TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Solution to Assignment 6

1. (Khalil 7.11)

(1) We have

$$G(j\omega) = \frac{1 - j\omega}{j\omega(1 + j\omega)} = \frac{1 - \omega^2 - 2j\omega}{j\omega(1 + \omega^2)}$$

and

$$\mathrm{Re}[G(j\omega)] = \frac{-2}{1+\omega^2}, \quad \mathrm{Im}[G(j\omega)] = \frac{-1+\omega^2}{\omega(1+\omega^2)}$$

For $\psi(y) = y^5$, we have $\Psi(a) = 5a^4/8$, thus $\mathrm{Im}[G(j\omega)] = 0$, and

$$\mathrm{Im}[G(j\omega_0)] = 0 \Longrightarrow \omega_0 = 1, \quad \mathrm{Re}[G(j\omega_0)] = -1$$

The equation $1 + G(j\omega_0)\Psi(a) = 0$ has a unique solution $a = \left(\frac{8}{5}\right)^{\frac{1}{4}} = 1.125$. There is a possibility of a periodic solution of amplitude close to 1.125 and frequency close to 1 rad/sec.

(3) We have

$$\begin{aligned} G(j\omega) &= \frac{1}{(1+j\omega)^6} = \frac{(1-j\omega)^6}{(1+\omega^2)^6} \\ &= \frac{1 + 6(-j\omega) + 15(-j\omega)^2 + 20(-j\omega)^3 + 15(-j\omega)^4 + 6(-j\omega)^5 + (-j\omega)^6}{(1+\omega^2)^6} \\ &= \frac{1 - 15\omega^2 + 15\omega^4 - \omega^6 + j\left[-6\omega + 20\omega^3 - 6\omega^5\right]}{(1+\omega^2)^6} \end{aligned}$$

and

$$\mathrm{Re}[G(j\omega)] = \frac{1 - 15\omega^2 + 15\omega^4 - \omega^6}{(1+\omega^2)^6}, \quad \mathrm{Im}[G(j\omega)] = \frac{-6\omega + 20\omega^3 - 6\omega^5}{(1+\omega^2)^6}$$

From Example 7.6 we know that $\Psi(a) = 4/\pi a$, and $\mathrm{Im}[G(j\omega_0)] = 0$.

$$\begin{aligned} \mathrm{Im}[G(j\omega_0)] &= 0 \Longrightarrow \omega_0^2 = 3 \text{ or } \omega_0^2 = \frac{1}{3} \\ \mathrm{Re}\left[G\left(j\sqrt{3}\right)\right] &= \frac{1}{64}, \quad \mathrm{Re}\left[G\left(j\sqrt{1/3}\right)\right] = -\frac{27}{64} \end{aligned}$$

For $\omega_0^2 = 3$, the equation $1 + G(j\omega_0)\Psi(a) = 0$ has no solution. For $\omega_0^2 = 1/3$, the equation $1 + G(j\omega_0)\Psi(a) = 0$ has a unique root $a = 27/16\pi$. Thus we expect that the system will have a periodic solution with amplitude close to $27/16\pi$ and frequency close to $1/\sqrt{3}$ rad/sec.

(4) We have

$$\begin{aligned} G(j\omega) &= \frac{j\omega+6}{j\omega(j\omega+2)(j\omega+3)} = \frac{-j(6+j\omega)(2-j\omega)(3-j\omega)}{\omega(4+\omega^2)(9+\omega^2)} \\ &= \frac{-\omega(24+\omega^2)-j(36-\omega^2)}{\omega(4+\omega^2)(9+\omega^2)} \end{aligned}$$

and

$$\mathrm{Re}[G(j\omega)] = \frac{-\omega(24+\omega^2)}{\omega(4+\omega^2)(9+\omega^2)}, \quad \mathrm{Im}[G(j\omega)] = \frac{-36+\omega^2}{\omega(4+\omega^2)(9+\omega^2)}$$

Also here $\Psi(a) = 4/\pi a$, which leads to

$$\mathrm{Im}[G(j\omega_0)] = 0 \Longrightarrow \omega_0 = 6 \ , \ \mathrm{Re}[G(j\omega_0)] = -1$$

The equation $1+G(j\omega_0)\Psi(a) = 0$ has a unique solution $a = 2/15\pi$. Thus we expect that the system will have a periodic solution with amplitude close to $2/15\pi$ and frequency close to 6 rad/sec.

2. (Khalil 7.14 part (c))
ψ is a special case of the piecewise linear function of Example 7.7 with $s_1 = b$, $s_2 = 0$ and $\delta = 1/b$. The describing function is

$$\Psi(a) = \frac{2b}{\pi}\left[\sin^{-1}\left(\frac{1}{ab}\right) + \frac{1}{ab}\sqrt{1-\left(\frac{1}{ab}\right)^2}\right]$$

We have

$$G(j\omega) = \frac{(1-\omega^2-2j\omega)(4-\omega^2-4j\omega)}{(1+\omega^2)^2(4+\omega^2)^2}$$

and

$$\mathrm{Re}[G(j\omega)] = \frac{4+\omega^2(\omega^2-13)}{(1+\omega^2)^2(4+\omega^2)^2}, \quad \mathrm{Im}[G(j\omega)] = \frac{-2\omega(6-3\omega^2)}{(1+\omega^2)^2(4+\omega^2)^2}$$

Since $\Psi(a)$ is real, $\mathrm{Im}(G(j\omega)] = 0$, and

$$\mathrm{Im}[G(j\omega_0)] = 0 \Longrightarrow \omega_0 = \sqrt{2} \ , \ \mathrm{Re}[G(j\omega_0)] = -1/18$$

From $1+G(j\omega_0)\Psi(a) = 0$ we have $\Psi(a) = 18$. Because $\Psi(a)$ starts from b at $a = 0$ and decrease after $a = 1/b$, (see Figure 7.16 in Khalil) the equation $\Psi(a) = 18$ has a solution if $b > 18$. The frequency of oscillation will be close to $\sqrt{2}$.

3. (Thermostat System)

(a) Using the notation in Khalil (Figure 7.1), we have that

$$\begin{aligned}
G(s) &= h_r(s)\,h_p(s) \\
&= \frac{K}{s}\frac{1}{1+Ts} \\
&= \frac{K}{s(1+Ts)}
\end{aligned}$$

where $K > 0$ and $T > 0$. It can be seen that $G(s)$ has law pass characteristic, by which we conclude that the describing function method can be applied. A bode diagram of $G(s)$ is shown in Figure 1.

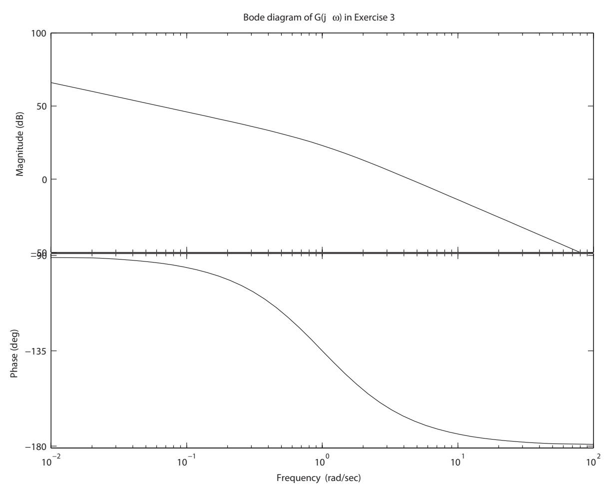

Figure 1: Bode plot of $G(j\omega)$

(b) Using $y = a\sin(\theta)$ as an argument it is seen from Figure 2 (where $a\sin(\alpha) = S$), that

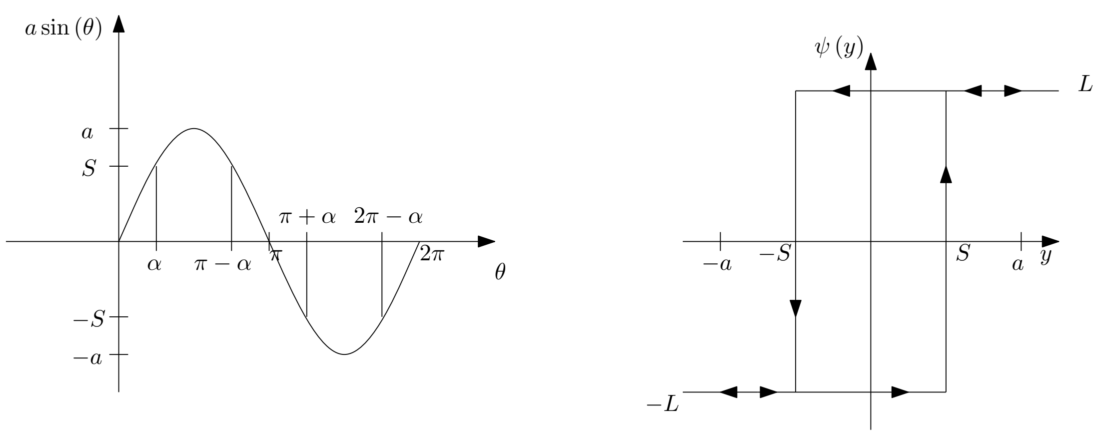

Figure 2: $\psi(y)$ and $a\sin(\theta)$

- When $0 \leq \theta \leq \alpha \to 0 \leq a\sin(\theta) \leq S$, and it is also rising. The function $\psi(a\sin(\theta))$ thus equals $-L$.
- When $\alpha < \theta < \pi - \alpha \to S \leq a\sin(\theta) \leq a$. The function $\psi(a\sin(\theta))$ thus equals L.
- When $\pi - \alpha \leq \theta \leq \pi + \alpha \to -S \leq a\sin(\theta) \leq S$, and it is also falling. The function $\psi(a\sin(\theta))$ thus equals L.
- When $\pi + \alpha < \theta < 2\pi - \alpha \to -a \leq a\sin(\theta) \leq -S$. The function $\psi(a\sin(\theta))$ thus equals $-L$.
- When $2\pi - \alpha \leq \theta \leq 2\pi \to -S \leq a\sin(\theta) \leq 0$, and it is also rising. The function $\psi(a\sin(\theta))$ thus equals $-L$.

$\psi(a\sin(\theta))$ thus becomes

$$\psi(a\sin(\theta)) = \begin{cases} -L & \text{when } 0 \leq \theta \leq \alpha \text{ and } \pi + \alpha \leq \theta \leq 2\pi \\ L & \text{when } \alpha < \theta < \pi + \alpha \end{cases}$$

where

$$a\sin(\alpha) = S$$

Since $\psi(y)$ is not memoryless, the theory from Appendix A is applied. The describing function is derived according to

$$\begin{aligned} z_{1s} &= \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin(\theta))\sin(\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^{\alpha} -L\sin(\theta)\,d\theta + \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} L\sin(\theta)\,d\theta + \frac{1}{\pi}\int_{\pi+\alpha}^{2\pi} -L\sin(\theta)\,d\theta \\ &= \frac{1}{\pi}(L\cos\alpha - L) + \frac{1}{\pi}(2L\cos\alpha) + \frac{1}{\pi}(L + L\cos\alpha) \\ &= \frac{4L}{\pi}\cos\alpha \end{aligned}$$

and

$$\begin{aligned} z_{1c} &= \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin(\theta))\cos(\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^{\alpha} -L\cos(\theta)\,d\theta + \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} L\cos(\theta)\,d\theta + \frac{1}{\pi}\int_{\pi+\alpha}^{2\pi} -L\cos(\theta)\,d\theta \\ &= \frac{1}{\pi}(-L\sin\alpha) + \frac{1}{\pi}(-2L\sin\alpha) + \frac{1}{\pi}(-L\sin\alpha) \\ &= -\frac{4L}{\pi}\sin\alpha \end{aligned}$$

and

$$
\begin{aligned}
z_1 &= \sqrt{z_{1s}^2 + z_{1c}^2} \\
&= \sqrt{\left(\frac{4L}{\pi}\cos\alpha\right)^2 + \left(-\frac{4L}{\pi}\sin\alpha\right)^2} \\
&= \sqrt{\frac{16L^2}{\pi^2}\cos^2\alpha + \frac{16L^2}{\pi^2}\sin^2\alpha} \\
&= \frac{4L}{\pi}\sqrt{\cos^2\alpha + \sin^2\alpha} \\
&= \frac{4L}{\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi &= \arctan\left(\frac{z_{1c}}{z_{1s}}\right) \\
&= \arctan\left(\frac{-\frac{4L}{\pi}\sin\alpha}{\frac{4L}{\pi}\cos\alpha}\right) \\
&= \arctan\left(-\frac{\sin\alpha}{\cos\alpha}\right) \\
&= \arctan\left(-\tan\left(\alpha\right)\right) \\
&= \arctan\left(\tan\left(-\alpha\right)\right) \\
&= -\alpha \\
&= -\arcsin\left(\frac{S}{a}\right)
\end{aligned}
$$

Using the preceding calculations the describing function is given by

$$
\begin{aligned}
\left|\Psi\left(a,\omega\right)\right| &= \frac{z_1}{a} \\
&= \frac{\frac{4L}{\pi}}{a} \\
&= \frac{4L}{\pi a}
\end{aligned}
$$

and

$$
\begin{aligned}
\angle\Psi\left(a,\omega\right) &= \varphi \\
&= -\arcsin\left(\frac{S}{a}\right)
\end{aligned}
$$

(c) In order to draw $-\frac{1}{\Psi(a,\omega)}$ in a Nichols diagram as a function of $\frac{a}{S}$, we calculate

$\left|-\frac{1}{\Psi(a,\omega)}\right|$ and $\angle - \frac{1}{\Psi(a,\omega)}$ as functions of $\frac{a}{S}$

$$
\begin{aligned}
\left|-\frac{1}{\Psi\left(a,\omega\right)}\right| &= \left|\frac{1}{\Psi\left(a,\omega\right)}\right| \\
&= \frac{1}{\left|\Psi\left(a,\omega\right)\right|} \\
&= \frac{1}{\frac{4L}{\pi a}} \\
&= \frac{\pi a}{4L} \qquad (1)\\
&= \frac{\pi S}{4L}\left(\frac{a}{S}\right) \\
&\downarrow \qquad (2)\\
\frac{L}{S}\left|-\frac{1}{\Psi\left(a,\omega\right)}\right| &= \frac{\pi}{4}\left(\frac{a}{S}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\angle - \frac{1}{\Psi\left(a,\omega\right)} &= \angle\left(-1\right) + \angle\frac{1}{\Psi\left(a,\omega\right)} \\
&= -180^0 - \angle\Psi\left(a,\omega\right) \\
&= -180^0 + \arcsin\left(\left(\frac{a}{S}\right)^{-1}\right) \qquad (3)
\end{aligned}
$$

Figure 3 shows a Nichols diagram of $-\frac{1}{\Psi(a,\omega)}$ where the magnitude and phase

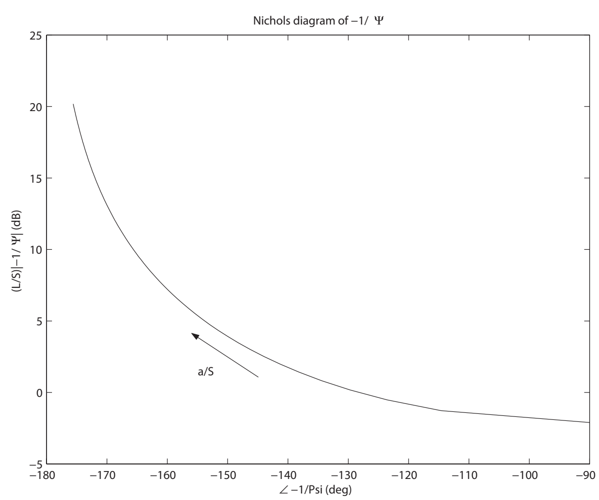

Figure 3: Nichols diagram of $-\frac{1}{\Psi}$ as a function of $\frac{a}{S}$

are normalized with respect to $\frac{a}{S}$, resulting in scaled magnitudes.

(d) Using the given constants the describing function is given by

$$\begin{aligned}\left|-\frac{1}{\Psi(a,\omega)}\right| &= \frac{\pi}{4}a \\ \angle -\frac{1}{\Psi(a,\omega)} &= -180^0 + \arcsin\left(a^{-1}\right)\end{aligned}$$

To establish existence of periodic solution in the system, we first of all need to solve the harmonic balance equation

$$h(j\omega)\,\Psi(a,\omega) + 1 = 0$$

It can be reformulated as

$$h(j\omega) = -\frac{1}{\Psi(a,\omega)}$$

which is used to investigate periodic solutions in a Nichols diagram. From Figure 4 it can be seen that a periodic solution exists since the harmonic

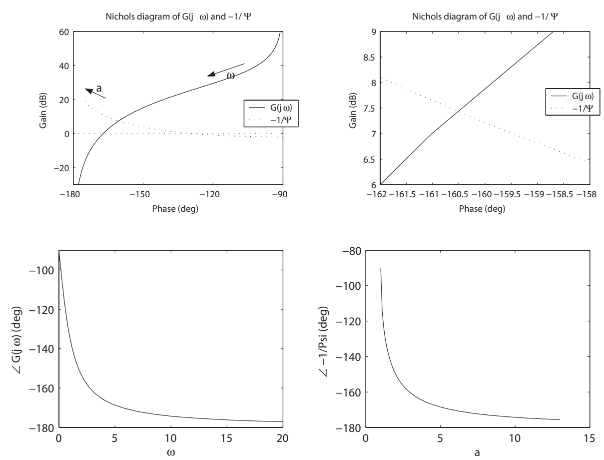

Figure 4: Graphical solution of the harmonic balance equation

balance equation has a solution at $-\frac{1}{\Psi} = h(j\omega)$ (In the diagram $h(j\omega)$ is called $G(j\omega)$). By further investigation, estimates of frequency and amplitude are found as

$$\begin{aligned}\omega &\approx 3 \\ a &\approx 3\end{aligned}$$

(e) A simulation of the system is shown in Figure 5 where it can be seen that $a \approx 3$ and

$$\begin{aligned}\Delta T &\approx 8 - 5.75 = 2.25 \\ &\Rightarrow f = \frac{1}{\Delta T} \approx 0.4 \\ &\Rightarrow \omega = 2\pi f \approx 2.79\end{aligned}$$

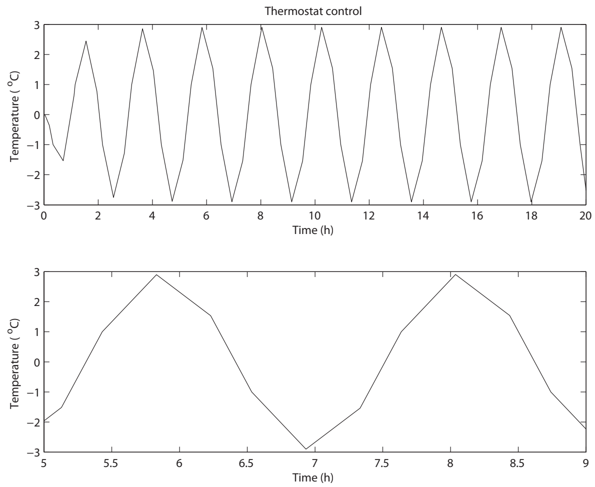

Figure 5: Simulation of the thermostat control system

which agrees with the results from the describing function method.

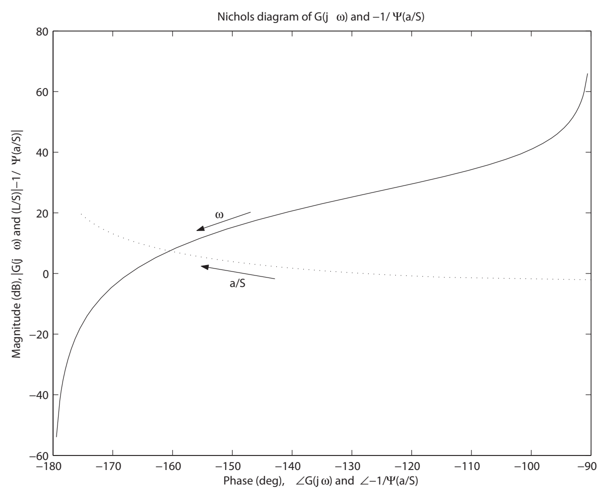

Figure 6: Nichols diagram of $G\left(j\omega\right)$ and $-\frac{1}{\Psi}$ when Ψ is expressed as a function of $\frac{\alpha}{S}$

(f) Figure 6 shows a Nichols diagram of $G\left(j\omega\right)$ and $-\frac{1}{\Psi}$ where Ψ is expressed as a function of $\frac{a}{S}$.
It can be recognized that there are several possibilities for reducing the amplitude a:

- moving $-\frac{1}{\Psi}$ to the left by reducing S (reducing S will only influence the phase, see (1) and (3))
- moving $-\frac{1}{\Psi}$ higher by reducing L (reducing L will only influence the magnitude, see (1) and (3))
- moving $G\left(j\omega\right)$ lower by reducing K

4. (Khalil 13.1) The system is given by

$$\begin{aligned} M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin(\delta) \\ \tau \dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD} \end{aligned}$$

which is rewritten in the form $\dot{x} = f(x) + g(x)u$ using

$$\begin{aligned} x_1 &= \delta \\ x_2 &= \dot{\delta} \\ x_3 &= E_q \\ u &= E_{FD} \end{aligned}$$

This results in the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\ \dot{x}_3 &= \tfrac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \end{aligned} \tag{4}$$

where it can be seen that

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 \\ \tfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\ \tfrac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1)\right) \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 0 \\ \tfrac{1}{\tau} \end{bmatrix} \end{aligned}$$

(a) The output is given by $y = \delta = x_1 = h(x)$. The relative degree is found as

$$\begin{aligned} y &= x_1 \\ \dot{y} &= \dot{x}_1 \\ &= x_2 \\ \ddot{y} &= \dot{x}_2 \\ &= \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\ \dddot{y} &= -\frac{D}{M}\dot{x}_2 - \frac{\eta_1}{M}\dot{x}_3 \sin(x_1) - \frac{\eta_1}{M}x_3 \frac{\partial \sin(x_1)}{\partial x_1}\dot{x}_1 \\ &= -\frac{D}{M}\frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\ &\quad -\frac{\eta_1}{\tau M}\sin(x_1)\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \\ &\quad -\frac{\eta_1}{M}x_3 \cos(x_1)\, x_2 \end{aligned}$$

And the relative degree of the system is $\rho = 3$.

We have our system on the form

$$\dot{x} = f(x) + g(x)u \tag{5}$$

and would like to transform it to a system on the form

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)]\\ y &= C_c\xi\end{aligned}$$

where η is the internal dynamics and ζ the external dynamics. They are both given through the diffeomorphism

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{6}$$

where

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n-\rho, \forall\, x \in D_0$$

Since $\rho = n$, one only needs the external part of the system on normal form, ζ. External variables of the normal form is given by evaluating the Lie Derivative of h with respect to f

$$\begin{aligned}\xi_1 &= h(x)\\ &= x_1\\ \xi_2 &= L_f h(x)\\ &= x_2\\ \xi_3 &= L_f^2 h(x)\\ &= \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right)\end{aligned}$$

The system (4) can therefore be written on normal form as

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)] \tag{7}$$

$$y = C_c\xi \tag{8}$$

where

$$\gamma(x) = L_g L_f^{\rho-1}h(x) \quad \text{and} \quad \alpha(x) = -\frac{L_f^{\rho}h(x)}{L_g L_f^{\rho-1}h(x)} = -\frac{L_f^{\rho}h(x)}{\gamma(x)} \tag{9}$$

This transformation is therefore only valid when $\gamma(x) \neq 0$, which means that

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\ &= -\frac{\eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \ \forall \ x \in D_0 \end{aligned}$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$. Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

(b) The output is given by $y = \delta + \zeta\dot{\delta} = x_1 + \zeta x_2 = h(x)$ where $\zeta \neq 0$. The relative degree is obtained from

$$\begin{aligned} y &= x_1 + \zeta x_2 \\ \dot{y} &= \dot{x}_1 + \zeta \dot{x}_2 \\ &= x_2 + \zeta \frac{1}{M} (P - D x_2 - \eta_1 x_3 \sin(x_1)) \\ &= \left(1 - \frac{\zeta D}{M}\right) x_2 - \frac{\zeta \eta_1}{M} x_3 \sin(x_1) + \zeta P \frac{1}{M} \\ \ddot{y} &= \frac{\partial \dot{y}}{\partial x} \dot{x} \\ &= \begin{bmatrix} -\frac{\zeta\eta_1}{M} x_3 \cos(x_1) & \left(1 - \frac{\zeta D}{M}\right) & -\frac{\zeta\eta_1}{M} \sin(x_1) \end{bmatrix} \dot{x} \\ &= \frac{\zeta \eta_1}{M} x_3 \cos(x_1) x_2 \\ &+ \left(1 - \frac{\zeta D}{M}\right) \frac{1}{M} (P - D x_2 - \eta_1 x_3 \sin(x_1)) \\ &- \frac{\zeta \eta_1}{\tau M} \sin(x_1) (-\eta_2 x_3 + \eta_3 \cos(x_1) + u) \end{aligned}$$

And the system thus has relative degree $\rho = 2$.
The region D_0 where the transformation is valid is where $L_g L_f^{\rho-1} h(x) \neq 0$

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^1 h(x) \\ &= -\frac{\gamma \eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \ \forall \ x \in D_0 \end{aligned}$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$.
Since $\rho < n$, both internal and external dynamics are needed. The external variables of the normal form is found by evaluating the Lie Derivative of h with

respect to f

$$
\begin{aligned}
\xi_1 &= h(x) = x_1 + \zeta x_2 \\
\xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix} 1 & \zeta & 0 \end{bmatrix} f(x) \\
&= x_2 + \frac{\zeta}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right)
\end{aligned}
$$

The internal dynamics $\eta = \phi(x)$ is chosen to satisfy $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ and the existence of $T^{-1}(x)$ in D_0. It can be verified that $\phi(x) = x_1$ meets these conditions. With $\phi(x) = x_1$ we have that

$$
\dot{\eta} = \dot{\phi}(x) = \dot{x}_1 = x_2 = \frac{1}{\zeta}(\xi_1 - \eta) = f_0(\eta, \xi)
$$

The system on normal form is thus

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \gamma(x)\left[u - \alpha(x)\right] \\
y &= C_c \xi
\end{aligned}
$$

The system is said to be minimum phase if the zero dynamics, $\dot{\eta} = f_0(\eta, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\eta} = f_0(\eta, 0) = -\frac{1}{\gamma}\eta$ it can be recognized that the origin of η is asymptotically stable.

5. (Khalil 13.2) The system is given by

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 - x_3 \\
\dot{x}_2 &= -x_1 x_3 - x_2 + u \\
\dot{x}_3 &= -x_1 + u \\
y &= x_3
\end{aligned}
$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$
\begin{aligned}
f(x) &= \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\
g(x) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

(a) The relative degree is obtained from

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x}_3 \\ &= -x_1 + u \end{aligned}$$

which shows that the system has relative degree 1 in R^3. Hence, the system is input-output linearizable.

(b) The external part of the normal form is given by

$$\xi_1 = h(x) = x_3$$

To find the internal dynamics we start by setting up the requirements on $\frac{\partial \phi_i}{\partial x}$

$$\begin{aligned} \frac{\partial \phi_1}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \end{bmatrix} g(x) \\ &= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} \\ &= 0 \\ \frac{\partial \phi_2}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{bmatrix} g(x) \\ &= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3} \\ &= 0 \end{aligned}$$

By choosing

$$\begin{aligned} \phi_1(x) &= x_1 \\ \phi_2(x) &= x_2 - x_3 \end{aligned}$$

we obtain a global diffeomorphism

$$\begin{aligned} T(x) &= \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x \end{aligned}$$

which is invertable. The system on normal form is

$$\begin{aligned} \dot{\eta}_1 &= \dot{x}_1 \\ &= -\eta_1 + \eta_2 \\ \dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3 \\ &= -x_1 x_3 - x_2 + u + x_1 - u \\ &= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1 \\ &= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \\ \dot{\xi}_1 &= -\eta_1 + u \end{aligned}$$

Since $L_g L_f^0 h(x) = L_g h(x) = 1$, the transformation is valid in $\mathcal{R}^3$.

(c) To investigate if the system is minimum phase, we analyze the zero dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)|_{\xi=0} \\ &= \left.\left[\begin{array}{c} -\eta_1 + \eta_2 \\ \eta_1 - \eta_2 - \xi_1 - \eta_1\xi_1 \end{array}\right]\right|_{\xi=0} \\ &= \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right]\eta = A\eta\end{aligned}$$

where it can be seen that $\text{eig}(A) = \begin{bmatrix}-2 & 0\end{bmatrix}^T$. Hence, the origin is not asymptotically stable, and the system is therefore not minimum phase.

6. The system is rewritten as

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

where

$$\begin{aligned}f(x) &= \left[\begin{array}{c} -x_1 \\ x_1x_2 \\ x_2 \end{array}\right] \\ g(x) &= \left[\begin{array}{c} e^{x_2} \\ 1 \\ 0 \end{array}\right] \\ h(x) &= x_3\end{aligned}$$

(a) The relative degree is found by derivative y with respect to time

$$\begin{aligned}y &= x_3 \\ \dot{y} &= \dot{x}_3 = x_2 \\ \ddot{y} &= \dot{x}_2 = x_1x_2 + u\end{aligned}$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in R^2$. The relative degree holds as long as $L_g L_f^{\rho-1} h(x) \neq 0$.

$$L_g L_f^{\rho-1} h(x) = L_g L_f h(x) = 1 \neq 0 \;\forall\; x \in \mathcal{R}^2$$

The relative degree thus holds over the entire $\mathcal{R}^3$ space.

(b) The system has a well defined relative degree ρ in the entire $\mathcal{R}^3$, and is therefore input-output linearizable in $\mathcal{R}^3$.

(c) The variables for the external dynamics are found according to

$$\xi_1 = h(x) = x_3$$
$$\xi_2 = L_f h(x) = \frac{\partial h(x)}{\partial x} f = x_2$$

The coordinates for the internal dynamics is chosen such that $T(x)$ is diffeomorphism on $\mathcal{R}^3$ and $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ on $\mathcal{R}^3$, where $\left[\eta, \xi^T\right]^T = [\phi(x), \psi(x)] = T(x)$. In addition to this we require $\phi(0) = 0$ in order to have the origin as equilibrium. We start by calculating

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2} \\ &= 0\end{aligned}$$

and based on these calculations we try

$$\frac{\partial \phi(x)}{\partial x_1} = 1$$
$$\frac{\partial \phi(x)}{\partial x_2} = -e^{x_2}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c is some constant. This constant is chosen to satisfy our requirement $\phi(0) = 0$

$$\begin{aligned}\phi(0) &= -e^0 + c \\ &= -1 + c \\ &\Rightarrow c = 1\end{aligned}$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

Consequently the inverse transformation exists. It follows that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is a diffeomorphism on $\mathcal{R}^3$ and $T(0) = T^{-1}(0) = 0$.

(d) The system may be rewritten as

$$\begin{aligned}
\dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2}\dot{x}_2 \\
&= -x_1 + e^{x_2}u - e^{x_2}\left(x_1 x_2 + u\right) \\
&= -x_1 - e^{x_2}x_1 x_2 \\
&= -\left(\eta + e^{x_2} - 1\right) - e^{x_2}\left(\eta + e^{x_2} - 1\right)x_2 \\
&= \left(1 - \eta - e^{\xi_2}\right) + \left(1 - \eta - e^{\xi_2}\right)e^{\xi_2}\xi_2 \\
&= \left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\xi} &= A_c\xi + B_c\gamma(x)\left(u - \alpha(x)\right) \\
y &= C_c\xi
\end{aligned}$$

where

$$\begin{aligned}
A_c &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C_c &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
\gamma(x) &= L_g L_f h(x) \\
&= 1 \\
\alpha(x) &= -\frac{L_f^2 h(x)}{L_g L_f h(x)} \\
&= -\frac{x_1 x_2}{1} \\
&= -x_1 x_2
\end{aligned}$$

(e) The zero dynamics is given by

$$\begin{aligned}
\dot{\eta} &= \left. f_0(\eta, \xi)\right|_{\xi=0} \\
&= \left.\left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)\right|_{\xi=0} \\
&= (1 - \eta - 1)(1 + 0) \\
&= -\eta
\end{aligned}$$

which has a globally asymptotically stable equilibrium at the origin.

(f) The external dynamics are given by

$$\dot{\xi} = A_c\xi + B_c\gamma(x)\left(u - \alpha(x)\right)$$

By choosing

$$u = \gamma^{-1}(x)\, v + \alpha(x)$$

the external dynamics are given by

$$\dot{\xi} = A_c\xi + B_c v$$

Since the system is controllable, $rank([B,\ AB]) = 2$, it can be stabilized (asymptotically stable) by a control input $v = -K\xi$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. u is now given by

$$u = -\gamma^{-1}(x)\, K\xi + \alpha(x)$$

Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the entire system is asymptotically stable.

(g) Let

$$\begin{aligned} \mathcal{R} &= \begin{bmatrix} r \\ \dot{r} \end{bmatrix} \\ e &= \xi - \mathcal{R} \end{aligned}$$

Then we can calculate

$$\begin{aligned} \dot{e} &= \dot{\xi} - \dot{\mathcal{R}} \\ &= (A_c\xi + B_c v) - (A_c\mathcal{R} + B_c\ddot{r}) \\ &= A_c(\xi - \mathcal{R}) + B_c(v - \ddot{r}) \\ &= A_c e + B_c(v - \ddot{r}) \\ &= A_c e + B_c(\gamma(x)[u - \alpha(x)] - \ddot{r}) \end{aligned}$$

where the simplified structure $\dot{\mathcal{R}} = A_c\mathcal{R} + B_c\ddot{r}$ is found using the known values of A_c and B_c (see part d).
We can choose the state feedback control

$$u = \gamma^{-1}(x)(v + \ddot{r}) + \alpha(x)$$

The resulting system is

$$\begin{aligned} \dot{\eta} &= f_0(\eta, e + \mathcal{R}) \\ \dot{e} &= A_c e + B_c v \end{aligned}$$

and since (A_c, B_c) is controllable, the loop is closed with $v = -Ke$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. This makes the external dynamics for e exponentially stable.
Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the overall closed-loop system is such that for sufficiently small initial conditions $e(0), \eta(0)$ and for $\mathcal{R}(t)$ with sufficiently small $\sup_{t\geq 0}\|\mathcal{R}(t)\|$, all solutions $(\eta(t), e(t))$ of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$.

7. (Khalil 14.31) The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 + a + \left(x_1 - a^{1/3}\right)^3 \\ \dot{x}_2 &= x_1 + u\end{aligned}$$

where the first system equation has the virtual input x_2. Choose

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3$$

such that $\dot{x}_1 = -x_1$. Then the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$ (which is positive definite and radially unbounded) will have $\dot{V}_1 = x_1\dot{x}_1 = -x_1^2$ which is negative definite.
Augment the virtual input with z, such that

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3 + z$$

then

$$z = x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3$$

and

$$\begin{aligned}\dot{z} &= \dot{x}_1 + \dot{x}_2 + 3\left(x_1 - a^{1/3}\right)^2 \dot{x}_1 \\ &= \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) + (x_1 + u) + 3\left(x_1 - a^{1/3}\right)^2 \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\ &= x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\end{aligned}$$

Calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1(-x_1 + z) = -x_1^2 + x_1 z$$

We may choose a Lyapunov function candidate for the overall system as $V_c = V_1 + \frac{1}{2}z^2$, then

$$\begin{aligned}\dot{V}_c &= \dot{V}_1 + z\dot{z} \\ &= -x_1^2 + x_1 z + z\left\{x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\} \\ &= -x_1^2 + z\underbrace{\left\{2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\}}_{\text{choose} \quad -z} \\ &= -x_1^2 - z^2\end{aligned}$$

where we have chosen

$$-z = 2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

i.e.

$$
\begin{aligned}
u &= -z - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -\overbrace{\left(x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)}^{z} - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -3x_1 - \left(2 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)
\end{aligned}
$$

such that $\dot{V}_c$ is negative definite. We already know that V_c is positive definite and radially unbounded. Hence, the overall system is globally asymptotically stable (GAS).

<u>Alternative solution</u>:
The system is in the form of (14.53)-(14.54) in Khalil with

$$
\begin{aligned}
f &= a + \left(x_1 - a^{1/3}\right)^3 \\
g &= 1 \\
f_a &= x_1 \\
g_a &= 1
\end{aligned}
$$

Take

$$
\begin{aligned}
\phi(x_1) &= -a - \left(x_1 - a^{1/3}\right)^3 - x_1 \\
V &= \frac{1}{2}x_1^2
\end{aligned}
$$

and use (14.56) in Khalil.

8. Consider $\dot{x}_1 = x_1 x_2 + x_1^2$ with x_2 as a virtual input. Choose a Lyapunov function candidate $V_1(x) = \frac{1}{2}x_1^2$ and calculate

$$
\dot{V}_1 = x_1\dot{x}_1 = x_1\left(x_1x_2 + x_1^2\right)
$$

We can enforce $\dot{V}_1 = -x_1^4$ which is negative definite, by choosing the input $x_2 = -x_1 - x_1^2$ (actually, any choice $x_2 = -x_1^{2k} - x_1^2$, $k = 1, 2, 3, \dots$ will be possible, to get a negative definite $\dot{V}_1$, but for simplification we choose k=1).
Augment the input with z, such that we have $x_2 = -x_1 - x_1^2 + z$ (i.e. $z = x_2 + x_1 + x_1^2$), then

$$
\dot{V}_1 = x_1\left(x_1x_2 + x_1^2\right) = x_1\left(x_1\left(-x_1 - x_1^2 + z\right) + x_1^2\right) = -x_1^4 + x_1^2 z
$$

Now choose a Lyapunov function candidate for the complete system $V_c = V_1 + \frac{1}{2}z^2$, which is positive definite and radially unbounded. Then

$$
\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \\
&= -x_1^4 + x_1 z + z\left(u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right) \\
&= -x_1^4 + z\underbrace{\left(x_1 + u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right)}_{\text{choose} \quad -z}
\end{aligned}
$$

We can enforce $\dot{V}_c = -x_1^2 - z^2$ (then $\dot{V}_c$ is negative definite), by choosing

$$\begin{aligned}
-z &= x_1 + u + (2x_1 + 1)\left(x_1 x_2 + x_1^2\right) \\
u &= -x_1 - (2x_1 + 1)\left(x_1 x_2 + x_1^2\right) - z
\end{aligned}$$

By inserting z, we get the expression for the stabilizing input

$$\begin{aligned}
u &= -x_1 - (2x_1 + 1)\left(x_1 x_2 + x_1^2\right) - x_2 + x_1 + x_1^2 \\
&= -(2x_1 + 1)\left(x_1 x_2 + x_1^2\right) - x_2 + x_1^2
\end{aligned}$$

Since $V_c(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_c(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0 \rightarrow x_2 = -x_1 - x_1^2$ and $x_1 = 0 \rightarrow -x_1 - x_1^2 = 0$, this means that also x_2 is asymptotically stabilized at the origin. In addition, since $V_c(x_1, z)$ is radially unbounded and there are no singularities in u, the equilibrium point $x = (0, 0)$ is globally asymptotically stable.