Stability of equilibrium points

Learning goals

After this set of video lectures you will...

- Understand how the need for stabilization of equilibrium points arises in control problems
- **Lyapunov stability properties**
  Know and understand the following stability definitions for time-invariant (autonomous) systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local versus global
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

Literature

The video lectures are based on

Khalil	Section 4.1
	Theorem 4.7, Section 4.3
	Corollary 4.3, Section 4.7

The control problem

- Given **a physical system** (a process) with
  - inputs
  - outputs

- Given **a set of specifications** of the desired system behavior

- **Model** the physical system by a set of differential equations

- **Design** a control law $\rightleftarrows$ - **Analysis** of the closed-loop system

- Implement the control law

The regulation problem: x_{ref} = constant

Process model

$$\dot{x} = f_p(t, x, u)$$

Control law design

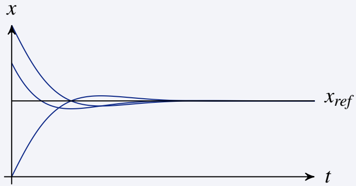
Find

$$u = \gamma(t, x)$$

such that the closed-loop (CL) system

$$\dot{x} = f_p(t, x, \gamma(t, x)) =: f(t, x)$$

has a desired behavior.



Desired CL system behavior?

- x_{ref} an equilibrium point
- convergence
- start close $\Rightarrow$ stay close

Asymptotic stabilization problem

Find $\gamma(t, e)$, $e = x - x_{ref}$, such that $e = 0$ is an asymptotically stable equilibrium point of $\dot{e} = f(t, e + x_{ref})$.

The tracking problem: $x_{ref}(t)$

Process

$$\dot{x} = f_p(t, x, u)$$

Control law design

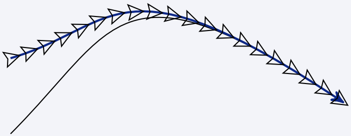
Find

$$u = \gamma(t, x)$$

such that the closed-loop (CL) system

$$\dot{x} = f_p(t, x, \gamma(t, x)) =: f(t, x)$$

has a desired behavior.



Desired CL system behavior?

- on trajectory $\Rightarrow$ stay on trajectory
- convergence to trajectory
- start close $\Rightarrow$ stay close

Asymptotic stabilization problem

Find $\gamma(t, e)$ such that $e = x - x_{ref}(t) = 0$ is an asymptotically stable equilibrium point of $\dot{e} = \bar{f}_p(t, e, \gamma(t, e)) = f(t, e)$.

Lyapunov stability

Aleksandr Mikhailovich Lyapunov

1857-1918

- Russian mathematician, mechanician and physicist
- Known for his development of the stability theory of dynamical systems

Lyapunov stability properties

Closed-loop system

$$\dot{x} = f(t, x)$$

$x = 0$ is the equilibrium point of interest

Lyapunov stability properties

Time-invariant systems

$$\dot{x} = f(x), \qquad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$$

f is locally Lipschitz

$x = 0$ is the equilibrium point of interest

We will define the following Lyapunov stability properties:

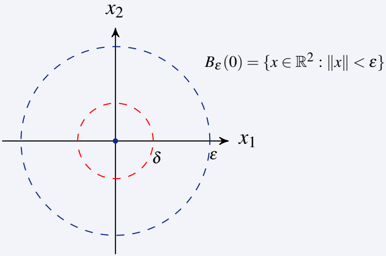
- Stability
- Asymptotic stability
- Exponential stability
- Global asymptotic stability
- Global exponential stability

Definition (Stability)

$x = 0$ is <u>stable</u> iff

$$\forall\ \varepsilon > 0 \quad \exists\ \delta(\varepsilon) > 0 \quad s.t. \quad \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$$

Visual interpretation ($n = 2$):

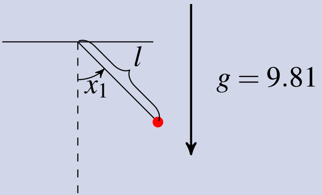


Note: $\forall\ \varepsilon$
Else: Unstable

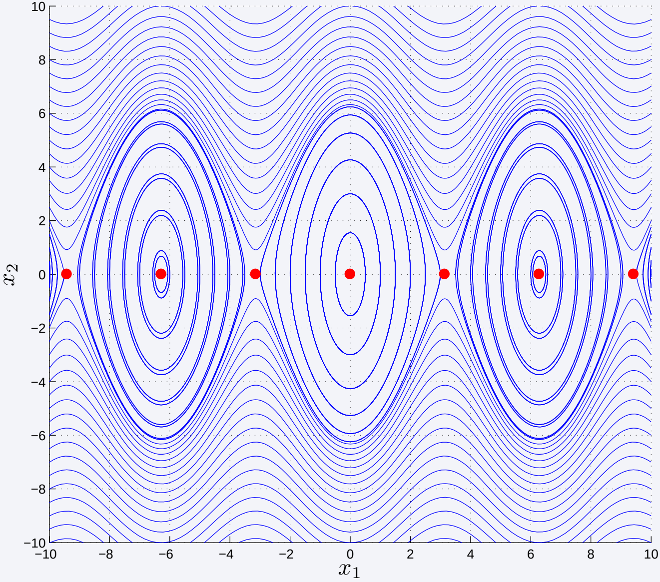
Stability: Example

Pendulum without friction

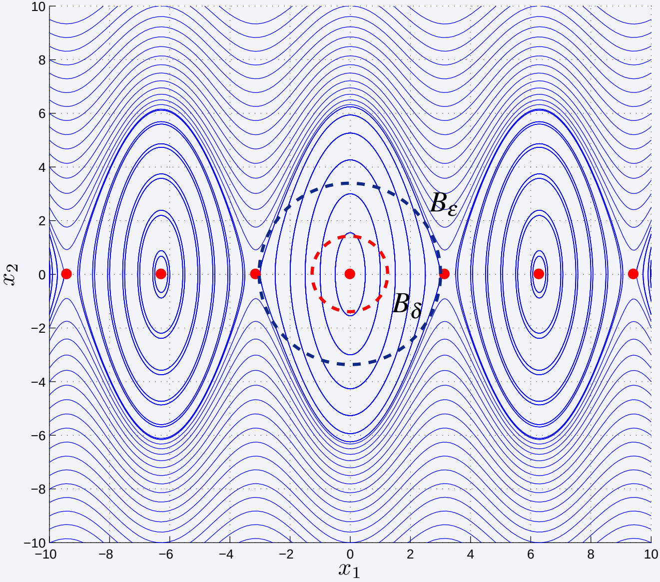
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$



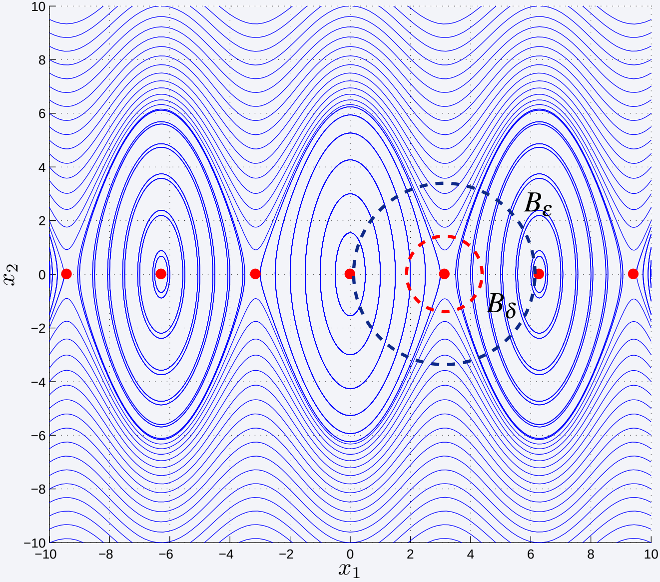
Pendulum without friction, cont.



Pendulum without friction, cont.



Pendulum without friction, cont.



Asymptotic stability

Asymptotic stability

The equilibrium point $x = 0$ is (locally) asymptotically stable iff

i) it is stable

ii) $\exists r > 0 \quad \text{s.t.} \quad \|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$ (Convergence)

Region of attraction

$$B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$$

Global asymptotic stability

The equilibrium point $x = 0$ is globally asymptotically stable iff

i) it is stable

ii) $\forall x(0) \qquad \lim_{t\to\infty} x(t) = 0$

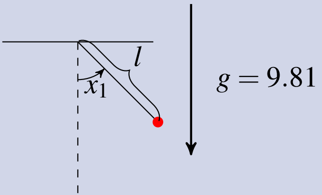
NB: This implies that $x = 0$ is the only equilibrium point.

Asymptotic stability: Pendulum with friction

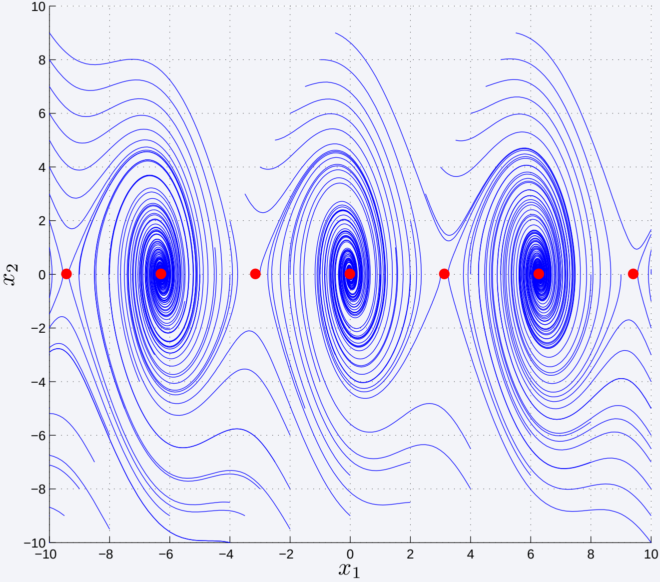
Pendulum with friction

$$\dot{x}_1 = x_2$$

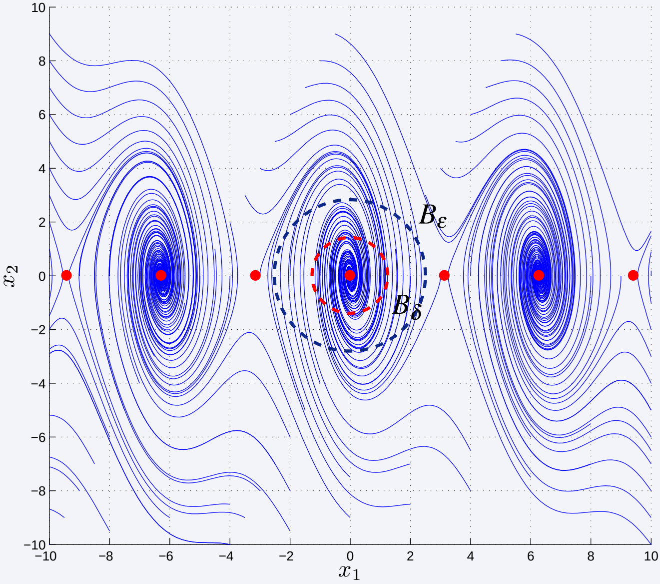
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



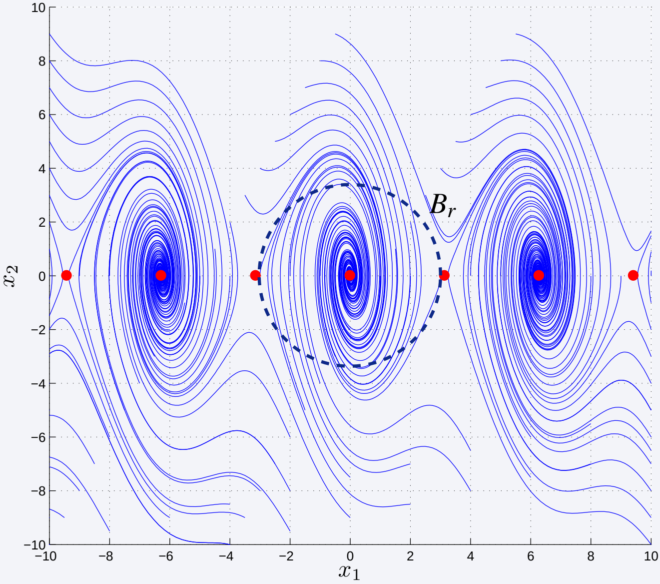
Pendulum with friction, cont.



Pendulum with friction, cont.



Pendulum with friction, cont.



Convergence

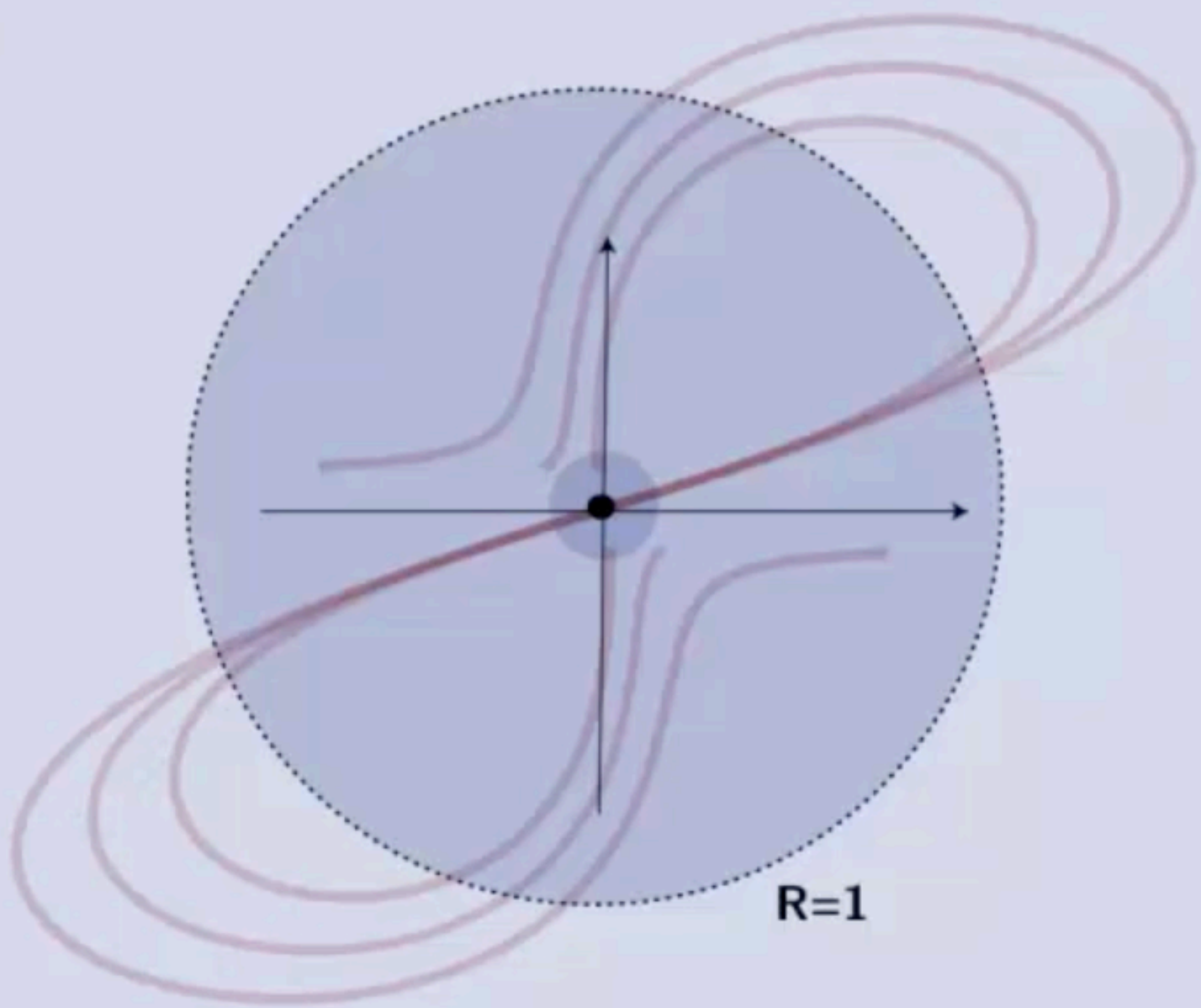
Convergence

$$\|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$$

NB!

Convergence $\not\Rightarrow$ Stability

Vinograd's counterexample:



Exponential stability

Exponential stability

The equilibrium point $x = 0$ is (locally) exponentially stable iff $\exists r, k, \lambda > 0$ such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \leq k \, \|x(0)\| \, e^{-\lambda t} \quad \forall t \geq 0$$

Region of attraction

$$B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$$

Exponential stability

Exponential stability

The equilibrium point $x = 0$ is (locally) exponentially stable iff $\exists r, k, \lambda > 0$ such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \leq k \|x(0)\| e^{-\lambda t} \quad \forall t \geq 0$$

Global exponential stability

The equilibrium point $x = 0$ is globally exponentially stable iff $\exists k, \lambda > 0$ such that

$$\forall x(0) \qquad \|x(t)\| \leq k \|x(0)\| e^{-\lambda t} \quad \forall t \geq 0$$

Remark

exponential stability $\Rightarrow$ asymptotic stability

Lyapunov stability properties

We have defined the following stability properties:

- Stability
- Asymptotic stability = Stability + Local convergence
- Exponential stability = Stability + Local exponential conv.
  - Region of attraction
- Global asymptotic stability = Stability + Global convergence
- Global exponential stability = Stability + Global exp. conv.

Lyapunov stability analysis

How do we analyze the Lyapunov stability properties?

- Definitions
  - If we have solution $x(t) = \dots$ OK
- Phase plane analysis ($\dim x = 2$)
  - Phase portrait
  - Local analysis

$$\text{Phase portrait of linearized system} \iff \text{Local phase portrait of nonlinear system}$$

- New method: Lyapunov's indirect method ($\dim x = n$)

Lyapunov's indirect method/Linearization method

Theorem 4.7 (Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \qquad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad C^1$$

1) Linearize the system about $x = 0$, $\dot{x} = Ax$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

2) Find the eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$

Lyapunov's indirect method cont.

Theorem 4.7 (Lyapunov's indirect method) cont.

3) a) $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally asymptotically stable

b) $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad x = 0$ is unstable

c) $\begin{array}{ll} \forall i & \mathrm{Re}(\lambda_i) \leq 0 \\ \exists i & \mathrm{Re}(\lambda_i) = 0 \end{array} \quad \Rightarrow$ No conclusion

Corollary 4.3

$\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally exponentially stable

Lyapunov's indirect method cont.

Theorem 4.7 (Lyapunov's indirect method) cont.

3)

a) $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally exponentially stable

b) $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad x = 0$ is unstable

c) $\begin{array}{ll} \forall i & \mathrm{Re}(\lambda_i) \le 0 \\ \exists i & \mathrm{Re}(\lambda_i) = 0 \end{array} \quad \Rightarrow$ No conclusion

Comments

+ Simple to use
÷ Not always conclusive
÷ Only <u>local</u> results

Lyapunov's indirect method: Example

Example

Given

$$\dot{x} = ax - x^3.$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's indirect method.

Corollary 4.3, Sec. 4.7

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n \text{ is } C^1$$

$$\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Leftrightarrow \quad x = 0 \text{ is (locally) exponentially stable}$$

Stability analysis of equilibrium points:

Lyapunov's direct method

Context

Previously:

- The control problem for
  - regulation
  - tracking

  lead to the asymptotic stabilization problem
- Definitions of stability for time-invariant systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local vs. global
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

This set of video lectures:

- **Lyapunov stability analysis** continued
  - Lyapunov's direct method

Learning goals and Literature

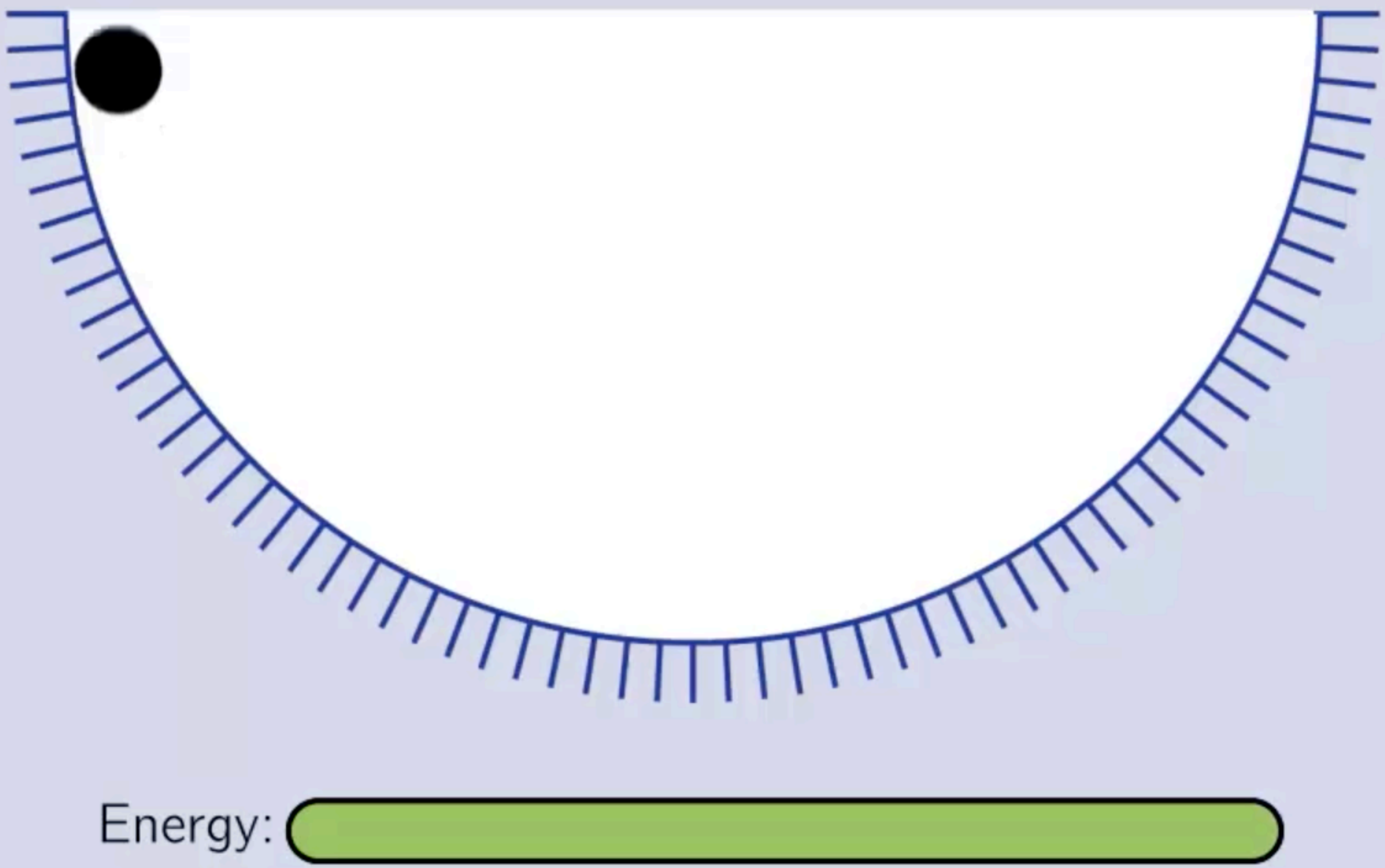
After this set of video lectures you will...

- Be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point.
- Know Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability

The video lectures are based on

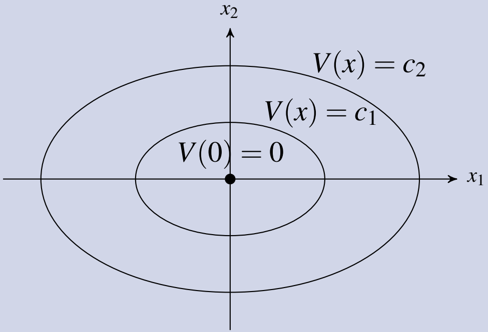
Khalil	Section 4.1
	Section 4.5: Theorem 4.10

Motivation



Motivation

Energy function $V(x)$



Level surfaces:

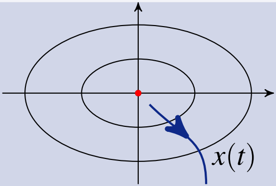
$$V(x) = c_i \quad (0 < c_1 < c_2 < c_3 \cdots)$$

Surfaces that represent constant energy levels.

Energy functions

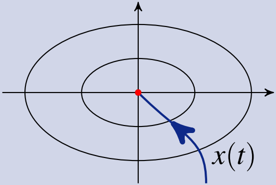
We consider the energy evolution of the system

$$\dot{x} = f(x), \quad x = 0 \text{ is an equilibrium point}$$



Energy increases along $x(t)$

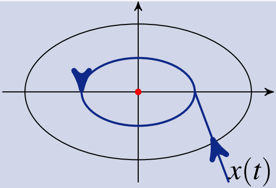
$$\frac{dV(x)}{dt} = \frac{dV}{dx} f(x) > 0$$



Energy decreases along $x(t)$

$$\frac{dV(x)}{dt} < 0$$

Energy functions cont.



Energy decreases or is constant

$$\frac{dV(x)}{dt} \leq 0$$

Aleksandr Mikhailovich Lyapunov (1857-1918)

- Russian mathematician, mechanician and physicist
- Known for his
  - development of the stability theory of dynamical systems

The general problem of the stability of motion (1892)

Lyapunov functions

The system

Consider the time-invariant system

$$\dot{x} = f(x)$$

where $f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz
$x = 0 \in \mathbb{D}$ is an equilibrium point of the system.

Lyapunov function candidate

Let $V : \mathbb{D} \to \mathbb{R}$ be a continuously differentiable (C^1) function

The derivative of V along the system trajectories is:

$$\dot{V} = \frac{dV(x)}{dt} = \frac{dV}{dx} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Lyapunov functions

Definition (Lyapunov function)

V is a Lyapunov function for $x = 0$ iff

i) V is C^1

ii) $V(0) = 0$
$V(x) > 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$

iii) $\dot{V}(0) = 0$
$\dot{V}(x) \leq 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$

If, moreover,

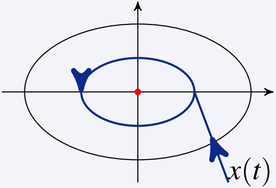
$\dot{V}(0) = 0$
$\dot{V}(x) < 0 \quad \text{in} \quad \mathbb{D}\backslash\{0\}$

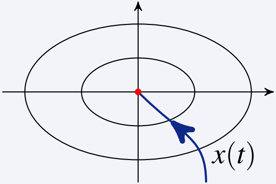
then V is a strict Lyapunov function for $x = 0$.

Lyapunov's direct method

Theorem 4.1

- If $\exists$ Lyapunov function for $x = 0$, then $x = 0$ is stable
- If $\exists$ strict Lyapunov function for $x = 0$, then $x = 0$ is asymptotically stable





How to apply Lyapunov’s direct method

How to apply Lyapunov’s direct method

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T Px$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2)$
     - *some* methods exist for choosing $V(x)$
2) Determine whether $V(x)$ is a Lyapunov function/a strict Lyapunov function for the equilibrium point.
3) If the answer is yes:

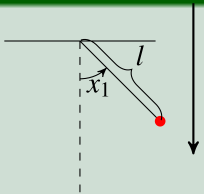
   The equilibrium point is stable/asymptotically stable

   If the answer is no:

Application of Lyapunov's direct method

Pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$



$g = 9.81$

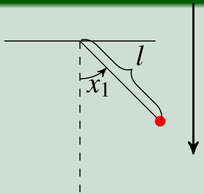
Pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \qquad m = 1$$

Investigate the stability properties of $x = 0$ using Lyapunov's direct method

Pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

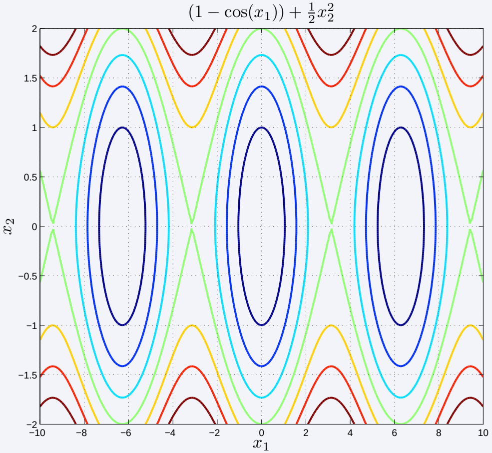


$g = 9.81$

Pendulum without friction: Level curves (contour plot)

Matlab

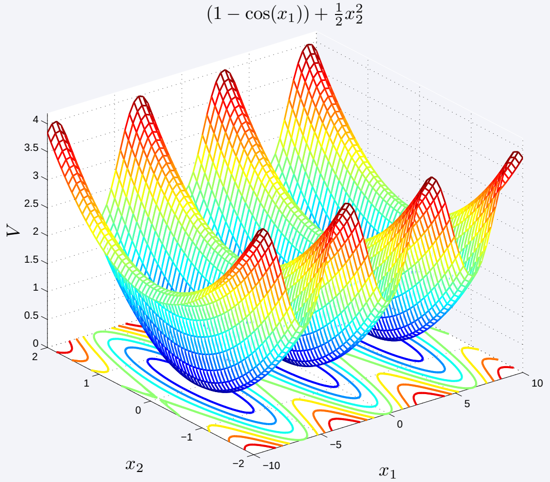
```
V=(1-cos(x1))+1/2*x2*x2
fcontour(V,[-10,10,-2,2])
```



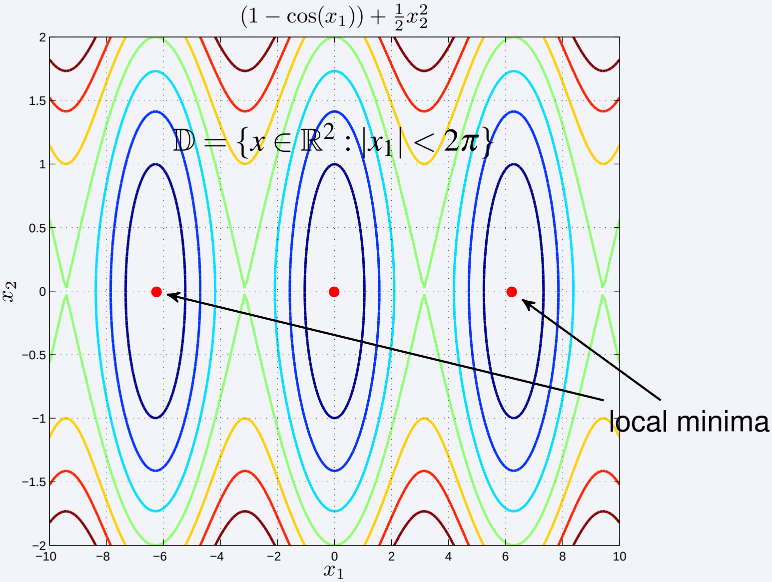
Pendulum without friction: Level curves (surface plot)

Matlab

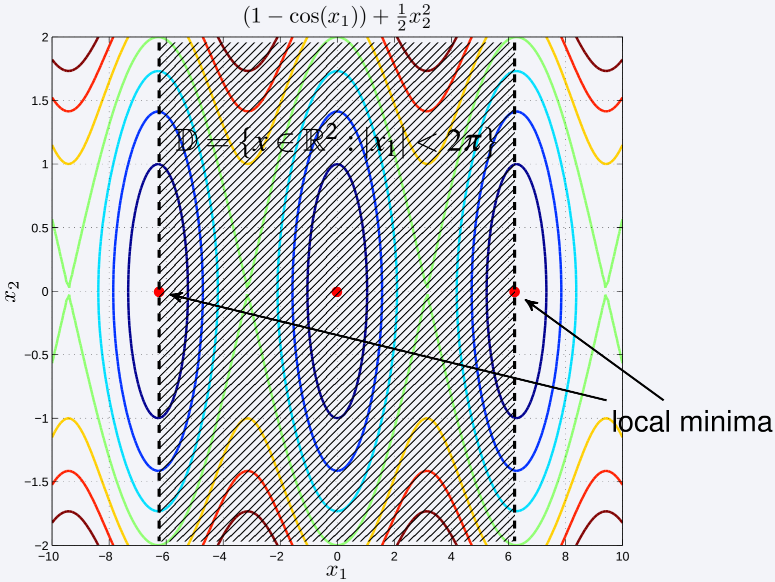
```
ezmeshc(V,[-10,10,-2,2])
```



Pendulum without friction: Domain of analysis



Pendulum without friction: Domain of analysis

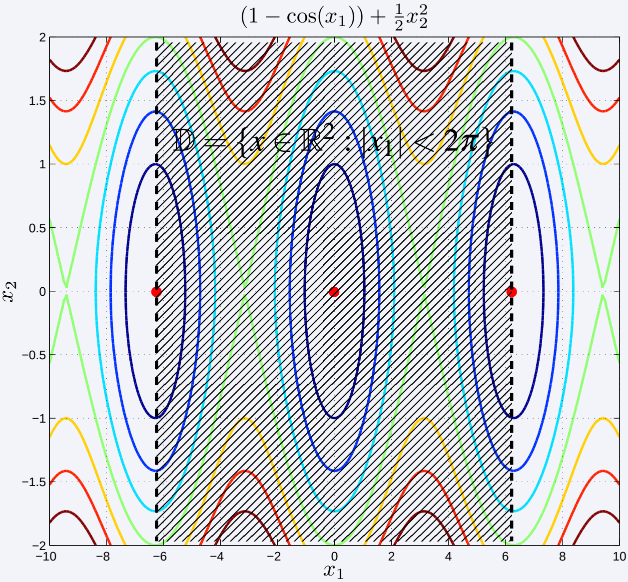


Application of Lyapunov's direct method

Pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \qquad m = 1$$

Pendulum with friction: Domain of analysis



Examples cont.

Example

Given

$$\dot{x} = -x^3$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's direct method.

Global asymptotic stability

Theorem 4.2: Global asymptotic stability

If

- $\exists$ a <u>strict</u> Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$

and

- V is radially unbounded

then $x = 0$ is globally asymptotically stable.

Radial unboundedness

Definition

$V(x)$ is radially unbounded iff

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

Necessary for global results

For C^1 functions V:

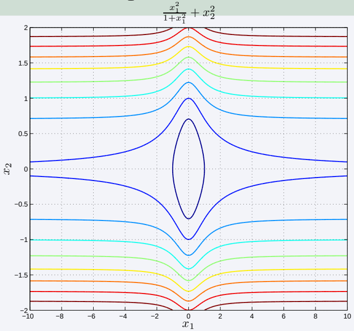
- Positive definite $\Rightarrow$ Level surfaces are closed for small values of c
- Radial unboundedness $\Rightarrow$ Level surfaces are closed $\forall c$

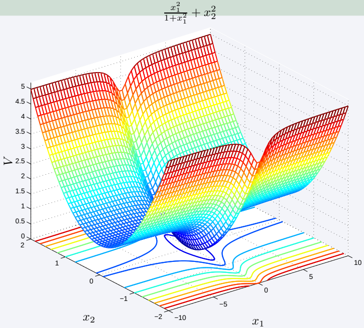
If the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$

Necessary for global results

Example

$$V(x) = \frac{x_1^2}{(1+x_1^2)} + x_2^2$$





Radial unboundedness

Definition

$V(x)$ is radially unbounded iff

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

Necessary for global results

For C^1 functions V:

- Positive definite $\Rightarrow$ Level surfaces are closed for small values of c
- Radial unboundedness $\Rightarrow$ Level surfaces are closed $\forall c$

If the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$

Exponential stability

The system

Consider the time-invariant system

$$\dot{x} = f(x)$$

where $f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz
$x = 0 \in \mathbb{D}$ is an equilibrium point of the system.

Exponential stability

Theorem 4.10: Exponential stability

If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable.

Exponential stability

Theorem 4.10: Exponential stability

If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x = 0$ is globally exponentially stable.

Exponential stability cont.

Convergence rate

It can be shown that

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| \, e^{-\frac{k_3}{k_2 a} t}$$

Exponential stability cont.

Example

Given

$$\dot{x} = -x - x^3$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's direct method.

How to apply Lyapunov's direct method

How to apply Lyapunov's direct method - revisited

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T P x$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2)$
     - *some* methods exist for choosing $V(x)$
2) Determine whether $V(x)$ satisfies the conditions of any of the Lyapunov theorems.
3) If the answer is yes:

   The equilibrium point is Stable/Asymptotically stable/Exponentially stable

   If the answer is no:

TTK4150 Nonlinear Control Systems

Session 2: Lectures 3 and 4

Stability

and

Stability analysis of equilibrium points:

Lyapunov's indirect method
Lyapunov's direct method

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Video lectures

In the video lectures you have learned:

- The control problem for
  - regulation
  - tracking

  lead to the asymptotic stabilization problem
- Definitions of stability (for time-invariant systems)
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local vs. global
- Lyapunov stability analysis (for time-invariant systems)
  - Lyapunov's indirect method
  - Lyapunov's direct method: Know Lyapunov's theorems for
    - stability
    - local and global asymptotic stability
    - local and global exponential stability

Working with error coordinates: shifting the desired equilibrium point to the origin

Exam 2010 - Problem 1

Consider the system:

$$\dot{x}_1 = x_2 - \frac{1}{8}(x_1 + x_2)^3$$
$$\dot{x}_2 = x_1 - \frac{1}{8}(x_1 + x_2)^3$$

a) Find *all* equilibrium points.

b) Linearize the system dynamics around each equilibrium point and classify the *qualitative behavior* of each equilibrium point.

e) Shift the equilibrium point $x^* = [-1, -1]^\mathsf{T}$ to the origin using a change of variables.
Write down the new set of equations.

Lyapunov´s indirect method

Exam 2010 - Problem 1

Consider the system:

$$\dot{x}_1 = x_2 - \frac{1}{8}(x_1 + x_2)^3$$
$$\dot{x}_2 = x_1 - \frac{1}{8}(x_1 + x_2)^3$$

e) Shift the equilibrium $x^* = [-1, -1]^T$ to the origin using a change of variables.
Write down the new set of equations.

f) Is the shifted equilibrium at the origin *locally* asymptotically stable?
Is the shifted equilibrium at the origin *globally* asymptotically stable?
'Yes' or 'no' answers are not sufficient.

Lyapunov's indirect method/Linearization method

Theorem 4.7 (Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \qquad f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad C^1$$

1) Linearize the system about $x = 0$, $\dot{x} = Ax$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

2) Find the eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$

Lyapunov's indirect method cont.

Theorem 4.7 (Lyapunov's indirect method) cont.

3) a) $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally asymptotically stable

b) $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad x = 0$ is unstable

c) $\begin{array}{ll} \forall i & \mathrm{Re}(\lambda_i) \leq 0 \\ \exists i & \mathrm{Re}(\lambda_i) = 0 \end{array}$ $\Rightarrow$ No conclusion

Corollary 4.3

$\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally exponentially stable

Comments

+ Simple to use
÷ Not always conclusive
÷ Only <u>local</u> results

Lyapunov´s direct method

Example Lyapunov´s direct method

Consider the system

$$\dot{x}_1 = -x_1^3 - x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

How to apply Lyapunov's direct method

How to apply Lyapunov's direct method - revisited

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T P x$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2)$
     - *some* methods exist for choosing $V(x)$
2) Determine whether $V(x)$ satisfies the conditions of any of the Lyapunov theorems.
3) If the answer is yes:

   The equilibrium point is Stable/Asymptotically stable/Exponentially stable

   If the answer is no:

Lyapunov functions

Definition (Lyapunov function)

V is a Lyapunov function for $x = 0$ iff

i) V is C^1

ii) $V(0) = 0$
$V(x) > 0$ in $\mathbb{D}\backslash\{0\}$

iii) $\dot{V}(0) = 0$
$\dot{V}(x) \leq 0$ in $\mathbb{D}\backslash\{0\}$

If, moreover,

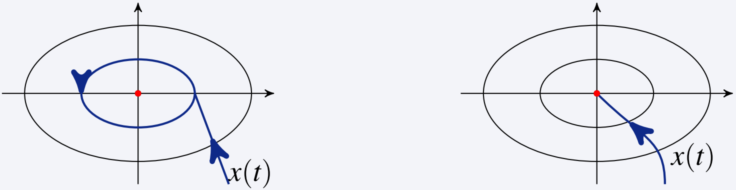
$\dot{V}(0) = 0$
$\dot{V}(x) < 0$ in $\mathbb{D}\backslash\{0\}$

then V is a strict Lyapunov function for $x = 0$.

Lyapunov's direct method

Theorem 4.1

- If $\exists$ Lyapunov function for $x=0$, then $x=0$ is stable
- If $\exists$ strict Lyapunov function for $x=0$, then $x=0$ is asymptotically stable



Global asymptotic stability

Theorem 4.2: Global asymptotic stability

If

- $\exists$ a strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$

and

- V is radially unbounded

then $x = 0$ is globally asymptotically stable.

Exponential stability

Theorem 4.10: Exponential stability

If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

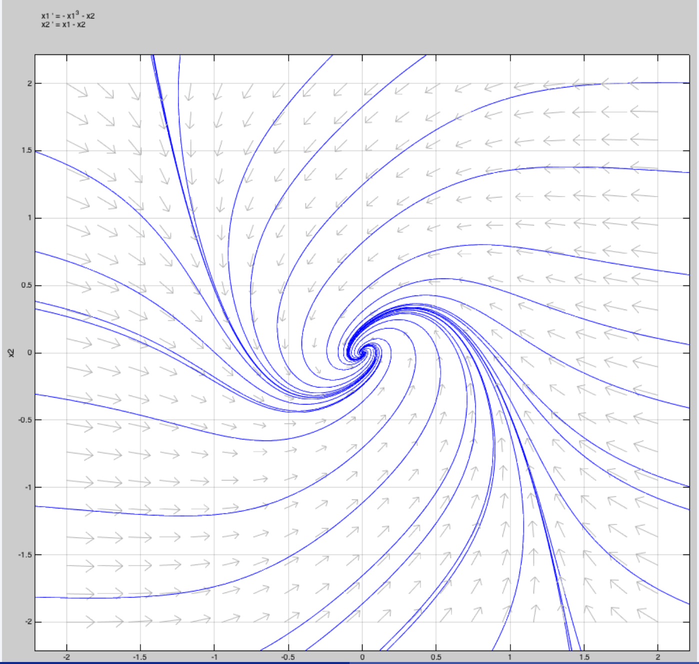
then $x = 0$ is exponentially stable.

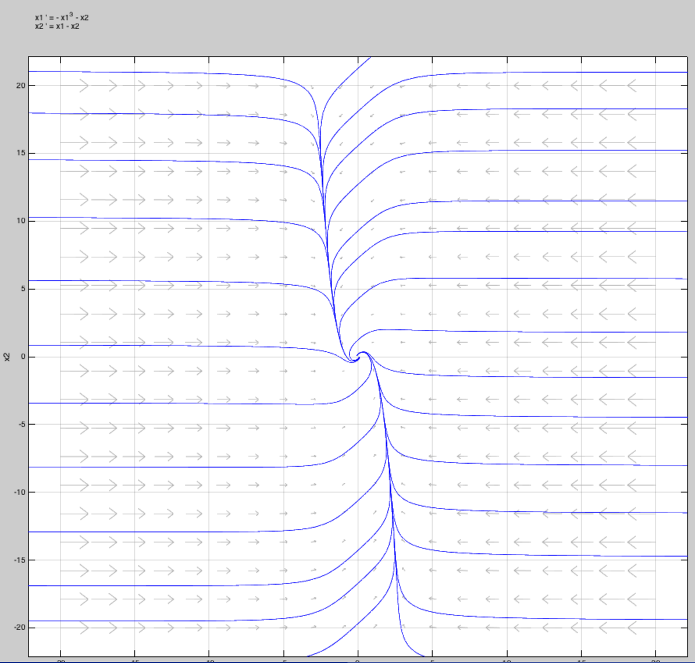
Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x = 0$ is globally exponentially stable.


x1 ' = - x1^3 - x2
x2 ' = x1 - x2
x2
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


x1 ' = - x1^3 - x2
x2 ' = x1 - x2
x2
20
15
10
5
0
-5
-10
-15
-20

Lyapunov´s direct method

Exam 2013 - Problem 3 (Slightly modified to be a time-invariant system)

Consider the system

$$\dot{x} = -(1 + g_0 x^2)x \tag{1}$$

where g_0 is a positive constant.

b) Show that $x = 0$ is a globally exponentially stable (GES) equilibrium point of the system (1).

Exponential stability

Theorem 4.10: Exponential stability

If there exist a function $V : \mathbb{D} \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x = 0$ is globally exponentially stable.

Lyapunov´s direct method

Exam 2014 - Problem 2 (Slightly modified)

Consider the system

$$\dot{x}_1 = -x_1 + 10x_1x_2$$
$$\dot{x}_2 = 3x_1^7 + 9u$$

Use Lyapunov-based methods to find a feedback control law $u = g(x)$ such that the origin becomes globally asymptotically stable. (*Hint: you may try* $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$)

Lyapunov´s direct method

Exam 2010 - Problem 2

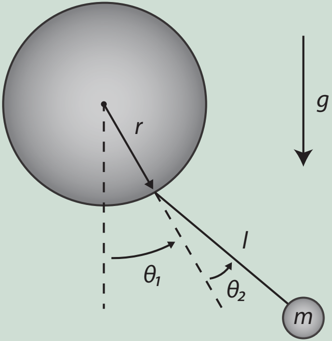


Figure : A planar pendulum suspended from a pinned wheel.

The figure displays a planar pendulum consisting of a massless rod of length l and a point mass at one end with mass m, that is suspended from a wheel with radius r, and that may spin freely about its center.

Lyapunov´s direct method

Exam 2010 - Problem 2

The moment of inertia of the wheel about its center is I. The system is placed in a constant gravitational field with strength g. The dynamic equations describing the motion of this system are

$$\begin{pmatrix} I+mr^2 & mlr\cos(\theta_1) \\ mlr\cos(\theta_1) & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -mgr\sin(\theta_1) - mgl\sin(\theta_1+\theta_2) - b_1\dot{\theta}_1 \\ mlr\sin(\theta_1)\dot{\theta}_1^2 - mgl\sin(\theta_1+\theta_2) - b_2\dot{\theta}_2 \end{pmatrix}$$

where $b_{1,2} > 0$. The system can be written in the form: (Verify this.)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = 0$$

where the coordinate vector is $q = [q_1, q_2]^T = [\theta_1, \theta_2]^T$, the inertia matrix $M(q)$ and the coriolis matrix $C(q,\dot{q})$ are

$$M(q) = \begin{pmatrix} I+mr^2 & mlr\cos(q_1) \\ mlr\cos(q_1) & ml^2 \end{pmatrix}, \; C(q,\dot{q}) = \begin{pmatrix} 0 & 0 \\ -mlr\sin(q_1)\dot{q}_1 & 0 \end{pmatrix}.$$

Lyapunov´s direct method

Exam 2010 - Problem 2

The damping matrix D is

$$D = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

and the gravitation vector $g(q)$ is

$$g(q) = \begin{pmatrix} mgr\sin(q_1) + mgl\sin(q_1+q_2) \\ mgl\sin(q_1+q_2) \end{pmatrix}.$$

For a physically correct model, the inertia matrix $M(q)$ is symmetric and positive definite for all $q \in \mathbb{R}$. Another important property for rigid body dynamics is the skew symmetry of the matrix $\dot{M} - 2C$, which means that

$$(\dot{M} - 2C) = -(\dot{M} - 2C)^T$$

and $z^T(\dot{M} - 2C)z = 0$, $\forall z = [\alpha, \beta]^T \in \mathbb{R}^2$. (Verify this.)

Lyapunov´s direct method

Exam 2010 - Problem 2

a) The potential energy of the system can be expressed as:

$$P(q) = -mgr\cos(q_1) - mgl\cos(q_1+q_2)$$

The gravitation vector $g(q)$ can be found as $g(q)^T = \frac{\partial}{\partial q}P(q)$. Verify the given expression for $g(q)$.

b) The total energy of the system can be found from:

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

A valid Lyapunov function candidate for the equilibrium point $(q^*,\dot{q}^*) = (0,0)$ can be found as:

$$V(q,\dot{q}) = E(q,\dot{q}) + P_0$$

Find the constant P_0.

Lyapunov´s direct method

Exam 2010 - Problem 2

c) By using the Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) + P_0,$$

determine whether the equilibrium point $(q^*, \dot{q}^*) = (0,0)$ is unstable, stable, asymptotically stable, or globally asymptotically stable. State the strongest conclusion that you can verify.

Lyapunov´s direct method

Exam 2010 - Problem 2

d) By adding (massless) motors to each joint of the pendulum system in Fig. 1, we augment the dynamics with a torque vector τ:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = \tau$$

Consider the Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}q^T K_p q$$

and the controller

$$\tau = -K_p q - K_d \dot{q} + g(q)$$

where $K_p, K_d \in \mathbb{R}^2$ are symmetric and positive definite. Determine whether the equilibrium $(q^*, \dot{q}^*) = (0,0)$ is unstable, stable, asymptotically stable, or globally asymptotically stable. State the strongest conclusion that you can verify.

Lyapunov´s direct method

Example Lyapunov´s direct method

Consider the system

$$\dot{x}_1 = -x_1^3 - x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Lyapunov´s direct method

Example Lyapunov´s direct method

Consider the system

$$\dot{x}_1 = -x_1^3 - 3x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Quadratic forms $x^T Px$

Positive/Negative definite quadratic forms

For the quadratic form

$$V(x) = x^T Px$$

where P is a **real, symmetric** matrix

- $V(x)$ is positive definite
  $\iff$ all eigenvalues of P are positive ($\lambda_i(P) > 0$)
  $\iff$ all upper left submatrices of P have positive determinants ($\det P_i > 0$)
- $V(x)$ is negative definite
  $\iff$ $-V(x)$ is positive definite

If $V(x) = x^T Px$ is positive definite (negative definite), we say that the matrix P is positive definite (negative definite) and write $P > 0$ ($P < 0$)

Quadratic forms $x^T Px$

Positive/Negative semidefinite quadratic forms

For the quadratic form

$$V(x) = x^T Px$$

where P is a **real, symmetric** matrix

- $V(x)$ is positive *semi*definite

  $\iff$ all eigenvalues of P are nonnegative ($\lambda_i(P) \geq 0$)

  $\iff$ all upper left submatrices of P have nonnegative determinants ($\det P_i \geq 0$)

- $V(x)$ is negative semidefinite

  $\iff$ $-V(x)$ is positive semidefinite

If $V(x) = x^T Px$ is positive semidefinite (negative semidefinite), we say that the matrix P is positive semidefinite (neg. semidef.) and write $P \geq 0$ ($P \leq 0$)

LFCs with quadratic terms $x^T P x$

Example

Consider the quadratic form

$$
\begin{aligned}
V(x) &= ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

Find the conditions on a for which $V(x)$ is positive definite, and for which $V(x)$ is negative definite.

Quadratic forms $x^T P x$

Rayleigh´s principle

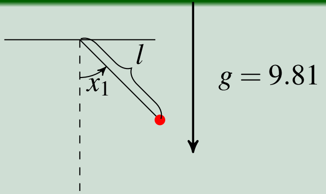
When P is a **real, symmetric** matrix, then

$$\lambda_{\min}(P)\,\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\,\|x\|^2$$

Example: LFCs with quadratic terms $\frac{1}{2}x^T Px$

Assignment: Pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



Use the generalized energy function

$$V(x,p) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x^T Px$$

as Lyapunov function candidate

Next lecture

- La Salle's theorem
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Convergence to other invariant sets than equilibrium points
  - Regions of attraction - find an estimate
- More methods for finding Lyapunov function candidates (LFCs)
- Recommended reading

Khalil	Section 4.1 p. 120-122
	Sections 4.2-4.3
	Section 8.2

TTK4150 Nonlinear Control Systems

Lecture 5

Stability analysis for time-invariant (autonomous) systems

-

continued

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Previously

Previously:

Lyapunov's direct method:

- Lyapunov functions - a generalization of energy functions
- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- How to apply Lyapunov's direct method

Outline I

1. Introduction
   - Previously
   - Today's goals
   - Literature
2. The Invariance Principle
   - Invariant sets
   - La Salle's theorem
   - Prove asymptotic stability when $\dot{V} \leq 0$
   - Estimate Region of attraction
   - Convergence to other invariant sets
3. Methods for choosing Lyapunov function candidates
   - Lyapunov functions for linear systems
4. How to handle terms in $\dot{V}$ with indeterminate sign
   - Tools for dominating cross-terms
5. Next lecture

Today's goals

After this lecture you should...

- Know La Salle's theorem, and how to use this
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Region of attraction - find an estimate
  - Convergence to other invariant sets than equilibrium points
- Know how to find Lyapunov functions for linear systems
- Know how to handle terms in $\dot{V}$ with indeterminate sign

Literature

Today's lecture is based on

Khalil	Section 4.1 pp. 120-122
	Sections 4.2-4.3
	Section 8.2

Part I

La Salle's theorem

Invariant sets

Definition (Invariant set)

A set M is an invariant set with respect to $\dot{x} = f(x)$ iff

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}$$

Definition (Positively invariant set)

A set M is a positively invariant set with respect to $\dot{x} = f(x)$ iff

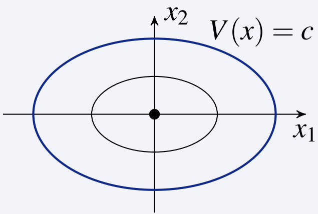
$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

Notation

Level surfaces (curves)

Lyapunov surfaces

$$V(x) = c$$

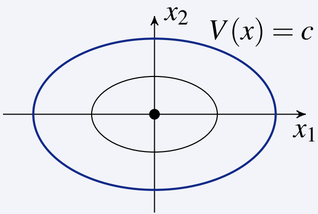


Notation

Level surfaces (curves)

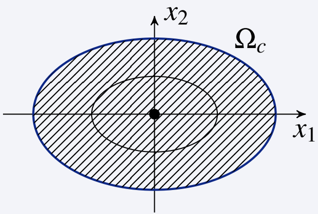
Lyapunov surfaces

$$V(x) = c$$



Level sets

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

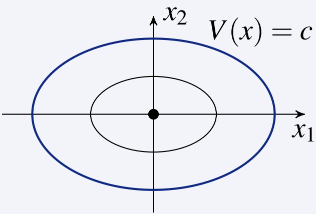


Notation

Level surfaces (curves)

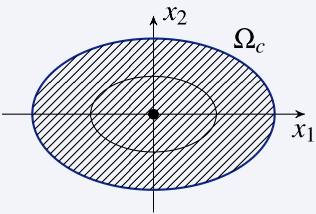
Lyapunov surfaces

$$V(x) = c$$



Level sets

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$



When V is a Lyapunov function then Ω_c is a (positively) invariant set for the system $\dot{x} = f(x)$

The invariance principle: La Salle's theorem

$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n$ locally Lipschitz

Theorem 4.4 (La Salle's theorem)

If $\exists\, V : \mathbb{D} \to \mathbb{R}$ such that

i) V is C^1

ii) $\exists\, c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$ is bounded

iii) $\dot{V}(x) \leq 0 \quad \forall\, x \in \Omega_c$

Let $E = \left\{x \in \Omega_c | \dot{V}(x) = 0\right\}$
Let M be the largest invariant set contained in E
Then

$$x(0) \in \Omega_c \Rightarrow x(t) \xrightarrow{t \to \infty} M$$

La Salle's theorem ii)

Note: V does not have to be positive definite

- V positive definite $\Rightarrow \Omega_c$ bounded for small c
- V radially unbounded $\Rightarrow \Omega_c$ bounded for $\forall\ c$

Special cases:

- Cor. 4.1 ($M = \{0\}$)
- Cor. 4.2 (Global version)

Application

Applications of La Salle's theorem:

- $\dot{V} \leq 0$ Prove asymptotic stability of equilibrium points
- Region of attraction - find an estimate
- Convergence to other invariant sets than equilibrium points

Examples

Example: $\dot{V} \leq 0$ Prove asymptotic stability of eq.point

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -b(x_2) - c(x_1)$$

$b, c \quad C^0$

$b(0) = c(0) = 0$

$x_1 c(x_1) > 0 \quad x_1 \neq 0 \quad x_1 \in (-a_1, a_1)$

$x_2 b(x_2) > 0 \quad x_2 \neq 0 \quad x_2 \in (-a_2, a_2)$

Analyze the stability properties of $x = 0$ using Lyapunov theory.

Pendulum with friction, $x =$ angle, $\dot{x} =$ angular velocity

$c(x_1) = \frac{g}{l} \sin x_1$

$b(x_2) = \frac{k}{m} x_2$

$a_1 = \pi$

$a_2 \to \infty$

Mass-spring-damper
$x =$ position, $\dot{x} =$ velocity

$c(x_1)$: spring force (Linear: $\frac{k}{m} x_1$)

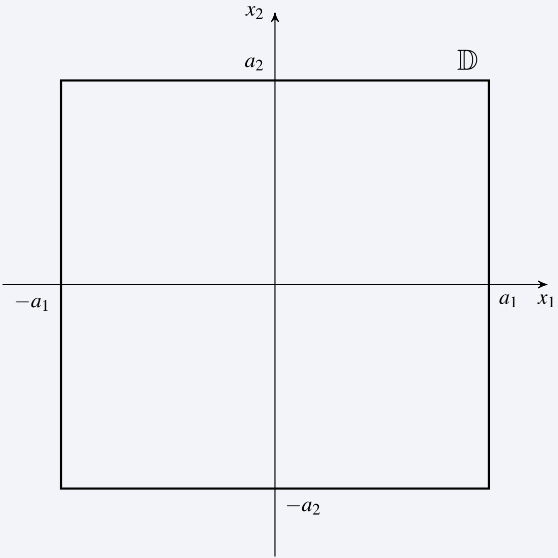
$b(x_2)$: damping force (Linear: $\frac{d}{m} x_2$)

RLC circuit
$x =$ charge, $\dot{x} =$ current

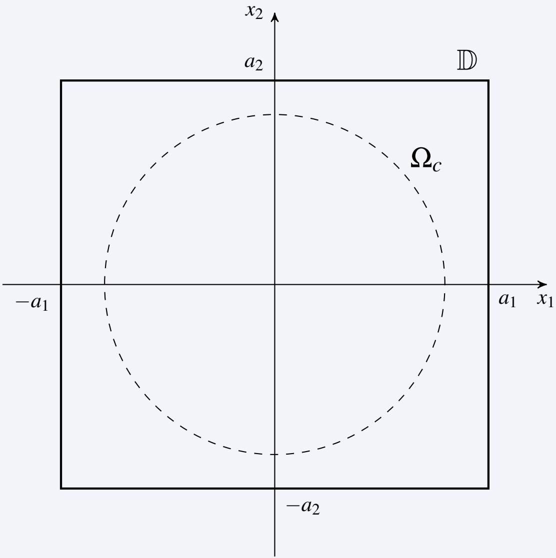
$c(x_1)$: Capacitor voltage (Linear: $\frac{1}{LC} x_1$)

$b(x_2)$: Resistor voltage (Linear: $\frac{R}{L} x_2$)

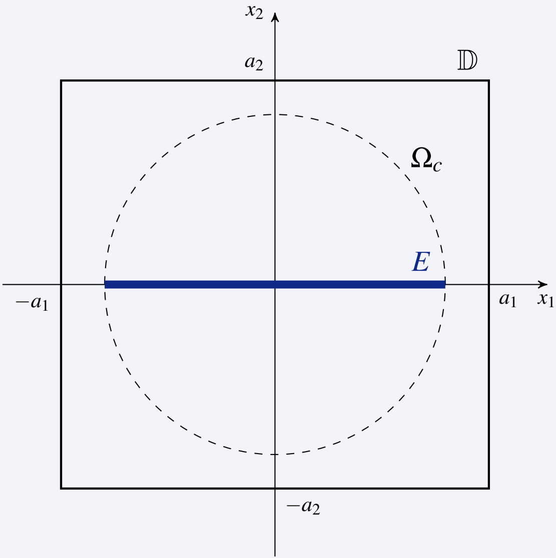
Example cont.



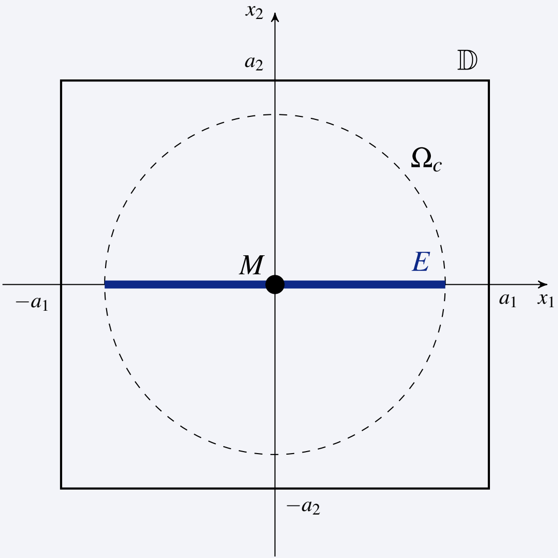
Example cont.



Example cont.



Example cont.



Region of attraction

Definition (The Region of attraction)

Let $x = 0$ be an *asymptotically stable* equilibrium point of the system $\dot{x} = f(x)$, where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz and $\mathbb{D} \subset \mathbb{R}^n$ is a domain that contains the origin.
Let $\phi(t, x_0)$ be the solution of $\dot{x} = f(x)$ that starts at initial state x_0 at time $t = 0$. The region of attraction of the origin, denoted by R_A, is defined by

$$R_A = \{x_0 \in \mathbb{D} | \phi(t, x_0) \text{ is defined } \forall\, t \geq 0 \text{ and } \phi(t, x_0) \to 0 \text{ as } t \to \infty\}$$

Region of attraction

Definition (The Region of attraction)

Let $x = 0$ be an *asymptotically stable* equilibrium point of the system $\dot{x} = f(x)$, where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz and $\mathbb{D} \subset \mathbb{R}^n$ is a domain that contains the origin.
Then the region of attraction R_A is the set of all points x_0 in $\mathbb{D}$ such that the solution of

$$\dot{x} = f(x) \qquad x(0) = x_0$$

is defined for all $t \geq 0$ and converges to the origin as $t \to \infty$

GAS

The origin $x = 0$ is globally asymptotically stable iff its region of attraction is the whole state space $\mathbb{R}^n$

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Is $\mathbb{D}$ an estimate of R_A?

Given a strict Lyapunov function

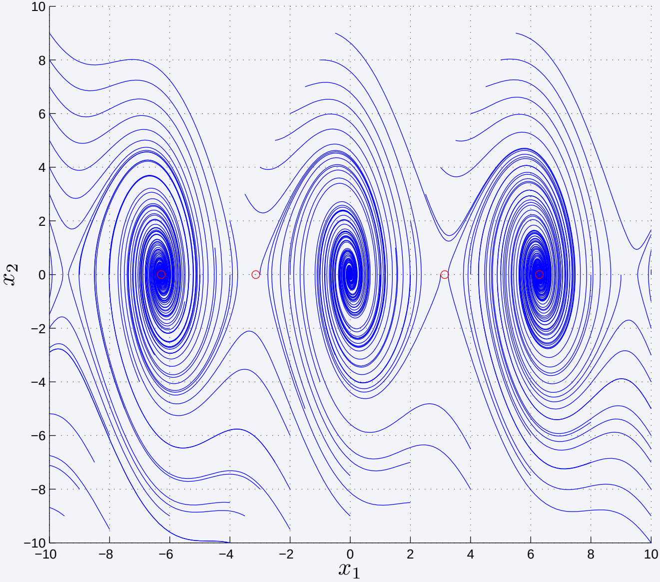
$$\left.\begin{array}{ll} V & \text{is } C^1 \\ V & \text{pos.def.} \\ \dot{V} & \text{neg.def.} \end{array}\right\} \quad \forall\, x \in \mathbb{D}$$

Example: Pendulum with friction

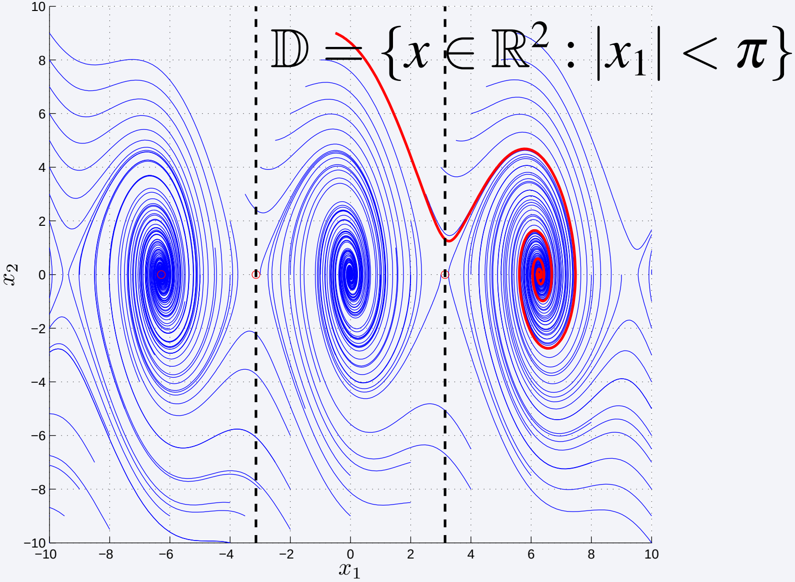
$$V(x) = \frac{g}{l}(1 - \cos x_1) + \frac{1}{2} x^T P x$$

$$\mathbb{D} = \{x \in \mathbb{R}^2 : |x_1| < \pi\}$$

Estimate the region of attraction



Estimate the region of attraction



An estimate of the region of attraction

Starting point:
You have proved asymptotic stability of the origin by either finding a strict Lyapunov function or by using La Salle's theorem

Estimate R_A using Ω_c

1) Choose the largest set

$$\Omega_c = \{x \in \mathbb{R}^n : \; V(x) \leq c\}$$

that is contained in $\mathbb{D}$ (where $V > 0$ and $\dot{V} < 0$ (Strict Lyapunov function), or in which $\dot{V} \leq 0$ (La Salle)) and which is bounded

2) Choose the connected component in this set that contains the origin.

Then this is a subset of the region of attraction of the origin, and can hence be used as an estimate.

Example: An estimate of the region of attraction

(Do not always trust your intuition)

Example

$$\dot{z}_1 = -z_1 + z_1^2 z_2$$
$$\dot{z}_2 = -z_2$$

Equilibrium point (0,0)
Lyapunov linearization method: Locally asymptotically stable
Corollary 4.3: Locally exponentially stable

Q: Is it <u>globally</u> asymptotically/exponentially stable?

Example: An estimate of the region of attraction

(Do not always trust your intuition)

Example

$$\dot{z}_1 = -z_1 + z_1^2 z_2$$
$$\dot{z}_2 = -z_2$$

Equilibrium point (0,0)
Lyapunov linearization method: Locally asymptotically stable
Corollary 4.3: Locally exponentially stable

Q: Is it <u>globally</u> asymptotically/exponentially stable?
Intuition may suggest yes...

Example cont.

Example cont.

For this particular system it is possible to find an analytical solution:

$$z_1(t) = \frac{2z_1(0)}{z_1(0)z_2(0)e^{-t} + [2 - z_1(0)z_2(0)]e^{t}} \tag{1}$$

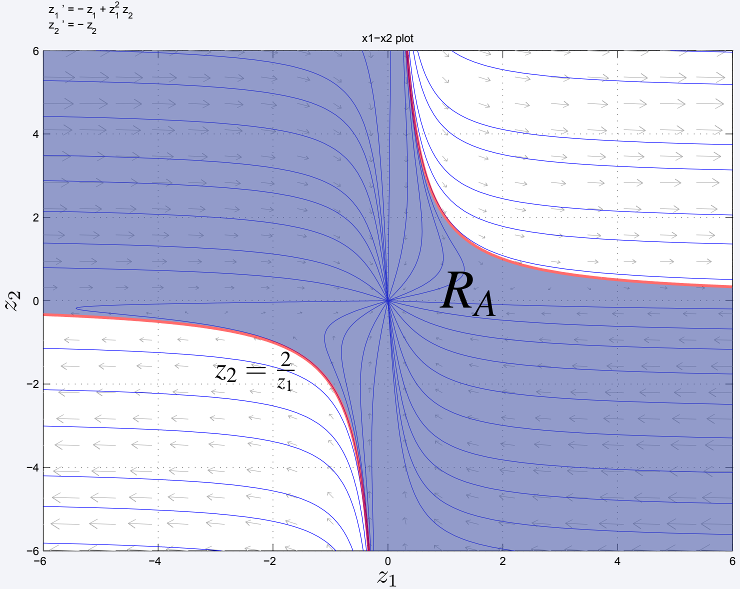
$$z_2(t) = z_2(0)e^{-t} \tag{2}$$

If $z_1(0)z_2(0) > 2$, the denominator in Eq. (1) becomes zero at the time

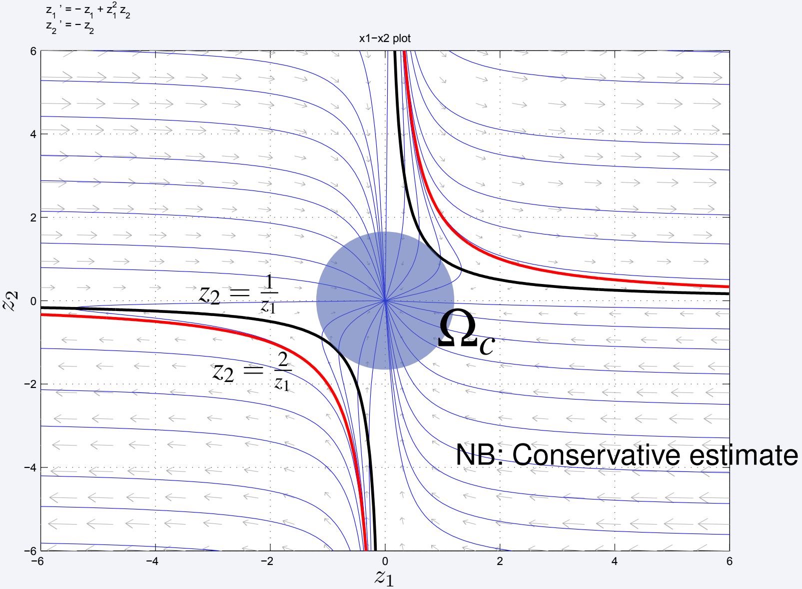
$$t_{esc} = \frac{1}{2}\ln\left(\frac{z_1(0)z_2(0)}{z_1(0)z_2(0) - 2}\right)$$

The equilibrium point is clearly not globally asymptotically stable. It is locally exponentially stable and the region of attraction is given by $z_1(0)z_2(0) < 2$.

Example: Region of attraction



Example: Estimate of region of attraction



Convergence to other invariant sets

Example

Consider the system

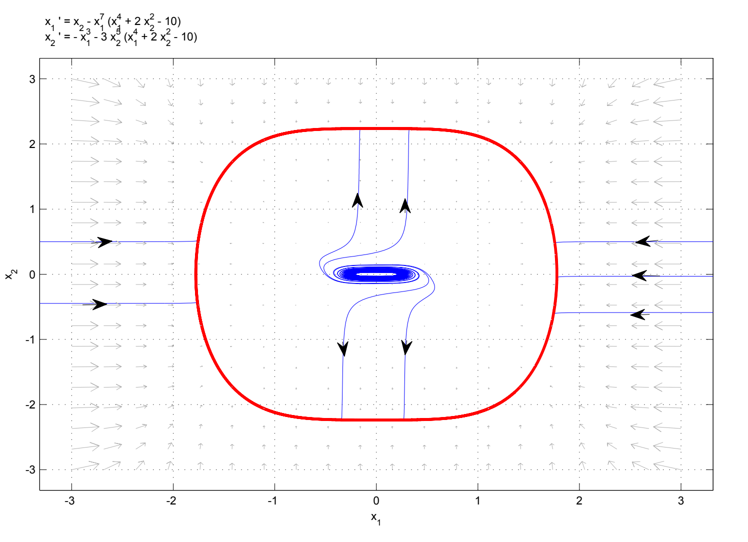
$$\dot{x}_1 = x_2 - x_1^7\left(x_1^4 + 2x_2^2 - 10\right)$$
$$\dot{x}_2 = -x_1^3 - 3x_2^5\left(x_1^4 + 2x_2^2 - 10\right)$$

Investigate the stability properties of the invariant set

$$Q = \{x \in \mathbb{R}^2 | x_1^4 + 2x_2^2 - 10 = 0\}$$

using

$$V(x) = \left(x_1^4 + 2x_2^2 - 10\right)^2$$


$x_1' = x_2 - x_1^7 (x_1^4 + 2 x_2^2 - 10)$
$x_2' = - x_1^3 - 3 x_2^5 (x_1^4 + 2 x_2^2 - 10)$
x_2
x_1
-3
-2
-1
0
1
2
3

Part II

Methods for choosing Lyapunov function candidates

Methods for choosing Lyapunov function candidates

Methods for choosing LFCs

- Total energy
- LFCs with quadratic terms $\frac{1}{2}x^T Px$
  - $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$
  - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$
  - $V(x) = \frac{1}{2}x^T Px$
- LFCs for linear time-invariant systems
- Krasovskii's method (Assignment)
- The variable gradient method (Assignment)
- $\vdots$

Linear time-invariant systems

LTI systems

The linear time-invariant system

$$\dot{x} = Ax \qquad (\det A \neq 0)$$

has one equilibrium point $x = 0$

Hurwitz

A is Hurwitz iff

$$\text{Re}(\lambda_i) < 0 \quad \forall\, i = 1, \ldots, n$$

LFC

Which Lyapunov function candidate do we choose?

Lyapunov functions for linear systems

Theorem 4.6

Given the system $\dot{x} = Ax$
Let $V(x) = x^T P x$ and choose $Q = Q^T$ positive definite.
Seek to find a solution $P = P^T$ of Lyapunov's matrix equation

$$A^T P + PA = -Q \tag{3}$$

- If (3) does not have a solution $P = P^T$, or the solution is not unique: $x = 0$ is not asymptotically stable
- If (3) has a unique solution $P = P^T$, but P is not positive definite: $x = 0$ is not asymptotically stable
- If (3) has a unique solution $P = P^T$, and P is positive definite: $x = 0$ is asymptotically stable

Example

Example

Consider the system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = 3x_1 - x_2$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Part III

How to handle terms in $\dot{V}$ with indeterminate sign

Handling terms with indeterminate sign

Terms in $\dot{V}$ with indeterminate sign

We have seen examples on how to

- Cancel
  - Adjust the a_i in $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$ such that cross-terms $x_i x_j$ in $\dot{V}$ cancel each other
  - Adjust the parameters in a non-diagonal P such that $V(x) = x^T P x > 0$ ($P = P^T > 0$) and $\dot{V} < 0$
- Dominate
  - Completion of squares
  - Write as $-x^T Q x$
  - Young's inequality
  - Cauchy-Schwarz inequality

Tools for dominating cross-terms

Completion of squares

$$(x \pm y)^2 \geq 0, \quad \forall x, y \in \mathbb{R}$$
$$\Updownarrow$$
$$x^2 \pm 2xy + y^2 \geq 0$$
$$\Updownarrow$$
$$x^2 + y^2 \geq \pm 2xy$$
$$\Rightarrow xy \leq |xy| \leq \frac{1}{2}(x^2 + y^2) \qquad \Rightarrow x_1 x_2 \leq \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}\|x\|_2^2$$

Tools for dominating cross-terms

Alternatively

Write $\dot{V}$ as $-x^T Qx$, where $Q = Q^T$ is positive definite

NB This is similar to completing the squares

Tools for dominating cross-terms cont.

Young's inequality $(x, y \in \mathbb{R})$

$$xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2, \quad \forall\, \varepsilon > 0 \quad \forall x, y \in \mathbb{R}$$

Proof:

$$\varepsilon (x - \tfrac{1}{2\varepsilon} y)^2 \geq 0$$

$$\Updownarrow$$

$$\varepsilon (x^2 - \tfrac{1}{\varepsilon} xy + \tfrac{1}{4\varepsilon^2} y^2) \geq 0$$

$$\Updownarrow$$

$$\varepsilon x^2 - xy + \tfrac{1}{4\varepsilon} y^2 \geq 0$$

Tools for dominating cross-terms

Completion of squares

$$\dot{V} = -x_1^2 + 6x_1x_2 - 20x_2^2$$

Handling terms with indeterminate sign

Cauchy-Schwarz inequality

$$|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \le \sqrt{(a_1^2 + a_2^2 + \cdots + a_n^2)}\,\|x\|_2$$

Example (See page 319)

$$x_1 - 2x_2 \le |x_1 - 2x_2| \le \sqrt{1^2 + (-2)^2}\,\|x\|_2 = \sqrt{5}\,\|x\|_2$$

Next lecture

Next lecture

- Lyapunov stability analysis for time-varying (nonautonomous) systems
- Recommended reading

Khalil	Sections 4.4-4.5
	Section 8.3

TTK4150 Nonlinear Control Systems

Lecture 6

Stability analysis of time-varying (nonautonomous) system

NTNU

Previous lectures

Previous lectures:

Lyapunov's direct method for time-invariant (autonomous) systems:

- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- La Salle's theorem
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Regions of attraction - find an estimate
  - Convergence to other invariant sets than equilibrium points
- Some methods for finding Lyapunov function candidates (LFCs)

Outline I

1. Introduction
   - Previous lecture
   - Today's goals
   - Literature
2. Time-varying systems
   - Time-varying systems and equilibrium points
3. Comparison functions
   - class $\mathscr{K}$ function
   - class $\mathscr{K}_\infty$ function
   - class $\mathscr{KL}$ function
4. Stability definitions
   - Stability definitions: $\varepsilon - \delta$-definitions
   - Stability definitions: Using class $\mathscr{K}$ and $\mathscr{KL}$ functions
5. Lyapunov's direct method for time-varying systems
   - Time-varying Lyapunov function candidates - Properties

Outline II

- Stability theorems
- Estimate of the Region of attraction
- Stability theorem: Exponential stability

6. Invariance-like results
   - Barbalat's lemma

7. Next lecture

Today's goals

After today you should...

Know **Lyapunov's direct method for time-varying systems**. In particular,

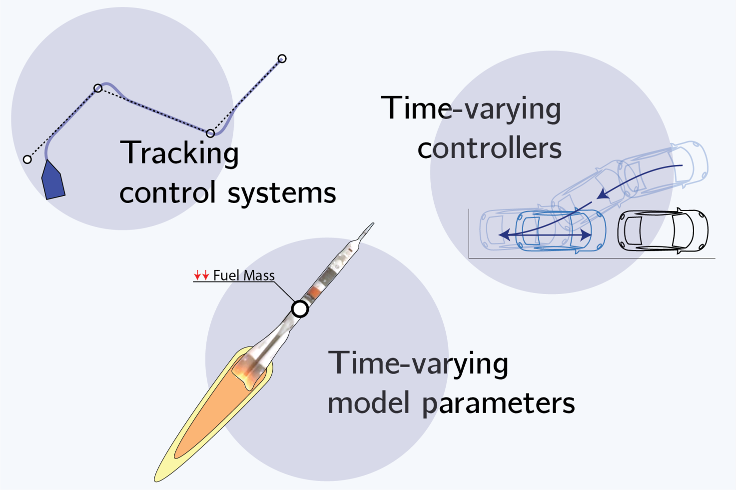
- Know comparison functions of class $\mathscr{K}$ and class $\mathscr{KL}$
- Know the stability definitions of time-varying systems (and how they deviate from the stability definitions of time-invariant systems)
- Be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point of a time-varying system.
- Be able to use Barbalat's lemma to analyze the convergence properties when $\dot{V}(t,x) \leq 0$

Literature

Today's lecture is based on

Khalil	Sections 4.4-4.5
	Section 8.3

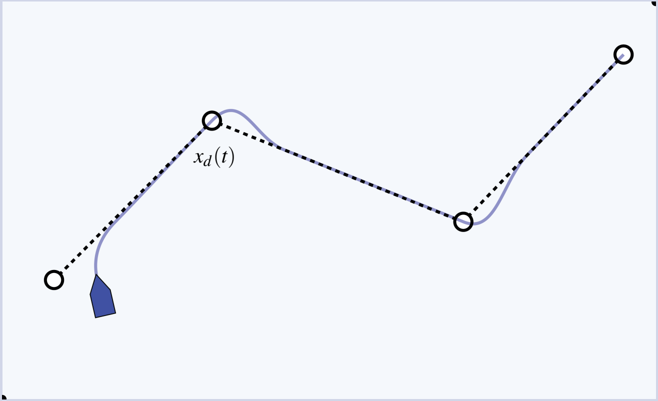
Time-varying systems



Need to analyze the stability properties of an equilibrium point of a time-varying system $\dot{x} = f(t, x)$

Tracking control systems

Tracking control systems



Tracking control systems

Tracking control systems

Time-varying model parameters

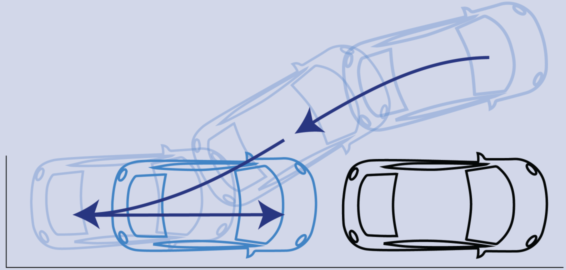
Time-varying model parameters

- Example: Spacecraft
    - At constant thrust, the mass $m(t)$ is decreasing at a constant rate

Time-varying controllers

Time-varying controllers

- Some systems cannot be stabilized by $u(x)$, but need $u(t,x)$
- Example: Point stabilization, i.e. parking, of a car



Time-varying systems

Time-varying systems

$$\dot{x} = f(t,x) \quad f : [0,\infty) \times \mathbb{D} \to \mathbb{R}^n$$

- $f(t,x)$ piecewise continuous in t
  locally Lipschitz in x
- $x = 0 \in \mathbb{D}$

Definition: Equilibrium point

x^* is an equilibrium point for $\dot{x} = f(t,x)$ at $t = 0$ iff

$$f(t,x^*) = 0 \quad \forall t \geq 0$$

Examples

Example

Find the equilibrium points x^* of the following systems at $t = 0$

a) $\dot{x} = -\dfrac{a(t)x}{1+x^2} \qquad a(t) > 0$

b) $\dot{x} = -\dfrac{a(t)x}{1+x^2} + b(t) \qquad \begin{array}{l} a(t) > 0 \\ b(t) \neq 0 \quad \forall t > 0, \quad b(0) = 0 \end{array}$

Translate a nonzero equilibrium point to the origin

We will analyse the stability properties of $x^* = 0$

Translate the equilibrium point of interest to the origin

We can alway translate a nonzero equilibrium point to the origin.

$$\dot{x} = f(t,x) \quad f(t,x^*) = 0 \quad \forall\, t \geq 0$$

Define the error variable

$$\begin{aligned} e &= x - x^* \\ \dot{e} &= \dot{x} - \dot{x}^* = f(t, e + x^*) = \bar{f}(t,e) \end{aligned}$$

We can now analyse

$$\dot{e} = \bar{f}(t,e) \quad e^* = 0 \quad \text{is an equilibrium point}$$

Comparison functions

Class $\mathscr{K}$ function

A continuous function $\alpha : [0,a) \to [0,\infty)$

- is a class $\mathscr{K}$ function
- belongs to class $\mathscr{K}$

iff $\begin{cases} \alpha(0) = 0 \\ \alpha(r) \text{ is strictly increasing, i.e. } \frac{\partial \alpha}{\partial r} > 0 \quad \forall r > 0 \end{cases}$

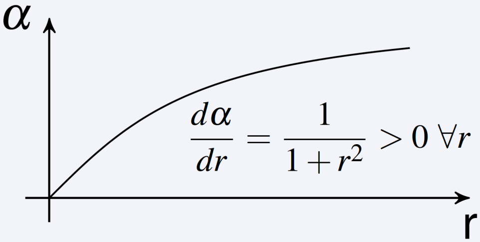


Figure : Example: $\alpha(r) = \arctan(r)$

Comparison functions

Class $\mathscr{K}_\infty$ function

If in addition

- $a \to \infty$
- $\alpha(r) \to \infty$ as $r \to \infty$

then

- α is a class $\mathscr{K}_\infty$ function / α belongs to class $\mathscr{K}_\infty$

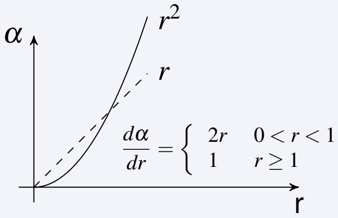


Figure : Example: $\alpha(r) = \min(r, r^2)$

Comparison functions

Class $\mathscr{K}_\infty$ function

If in addition

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- $\alpha(r) \to \infty$ as $r \to \infty$

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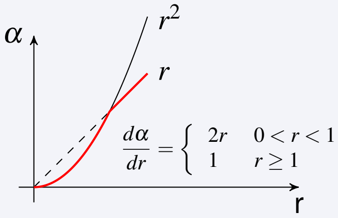


Figure : Example: $\alpha(r) = \min(r, r^2)$

Comparison functions

Class $\mathscr{KL}$ function

A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$

- is a class $\mathscr{KL}$ function
- belongs to class $\mathscr{KL}$

if, for each fixed s

$$\beta(r,s) \text{ is a class } \mathscr{K} \text{ function with respect to } r$$

and, for each fixed r

- $\beta(r,s)$ is decreasing with respect to s
- $\beta(r,s) \to 0$ as $s \to \infty$

Properties

Read Lemma 4.2

System behavior depends on t_0

Initial value problem (IVP)

$$\left.\begin{array}{ll}\dot{x} = f(x) & x(t_0) = x_0\end{array}\right\} \text{Time-invariant IVP} \quad x(t) = \varphi(t - t_0, x_0)$$

$$\left.\begin{array}{ll}\dot{x} = f(t,x) & x(t_0) = x_0\end{array}\right\} \text{Time-varying IVP} \quad x(t) = \varphi(t - t_0, x_0, t_0)$$

NB

The solutions of time-varying systems
in general depend on t_0

NB

The stability properties of time-varying systems
in general depend on t_0

Stability, uniform stability and instability

Stability definitions

The equilibrium point $x^* = 0$ is

- Stable, iff

  $\forall\ \varepsilon > 0, \quad \exists \delta(\varepsilon, t_0) > 0$ such that

  $$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall\ t \geq t_0 \geq 0$$

- Uniformly stable, iff

  $\forall\ \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0$ such that

  $$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall\ t \geq t_0 \geq 0$$

- Unstable, iff it is not stable

Asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

- Asymptotically stable, iff
  - it is stable
  - $\exists\ c(t_0) > 0$ such that $\|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t\to\infty} 0$
- Uniformly asymptotically stable, iff
  - it is uniformly stable
  - $\exists\ c > 0$ such that $\|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t\to\infty} 0$ uniformly in t_0

Convergence vs Uniform convergence cont.

Example

Given

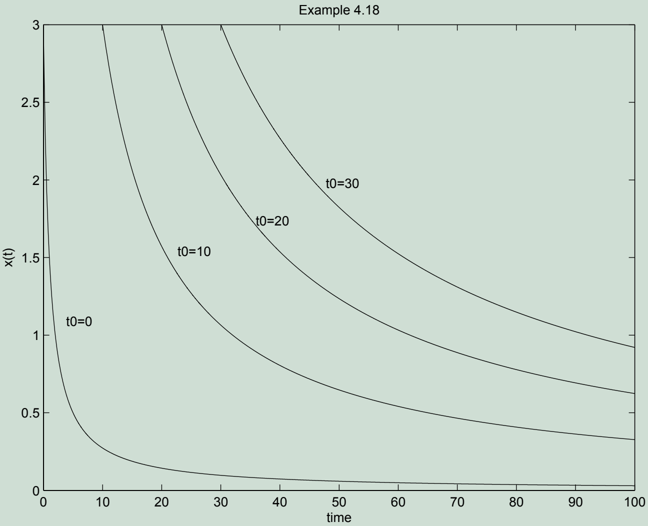
$$\dot{x} = -\frac{x}{1+t} \quad x(t_0) = x_0$$

Equilibrium point $x^* = 0$

Stability properties? Convergence properties?

Example: Non-uniform convergence

Note: The convergence rate depends on t_0. ($x_0 = 3$)



Global uniform asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

- Globally uniformly asymptotically stable, iff
  - it is uniformly stable, with $\delta(\varepsilon) \xrightarrow{\varepsilon \to \infty} \infty$
  - $\forall\, c > 0 \quad \|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t \to \infty} 0$ uniformly in t_0

Global uniform asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

- Globally uniformly asymptotically stable, iff
  - it is uniformly stable, with $\delta(\varepsilon) \xrightarrow{\varepsilon \to \infty} \infty$
  - $\forall \; c > 0 \quad \|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t \to \infty} 0$ uniformly in t_0

Equivalent stability definitions

Equivalent stability definitions using class $\mathcal{K}$ and $\mathcal{KL}$ functions

The equilibrium point $x^* = 0$ is

- uniformly stable, iff

$\exists$ class $\mathcal{K}$ function α such that
$\exists c > 0$

$$\|x(t)\| \leq \alpha(\|x(t_0)\|)$$

$$\forall t \geq t_0 \geq 0, \quad \forall\ \|x(t_0)\| < c$$

- uniformly asymptotically stable, iff

$\exists$ class $\mathcal{KL}$ function β such that
$\exists c > 0$

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

$$\forall t \geq t_0 \geq 0, \quad \forall\ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable, iff

$\exists$ class $\mathcal{KL}$ function β such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

$$\forall t \geq t_0 \geq 0, \quad \forall\ \|x(t_0)\|$$

Exponential stability

Definition (Exponential stability)

The equilibrium point $x^* = 0$ is exponentially stable, iff

$$\exists c, k, \lambda > 0 \quad \text{s.t.} \quad \|x(t)\| \leq k\,\|x(t_0)\|\, e^{-\lambda(t-t_0)} \qquad \begin{array}{l} t \geq t_0 \geq 0 \\ \|x(t_0)\| \leq c \end{array}$$

Exponential stability

Definition (Exponential stability)

The equilibrium point $x^* = 0$ is exponentially stable, iff

$$\exists c, k, \lambda > 0 \quad \text{s.t.} \quad \|x(t)\| \leq k\,\|x(t_0)\|\,e^{-\lambda(t-t_0)} \qquad \begin{array}{l} t \geq t_0 \geq 0 \\ \|x(t_0)\| \leq c \end{array}$$

Global exponential stability (GES)

If satisfied $\forall c$, then globally exponentially stable

Exponential stability

Definition (Exponential stability)

The equilibrium point $x^* = 0$ is exponentially stable, iff

$$\exists c, k, \lambda > 0 \quad \text{s.t.} \quad \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} \quad \begin{array}{l} t \geq t_0 \geq 0 \\ \|x(t_0)\| \leq c \end{array}$$

Global exponential stability (GES)

If satisfied $\forall c$, then globally exponentially stable

Exponential stability $\Rightarrow$ Uniform asymptotic stability

Special case of uniform asymptotic stability when

$$\beta(r, s) = kre^{-\lambda s}$$

Time-varying Lyapunov function candidates

Time-varying generalized energy function $V(t,x)$

Definition: Positive definite

- $V(t,x)$ is positive definite iff

$$\left.\begin{array}{l} V(t,0)=0 \\ V(t,x)\geq W_1(x) \end{array}\right\} \forall t\geq 0, \text{ for some positive definite } W_1(x)$$

- $V(t,x)$ is positive semidefinite if $W_1(x)$ positive semidefinite
- $V(t,x)$ is radially unbounded if $W_1(x)$ is radially unbounded

Definition: Negative definite

- $V(t,x)$ is negative (semi-)definite iff $-V(t,x)$ is positive (semi-)definite

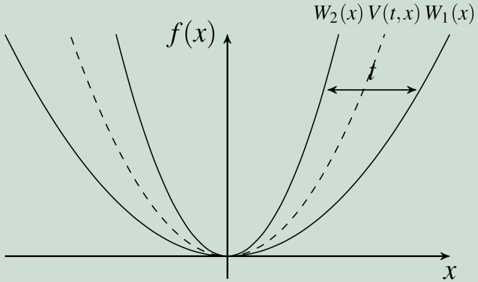
Time-varying Lyapunov function candidates

Definition: Decrescent

- $V(t,x)$ is decrescent iff

$$\left.\begin{array}{l} V(t,0)=0 \\ V(t,x)\leq W_2(x) \end{array}\right\} \forall t\geq 0, \text{ for some positive definite } W_2(x)$$

Positive definite and decrescent $V(t,x)$



Examples

Example

a) $V_A(t,x) = (t+1)(x_1^2 + x_2^2)$

b) $V_B(t,x) = e^{-t}(x_1^2 + x_2^2)$

c) $V_C(t,x) = \frac{1}{1+\cos^2 t}(x_1^2 + x_2^2)$

Q: Positive definite? Positive semidefinite? Radially unbounded? Decrescent?

Stability theorems

$\dot{x} = f(t,x)$ $\quad f : [0,\infty) \times \mathbb{D} \to \mathbb{R}^n$ piecewise continuous in t
locally Lipschitz in x

Stability theorem (Theorems 4.8 - 4.9)

Let $V : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ be a C^1 function

The equilibrium point $x^* = 0$ is

	Stable	Uniformly stable	UAS	GUAS
V	Pos.def.	Pos.def. Decrescent	Pos.def. Decrescent	Pos.def. Decrescent Rad. unb.
$\dot{V}$	Neg.semidef	Neg.semidef.	Neg.def.	Neg.def.
	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D} = \mathbb{R}^n$

Region of attraction

When $x^* = 0$ is uniformly asymptotically stable (UAS)

Estimate of the Region of attraction

Choose r, c such that

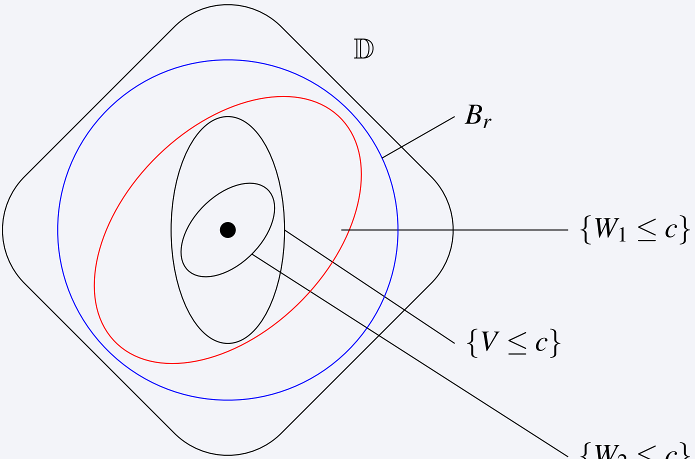
$$B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathbb{D}$$
$$c < \min_{\|x\|=r} W_1(x)$$

then

$$\{x \in B_r : W_2(x) \leq c\}$$

is a region of attraction for $x^* = 0$.

Region of attraction



Exponential stability

Exponential stability (Theorem 4.10)

Let $V : [0,\infty) \times \mathbb{D} \to \mathbb{R} \quad C^1$

If there exist constants $a, k_1, k_2, k_3 > 0$ such that

- $k_1 \|x\|^a \leq V(t,x) \leq k_2 \|x\|^a, \quad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$
- $\dot{V}(t,x) \leq -k_3 \|x\|^a, \qquad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$

then $x^* = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x^* = 0$ is globally exponentially stable.

Examples

Example

Consider the system

$$\dot{x}_1 = -x_1 - e^{-2t}x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Determine the stability properties of $x^* = 0$ using

$$V(t,x) = x_1^2 + (1+e^{-2t})x_2^2$$

Read: Ex 4.19 and Ex 4.20

Inavariance-like theorems (Sec. 8.3)

Time-invariant systems

$$\dot{V}(x) \leq 0 \Rightarrow \text{La Salle } E = \{x \in \Omega_c : \dot{V}(x) = 0\}$$
$$x(t) \rightarrow \text{largest invariant set in E.}$$

Time-varying systems

$$\dot{V}(t,x) \leq 0 \Rightarrow \text{?}$$

Note

- $\dot{f} \rightarrow 0 \not\Rightarrow f$ converges to a limit

  Ex: $f(t) = \sin(10 \log t)$
- f converges to a limit $\not\Rightarrow \dot{f} \rightarrow 0$

  Ex: $f(t) = e^{-t} \sin(e^{2t})$

Inavariance-like theorems (Sec. 8.3)

Time-invariant systems

$$\dot V(x) \leq 0 \Rightarrow \text{La Salle } E = \{x \in \Omega_c : \dot V(x) = 0\}$$
$$x(t) \rightarrow \text{largest invariant set in E.}$$

Time-varying systems

$$\dot V(t,x) \leq 0 \Rightarrow \;?$$

Note

- $\dot f \rightarrow 0 \not\Rightarrow f$ converges to a limit

  Ex: $f(t) = \sin(10 \log t)$
- f converges to a limit $\not\Rightarrow \dot f \rightarrow 0$

  Ex: $f(t) = e^{-t} \sin(e^{2t})$

Barbalat's lemma

Calculus

If $f(t)$ is lower bounded and $\dot{f} \leq 0$

then

f converges to a limit.

Barbalat's lemma (Khalil Lemma 8.2)

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$

If $\lim_{t \to \infty} \int_0^t \varphi(\tau) d\tau$ exists and is finite, then

$$\varphi(t) \to 0 \text{ as } t \to \infty$$

Barbalat's lemma cont.

Barbalat's lemma - rephrased

Let $\varphi = \dot{f}$. We can then rephrase the lemma:

Let $\dot{f} : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$
If $\lim_{t\to\infty} f(t)$ exists and is finite, then

$$\dot{f} \to 0 \text{ as } t \to \infty$$

Barbalat's lemma

Definition: Uniformly continuous

$\varphi : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $[0,\infty)$ iff

$$\forall\, \varepsilon\ \exists\ \delta(\varepsilon) \text{ s.t. } |t - t_1| < \delta \Rightarrow |\varphi(t) - \varphi(t_1)| < \varepsilon\ \forall\ t, t_1 \in [0,\infty)$$

Sufficient condition

$\frac{dg}{dt}$ is bounded (uniformly in t) $\Rightarrow$ g is uniformly continuous on $[0,\infty)$

Barbalat's lemma

With

$$f(t) = V(t, x(t)) \quad C^1$$
$$\varphi(t) = \dot{V}(t, x(t))$$

Barbalat's lemma gives

Barbalat's lemma

If

- V is lower bounded (e.g. $V \geq 0$)
- $\dot{V} \leq 0$
- $\ddot{V}$ is uniformly bounded

then

$$\dot{V} \to 0 \quad \text{as} \quad t \to \infty$$

Example

Adaptive control example

Consider the system

$$\dot{e} = -e + \theta\omega(t) \qquad \dot{\theta} = -e\omega(t)$$

$e = y - y_d(t)$ tracking error

$\theta =$ parameter estimation error

$\omega(t) =$ continuous, bounded function

Analyse the stability properties of the system.

Theorem 8.4

Barbalat's lemma gives Theorem 8.4. Read on your own.

Summary

Lyapunov's direct method for time-varying systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability
- Barbalat's lemma

Next lecture

- Learn that there also exist other stability concepts than Lyapunov stability.
- Recommended reading
  Khalil Section 4.9
  Sections 5.1 and 5.4
  (5.2 - 5.3 and Ex. 5.14 are additional material)

TTK4150 Nonlinear Control Systems

Lecture 7

Input-to-State Stability (ISS)

and

Input-Output Stability (IOS)

NTNU

Previous lecture

Previous lecture:

Lyapunov's direct method for time-varying systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability (ES $\Rightarrow$ UAS)
- Barbalat's lemma

Previous lecture

Previous lecture:

Lyapunov's direct method for time-varying systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability (ES $\Rightarrow$ UAS)
- Barbalat's lemma

In Lectures 3-6 we have been working with

Lyapunov stability

- Definitions
- Stability analysis

Outline I

1. Introduction
   - Previous lecture
   - Today's goals
   - Literature
2. Input-to-State Stability
   - Systems with inputs
   - Motivation for ISS
   - Definition of ISS
   - How to check ISS
   - ISS vs. Lyapunov stability properties
   - How do we use ISS?
3. Stability of cascades
   - Application example
4. Input-output stability
   - Introduction

Outline II

- $\mathscr{L}_p$ Norms and spaces
- Definition
- Causal operators
- Small gain theorem

Today's goals

After today you should...

- Know that there exists other stability concepts than Lyapunov stability

In particular

- Understand the motivation and the definition of Input-to-State stability (ISS)
- Be able to analyze ISS using ISS-Lyapunov functions
- Know some relations between ISS and Lyapunov stability

- Know the definition of Input-Output Stability (IOS)
- Be able to analyze IOS using the definition
- Know the small-gain theorem

Literature

Today's lecture is based on

Khalil Section 4.9

Background material:

- Paper and lecture by E.D. Sontag:
  The ISS Philosophy as a Unifying Framework for Stability-Like Behavior
- Mini-course by A. Loria:
  Cascaded nonlinear time-varying systems: analysis and design

Sections 5.1 and 5.4

(5.2 - 5.3 and Ex. 5.14 are additional material)

Part I

Input-to-State stability (ISS)

Systems with inputs

System

We want to analyse systems described by

$$\dot{x} = f(t, x, u) \qquad (\Sigma)$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Systems with inputs

System

We want to analyse systems described by

$$\dot{x} = f(t, x, u) \qquad (\Sigma)$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Input

$u(t)$ pieciewise continuous, bounded function

- disturbance
- modelling error

Systems with inputs

System

We want to analyse systems described by

$$\dot{x} = f(t, x, u) \qquad (\Sigma)$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Input

$u(t)$ pieciewise continuous, bounded function

- disturbance
- modelling error

When $u(t) = 0$

$$\dot{x} = f(t, x, 0)$$

$x = 0$ is GUAS (0-GUAS)

Systems with inputs

System

We want to analyse systems described by

$$\dot{x} = f(t,x,u) \qquad (\Sigma)$$

$f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Input

$u(t)$ pieciewise continuous, bounded function

- disturbance
- modelling error

When $u(t) = 0$

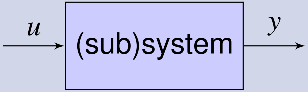
$$\dot{x} = f(t,x,0)$$

$x = 0$ is GUAS (0-GUAS)

What if $u(t) \neq 0$?

Motivation

Motivation



- Adding to control system theorist's "toolkit" for studying systems via decomposition
- Quantify response to external signals
- Unify state-space and i/o stability theory

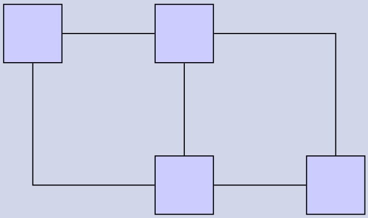
Motivation: Decomposition

Motivation: Decomposition (Cascades)

Even if the original system has no inputs

$$\dot{x} = f(x)$$

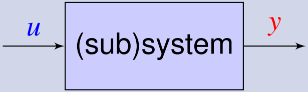
we may study "systems with i/o signal"

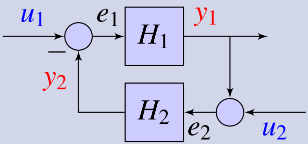


(Otherwise, how do we interconnect them?)

Motivation: Response to external signals

Motivation: Response to external signals



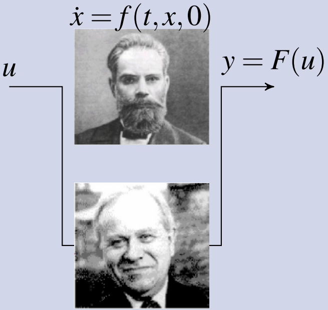


$u = (u_1, u_2)$ = noise, disturbance, modelling error, ...
$y = (y_1, y_2)$ = distance to desired states, tracking error, ...

Motivation: Unify state-space and i/o stability theory

Motivation: Merge Lyapunov/Zames

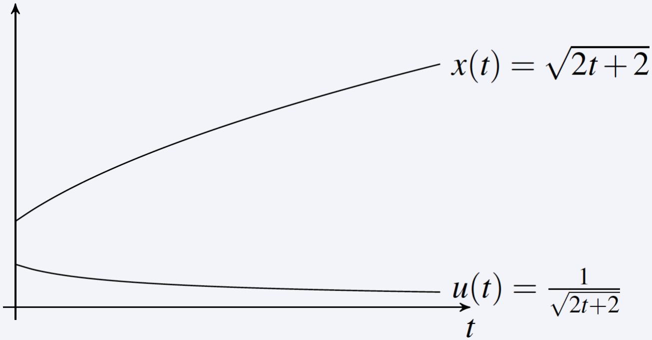
- We have Lyapunov theory for systems without inputs and outputs
- We have a rich theory for stability of input/output operators developed by George Zames, and many others
- ISS allows us to combine features of both



Motivation: $\dot{x} = f(x,0)$ Stable is not enough

For linear $\dot{x} = Ax + Bu$, A Hurwitz $\Rightarrow (u \to 0 \Rightarrow x \to 0)$
i.e. Bounded Input Bounded State (BIBS)

This is NOT true for nonlinear systems. Ex: $\dot{x} = -x + (x^2+1)u$



even though $\dot{x} = f(x,0)$ is GES: $\dot{x} = -x$.

Motivation: Require I/O boundedness

We must bound the solution $\|x(t,x_0,u)\|$ in a "nonlinear gain" sense

$$\|x(t)\| \text{ ("ultimately") } \leq \gamma(\|u(\cdot)\|_\infty)$$

$\gamma \in \mathscr{K}_\infty$:
$\gamma(0) = 0$
$C^0, \nearrow +\infty$

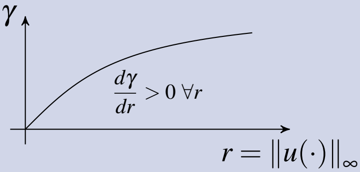


Figure : Example class $\mathscr{K}_\infty$ function γ

Motivation: $\dot{x} = f(x,0)$ GUAS

Repetition (from last lecture):

Global uniform asymptotic stability (GUAS) of the origin means

$$\exists \text{ class } \mathcal{KL} \text{ function } \beta \text{ s.t. } \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \begin{array}{l} \forall t \geq t_0 \geq 0 \\ \forall \ \|x(t_0)\| \end{array}$$

$\|x(t)\| \leq \beta(\|x(t_0)\|, 0) \rightsquigarrow$ stability (small overshoot)

$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \xrightarrow{(t-t_0)\to\infty} 0 \rightsquigarrow$ convergence

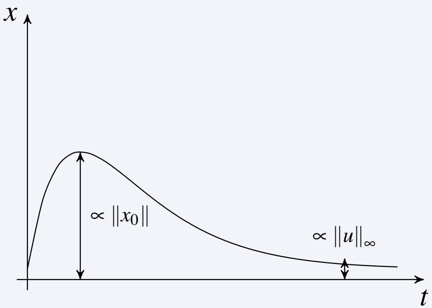
Definition of ISS

Original definition

$\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t.

$$\|x(t,x_0,u)\| \leq \max\{\beta(\|x(t_0)\|, t-t_0), \gamma(\|u\|_\infty)\}$$

Transient (overshoot) depends on x_0
When $(t-t_0)$ is large $x(t)$ bounded by $\gamma(\|u\|_\infty)$ independent of x_0



Definition of ISS: Khalil

An alternative definition is found in Khalil

Definition

Consider

$$\Sigma : \dot{x} = f(t,x,u)$$

The system Σ is ISS if $\exists \beta \in \mathscr{KL}$ and $\exists \gamma \in \mathscr{K}$ such that for any $t_0 \geq 0$, any $x_0 = x(t_0) \in \mathbb{R}^n$ and any bounded input $u(t)$, the solution $x(t)$ exists $\forall t \geq t_0$ and satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)$$

Linear case, for comparison

Example: Linear case

Given a stable linear system:
(i.e. the matrix A is Hurwitz: $\mathrm{Re}(\lambda_i(A)) < 0 \quad \forall i = 1, \ldots, n$)

$$\dot{x} = Ax + Bu$$

Is this an input-to-state stable system?

Linear case, for comparison

Example: Linear case

Given a stable linear system:
(i.e. the matrix A is Hurwitz: Re$(\lambda_i(A)) < 0 \quad \forall i = 1, \ldots, n$)

$$\dot{x} = Ax + Bu$$

Is this an input-to-state stable system?

Well-known that the system solution is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\|x(t)\| \leq \left\|e^{A(t-t_0)}\right\| \|x(t_0)\| + \int_{t_0}^{t} \left\|e^{A(t-\tau)}\right\| \|B\| \|u(\tau)\| d\tau$$

Theorem 4.11: A Hurwitz $\Leftrightarrow \left\|e^{A(t-t_0)}\right\| \leq ke^{-\lambda(t-t_0)} \quad k, \lambda > 0$

Linear case, for comparison

$$\|x(t)\| \le k e^{-\lambda(t-t_0)} \|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \le \tau \le t} \|u(\tau)\|$$

$$\rightsquigarrow \boxed{\|x(t)\| \le \sigma(t-t_0)\|x(t_0)\| + \bar{\gamma}\sup_{t_0\le\tau\le t}\|u(\tau)\|}$$

$$\sigma(t-t_0) = k e^{-\lambda(t-t_0)} \xrightarrow{(t-t_0)\to\infty} 0$$

$$\gamma(r) = \bar{\gamma} r = \frac{k\|B\|}{\lambda} r$$

This is a particular case of the ISS estimate

$$\boxed{\|x(t,x_0,u)\| \le \beta(\|x(t_0)\|, t-t_0) + \gamma(\sup_{t_0\le\tau\le t}\|u(\tau)\|)}$$

How to check ISS?

Definition: ISS Lyapunov function (ISS-LF)

$V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for Σ iff

i) V is C^1

$\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ s.t.

ii) $\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$

iii) $\dot{V}(t,x) = \dfrac{\partial V}{\partial x} f + \dfrac{\partial V}{\partial t} \le -W_3(x) \quad \|x\| \ge \rho(\|u\|) > 0$

where $W_3(x)$ is a C^0 positive definite function on $\mathbb{R}^n$.

A Lyapunov-like theorem for ISS

Theorem 4.19

$\exists$ ISS-LF for $\Sigma \Rightarrow \Sigma$ is ISS

Sontag & Wang 1995

For time-invariant systems: Σ is ISS $\Leftrightarrow \exists$ ISS-LF for Σ

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

Examples

Example

Consider the ISS-LFC

$$V(t,x) = (1 + \frac{1}{1+t})x^2$$

Does there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ s.t. $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$

Examples

Example

$$\dot{x} = -x^3 + x^2 u$$

The system is 0-GUAS
(When $u = 0$: the origin of $\dot{x} = -x^3$ is GUAS)

Determine the system's ISS properties using the ISS-LFC $V(x) = \frac{1}{2}x^2$

Examples

Example

$$\dot{x} = -x^3 + x^2 u$$

The system is 0-GUAS
(When $u = 0$: the origin of $\dot{x} = -x^3$ is GUAS)

Determine the system's ISS properties using the ISS-LFC $V(x) = \frac{1}{2}x^2$

Read

Read Examples 4.25 - 4.27

ISS vs. Lyapunov stability properties

ISS vs. 0-GUAS

$$\Sigma \text{ is ISS} \Rightarrow \Sigma \text{ is 0-GUAS}$$

$$\Updownarrow$$

$$\neg(\Sigma \text{ is 0-GUAS}) \Rightarrow \neg(\Sigma \text{ is ISS})$$

ISS vs. 0-GES (Lemma 4.6)

$\Sigma : \dot{x} = f(t,x,u)$ f is C^1 and globally Lipschitz in (x,u)

$$\Sigma \text{ is 0-GES} \Rightarrow \Sigma \text{ is ISS}$$

How do we use ISS: Cascades

Stability of cascades

$$\boxed{\Sigma_2} \longrightarrow \boxed{\Sigma_1}$$

$$\begin{aligned} \Sigma_1: \quad \dot{x}_1 &= f_1(t, x_1, x_2) \\ \Sigma_2: \quad \dot{x}_2 &= f_2(t, x_2) \end{aligned}$$

$f_1 : [0,\infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $f_2 : [0,\infty) \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ are piecewise continuous in t and locally Lipschitz in x

Lemma 4.7

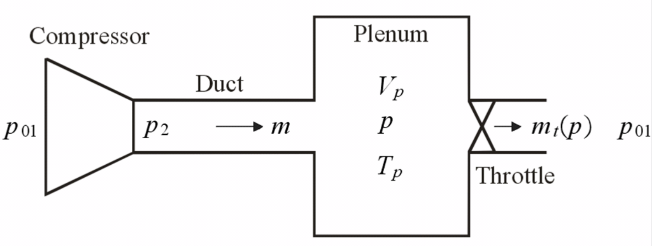
$$\underbrace{\boxed{\text{GUAS}} \longrightarrow \boxed{\text{ISS}}}_{\text{GUAS}}$$

Example

Example

$$\dot{x}_1 = -x_1^3 + x_1^2 x_2$$
$$\dot{x}_2 = -kx_2 \qquad k > 0$$

Application example: Compressor



$$\dot{m} = \frac{A_1}{L_c}\left(p_2(m,\omega) - p\right)$$

$$\dot{p} = \frac{a_{01}^2}{V_p}\left(m - m_t(p)\right)$$

$$\dot{\omega} = \frac{1}{J}\left(\tau_d - \sigma r_2^2 |m| \omega\right)$$

- Objective: Active surge control
  - High efficiency
  - Avoid surging: pressure and mass flow oscillations
- Need mass flow observer
  - Bøhagen & Gravdahl (2003) - reduced order observer

Compressor application cont.

- Suggested observer

$$\dot{z} = \frac{A_1}{L_c}(p_2 - p - u) + k_{\tilde{m}}\left(m_t(p) - \hat{m}\right)$$
$$\dot{\hat{m}} = z + k_{\tilde{m}} \frac{V_p}{a_{01}^2} p$$

- Observer error is GES

$$\dot{\tilde{m}} = -k_{\tilde{m}} \tilde{m}$$

- CE control yields the cascade

$$\Sigma_1 : \quad \dot{x}_1 = f_1(x_1) + g(x_1, x_2)$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(x_2)$$

- Interconnection

$$|g(x_1, x_2)| \leq g_1 |x_2|$$

- Hence, Σ_1 is ISS wrt x_2

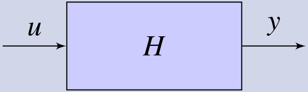
$\Rightarrow$ The cascade is GUAS

Part II

Input-output stability (IOS)

Introduction

Input-output models



We consider systems on the form

$$y = Hu$$

$u : [0, \infty) \to \mathbb{R}^m$ piecewise continuous
$y : [0, \infty) \to \mathbb{R}^q$ piecewise continuous

Input-output stability

How do we analyze stability of such systems?

$\mathscr{L}_p$ Norms and spaces

We need a measure of the size of a signal ($u(t)$ and $y(t)$)

Recall from Lecture 1: Norm

Norms on $C[0,\infty)$

$$\left.\begin{aligned} \|f\|_{\mathscr{L}_p} &= \left(\int_0^\infty |f(t)|^p \, dt\right)^{\frac{1}{p}} \\ \|f\|_{\mathscr{L}_\infty} &= \sup_{0\le t\le\infty} |f(t)| \end{aligned}\right\} \quad \mathscr{L}_p - \text{norms}$$

$\mathscr{L}_p$-space

$$(C[0,\infty), \mathscr{L}_p - \text{norm})$$

- $f \in \mathscr{L}_p \Leftrightarrow \|f\|_{\mathscr{L}_p}$ is bounded $\quad (\exists\, c : \|f\|_{\mathscr{L}_p} \le c)$

$\mathscr{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$

$\mathscr{L}_p^m$ space

$$u \in \mathscr{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_{\bar{p}}^p \, dt \right)^{\frac{1}{p}} < \infty$$

$\mathscr{L}_2^m$ space (with $\bar{p} = 2$)

$$u \in \mathscr{L}_2^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_2} = \sqrt{\int_0^\infty u^T(t) u(t) \, dt} < \infty$$

$\mathscr{L}_\infty^m$ space

$$u \in \mathscr{L}_\infty^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_\infty} = \sup_{t \ge 0} \|u(t)\|_{\bar{p}} < \infty$$

$\mathcal{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$

$\mathcal{L}_p^m$ space

$$u \in \mathcal{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|_{\bar{p}}^p \, dt \right)^{\frac{1}{p}} < \infty$$

Any $\bar{p}$-norm on $\mathbb{R}^m$

$\mathcal{L}_2^m$ space (with $\bar{p} = 2$)

$$u \in \mathcal{L}_2^m \quad \Leftrightarrow \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t) u(t) \, dt} < \infty$$

$\mathcal{L}_\infty^m$ space

$$u \in \mathcal{L}_\infty^m \quad \Leftrightarrow \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t \ge 0} \|u(t)\|_{\bar{p}} < \infty$$

$\mathscr{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0,\infty) \to \mathbb{R}^m$

$\mathscr{L}_p^m$ space

$$u \in \mathscr{L}_p^m \quad 1 \leq p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_p^p \, dt\right)^{\frac{1}{p}} < \infty$$

$\mathscr{L}_2$: "Space of piecewise continuous, square-integrable functions"
$\mathscr{L}_\infty$: "Space of piecewise continuous, bounded functions"

Notation

$u \in \mathscr{L}_p^m \qquad u \in \mathscr{L}_p \qquad u \in \mathscr{L}^m \qquad u \in \mathscr{L}$

$\mathscr{L}_{pe}^m$ - space

To be able to handle unbounded signals we introduce an extended space:

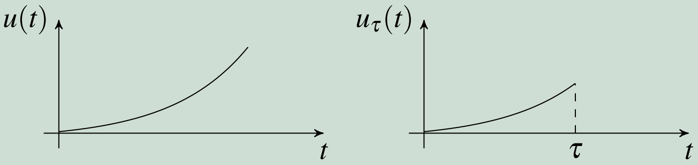
$\mathscr{L}_{pe}^m$ - space

$$u \in \mathscr{L}_{pe}^m \Leftrightarrow u_\tau \in \mathscr{L}_p^m \quad \forall\ \tau \in [0,\infty)$$

where

$$u_\tau(t) = \begin{cases} u(t), & t \in [0,\tau] \\ 0, & t > \tau \end{cases} \quad \text{truncation}$$

$u(t) = e^t$



$\mathscr{L}_{pe}^m$ - space

To be able to handle unbounded signals we introduce an extended space:

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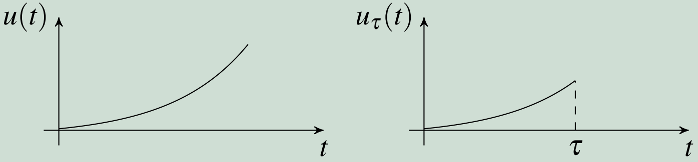
where

$$u_\tau(t) = \begin{cases} u(t), & t \in [0,\tau] \\ 0, & t > \tau \end{cases} \quad \text{truncation}$$

$u(t) = e^t$



Is $u(t) \in \mathscr{L}_\infty^1$?

$\mathscr{L}_{pe}^m$ - space

To be able to handle unbounded signals we introduce an extended space:

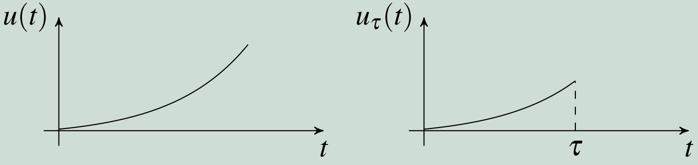
$\mathscr{L}_{pe}^m$ - space

$$u \in \mathscr{L}_{pe}^m \Leftrightarrow u_\tau \in \mathscr{L}_p^m \quad \forall\ \tau \in [0,\infty)$$

where

$$u_\tau(t) = \begin{cases} u(t), & t \in [0,\tau] \\ 0, & t > \tau \end{cases} \quad \text{truncation}$$

$u(t) = e^t$



Is $u_\tau(t) \in \mathscr{L}_\infty^1$?

Input-output stability

Consider the mapping

$$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$$

$\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is $\underline{\mathscr{L}_p \text{ stable}}$ iff

i) $\exists \alpha$ class $\mathscr{K}$ $\quad \alpha : [0,\infty) \to [0,\infty)$

ii) $\exists$ constant $\beta \geq 0$

such that

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \alpha(\|u_\tau\|_{\mathscr{L}_p}) + \beta \qquad \forall u \in \mathscr{L}_{pe}^m \text{ and } \tau \in [0,\infty)$$

BIBO stability $\equiv$ $\mathscr{L}_\infty$ stability

Input-output stability cont.

Finite-gain $\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is finite-gain $\mathscr{L}_p$ stable iff

$$\exists \text{ constants } \gamma, \beta \geq 0$$

such that

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \gamma \, \|u_\tau\|_{\mathscr{L}_p} + \beta$$

$\mathscr{L}_p$ gain

Bias term

Causal

Definition (causal)

$H : \mathscr{L}_e^m \to \mathscr{L}_e^q$ is causal iff

$$(Hu)_\tau = (Hu_\tau)_\tau$$

If H is causal and $\mathscr{L}_p$ stable, then

$$u \in \mathscr{L}_p^m \Rightarrow Hu \in \mathscr{L}_p^q$$

and

$$\|(Hu)\|_{\mathscr{L}_p} \leq \alpha(\|u\|_{\mathscr{L}_p}) + \beta$$

If H is causal and finite-gain $\mathscr{L}_p$ stable, then

$$u \in \mathscr{L}_p^m \Rightarrow Hu \in \mathscr{L}_p^q$$

and

$$\|(Hu)\|_{\mathscr{L}_p} \leq \gamma\|u\|_{\mathscr{L}_p} + \beta$$

Examples

Example

Given

$$y = u^{\frac{1}{3}},$$

is it BIBO stable?

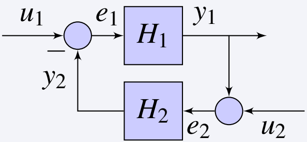
Read

Read Examples 5.1 and 5.3

Read Definition 5.2 page 201

Small gain theorem

Feedback interconnection



$$H_1 : \mathscr{L}_e^m \to \mathscr{L}_e^q \qquad H_2 : \mathscr{L}_e^q \to \mathscr{L}_e^m$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Stability of feedback interconnection

The feedback interconnection where H_1 and H_2 are finite-gain $\mathscr{L}$-stable, i.e.

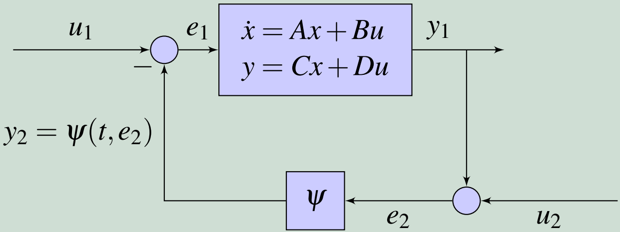
$$\|y_{1\tau}\|_{\mathscr{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathscr{L}} + \beta_1$$
$$\|y_{2\tau}\|_{\mathscr{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathscr{L}} + \beta_2$$

is finite-gain $\mathscr{L}$-stable if

$$\gamma_1 \gamma_2 < 1$$

Example

Example



A Hurwitz
$G(s) = C(sI - A)^{-1}B + D$

Analyse the input-output stability properties of the interconnection.

Next lecture

- How to analyze the passivity properties of a system by using the definition of passivity for
  - Memoryless functions
  - Dynamical systems
- The relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IO stability)
- Passivity theorems (for feedback interconnections)
- Recommended reading

  Khalil **Chapter 6**
  Sections 6.1 and 6.2
  (Section 6.3 is additional material)
  Sections 6.4 - 6.5, page 254
  (Pages 254-259, including Ex. 6.12, is add. material)

Next week

Study week:
Start working on solving exam sets from previous years

- test your understanding and skills for important parts of the syllabus
- review and deepen your understanding and skills of the topics that we have worked on through Weeks 34 - 40

TTK4150 Nonlinear Control Systems

Lecture 8

Passivity

NTNU

Previous lecture

Previous lecture:

- Introduced other stability concepts than Lyapunov stability.

In particular

- Motivation and definition of Input-to-State stability (ISS)
- ISS analysis using ISS-Lyapunov functions
- Relations between ISS and Lyapunov stability

- Definition of Input-Output Stability (IOS)
- How to analyze IOS using the definition
- Small-gain theorem

Outline

1. Introduction
   - Previous lecture
   - Today's goals
   - Literature
2. Passivity for memoryless functions
   - Passive/Lossless
   - Input strictly passive
   - Output strictly passive
3. Passivity for dynamical systems
   - Motivating example
   - Definitions
4. Lyapunov and $\mathscr{L}_2$ stability of passive systems
   - Relations between Passivity properties and Lyapunov stability
   - Relations between Passivity properties and $\mathscr{L}_2$ stability
   - Relations between Passivity properties and Asymptotic stability
5. Passivity theorems
   - Passivity and Lyapunov stability of feedback connection
   - $\mathscr{L}_2$-stability of feedback connection
   - Asymptotic stability of feedback connection
6. Next lecture

Today's goals

After today you should...

- Be able to analyze the passivity properties of a system by using the definition of passivity for
  - Memoryless functions
  - Dynamical systems
- Understand the relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IOS)
- Know the passivity theorems
  (for feedback connections)

Literature

Today's lecture is based on

Khalil **Chapter 6**

Sections 6.1 and 6.2

(Section 6.3 is additional material)

Sections 6.4 - 6.5, page 254

(Pages 254-259, incl. Ex. 6.12, is additional material)

Passivity

What is passivity?

- A tool (not a stability concept) for analysis and design of control systems
- Based on an Input-Output description of systems
- Has an interesting energy interpretation
  (allows the control engineer to relate a set of efficient mathematical tools to well known physical phenomena)

Main use:

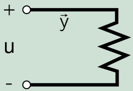
- Relates nicely to
  - Lyapunov stability
  - $\mathcal{L}_2$ stability
- Can provide a somewhat systematic way to build Lyapunov functions
- Can give conclusions about properties of feedback connections (based on the properties of each subsystem)

Passivity for memoryless functions

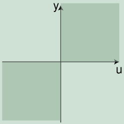
Memoryless functions

$$y = h(t,u) \qquad h : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^p$$

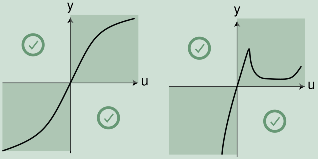
Example: An electric circuit



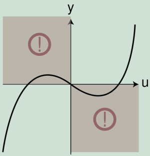
A passive resistor



u-y-characteristic lies in the first-third quadrant



Examples of nonlinear passive resistors characteristics



Nonpassive resistor

Passive elements

Passive elements in electric circuits

Passive element: The element cannot generate energy

$$P = u \cdot i = u \cdot y$$

$P > 0$ The element absorbs energy

$P < 0$ The element generates energy

$$\text{Passive} \quad \Leftrightarrow \quad P \geq 0 \quad \Updownarrow \quad u \cdot i \geq 0$$

Generalized definition

The system is passive if

$$u^T y \geq 0$$

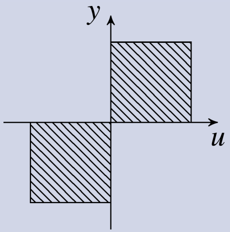
Passivity definition

Definition: Passive/Lossless

The memoryless system $y = h(t,u)$ is

- Passive if $u^T y \geq 0$

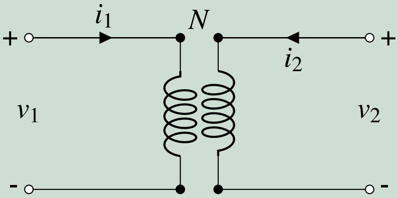
  i.e. $h \in$ sector $[0,\infty]$



- Lossless if $u^T y = 0$

Example

Ideal transformer



$$y = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \quad u = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \quad y = Su \quad S = \begin{bmatrix} 0 & -N \\ N & 0 \end{bmatrix}$$

Analyse the passivity properties of the ideal transformer

Input strictly passive

Definition: Input strictly passive

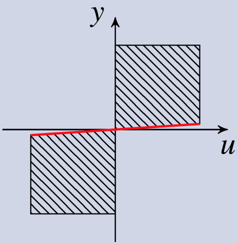
The memoryless system $y = h(t,u)$ is input strictly passive iff

$$u^T y \geq u^T \varphi(u)$$

and

$$u^T \varphi(u) > 0 \quad \forall\, u \neq 0$$

i.e. $h \in$ sector $(0, \infty]$



Output strictly passive

Output strictly passive

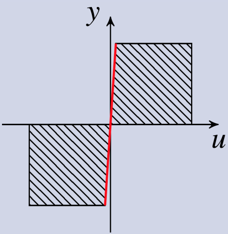
The memoryless system $y = h(t,u)$ is output strictly passive iff

$$u^T y \geq y^T \rho(y)$$

and

$$y^T \rho(y) > 0 \quad \forall\, y \neq 0$$

i.e. $h \in$ sector $[0, \infty)$



Passivity for dynamical systems

Dynamical systems

We consider dynamical systems

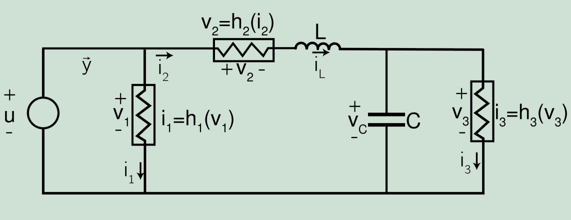
$$\Sigma \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz
$h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ continuous

$f(0,0) = 0$ and $h(0,0) = 0$

Motivating example

Motivating example: Electric circuit



$$x = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

Kirchoff's laws give

$$\begin{aligned} L\dot{x}_1 &= u - h_2(x_1) - x_2 \\ C\dot{x}_2 &= x_1 - h_3(x_2) \\ y &= x_1 + h_1(u) \end{aligned}$$

Passive circuit

Passive electric circuits

Passive circuits cannot generate electric energy i.e.

$$\text{change of stored energy} \quad \leq \quad \text{energy supplied}$$

$$V(x(t)) - V(x(0)) \quad \leq \quad \int_0^t u(s)y(s)ds$$

Generalized definition

The system is passive iff

$$u(t)^T y(t) \geq \dot{V}(x(t), u(t)) \quad \forall\, t \geq 0$$

Passivity definitions for dynamical systems

Definition

The dynamical system is

- passive if

  $\exists C^1$ positive semidefinite function $V(x) : \mathbb{R}^n \to \mathbb{R}$

  (Storage function)

  such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Moreover, it is

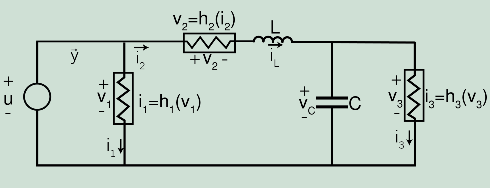
- lossless if

$$u^T y = \dot{V}$$

Example

Example: Electric circuit

Given the electric circuit



The energy stored in the RLC network is

$$V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$$

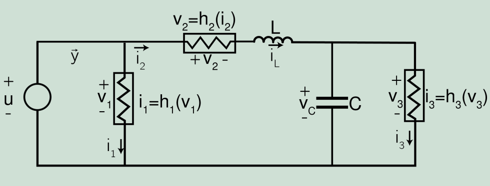
Choose input = input voltage u
output = current $y = x_1 + h_1(u)$

Analyse the passivity properties of the RLC network

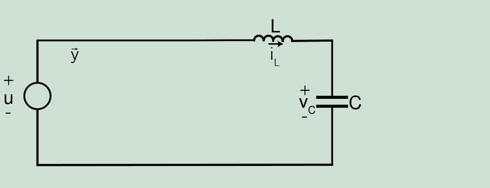
Example

Example: Electric circuit

Given the electric circuit



Analyse the passivity properties when $h_1 = h_2 = h_3 = 0$:



Strict passivity

Definition continued

- Input strictly passive if

$$u^T y \geq \dot{V} + u^T \varphi(u), \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$$

- Output strictly passive if

$$u^T y \geq \dot{V} + y^T \rho(y), \quad y^T \rho(y) > 0 \quad \forall y \neq 0$$

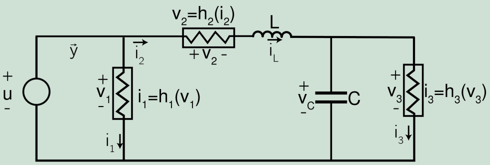
- (State) Strictly passive if

$$u^T y \geq \dot{V} + \psi(x), \quad \psi(x) \quad \text{positive definite function}$$

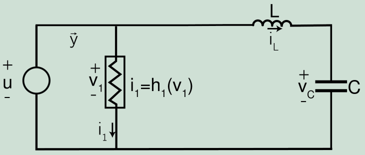
Example

Example: Electric circuit

Given the electric circuit



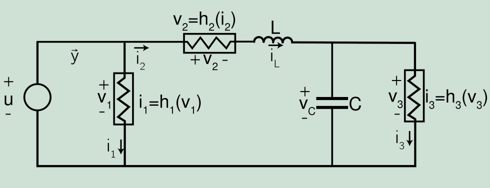
Analyse the passivity properties when $h_2 = h_3 = 0$:



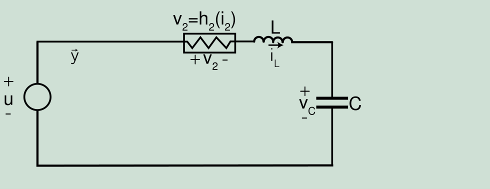
Example

Example: Electric circuit

Given the electric circuit



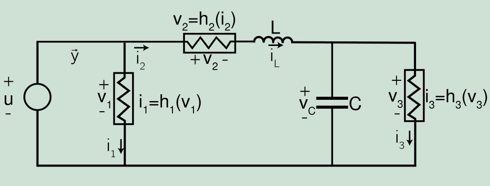
Analyse the passivity properties when $h_1 = h_3 = 0$:



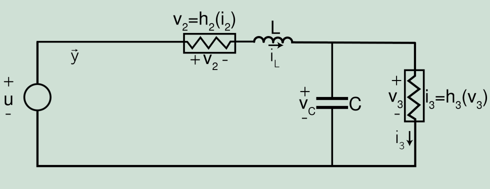
Example

Example: Electric circuit

Given the electric circuit



Analyse the passivity properties when $h_1 = 0$:



Relations between Passivity properties and Lyapunov stability

Dynamical systems

$$\Sigma \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz
$h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ continuous

$f(0,0) = 0$ and $h(0,0) = 0$

Lemma 6.6 (Lyapunov stable (0-stable))

If Σ is passive with a *positive definite* storage function $V(x)$, then

the origin of $\dot{x} = f(x,0)$ is stable

Relations between Passivity properties and $\mathscr{L}_2$ stability

Lemma 6.5 (Finite-gain $\mathscr{L}_2$ stable)

If Σ is output strictly passive with $\rho(y) = \delta y$, $\delta > 0$, then

Σ is finite-gain $\mathscr{L}_2$ stable
with $\mathscr{L}_2$-gain $\gamma \leq \frac{1}{\delta}$

Asymptotic stability of passive systems

Lemma 6.7 (Asymptotically stable (0-AS))

The origin of $\dot{x} = f(x,0)$ is asymptotically stable if Σ is either

- state strictly passive

or

- output strictly passive
  zero-state observable

If furthermore $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

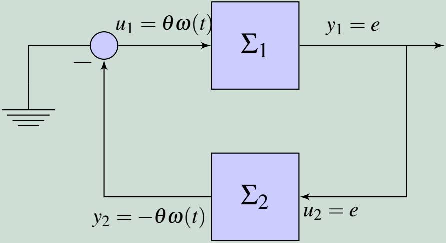
Definition: Zero-state observability

Σ is zero-state observable iff
no solution of $\dot{x} = f(x,0)$ can stay identically in
$S = \{x \in \mathbb{R}^n | h(x,0) = 0\}$ other than the trivial solution $x(t) = 0$.

Example: Adaptive control system

Adaptive control system

$$\dot{e} = -e + \theta\omega(t)$$
$$\dot{\theta} = -e\omega(t)$$



Subsystem Σ_1

$$\dot{x}_1 = -x_1 + u_1$$
$$y_1 = ?$$

- Investigate the passivity properties of subsystem Σ_1
- What can thus be concluded about the stability properties of subsystem Σ_1

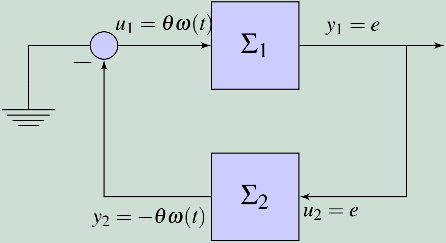
Example: Adaptive control system cont.

Adaptive control system, cont.

$$\dot{e} = -e + \theta\omega(t)$$
$$\dot{\theta} = -e\omega(t)$$

Subsystem Σ_2

$$\dot{x}_2 = -u_2\omega(t)$$
$$y_2 = ?$$

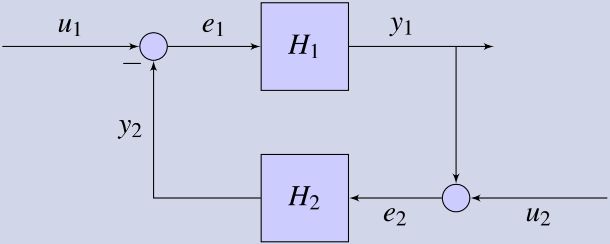


- Investigate the passivity properties of subsystem Σ_2
- What can thus be concluded about the stability properties of subsystem Σ_2

Passivity theorems
Feedback systems

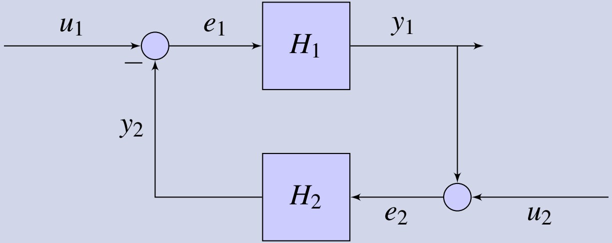
Feedback systems

Σ



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Passivity theorems
Passivity and Lyapunov stability of feedback connection



Theorem 6.1: Passivity of feedback connection

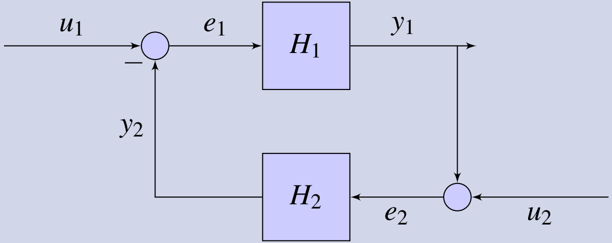
H_1 passive and H_2 passive $\Rightarrow \Sigma$ passive (with $V = V_1 + V_2$)

Lyapunov stability of feedback connection

Theorem 6.1 + Lemma 6.6 gives Lyapunov stability (0-stability) of the feedback connection when V is positive definite

Passivity theorems

$\mathscr{L}_2$-stability of feedback connection



Theorem 6.2: $\mathscr{L}_2$-stability of feedback connection

If H_1 and H_2 satisfy

$$e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i \qquad i = 1, 2$$

and

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0$$

then Σ is finite-gain $\mathscr{L}_2$-stable.

Passivity theorems

Asymptotic stability of feedback connection

Theorem 6.3: Asymptotic stability of feedback connection

If

- H_1 and H_2 state strictly passive

or

- H_1 and H_2 output strictly passive and zero-state observable

or

- H_1 state strictly passive
  H_2 output strictly passive and zero-state observable
  or opposite

then Σ is 0-AS

If furthermore V_1 and V_2 are radially unbounded then Σ is 0-GAS.

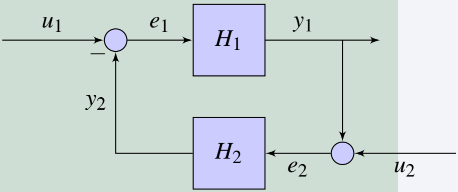
Example: DC Motor control system

Example: DC Motor control system (Boyd, 1997)

A DC motor is characterized by

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\omega + u$$

where θ is the shaft angle and u is the input voltage.



The dynamic controller

$$\dot{z} = 2(\theta - z) - \mathrm{sat}(\theta - z)$$
$$u = z - 2\theta$$

is used to control the shaft position. Use passivity analysis to prove that $\theta(t)$ and $\omega(t)$ converge to zero as $t \to \infty$

Hint

Using the state transformation $x = z - \theta$ the dynamic controller can be rewritten as

$$\dot{x} = -2x + \mathrm{sat}(x) - \omega$$
$$u = x - \theta$$

Next lecture

Next lecture: **Passivity-based control**

Khalil **Chapter 6**

Sections 6.4 and 6.5

(Pages 254-259, including Ex. 6.12, is additional material)

Chapter 14

Section 14.4

TTK4150 Nonlinear Control Systems

Lecture 9

Passivity-based control design

NTNU

Previous lecture

Previous lecture:

Passivity

- How to analyze the passivity properties of a system by using the definitions of passivity for
  - Memoryless functions
  - Dynamical systems
- Understand the relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IOS)
- The passivity theorems
  (for feedback connections)

Outline I

1. Introduction
   - Previous lecture
   - Today's goals
   - Literature
2. Energy-based Lyapunov control design
   - Lyapunov control design
   - Example: Dynamic positioning system for ships
   - Example: Two-link robot manipulator
3. Control design using passivity theorems
   - 2 useful results: Passivity of LTI systems
   - PID feedback control
   - Example: Motor control
   - A new passivity theorem for stabilization: Theorem 14.4
4. How to achieve passivity
   - Choice of y

Outline II

- Choice of u (Feedback passivation)

5 Summary

Today's goals

After today you should...

Be able to design a passivity-based feedback control law

- Energy-based Lyapunov control design
- Using passivity theorems

Literature

Today's lecture is based on

Khalil	**Chapter 6**
	Sections 6.4 - 6.5
	(Pages 254-259, incl. Ex. 6.12, is additional material)
	Chapter 14
	Section 14.4
Lozano et al.	**Dissipative Systems Analysis and Control**
	Sections 2.3 - 2.4

Lyapunov Control Design

Lyapunov Control Design

- Alternative A)

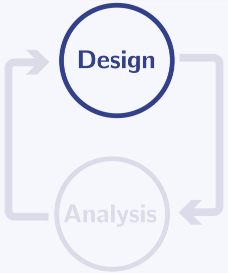
  Propose a control law $u = g(t,x)$

  Analyze the resulting system by Lyapunov's Direct Method (incl. La Salle/Barbalat)

- Alternative B)

  Propose a Lyapunov function candidate

  Find a control law $u = g(t,x)$ that makes this LFC a (strict) Lyapunov function



Lyapunov Control Design

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- Alternative B)

  Propose a Lyapunov function candidate

  Find a control law $u = g(t,x)$ that makes this LFC a (strict) Lyapunov function

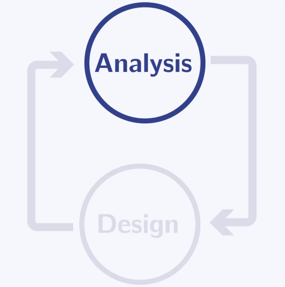
Energy-based Lyapunov Control Design

Energy-based Lyapunov Control Design

- Alternative B)

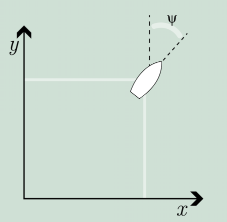
  Propose a Lyapunov function candidate
  = *Desired energy of the closed-loop system*

  Find a control law $u = g(t,x)$ that makes this LFC a (strict) Lyapunov function



Example: Dynamic positioning system for ships

Dynamic positioning system for ships



$$\eta = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

System model:

$$M(\eta)\ddot{\eta} + C(\eta, \dot{\eta})\dot{\eta} + D(\eta)\dot{\eta} = \tau$$

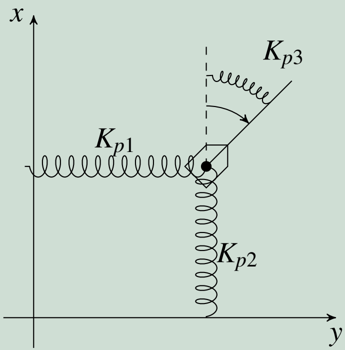
System properties:

$$M = M^T > 0$$
$$z^T D z > 0 \quad \forall z \neq 0$$
$$z^T\left(\tfrac{1}{2}\dot{M} - C\right)z = 0 \quad \forall\, z \in \mathbb{R}^3$$

Find a control law $\tau = g(t, (\eta, \dot{\eta}))$ that makes the origin $(\eta, \dot{\eta}) = (0, 0)$ an asymptotically stable equilibrium point.

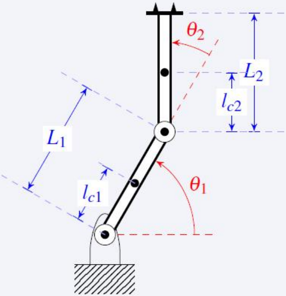
Example cont.

Desired energy of the closed-loop system



- We shape the (potential) energy
- We add virtual spring forces

Example: Two-link robot manipulator



Dynamic model:

General robot manipulator:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

System properties:

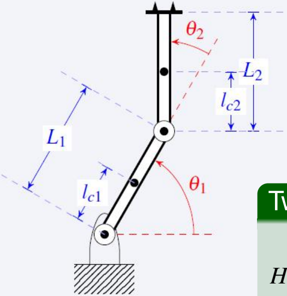
$$M(q) = M^T(q) > 0 \quad \forall q \in \mathbb{R}^m$$
$$z^T(\tfrac{1}{2}\dot{M} - C)z = 0 \quad \forall\, z, q, \dot{q} \in \mathbb{R}^m$$

Dynamic model:

Two-link robot manipulator:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Example: Two-link robot manipulator



Control problem:

Find a feedback control law that stabilizes a constant desired configuration θ_d.
Let $q = \theta - \theta_d$, i.e. stabilize $(q, \dot{q}) = (0, 0)$

Two-link robot manipulator

$$H_{11} = m_1 l_{c_1}^2 + I_1 + m_2 \left[l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cos q_2 \right] + I_2$$

$$H_{22} = m_2 l_{c_2}^2 + I_2$$

$$H_{12} = H_{21} = m_2 l_1 l_{c_2} \cos q_2 + m_2 l_{c_2}^2 + I_2$$

$$h = m_2 l_1 l_{c2} \sin q_2$$

$$g_1 = m_1 l_{c1} g \cos q_1 + m_2 g \left[l_{c_2} \cos (q_1 + q_2) + l_1 \cos q_1 \right]$$

$$g_2 = m_2 l_{c_2} g \cos (q_1 + q_2)$$

Control design using passivity theorems

Two useful results to include linear systems/controllers:

Result 1 (Theorem 2.3, Lozano et al.)

LTI system $y(s) = h(s)u(s)$ $h(s)$ rational transfer function

$\text{Re}(p_i) < 0, \forall i$

1) The system is passive $\Leftrightarrow \text{Re}[h(j\omega)] \geq 0, \quad \forall \omega$

2) The system is input strictly passive ($\varphi(u) = \delta u$)

$$\Updownarrow$$

$$\text{Re}[h(j\omega)] \geq \delta > 0, \quad \forall\ \omega$$

3) The system is output strictly passive ($\rho(y) = \varepsilon y$)

$$\Updownarrow$$

$$\exists \varepsilon > 0 \text{ s.t. } \text{Re}[h(j\omega)] \geq \varepsilon |h(j\omega)|^2$$

Control design using passivity theorems

Example: Time constant

Example: Time constant

Consider the system $y(s) = \frac{1}{1+Ts}u(s), \quad T > 0$

Analyze the passivity properties of this LTI system

Control design using passivity theorems

Two useful results to include linear systems/controllers:

Result 2 (Proposition 2.1, Lozano et al.)

Let

$$h(s) = \frac{(s-z_1)(s-z_2)\cdots}{s(s-p_1)(s-p_2)\cdots} \qquad \text{Re}(z_i) < 0, \quad \text{Re}(p_i) < 0$$

$$\text{The system } y(s) = h(s)u(s) \text{ is passive}$$

$$\Updownarrow$$

$$\text{Re}[h(j\omega)] \geq 0 \quad \forall\ \omega$$

Control design using passivity theorems

Example: PID controllers

Example: PID controllers

PID controller
with bounded derivative action:

$$h_{r1}(s) = K_p \frac{1+T_i s}{T_i s} \cdot \frac{1+T_d s}{1+\alpha T_d s} \qquad \begin{array}{l} 0 \le T_d < T_i \\ 0 \le \alpha \le 1 \end{array} \qquad K_p > 0$$

PID controller
with bounded integral action and bounded derivative action:

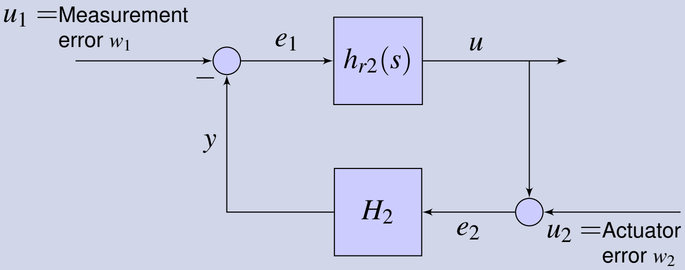
$$h_{r2}(s) = K_p \beta \frac{1+T_i s}{1+\beta T_i s} \cdot \frac{1+T_d s}{1+\alpha T_d s} \qquad \begin{array}{l} 0 \le T_d < T_i \\ 0 < \alpha \le 1 \\ 1 \le \beta < \infty \end{array} \qquad K_p > 0$$

Analyze the passivity properties of these two PID controllers

PID feedback control

Analysis/Design using passivity theorems

PID feedback control: Analysis/Design using passivity theorems



Example: Motor control

Analysis/Design using passivity theorems

Example: Motor control

Motor and load with elastic transmission:

$$J_m \ddot{\theta}_m = \phi_K(\Delta\theta) + \phi_D(\Delta\dot{\theta}) + T_m + F(\dot{\theta}_m)$$
$$J_L(\ddot{\theta}_m + \Delta\ddot{\theta}) = -(\phi_K(\Delta\theta) + \phi_D(\Delta\dot{\theta}))$$

θ_m motor angle
$\Delta\theta$ angular deflection through spring
$\Delta\theta = \theta_L - \theta_m$

$\phi_K \in$ sector $[0, \infty)$
$\phi_D \in$ sector $[0, \infty)$

$$F(\dot{\theta}_m) = \begin{cases} F_0, & \dot{\theta}_m < 0 \\ -F_0, & \dot{\theta}_m > 0 \end{cases}$$

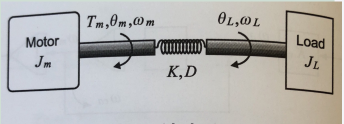
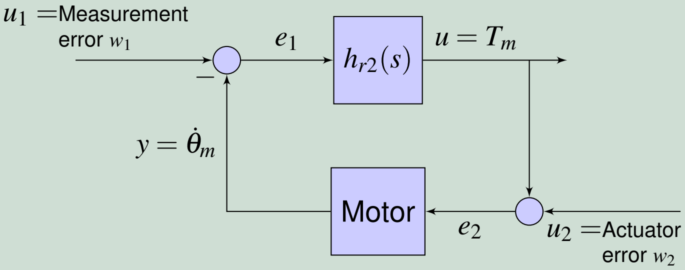


Figure 3.10: Motor with elastic transmission

Example: Motor control

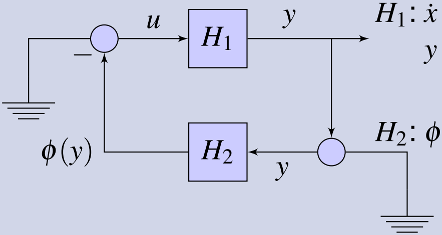
Analysis/Design using passivity theorems

Motor control: Analysis/Design using passivity theorems



A new passivity theorem for stabilization

Theorem 14.4



H_1: $\dot{x} = f(x,u)$ locally Lipschitz
$y = h(x)$ continuous
$f(0,0) = 0 \quad h(0) = 0$

H_2: $\phi(y)$ memoryless function
locally Lipschitz
$\phi(0) = 0$

If

i) H_1 is
- passive with V positive definite and radially unbounded
- zero-state observable

ii) H_2 satisfies $y^T\phi(y) > 0, \quad y \neq 0$

then the origin is *globally asymptotically stable*.

How to achieve passivity

Choice of y

Let

$$\dot{x} = f(x) + G(x)u \qquad \text{(affine system)}$$

If $\exists\ V(x)$

- C^1
- positive definite
- radially unbounded
- $\dfrac{\partial V}{\partial x} f(x) \leq 0$

Choose $y = \left[\dfrac{\partial V}{\partial x} G(x)\right]^T$

Then $u \mapsto y$ is passive

How to achieve passivity

Choice of u (Feedback passivation)

Let

$$\dot{x} = f(x) + G(x)u$$

Choose

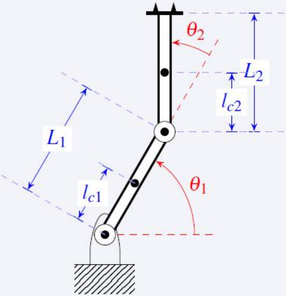
$$\begin{aligned} u &= \alpha(x) + \beta(x)v \\ y &= h(x) \end{aligned}$$

such that

$$\begin{aligned} \dot{x} &= f(x) + G(x)\alpha(x) + G(x)\beta(x)v \\ y &= h(x) \end{aligned}$$

has desired passivity properties $v \mapsto y$

Example: Robot manipulator



Dynamic model:

General robot manipulator:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

System properties:

$$M(q) = M^T(q) > 0 \quad \forall q \in \mathbb{R}^m$$
$$z^T\left(\tfrac{1}{2}\dot{M} - C\right)z = 0 \quad \forall\, z, q, \dot{q} \in \mathbb{R}^m$$

Control problem:

Find a feedback control law that stabilizes $(q, \dot{q}) = (0, 0)$ using passivity based control design.

Summary

Passivity

- Relates nicely to Lyapunov stability and $\mathscr{L}_2$ stability
- Can provide a somewhat systematic way to build Lyapunov functions

  Choosing the LFC thinking in terms of energy of the controlled system. Typically potential energy shaping, possibly kinetic energy shaping (acceleration feedback). This is also denoted Energy-based control.
- Can give conclusions about properties of feedback connections based on the properties of each subsystem

  This allows for modular analysis and design, something which simplifies the design process. (Resembling the cascade results for ISS systems, page 179-180)

Summary

Passivity

- **Robustness:** If the model possesses the same passivity properties regardless of the numerical values of the physical parameters, and a controller is designed so that stability relies on the passivity properties only, the closed-loop system will be stable regardless of the values of the physical parameters
- **A tool for choosing where to place sensors:** Passivity considerations are helpful as a guide for the choice of a suitable variable y for output feedback. This is helpful for selecting where to place sensors for feedback control.
- **A tool for choosing where to place actuators:** A guide for choice of location of actuators

Next lecture

Feedback linearization

Recommended reading

Khalil	**Chapter 13** Sections 13.1,13.2 and 13.4 Example 13.16 - page 538 is additional material

TTK4150 Nonlinear Control Systems

Lecture 10

Feedback Linearization

NTNU

Previous lecture

Previous lectures:

- Passivity
- Passivity based control
  - Energy-based Lyapunov Control Design
  - Control design using passivity theorems

Previous lectures on nonlinear control design

Previous lectures on nonlinear control design:

- Lyapunov based control design
- Cascaded control: Lemma 4.7 allows for modular design
  (And background material, Sontag and Loria)
- Passivity-based control design

Outline I

1. Motivation
2. Input-state linearization
   - Introduction
   - Application example
3. Input-output linearization
   - Introduction
   - The method
     - Step 1 - Find the relative degree
     - Step 2 - Write the system in normal form
     - Step 3 - Choose u to cancel the nonlinearities
     - Step 4 - Analyze the zero-dynamics
     - Step 5 - Choose v to solve the control problem
     - Summary: Input-output linearization for stabilization
     - Summary: Input-output linearization for tracking control
   - Application example
   - Advantages/shortcomings

Today's goals

After today you should...

- Know the concepts of relative degree, normal form, external dynamics, internal dynamics and zero dynamics.
- Be able to design a stabilizing control law using the input-output linearization method, including
    1) Finding the relative degree
    2) Writing the system in normal form
    3) Creating a linear input-output relation by feedback control
    4) Analyzing the zero dynamics
    5) Choosing the transformed input variable v to stabilize the origin of the system, locally or globally
- Be able to design a control law that solves the local tracking control problem, using the input-output linearization method
- Be able to discuss the advantages and the disadvantages of the input-output linearization method

Literature

Today's lecture is based on

Khalil	**Chapter 13**
	Sections 13.1,13.2 and 13.4
	Example 13.16 - page 538 is additional material

Motivation

A number of methods exist for control design for linear systems.

It would therefore be nice if we could obtain a linear system instead of the nonlinear system we are dealing with.

Jacobian linearization: An approximation

Motivation

A number of methods exist for control design for linear systems.

It would therefore be nice if we could obtain a linear system instead of the nonlinear system we are dealing with.

Jacobian linearization: An approximation

Question

Is it possible to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one?

Input-state linearization (Full-state linearization)

Given a nonlinear system

$$\boxed{\dot{x} = f(x) + G(x)u}$$

where $f(0) = 0$, and $f : \mathbb{D} \to \mathbb{R}^n$ and $G : \mathbb{D} \to \mathbb{R}^{n\times p}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Find a state transformation

$$z = T(x)$$

and an input transformation

$$u = \alpha(x) + \beta(x)v$$

such that the corresponding closed-loop system

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v$$

written in the coordinates $z = T(x)$ is linear and controllable

Input-state linearization (cont.)

i.e.

$$\boxed{\dot{z} = \frac{d}{dt}T(x) = \frac{\partial T}{\partial x}\dot{x} = Az + Bv}$$

where

$$\left[\frac{\partial T}{\partial x}(f(x) + G(x)\alpha(x))\right]_{x=T^{-1}(z)} = Az$$
$$\left[\frac{\partial T}{\partial x}G(x)\beta(x)\right]_{x=T^{-1}(z)} = B$$

and

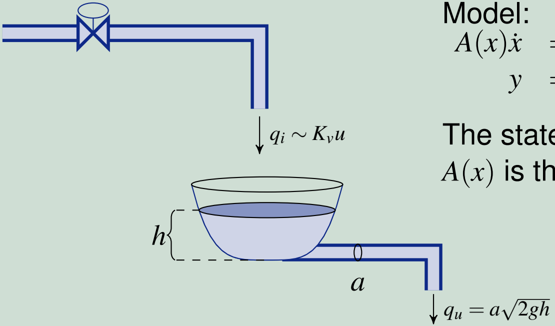
$$\text{rank}[B \quad AB \quad \cdots \quad A^{n-1}B] = n$$

Example: Pendulum equation

See pages 505 - 507

Example
Fluid level control

Fluid level control



Model:

$$\begin{aligned} A(x)\dot{x} &= -a\sqrt{2gx} + K_v u \\ y &= x \end{aligned}$$

The state x is the water level
$A(x)$ is the cross section

Find a state transformation $z = T(x)$, an input transformation $u = \alpha(x) + \beta(x)v$, and a control law $v(z)$ which

a) asymptotically stabilizes $h_d =$ constant

b) asymptotically tracks $h_d(t)$

Input-output linearization
The system

Given a nonlinear system

$$\boxed{\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Note: $\dim u = \dim y = 1$

The method
Notation

Lie derivative of scalar function h along vector field f

$$L_f h = \frac{\partial h}{\partial x} f$$

$$L_f^2 h = L_f\left(L_f h\right) = \frac{\partial L_f h}{\partial x} f$$

$$\vdots$$

$$L_f^0 h = h$$

$$L_f^i h = L_f(L_f^{i-1} h), \quad i = 1, 2, \ldots$$

The method

Step 1 - Find the relative degree

Given the system

$$\boxed{\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Example:

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_3 + x_2x_3 + u \\ y &= x_2 \end{aligned}$$

1) Differentiate y until u appears

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial x} g \cdot u \\ &= L_f h + L_g h \cdot u \end{aligned}$$

The method
Step 1 - Find the relative degree

Suppose $L_g h = 0 \quad \forall\, x \in \mathbb{D}_0 \subset \mathbb{D}$

$$\ddot{y} = \frac{\partial (L_f h)}{\partial x}\dot{x} = \underbrace{\frac{\partial (L_f h)}{\partial x} f}_{L_f^2 h} + \underbrace{\frac{\partial (L_f h)}{\partial x} g(x) u}_{L_g L_f h \cdot u}$$

$$\vdots$$

$$y^{(i)} = L_f^i h + L_g L_f^{i-1} h \cdot u$$

Relative degree

The system has <u>relative degree</u> ρ in a region $\mathbb{D}_0 \subset \mathbb{D} \subset \mathbb{R}^n$ if

$$\left.\begin{array}{ll} L_g L_f^{i-1} h = 0, & 1 \le i \le \rho - 1 \\ L_g L_f^{\rho-1} h \neq 0 & \end{array}\right\} \forall\, x \in \mathbb{D}_0$$

The method
Step 1 - Find the relative degree

Terminology - Relation to linear systems

Let

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with $\dim u = \dim y = 1$.
The system transfer function is

$$h(s) = C(sI - A)^{-1}B = K\frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Then $\rho = n - m$

The method

Step 1 - Find the relative degree

When is input-output linearization possible?

Q When is it possible to perform an input-output linearization?

A If the relative degree is well defined in the region of interest $\mathbb{D}_0$ (Theorem 13.1), then the system can be input-output linearized.

The method

Step 1 - Find the relative degree

Example

Given
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2 + u$$

find the relative degree ρ at $x_0 = 0$ when

a) $y = x_1$

b) $y = x_2^2$

Example

Given
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (1 + x_1)u$$
$$\dot{x}_3 = u$$

find the relative degree ρ at $x_0 = 0$ when

c) $y = x_1$

The method

Step 2 - Write the system in normal form

Suppose ρ is well defined in $\mathbb{D}_0$

2a) Derive the external dynamics

2a) Let

$$\begin{aligned}\psi_1 &= y\\ \psi_2 &= \dot{y}\\ &\vdots\\ \psi_\rho &= y^{(\rho-1)}\end{aligned}$$

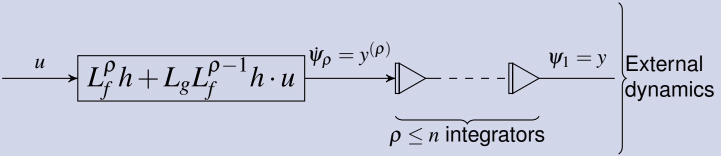
Then

$$\begin{aligned}\dot{\psi}_1 &= \psi_2\\ \dot{\psi}_2 &= \psi_3\\ &\vdots\\ \dot{\psi}_\rho &= L_f^\rho h + L_g L_f^{\rho-1} h \cdot u\end{aligned}$$

The method

Step 2a - Derive the external dynamics

The external dynamics



If $\rho < n$

$\dot{\varphi} = ?$ Internal dynamics

The method
Step 2b - Derive the internal dynamics

2b) Derive the internal dynamics

2b) Choose $n-\rho$ coordinates $\varphi_1, \ldots, \varphi_{n-\rho}$

$\rightarrow$ Coordinate transformation

$$z = T(x) = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-\rho} \\ \psi_1 \\ \vdots \\ \psi_\rho \end{bmatrix}$$

normal coordinates/states

The method

Step 2b - Derive the internal dynamics

Choose $\varphi_1, \ldots, \varphi_{n-\rho}$ such that

T is a diffeomorphism

and

$$L_g \varphi_1 = 0$$
$$\vdots$$
$$L_g \varphi_{n-\rho} = 0$$

and

$$\varphi_i(0) = 0$$

When are these conditions possible to fulfill?

Always possible when the systen has a well-defined relative degree $\rho \leq n$ (Theorem 13.1)

Useful fact

The Jacobian matrix

$$\left.\frac{\partial T}{\partial x}\right|_{x_0}$$

is nonsingular

$\Rightarrow$

T is a diffeomorphism in a neighborhood of x_0

The method

Step 2b - Derive the internal dynamics

$$\begin{aligned}\dot{\varphi}_j &= \frac{\partial \varphi_j}{\partial x}\dot{x} = L_f\varphi_j + \underbrace{L_g\varphi_j}_{=0}\cdot u \\ &= f_{0_j}(\varphi_i, \psi_i)\end{aligned}$$

Write the system in normal form:

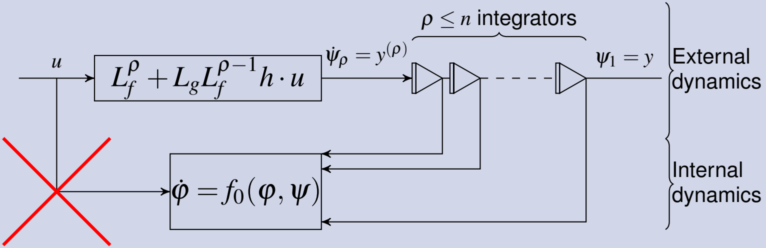
$$\left.\begin{aligned}\dot{\varphi}_1 &= f_{0_1}(\varphi_i, \psi_i) \\ &\vdots \\ \dot{\varphi}_{n-\rho} &= f_{0_{n-\rho}}(\varphi_i, \psi_i)\end{aligned}\right\} \text{Internal dynamics } \dot{\varphi} = f_0(\varphi, \psi)$$

$$\left.\begin{aligned}\dot{\psi}_1 &= \psi_2 \\ \dot{\psi}_2 &= \psi_3 \\ &\vdots \\ \dot{\psi}_\rho &= L_f^\rho h + L_g L_f^{\rho-1} h \cdot u\end{aligned}\right\} \text{External dynamics}$$

The method

Step 2 - Write the system in normal form

The system in normal form



The method

Step 3 - Choose u to cancel the nonlinearities

3) Choose u to cancel the nonlinearities

Create a linear input-output relation by feedback control:

Choose u to cancel the nonlinearities

(an input transformation/a linearizing inner feedback control loop)

$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}\left(-L_f^{\rho} h + v\right)}$$

$$\Downarrow$$

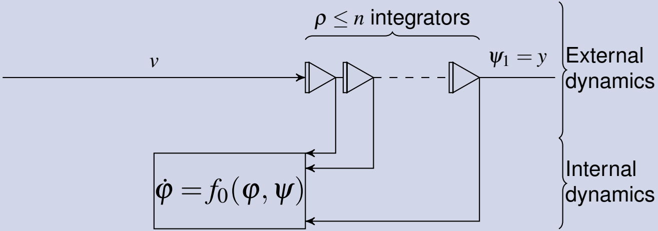
$$\dot{\varphi} = f_0(\varphi, \psi)$$

$$\left.\begin{aligned} \dot{\psi}_1 &= \psi_2 \\ &\vdots \\ \dot{\psi}_\rho &= v \end{aligned}\right\} y^{(\rho)} = v$$

The method

Step 3 - Choose u to cancel the nonlinearities

A linear input-output relationship



The method

Step 4 - Analyze the zero-dynamics

Definition: Zero-dynamics

The zero-dynamics is the internal dynamics when the system output is kept at zero by the input

i.e.

$$y \equiv 0$$

$$\vdots$$

$$y^{(\rho)} \equiv 0 \qquad \text{(i.e.} \quad u_0(x) = -\frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)})$$

$$\Updownarrow$$

$$\psi_i = 0 \quad i = 1, \ldots, \rho$$

The zero-dynamics:

$$\boxed{\dot{\varphi} = f_0(\varphi, 0)}$$

The method

Step 4 - Analyze the zero-dynamics

Definition: Minimum phase systems

The system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

is minimum phase if (the origin of) the zero-dynamics is asymptotically stable.

Terminology - Relation to linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \rightarrow \quad h(s) = K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Zero-dynamics:

$$\dot{\varphi} = Q\varphi, \qquad \lambda_j(Q) = z_j, \quad j = 1, \ldots, m = n - \rho$$

The method

Step 5 - Choose v to solve the control problem

Asymptotic stabilization

Control objective:

$$x = 0 \text{ asymptotically stable}$$

$$\Updownarrow$$

$$z = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = 0 \text{ asymptotically stable}$$

We have

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi}_1 &= \psi_2 \\ &\vdots \\ \dot{\psi}_\rho &= v \end{aligned}$$

Special case of

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi} &= A\psi + Bv \end{aligned}$$

(A, B) controllable

The method

Step 5 - Choose v to solve the control problem

Recall: We have a special case of

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi} &= A\psi + Bv \end{aligned}$$

(A, B) controllable

$$\boxed{v = -K\psi \qquad K \text{ is chosen such that } (A - BK) \text{ is Hurwitz}}$$

$$\Downarrow$$

$$\dot{\varphi} = f_0(\varphi, \psi)$$

$$\dot{\psi} = (A - BK)\psi \; \} \text{ Exponentially stable}$$

The method

Step 5 - Choose v to solve the control problem

Recall:

$$\dot{\varphi} = f_0(\varphi, \psi)$$

$$\dot{\psi} = (A - BK)\psi \quad \} \text{Exponentially stable}$$

Lemma 13.1

If the origin $\varphi = 0$ of $\dot{\varphi} = f_0(\varphi, 0)$ is asymptotically stable, then the origin $(\varphi, \psi) = (0, 0)$ of

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi} &= (A - BK)\psi \end{aligned}$$

is asymptotically stable.

NB

Only *local* asymptotic stability can be concluded

The method

Step 5 - Choose v to solve the control problem

Summary Step 5 - Asymptotic stabilization

Choose

$$v = -K\psi$$

such that $(A - BK)$ is Hurwitz.

Choose for instance

$$\begin{aligned} v &= -k_0\psi_1 - k_1\psi_2 - \cdots - k_{\rho-1}\psi_\rho \\ &= -k_0 y - k_1\dot{y} - \cdots - k_{\rho-1}y^{(\rho-1)} \end{aligned}$$

such that

$$s^\rho + k_{\rho-1}s^{\rho-1} + \cdots + k_1 s + k_0$$

has all its roots strictly in the left-half plane.

The method

Summary: Input-output linearization for stabilization

If the system

$$\boxed{\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$, has a well-defined relative degree $\rho \in \mathbb{D}_0 \subset \mathbb{D}$, $1 \leq \rho \leq n$ and is minimum phase then the control law

$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}\left(-L_f^{\rho} h + v\right)}$$

makes $x = 0$ locally asymptotically stable.

The method

Input-output linearization for stabilization - global result

Global result (Lemma 13.2)

If $\dot{\varphi} = f_0(\varphi, \psi)$ is ISS then the origin $(\varphi, \psi) = (0, 0)$ of

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi} &= (A - BK)\psi \end{aligned}$$

is globally asymptotically stable.

Proof: Satisfies conditions of Lemma 4.7 (Cascaded control)

The method

Step 5 - Choose v to solve the control problem

Tracking control

Reference trajectory: $y_d(t), \dot{y}_d(t), \ldots, y_d^{(\rho)}$ bounded

$y_d^{(\rho)}$ piecewise continuous

Define

$$e = \begin{bmatrix} \psi_1 - y_d \\ \vdots \\ \psi_\rho - y_d^{(\rho-1)} \end{bmatrix} = \begin{bmatrix} y - y_d \\ \dot{y} - \dot{y}_d \\ \vdots \\ y^{(\rho-1)} - y_d^{(\rho-1)} \end{bmatrix} = \psi - R, \quad R = \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\rho-1)} \end{bmatrix}$$

Choose

$$\boxed{v = -Ke + y_d^{(\rho)}}$$

where K is chosen such that $(A - BK)$ is Hurwitz.

The method
Step 5 - Choose v

We obtain

$$\boxed{\begin{aligned} \dot{\varphi} &= f_0(\varphi, e+R) \\ \dot{e} &= (A-BK)e \quad \text{exponentially stable} \end{aligned}}$$

Tracking control (local)

If the origin $\varphi = 0$ of $\dot{\varphi} = f_0(\varphi, 0)$ is asymptotically stable, then, for sufficiently small $e(0), \varphi(0)$ and $R(t)$

$$e(t) \xrightarrow{\exp} 0$$
$$\varphi(t) \text{ is bounded}$$

i.e. the control law solves local tracking control problem.

The method

Step 5 - Choose v to solve the control problem

Summary Step 5 - Tracking control

Choose

$$v = -Ke + y_d^{(\rho)}$$

such that $(A - BK)$ is Hurwitz.

Choose for instance

$$\begin{aligned} v &= -k_0 e_1 - k_1 e_2 - \cdots - k_{\rho-1} e_\rho + y_d^{(\rho)} \\ &= -k_0 (y - y_d) - k_1 (\dot{y} - \dot{y}_d) - \cdots - k_{\rho-1} (y^{(\rho-1)} - y_d^{(\rho-1)}) + y_d^{(\rho)} \end{aligned}$$

such that

$$s^\rho + k_{\rho-1} s^{\rho-1} + \cdots + k_1 s + k_0$$

has all its roots strictly in the left-half plane.

The method

Summary: Input-output linearization for tracking control

If the system

$$\boxed{\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$, has a well-defined relative degree $\rho \in \mathbb{D}_0 \subset \mathbb{D}$, $1 \le \rho \le n$ and is minimum phase then the control law

$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}(-L_f^{\rho} h + v)}$$

ensures that if $e(0)$ and $\varphi(0)$ and $R(t)$ are sufficiently small, then

$$\boxed{\begin{aligned}&e(t) \xrightarrow{\exp} 0 \\ &\varphi(t) \text{ is bounded}\end{aligned}}$$

The method

Input-output linearization for tracking control - Global results

If $\rho = n$ then no internal dynamics

$$\Downarrow$$

System dynamics is then

$$\dot{e} = (A - BK)e \quad \text{GES}$$

If $\dot{\varphi} = f_0(\varphi, \psi)$ is ISS then

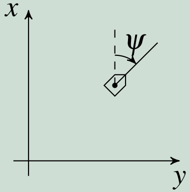
$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}\left(-L_f^{\rho} h + v\right)}$$

gives

$$\boxed{\begin{array}{ll} e(t) \xrightarrow{\exp} 0 & \\ \varphi(t) \text{ is bounded} & \forall\ e(0), \varphi(0), R(t) \end{array}}$$

Example: Dynamic positioning system for ships

Dynamic positioning system for ships



$$\eta = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

System model:

$$M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} + D(\eta)\dot{\eta} = \tau$$
$$y = \eta$$

System properties:

$$M = M^T > 0$$
$$z^T D z > 0 \quad z \neq 0$$
$$z^T(\tfrac{1}{2}\dot{M} - C)z = 0 \quad \forall\, z \in \mathbb{R}^3$$

Control problem: Design a control law $\tau = g(t,(\eta,\dot{\eta}))$ using input-output linearization, that makes the origin $(\eta,\dot{\eta}) = (0,0)$ an asymptotically stable equilibrium point.

Advantages/shortcomings

Advantages/shortcomings

- Cancels all dynamics $L_f h$
  - ÷ Does not take advantage of stabilizing terms
  - ÷ Robustness to modelling errors is questionable
- Requires well-defined relative degree
- Requires minimum phase system

+ Exponential convergence
+ We can use linear control design methods
+ Easy tuning

Next lecture

Adaptive Control

Recommended reading:

Slotine and Li: Applied Nonlinear Control	**Chapter 8** Pages 311-316 Section 8.2 pp. 326-330, p. 333 Example 8.4 pp. 334-335 **Chapter 9.2.1**
Lavretsky and Wise: Robust and Adaptive Control	**Chapter 7**

TTK4150 Nonlinear Control Systems

Lecture 11

Adaptive Control

Previous lectures on nonlinear control design

Previous lectures on control design:

- Lyapunov based control design
- Cascaded control: Lemma 4.7 allows for modular design
  (And background material, Sontag and Loria)
- Passivity-based control design
- Input-output linearization

Outline I

1. Introduction
   - Today's goals
   - Literature
2. Adaptive control
3. Model Reference Adaptive Control
   - MRAC for SISO linear systems
   - MRAC for SISO nonlinear systems
4. Adaptive tracking control for MIMO nonlinear systems

Today's goals

Adaptive control

After today you should...

- Be able to design a model reference adaptive controller (MRAC) for SISO linear systems
- Be able to design a MRAC controller for SISO nonlinear systems
- Be able to design an adaptive tracking controller for a class of MIMO nonlinear systems

Literature

Today's lecture is based on

Slotine and Li: Applied Nonlinear Control	**Chapter 8:** Pages 311-316 Section 8.2 pp. 326-330, p. 333 Example 8.4 pp. 334-335 **Chapter 9:** Section 9.2.1
Lavretsky and Wise: Robust and Adaptive Control	**Chapter 7**

Parametric uncertainty

Parametric uncertainty

Uncertain plant parameters (parametric uncertainty) occur in many practical problems

- Robot manipulators: Loads of various sizes, weights and mass distribution

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- Marine craft: Uncertain hydrodynamic parameters, varying loads
- Aircraft: System parameters vary with altitude, speed and configuration

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- Metallurgical and chemical processes: Model parameters vary and can be hard to obtain

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- Marine craft: Uncertain hydrodynamic parameters, varying loads
- Aircraft: System parameters vary with altitude, speed and configuration
- Metallurgical and chemical processes: Model parameters vary and can be hard to obtain
- Power systems: Loading conditions

Robustness

ISS gives robustness

Adaptive Control

Can we do better?

- Constant (or slowly varying) uncertain parameters

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- Constant (or slowly varying) uncertain parameters
- Matched uncertainties

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Objective:

Maintain consistent performance of a control system in the presence of uncertainty or unknown variation in plant parameters

Adaptive Control

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Adaptive control = Control with on-line parameter estimation

Adaptive Control

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Objective:

Maintain consistent performance of a control system in the presence of uncertainty or unknown variation in plant parameters

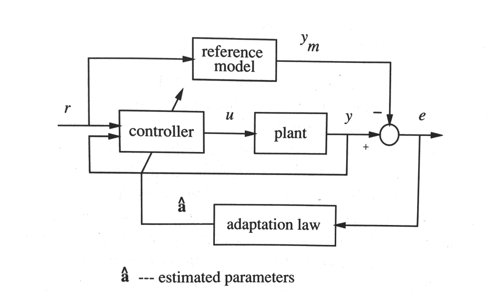
Adaptive control = Control with on-line parameter estimation

Direct adaptive control:

Achieve the tracking control objectives without necessarily identifying on-line the true values of the parameters (as opposed to Indirect adaptive control)

Model Reference Adaptive Control (MRAC)

MRAC



MRAC for SISO linear systems

SISO linear systems

$$\dot{x} = a_p x + b_p u$$

where a_p, b_p are uncertain/unknown constants.
We know the sign of b_p.

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MRAC

1) Specify the desired closed-loop behavior by a reference model

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1) Specify the desired closed-loop behavior by a reference model
2) Choose a control law
   a) such that the plant output tracks the reference model output *when the parameters are exactly known.*
   b) In this control law, replace the parameters with their estimates

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   b) Lyapunov analysis of error dynamics $\rightarrow$ adaptation law

MRAC for SISO linear systems

Barbalat's lemma

If

- V is lower bounded (e.g. $V \geq 0$)
- $\dot{V} \leq 0$
- $\ddot{V}$ is uniformly bounded

then

$$\dot{V} \to 0 \quad \text{as} \quad t \to \infty$$

MRAC for SISO linear systems

Barbalat's lemma

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- $\dot{V} \leq 0$
- $\ddot{V}$ is uniformly bounded

then

$$\dot{V} \to 0 \quad \text{as} \quad t \to \infty$$

The adaptive controller

$$\begin{aligned} u &= \hat{a}_x x + \hat{a}_r r \\ \dot{\hat{a}}_x &= -\gamma_x \operatorname{sgn}(b_p) x e \\ \dot{\hat{a}}_r &= -\gamma_r \operatorname{sgn}(b_p) r e \end{aligned}$$

provides the desired closed-loop tracking performance.

MRAC for SISO linear systems

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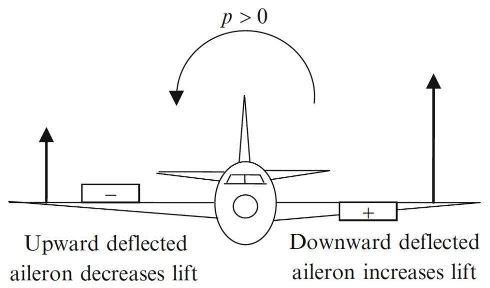
MRAC for SISO linear systems

Aircraft Roll Dynamics

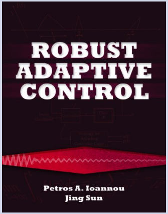
$$\dot{p} = L_p p + L_\delta \delta$$

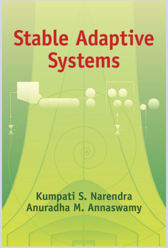
p : aircraft roll rate [rad/s]
δ : total differential aileron-spoiler deflection [rad]
L_p : roll damping derivative (Uncertain value)
L_δ : rolling moment derivative (Uncertain value. Known sign)

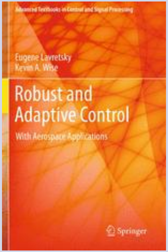


Adaptive control for linear systems

Adaptive control for linear systems







MRAC for SISO nonlinear systems

SISO nonlinear systems

$$\dot{x} = a_p x + c_p f(x) + b_p u$$

where a_p, b_p, c_p are uncertain/unknown constants.
We know the sign of b_p. The function $f(x)$ is locally Lipschitz.

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We know the sign of b_p. The function $f(x)$ is locally Lipschitz.

Note: Linear in the unknown parameters

$$\dot{x} = \begin{bmatrix} x & f(x) & u \end{bmatrix} \begin{bmatrix} a_p \\ c_p \\ b_p \end{bmatrix}$$

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MRAC for SISO nonlinear systems

Example

Plant:

$$\dot{x} = x + x^2 + 3u$$

Reference model:

$$\dot{x}_r = -4x_r + 4r(t)$$

MRAC for SISO nonlinear systems

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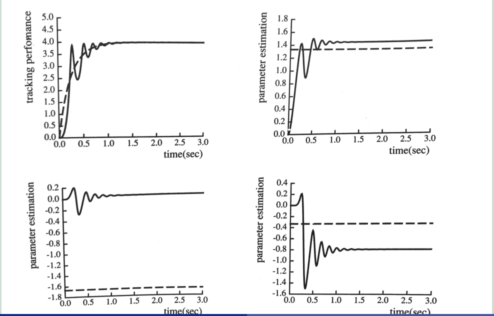
$r(t) = 4$



MRAC for SISO nonlinear systems

Example

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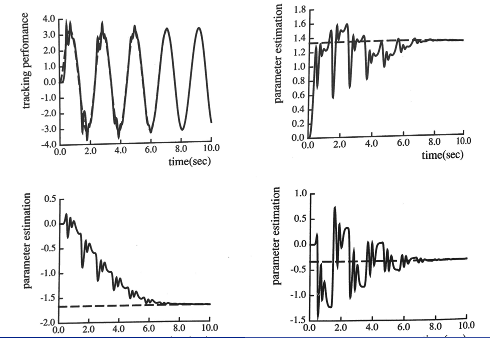
Reference model:

$$\dot{x}_r = -4x_r + 4r(t)$$

$r(t) = 4\sin(3t)$



A class of MIMO nonlinear systems

Mechanical systems

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+D(q)\dot{q}+g(q)=u$$

$M>0$, $\dot{M}-2C$ is skew symmetric, $z^T Dz>0 \quad \forall z\neq 0$.
The matrix coefficients are uncertain/unknown constants.

A class of MIMO nonlinear systems

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$M>0$, $\dot{M}-2C$ is skew symmetric, $z^TDz>0 \quad \forall z\neq 0$.
The matrix coefficients are uncertain/unknown constants.

Note: Linear in the unknown parameters

The terms all depend linearly on the unknown parameter vector a, i.e.

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+D(q)\dot{q}+g(q)=Y(q,\dot{q},\ddot{q})a$$

The matrix Y is denoted the Regression matrix.

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded

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Choice of tracking control law:
Input-output linearization/Computed torque:

$$u = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q} + D(q)\dot{q} + g(q) - M(q)K_p(q - q_d) - M(q)K_d(\dot{q} - \dot{q}_d)$$

Making this control law adaptive is possible, but not trivial

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded
2) Choose a control law
   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*

Choice of tracking control law:
First modification:

$$u = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + D(q)\dot{q}_d + g(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d)$$

Adaptive tracking control

Modified control law

Introduce

$$s = \dot{e} + \Lambda e \qquad \Lambda = \Lambda^T > 0$$

Interpretation:

$$s = \dot{q} - \dot{q}_r$$

where

$$\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$$

Adaptive tracking control

Modified control law

Introduce

$$s = \dot{e} + \Lambda e \qquad \Lambda = \Lambda^T > 0$$

Interpretation:

$$s = \dot{q} - \dot{q}_r$$

where

$$\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$$

Choice of tracking control law:

Second modification:

$$u = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + D(q)\dot{q}_r + g(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d)$$

Adaptive tracking control

Adaptive tracking control

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   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*

Resulting tracking control law:
Makes q track $q_d(t)$ when the parameters are exactly known:

$$u = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + D(q)\dot{q}_r + g(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d)$$

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded
2) Choose a control law
   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*
   b) In this control law, replace the parameters with their estimates

Adaptive tracking control law:
Replace the parameters with their estimates:

$$
\begin{aligned}
u &= \hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{D}(q)\dot{q}_r + \hat{g}(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d) \\
&= Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\hat{a} - K_p e - K_d \dot{e}
\end{aligned}
$$

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded
2) Choose a control law
   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*
   b) In this control law, replace the parameters with their estimates
3) Choose an adaptation law such that tracking is achieved asymptotically

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded
2) Choose a control law
   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*
   b) In this control law, replace the parameters with their estimates
3) Choose an adaptation law such that tracking is achieved asymptotically
   a) Express the system in error variables

Adaptive tracking control

Adaptive tracking control

1) Given the desired trajectory: $q_d(t)$, $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded
2) Choose a control law
   a) such that q tracks $q_d(t)$ *when the parameters are exactly known.*
   b) In this control law, replace the parameters with their estimates
3) Choose an adaptation law such that tracking is achieved asymptotically
   a) Express the system in error variables
   b) Lyapunov analysis of error dynamics $\rightarrow$ adaptation law

Adaptive tracking control

To sum up:

The adaptive tracking control law

$$u = \hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{D}(q)\dot{q}_r + \hat{g}(q) - K_p(q - q_d) - K_d(\dot{q} - \dot{q}_d)$$
$$\dot{\hat{a}} = -\Gamma Y^T(q,\dot{q},\dot{q}_r,\ddot{q}_r)s$$

where

$$s = \dot{q} - \dot{q}_r$$
$$\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$$
$$\Lambda = K_d^{-1}K_p$$

makes the tracking error converge to zero:

$$e = q - q_d \to 0, \quad \dot{e} = \dot{q} - \dot{q}_d \to 0$$

and the estimation error $\tilde{a} = \hat{a} - a$ bounded.

Next lecture

Backstepping

Recommended reading:

Khalil	**Chapter 14**
	Sections 14.2 pages 589-598

TTK4150 Nonlinear Control Systems

Lecture 12

Backstepping

NTNU

Previous lectures on nonlinear control design

Previous lectures on control design:

- Lyapunov based control design
- Cascaded control: Lemma 4.7 allows for modular design
- Passivity-based control design
- Input-output linearization
- Adaptive control

Today's goals

Backstepping

After today you should...

- Be able to design a stabilizing control law using the integrator backstepping method
- Be able to discuss the advantages and disadvantages of this method

Literature

Today's lecture is based on

Khalil **Chapter 14**
Section 14.3 pages 589-598

Outline I

1. The system
   - Cascade structure
2. The backstepping method
   - Step 1 - Find a stabilizing function for Σ_1
   - Step 2 - Design the actual control input u
     - Introduce the error variable as a new state
     - Write the system equations in the new coordinates
     - Choose the Lyapunov function candidate $V_c = V(\eta) + \frac{1}{2}z^2$
     - Choose u such that $\dot{V}_c < 0$ (in (η, z))
3. Examples
   - Application example
4. Kahoot

The system to be controlled

The system

Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} = f\left(\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\right) + g\left(\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\right)\cdot x_{n+1}$$

$$\dot{x}_{n+1} = u$$

f, g sufficiently smooth (C^k) in a set $\mathbb{D} \subseteq \mathbb{R}^n$ that contains $x = 0$, and $f(0) = 0$

Control task

Find a control law $u = \gamma(x)$ that stabilizes $x = \begin{bmatrix}x_1\\ \vdots\\ x_{n+1}\end{bmatrix} = 0$

The system to be controlled
A cascade structure

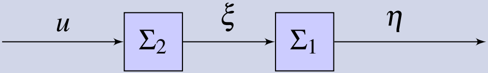
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} = f\left(\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\right) + g\left(\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\right)\cdot x_{n+1}$$

$$\dot{x}_{n+1} = u$$

Cascade structure

We can identify a cascade structure:

$$\Sigma_1 : \quad \dot{\eta} = f(\eta) + g(\eta)\xi$$
$$\Sigma_2 : \quad \dot{\xi} = u$$



The backstepping method

Step 1 - Find a stabilizing function for Σ_1

Step 1 - Find a stabilizing function for Σ_1

Regard ξ as a *virtual control input* to Σ_1

- Find a *stabilizing function*

$$\xi = \varphi(\eta), \quad \varphi(0) = 0$$

  such that $\eta = 0$ is an asymptotically stable equilibrium point of

$$\dot{\eta} = f(\eta) + g(\eta)\varphi(\eta)$$

- and find a corresponding Lyapunov function to prove this

$$\begin{aligned} &V(\eta) > 0, \quad C^1 \\ &\frac{\partial V}{\partial x}\left[f(\eta) + g(\eta)\varphi(\eta)\right] < 0, \quad \forall\, \eta \in \mathbb{D} \end{aligned}$$

The backstepping method

Step 2 - Design the actual control input u

Step 2 - Design the actual control input u

Design the actual control input u to stabilize the full system:

1. Introduce the error variable as a new state (replacing ξ)

$$z = \xi - \varphi(\eta)$$

2. Write the system equations in the new coordinates $\begin{bmatrix} \eta \\ z \end{bmatrix}$

$$\begin{aligned} \dot{\eta} &= f(\eta) + g(\eta)(z + \varphi(\eta)) \\ \dot{z} &= \dot{\xi} - \dot{\varphi} \\ &\Downarrow \\ \dot{\eta} &= f(\eta) + g(\eta)\varphi(\eta) + g(\eta)z \\ \dot{z} &= u - \dot{\varphi} \end{aligned}$$

The backstepping method

Step 2 - Design the actual control input u

Step 2 - Design the actual control input u

3. Choose the Lyapunov function candidate

$$V_c(\eta, z) = V(\eta) + \frac{1}{2}z^2$$

4. Find a control law u which asymptotically stabilizes

$$\begin{bmatrix} \eta \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

based on $V_c = V(\eta) + \frac{1}{2}z^2$

Because $[\eta, \xi]^T \mapsto [\eta, z]^T$ is a diffeomorphism:

$[\eta, z]^T = 0$ asymptotically stable $\Leftrightarrow$ $[\eta, \xi]^T = 0$ asymptotically stable

The backstepping method
Step 2 - Design the actual control input u

Step 2 - Design the actual control input u

4. Choose u such that $\dot{V}_c < 0$ (in (η, z)):

$$u = -\frac{\partial V}{\partial \eta} g(\eta) + \dot{\varphi} - kz \qquad k > 0$$

$$\dot{V}_c = \underbrace{\frac{\partial V}{\partial \eta}\left[f(\eta) + g(\eta)\varphi(\eta)\right]}_{<0 \text{ in } \eta} \underbrace{-kz^2}_{<0 \text{ in } z} < 0$$

Conclusion

$$u = -\frac{\partial V}{\partial \eta} g(\eta) + \frac{\partial \varphi}{\partial \eta}\left[f(\eta) + g(\eta)\xi\right] - k\left[\xi - \varphi(\eta)\right]$$

$\Rightarrow$ $(\eta, \xi) = (0, 0)$ is asymptotically stable

(*Globally* asymptotically stable if $\mathbb{D} = \mathbb{R}^n$ and V is radially unbounded in η)

Examples

Read examples 14.8 - 14.9

Example

Consider the system

$$\dot{x}_1 = \sin x_1 - x_1^3 + x_2$$
$$\dot{x}_2 = u$$

Use the backstepping method to design a stabilizing control law

Application: Active suspension

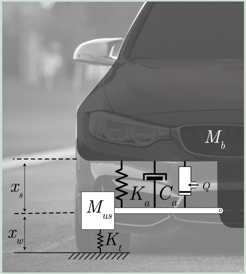
Example: Active suspension

When designing vehicle suspension systems for cars, there is a dual objective:

- Minimize the vertical acceleration of the car body (for passenger comfort)
- Maximize tire contact with the road surface (for handling)

To this end *active* suspension systems with hydraulic actuators are developed.

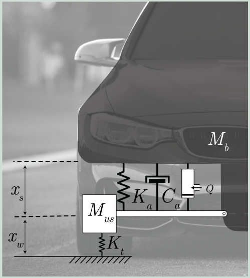
Active suspensions should be designed to behave differently on smooth and rough roads. Thic can be achieved by introducing nonlinearities in the controller which make the suspension stiffer near its travel limits:



Application: Active suspension cont.

Example: Active suspension cont.

Active suspension design:



The fluid flow is adjusted by a current controlled valve:

$$\dot{d}_v = -c_v d_v + k_v i_v$$

The resulting flow is (advanced valve, cancels the square-root nonlinearity):

$$\dot{Q} = -c_f Q + k_f i_v$$

Application: Active suspension cont.

Example: Active suspension cont.

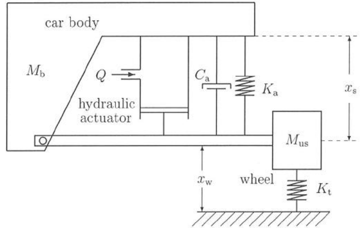


Figure 2.6: Quarter-car model for active suspension design with *parallel* connection of hydraulic actuator with passive spring/damper.

In this parallell configuration, neglecting leakage and compressability, the suspension travel x_s is related to the fluid flow Q through the equation

$$\dot{x}_s = \frac{1}{A}Q$$

Application: Active suspension cont.

Example: Active suspension cont.

The system equations are thus

$$\dot{x}_s = \frac{1}{A} Q$$
$$\dot{Q} = -c_f Q + k_f i_v$$

To apply backstepping, we view the flow Q as a virtual control, and design for it a nonlinear stabilizing function $\varphi(x_s)$ which will stiffen the suspension near its travel limits:

$$Q_{\text{des}} = \varphi(x_s) = -A(c_1 x_s + k_1 x_s^3)$$

Find a stabilizing controller for i_v

Advantages/Drawbacks

Advantages/Drawbacks of the backstepping method:

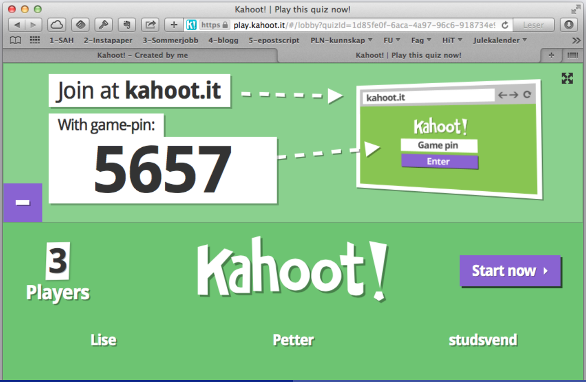
+ Does not cancel stabilizing terms
+ Simple proofs
+ Achieves a cascaded systems structure
- Complex controllers (Modelling robustness issues)

- Tip: You may use the method to achieve a cascaded systems structure with an asymptotically stable nominal system. Then you can utilize cascaded control systems theory to derive the control law and prove stability of the total system.

Kahoot

Sign in:

Go to www.kahoot.it
or
Open your Kahoot app



Exam

This was the final lecture
First exams then...

Happy holidays

Why isn’t it enough to know linear control theory?

Learning goals and Literature

After this set of video lectures you should...

- Know the basic differences between linear and nonlinear systems
- Recognize the need for new analysis and control design methods
- Know when to use nonlinear methods for analysis and design
- Know how to calculate equilibrium points

The video lectures are based on

Khalil   Chapter 1

Nonlinear dynamic systems

Process / System / Mechanism

Nonlinear dynamic systems

$$\dot{u} = \frac{m_{22}}{m_{11}} vr - \frac{d_{11}}{m_{11}} u + \frac{1}{m_{11}} u_1$$
$$\dot{v} = -\frac{m_{11}}{m_{22}} ur - \frac{d_{22}}{m_{22}} v$$
$$\dot{r} = \frac{m_{11} - m_{22}}{m_{33}} uv - \frac{d_{33}}{m_{33}} r + \frac{1}{m_{33}} u_2$$
$$\dot{x} = u \cos\psi - v \sin\psi$$
$$\dot{y} = u \sin\psi + v \cos\psi$$
$$\dot{\psi} = r$$

Mathematical Model

$$m \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = F(x(t))$$

ODE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

PDE

Nonlinear dynamic systems

Linear systems (LTI)

$$\dot{x} = ax + bu$$

$$u = -K_p(x + \frac{1}{T_i}\int x d\tau + T_d\dot{x})$$

$$z \triangleq [x,\ \dot{x}]^T$$

$$\downarrow$$

$$\dot{z} = Az$$

Solution:
$z(t) = e^{At}z(0)$

Analysis:
Laplace transform $\rightarrow$
Transfer function

Nonlinear systems

$$\dot{x} = f_p(t, x, u)$$

$$u = \gamma(t, x)$$

$$\downarrow$$

$$\dot{x} = f(t, x)$$

Solution:
Generally no analytical solution

Analysis:
No Laplace transform
$\Rightarrow$ need new methods

Nonlinear dynamic systems

We consider nonlinear dynamic systems in the form

$$\begin{aligned}
\dot{x} &= f_p(t, x, u) \\
\dot{x}_1 &= f_{p_1}(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
&\vdots \\
\dot{x}_n &= f_{p_n}(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
&\downarrow u = \gamma(t, x) \\
\dot{x} &= f(t, x) \qquad \text{Time-varying / Nonautonomous}
\end{aligned}$$

Special case

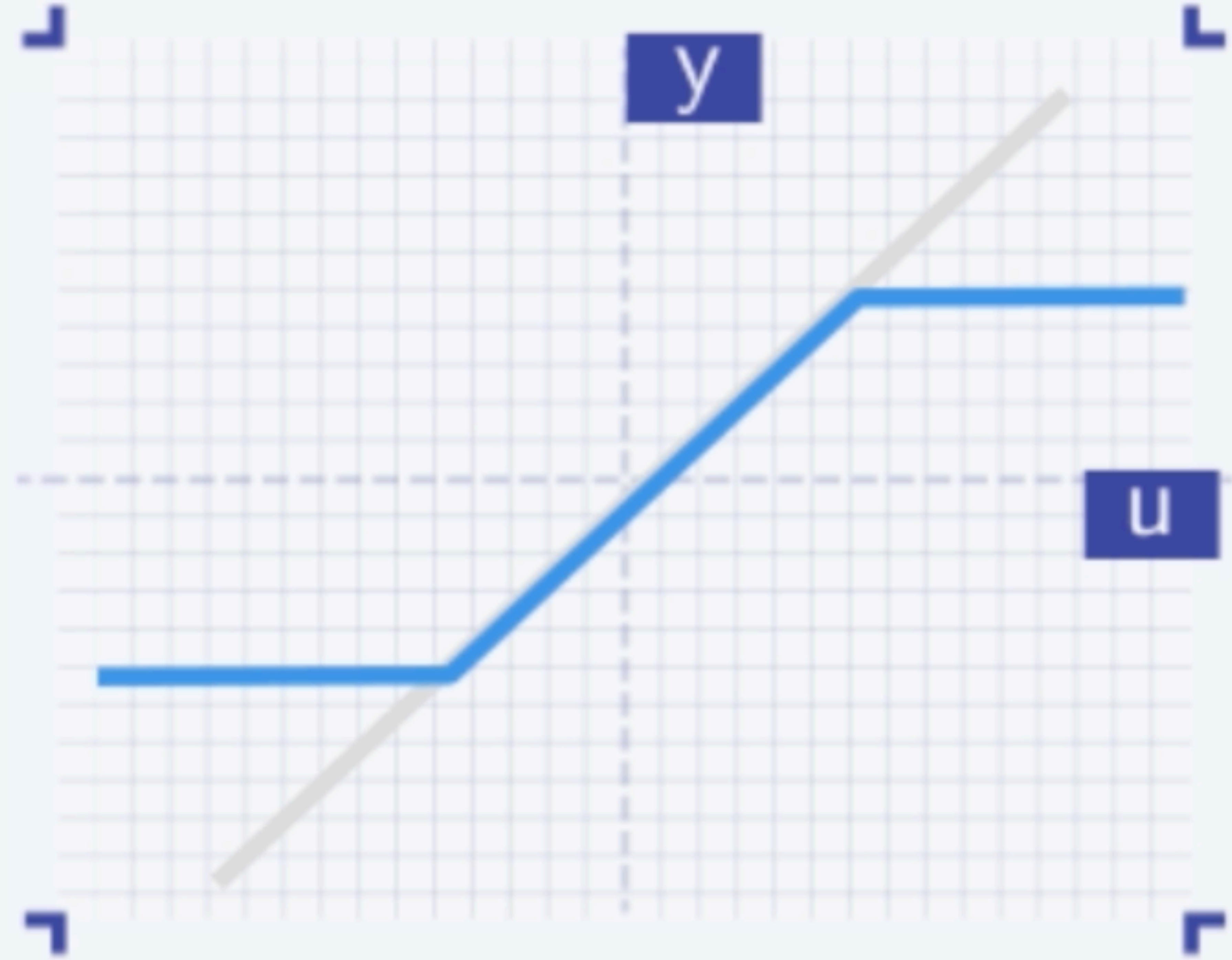
$$\dot{x} = f(x) \qquad \text{Time-invariant / Autonomous}$$

Examples of nonlinear systems

Systems with input saturation

$$\dot{x} = Ax + B\text{sat}(u)$$
$$u = PID(x)$$

$$\text{sat}(u) = \begin{cases} u & \text{if } |u| \leq 1 \\ \text{sgn}(u) & \text{if } |u| > 1 \end{cases}$$



Systems with essential nonlinearities in the model

- Electric motors in hybrid cars

$$
\begin{aligned}
\dot{\lambda}_r &= \left(-\frac{R_r}{L_r} I + n_p \omega J\right)\lambda_r + \frac{R_r}{L_r} M i_s \\
\dot{i}_s &= -\frac{M}{\sigma L_s L_r}\left(-\frac{R_r}{L_r} I + n_p \omega J\right)\lambda_r - \frac{1}{\sigma L_s}\left(R_s + \frac{M^2 R_r}{L_r^2}\right) i_s \\
\dot{\Theta} &= \omega \\
\dot{\omega} &= \mu i_s^T J \lambda_r - \frac{\alpha\sigma}{m} + \frac{T_L}{m}
\end{aligned}
$$

Examples of nonlinear systems

- Systems with highly complex dynamics
    - Weather prediction
    - Lorenz attractor

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

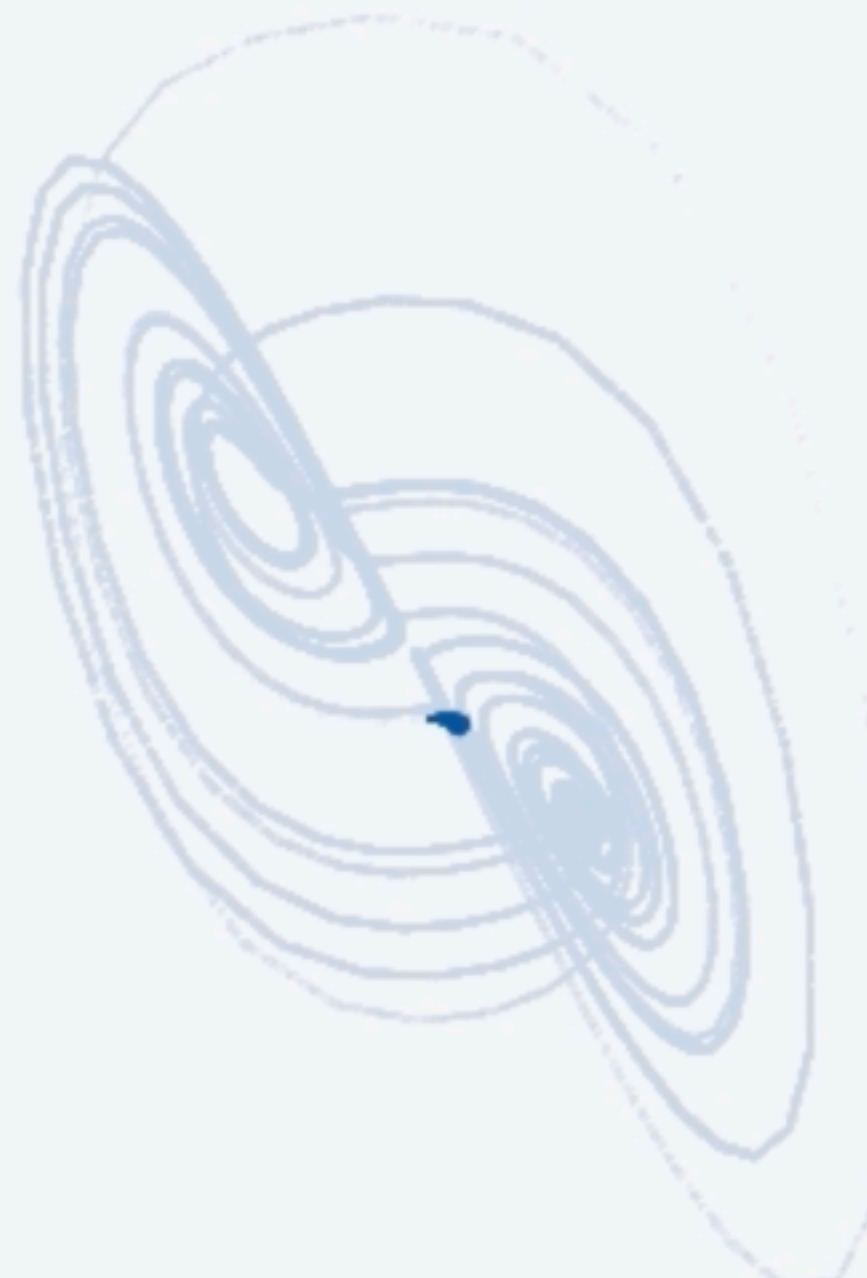


Figure : Lorenz attractor

Examples of nonlinear systems

- Systems "pushing the limit"
  - Systems operating in extreme conditions (where nonlinear effects become significant)

Examples of nonlinear systems

- Systems with complex non-local dynamics
  - HIV virus dynamics

$$
\begin{aligned}
\dot{T} &= s - dT - \beta v T \\
\dot{T}_1 &= q_1 \beta v T - \mu_1 T_1 - k_1 T_1 \\
\dot{T}_2 &= q_2 \beta v T + k_1 T_1 - \mu_2 T_2 \\
\dot{v} &= k_2 T_2 - c v
\end{aligned}
$$

T number of healthy immune cells
T_1 number of infected, but non-reproducing cells
T_2 number of infected and virus reproducing cells
v number of free virus particles

The Superposition Principle

- For linear ODEs

Formulation 1

The response to several inputs working together is equal to the sum of the response of that circuit to the inputs working separately

Formulation 2

If input u_1 produces output y_1, and input u_2 produces output y_2, then input $u = u_1 + u_2$ produces output $y = y_1 + y_2$.

- For nonlinear systems

No superposition principle $\Rightarrow$ More complex behavior

The Superposition Principle

Leo Tolstoy´s Anna Karenina

All happy families are alike;
each unhappy family is unhappy in its own way.

The Superposition Principle

Leo Tolstoy´s Anna Karenina

All happy families are alike;
each unhappy family is unhappy in its own way.

Slight adaptation

All linear systems are alike;
each nonlinear system is nonlinear in its own way.

Invariant sets

We study the system's behavior in and around invariant sets

Equilibrium point

x^* is an equilibrium point of $\dot{x} = f(x)$ iff

$$f(x^*) \equiv 0$$

An equilibrium point is a special case of

Invariant sets

A set M is an invariant set of $\dot{x} = f(x)$ iff

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \quad \forall t \in \mathbb{R}$$

Invariant sets: Linear systems

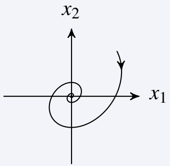
Linear systems

- One equilibrium point

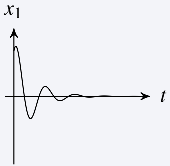
$$\dot{x} = Ax = 0$$
$$\Updownarrow$$
$$x^* = 0$$

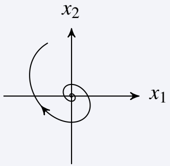
- May in addition have periodic solutions

$$x(t+T) = x(t)$$

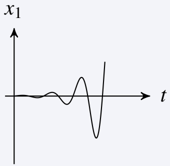


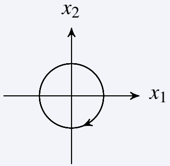
$x^* = 0$ as. stable



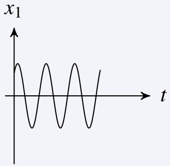


$x^* = 0$ unstable



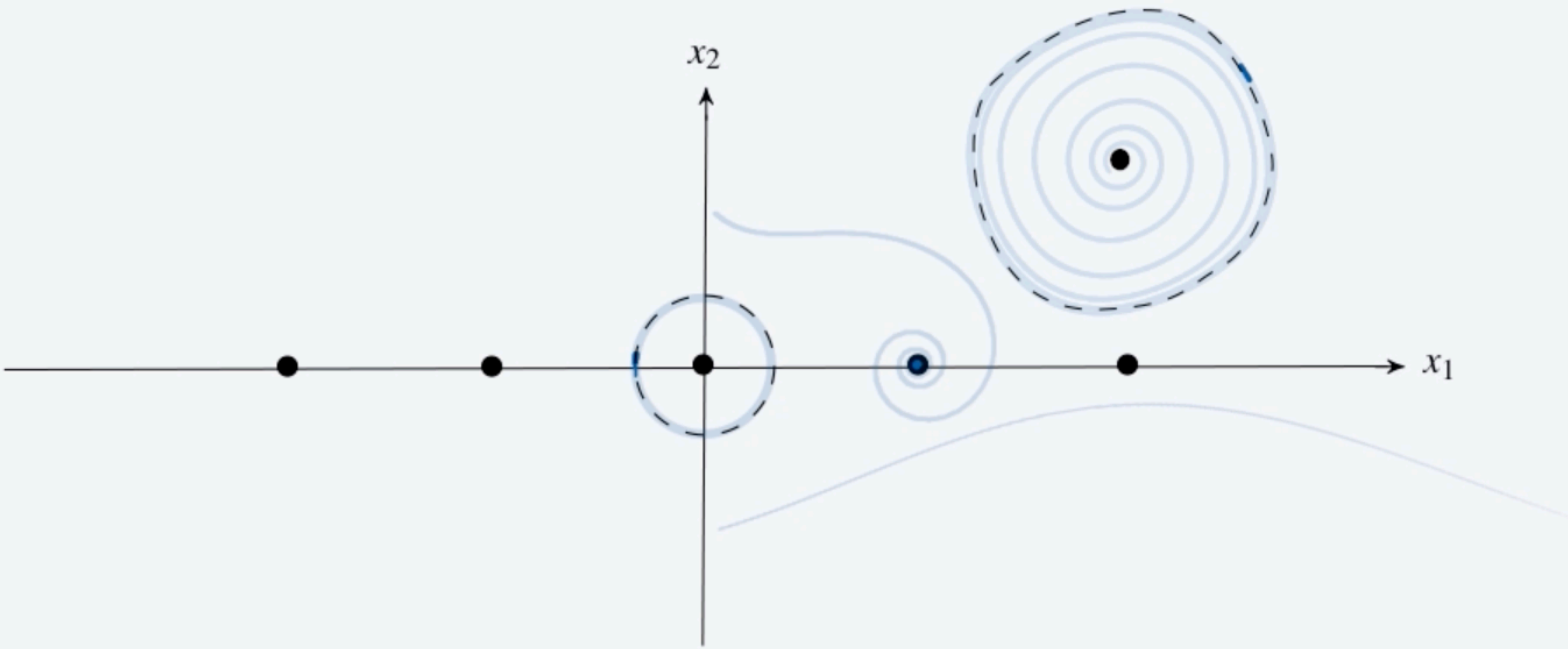


$x^* = 0$ marginally stable



Invariant sets: Nonlinear systems

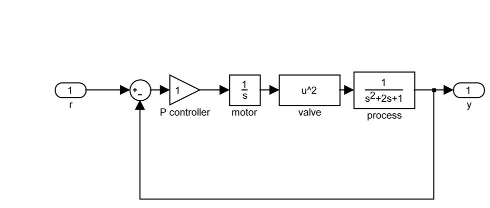
- Equilibrium points, one or more
- Periodic solutions
  - Limit cycles
- General invariant sets (Chaos)



Stability may depend on the magnitude of r

- $\neq$ Linear systems: Stability properties local $\sim$ global

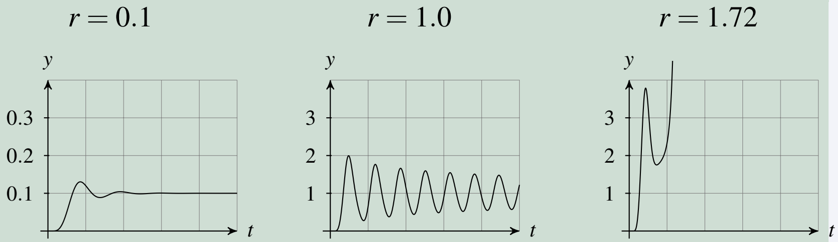
Example: Motor valve process



- Valve characteristic: $f(\theta) = \theta^2$
- Reference signals of different magnitude
  - $r = 0.1$
  - $r = 1$
  - $r = 1.72$

Common nonlinear phenomena

Step responses ($x_0 = 0$)



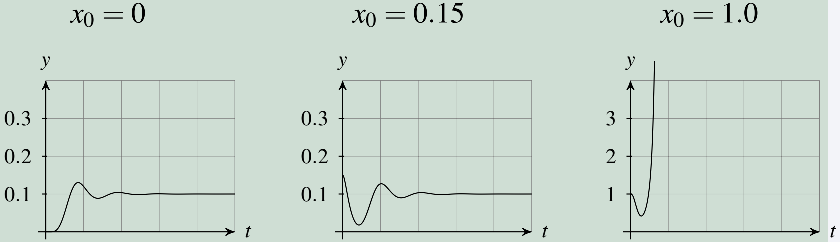
Stability depends on the magnitude of r!

Common nonlinear phenomena

Stability may depend on initial conditions

$x_0 = x(t_0)$

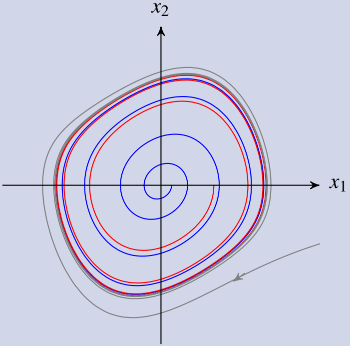
Step responses $r = 0.1$, for different x_0



Common nonlinear phenomena

Stable periodic solutions (Limit cycles)

- No external periodic input
- Frequency and amplitude independent of initial state

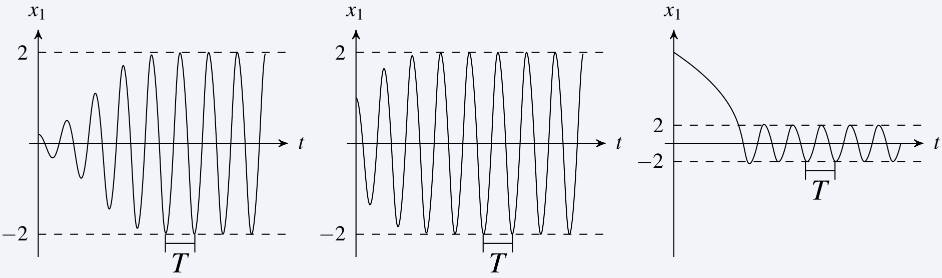


Example: Van der Pol

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$$

Common nonlinear phenomena

Simulation results:



Frequency and amplitude independent on initial conditions!

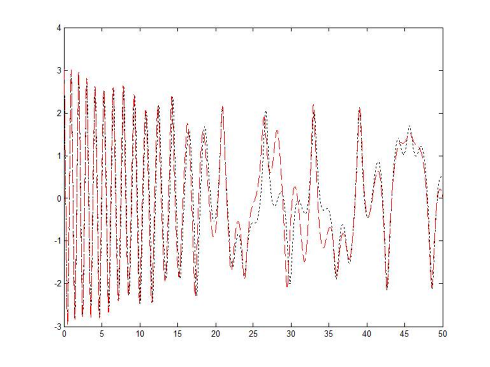
Common nonlinear phenomena

Chaos and Bifurcations

Example

$\ddot{x} + 0.1\dot{x} + x^5 = 6\sin t$

- - - - $x(0) = 2, \quad \dot{x}(0) = 3$
- - - - $x(0) = 2.01, \quad \dot{x}(0) = 3.01$



When to use nonlinear analysis/design

When should we use nonlinear analysis/design?

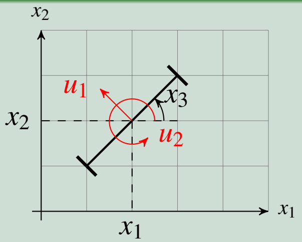
- Nontrivial question
- When you recognize essential nonlinear phenomena (that cannot be properly described by a linear model)
- When the workspace is large
- When the nonlinearities are hard

The nonlinear phenomena are dominant

Example: Car

Kinematic model:

$$\dot{x}_1 = \sin(x_3)u_1$$
$$\dot{x}_2 = \cos(x_3)u_1$$
$$\dot{x}_3 = u_2$$



- Control problem: Parallel parking

  Find $u = [u_1, u_2]^T$ that makes $x = [0,0,0]^T$ as. stable.

- Linearized model

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = u_1$$
$$\dot{x}_3 = u_2$$

Not controllable

When to use nonlinear analysis/design

When should we use nonlinear analysis/design?

- Nontrivial question
- When you recognize essential nonlinear phenomena (that cannot be properly described by a linear model)
- When the workspace is large
- When the nonlinearities are hard

TTK4150 Nonlinear Control Systems

Session 1: Lectures 1 and 2

Course Introduction

NTNU

People

Esten Ingar Grøtli - Lecturer

Marianna Wrzos-Kaminska - TA

Student assistants

- Julia Maria Graham
- Vebjørn Johansen Rognli
- Lars Lødemel Sandberg
- Magne Sirnes

Blackboard: TTK4150 Nonlinear Control Systems

Contents

- **Course Information**
  - Course information, Lecture Plan, Terminology, Errata Khalil
- **Lecture Material**
  - Week X:
    - Lecture notes.pdf
- **Assignments and Previous Exams**
  - Information about Assignments
    - Assignments: Assignment 1
  - Previous exam sets with solutions
- **Links to other resources**
  - MIT course: Dynamics of nonlinear systems
  - Khalil´s homepage of Nonlinear Systems
    - Lecture notes
    - Additional exercises with solutions

Exam date: December 16, 09:00-13:00 (4 hours exam)

Reference group

Reference group

- Reference groups are an important part of NTNUs quality assurance of education
- Every time a course has been given, the course coordinator writes a course report to describe the state of the course, including feedback from the students and a plan for improvements of the course.
- Feedback from the students are given through reference groups
- Please see https://innsida.ntnu.no/wiki/-/wiki/English/Quality+assurance+of+education for more information

Reference group

Reference group

- The reference group
  - Consists of at least three students - a representative selection of the students
  - Has at least three meetings: one at the beginning, one in the middle, and one at the course's end.
  - Has an ongoing dialogue with other students throughout the semester
  - Writes a reference group report, that summarizes the student's opinions and recommendations for improvement of the course, to be included in the course report

Course Goals

Course goals

Build a toolbox for nonlinear control analysis and design, such that given a control problem* you are able to

- make an intelligent choice of method(s) from this toolbox, weighing the advantages and short-comings of the different methods
- apply the method(s) to solve the problem
- possibly combine and adapt the methods to fit the given problem

***Example:** Design a feedback control law that makes the tool of robot manipulator follow a desired (time-varying) trajectory to spray paint a car.

At the end of this course you will for instance be able to...

Example

- Given a satellite, a chemical process, a ship etc. use a number of methods to
    1. Develop different controllers
    2. Make a good choice between these
- Decide how close the system states have to be to their reference values when you start up the control system, in order to guarantee that the system will converge to these reference values (Region of attraction)
- Be able to use energy considerations to decide where the actuators and sensors of a system preferably should be placed (Passivity-based control)
- Know how to design a controller to achieve robustness to actuator and measurement noise (Passivity-based control, Input-to-state stability)

In order to get there, we need some mathematical tools...

Lecture 1 Part I

Mathematical tools

A mathematical toolbox

In the video lectures you have learned about

- Metrics
- Norms
  - The set of n-dimensional vectors and p-norms
  - The set of real-valued continuous functions, corresponding norms and the $\mathscr{L}_p$-space
  - Matrix norms
- Inner Products
  - Relations between inner products and norms
    - Schwarz' inequality
    - Hölder's inequality
- Differentiable functions

Norms on $C[a,b]$

Norms on $C[a,b]$

$$\left.\begin{aligned} \|f\|_{\mathscr{L}_p} &= \left(\int_a^b |f(\tau)|^p d\tau\right)^{\frac{1}{p}}, \quad p \in [1,\infty] \\ \|f\|_{\mathscr{L}_\infty} &= \sup_{a\le t\le b} |f(t)| \end{aligned}\right\} \quad \mathscr{L}_p - \text{norms}$$

$\mathscr{L}_p$-space

$(C[0,\infty), \mathscr{L}_p - \text{norm})$ is a Banach space

- $f \in \mathscr{L}_p \Leftrightarrow \|f\|_{\mathscr{L}_p}$ is bounded, i.e. $\exists\, c : \|f\|_{\mathscr{L}_p} \le c$

Norms: Example

An example on $\mathscr{L}_p$-space $(C[0,\infty), \mathscr{L}_p - \text{norm})$

Let

$$f(t) = \frac{1}{1+t}$$

Is $f(t) \in \mathscr{L}_\infty, \mathscr{L}_2, \mathscr{L}_1$?

Norms: Example

Relating the magnitude of vector elements x_i with the magnitude (norm) of the vector x

Consider

$$2x_1 + x_2$$

Find an upper bound on these terms, relating their magnitude to the Euclidean norm of x

Schwarz' inequality

Induced norm

Inner product $\overset{Induces}{\rightarrow}$ Norm

- $\|x\| = < x, x >^{\frac{1}{2}}$

A useful property (holds only for the induced norm):

Schwarz' inequality

Let $< x, y >$ be an inner product on a set X. Then

$$|< x, y >| \leq \|x\| \cdot \|y\|$$

when $\|\cdot\|$ is defined as above.

Lecture 1 Part II

Why isn’t it enough to know linear control theory?

Why isn't it enough to know linear control theory?

In the video lectures you have learned to

- Know the basic differences between linear and nonlinear systems
- Recognize the need for new analysis and control design methods
- Know when to use nonlinear methods for analysis and design
- Know how to calculate equilibrium points

Invariant sets

We study the system's behavior in and around invariant sets

Equilibrium point

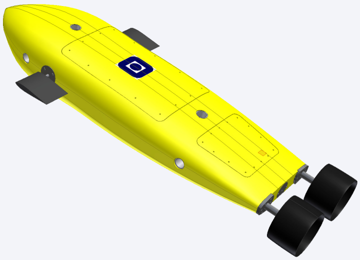
x^* is an equilibrium point of $\dot{x} = f(x)$ iff

$$f(x^*) \equiv 0$$

AUV example: Equilibrium points

Simple surge velocity model

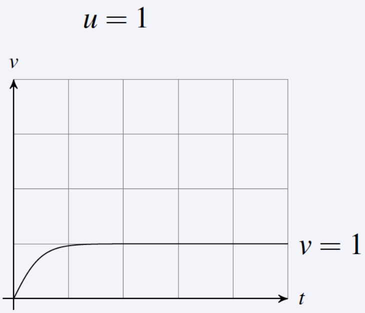
$$\underset{\substack{\uparrow \\ \text{Acceleration}}}{\dot{v}} + \underset{\substack{\uparrow \\ \text{Viscous damping}}}{v|v|} = \underset{\substack{\uparrow \\ \text{Forward thrust}}}{u}$$

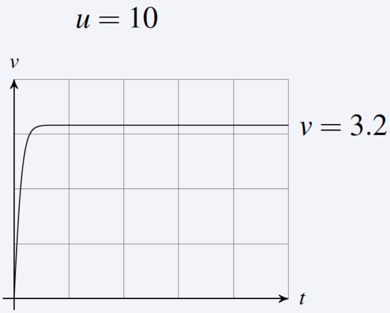


Example

- Find the system's equilibrium point when
  1) $u = 1$
  2) $u = 10$

AUV example: Simulation results





Exam 2010

Problem 1

Consider the system:

$$\dot{x}_1 = x_2 - \frac{1}{8}(x_1 + x_2)^3$$
$$\dot{x}_2 = x_1 - \frac{1}{8}(x_1 + x_2)^3$$

a) Find *all* equilibrium points.

Exam 2013

Problem 1

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1) - \left(5 + x_2^2 + x_2^4\right) x_2$$

a) Find all equilibrium points and classify the qualitative behavior of each of them.

b) Prove that there are no periodic orbits in $\mathbb{R}^2$.

Exam 2013

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Lecture 2

Fundamental properties

Second-order time-invariant systems

Fundamental properties
Second-order time-invariant systems

In the video lectures you have learned

- How to validate a mathematical model by ensuring the existence and uniqueness of the solution of an initial value problem
- How to construct phase portraits and interpret them
  - Analytical method
  - Computer simulations
- How to do a local analysis, i.e. describe the system behaviour near an equilibrium point
- How to describe a periodic orbit and a limit cycle
- How to tell whether a periodic orbit may or may not exist for a 2D system

How to do a phase plane analysis

Summary: How to do a phase plane analysis

1. Find the equilibrium points of the system
2. Perform a local analysis (about each equilibrium point)
    1. Linearize about the equilibrium point
    2. Find the eigenvalues $\lambda(A)$
    3. Classify the equilibrium point

    In order to obtain qualitative knowledge about the system behavior locally around the equilibria. (This will guide you in the next step)
3. Construct a phase portrait using
    a) the analytical method
    b) computer simulations
4. Try to find possible periodic orbits and limit cycles

Find the equilibrium points

Covered previously:

Equilibrium point

x^* is an equilibrium point of $\dot{x} = f(x)$ iff

$$f(x^*) \equiv 0$$

Pendulum without friction

$$ml\ddot{\theta} = -mg\sin\theta$$

Find all equilibrium points of this system

Local analysis: Pendulum example

Example

Perform a local analysis of the pendulum without friction, given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 \end{aligned}$$

1. Linearize about the equilibrium point
2. Find the eigenvalues $\lambda(A)$
3. Classify the (isolated) equilibrium points

Bendixson (negative) Criterion: Example

Example

Consider the system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

Under which conditions on $f(x)$ can the system not have any periodic orbits?

Bendixson (negative) Criterion

Lemma 2.2 Bendixson (negative) Criterion

Consider the system

$$\dot{x} = f(x), \qquad f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is } C^1$$

If, on a simply connected region D of the plane, the expression

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is not identically zero and does not change sign, then the system has no periodic orbits lying entirely in D.

Existence of periodic orbits: Example

Example

Consider the electronic oscillator, given by

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2)\\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2)\end{aligned}$$

- Find the equilibrium point(s) of the system
- Classify the equlibrium point(s) of the system
- By the index theorem, where in the state space can a periodic orbit exist?
- Using the Bendixson criterion, find a region in the state space that does not contain a periodic orbit
- Using the Poincaré-Bendixson criterion, find a region of the state space which does contain a periodic orbit

The index theorem

Corollary 2.1 The index theorem

C is a periodic orbit $\quad\Rightarrow\quad \sum_i I$(eq.point i) within $C = +1$

I(node, focus, center) = +1

I (saddle point) = -1

Poincaré-Bendixson criterion

Lemma 2.1 (Poincaré-Bendixson criterion)

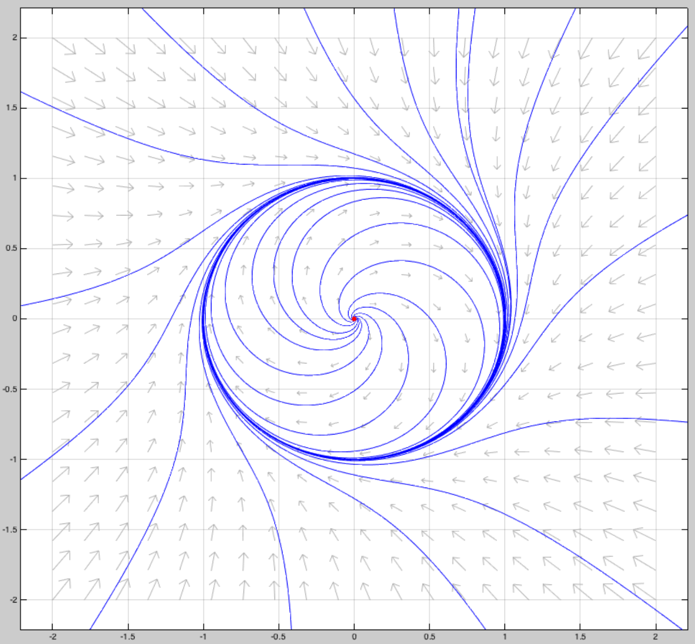
Consider the system

$$\dot{x} = f(x), \qquad f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is } C^1$$

Let M be a closed bounded subset of the plane such that.

- M contains no equilibrium points of the system, or it contains only one equilibrium point with the property that the eigenvalues of the Jacobian matrix at this point have positive real parts (unstable focus or unstable node)
- Every trajectory starting in M stays in M for all future time

Then M contains a periodic orbit of the system.


2
1.5
1
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-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

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Exam 2013

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