Introduction to Differential-Algebraic Equations (DAEs)

Sébastien Gros
NTNU, Eng. Cybernetics

Slides for TTK4130
2021

Objectives of the slides

Learn the basics of DAEs

- ✓ understand what a DAE is
- ✓ identify the different forms of DAEs
- ✓ understand the Tikhonov theorem
- ✓ understand why there are “easy” and “hard” DAEs
- ✓ understand differential index & index reductions

Simulating ODEs

Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$

Simulating ODEs

Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$
wherever it leads

Simulating ODEs

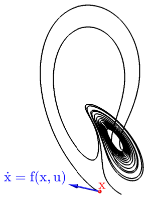
Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads



Simulating ODEs

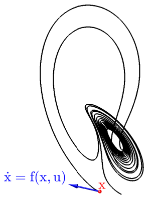
Most common form of ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right) \tag{1}$$

... delivers $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given

State derivative $\dot{\mathbf{x}}$ tells us "where" the state is going from $\mathbf{x}, \mathbf{u}$

Then we "can" simulate (1), i.e. "follow" $\dot{\mathbf{x}}$ wherever it leads



Implicit ODEs

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$$

...simulation similarly requires one to be able to compute $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. try $e^{\dot{x}} + \dot{x}^3 - u = 0$
    - has a well-defined solution $\dot{x}$ for any u
    - cannot write an expression $\dot{x} = \ldots$

Implicit ODEs

How to “solve” an implicit ODE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. try $e^{\dot{x}} + \dot{x}^3 - u = 0$
  - has a well-defined solution $\dot{x}$ for any u
  - cannot write an expression $\dot{x} = \ldots$
- Numerical methods are often needed. More on that soon!

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. try $e^{\dot{x}} + \dot{x}^3 - u = 0$
  - has a well-defined solution $\dot{x}$ for any u
  - cannot write an expression $\dot{x} = ...$
- Numerical methods are often needed. More on that soon!
- But if (2) defines $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$, then *there exists a function* $\mathbf{f}$ such that:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

even if we cannot *write* $\mathbf{f}$ in terms of *mathematical expressions* (e.g. $\sin, \cos, .^2$),

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. try $e^{\dot{x}} + \dot{x}^3 - u = 0$
  - has a well-defined solution $\dot{x}$ for any u
  - cannot write an expression $\dot{x} = ...$
- Numerical methods are often needed. More on that soon!
- But if (2) defines $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$, then *there exists a function* $\mathbf{f}$ such that:

  $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

  even if we cannot *write* $\mathbf{f}$ in terms of *mathematical expressions* (e.g. $\sin, \cos, .^2$),
- Then

  $$\mathbf{F}(\mathbf{f}(\mathbf{x}, \mathbf{u}), \mathbf{x}, \mathbf{u}) = 0, \qquad \text{for all } \mathbf{x}, \mathbf{u}$$

Implicit ODEs

How to "solve" an implicit ODE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \tag{2}$$

- In order to simulate (2) we need $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given)
- Equation (2) provides $\dot{\mathbf{x}}$ (for $\mathbf{x}, \mathbf{u}$ given) implicitly
- In order to get $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given, we need to solve (2) for $\dot{\mathbf{x}}$
- Pen-and-paper solution may not exist! E.g. try $e^{\dot{x}} + \dot{x}^3 - u = 0$
  - has a well-defined solution $\dot{x}$ for any u
  - cannot write an expression $\dot{x} = \dots$
- Numerical methods are often needed. More on that soon!
- But if (2) defines $\dot{\mathbf{x}}$ for any $\mathbf{x}, \mathbf{u}$, then *there exists a function* $\mathbf{f}$ such that:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

even if we cannot *write* $\mathbf{f}$ in terms of *mathematical expressions* (e.g. $\sin, \cos, .^2$),

- Then

$$\mathbf{F}(\mathbf{f}(\mathbf{x}, \mathbf{u}), \mathbf{x}, \mathbf{u}) = 0, \qquad \text{for all } \mathbf{x}, \mathbf{u}$$

When does (2) define $\dot{\mathbf{x}}$ properly? I.e. when does $\mathbf{f}$ exist?

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Implicit Function Theorem (simplified version)

Let $\mathbf{F}(\mathbf{y}, \mathbf{x})$ be smooth

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Implicit Function Theorem (simplified version)

Let $\mathbf{F}(\mathbf{y}, \mathbf{x})$ be smooth, and suppose that

$$\frac{\partial \mathbf{F}(\mathbf{y}, \mathbf{x})}{\partial \mathbf{y}} \text{ is full rank for all } \mathbf{y},\ \mathbf{x} \text{ s.t. } \mathbf{F}(\mathbf{y}, \mathbf{x}) = 0$$

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Implicit Function Theorem (simplified version)

Let $\mathbf{F}(\mathbf{y}, \mathbf{x})$ be smooth, and suppose that

$$\frac{\partial \mathbf{F}(\mathbf{y}, \mathbf{x})}{\partial \mathbf{y}} \text{ is full rank for all } \mathbf{y},\ \mathbf{x} \text{ s.t. } \mathbf{F}(\mathbf{y}, \mathbf{x}) = 0$$

then there exists $\mathbf{f}(\mathbf{x})$ such that

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Implicit Function Theorem (simplified version)

Let $\mathbf{F}(\mathbf{y}, \mathbf{x})$ be smooth, and suppose that

$$\frac{\partial \mathbf{F}(\mathbf{y}, \mathbf{x})}{\partial \mathbf{y}} \text{ is full rank for all } \mathbf{y},\ \mathbf{x} \text{ s.t. } \mathbf{F}(\mathbf{y}, \mathbf{x}) = 0$$

then there exists $\mathbf{f}(\mathbf{x})$ such that

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

Solving ODEs - Implicit Function Theorem

Consider function $\mathbf{f}(\mathbf{x})$ implicitly given by:

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0$$

... when is $\mathbf{f}(\mathbf{x})$ well defined?

Implicit Function Theorem a "practical version"

If $\mathbf{F}(\mathbf{y}, \mathbf{x})$ is smooth, and

$$\frac{\partial \mathbf{F}(\mathbf{y}, \mathbf{x})}{\partial \mathbf{y}} \text{ is always full rank}$$

... then $\mathbf{f}(\mathbf{x})$ exists.

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$?

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$?

$$\dot{x} = \pm\sqrt{\frac{x}{u}}, \quad \text{if} \quad u \neq 0$$

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$?

$$\dot{x} = \pm\sqrt{\frac{x}{u}}, \quad \text{if} \quad u \neq 0$$

Use IFT:

Equation (3) can be solved for $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given if:

$$\frac{\partial \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u})}{\partial \dot{\mathbf{x}}}$$

... is always full rank

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$?

$$\dot{x} = \pm\sqrt{\frac{x}{u}}, \quad \text{if} \quad u \neq 0$$

Use IFT:

Equation (3) can be solved for $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given if:

$$\frac{\partial \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u})}{\partial \dot{\mathbf{x}}}$$

... is always full rank

What about e.g. $\mathbf{F}(\dot{x}, x, u) = u\dot{x}^2 - x = 0$?

$$\frac{\partial \mathbf{F}}{\partial \dot{x}} = 2\dot{x}u$$

... this is "full rank" (i.e. $\neq 0$ here) if $u \neq 0$ and $\dot{x} \neq 0$

Solving ODEs - Implicit Function Theorem (cont')

Can we solve

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0 \qquad (3)$$

... for $\dot{\mathbf{x}}$?

What about e.g. $u\dot{x}^2 - x = 0$?

$$\dot{x} = \pm\sqrt{\frac{x}{u}}, \quad \text{if} \quad u \neq 0$$

Use IFT:

Equation (3) can be solved for $\dot{\mathbf{x}}$ for $\mathbf{x}, \mathbf{u}$ given if:

$$\frac{\partial \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right)}{\partial \dot{\mathbf{x}}}$$

... is always full rank

What about e.g. $\mathbf{F}\left(\dot{x}, x, u\right) = u\dot{x}^2 - x = 0$?

$$\frac{\partial \mathbf{F}}{\partial \dot{x}} = 2\dot{x}u$$

... this is "full rank" (i.e. $\neq 0$ here) if $u \neq 0$ and $\dot{x} \neq 0$

Now we know when an ODE has a "well-posed $\dot{\mathbf{x}}$"

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}\right) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

because $\dot{\mathbf{x}}_2$ does not appear, (4) does not deliver $\dot{\mathbf{x}}$...

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

because $\dot{\mathbf{x}}_2$ does not appear, (4) does not deliver $\dot{\mathbf{x}}$...

Jacobian is rank deficient because:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

because $\dot{\mathbf{x}}_2$ does not appear, (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian is rank deficient because:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

because $\dot{\mathbf{x}}_2$ does not appear, (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian is rank deficient because:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

Actually (4) yields

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1$$
$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

E.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{bmatrix} = 0 \qquad (4)$$

State is

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

because $\dot{\mathbf{x}}_2$ does not appear, (4) does not deliver $\dot{\mathbf{x}}$...

... however (4) provides well-defined trajectories!

Jacobian is rank deficient because:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ (4) is a DAE

Actually (4) yields

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1$$

$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

In applications, DAEs are often differential equations where **some states derivatives do not appear** as in e.g. (4)

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{bmatrix} = 0$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{bmatrix} = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array}\right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

This is a DAE even though both $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ appear

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 - \mathbf{x}_1 + \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \mathbf{x}_2 + u \end{array} \right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$

has solution

$$\mathbf{x}_1 = -\dot{u}$$
$$\mathbf{x}_2 = -\mathbf{x}_1 - u$$

DAE well defined only for u continuous!

This is a DAE even though both $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ appear

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated → DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{array} \right] = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \left[\begin{array}{cc} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{array} \right]$$

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix}$$

For $\mathbf{x}_2(0) = \mathbf{x}_1(0)$, has solution

$$\dot{\mathbf{x}}_1 = u - \mathbf{x}_1$$
$$\mathbf{x}_2 = \mathbf{x}_1$$

Is it a DAE or an ODE?

What are DAEs?

Definition

An implicit differential equation $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$ is a DAE if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is rank deficient along the trajectory $\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}$.

A state does not appear time differentiated $\rightarrow$ DAE, but also...

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 + \mathbf{x}_1 - u \\ (\mathbf{x}_1 - \mathbf{x}_2)\,\dot{\mathbf{x}}_2 + \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix} = 0$$

yields

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x}_1 - \mathbf{x}_2 \end{bmatrix}$$

For $\mathbf{x}_2(0) = \mathbf{x}_1(0)$, has solution

$$\dot{\mathbf{x}}_1 = u - \mathbf{x}_1$$
$$\mathbf{x}_2 = \mathbf{x}_1$$

Is it a DAE or an ODE? It can be both!

A differential equations can be both an ODE & DAE, even jump back-and-forth. Avoided in practice, i.e. we like $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ having fixed rank

A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}(\dot{x}, x, z, u) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \, \dot{\text{states}}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

because $\text{states} = \left[\begin{array}{c} \mathbf{x} \\ \mathbf{z} \end{array}\right]$

A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, u\right) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}\left(\dot{x}, x, z, u\right) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \text{ states}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

because $\text{states} = \left[\begin{array}{c} \mathbf{x} \\ \mathbf{z} \end{array}\right]$

- Fully-Implicit DAEs

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0$$

A bit of notation

- If some states do not appear time differentiated, we highlight them as "$\mathbf{z}$", e.g.

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, u) = \left[\begin{array}{c} \dot{\mathbf{x}}_1 + \mathbf{x}_2 + u \\ \mathbf{x}_1 + \mathbf{x}_2 + u \end{array}\right] = 0 \quad \longrightarrow \quad \mathbf{F}(\dot{x}, x, z, u) = \left[\begin{array}{c} \dot{x} + z + u \\ x + z + u \end{array}\right] = 0$$

- Then the DAE definition "works" and is to be understood as:

$$\frac{\partial \mathbf{F}}{\partial \text{ states}} = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{z}}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \mathbf{0} \end{array}\right] \quad \text{is rank deficient}$$

because $\text{states} = \left[\begin{array}{c} \mathbf{x} \\ \mathbf{z} \end{array}\right]$

- Fully-Implicit DAEs

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}) = 0$$

- Semi-explicit DAEs (most common form)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

$\longrightarrow$ "explicit ODE + algebraic equations"

Conversion semi-explicit $\leftrightarrow$ fully-implicit

Semi-explicit DAEs		Fully-Implicit DAEs
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$ $0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$	$\longrightarrow$	
	$\longleftarrow$	$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$

Conversion semi-explicit $\leftrightarrow$ fully-implicit

Semi-explicit DAEs		Fully-Implicit DAEs
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$ $0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$	$\longrightarrow$	$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = \left[\begin{array}{c} \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{array}\right] = 0$
	$\longleftarrow$	$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$

Conversion semi-explicit $\leftrightarrow$ fully-implicit

Semi-explicit DAEs		Fully-Implicit DAEs
$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u})\\ 0 &= \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u})\end{aligned}$	$\longrightarrow$	$\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = \left[\begin{array}{c}\dot{\mathbf{x}} - \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u})\\ \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u})\end{array}\right] = 0$
$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v}\\ 0 &= \mathbf{F}(\mathbf{v},\mathbf{z},\mathbf{x},\mathbf{u})\end{aligned}$	$\longleftarrow$	$\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = 0$

Conversion semi-explicit $\leftrightarrow$ fully-implicit

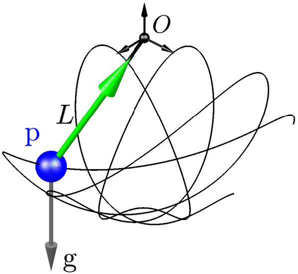
Semi-explicit DAEs		Fully-Implicit DAEs
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u})$ $0 = \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u})$	$\longrightarrow$	$\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u}) \\ \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u}) \end{bmatrix} = 0$
$\dot{\mathbf{x}} = \mathbf{v}$ $0 = \mathbf{F}(\mathbf{v},\mathbf{z},\mathbf{x},\mathbf{u})$	$\longleftarrow$	$\mathbf{F}(\dot{\mathbf{x}},\mathbf{z},\mathbf{x},\mathbf{u}) = 0$

Fully-implicit or semi-explicit DAEs are not *really* different.

- we like semi-explicit DAEs for their neat structure
- fully-implicit $\rightarrow$ semi-explicit adds variables $\mathbf{v}$, can be counter-productive

DAEs in mechanics - Example

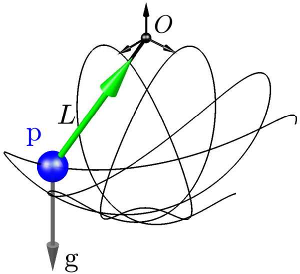
Pendulum simulation



- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass

DAEs in mechanics - Example

Pendulum simulation

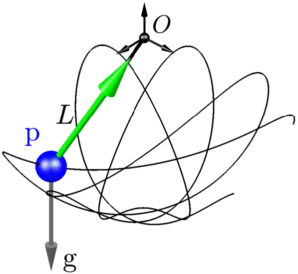


- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

DAEs in mechanics - Example

Pendulum simulation



- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

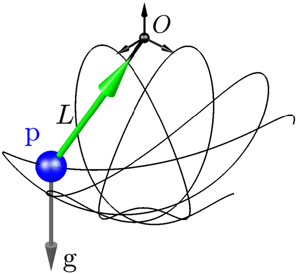
$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

DAEs in mechanics - Example

Pendulum simulation



- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

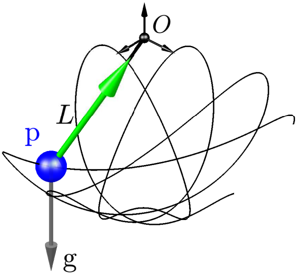
$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

- Motion:

$$m\ddot{\mathbf{p}} = -z\mathbf{p} - m\mathbf{g}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

DAEs in mechanics - Example

Pendulum simulation



Semi-explicit DAE:

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z \mathbf{p} - \mathbf{g}$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

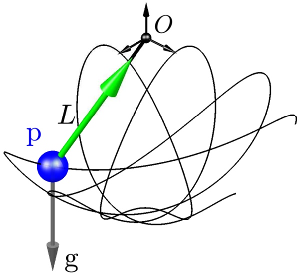
$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

- Motion:

$$m\ddot{\mathbf{p}} = -z\mathbf{p} - m\mathbf{g}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

DAEs in mechanics - Example

Pendulum simulation



Semi-explicit DAE:

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

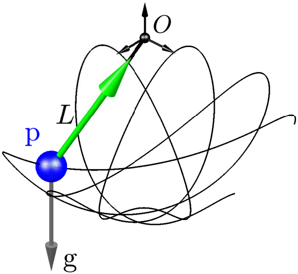
$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

- Motion:

$$m\ddot{\mathbf{p}} = -z\mathbf{p} - m\mathbf{g}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

DAEs in mechanics - Example

Pendulum simulation



Semi-explicit DAE:

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

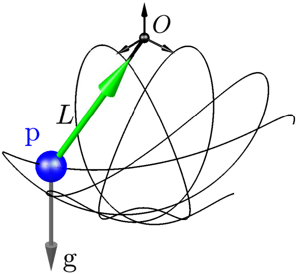
- Motion:

$$m\ddot{\mathbf{p}} = -z\mathbf{p} - m\mathbf{g}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

- Algebraic variable z "adjusts" acceleration to keep $\mathbf{p}$ at distance L from O

DAEs in mechanics - Example

Pendulum simulation



Semi-explicit DAE:

$$\dot{\mathbf{p}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

- Cartesian position $\mathbf{p} \in \mathbb{R}^3$, unit mass
- Lagrange function:

$$\mathcal{L} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} - mg\mathbf{p}_3 - z\mathbf{c}(\mathbf{p})$$

- Constraint:

$$\mathbf{c}(\mathbf{p}) = \mathbf{p}^\top \mathbf{p} - L^2$$

- Motion:

$$m\ddot{\mathbf{p}} = -z\mathbf{p} - m\mathbf{g}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

- Algebraic variable z "adjusts" acceleration to keep $\mathbf{p}$ at distance L from O
- DAE must hold this specification as a constraint

Tikhonov Theorem - What is it about?

Consider ODE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})\end{aligned}$$

with $\epsilon << 1$

Consider DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})\end{aligned}$$

approximation $\epsilon = 0$

Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

Tikhonov Theorem - What is it about?

Consider ODE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ \epsilon \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{aligned}$$

with $\epsilon << 1$

Consider DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{aligned}$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

Example

ODE:

$$\begin{aligned} \dot{x} &= -x + z \\ \epsilon \dot{z} &= x - 2z + u \end{aligned}$$

DAE approximation:

$$\begin{aligned} \dot{x} &= -x + z \\ 0 &= x - 2z + u \end{aligned}$$

Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

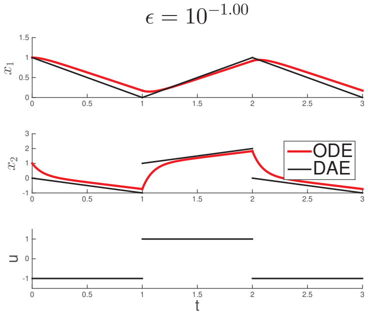
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

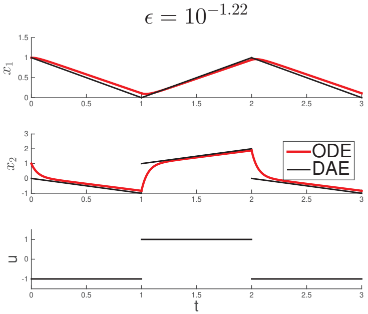
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon \dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

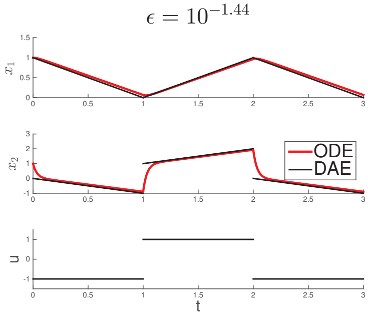
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

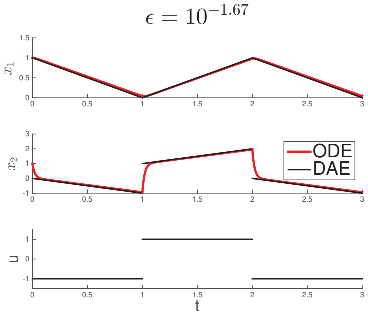
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

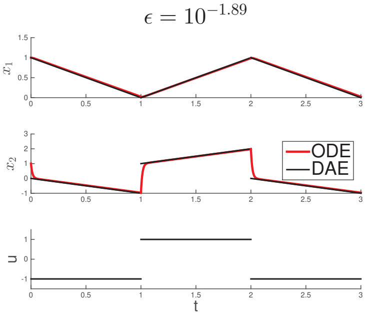
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

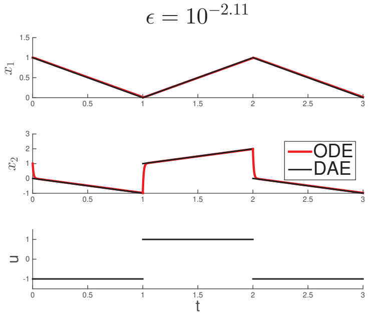
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

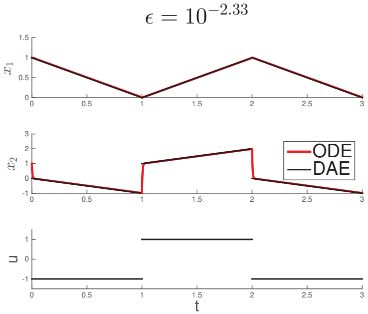
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

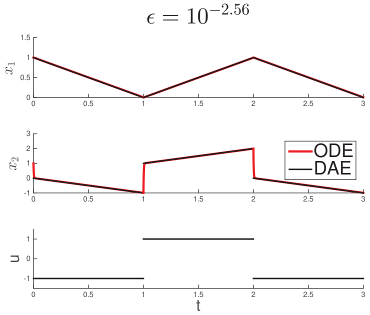
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

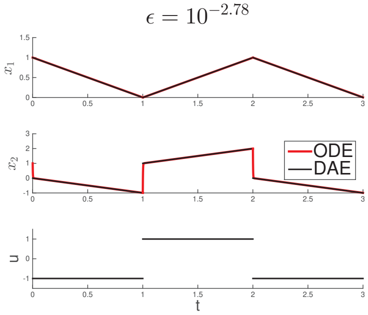
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

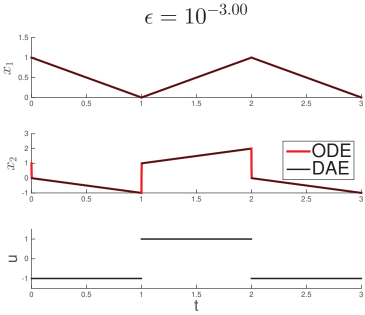
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



Tikhonov Theorem - What is it about?

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Consider DAE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

approximation $\epsilon = 0$

Is the DAE a good approximation of the ODE?

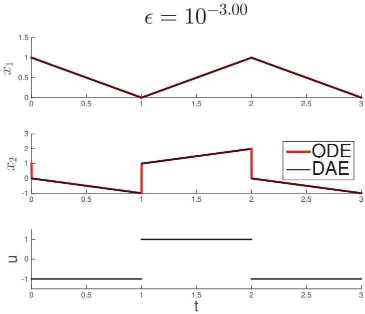
Example

ODE:

$$\dot{x} = -x + z$$
$$\epsilon\dot{z} = x - 2z + u$$

DAE approximation:

$$\dot{x} = -x + z$$
$$0 = x - 2z + u$$



For $\epsilon \to 0$, DAE matches ODE almost everywhere

Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

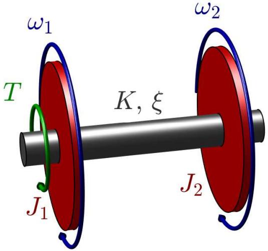
Tikhonov Theorem - Mechanical example

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$

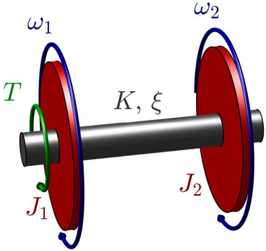
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$

ODE:

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
J_1 \dot{\omega}_1 &= T - K(\theta_1 - \theta_2) - \xi K(\omega_1 - \omega_2) \\
\dot{\theta}_2 &= \omega_2 \\
J_2 \dot{\omega}_2 &= -K(\theta_2 - \theta_1) - \xi K(\omega_2 - \omega_1)
\end{aligned}$$

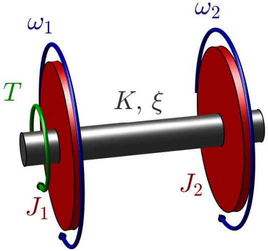
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$

ODE:

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
J_1 \dot{\omega}_1 &= T - K(\theta_1 - \theta_2) - \xi K(\omega_1 - \omega_2) \\
\dot{\theta}_2 &= \omega_2 \\
J_2 \dot{\omega}_2 &= -K(\theta_2 - \theta_1) - \xi K(\omega_2 - \omega_1)
\end{aligned}$$

- If shaft is highly rigid ($K >> J_1$), modelling its dynamics is an "overkilll"
- If neglected $\omega = \omega_1 = \omega_2$ and

$$\dot{\omega} = \frac{1}{J_1 + J_2} T$$

- Tikhonov provides a systematic way of neglecting fast dynamics

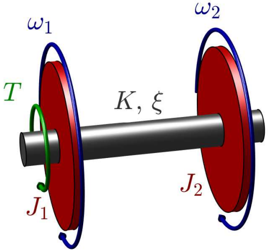
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$

ODE:

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\frac{J_1}{K}\dot{\omega}_1 &= \frac{T}{K} + \Delta + \xi\eta \\
\dot{\theta}_2 &= \omega_2 \\
J_2\dot{\omega}_2 &= -K\Delta - \xi K\eta
\end{aligned}$$

where $\Delta = \theta_2 - \theta_1$ and $\eta = \omega_2 - \omega_1$.

- If shaft is highly rigid ($K >> J_1$), modelling its dynamics is an "overkilll"
- If neglected $\omega = \omega_1 = \omega_2$ and

$$\dot{\omega} = \frac{1}{J_1 + J_2} T$$

- Tikhonov provides a systematic way of neglecting fast dynamics

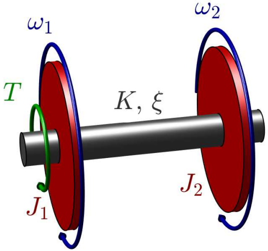
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$

ODE:

$$\dot{\Delta} = \eta$$
$$\frac{J_1}{K}\dot{\eta} = -\frac{T}{K} + \left(1 + \frac{J_1}{J_2}\right)(-\Delta - \xi\eta)$$
$$\dot{\theta}_2 = \omega_2$$
$$J_2\dot{\omega}_2 = -K\Delta - \xi K\eta$$

where $\Delta = \theta_2 - \theta_1$ and $\eta = \omega_2 - \omega_1$.

- If shaft is highly rigid ($K >> J_1$), modelling its dynamics is an "overkilll"
- If neglected $\omega = \omega_1 = \omega_2$ and

$$\dot{\omega} = \frac{1}{J_1 + J_2} T$$

- Tikhonov provides a systematic way of neglecting fast dynamics

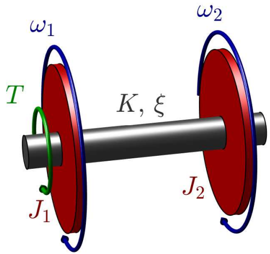
Tikhonov Theorem - Mechanical example

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$

ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} \omega_2 \\ -\frac{K}{J_2}\Delta - \frac{\xi K}{J_2}\eta \end{bmatrix}$$

$$\epsilon\dot{\mathbf{z}} = \begin{bmatrix} \frac{J_1}{K}\eta \\ -\frac{T}{K} + \left(1 + \frac{J_1}{J_2}\right)(-\Delta - \xi\eta) \end{bmatrix}$$

where $\mathbf{z} = \begin{bmatrix} \Delta \\ \eta \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \theta_2 \\ \omega_2 \end{bmatrix}$ and $\epsilon \equiv \frac{J_1}{K}$.

- If shaft is highly rigid ($K >> J_1$), modelling its dynamics is an "overkilll"
- If neglected $\omega = \omega_1 = \omega_2$ and

$$\dot{\omega} = \frac{1}{J_1 + J_2} T$$

- Tikhonov provides a systematic way of neglecting fast dynamics

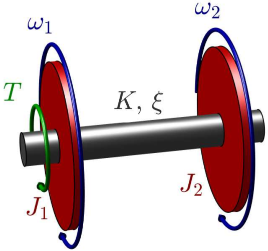
Tikhonov Theorem - Mechanical example

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$

ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} \omega_2 \\ -\frac{K}{J_2}\Delta - \frac{\xi K}{J_2}\eta \end{bmatrix}$$

$$\epsilon\dot{\mathbf{z}} = \begin{bmatrix} \frac{J_1}{K}\eta \\ -\frac{T}{K} + \left(1 + \frac{J_1}{J_2}\right)(-\Delta - \xi\eta) \end{bmatrix}$$

where $\mathbf{z} = \begin{bmatrix} \Delta \\ \eta \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \theta_2 \\ \omega_2 \end{bmatrix}$ and $\epsilon \equiv \frac{J_1}{K}$.

DAE approximation

$$\dot{\mathbf{x}} = \begin{bmatrix} \omega_2 \\ -\frac{K}{J_2}\Delta \end{bmatrix} = \begin{bmatrix} \omega_2 \\ \frac{1}{J_1+J_2}T \end{bmatrix}$$

$$\eta = 0$$

$$\Delta = -\frac{J_2}{K(J_1 + J_2)}T$$

... approximates "fast" dynamics by pretending they decay to their steady-state

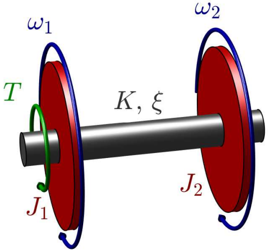
Tikhonov Theorem - Mechanical example

Consider ODE:

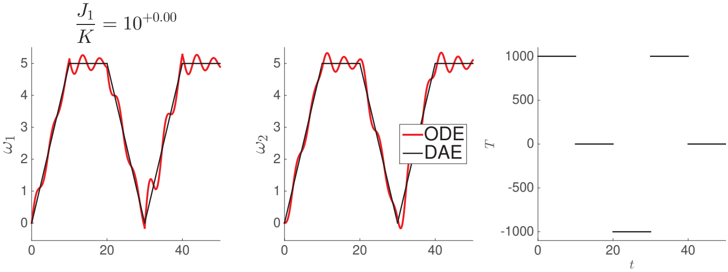
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$



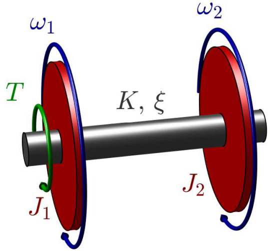
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

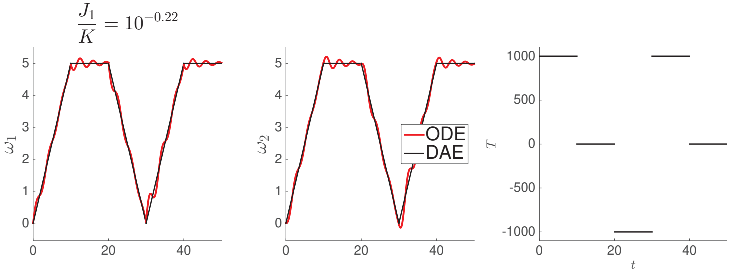
Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

with $\epsilon << 1$



with $J_1 << K$



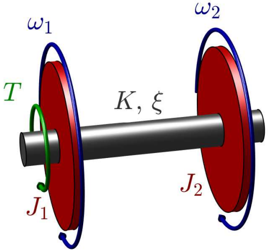
Tikhonov Theorem - Mechanical example

Systems often have "slow" & "fast" dynamics, e.g.

Consider ODE:

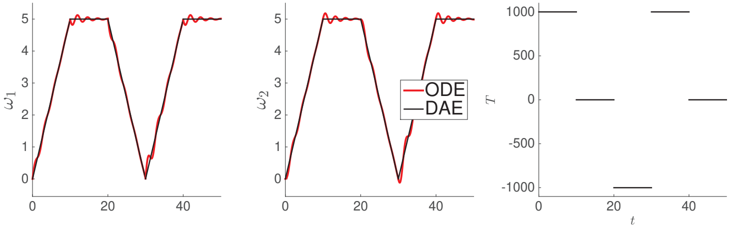
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$

$$\frac{J_1}{K} = 10^{-0.44}$$



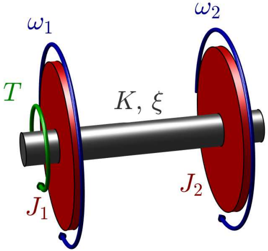
Tikhonov Theorem - Mechanical example

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

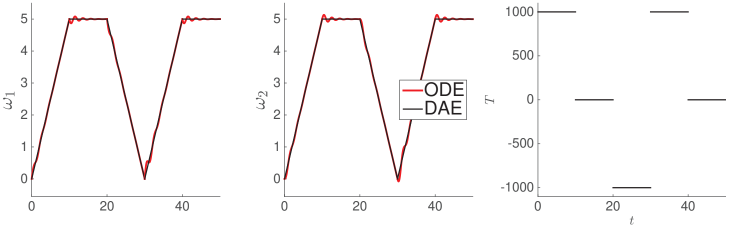
with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$

$$\frac{J_1}{K} = 10^{-0.67}$$



Tikhonov Theorem - Mechanical example

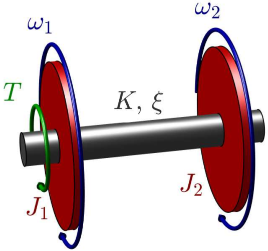
Systems often have “slow” & “fast” dynamics, e.g.

Consider ODE:

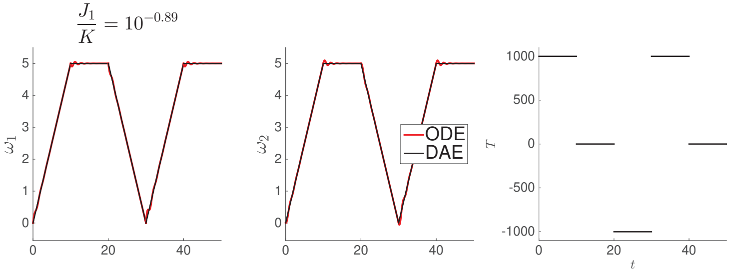
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

with $\epsilon << 1$



with $J_1 << K$



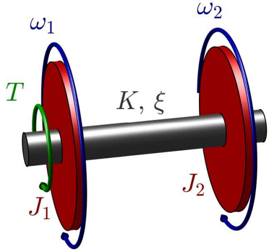
Tikhonov Theorem - Mechanical example

Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$\epsilon \dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

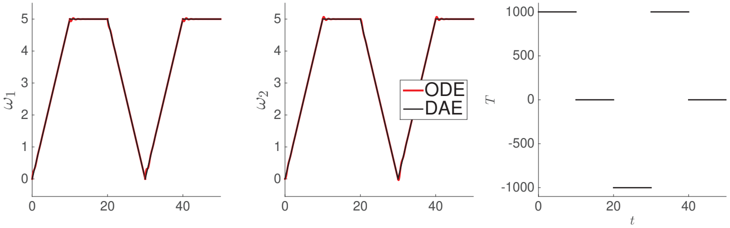
with $\epsilon << 1$

Systems often have "slow" & "fast" dynamics, e.g.



with $J_1 << K$

$$\frac{J_1}{K} = 10^{-1.11}$$



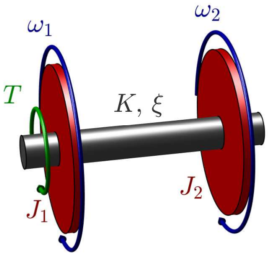
Tikhonov Theorem - Mechanical example

Systems often have “slow” & “fast” dynamics, e.g.

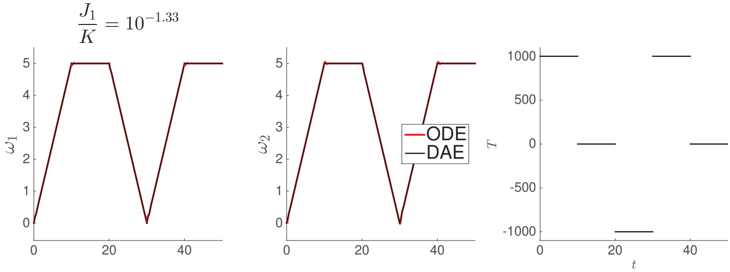
Consider ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{u})$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x},\mathbf{z},\mathbf{u})$$

with $\epsilon << 1$



with $J_1 << K$



Tikhonov Theorem - What does it say?

Tikhonov Theorem

Consider the ordinary differential equation (ODE):

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}\right) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}\right)\end{aligned}$$

of solution $\mathbf{x}_\epsilon(t)$, $\mathbf{z}_\epsilon(t)$ $\quad(\epsilon > 0)$

Tikhonov Theorem - What does it say?

Tikhonov Theorem

Consider the ordinary differential equation (ODE):

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}\right) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}\right) \end{aligned} \qquad \text{of solution } \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \qquad (\epsilon > 0)$$

Suppose:

- the dynamics $\dot{\mathbf{z}} = \mathbf{g}\left(\mathbf{x}, \mathbf{z}\right)$ are stable $\forall\, \mathbf{x}$
- matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank everywhere

Tikhonov Theorem - What does it say?

Tikhonov Theorem

Consider the ordinary differential equation (ODE):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) \qquad \epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z}) \qquad \text{of solution } \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \qquad (\epsilon > 0)$$

Suppose:

- the dynamics $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z})$ are stable $\forall\, \mathbf{x}$
- matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank everywhere

then

$$\lim_{\epsilon \to 0} \quad \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \quad = \quad \mathbf{x}_0(t),\ \mathbf{z}_0(t) \qquad \text{almost everywhere}$$

Tikhonov Theorem - What does it say?

Tikhonov Theorem

Consider the ordinary differential equation (ODE):

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z})\end{aligned}$$

of solution $\mathbf{x}_\epsilon(t)$, $\mathbf{z}_\epsilon(t)$ $(\epsilon > 0)$

Suppose:

- the dynamics $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z})$ are stable $\forall\, \mathbf{x}$
- matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank everywhere

then

$$\lim_{\epsilon \to 0} \quad \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \quad = \quad \mathbf{x}_0(t),\ \mathbf{z}_0(t)$$

almost everywhere

where $\mathbf{x}_0(t)$, $\mathbf{z}_0(t)$ is solution of DAE

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z})\end{aligned}$$

Tikhonov Theorem - What does it say?

Tikhonov Theorem

Consider the ordinary differential equation (ODE):

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ \epsilon \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}) \end{aligned} \qquad \text{of solution } \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \qquad (\epsilon > 0)$$

Suppose:

- the dynamics $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{x}, \mathbf{z})$ are stable $\forall\, \mathbf{x}$
- matrix $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank everywhere

then

$$\lim_{\epsilon \to 0} \quad \mathbf{x}_\epsilon(t),\ \mathbf{z}_\epsilon(t) \quad = \quad \mathbf{x}_0(t),\ \mathbf{z}_0(t) \qquad \text{almost everywhere}$$

where $\mathbf{x}_0(t)$, $\mathbf{z}_0(t)$ is solution of DAE

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}) \end{aligned}$$

In other words:

- Fast dynamics should "decay"
- $\mathbf{g}(\mathbf{x}, \mathbf{z}) = 0$ must be "solvable for $\mathbf{z}$" (see Implicit Function Theorem)

Simulating a DAE - "Easy" DAEs

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

What does it mean to be able to simulate a DAE "easily"?

Simulating a DAE - “Easy” DAEs

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

What does it mean to be able to simulate a DAE “easily”?

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

Simulating a DAE - "Easy" DAEs

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

What does it mean to be able to simulate a DAE "easily"?

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

Simulating a DAE - "Easy" DAEs

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

What does it mean to be able to simulate a DAE "easily"?

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

As long as the DAE delivers $\dot{\mathbf{x}}, \mathbf{z}$, simulating is "just" about following $\dot{\mathbf{x}}$.

Simulating a DAE - "Easy" DAEs

Simulating an ODE $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0$, requires solving for $\dot{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{u}$ on the trajectory

What does it mean to be able to simulate a DAE "easily"?

- for a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5a}$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \tag{5b}$$

  (5a) delivers $\dot{\mathbf{x}}$ if $\mathbf{z}$ is known, $\mathbf{z}$ must be provided by (5b)

- for a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

  both $\dot{\mathbf{x}}, \mathbf{z}$ must be provided by $\mathbf{F} = 0$

As long as the DAE delivers $\dot{\mathbf{x}}, \mathbf{z}$, simulating is "just" about following $\dot{\mathbf{x}}$.

When can the algebraic equation

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\mathbf{z}$?

When can the DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$?

"Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{aligned}$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$?

"Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$?

Reminder - IFT

If $\mathbf{F}(\mathbf{y}, \mathbf{x})$ is smooth, and $\frac{\partial \mathbf{F}(\mathbf{y}, \mathbf{x})}{\partial \mathbf{y}}$ always full rank then $\mathbf{y} = \mathbf{f}(\mathbf{x})$ exists such that

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0 \quad \text{for all } \mathbf{x}$$

"Easy" DAEs - Semi-explicit case

When can a semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

be solved for $\dot{\mathbf{x}}$, $\mathbf{z}$?

Reminder - IFT

If $\mathbf{F}(\mathbf{y}, \mathbf{x})$ is smooth, and $\frac{\partial \mathbf{F}(\mathbf{y},\mathbf{x})}{\partial \mathbf{y}}$ always full rank then $\mathbf{y} = \mathbf{f}(\mathbf{x})$ exists such that

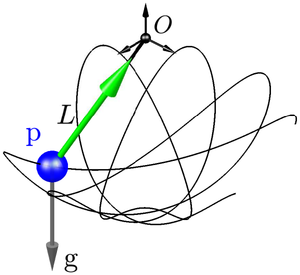
$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0 \quad \text{for all } \mathbf{x}$$

Semi-explicit DAE case

- Getting $\dot{\mathbf{x}}$ from the first equation is trivial
- **I**mplicit **F**unction **T**heorem says that we can solve $\mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) = 0$ for $\mathbf{z}$

$$\underline{\text{if}} \quad \text{(square) Jacobian} \quad \frac{\partial \mathbf{g}}{\partial \mathbf{z}} \quad \text{is full rank}$$

Back to our mechanical example



Semi-explicit DAE:

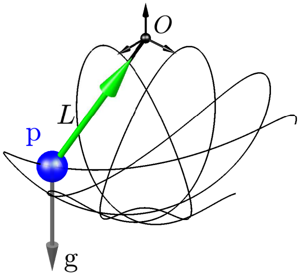
$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\tfrac{1}{m} z\mathbf{p} - \mathbf{g}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \tag{6b}$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\tfrac{1}{m} z \mathbf{p} - \mathbf{g} \end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

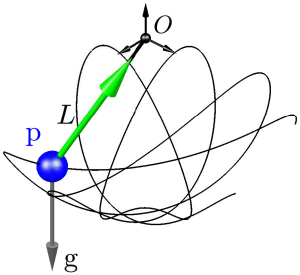
$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\; \} \equiv \mathbf{g} \tag{6b}$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\tfrac{1}{m}z\mathbf{p} - \mathbf{g}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top\mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \tag{6b}$$

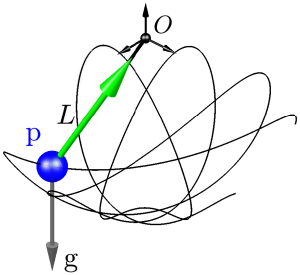
State

$$\mathbf{x} = \begin{bmatrix}\mathbf{p}\\ \mathbf{v}\end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top\mathbf{p} - L^2$

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\tfrac{1}{m} z \mathbf{p} - \mathbf{g} \end{aligned}\right\} \equiv \mathbf{f} \qquad (6a)$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \qquad (6b)$$

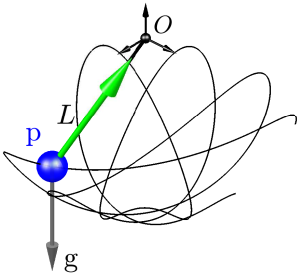
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, z does not enter in $\mathbf{g}$!

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\tfrac{1}{m} z \mathbf{p} - \mathbf{g}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \tag{6b}$$

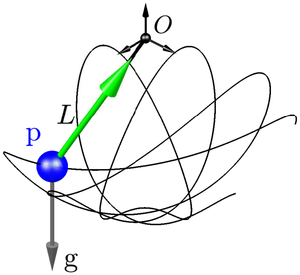
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, z does not enter in $\mathbf{g}$!
- (6b) cannot be solved for z...

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{array}{l}\dot{\mathbf{p}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}\end{array}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\Big\} \equiv \mathbf{g} \tag{6b}$$

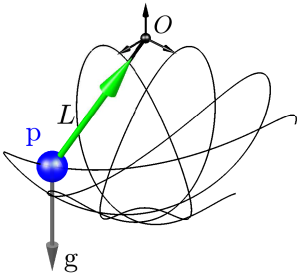
State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, z does not enter in $\mathbf{g}$!
- (6b) cannot be solved for z...
- (6a) delivers $\dot{\mathbf{x}}$ if we know z...

Back to our mechanical example



Semi-explicit DAE:

$$\left.\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\tfrac{1}{m} z\mathbf{p} - \mathbf{g}\end{aligned}\right\} \equiv \mathbf{f} \tag{6a}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \;\} \equiv \mathbf{g} \tag{6b}$$

State

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad z$$

Is that an "easy" DAE?

- Algebraic part is $\mathbf{g} = \mathbf{p}^\top \mathbf{p} - L^2$
- Jacobian $\frac{\partial \mathbf{g}}{\partial z} = 0$. In fact, z does not enter in $\mathbf{g}$!
- (6b) cannot be solved for z...
- (6a) delivers $\dot{\mathbf{x}}$ if we know z...

It is not an "easy" DAE

“Easy” DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$?

"Easy" DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$?

Reminder - IFT

If $\mathbf{F}\left(\mathbf{y}, \mathbf{x}\right)$ is smooth, and $\frac{\partial \mathbf{F}(\mathbf{y},\mathbf{x})}{\partial \mathbf{y}}$ always full rank then $\mathbf{y} = \mathbf{f}\left(\mathbf{x}\right)$ exists such that

$$\mathbf{F}\left(\mathbf{f}\left(\mathbf{x}\right), \mathbf{x}\right) = 0 \quad \text{for all } \mathbf{x}$$

"Easy" DAEs - Fully-implicit case

When can a fully-implicit DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

be solved for both $\dot{\mathbf{x}}$, $\mathbf{z}$?

Reminder - IFT

If $\mathbf{F}(\mathbf{y}, \mathbf{x})$ is smooth, and $\frac{\partial \mathbf{F}(\mathbf{y},\mathbf{x})}{\partial \mathbf{y}}$ always full rank then $\mathbf{y} = \mathbf{f}(\mathbf{x})$ exists such that

$$\mathbf{F}(\mathbf{f}(\mathbf{x}), \mathbf{x}) = 0 \quad \text{for all } \mathbf{x}$$

Fully-Implicit DAE case

Implicit **F**unction **T**heorem says that we can solve $\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$ for $\dot{\mathbf{x}}$, $\mathbf{z}$

$$\underline{\text{if}} \quad \text{(square) Jacobian } \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix} \quad \text{is full rank}$$

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

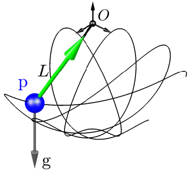
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



“Hard” DAEs

Definition (informal): “hard” DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

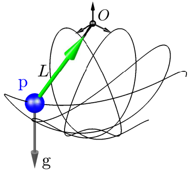
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

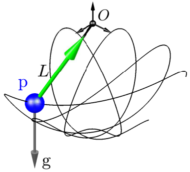
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

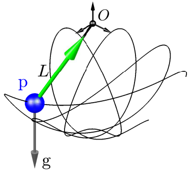
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}}$$

"Hard" DAEs

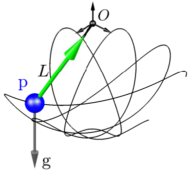
Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z \mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}}$$

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

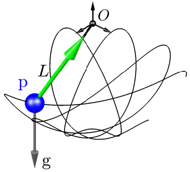
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p}$$

“Hard” DAEs

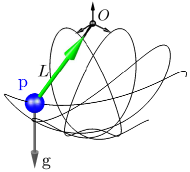
Definition (informal): “hard” DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z “adjusts” $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of “influence”:

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

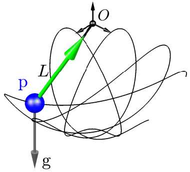
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects z to (7c) via **2 integrations**

"Hard" DAEs

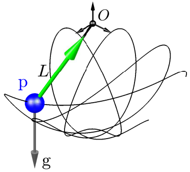
Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects z to (7c) via **2 integrations**

- How to reconnect z to (7c) *algebraically*? (equation we could "solve for z"...)

"Hard" DAEs

Definition (informal): "hard" DAEs do not *readily* deliver $\dot{\mathbf{x}}$, $\mathbf{z}$...

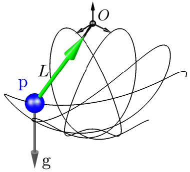
... but they (may) deliver well-defined trajectories nonetheless!!

Example: how do we get a trajectory from

$$\dot{\mathbf{p}} = \mathbf{v} \tag{7a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{7b}$$

$$0 = \mathbf{p}^\top \mathbf{p} - L^2 \tag{7c}$$



Reminder:

- z "adjusts" $\dot{\mathbf{v}}$ so that (7c) holds
- Chain of "influence":

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} = \dot{\mathbf{p}} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} \text{(7c)}$$

connects z to (7c) via **2 integrations**

- How to reconnect z to (7c) *algebraically*? (equation we could "solve for z"...)

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ twice on (7c) to "rewind" the chain to z

Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to z

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top\mathbf{p} - L^2$, to "rewind" the chain to z

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top\mathbf{p} - L^2$$

$$\boxed{\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2 \end{aligned}} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \boxed{\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top\mathbf{v} \end{aligned}} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to z

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned}$$

Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to z

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned}$$

Yields "easy" DAE:

$$\dot{\mathbf{p}} = \mathbf{v} \tag{8a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g} \tag{8b}$$

$$0 = -\frac{1}{m}\mathbf{p}^\top \mathbf{p} z - \mathbf{p}^\top \mathbf{g} + \mathbf{v}^\top \mathbf{v}^\top \tag{8c}$$

as (8c) delivers z (if $\mathbf{p} \neq 0$)

Solving "Hard" DAEs

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on $0 = \mathbf{p}^\top \mathbf{p} - L^2$, to "rewind" the chain to z

$$z \xrightarrow{\text{alg.}} \dot{\mathbf{v}} \xrightarrow{\int \mathrm{d}t} \mathbf{v} \xrightarrow{\int \mathrm{d}t} \mathbf{p} \xrightarrow{\text{alg.}} 0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{v} \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned}$$

Yields "easy" DAE:

$$\dot{\mathbf{p}} = \mathbf{v} \tag{8a}$$

$$\dot{\mathbf{v}} = -\frac{1}{m} z \mathbf{p} - \mathbf{g} \tag{8b}$$

$$0 = -\frac{1}{m} \mathbf{p}^\top \mathbf{p} z - \mathbf{p}^\top \mathbf{g} + \mathbf{v}^\top \mathbf{v}^\top \tag{8c}$$

as (8c) delivers z (if $\mathbf{p} \neq 0$)

$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$

One more $\frac{\mathrm{d}}{\mathrm{d}t}$ on (8c) **turns** (8) **into an ODE**

Differential Index

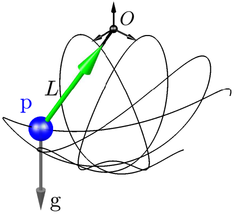
Definition

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Differential Index

Definition

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE



Pendulum example

Index 3

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned}$$

Physical model
"Hard" DAE

$$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

Index 1

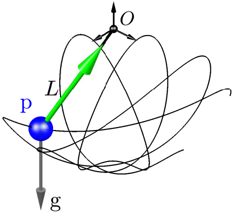
$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z \mathbf{p} - \mathbf{g} \\ 0 &= 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v} \end{aligned}$$

2 time-differentiations
→ "Easy" DAE

$$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

ODE

Differential Index



Definition

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Pendulum example

Index 3

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

Physical model
"Hard" DAE

$$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

Index 1

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$
$$0 = 2\mathbf{p}^\top \dot{\mathbf{v}} + 2\mathbf{v}^\top \mathbf{v}$$

2 time-differentiations
$\rightarrow$ "Easy" DAE

$$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

ODE

The transformation **index-n** $\xrightarrow{\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}}$ **index-1** is called **index reduction**

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right) \end{aligned}$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

Index-1 fully-implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

Index-1 fully-implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + +\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

Index-1 fully-implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

Index-1 DAEs

The differential index of a DAE is the number of $\frac{\mathrm{d}}{\mathrm{d}t}$ needed to turn it into an ODE

Index-1 semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\end{aligned}$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g} = 0$ yields:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)\\ \dot{\mathbf{z}} &= -\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} + + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ must be full rank.

Index-1 fully-implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}\right) = 0$$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F} = 0$ yields:

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\ddot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\dot{\mathbf{z}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

and should give us the ODE:

$$\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\dot{\mathbf{u}} = 0$$

where $\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}$ must be full rank.

Delivers $\ddot{\mathbf{x}}$ and $\dot{\mathbf{z}}$.

Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by 1^{st} equations
- DAE can be "easily" simulated

Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by 1^{st} equations
- DAE can be "easily" simulated

Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

Index-1 semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})\end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by 1^{st} equations
- DAE can be "easily" simulated

Index-1 fully-implicit DAE:

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

Index-1 DAEs are "easy" DAEs - Connection to IFT

Index-1 DAEs readily deliver $\dot{\mathbf{x}}, \mathbf{z}$ and are therefore "easy"

Index-1 semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}) \end{aligned}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank.

Then

- IFT: $\mathbf{g} = 0$ can be solved for $\mathbf{z}$
- $\dot{\mathbf{x}}$ is delivered by 1^{st} equations
- DAE can be "easily" simulated

Index-1 fully-implicit DAE:

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{z}, \mathbf{x}, \mathbf{u}) = 0$$

where $\begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{bmatrix}$ is full rank.

Then

- IFT: $\mathbf{F} = 0$ can be solved for $\dot{\mathbf{x}}, \mathbf{z}$
- DAE can be "easily" simulated

Index reduction for semi-explicit DAEs - A general view

High-index semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

Algorithm

Index reduction for semi-explicit DAEs - A general view

High-index semi-explicit DAE

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x},\mathbf{z},\mathbf{u}\right)\\ 0 &= \mathbf{g}\left(\mathbf{x},\mathbf{z},\mathbf{u}\right)\end{aligned}$$

Algorithm

1. Check if the DAE system is index 1 (i.e. $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ full rank). If yes, stop.
2. Identify a subset of algebraic equations that **can be solved** for a subset of algebraic variables.
3. Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on the remaining algebraic equations that contain the differential variables $\mathbf{x}_j$.
4. Terms $\dot{\mathbf{x}}_j$ will appear in these differentiated equations.
5. Substitute the $\dot{\mathbf{x}}_j$ with $\mathbf{f}_j\left(\mathbf{x},\mathbf{z},\mathbf{u}\right)$. This leads to new algebraic equations.
6. With this new DAE system, go to step 1.

Index reduction for semi-explicit DAEs - A general view

High-index semi-explicit DAE

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \\ 0 &= \mathbf{g}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right) \end{aligned}$$

Algorithm

1. Check if the DAE system is index 1 (i.e. $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ full rank). If yes, stop.
2. Identify a subset of algebraic equations that **can be solved** for a subset of algebraic variables.
3. Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on the remaining algebraic equations that contain the differential variables $\mathbf{x}_j$.
4. Terms $\dot{\mathbf{x}}_j$ will appear in these differentiated equations.
5. Substitute the $\dot{\mathbf{x}}_j$ with $\mathbf{f}_j\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$. This leads to new algebraic equations.
6. With this new DAE system, go to step 1.

Index reduction for semi-explicit DAEs - A general view

High-index semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
$$0 = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$

Algorithm

1. Check if the DAE system is index 1 (i.e. $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ full rank). If yes, stop.
2. Identify a subset of algebraic equations that **can be solved** for a subset of algebraic variables.
3. Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ on the remaining algebraic equations that contain the differential variables $\mathbf{x}_j$.
4. Terms $\dot{\mathbf{x}}_j$ will appear in these differentiated equations.
5. Substitute the $\dot{\mathbf{x}}_j$ with $\mathbf{f}_j(\mathbf{x}, \mathbf{z}, \mathbf{u})$. This leads to new algebraic equations.
6. With this new DAE system, go to step 1.

Writing a general-purpose "Index-reduction algorithm" can be very tricky, as one of the steps is not easily automated

Differential index & Index reduction - Examples

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \qquad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \qquad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

DAE index-1 **ODE** **ODE**

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

DAE index-1 | **ODE** | **ODE**

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_2 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}}$$

Differential index & Index reduction - Examples

DAE index-1 **ODE** **ODE**

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_2 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_1 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

DAE index-1 | **ODE** | **ODE**

$$\begin{aligned} z &= u - x_2 \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= u - x_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{z} &= \dot{u} - \dot{x}_2 \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= u - x_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{z} &= \dot{u} - x_1 \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= u - x_2 \end{aligned}$$

$$\begin{aligned} \dot{x}_1 - z &= 0 \\ \dot{x}_2 - x_1 &= 0 \\ x_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{x}_1 - z &= 0 \\ \dot{x}_2 - x_1 &= 0 \\ \dot{x}_2 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{x}_1 - z &= 0 \\ \dot{x}_2 - x_1 &= 0 \\ x_1 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned} \dot{x}_1 - z &= 0 \\ \dot{x}_2 - x_1 &= 0 \\ \dot{x}_1 - \ddot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}}$$

Differential index & Index reduction - Examples

DAE index-1 **ODE** **ODE**

$$\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_2 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_1 - \dot{u} &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_1 - \ddot{u} &= 0 \end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ z - \ddot{u} &= 0 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

Differential index & Index reduction - Examples

DAE index-1 **ODE** **ODE**

$$\begin{aligned}\mathbf{z} &= u - \mathbf{x}_2\\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1\\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned}\dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2\\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1\\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2\end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned}\dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1\\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1\\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{x}}_1 - z &= 0\\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0\\ \mathbf{x}_2 - u &= 0\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned}\dot{\mathbf{x}}_1 - z &= 0\\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0\\ \dot{\mathbf{x}}_2 - \dot{u} &= 0\end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned}\dot{\mathbf{x}}_1 - z &= 0\\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0\\ \mathbf{x}_1 - \dot{u} &= 0\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned}\dot{\mathbf{x}}_1 - z &= 0\\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0\\ \dot{\mathbf{x}}_1 - \ddot{u} &= 0\end{aligned} \quad \xrightarrow{\text{Alg.}} \quad \begin{aligned}\dot{\mathbf{x}}_1 - z &= 0\\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0\\ z - \ddot{u} &= 0\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \begin{aligned}\dot{\mathbf{x}}_1 &= z\\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1\\ \dot{z} &= \dddot{u}\end{aligned}$$

Differential index & Index reduction - Examples

DAE index-1 $\xrightarrow{\frac{d}{dt}}$ **ODE** $\xrightarrow{\text{Alg.}}$ **ODE**

$$
\begin{aligned} \mathbf{z} &= u - \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}
\quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad
\begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}
\quad \xrightarrow{\text{Alg.}} \quad
\begin{aligned} \dot{\mathbf{z}} &= \dot{u} - \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= u - \mathbf{x}_2 \end{aligned}
$$

DAE index 3 $\xrightarrow{\frac{d}{dt}}$ **DAE index 2** $\xrightarrow{\text{Alg.}}$ **DAE index 2**

$$
\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_2 - u &= 0 \end{aligned}
\quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad
\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_2 - \dot{u} &= 0 \end{aligned}
\quad \xrightarrow{\text{Alg.}} \quad
\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \mathbf{x}_1 - \dot{u} &= 0 \end{aligned}
\quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}
$$

DAE index 1 $\xrightarrow{\text{Alg.}}$ **DAE index 1** $\xrightarrow{\frac{d}{dt}}$ **ODE**

$$
\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ \dot{\mathbf{x}}_1 - \ddot{u} &= 0 \end{aligned}
\quad \xrightarrow{\text{Alg.}} \quad
\begin{aligned} \dot{\mathbf{x}}_1 - z &= 0 \\ \dot{\mathbf{x}}_2 - \mathbf{x}_1 &= 0 \\ z - \ddot{u} &= 0 \end{aligned}
\quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad
\begin{aligned} \dot{\mathbf{x}}_1 &= z \\ \dot{\mathbf{x}}_2 &= \mathbf{x}_1 \\ \dot{z} &= \dddot{u} \end{aligned}
$$

Consistency conditions - Pendulum example

Pendulum equations:

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

Consistency conditions - Pendulum example

Pendulum equations:

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m}z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m}z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned}$$

Consistency conditions - Pendulum example

Pendulum equations:

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2 \end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g} \\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v} \end{aligned}$$

Original model imposes

$$\mathbf{p}^\top \mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top \mathbf{p}\right) = 0 \tag{10}$$

Does (10) $\Rightarrow$ (9)**?** I.e. does index-1 model match original model?

Consistency conditions - Pendulum example

Pendulum equations:

$$\dot{\mathbf{p}} = \mathbf{v}, \quad \dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}, \quad 0 = \mathbf{p}^\top \mathbf{p} - L^2 \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \dot{\mathbf{p}} = \mathbf{v}, \quad \dot{\mathbf{v}} = -\frac{1}{m} z\mathbf{p} - \mathbf{g}, \quad 0 = \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v}$$

Original model imposes

$$\mathbf{p}^\top \mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top \mathbf{p}\right) = 0 \tag{10}$$

Does (10) $\Rightarrow$ (9)**?** I.e. does index-1 model match original model?

Let's write $c(\mathbf{p}(t)) = \mathbf{p}(t)^\top \mathbf{p}(t) - L^2$

Consistency conditions - Pendulum example

Pendulum equations:

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m}z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top\mathbf{p} - L^2\end{aligned} \qquad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \qquad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m}z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top\dot{\mathbf{v}} + \mathbf{v}^\top\mathbf{v}\end{aligned}$$

Original model imposes

$$\mathbf{p}^\top\mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top\mathbf{p}\right) = 0 \tag{10}$$

Does (10) $\Rightarrow$ (9)**?** I.e. does index-1 model match original model?

Let's write $c(\mathbf{p}(t)) = \mathbf{p}(t)^\top\mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c} = 0 \quad \Rightarrow \quad c(\mathbf{p}(t)) = c(\mathbf{p}(0)) + \dot{c}(\mathbf{p}(0), \dot{\mathbf{p}}(0))t$$

Consistency conditions - Pendulum example

Pendulum equations:

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z \mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \mathbf{p} - L^2$$

$$\xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}}$$

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = -\frac{1}{m} z \mathbf{p} - \mathbf{g}$$
$$0 = \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v}$$

Original model imposes

$$\mathbf{p}^\top \mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\mathbf{p}^\top \mathbf{p} \right) = 0 \tag{10}$$

Does (10) $\Rightarrow$ (9)**?** I.e. does index-1 model match original model?

Let's write $c(\mathbf{p}(t)) = \mathbf{p}(t)^\top \mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c} = 0 \quad \Rightarrow \quad c(\mathbf{p}(t)) = c(\mathbf{p}(0)) + \dot{c}(\mathbf{p}(0), \dot{\mathbf{p}}(0))t$$

Hence index-1 model imposes $c(t) = 0$ for all t iff:

$$c(\mathbf{p}(0)) \qquad \dot{c}(\mathbf{p}(0), \dot{\mathbf{p}}(0)) = 0$$

Consistency conditions - Pendulum example

Pendulum equations:

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top \mathbf{p} - L^2\end{aligned} \quad \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v}\\ \dot{\mathbf{v}} &= -\frac{1}{m} z\mathbf{p} - \mathbf{g}\\ 0 &= \mathbf{p}^\top \dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{v}\end{aligned}$$

Original model imposes

$$\mathbf{p}^\top \mathbf{p} = L^2 \tag{9}$$

Index-1 model imposes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathbf{p}^\top \mathbf{p}\right) = 0 \tag{10}$$

Does (10) $\Rightarrow$ (9)**?** I.e. does index-1 model match original model?

Let's write $c(\mathbf{p}(t)) = \mathbf{p}(t)^\top \mathbf{p}(t) - L^2$ then index-1 model yields:

$$\ddot{c} = 0 \quad \Rightarrow \quad c(\mathbf{p}(t)) = c(\mathbf{p}(0)) + \dot{c}(\mathbf{p}(0), \dot{\mathbf{p}}(0))t$$

Hence index-1 model imposes $c(t) = 0$ for all t iff:

$$c(\mathbf{p}(0)) \qquad \dot{c}(\mathbf{p}(0), \dot{\mathbf{p}}(0)) = 0$$

These are called consistency conditions. Must be satisfied by $\mathbf{x}(0)$.

Consistency conditions