TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 4

Hand-out time:	Monday, September 30, 2013, at 12:00
Hand-in deadline:	Friday, October 11, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Optimal set-point control (computer exercise)

For this exercise, you are allowed and advised to use MATLAB and Simulink.

In this exercise, we will design a controller for a "floating camera", which is used for big television events. The camera is attached to two cables. Pulling on the cables makes the camera move vertically or swing in a horizontal direction. We consider the camera system in Fig. 1. Each cable goes via a motor to a pulley system, which is connected to a counterweight. Each motor can apply a force to the cable to influence the motion of the camera.

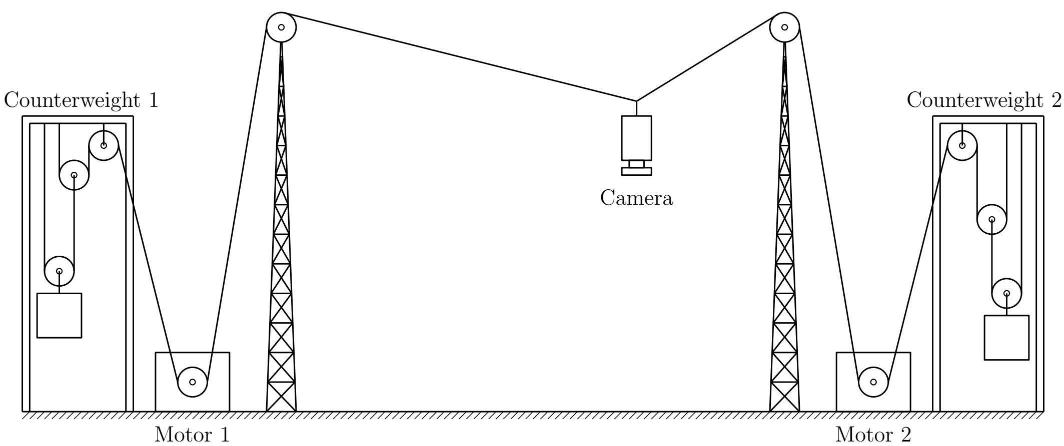

Fig. 1: Camera system.

The mass of the camera is given by m_c. The mass of both counterweights is the same and is given by m_w. The force that the motors apply to each cable is given by F_1 for motor 1, and F_2 for motor 2. The masts on which the cables for the camera system are attached have a height of $2l_y$. The distance between the masts is $2l_x$. The horizontal position of the camera is indicated by x. If the camera is at a distance of l_x from each mast, x is equal to zero. The vertical position of the camera is indicated by y. If the camera is of a distance l_y from the ground (and the top of the mast), y is equal to zero; see Fig. 2.

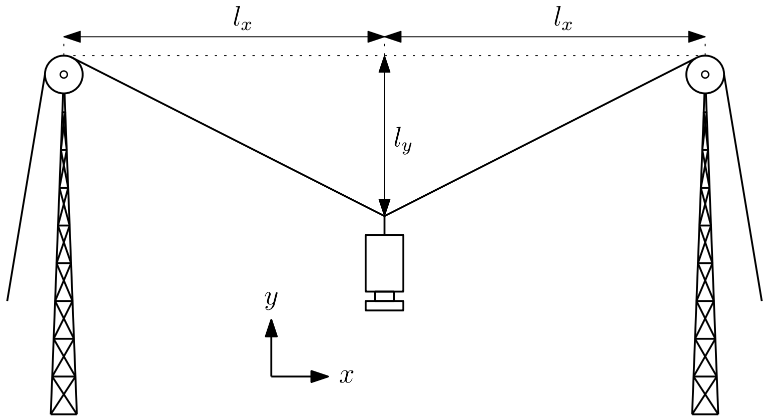

Fig. 2: Position of camera for $x = y = 0$.

The equations of motion of the camera system are highly nonlinear. However, after a few simplifications, the equations of motion for $-l_x \le x \le l_x$ and $-l_y \le y \le l_y$ are given by

$$\begin{aligned} \left(m_c + \frac{m_w l_x^2}{8r^2}\right)\ddot{x} &= -\frac{l_x + x}{r_1(x,y)}\left(\frac{m_w g}{4} + F_1\right) + \frac{l_x - x}{r_2(x,y)}\left(\frac{m_w g}{4} + F_2\right) \\ \left(m_c + \frac{m_w l_y^2}{8r^2}\right)\ddot{y} &= -m_c g + \frac{l_y - y}{r_1(x,y)}\left(\frac{m_w g}{4} + F_1\right) + \frac{l_y - y}{r_2(x,y)}\left(\frac{m_w g}{4} + F_2\right), \end{aligned} \tag{1}$$

with $r = \sqrt{l_x^2 + l_y^2}$, where g is the gravitational constant,

$$r_1(x,y) = \sqrt{(l_x + x)^2 + (l_y - y)^2} \qquad \text{and} \qquad r_2(x,y) = \sqrt{(l_x - x)^2 + (l_y - y)^2}.$$

Step 1: Linearize the system about an equilibrium point

a) Write the equations of motion in (1) as a set of first-order differential equations, i.e., write system (1) in the form

$$\begin{aligned} \dot{x} &= f_1(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ \ddot{x} &= f_2(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ \dot{y} &= f_3(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ \ddot{y} &= f_4(x, \dot{x}, y, \dot{y}, F_1, F_2), \end{aligned} \tag{2}$$

where f_1, f_2, f_3 and f_4 are functions that need to be determined.

The mass of the camera m_c, the distances l_x and l_y and the gravitational constant g are given by

$$m_c = 3, \qquad l_x = 16, \qquad l_y = 12 \qquad \text{and} \qquad g = 9.81.$$

b) Determine the mass m_w of the counterweights such that $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$ is an equilibrium point of the system, i.e., determine the mass m_w such that

$$\begin{aligned} 0 &= f_1(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ 0 &= f_2(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ 0 &= f_3(x, \dot{x}, y, \dot{y}, F_1, F_2), \\ 0 &= f_4(x, \dot{x}, y, \dot{y}, F_1, F_2), \end{aligned}$$

for $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$. We will use this value of m_w in the remaining part of the exercise.

c) Linearize the equations of motion in (2) about the equilibrium point $x = \dot{x} = y = \dot{y} = F_1 = F_2 = 0$. Show that the linearized system can be written in the form

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}, \tag{3}$$

with

$$\mathbf{z} = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

and matrices

$$\mathbf{A} \approx \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.2323 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.4550 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} \approx \begin{bmatrix} 0 & 0 \\ -0.2105 & 0.2105 \\ 0 & 0 \\ 0.1739 & 0.1739 \end{bmatrix}.$$

Hint: note that

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{r_1(x,y)}\right) &= -\frac{l_x + x}{r_1^3(x,y)}, & \frac{\partial}{\partial x}\left(\frac{1}{r_2(x,y)}\right) &= \frac{l_x - x}{r_2^3(x,y)}, \\ \frac{\partial}{\partial y}\left(\frac{1}{r_1(x,y)}\right) &= \frac{l_y - y}{r_1^3(x,y)}, & \frac{\partial}{\partial y}\left(\frac{1}{r_2(x,y)}\right) &= \frac{l_y - y}{r_2^3(x,y)}. \end{aligned}$$

d) Check if the linearized system in (3) is controllable by computing the rank of the controllability matrix. Motivate your answer.

Step 2: Determine the relation between the states and the set-point variables

We will design a controller based on a linear quadratic regulator, such that x and y

will go to fixed set-point values x_c and y_c. We will refer to x and y as set-point variables. The set-point variables x and $y$are collected in the vector

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

e) Determine the matrix $\mathbf{C}$ such that

$$\mathbf{r} = \mathbf{C}\mathbf{z}. \tag{4}$$

Combining the equations in (3) and (4), we obtain the system

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}, \\ \mathbf{r} &= \mathbf{C}\mathbf{z}. \end{aligned} \tag{5}$$

f) Check if the system in (5) is observable by computing the rank of the observability matrix. Motivate your answer.

Step 3: Determine the feedforward controller by taking the inverse of the system

Suppose that the states x and y are equal to the set-point values x_c and y_c for all time, that is, $\mathbf{r} = \mathbf{r}_c$, with

$$\mathbf{r}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$$

If we take the time-derivative of both sides of the equation $\mathbf{r}(t) = \mathbf{r}_c$, it follows that

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x}_c = 0 \\ \dot{y}_c = 0 \end{bmatrix} = \dot{\mathbf{r}}_c,$$

because x_c and y_c are constants and the time-derivative of a constant is zero. Therefore, the state $\mathbf{z}$ of system (3) that corresponds to the set point $\mathbf{r}_c$ is given by

$$\mathbf{z}_c = \begin{bmatrix} x_c \\ 0 \\ y_c \\ 0 \end{bmatrix}.$$

g) Determine the matrix $\mathbf{E}$ such that

$$\mathbf{z}_c = \mathbf{E}\mathbf{r_c}. \tag{6}$$

Let the input that corresponds to the state vector $\mathbf{z}_c$ be given by

$$\mathbf{u}_c = \begin{bmatrix} F_{1c} \\ F_{2c} \end{bmatrix},$$

where F_{1c} and F_{2c} are constants (dependent on x_c and y_c) that need to be determined. Substituting $\mathbf{r}_c$, $\mathbf{z}_c$ and $\mathbf{u}_c$ in the equations in (5), gives

$$\begin{aligned} \mathbf{0} &= \mathbf{A}\mathbf{z}_c + \mathbf{B}\mathbf{u}_c, \\ \mathbf{r}_c &= \mathbf{C}\mathbf{z}_c. \end{aligned} \tag{7}$$

The feedforward controller has the form

$$\mathbf{u}_c = \mathbf{F}\mathbf{r}_c. \tag{8}$$

h) Use the equations in (7) to show that the matrix $\mathbf{F}$ in (8) is given by

$$\mathbf{F} \approx \begin{bmatrix} -0.5518 & 1.3080 \\ 0.5518 & 1.3080 \end{bmatrix}.$$

Step 4: Determine the optimal feedback controller

We define the coordinate transformation

$$\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{z}_c, \qquad \tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_c \qquad \text{and} \qquad \tilde{\mathbf{r}} = \mathbf{r} - \mathbf{r}_c. \tag{9}$$

i) Using (5), (7) and the coordinate transformation in (9), show that the following equations hold:

$$\begin{aligned} \dot{\tilde{\mathbf{z}}} &= \mathbf{A}\tilde{\mathbf{z}} + \mathbf{B}\tilde{\mathbf{u}}, \\ \tilde{\mathbf{r}} &= \mathbf{C}\tilde{\mathbf{z}}. \end{aligned} \tag{10}$$

We will refer to the system in (10) as the error system. We will design a linear quadratic regulator for error system (10), such that $\tilde{\mathbf{r}}$ goes to zero as time goes to infinity. Note that if $\tilde{\mathbf{r}}$ goes to zero, then from (9) it follows that $\mathbf{r}$ goes to $\mathbf{r}_c$, which implies that the set-point variables will go the the set-point values. The regulator minimizes the cost function

$$J_{LQR} = \int_0^\infty \tilde{\mathbf{r}}^T(t)\mathbf{Q}\tilde{\mathbf{r}}(t) + \tilde{\mathbf{u}}^T(t)\mathbf{R}\tilde{\mathbf{u}}(t)dt, \tag{11}$$

with

$$\mathbf{Q} = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The optimal regulator is given by

$$\tilde{\mathbf{u}} = -\mathbf{K}\tilde{\mathbf{z}}, \tag{12}$$

where the matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{W},$$

where $\mathbf{W}$ is the positive definite solution of the Riccati equation

$$\mathbf{A}^T\mathbf{W} + \mathbf{W}\mathbf{A} + \mathbf{C}^T\mathbf{Q}\mathbf{C} - \mathbf{W}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{W} = \mathbf{0}.$$

j) Show that the matrix $\mathbf{K}$ is given by

$$\mathbf{K} \approx \begin{bmatrix} -1.9591 & -3.0505 & 1.4688 & 2.9062 \\ 1.9591 & 3.0505 & 1.4688 & 2.9062 \end{bmatrix}.$$

Note that compared to a "normal" linear-quadratic-regulator cost function, the set-point error $\tilde{\mathbf{r}}$ is part of the cost function in (11) instead of the state error $\tilde{\mathbf{z}}$. The second equation in (10) can be used to write the cost function in (11) as a function of state error $\tilde{\mathbf{z}}$ instead of the set-point error $\tilde{\mathbf{r}}$.

Step 5: Determine the control input

We combine the feedforward controller in Step 3 with the feedback controller in Step 4.

k) Combine (6), (8), (9) and (12) to show that the control input is given by

$$\mathbf{u} = -\mathbf{M}_1\mathbf{z} + \mathbf{M}_2\mathbf{r}_c, \tag{13}$$

with matrices

$$\mathbf{M}_1 \approx \begin{bmatrix} -1.9591 & -3.0505 & 1.4688 & 2.9062 \\ 1.9591 & 3.0505 & 1.4688 & 2.9062 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_2 \approx \begin{bmatrix} -2.5109 & 2.7768 \\ 2.5109 & 2.7768 \end{bmatrix}.$$

Step 6*: Add an integral effect to the controller

Instead of the feedback controller in (12), we will design another feedback controller that has an integral effect. First, we extent the error system in (10) with two additional states:

$$\tilde{\mathbf{R}} = \int_0^t \tilde{\mathbf{r}}(\tau)d\tau = \begin{bmatrix} \int_0^t \tilde{x}(\tau)d\tau \\ \int_0^t \tilde{y}(\tau)d\tau \end{bmatrix}.$$

The extended state vector and set-point error vector are given by

$$\underline{\tilde{\mathbf{z}}} = \begin{bmatrix} \tilde{\mathbf{z}} \\ \tilde{\mathbf{R}} \end{bmatrix} \qquad \text{and} \qquad \underline{\tilde{\mathbf{r}}} = \begin{bmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{R}} \end{bmatrix}.$$

The extended error system is given by

$$\begin{aligned} \dot{\underline{\tilde{\mathbf{z}}}} &= \underline{\mathbf{A}}\underline{\tilde{\mathbf{z}}} + \underline{\mathbf{B}}\tilde{\mathbf{u}}, \\ \underline{\tilde{\mathbf{r}}} &= \underline{\mathbf{C}}\underline{\tilde{\mathbf{z}}}. \end{aligned} \tag{14}$$

l) Show that the matrices in (14) are given by

$$\underline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \qquad \underline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix.

Similar to Step 4, we design a regulator that minimizes the cost function

$$J_{LQR} = \int_0^\infty \underline{\tilde{\mathbf{r}}}^T(t)\underline{\mathbf{Q}}\underline{\tilde{\mathbf{r}}}(t) + \tilde{\mathbf{u}}^T(t)\underline{\mathbf{R}}\tilde{\mathbf{u}}(t)dt,$$

with

$$\underline{\mathbf{Q}} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{R}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the first two elements on the diagonal of $\underline{\mathbf{Q}}$ correspond to the set-point values x and y, and that the last two elements on the diagonal of $\underline{\mathbf{Q}}$ correspond to the integrated set-point values X and Y.

The optimal regulator is given by

$$\tilde{\mathbf{u}} = -\underline{\mathbf{K}}\underline{\tilde{\mathbf{z}}},$$

where the matrix $\underline{\mathbf{K}}$ is given by

$$\underline{\mathbf{K}} = \underline{\mathbf{R}}^{-1}\underline{\mathbf{B}}^T\underline{\mathbf{W}},$$

where $\underline{\mathbf{W}}$ is the positive definite solution of the Riccati equation

$$\underline{\mathbf{A}}^T\underline{\mathbf{W}} + \underline{\mathbf{W}}\underline{\mathbf{A}} + \underline{\mathbf{C}}^T\underline{\mathbf{Q}}\underline{\mathbf{C}} - \underline{\mathbf{W}}\underline{\mathbf{B}}\underline{\mathbf{R}}^{-1}\underline{\mathbf{B}}^T\underline{\mathbf{W}} = \mathbf{0}.$$

m) Show that the matrix $\underline{\mathbf{K}}$ is given by

$$\underline{\mathbf{K}} \approx \begin{bmatrix} -2.8390 & -3.6722 & 2.2730 & 3.6152 & -0.7071 & 0.7071 \\ 2.8390 & 3.6722 & 2.2730 & 3.6152 & 0.7071 & 0.7071 \end{bmatrix}.$$

n) Show that the control input with integral effect is given by

$$\mathbf{u} = -\underline{\mathbf{M}}_1\mathbf{z} + \underline{\mathbf{M}}_2\mathbf{r}_c - \underline{\mathbf{M}}_3\int_0^t \mathbf{C}\mathbf{z}(\tau) - \mathbf{r}_c d\tau, \tag{15}$$

with matrices

$$\underline{\mathbf{M}}_1 \approx \begin{bmatrix} -2.8390 & -3.6722 & 2.2730 & 3.6152 \\ 2.8390 & 3.6722 & 2.2730 & 3.6152 \end{bmatrix}, \qquad \underline{\mathbf{M}}_2 \approx \begin{bmatrix} -3.3908 & 3.5810 \\ 3.3908 & 3.5810 \end{bmatrix},$$

and

$$\underline{\mathbf{M}}_3 \approx \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}.$$

Step 7**: Implementation of the controllers

On It's Learning, the MATLAB file *camera_run.m* and the Simulink file *camera_sim.mdl* are posted. The file *camera_sim.mdl* contains a model of the camera system (without simplifications). The file *camera_run.m* is an initialization file that calls the Simulink model in the *camera_sim.mdl* file.

o) Implement the two controllers in (13) and (15) and plot the corresponding states x and y as a function of time for the initial conditions

$$x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$$

and the set-point values

$$x_c = 8 \qquad \text{and} \qquad y_c = 9.$$

p) Compare the results for both controllers. What causes the differences? Motivate your answer.

q) How do the matrices $\mathbf{Q}$ and $\mathbf{R}$, respectively $\underline{\mathbf{Q}}$ and $\underline{\mathbf{R}}$, need to be tuned such that the set-point variables x and y converge faster to the set-point values x_c and y_c? Illustrate this with simulation results.

Problem 2: Output-feedback controllers

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{w},\\ y(t) &= \mathbf{C}\mathbf{x}(t) + v,\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\mathbf{w}$ and v are constant disturbances given by

$$\mathbf{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \text{and} \qquad v = 3.$$

For the first part of this exercise, consider the following static output-feedback controller, which is also know as a P-controller:

$$u(t) = -k_p y(t), \tag{16}$$

where k_p is a constant.

a) The controller (16) can be written as $u(t) = -\mathbf{K}_p\mathbf{x}(t) + q_p$, where $\mathbf{K}_p$ is a constant matrix and q_p is a constant. Compute $\mathbf{K}_p$ and q as a function of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{w}$, v and k_p.

b) Does there exist a value of k_p such that the closed-loop system is asymptotically stable (i.e. such that the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ all have a negative real part)? If so, give a value of k_p for which this is the case.

Next, we add derivative output feedback to the controller, which results in the following PD-controller:

$$u(t) = -k_p y(t) - k_d \dot{y}(t), \tag{17}$$

where k_p and k_d are constants.

c) The controller (17) can be written as $u(t) = -\mathbf{K}_{pd}\mathbf{x}(t) + q_{pd}$, where $\mathbf{K}_{pd}$ is a constant matrix and q_{pd} is a constant. Compute $\mathbf{K}_{pd}$ and q_{pd} as a function of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{w}$, v, k_p and k_d.

d) Calculate the constants k_p and k_d such that the poles of the closed-loop system (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}_{pd}$) are equal to $-1 \pm i$, with $i = \sqrt{-1}$.

e) Using the values of k_p and k_d obtained in d), what is the steady-state output of the system (i.e. what is the value of $y(t)$ as $t \to \infty$)?

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*Write your **name** and **student number** on your solutions before you hand them in!*
(Your student number has six digits. You can find it on your semester card.)
