance

The factor $\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]$ from the expression above is by definition the autocovariance of the input signal $\mathbb{u}(t)$:

$$
\mathcal{A}_\mathbf{u}(t, \tau) \triangleq \mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]
$$

White noise

White noise is a theoretical signal that is *completely uncorrelated* to itself over time. The autocovariance of white noise is:

$$\mathcal{A}_n(t,\tau) = \delta(t-\tau)q_n(\tau)$$

where $\delta(t)$ represents Dirac's function and $q_n(t) > 0$.
Knowing the white noise $\mathrm{n}(t)$ at the instant t_1 does not inform us in any way whatsoever about its value at time t_2:

$$\mathcal{A}_n(t,\tau) = \mathrm{E}[(\mathrm{n}(t) - m_n(t))(\mathrm{n}(\tau) - m_n(\tau))] = 0, \quad t \neq \tau$$

At $\tau = t$, the autocovariance reduces to a simple *variance*. This variance is given by

$$\mathcal{A}_n(t,t) = \mathrm{E}[(\mathrm{n}(t) - m_n(t))^2] = \delta(0)q_n(t), \quad t = \tau$$

Remember

White noise is a theoretical construct aimed at simplifying analysis and modeling
- No physical signal has infinite variance.

Autocovariance with $\mathbb{u}$ modeled as **white noise**.

For the random input used in the present process we have:

$$\mathcal{A}_{\mathbf{u}}(t,\tau) \triangleq \mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]$$

Assuming that $\mathbb{u}(t)$ represents white noise permits the simplification

$$\mathcal{A}_{\mathbf{u}}(t,\tau) = \delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau), \quad \mathbf{Q}_{\mathbf{u}} \succ \mathbf{0}$$

Covariance computation

The particular properties of white noise permit significant simplifications to the analysis:

$$\begin{aligned}\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathcal{A}_{\mathbf{u}}(t,\tau)\mathbf{B}^{\mathsf{T}}\,d\tau \\ &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau)\mathbf{B}^{\mathsf{T}}\,d\tau\end{aligned}$$

We use the half-maximum convention on Heaviside's function $\Theta(0) = 1/2$ to arrive at:

$$\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} = \int_0^\infty \Theta(t-\tau)e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau)\mathbf{B}^{\mathsf{T}}\,d\tau = \frac{1}{2}\mathbf{B}\mathbf{Q}_{\mathbf{u}}(t)\mathbf{B}^{\mathsf{T}}$$

Covariance computation

We previously found the following expression for the covariance update equation:

$$\dot{\mathcal{C}}_{\mathbf{x}} = \mathbf{A}\mathcal{C}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{x} - \mathbf{m_x})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{u} - \mathbf{m_u})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}$$

We then found that:

$$\mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{u} - \mathbf{m_u})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} = \frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}$$

Since $\mathbf{Q_u}$ is symmetric, it follows that:

$$\mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{x} - \mathbf{m_x})^{\mathsf{T}}] = \left(\frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}\right)^{T} = \frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}$$

Covariance update equation

$$\dot{\mathcal{C}}_{\mathbf{x}} - \mathbf{A}\mathcal{C}_{\mathbf{x}} - \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} = \mathbf{B}\mathbf{Q_u}(t)\mathbf{B}^{\mathsf{T}}$$

It will be assumed in the following that $\mathbf{Q_u}$ is a constant matrix, although this need not be the case.

Uncertain state-space model

It is in fact possible to say quite a lot about what to *expect* from the random process given below, even though both $\mathbb{u}$ and $\mathbb{x}_0$ are *random*.

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{y} = \mathbf{C}\mathbb{x}$$

Results

The following quantities are assumed *known*.

Means : $\mathrm{E}[\mathbb{u}(t)] = \mathbf{m_u}(t)$ and $\mathrm{E}[\mathbb{x}_0] = \mathbf{m}_{\mathbf{x}_0}$.

Covariances : $\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{u} - \mathbf{m_u})^\mathsf{T}] = \delta(0)\mathbf{Q_u}$ and $\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})^\mathsf{T}] = \mathcal{C}_\mathbf{x}(0)$.

Adopting the assumption that $\mathbb{u}(t)$ is well represented by white noise informs us what to *expect* from the uncertain model. Verify, and note **linearity**, of the following.

The **means** are given by

$$\begin{aligned}\dot{\mathbf{m}}_\mathbf{x} &= \mathbf{A}\mathbf{m}_\mathbf{x} + \mathbf{B}\mathbf{m}_\mathbf{u}, \quad \mathbf{m}_\mathbf{x}(0) = \mathbf{m}_{\mathbf{x}_0}\\ \mathbf{m}_\mathbf{y} &= \mathbf{C}\mathbf{m}_\mathbf{x}\end{aligned}$$

The **covariance matrices** follow from

$$\begin{aligned}\dot{\mathcal{C}}_\mathbf{x} &= \mathbf{A}\mathcal{C}_\mathbf{x} + \mathcal{C}_\mathbf{x}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{Q_u}\mathbf{B}^\mathsf{T}, \quad \mathcal{C}_\mathbf{x}(0) = \mathcal{C}_{\mathbf{x}_0}\\ \mathcal{C}_\mathbf{y} &= \mathbf{C}\mathcal{C}_\mathbf{x}\mathbf{C}^\mathsf{T}\end{aligned}$$

Here $\mathcal{C}_\mathbf{y} \triangleq \mathrm{E}[(\mathbb{y} - \mathbf{m_y})(\mathbb{y} - \mathbf{m_y})^\mathsf{T}] = \mathbf{C}\mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{x} - \mathbf{m_x})^\mathsf{T}]\mathbf{C}^\mathsf{T} = \mathbf{C}\mathcal{C}_\mathbf{x}\mathbf{C}^\mathsf{T}$.

Topic

1. Random state space systems

2. Optimal estimation preview

Physical model

Let a general plant model be given by a random process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w}, \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}$$

The random signals giving rise to the uncertainty are

Noise $\mathbb{v}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}_{\mathbf{v}}$.

Disturbance $\mathbb{w}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{w}}(t,\tau) = \mathrm{E}[\mathbb{w}(\tau)\mathbb{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}_{\mathbf{w}}$

The noise and disturbance are assumed to be *uncorrelated* implying that $\mathcal{A}_{\mathbf{vw}}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{w}(\tau)^{\mathsf{T}}] \equiv \mathbf{0}$.

Luenberger observer

It will be of interest to perform estimation on the random process representing the plant. Let a *Luenberger observer* be given by

$$\dot{\hat{\mathbb{x}}} = \mathbf{A}\hat{\mathbb{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(t)(\mathbb{y} - \mathbf{C}\hat{\mathbb{x}})$$

Note that the estimate is not deterministic since it is perturbed by the random process $\mathbb{y}$. We let $\mathbf{L}(t)$ be undetermined for now.

Dynamics of the estimation error

The random estimation error is defined by $\mathbb{e} = \mathbb{x} - \hat{\mathbb{x}}$. Verify that

$$\dot{\mathbb{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbb{e} + \mathbf{G}\mathbb{w} - \mathbf{L}\mathbb{v}$$

Unbiased estimation

At $t = 0$ the observer is initialized at the *mean* of the true state vector so that $\hat{\mathbf{x}}_0 = \mathrm{E}[\mathbb{x}_0]$. Taking expectations, noting the unbiased noise and disturbance, shows that no mean error is committed

$$\dot{\mathbf{m}}_\mathbf{e} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbf{m}_\mathbf{e}, \quad \mathbf{m}_\mathbf{e}(0) = \mathrm{E}[\mathbb{x}_0] - \hat{\mathbf{x}}_0 = \mathbf{0} \quad \Rightarrow \mathbf{m}_\mathbf{e}(t) = \mathbf{0}$$

This result implies that the estimate is *unbiased*.

Covariance dynamics

The covariance matrix for the estimation error is equipped with the special notation

$$\mathbf{P}(t) \triangleq \mathrm{E}[\mathbb{e}(t)\mathbb{e}(t)^\mathsf{T}]$$

The matrix **P** quantifies the uncertainty in the estimate; low variances (found along the diagonal) imply good estimates!

The Kalman filter gives the gain matrix **K** that reduces the uncertainty in the estimate, tr(**P**), at the fastest rate.

TTK4115

Lecture 11

The Kalman filter part 1

Morten. O. Alver (based on material by Morten D. Pedersen)

This lecture

1. Kalman filtering in continuous time

2. Colored noise

3. Diagonalization of noise/disturbance terms

Topic

1. Kalman filtering in continuous time

2. Colored noise

3. Diagonalization of noise/disturbance terms

Physical model

Let a general plant model be given by a random process

$$\begin{aligned}\dot{\mathbb{x}} &= \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w} \\ \mathbb{y} &= \mathbf{C}\mathbb{x} + \mathbb{v}\end{aligned}$$

The random signals giving rise to the uncertainty are

Noise $\mathbb{v}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathsf{E}[\mathbb{v}(t)\mathbb{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}_{\mathbf{v}}$.

Disturbance $\mathbb{w}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{w}}(t,\tau) = \mathsf{E}[\mathbb{w}(\tau)\mathbb{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}_{\mathbf{w}}$

The noise and disturbance are assumed to be *uncorrelated*, implying that $\mathcal{A}_{\mathbf{vw}}(t,\tau) = \mathsf{E}[\mathbb{v}(t)\mathbb{w}(\tau)^{\mathsf{T}}] \equiv \mathbf{0}$.

Observer

We want to do state estimation on the random process representing the plant. Let a *Luenberger observer* be given by

$$\dot{\hat{\mathbb{x}}} = \mathbf{A}\hat{\mathbb{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(t)(\mathbb{y} - \mathbf{C}\hat{\mathbb{x}})$$

Note that the estimate is not deterministic since it is perturbed by the random process $\mathbb{y}$.

Dynamics of the estimation error

The random estimation error is defined by $\mathbb{e} = \mathbb{x} - \hat{\mathbb{x}}$. The rate of change of $\mathbb{e}$ is:

$$\dot{\mathbb{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbb{e} + \mathbf{G}\mathbb{w} - \mathbf{L}\mathbb{v}$$

Unbiased estimation

At $t = 0$ the observer is initialized at the *mean* of the true state vector so that $\hat{\mathbf{x}}_0 = \mathrm{E}[\mathbb{x}_0]$. Taking expectations, noting the unbiased noise and disturbance, shows that no mean error is committed

$$\dot{\mathbf{m}}_\mathbf{e} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbf{m}_\mathbf{e}, \quad \mathbf{m}_\mathbf{e}(0) = \mathrm{E}[\mathbb{x}_0] - \hat{\mathbf{x}}_0 = \mathbf{0} \quad \Rightarrow \mathbf{m}_\mathbf{e}(t) = \mathbf{0}$$

This result implies that the estimate is *unbiased*.

Covariance dynamics

The covariance matrix for the estimation error is equipped with the special notation

$$\mathbf{P}(t) \triangleq \mathrm{E}[\mathbb{e}(t)\mathbb{e}(t)^\mathsf{T}]$$

The matrix **P** quantifies the uncertainty in the estimate; low variances (found along the diagonal) imply good estimates!

Reminder: covariance of model states

We earlier found that the covariance update equation for this model:

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{x}(0) = \mathbb{x}_0, \quad \mathbb{y} = \mathbf{C}\mathbb{x}$$

Is on this form:

$$\dot{\mathcal{C}}_{\mathbf{x}} = \mathbf{A}\mathcal{C}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m}_{\mathbf{u}})(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\mathbb{u} - \mathbf{m}_{\mathbf{u}})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}$$

Covariance of estimation error

Now we want to find the covariance update equation for:

$$\dot{\mathbb{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbb{e} + \mathbf{G}\mathbb{w} - \mathbf{L}\mathbb{v}$$

The rate of change of $\mathbf{P}(t) \triangleq \mathrm{E}[\mathbb{e}(t)\mathbb{e}(t)^{\mathsf{T}}]$ is:

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC})^{\mathsf{T}} + \mathrm{E}[\mathbb{e}(\mathbb{w}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} - \mathbb{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}})] + \mathrm{E}[\mathbb{e}(\mathbb{w}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} - \mathbb{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}})]^{\mathsf{T}}$$

The covariance matrices $\mathrm{E}[\mathbb{w}\mathbb{e}^{\mathsf{T}}]$ and $\mathrm{E}[\mathbb{v}\mathbb{e}^{\mathsf{T}}]$ must now be found.

Dynamics of the estimation error

Since $\mathbf{L}(t)$ is time-varying, a transition matrix[1] satisfying $\dot{\boldsymbol{\Phi}}(t,\tau) = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\boldsymbol{\Phi}(t,\tau)$ and $\boldsymbol{\Phi}(t,t) = \mathbb{I}$ is used to recover the solution of $\dot{\mathbb{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbb{e} + \mathbf{G}\mathbb{w} - \mathbf{L}\mathbb{v}$, viz.

$$\mathbb{e}(t) = \boldsymbol{\Phi}(t,0)\mathbb{e}_0 + \int_0^t \boldsymbol{\Phi}(t,\tau)(\mathbf{G}\mathbb{w}(\tau) - \mathbf{L}(\tau)\mathbb{v}(\tau))\, d\tau$$

Covariance dynamics

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC})^\mathsf{T} + \mathrm{E}[\mathbb{e}(\mathbb{w}^\mathsf{T}\mathbf{G}^\mathsf{T} - \mathbb{v}^\mathsf{T}\mathbf{L}^\mathsf{T})] + \mathrm{E}[\mathbb{e}(\mathbb{w}^\mathsf{T}\mathbf{G}^\mathsf{T} - \mathbb{v}^\mathsf{T}\mathbf{L}^\mathsf{T})]^\mathsf{T}$$

Computation

Using a similar procedure as before, it follows that

$$\begin{aligned}
&\mathrm{E}[\mathbb{e}(t)(\mathbb{w}(t)^\mathsf{T}\mathbf{G}^\mathsf{T} - \mathbb{v}(t)^\mathsf{T}\mathbf{L}(t)^\mathsf{T})] \\
&\quad = \int_0^\infty \Theta(t-\tau)\boldsymbol{\Phi}(t,\tau)\mathrm{E}[(\mathbf{G}\mathbb{w}(\tau) - \mathbf{L}(\tau)\mathbb{v}(\tau))(\mathbb{w}(t)\mathbf{G}^\mathsf{T} - \mathbb{v}(t)^\mathsf{T}\mathbf{L}(t)^\mathsf{T}))]\, d\tau \\
&\quad = \int_0^\infty \Theta(t-\tau)\boldsymbol{\Phi}(t,\tau)(\mathbf{G}\mathcal{A}_\mathbf{w}(t,\tau)\mathbf{G}^\mathsf{T} + \mathbf{L}\mathcal{A}_\mathbf{v}(t,\tau)\mathbf{L}^\mathsf{T})\, d\tau = \frac{1}{2}\mathbf{G}\mathbf{Q}_\mathbf{w}\mathbf{G}^\mathsf{T} + \frac{1}{2}\mathbf{L}\mathbf{R}_\mathbf{v}\mathbf{L}^\mathsf{T}
\end{aligned}$$

Causality justifies the assumptions $\mathrm{E}[\mathbb{w}\mathbb{e}_0^\mathsf{T}] = \mathbf{0}$ and $\mathrm{E}[\mathbb{v}\mathbb{e}_0^\mathsf{T}] = \mathbf{0}$.

---

[1] Reduces to $\boldsymbol{\Phi}(t,\tau) = e^{(\mathbf{A}-\mathbf{LC})(t-\tau)}$ for constant $\mathbf{L}$.

Covariance dynamics

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC})^{\mathsf{T}} + \mathsf{E}[\mathbb{e}(\mathbb{w}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} - \mathbb{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}})] + \mathsf{E}[\mathbb{e}(\mathbb{w}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} - \mathbb{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}})]^{\mathsf{T}}$$

Covariance dynamics

The following equation describes the covariance dynamics of the random estimate error $\mathbb{e}$, viz.

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC})^{\mathsf{T}} + \mathbf{G}\mathbf{Q}_{\mathbf{w}}\mathbf{G}^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}}$$

Estimation performance

The variance of the i'th estimation error at time t is given by

$$\sigma_i^2(t) = \mathrm{E}[\mathbb{e}_i(t)\mathbb{e}_i(t)] = P_{ii}(t)$$

Let the mean-square errors serve as a measure of the overall estimation performance

$$J_{\text{mse}} = \sum_{i}^{n} \sigma_i^2 = \mathrm{tr}(\mathbf{P}) > 0$$

Kalman Gain

We now ensure that J_{mse} decreases at the fastest possible rate by optimizing with respect to the observer gain $\mathbf{L}(t)$. We need to calculate:

$$\frac{\partial \mathrm{tr}(\dot{\mathbf{P}})}{\partial \mathbf{L}}$$

which is *the derivative of a scalar (the trace) with respect to a matrix*. Then we need to find the value of $\mathbf{L}$ for which the derivative is 0.

Kalman gain

Matrix differentiation rules

These are the differentiation rules that we will use:

$$\frac{\partial}{\partial \mathbf{X}}\text{tr}(\mathbf{XA}) = \mathbf{A}^\mathsf{T}$$

$$\frac{\partial}{\partial \mathbf{X}}\text{tr}(\mathbf{AX}^\mathsf{T}) = \mathbf{A}$$

$$\frac{\partial}{\partial \mathbf{X}}\text{tr}(\mathbf{XBX}^\mathsf{T}) = \mathbf{XB}^\mathsf{T} + \mathbf{XB}$$

Kalman Gain

Using the matrix differentiation rules, we get the result:

$$\frac{\partial \text{tr}(\dot{\mathbf{P}})}{\partial \mathbf{L}} = \frac{\partial}{\partial \mathbf{L}}\text{tr}((\mathbf{A} - \mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{LC})^\mathsf{T} + \mathbf{GQ_wG}^\mathsf{T} + \mathbf{LR_vL}^\mathsf{T}) = -2\mathbf{PC}^\mathsf{T} + 2\mathbf{LR_v}$$

Then we set $\frac{\partial \text{tr}(\dot{\mathbf{P}})}{\partial \mathbf{L}} = \mathbf{0}$ to find the value of $\mathbf{L}(t)$ that gives the optimum:

$$\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^\mathsf{T}\mathbf{R_v}^{-1}$$

This value for the observer feeback gain is called the *Kalman Gain*.

Kalman Gain

$$\frac{\partial \mathrm{tr}(\dot{\mathbf{P}})}{\partial \mathbf{L}} = \frac{\partial}{\partial \mathbf{L}} \mathrm{tr}((\mathbf{A}-\mathbf{LC})\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{LC})^\mathsf{T} + \mathbf{G}\mathbf{Q}_\mathbf{w}\mathbf{G}^\mathsf{T} + \mathbf{L}\mathbf{R}_\mathbf{v}\mathbf{L}^\mathsf{T}) = -2\mathbf{P}\mathbf{C}^\mathsf{T} + 2\mathbf{L}\mathbf{R}_\mathbf{v} = \mathbf{0}$$

Differentiating again with respect to **L** shows that a minimum has indeed been found; the Hessian is positive definite.

$$\frac{1}{2}\frac{\partial^2 \mathrm{tr}(\dot{\mathbf{P}})}{\partial \mathbf{L}^2} = \mathbf{R}_\mathbf{v} \succ \mathbf{0}$$

Optimal covariance dynamics

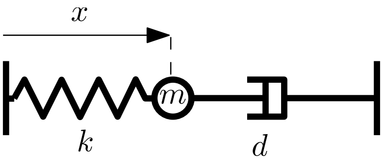
Upon selection of **L**, the covariance follows from

$$\dot{\mathbf{P}} = \mathbf{AP} + \mathbf{PA}^\mathsf{T} + \mathbf{G}\mathbf{Q}_\mathbf{w}\mathbf{G}^\mathsf{T} - \mathbf{P}\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}\mathbf{CP}$$

This is known as the *Matrix Riccati Equation*. The covariance matrix will converge assuming stationarity of the random processes. Hence $\mathbf{P}(t) \to \mathbf{P}_\infty,\; t \to \infty$. An optimal *stationary* observer gain can be obtained by solving

$$\mathbf{AP}_\infty + \mathbf{P}_\infty\mathbf{A}^\mathsf{T} + \mathbf{G}\mathbf{Q}_\mathbf{w}\mathbf{G}^\mathsf{T} - \mathbf{P}_\infty\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}\mathbf{C}\mathbf{P}_\infty = \mathbf{0}, \quad \mathbf{L}_\infty = \mathbf{P}_\infty\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}$$

Example: Mass spring damper



State space equation

We now study a mass spring damper system with a disturbance and measurement noise:

$$\overbrace{\begin{bmatrix} \dot{\mathbb{x}}_1 \\ \dot{\mathbb{x}}_2 \end{bmatrix}}^{\dot{\mathbb{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbb{x}_1 \\ \mathbb{x}_2 \end{bmatrix}}^{\mathbb{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\mathbf{G}} \mathbb{w}$$

$$\mathbb{y} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \mathbb{x}_1 \\ \mathbb{x}_2 \end{bmatrix}}_{\mathbb{x}} + \mathbb{v}$$

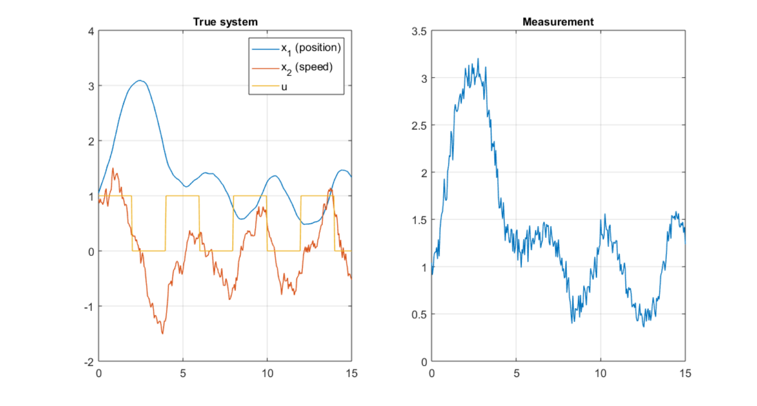
Noise $\mathbb{v}$: $\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R_v}$
Disturbance $\mathbb{w}$: $\mathcal{A}_{\mathbf{w}}(t,\tau) = \mathrm{E}[\mathbb{w}(\tau)\mathbb{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q_w}$
Parameter values : $m = 1$, $k = 0.5$, $d = 0.5$, $\mathbf{R_v} = 0.01$ and $\mathbf{Q_w} = 4$.

Example: Mass spring damper

Simulation of system with disturbance and measurement noise

We impose a regular input signal for 2 seconds at a time.

The figure shows one realization of the true system's state values and the noisy measurement:



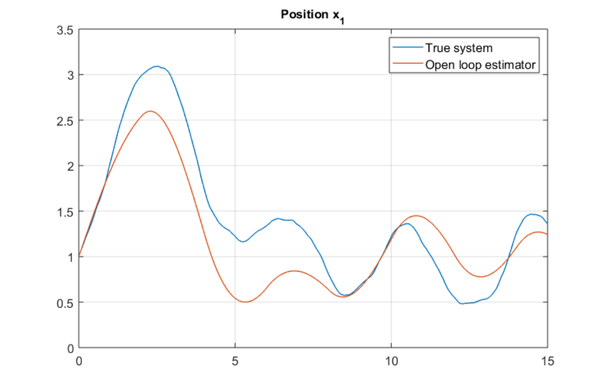
Example: Mass spring damper

Open loop estimator

An open loop estimator of the form

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

performs poorly due to the process disturbance:



Example: Mass spring damper

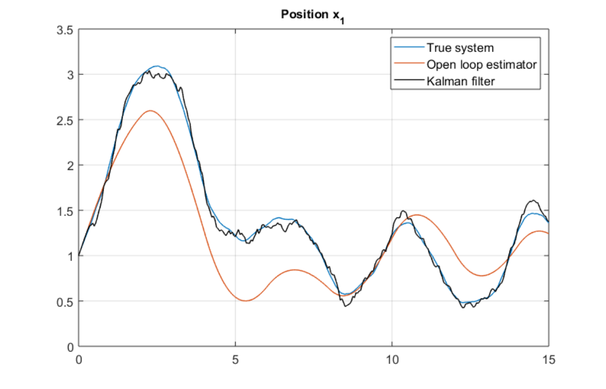
Kalman filter

We then run an observer:

$$\dot{\hat{\mathbb{x}}} = \mathbf{A}\hat{\mathbb{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(t)(\mathbb{y} - \mathbf{C}\hat{\mathbb{x}})$$

using the Kalman gain: $\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}$, with $\mathbf{P}$ calculated according to:

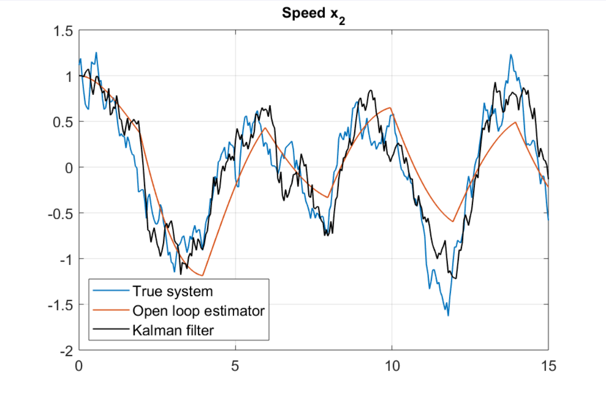
$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{G}\mathbf{Q}_\mathbf{w}\mathbf{G}^\mathsf{T} - \mathbf{P}\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}\mathbf{C}\mathbf{P}$$



Example: Mass spring damper

Kalman filter

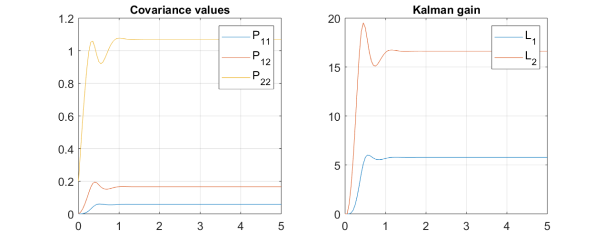
Plotting the true, open loop estimate and Kalman filter estimate of the speed (the unmeasured state), we see again that the observer tracks better:



Example: Mass spring damper

Kalman filter

The $\mathbf{P}(t)$ matrix and Kalman gain $\mathbf{L}(t)$ reach constant values after a short time:



The Kalman gain represents the optimal weighting between the uncertainty of the model and of the measurements:

- More noise (higher value of $\mathbf{R_v}$) gives lower Kalman gain (less emphasis on measurements).
- Stronger disturbance (higher value of $\mathbf{Q_w}$) gives higher Kalman gain (less emphasis on the model).

LQR and Kalman filter duality

Dual dynamics

Recall the *dual* system

$$\dot{\mathbf{z}} = \mathbf{A}^\mathsf{T}\mathbf{z} + \mathbf{C}^\mathsf{T}\mathbf{u}_{\text{dual}}, \quad \mathbf{y}_{\text{dual}} = \mathbf{B}^\mathsf{T}\mathbf{z}$$

LQR

The optimal (output-weighted) feedback gain is well known to be given by

$$\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{Q}_\mathbf{y}\mathbf{C} - \mathbf{S}\mathbf{B}\mathbf{R}_\mathbf{u}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S} = \mathbf{0}, \quad \mathbf{K} = \mathbf{R}_\mathbf{u}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}$$

Duality[2]

The stationary *Kalman gain* can be construed as the optimal *feedback gain* for the *dual system*. Letting $\mathbf{G} = \mathbf{B}$ (often a convenient choice for matched disturbances[3]), compare:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{Q}_\mathbf{w}\mathbf{B}^\mathsf{T} - \mathbf{P}\mathbf{C}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}\mathbf{C}\mathbf{P} = \mathbf{0}, \quad \mathbf{L}^\mathsf{T} = \mathbf{R}_\mathbf{v}^{-1}\mathbf{C}\mathbf{P}$$
$$\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{Q}_\mathbf{y}\mathbf{C} - \mathbf{S}\mathbf{B}\mathbf{R}_\mathbf{u}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S} = \mathbf{0}, \quad \mathbf{K} = \mathbf{R}_\mathbf{u}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}$$

The LQR problem requires a controllable plant, which must hold for the dual plant. This entails that the pair $(\mathbf{A}, \mathbf{C})$ must be observable in order to permit computation of $\mathbf{L}$.

[2] The matlab code is simply `L = (lqr(A',C',B*Qw*B',Rv))'`.

[3] Disturbances that can be canceled directly through control.

LQR **stability**

The closed loop plant is governed by $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ where

$$\mathbf{A}^{\mathsf{T}}\mathbf{S} + \mathbf{SA} + \mathbf{C}^{\mathsf{T}}\mathbf{Q_y C} - \mathbf{SBR_u^{-1}B}^{\mathsf{T}}\mathbf{S} = \mathbf{0}, \quad \mathbf{K} = \mathbf{R_u^{-1}B}^{\mathsf{T}}\mathbf{S}$$

Let a Lyapunov function be given by $V(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{Sx} > 0$. Then we compute $\dot{V}(\mathbf{x})$:

$$\begin{aligned}
\dot{V} = \mathbf{x}^{\mathsf{T}}[\mathbf{S}(\mathbf{A} - \mathbf{BK}) + (\mathbf{A} - \mathbf{BK})^{\mathsf{T}}\mathbf{S}]\mathbf{x} \\
= \mathbf{x}^{\mathsf{T}}[\mathbf{A}^{\mathsf{T}}\mathbf{S} + \mathbf{SA} - 2\mathbf{SBR_u^{-1}B}^{\mathsf{T}}\mathbf{S}]\mathbf{x} \\
= -\mathbf{x}^{\mathsf{T}}[\mathbf{C}^{\mathsf{T}}\mathbf{Q_y C} + \mathbf{SBR_u^{-1}B}^{\mathsf{T}}\mathbf{S}]\mathbf{x} = -[\mathbf{y}^{\mathsf{T}}\mathbf{Q_y y} + \mathbf{u}^{\mathsf{T}}\mathbf{Ru}] < 0
\end{aligned}$$

The Lyapunov function is decreasing, indicating stability of the closed-loop plant (provided it is controllable).

KF **stability**

The estimation error of a stationary Kalman filter is governed by

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

where

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{Q}_{\mathbf{w}}\mathbf{B}^{\mathsf{T}} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C}\mathbf{P} = \mathbf{0}, \quad \mathbf{L} = \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}$$

A deterministic model is used here to avoid the ambiguities of a random Lyapunov function.

Let a Lyapunov function be given by $V(\mathbf{e}) = \mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}\mathbf{e} > 0$. Then we compute $\dot{V}(\mathbf{x})$:

$$\begin{aligned}
\dot{V} = \mathbf{e}^{\mathsf{T}}[\mathbf{P}^{-1}(\mathbf{A} - \mathbf{L}\mathbf{C}) + (\mathbf{A} - \mathbf{L}\mathbf{C})^{\mathsf{T}}\mathbf{P}^{-1}]\mathbf{e} \\
= \mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}[\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{L}\mathbf{C}\mathbf{P} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}]\mathbf{P}^{-1}\mathbf{e} \\
= \mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}[\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A} - 2\mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C}\mathbf{P}]\mathbf{P}^{-1}\mathbf{e} \\
= -\mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}[\mathbf{B}\mathbf{Q}_{\mathbf{w}}\mathbf{B}^{\mathsf{T}} + \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C}\mathbf{P}]\mathbf{P}^{-1}\mathbf{e} < 0
\end{aligned}$$

This indicates stability since the left-hand side is negative, forcing the Lyapunov function to decrease. (A more detailed analysis is found in Hespanha 2009).

Topic

1. Kalman filtering in continuous time

2. Colored noise

3. Diagonalization of noise/disturbance terms

Colored noise

In the development of the continuous-time Kalman Filter, a crucial assumption was that $\mathbb{v}(t)$ and $\mathbb{w}(t)$ were *white* leading to the simplified autocovariances

$$\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathsf{E}[(\mathbb{v}(t) - \mathbf{m}_{\mathbf{v}}(t))(\mathbb{v}(\tau) - \mathbf{m}_{\mathbf{v}}(\tau))^{\mathsf{T}}] = \mathbf{0}, \quad t \neq \tau$$
$$\mathcal{A}_{\mathbf{w}}(t,\tau) = \mathsf{E}[(\mathbb{w}(t) - \mathbf{m}_{\mathbf{w}}(t))(\mathbb{w}(\tau) - \mathbf{m}_{\mathbf{w}}(\tau))^{\mathsf{T}}] = \mathbf{0}, \quad t \neq \tau$$

What if the noise affecting our system is not white, but colored?

Coloration

Colored noise can be obtained by passing white noise through a linear plant. Let $\mathbb{u}(t) \in \mathbb{R}$ be a white noise with zero mean and autocorrelation $\mathcal{R}_u(\tau) = \mathsf{E}[\mathbb{u}(t)\mathbb{u}(t+\tau)]$. Assuming a stable process initialized a long time ago, the output from a linear filter $H(s)$ is

$$\mathbb{y}(t) = \int_{-\infty}^{\infty} H(t-\tau)\mathbb{u}(\tau)\, d\tau$$

where the causal impulse response is given by

$$H(t) = \Theta(t)\left[\mathbf{c}e^{\mathbf{A}t}\mathbf{b} + d\delta(t)\right]$$

We say that H colors $\mathbb{y}$.

Stationarity

If the statistics of a random process remain constant over time, it is said to be *stationary*. For some random variable $\mathbb{r}(t)$, this implies that

$$\mathsf{E}[\mathbb{r}(t)] = m_r, \quad \mathsf{E}[\mathbb{r}(t)\mathbb{r}(t+\tau)] = \mathsf{E}[\mathbb{r}(t)\mathbb{r}(t-\tau)] = \mathcal{A}_r(\tau)$$

If the process is zero-mean, the autocovariance reduces to the *autocorrelation*, viz.

$$m_r = 0 \ \Rightarrow \mathcal{A}_r(\tau) = \mathcal{R}_r(\tau)$$

Autocorrelation of $\mathbb{y}$ from $\mathbb{u}$

The autocorrelation of $\mathbb{y}(t)$ can be related to the autocorrelation of $\mathbb{u}(t)$:

$$\mathcal{R}_{uy}(\tau) = \mathsf{E}[\mathbb{y}(t)\mathbb{u}(t+\tau)] = \int_{-\infty}^{\infty} H(\alpha)\underbrace{\mathsf{E}[\mathbb{u}(t-\alpha)\mathbb{u}(t+\tau)]}_{\mathcal{R}_u(\tau+\alpha)}\, d\alpha = H(-\tau) * \mathcal{R}_u(\tau)$$

and

$$\mathcal{R}_{y}(\tau) = \mathsf{E}[\mathbb{y}(t)\mathbb{y}(t-\tau)] = \int_{-\infty}^{\infty} H(\alpha)\underbrace{\mathsf{E}[\mathbb{u}(t-\alpha)\mathbb{y}(t-\tau)]}_{\mathcal{R}_{uy}(\tau-\alpha)}\, d\alpha = H(\tau) * \mathcal{R}_{uy}(\tau)$$

Together, it follows that

$$\mathcal{R}_y(\tau) = H(\tau) * \mathcal{R}_{uy}(\tau) = H(\tau) * [H(-\tau) * \mathcal{R}_u(\tau)] = [H(\tau) * H(-\tau)] * \mathcal{R}_u(\tau)$$

Summary

Assuming stationarity, the filter H produces the autocorrelation $\mathcal{R}_y(\tau)$ from $\mathcal{R}_u(\tau)$ by blending past and present values through convolution

$$\mathcal{R}_y(\tau) = \rho(\tau) * \mathcal{R}_u(\tau), \quad \rho(\tau) \triangleq \int_{-\infty}^{\infty} H(\tau-\alpha)H(-\alpha)\, d\alpha = \int_{-\infty}^{\infty} H(\tau+\beta)H(\beta)\, d\beta$$

Note that $\rho(-\tau) = \rho(\tau)$ since $\rho(\tau) = H(\tau) * H(-\tau)$.

The spectrum of noise

Colors are defined by their spectral power content. Stationary random signals are no different. The **Fourier transform** is the appropriate tool for this analysis.

Fourier transformation

The Fourier transform is defined by

$$\hat{f}(\jmath\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\jmath\omega t}\,dt, \quad \left(f(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt\right)$$

The Laplace transform for a plant is shown to the right. If the plant is BIBO stable[4] and $f(t) = 0$ for $t < 0$ the Fourier transform can be obtained by evaluating the Laplace transform along the imaginary axis:

$$\hat{f}(\jmath\omega) = f(s)|_{s=\jmath\omega}$$

Inverse Fourier transformation

An *inverse* Fourier-transform follows from

$$f(t) = \mathcal{F}^{-1}\{\hat{f}(\jmath\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\jmath\omega)e^{\jmath\omega t}\,d\omega$$

Given BIBO stability, we have the *equality* $\mathcal{F}^{-1}\{\hat{f}(\jmath\omega)\} = \mathcal{L}^{-1}\{f(s)\}$.

[4] No poles in the RHP.

The **Wiener–Khinchin–Einstein**[5] theorem

The following result applies to stationary processes. Since

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \hat{f}_1(j\omega)\hat{f}_2(j\omega)$$

the following result holds for the autocorrelation of $\mathbb{y}$ obtained by filtering $\mathbb{y}$ through H, viz.

$$\mathcal{F}\{\mathcal{R}_y(\tau)\} = \mathcal{F}\{\rho(\tau) * \mathcal{R}_u(\tau)\} = \hat{\rho}(j\omega)\mathcal{F}\{\mathcal{R}_u(\tau)\}$$

Furthermore, since $\mathcal{F}\{f(-t)\} = \hat{f}(-\jmath\omega)$, one has

$$\hat{\rho}(j\omega) = \mathcal{F}\{H(\tau) * H(-\tau)\} = \hat{H}(\jmath\omega)\hat{H}(-\jmath\omega)$$

Power spectral density

The **spectral density** of a zero-mean stationary random process $\mathbb{r}(t)$ can be *defined* as

$$\mathcal{S}_r(\omega) = \mathcal{F}\{\mathsf{E}[\mathbb{r}(t)\mathbb{r}(t+\tau)]\} = \mathcal{F}\{\mathcal{R}_r(\tau)\}$$

Therefore we have:

$$\mathcal{S}_y(\omega) = \hat{H}(\jmath\omega)\hat{H}(-\jmath\omega)\mathcal{S}_u(\omega) = |\hat{H}(\jmath\omega)|^2\mathcal{S}_u(\omega)$$

[5] Einstein was first in 1914!

Motivating the notion of **white**

White light is special in being made up of a (somewhat) uniform distribution of spectral intensities. Its power spectral density can be seen as *constant*.

$$S_w(\omega) = q$$

The inverse Fourier transform gives the autocorrelation of white light as

$$\mathcal{R}_w(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_w(\omega) e^{\jmath\omega\tau}\, d\omega = \frac{q}{2\pi}\int_{-\infty}^{\infty} e^{\jmath\omega\tau}\, d\omega = q\delta(\tau)$$

White noise $\rightarrow \hat{H}(\jmath\omega) \rightarrow$ **Colored noise**

Passing white light through a filter alters its spectral content and gives it color. Let $\mathbb{u}(t)$ be white noise with zero mean $m_u = 0$, autocorrelation $\mathcal{R}_u(\tau) = q\delta(\tau)$ and spectral density $\mathcal{S}_u(\omega)$. Let $\mathbb{u}(t)$ be filtered by H.
The output $\mathbb{y}(t)$ then has a *colored* spectrum:

$$\mathcal{S}_y(\omega) = |\hat{H}(\jmath\omega)|^2 \mathcal{S}_u(\omega) = |\hat{H}(\jmath\omega)|^2 q$$

This operation is referred to as **spectral factorization**.

Filters & Colors

Color	$H(s)$	$\mathcal{S}(\omega)$	$\mathcal{R}(\tau)$
White	1	1	$\delta(\tau)$
Brown	$\lim\limits_{\epsilon\to 0} \frac{1}{s+\epsilon}$	$\lim\limits_{\epsilon\to 0} \frac{1}{\omega^2+\epsilon^2}$	$\lim\limits_{\epsilon\to 0} \frac{e^{-\epsilon\lvert\tau\rvert}}{2\epsilon}$
Violet	$\lim\limits_{\epsilon\to 0} \frac{s}{\epsilon s+1}$	$\lim\limits_{\epsilon\to 0} \frac{\omega^2}{\omega^2\epsilon^2+1}$	$\lim\limits_{\epsilon\to 0} \frac{2\epsilon\delta(\tau)-\Theta(\lvert\tau\rvert)e^{-\frac{\lvert\tau\rvert}{\epsilon}}}{2\epsilon^3}$
Band-limited	$-$	$\Theta(\omega+\omega_c)-\Theta(\omega-\omega_c)$	$\frac{\sin(\tau\omega_c)}{\pi\tau}$
Low-passed	$\frac{1}{s/\omega_c+1}$	$\frac{1}{(\omega/\omega_c)^2+1}$	$\frac{\omega_c}{2}e^{-\omega_c\lvert\tau\rvert}$

Key idea

By passing white noise through one (or more) linear filters, an assortment of colors can be simulated. This technique permits extension of the Kalman filter to cases where the input is not **white** but **colored**.

Model augmentation

The general plant model used by the Kalman filter was given by

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w}, \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}$$

Here, $\mathbb{v}$ and $\mathbb{w}$ were assumed white. If $\mathbb{w}$ is colored, an augmented state-space can be employed. The notation $\mathbb{v}$ and $\mathbb{w}$ is reserved for white processes. So let the colored disturbance be denoted $\mathbb{d}$, leading to $\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{d}$.

Suppose that an element of $\mathbb{d}(t)$ is observed to have the spectrum $\mathcal{S}_d(\omega) = q_w|\hat{H}_d(j\omega)|^2$. The *shaping filter* can be *realized* as

$$\hat{H}_d(\jmath\omega) = H_d(s)|_{s=\jmath\omega}, \quad H_d(s) = \mathbf{c}_d(s\mathbb{I} - \mathbf{A}_d)^{-1}\mathbf{b}_d + d_d$$

In the time-domain, the colored noise is therefore simulated by

$$\dot{\mathbb{x}}_d(t) = \mathbf{A}_d\mathbb{x}_d(t) + \mathbf{b}_d\mathbb{w}(t), \quad \mathbb{d}(t) = \mathbf{c}_d\mathbb{x}_d(t) + d_d\mathbb{w}(t)$$

where $\mathbb{w}$ is a zero-mean white process with variance $\delta(0)q_w$.

Model augmentation

Let the disturbance be modeled (colored) by

$$\dot{\mathbb{x}}_d = \mathbf{A}_d \mathbb{x}_d + \mathbf{B}_d \mathbb{w}, \quad \mathbb{d} = \mathbf{C}_d \mathbb{x}_d + \mathbf{D}_d \mathbb{w}$$

where $\mathbb{w}$ is white with zero mean and variance given by $\mathrm{E}[\mathbb{w}\mathbb{w}^\mathsf{T}] = \delta(0)\mathbf{Q_w}$. Then, an augmented state-space model becomes

$$\begin{bmatrix} \dot{\mathbb{x}} \\ \dot{\mathbb{x}}_d \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{GC}_d \\ \mathbf{0} & \mathbf{A}_d \end{bmatrix} \begin{bmatrix} \mathbb{x} \\ \mathbb{x}_d \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbb{u} + \begin{bmatrix} \mathbf{GD}_d \\ \mathbf{B}_d \end{bmatrix} \mathbb{w}$$

Note that augmentation need only be done for non-white disturbance processes.

An example

A wind turbine with three blades will experience significant disturbances at the so-called 3P-frequency due to tower-passing. A model of the rotor's velocity around a stable operating point can be furnished by

$$\tau \dot{\mathbb{x}}_1 + \mathbb{x}_1 = \mathbb{Q}$$

The aerodynamic torque driving the plant $\mathbb{Q}$ is here modeled as a random disturbance given by the sum of a slowly-varying component and a periodic disturbance

$$\mathbb{Q} = \mathbb{Q}_0 + \mathbb{Q}_{3P}$$

We wish to estimate $\mathbb{Q}_0$ given the measurement $\mathbb{y} = \mathbb{x} + \mathbb{v}$ where $\mathbb{v}$ is a zero-mean white process of intensity r_v.

The two torque disturbances affect the rotor in the same way, so it would at first glance appear difficult to tease them apart. But, by assuming that they are *shaped* differently[6], progress can be made.

[6] Implying different spectra.

Spectral densities

The slowly varying torque component is well modeled by a random walk $\mathbb{Q}_0 = \mathbb{x}_2$. The random walk can be simulated by

$$\dot{\mathbb{x}}_2 = k_0 \mathbb{w}_1, \quad \mathcal{S}_0(\omega) = \frac{k_0^2}{\omega^2}$$

where $\mathbb{w}_1$ is unbiased white noise of unit intensity (the scaling is done with k_0).

Since the $\mathbb{Q}_{3\mathrm{P}}$ component occurs around the frequency $\omega_{3\mathrm{P}} = 3\Omega$ where Ω represents the rotor's angular velocity, a natural spectrum is furnished by

$$\mathcal{S}_{3\mathrm{P}}(\omega) = \frac{k_{3\mathrm{P}}^2 \omega^2}{\left(\omega^2 - \omega_{3\mathrm{P}}^2\right)^2} = \frac{k_{3\mathrm{P}}^2 \jmath\omega}{(\jmath\omega)^2 + \omega_{3\mathrm{P}}^2} \frac{-k_{3\mathrm{P}}^2 \jmath\omega}{(-\jmath\omega)^2 + \omega_{3\mathrm{P}}^2} = \hat{H}(\jmath\omega)\hat{H}(-\jmath\omega)$$

The appropriate shaping filter is clearly

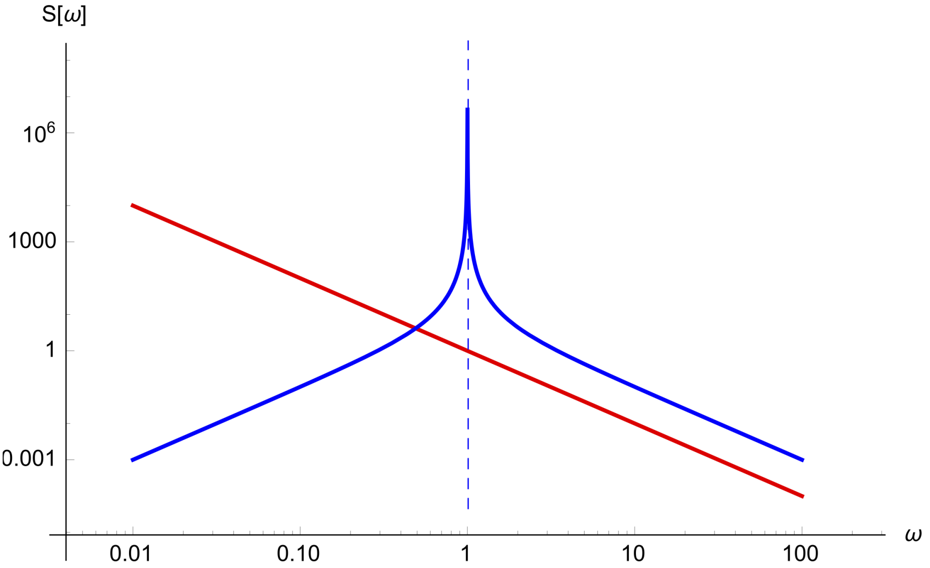
$$H(s) = \frac{k_{3\mathrm{P}}\, s}{s^2 + \omega_{3\mathrm{P}}^2}$$

Hence the disturbance model

$$\begin{bmatrix} \dot{\mathbb{x}}_3 \\ \dot{\mathbb{x}}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_{3\mathrm{P}}^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{x}_3 \\ \mathbb{x}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ k_{3\mathrm{P}} \end{bmatrix} \mathbb{w}_2, \quad \mathbb{Q}_{3\mathrm{P}} = \mathbb{x}_4$$

where $\mathbb{w}_2$ is unbiased white noise of unit intensity (the scaling is done with $k_{3\mathrm{P}}$).

Spectra of brown and monochrome noise



Augmented model

The augmented random process becomes

$$\begin{bmatrix} \dot{\mathbb{x}}_1 \\ \dot{\mathbb{x}}_2 \\ \dot{\mathbb{x}}_3 \\ \dot{\mathbb{x}}_4 \end{bmatrix} = \overbrace{\begin{bmatrix} -\tau^{-1} & \tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{3P}^2 & 0 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbb{x}_1 \\ \mathbb{x}_2 \\ \mathbb{x}_3 \\ \mathbb{x}_4 \end{bmatrix}}^{\mathbb{x}} + \overbrace{\begin{bmatrix} 0 & 0 \\ k_0 & 0 \\ 0 & 0 \\ 0 & k_{3P} \end{bmatrix}}^{\mathbf{G}} \overbrace{\begin{bmatrix} \mathbb{w}_1 \\ \mathbb{w}_2 \end{bmatrix}}^{\mathbb{w}}$$

$$\mathbb{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbb{x}_1 \\ \mathbb{x}_2 \\ \mathbb{x}_3 \\ \mathbb{x}_4 \end{bmatrix} + \mathbb{v}, \quad \mathbf{Q_w} = \mathbb{I}$$

The pair $(\mathbf{A}, \mathbf{C})$ is required to be observable for the Kalman filter to apply. This is indeed the case.

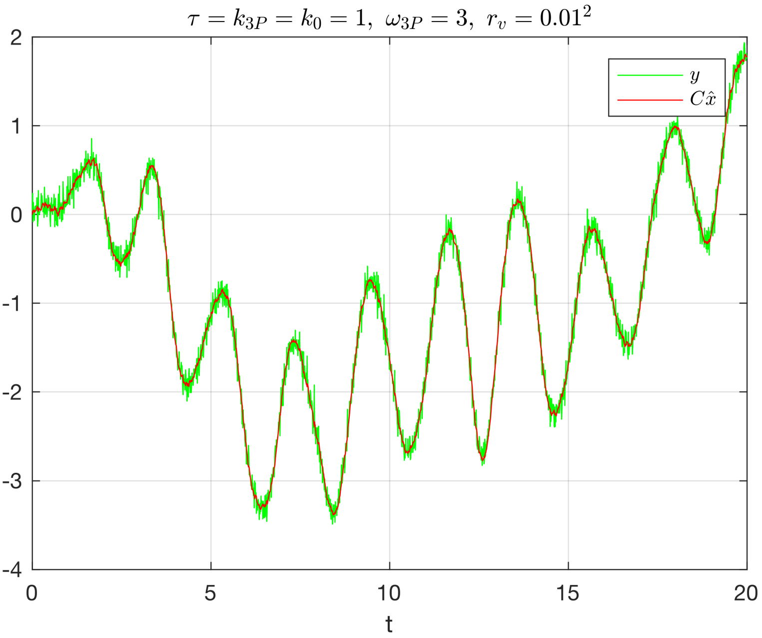
In order to arrive at the mean torque-component, the following estimate is used

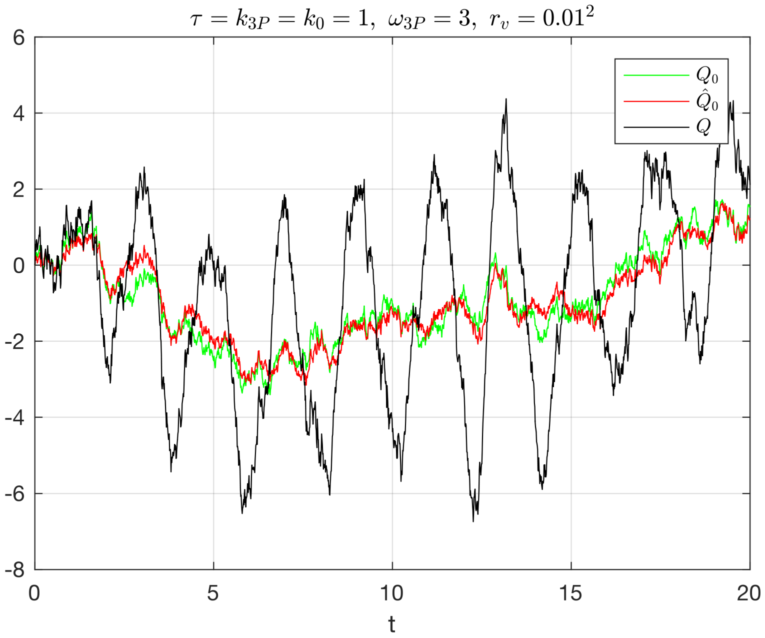
$$\hat{\mathbb{Q}}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbb{x}}_1 \\ \hat{\mathbb{x}}_2 \\ \hat{\mathbb{x}}_3 \\ \hat{\mathbb{x}}_4 \end{bmatrix}$$

Optimal estimation

An optimal stationary estimator for the states of the uncertain plant can now be obtained from the solution of

$$\mathbf{AP} + \mathbf{PA}^\mathsf{T} + \mathbf{GG}^\mathsf{T} - \frac{1}{r_v}\mathbf{PC}^\mathsf{T}\mathbf{CP} = \mathbf{0}, \quad \mathbf{L} = \frac{1}{r_v}\mathbf{PC}^\mathsf{T}$$


$\tau = k_{3P} = k_0 = 1,\ \omega_{3P} = 3,\ r_v = 0.01^2$
y
$C\hat{x}$
2
1
0
-1
-2
-3
-4
0
5
10
15
20
t


$\tau = k_{3P} = k_0 = 1,\ \omega_{3P} = 3,\ r_v = 0.01^2$
Q_0
$\hat{Q}_0$
Q
6
4
2
0
-2
-4
-6
-8
0
5
10
15
20
t

Topic

1. Kalman filtering in continuous time

2. Colored noise

3. Diagonalization of noise/disturbance terms

Physical model

The plant model is described by the random process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w}, \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}$$

where $\mathcal{C}_{\mathbf{w}} = \delta(0)\mathbf{Q}_{\mathbf{w}}$ and $\mathcal{C}_{\mathbf{v}} = \delta(0)\mathbf{R}_{\mathbf{v}}$. If there are off-diagonal elements in $\mathbf{Q}_{\mathbf{w}}$ and $\mathbf{R}_{\mathbf{v}}$, the elements of the respective random vectors are <u>correlated</u>.

Diagonalization

It is in practice useful to represent the noise $\mathbb{v}$ and disturbance $\mathbb{w}$ in terms of uncorrelated sequences. This is achieved by diagonalizing the covariance matrices. Suppose that $\mathbf{M}$ is a symmetric matrix. Let $\mathbf{\Lambda}_M$ denote a diagonal matrix of real[7] eigenvalues and let $\mathbf{E}_M$ describe the corresponding matrix of orthonormal[8] eigenvectors. Then, the matrix can be represented as

$$\mathbf{M} = \mathbf{E}_M \mathbf{\Lambda}_M \mathbf{E}_M^\mathsf{T}, \quad \mathbf{E}_M^\mathsf{T} \mathbf{E}_M = \mathbb{I}$$

Diagonalized representation

Let the covariance matrices be diagonalized

$$\mathbf{R}_\mathbf{v} = \mathbf{E}_\mathbf{v} \mathbf{\Lambda}_\mathbf{v} \mathbf{E}_\mathbf{v}^\mathsf{T}, \quad \mathbf{Q}_\mathbf{w} = \mathbf{E}_\mathbf{w} \mathbf{\Lambda}_\mathbf{w} \mathbf{E}_\mathbf{w}^\mathsf{T}$$

The model can now be simulated with

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{E}_\mathbf{w}\mathbb{w}', \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbf{E}_\mathbf{v}\mathbb{v}'$$

where the covariances have been diagonalized so that $\mathcal{C}_{\mathbf{w}'} = \delta(0)\mathbf{\Lambda}_\mathbf{w}$ and $\mathcal{C}_{\mathbf{v}'} = \delta(0)\mathbf{\Lambda}_\mathbf{v}$. The entries in $\mathbb{v}'$ and $\mathbb{w}'$ now represent *independent* processes. The variance of each entry can be read off the diagonals in the eigenvalue matrices. This permits far easier simulation.

---

[7] The eigenvalues of a symmetric matrix are always real.

[8] The orthonormal column vectors $\mathbf{e}_i$ making up $\mathbf{E}$ satisfy $\mathbf{e}_i^\mathsf{T}\mathbf{e}_j = \delta[i, j]$. Symmetric matrices always have orthogonal eigenvectors, the rest is a matter of scaling.

TTK4115

Lecture 12

The Kalman filter part 2

Morten. O. Alver (based on material by Morten D. Pedersen)

This lecture

1. Discrete time modeling

2. Kalman filtering in discrete time

3. Time varying models

Topic

1. Discrete time modeling

2. Kalman filtering in discrete time

3. Time varying models

Discrete time Kalman filter

Measurements $\mathbb{y}$ are typically obtained through sampling at discrete intervals in time $t = kT,\ k = 0, 1, 2, \ldots$. Furthermore, estimates $\hat{\mathbb{x}}$ will typically be requested at discrete intervals. For these reasons (and others), the discrete Kalman filter is the version that sees most frequent use (by far).

Discrete time analysis

The passage from continuous to discrete time introduces a range of changes, some of which are quite subtle.

Continuous time random process

The continuous time plant model is given by the random process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w}, \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}$$

where the noise and disturbance are unbiased ($\mathbf{m}_\mathbf{v} = \mathbf{0},\ \mathbf{m}_\mathbf{w} = \mathbf{0}$) and white

$$\mathcal{A}_\mathbf{v}(t,\tau) = \mathsf{E}[\mathbb{v}(t)\mathbb{v}(\tau)^\mathsf{T}] = \delta(t-\tau)\mathbf{R}, \quad \mathcal{A}_\mathbf{w}(t,\tau) = \mathsf{E}[\mathbb{w}(\tau)\mathbb{w}(t)^\mathsf{T}] = \delta(t-\tau)\mathbf{Q}$$

Exact solution

Knowing the solution permits exact discretization. For the process model given above, an **exact** solution is furnished by

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{G}\mathbb{w}(\tau)\,d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t) + \mathbb{v}(t)$$

Exact solution

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{G}\mathbb{w}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t) + \mathbb{v}(t)$$

Discretization

Starting the solution at $t = kT$ and terminating it at $t = (k+1)T$ produces

$$\mathbb{x}[k+1] = e^{\mathbf{A}T}\mathbb{x}[k] + \int_0^T e^{\mathbf{A}\alpha}\mathbf{B}\mathbf{u}((k+1)T - \alpha)\, d\alpha + \int_0^T e^{\mathbf{A}\alpha}\mathbf{G}\mathbb{w}((k+1)T - \alpha)\, d\alpha$$

A calculation variable $\alpha = (k+1)T - \tau$ is here introduced to make life easier.

$$\bar{\mathbf{A}} \triangleq e^{\mathbf{A}T}, \quad \bar{\mathbf{B}} \triangleq \int_0^T e^{\mathbf{A}\alpha}\mathbf{B}\, d\alpha, \quad \bar{\mathbb{w}}[k] \triangleq \int_0^T e^{\mathbf{A}\alpha}\mathbf{G}\mathbb{w}((k+1)T - \alpha)\, d\alpha$$

Assuming that the deterministic input $\mathbf{u}(t)$ varies little over $(k+1)T \leq t \leq (k+1)T$ yields the discretized model

$$\mathbb{x}[k+1] = \bar{\mathbf{A}}\mathbb{x}[k] + \bar{\mathbf{B}}\mathbf{u}[k] + \bar{\mathbb{w}}[k]$$

Note that the discretized noise contribution is quite different from the continuous time variety, $\bar{\mathbb{w}}[k] \neq \mathbb{w}(kT)$.

Discrete time white **disturbances**

The discrete time white disturbance signal is now subjected to a closer examination.

$$\bar{\mathbb{w}}[k] \triangleq \int_0^T e^{\mathbf{A}\alpha}\mathbf{G}\mathbb{w}((k+1)T-\alpha)\,d\alpha$$

It is straightforward to verify that $\bar{\mathbb{w}}[k]$ inherits the unbiased nature of $\mathbb{w}(t)$. But, the autocovariance (incl. variance) changes in a subtle fashion. The discrete time autocovariance of $\bar{\mathbb{w}}[k]$ is given by

$$\begin{aligned}\bar{\mathcal{A}}_{\mathbf{w}}[k,l] &= \mathsf{E}[\bar{\mathbb{w}}[k]\bar{\mathbb{w}}[l]^{\mathsf{T}}] \\ &= \int_0^T\int_0^T e^{\mathbf{A}\alpha_1}\mathbf{G}\underbrace{\mathsf{E}[\mathbb{w}((k+1)T-\alpha_1)\mathbb{w}((l+1)T-\alpha_2)^{\mathsf{T}}]}_{\mathcal{A}_{\mathbf{w}}((k+1)T-\alpha_1,(l+1)T-\alpha_2)=\delta((l-k)T+\alpha_1-\alpha_2)\mathbf{Q}_{\mathbf{w}}}\mathbf{G}^{\mathsf{T}}e^{\mathbf{A}^{\mathsf{T}}\alpha_2}\,d\alpha_1\,d\alpha_2\end{aligned}$$

Kronecker's δ-function satisfies

$$\delta[k,l] = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

Noting that $\delta((l-k)T+\alpha_1-\alpha_2) = \delta[k,l]\delta(\alpha_1-\alpha_2)$ the result follows

$$\bar{\mathcal{A}}_{\mathbf{w}}[k,l] = \delta[k,l]\bar{\mathbf{Q}}_{\mathbf{w}}, \quad \bar{\mathbf{Q}}_{\mathbf{w}} \triangleq \int_0^T e^{\mathbf{A}\alpha}\mathbf{G}\mathbf{Q}_{\mathbf{w}}\mathbf{G}^{\mathsf{T}}e^{\mathbf{A}^{\mathsf{T}}\alpha}\,d\alpha$$

Exact discretization has rendered the infinite variance of $\mathbb{w}(t)$ *finite* and equal to $\bar{\mathcal{A}}_{\mathbf{w}}[k,k] = \bar{\mathbf{Q}}_{\mathbf{w}}$ in discrete time (this in fact a consequence of the *central limit theorem*).

Discrete time white **noise**

The measurement model in continuous time is given by

$$\mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}, \quad \mathcal{A}_\mathbf{v}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{v}(\tau)^\mathsf{T}] = \delta(t-\tau)\mathbf{R}_\mathbf{v}$$

A naïve conversion to discrete time would suggest

$$\mathbb{y}[k] = \mathbf{C}\mathbb{x}[k] + \mathbb{v}[k], \quad \bar{\mathbf{R}}_\mathbf{v} = \mathrm{E}[\mathbb{v}[k]\mathbb{v}[k]^\mathsf{T}] = \delta(0)\mathbf{R}_\mathbf{v}$$

This interpretation leads to extreme exaggerations of noise in discrete time (but is suitable in continuous time).

Averaging convention

Rather than interpreting measurement noise as occuring at the instant of sampling, it can be interpreted in a *averaged* sense. This idea is captured in the convention

$$\bar{\mathbb{v}}[k] \triangleq \frac{1}{T}\int_0^T \mathbb{v}(kT-\alpha)\, d\alpha$$

The discrete time noise vector inherits the unbiased nature of the continuous time signal, whilst the autocovariance transforms to

$$\bar{\mathcal{A}}_\mathbf{v}[k,l] = \mathrm{E}[\bar{\mathbb{v}}[k]\bar{\mathbb{v}}[l]^\mathsf{T}] = \frac{1}{T^2}\int_0^T\int_0^T \mathrm{E}[\mathbb{v}(kT-\alpha_1)\mathbb{v}(lT-\alpha_2)^\mathsf{T}]\, d\alpha_1 d\alpha_2 = \delta[k,l]\bar{\mathbf{R}}_\mathbf{v}, \quad \bar{\mathbf{R}}_\mathbf{v} \triangleq \mathbf{R}_\mathbf{v}/T$$

Discrete time random process

The discrete time plant model is given by the random process

$$\mathbb{x}[k+1] = \bar{\mathbf{A}}\mathbb{x}[k] + \bar{\mathbf{B}}\mathbf{u}[k] + \bar{\mathbb{w}}[k], \quad \mathbb{y}[k] = \mathbf{C}\mathbb{x}[k] + \bar{\mathbb{v}}[k]$$

where the noise and disturbance are unbiased ($\mathbf{m_v} = \mathbf{0},\ \mathbf{m_w} = \mathbf{0}$) and white

$$\bar{\mathcal{A}}_{\mathbf{v}}[k,l] = \mathrm{E}[\bar{\mathbb{v}}[k]\bar{\mathbb{v}}[l]^{\mathsf{T}}] = \delta[k,l]\bar{\mathbf{R}}_{\mathbf{v}}, \quad \bar{\mathcal{A}}_{\mathbf{w}}[k,l] = \mathrm{E}[\bar{\mathbb{w}}[k]\bar{\mathbb{w}}[l]^{\mathsf{T}}] = \delta[k,l]\bar{\mathbf{Q}}_{\mathbf{w}}$$

It will be assumed that the noise and disturbance processes are uncorrelated $\mathrm{E}[\bar{\mathbb{v}}[k]\bar{\mathbb{w}}[l]^{\mathsf{T}}] = \mathbf{0}$.

Continuous to discrete conversion - sampling time T.

Transition matrix: Obtained from exact discretization.

$$\bar{\mathbf{A}} = e^{\mathbf{A}T}$$

Input matrix: Obtained from exact discretization & assumption of constant $\mathbf{u}$ over sampling period.

$$\bar{\mathbf{B}} = \int_0^T e^{\mathbf{A}\alpha}\mathbf{B}\, d\alpha$$

Disturbance covariance: Obtained from exact discretization.

$$\bar{\mathbf{Q}}_{\mathbf{w}} = \int_0^T e^{\mathbf{A}\alpha}\mathbf{G}\mathbf{Q}_{\mathbf{w}}\mathbf{G}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}\alpha}\, d\alpha$$

Noise covariance: Obtained through an averaging convention.

$$\bar{\mathbf{R}}_{\mathbf{v}} = \mathbf{R}_{\mathbf{v}}/T$$

Van Loan's method[1]

The integrals in the preceding slide are often quite intractable. It is however possible to arrive at the correct matrices without integrating. This is done with *Van Loan's method*. The key result is

$$\exp\left(\begin{bmatrix} \mathbf{A} & \mathbf{G}\mathbf{Q_w}\mathbf{G}^\mathsf{T} \\ \mathbf{0} & -\mathbf{A}^\mathsf{T} \end{bmatrix} T\right) = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix}, \quad \bar{\mathbf{A}} = \mathbf{M}_{11}, \quad \bar{\mathbf{Q}}_\mathbf{w} = \mathbf{M}_{12}\mathbf{M}_{11}^\mathsf{T}$$

Matrix exponentials are readily computed numerically, obviating the need for integration. The input matrix can be computed from

$$\exp\left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} T\right) = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{0} & \mathbb{I} \end{bmatrix}, \quad \bar{\mathbf{A}} = \mathbf{N}_{11}, \quad \bar{\mathbf{B}} = \mathbf{N}_{12}$$

[1] Van Loan C.F. (1978), *Computing Integrals Involving the Matrix Exponential*, IEEE Transactions on Automatic Control, Vol. 23, No. 3. See also page 126 in the Brown & Hwang book.

Topic

1. Discrete time modeling

2. Kalman filtering in discrete time

3. Time varying models

Discrete observer

Discrete time requires a sligthly more explicit observer design. The estimate is generated in two distinct phases:

---

1 - A priori (denoted $\hat{\mathbb{x}}^-[k]$): The best guess for $\mathbb{x}[k]$ **prior** to incorporation of the measurement $\mathbb{y}[k]$. The deterministic model is to arrive at this estimate.

$$\hat{\mathbb{x}}^-[k] = \bar{\mathbf{A}}\hat{\mathbb{x}}[k-1] + \bar{\mathbf{B}}\mathbf{u}[k-1]$$

---

2 - A posteriori (denoted $\hat{\mathbb{x}}[k]$): The best guess for $\mathbb{x}[k]$ **after** incorporation of the measurement $\mathbb{y}[k]$. A linear blend of what the model suggests ($\hat{\mathbb{x}}^-[k]$) and the new measurement $\mathbb{y}[k]$ is used to arrive at this final estimate. The Kalman gain $\mathbf{L}[k]$ serves as the *blending factor*, viz.

$$\hat{\mathbb{x}}[k] = \hat{\mathbb{x}}^-[k] + \mathbf{L}[k](\mathbb{y}[k] - \mathbf{C}\hat{\mathbb{x}}^-[k])$$

Kalman gain

The Kalman gain is (as for the continuous time case) designed to minimize the mean-square error of the estimate at time k.

$$J[k] = \mathrm{tr}(\mathbf{P}[k]), \quad \mathbf{P}[k] \triangleq \mathrm{E}[(\mathbb{x}[k] - \hat{\mathbb{x}}[k])(\mathbb{x}[k] - \hat{\mathbb{x}}[k])^\mathsf{T}]$$

A **priori** error and covariance matrix

The a priori and a posteriori estimation errors and covariance matrices are given by

$$\mathbb{e}^-[k] \triangleq \mathbb{x}[k] - \hat{\mathbb{x}}^-[k], \quad \mathbf{P}^-[k] \triangleq \mathsf{E}[\mathbb{e}^-[k]\mathbb{e}^-[k]^\mathsf{T}]$$

$$\mathbb{e}[k] \triangleq \mathbb{x}[k] - \hat{\mathbb{x}}[k], \quad \mathbf{P}[k] \triangleq \mathsf{E}[\mathbb{e}_k \mathbb{e}_k^\mathsf{T}]$$

The process model produces the following state at k

$$\mathbb{x}[k] = \bar{\mathbf{A}}\mathbb{x}[k-1] + \bar{\mathbf{B}}\mathbf{u}[k-1] + \bar{\mathbb{w}}[k-1]$$

whereas the a priori estimate reads as

$$\hat{\mathbb{x}}^-[k] = \bar{\mathbf{A}}\hat{\mathbb{x}}[k-1] + \bar{\mathbf{B}}\mathbf{u}[k-1]$$

This permits the following expression for the **a priori** error

$$\mathbb{e}^-[k] = \bar{\mathbf{A}}\mathbb{e}[k-1] + \bar{\mathbb{w}}[k-1]$$

The **a priori** covariance matrix follows as

$$\mathbf{P}^-[k] = \mathsf{E}[(\bar{\mathbf{A}}\mathbb{e}[k-1] + \bar{\mathbb{w}}[k-1])(\bar{\mathbf{A}}\mathbb{e}[k-1] + \bar{\mathbb{w}}[k-1])^\mathsf{T}] = \bar{\mathbf{A}}\mathbf{P}[k-1]\bar{\mathbf{A}}^\mathsf{T} + \bar{\mathbf{Q}}_\mathbf{w}$$

The disturbance at k is uncorrelated to the a-posteriori estimate at k, hence $\mathsf{E}[\mathbb{e}[k]\bar{\mathbb{w}}[k]^\mathsf{T}] = \mathbf{0}$.

A **posteriori** error and covariance matrix

The a priori and a posteriori estimation errors and covariance matrices are given by

$$\mathbb{e}^-[k] \triangleq \mathbb{x}[k] - \hat{\mathbb{x}}^-[k], \quad \mathbf{P}^-[k] \triangleq \mathsf{E}[\mathbb{e}^-[k]\mathbb{e}^-[k]^\mathsf{T}]$$
$$\mathbb{e}[k] \triangleq \mathbb{x}[k] - \hat{\mathbb{x}}[k], \quad \mathbf{P}[k] \triangleq \mathsf{E}[\mathbb{e}_k\mathbb{e}_k^\mathsf{T}]$$

The a posteriori estimate can be expanded to read

$$\hat{\mathbb{x}}[k] = \hat{\mathbb{x}}^-[k] + \mathbf{L}[k](\mathbb{y}[k] - \mathbf{C}\hat{\mathbb{x}}^-[k]) = \hat{\mathbb{x}}^-[k] + \mathbf{L}[k]\mathbf{C}\mathbb{e}^-[k] + \mathbf{L}[k]\bar{\mathbb{v}}[k]$$

This permits the following expression for the **a posteriori** error

$$\mathbb{e}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbb{e}^-[k] - \mathbf{L}[k]\bar{\mathbb{v}}[k]$$

The **a posteriori** covariance matrix follows as

$$\begin{aligned}\mathbf{P}[k] = \mathsf{E}[((\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbb{e}^-[k] - \mathbf{L}[k]\bar{\mathbb{v}}[k])((\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbb{e}^-[k] - \mathbf{L}[k]\bar{\mathbb{v}}[k])^\mathsf{T}]\\ = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\bar{\mathbf{R}}_\mathbf{v}[k]\mathbf{L}[k]^\mathsf{T}\end{aligned}$$

The noise at k is uncorrelated to the a-priori estimate at k, hence $\mathsf{E}[\mathbb{e}^-[k]\bar{\mathbb{v}}[k]^\mathsf{T}] = \mathbf{0}$.

Optimal estimation

The **a posteriori** covariance matrix describes the covariance of the final estimate error $\mathbb{e}[k] = \mathbb{x}[k] - \hat{\mathbb{x}}[k]$. We now seek to minimize the mean-square error

$$J[k] = \mathrm{tr}(\mathbf{P}[k])$$

We need to differentiate w.r.t. to the Kalman gain and solve for the extremum.

Matrix differentiation rules

These are the differentiation rules that we will use:

$$\frac{\partial}{\partial \mathbf{X}} \mathrm{tr}(\mathbf{XA}) = \mathbf{A}^\mathsf{T}$$
$$\frac{\partial}{\partial \mathbf{X}} \mathrm{tr}(\mathbf{AX}^\mathsf{T}) = \mathbf{A}$$
$$\frac{\partial}{\partial \mathbf{X}} \mathrm{tr}(\mathbf{XBX}^\mathsf{T}) = \mathbf{XB}^\mathsf{T} + \mathbf{XB}$$

Optimal estimation:

These are the differentiation rules that we will use:

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{XA}) = \mathbf{A}^\mathsf{T}, \quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{AX}^\mathsf{T}) = \mathbf{A}, \quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{XBX}^\mathsf{T}) = \mathbf{XB}^\mathsf{T} + \mathbf{XB}$$

The expression for $\mathbf{P}[k]$ is:

$$\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\bar{\mathbf{R}}_\mathbf{v}[k]\mathbf{L}[k]^\mathsf{T}$$

Optimal estimation

The **a posteriori** covariance matrix describes the covariance of the final estimate error $\mathbb{e}[k] = \mathbb{x}[k] - \hat{\mathbb{x}}[k]$. We now seek to minimize the mean-square error

$$J[k] = \mathrm{tr}(\mathbf{P}[k])$$

Differentiation w.r.t. to the Kalman gain and solving for the extremum yields:

$$\frac{\partial \mathrm{tr}(\mathbf{P}[k])}{\partial \mathbf{L}[k]} = \frac{\partial}{\partial \mathbf{L}[k]} \mathrm{tr}\left((\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\bar{\mathbf{R}}_\mathbf{v}[k]\mathbf{L}[k]^\mathsf{T} \right)$$
$$= -2\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + 2\mathbf{L}[k](\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \bar{\mathbf{R}}_\mathbf{v}) = \mathbf{0}$$

The *Kalman gain* thus follows as

$$\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \bar{\mathbf{R}}_\mathbf{v})^{-1}$$

Discrete Kalman filter algorithm

The filter is initialized at

$$\hat{\mathbf{x}}^-[0] = \mathrm{E}[\mathbb{x}(0)] = \mathbf{m}_{\mathbf{x}_0}$$
$$\mathbf{P}^-[0] = \mathrm{E}[\mathbb{e}^-[0]\mathbb{e}^-[0]^\mathsf{T}] = \mathrm{E}[(\mathbb{x}[0] - \mathbf{m}_{\mathbf{x}_0})(\mathbb{x}[0] - \mathbf{m}_{\mathbf{x}_0})^\mathsf{T}] = \mathcal{C}_{\mathbf{x}_0}$$

The recursive algorithm running over $k = 0 \dots K$ is summarized by

1 - Compute Kalman gain

$$\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \bar{\mathbf{R}}_\mathbf{v})^{-1}$$

2 - Update estimate with measurement

$$\hat{\mathbb{x}}[k] = \hat{\mathbb{x}}^-[k] + \mathbf{L}[k](\mathbb{y}[k] - \mathbf{C}\hat{\mathbb{x}}^-[k])$$

3 - Update error covariance matrix

$$\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\bar{\mathbf{R}}_\mathbf{v}[k]\mathbf{L}[k]^\mathsf{T}$$

4 - Project ahead

$$\hat{\mathbb{x}}^-[k+1] = \bar{\mathbf{A}}\hat{\mathbb{x}}[k] + \bar{\mathbf{B}}\mathbf{u}[k]$$
$$\mathbf{P}^-[k+1] = \bar{\mathbf{A}}\mathbf{P}[k]\bar{\mathbf{A}}^\mathsf{T} + \bar{\mathbf{Q}}_\mathbf{w}$$

Example: handheld GPS

Problem

GPS measurements are typically available at a sample time $T \sim 1[s]$. It is assumed that the horisontal measurements are approximately normally distributed around the true position $\mathbb{p} = [\mathbb{p}_1\ \mathbb{p}_2]^T$ with a standard deviation $\sigma_v \sim 5[m]$. A measurement model is thus

$$\mathbb{y}[k] = \mathbb{p}[k] + \mathbb{v}[k], \quad \bar{\mathbf{R}} = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

---

1. How can one improve upon the direct measurement?
2. Is it possible to obtain velocity estimates?

Solution

The desired improvements can be had by incorporating system knowledge. The position of the handheld GPS unit will change in a manner that cannot be predicted exactly. We assume instead that the user moves in accordance with the random model

$$\tau\ddot{\mathbb{p}}_1 + \dot{\mathbb{p}}_1 = \mathbb{w}_1$$
$$\tau\ddot{\mathbb{p}}_2 + \dot{\mathbb{p}}_2 = \mathbb{w}_2$$

Physically, this model represents a mass-damper perturbed by an unknown force.

Note that the velocities $\dot{\mathbb{p}}$ enter as states of the model and can therefore be estimated.

Example: handheld GPS

Continuous time random process

The intensities of the disturbance signals and the time-constant τ should be tuned through practical experiments. A useful model structure can however be supplied as

$$\overbrace{\begin{bmatrix} \dot{\mathbb{p}}_1 \\ \dot{\mathbb{p}}_2 \\ \ddot{\mathbb{p}}_1 \\ \ddot{\mathbb{p}}_2 \end{bmatrix}}^{\dot{\mathbb{x}}} = \overbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\tau^{-1} & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} \mathbb{p}_1 \\ \mathbb{p}_2 \\ \dot{\mathbb{p}}_1 \\ \dot{\mathbb{p}}_2 \end{bmatrix}}^{\mathbb{x}} + \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tau^{-1} & 0 \\ 0 & \tau^{-1} \end{bmatrix}}^{\mathbf{G}} \overbrace{\begin{bmatrix} \mathbb{w}_1 \\ \mathbb{w}_2 \end{bmatrix}}^{\mathbb{w}}$$

$$\underbrace{\begin{bmatrix} \mathbb{y}_1 \\ \mathbb{y}_2 \end{bmatrix}}_{\mathbb{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbb{p}_1 \\ \mathbb{p}_2 \\ \dot{\mathbb{p}}_1 \\ \dot{\mathbb{p}}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbb{v}_1 \\ \mathbb{v}_2 \end{bmatrix}}_{\mathbb{v}}$$

where

$$\mathbf{Q} = q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} T$$

Discrete time random process

The Handheld GPS problem is solved using a discrete time Kalman filter. Tuning constants are chosen as $\tau = 200$ and $q = 25^2$.

Using Van Loan's method the discrete time system matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{Q}}$ can be found precisely:

$$\bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0.9975 & 0 \\ 0 & 1 & 0 & 0.9975 \\ 0 & 0 & 0.995 & 0 \\ 0 & 0 & 0 & 0.995 \end{bmatrix}, \quad \bar{\mathbf{Q}} = \begin{bmatrix} 0.0052 & 0 & 0.0078 & 0 \\ 0 & 0.0052 & 0 & 0.0078 \\ 0.0078 & 0 & 0.0155 & 0 \\ 0 & 0.0078 & 0 & 0.0155 \end{bmatrix}$$

Note: the discrete system, instead of 2 uncorrelated disturbance signals, has 4 signals with off-diagonal correlations. This is because the disturbances propagate through the system during the time period T.

The final model reads as

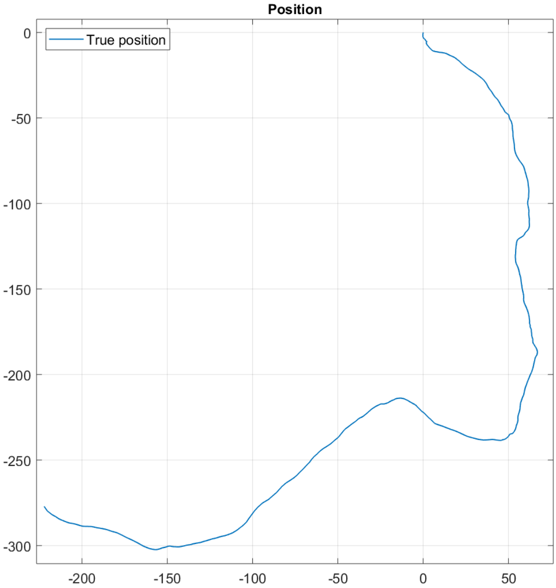
$$\mathbb{x}[k+1] = \bar{\mathbf{A}}\mathbb{x}[k] + \bar{\mathbb{w}}[k], \quad \mathbb{y}[k] = \mathbf{C}\mathbb{x}[k] + \bar{\mathbb{v}}[k]$$

where $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ describe the respective covariances of the disturbance and noise signals.

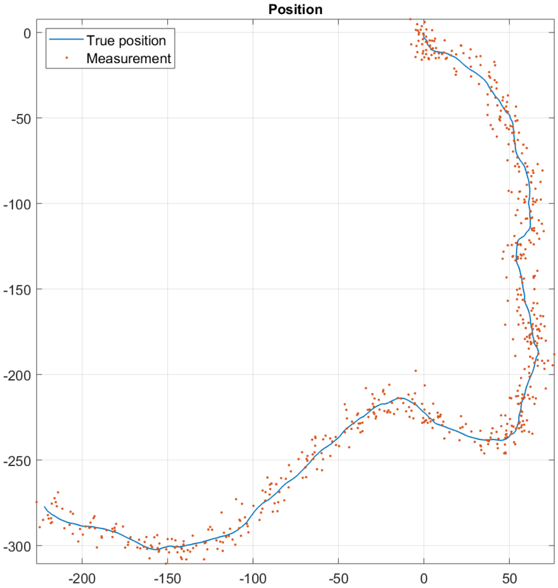
Matlab Demo

Model setup, discretization, simulation and plotting is done in the *handheld_GPS.m* Matlab script (available on Blackboard).

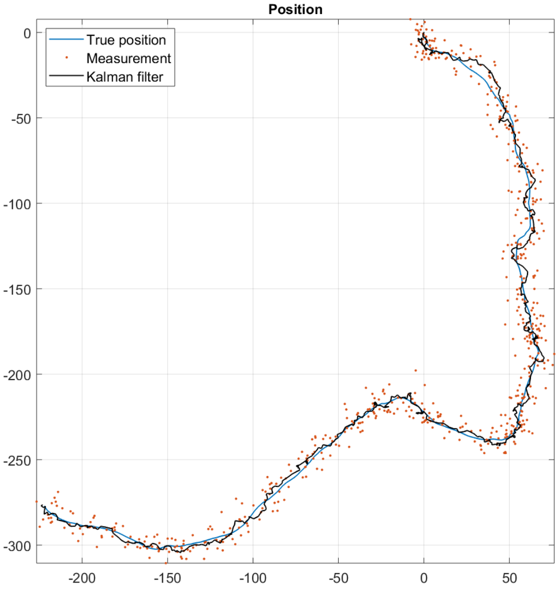
Example realization of random walk



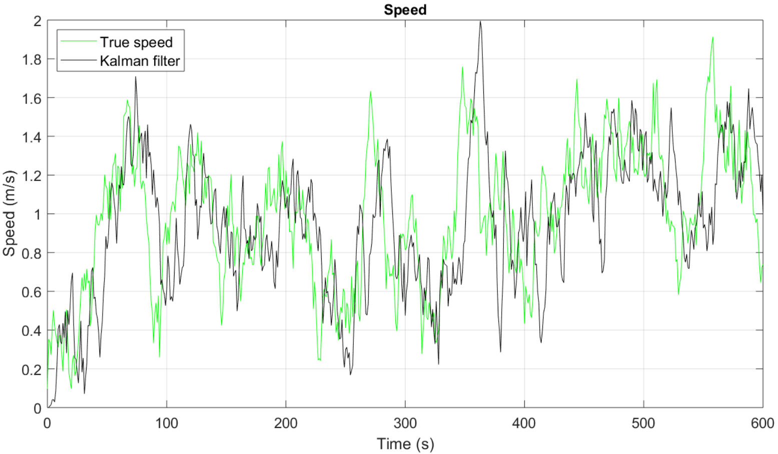
Noisy measurements of true position



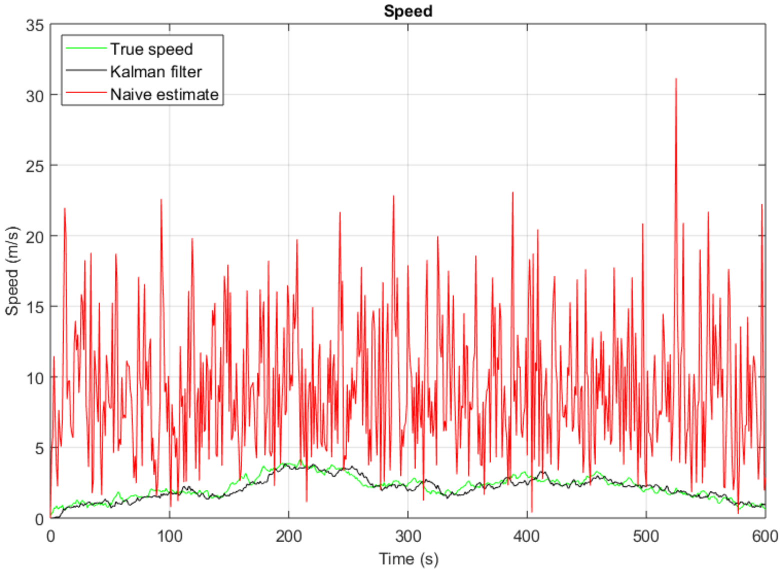
Kalman filter estimate of position



True speed vs. estimated velocity



Comparison to "naive" velocity estimate



Topic

1. Discrete time modeling

2. Kalman filtering in discrete time

3. Time varying models

Time varying models

The continuous and discrete time Kalman filters are not limited to time-invariant plants. They can in fact serve as optimal estimators for time-varying systems!

LTV system

Let a linear time-varying random process be given by

$$\dot{\mathbb{x}}(t) = \mathbf{A}(t)\mathbb{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbb{w}(t), \quad \mathbb{y}(t) = \mathbf{C}(t)\mathbb{x}(t) + \mathbb{v}(t)$$

where the noise and disturbance are unbiased ($\mathbf{m_v} = \mathbf{0},\ \mathbf{m_w} = \mathbf{0}$) and white

$$\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathsf{E}[\mathbb{v}(t)\mathbb{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}(t), \quad \mathcal{A}_{\mathbf{w}}(t,\tau) = \mathsf{E}[\mathbb{w}(\tau)\mathbb{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}(t)$$

Optimal estimator[2]

An optimal estimator for the LTV process is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)(\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)), \quad \mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}(t)\mathbf{R}_{\mathbf{v}}^{-1}(t)$$

The covariance matrix is here computed by solving the Riccati Equation

$$\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t)^{\mathsf{T}} + \mathbf{G}(t)\mathbf{Q}_{\mathbf{w}}(t)\mathbf{G}(t)^{\mathsf{T}} - \mathbf{P}(t)\mathbf{C}(t)^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}(t)^{-1}\mathbf{C}(t)\mathbf{P}(t)$$

[2] Simulation of the continuous time Riccati equation can be challenging. This is one of the reasons that a discrete time formulation is preferred.

Time varying models

The continuous and discrete time Kalman filters are not limited to time-invariant plants. They can in fact serve as optimal estimators for time-varying systems!

DLTV system

Let a discrete time-varying random process plant model be given by

$$\mathbb{x}[k+1] = \bar{\mathbf{A}}[k]\mathbb{x}[k] + \bar{\mathbf{B}}[k]\mathbf{u}[k] + \bar{\mathbb{w}}[k], \quad \mathbb{y}[k] = \mathbf{C}[k]\mathbb{x}[k] + \bar{\mathbb{v}}[k]$$

where the noise and disturbance are unbiased ($\mathbf{m_v} = \mathbf{0},\ \mathbf{m_w} = \mathbf{0}$) and white

$$\bar{\mathcal{A}}_{\mathbf{v}}[k,l] = \mathrm{E}[\bar{\mathbb{v}}[k]\bar{\mathbb{v}}[l]^{\mathsf{T}}] = \delta[k,l]\bar{\mathbf{R}}_{\mathbf{v}}[k], \quad \bar{\mathcal{A}}_{\mathbf{w}}[k,l] = \mathrm{E}[\bar{\mathbb{w}}[k]\bar{\mathbb{w}}[l]^{\mathsf{T}}] = \delta[k,l]\bar{\mathbf{Q}}_{\mathbf{w}}[k]$$

Optimal estimator

The optimal estimator for the preceding system is furnished, quite simply, by letting the matrices in the Kalman filter algorithm be time-varying.

Kalman filter algorithm, general case

The filter is initialized at

$$\hat{\mathbf{x}}^-[0] = \mathrm{E}[\mathbb{x}(0)] = \mathbf{m}_{\mathbf{x}_0}$$

$$\mathbf{P}^-[0] = \mathrm{E}[\mathbb{e}^-[0]\mathbb{e}^-[0]^\mathsf{T}] = \mathrm{E}[(\mathbb{x}[0] - \mathbf{m}_{\mathbf{x}_0})(\mathbb{x}[0] - \mathbf{m}_{\mathbf{x}_0})^\mathsf{T}] = \mathcal{C}_{\mathbf{x}_0}$$

The recursive algorithm running over $k = 0 \ldots K$ is summarized by

1 - Compute Kalman gain

$$\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}[k]^\mathsf{T}(\mathbf{C}[k]\mathbf{P}^-[k]\mathbf{C}[k]^\mathsf{T} + \bar{\mathbf{R}}_\mathbf{v}[k])^{-1}$$

---

2 - Update estimate with measurement

$$\hat{\mathbf{x}}[k] = \hat{\mathbf{x}}^-[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{C}[k]\hat{\mathbf{x}}^-[k])$$

---

3 - Update error covariance matrix

$$\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C}[k])\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C}[k])^\mathsf{T} + \mathbf{L}[k]\bar{\mathbf{R}}_\mathbf{v}[k]\mathbf{L}[k]^\mathsf{T}$$

---

4 - Project ahead

$$\hat{\mathbf{x}}^-[k+1] = \bar{\mathbf{A}}[k]\hat{\mathbf{x}}[k] + \bar{\mathbf{B}}[k]\mathbf{u}[k]$$

$$\mathbf{P}^-[k+1] = \bar{\mathbf{A}}[k]\mathbf{P}[k]\bar{\mathbf{A}}[k]^\mathsf{T} + \bar{\mathbf{Q}}_\mathbf{w}[k]$$

...repeat with $k = k + 1$...