TTT4120 Digital Signal Processing
Fall 2020

Modeling of Stochastic Processes:
Parametric Spectral Estimation

Prof. Stefan Werner

stefan.werner@ntnu.no

Office B329

Department of Electronic Systems
© Stefan Werner

1

Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 12.2   Innovations representation of stationary random processes
  - 14.3   Parametric methods for spectral estimation
- A comprehensive overview of topics treated in the lecture, see "Introdukjon til statistisk signalbehandling" on Blackboard

  *Level of detail is defined by lectures and problem sets
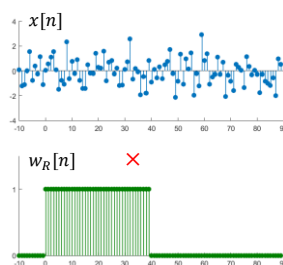
2

2

Contents and learning outcomes

- Non-parametric versus parametric models
- Innovations representation
- Rational power spectra: AR, MA, ARMA
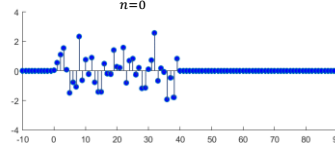- AR models and Yule-Walker equations

3

3

Nonparametric PSD estimation

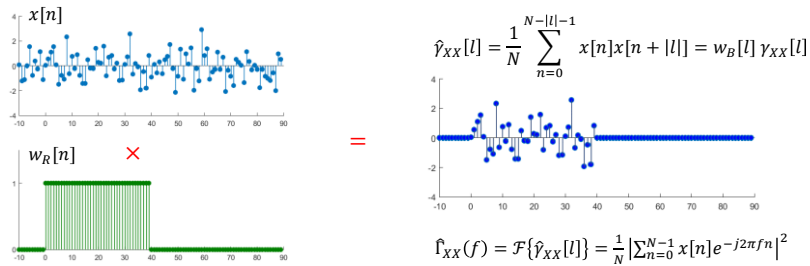$$\hat{\gamma}_{XX}[l] = \frac{1}{N} \sum_{n=0}^{N-|l|-1} x[n]x[n+|l|] = w_B[l]\,\gamma_{XX}[l]$$

$$\hat{\Gamma}_{XX}(f) = \mathcal{F}\{\hat{\gamma}_{XX}[l]\} = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j2\pi f n}\right|^2$$

- Nonparametric power spectrum estimation: Assume that stochastic process $X[n]$ is wide-sense stationarity and ergodic
  - $+$ Simple and easy to compute using the FFT_N
  - $-$ Requires long data records for good frequency resolution
  - $-$ Spectral leakage due to windowing $\Rightarrow$ Can mask weak signals

4

4

Nonparametric PSD estimation…

$x[n]$

$$\hat{\gamma}_{XX}[l] = \frac{1}{N} \sum_{n=0}^{N-|l|-1} x[n]x[n+|l|] = w_B[l]\, \gamma_{XX}[l]$$

$w_R[n]$ ✗

$$=$$

$$\hat{\Gamma}_{XX}(f) = \mathcal{F}\{\hat{\gamma}_{XX}[l]\} = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n]e^{-j2\pi f n} \right|^2$$

- Basic limitations of nonparametric spectrum estimation
  - Inherent assumption that $\hat{\gamma}_{XX}[l] = 0$ for some $l \geq N$
  - Inherent assumption that the data is periodic with N

5

5

Parametric PSD estimation

- Consider methods that can extrapolate the values of the autocorrelation function for $l \geq N$

$$\hat{\gamma}_{XX}[l], |l| \leq N - 1 \implies \hat{\gamma}_{XX}[l], |l| \geq N$$

- Requires *a priori* information on how data signal is generated
- A parametric model for the signal generation is constructed
  - Sufficient to find the values of the model parameters
  - Often provides us with a better description of the process, whenever the model is close to reality
  - Eliminates need for windowing and assumption that $\hat{\gamma}_{XX}[m] = 0$ for some $m \geq N$

6

6

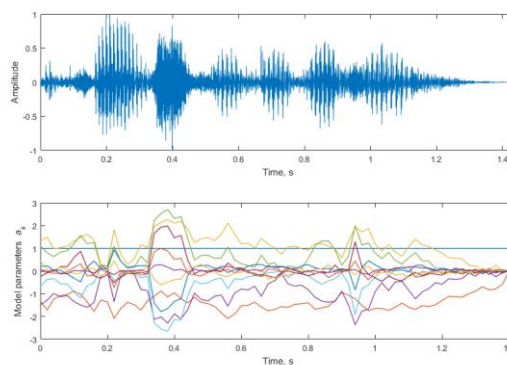## Parametric PSD estimation…

- Parametric modeling does not provide an exact representation
  - approximation characterized by a few parameters
  - enhanced spectral resolution: especially for finite data records, e.g., due to time-variant or transient phenomena
  - Efficient signal compression (e.g., LPC of speech)
- Different parametric models
  - rational models
  - all-pole models lead to linear equation systems
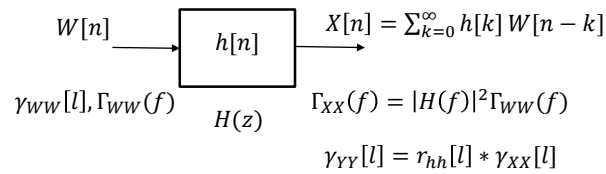
7

7

Parametric PSD estimation…

- Example: Model a speech signal (parameters change every 20ms)

8

8

4

Innovations representations

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = \sum_{k=0}^{\infty} h[k]\, W[n-k]$$

$$\gamma_{WW}[l], \Gamma_{WW}(f) \qquad H(z) \qquad \Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f)$$
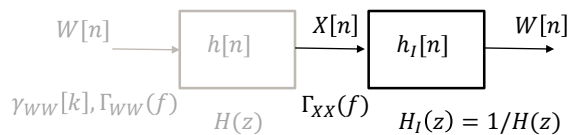
$$\gamma_{YY}[l] = r_{hh}[l] * \gamma_{XX}[l]$$

- Wide-sense stationary random processes can be represented as the output of a causal and causually invertible system excited by a white noise process
- This representation is called the Wold representation

9

9

Innovations representations…

$$W[n] \longrightarrow \boxed{h[n]} \xrightarrow{X[n]} \boxed{h_I[n]} \xrightarrow{W[n]}$$

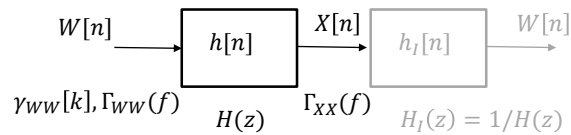$$\gamma_{WW}[k], \Gamma_{WW}(f) \qquad H(z) \qquad \Gamma_{XX}(f) \qquad H_I(z) = 1/H(z)$$

- Consequently, a WSS random process can be represented by the output of the inverse system, which is a white process
  - a random process can be transformed into a white process by passing $X[n]$ through a linear filter
  - $H_I(z)$ is a called a whitening filter

10

10

Innovations representations…

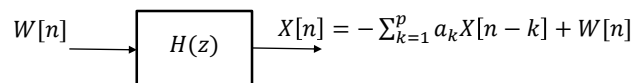- Restrict our attention to cases where the PSD of $X[n]$ is rational

$$\Gamma_{XX}(f) = \left.\frac{\sigma_W^2 B(z)B(z^{-1})}{A(z)A(z^{-1})}\right|_{z=e^{j2\pi f}} \text{ or } H(z) = \frac{B(z)}{A(z)}$$

- $H(z)$ is causal stable and minimum-phase $\Rightarrow H_I(z)$ is also causal stable and minimum phase
- By knowing $H(z)$, described by a few parameters, we can go from the statistical properties of $W[n]$ to $X[n]$, and vice versa

11

11

Model types: AR process

- For an autoregressive (AR) process, filter $H(z)$ has only poles,

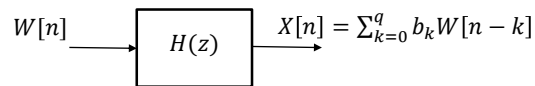$$H(z) = \frac{1}{A(z)} = \frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

and is described in time domain as

$$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

12

12

Model types: MA process

$W[n]$ → $H(z)$ → $X[n] = \sum_{k=0}^{q} b_k W[n-k]$

- For a moving average (MA) process, filter $H(z)$ has only zeros,

$$H(z) = B(z) = \sum_{k=0}^{q} b_k z^{-k}$$

and is described in time domain as

$$X[n] = \sum_{k=0}^{q} b_k W[n-k]$$

13

13

Model types: ARMA process

$W[n]$ → $H(z)$ → $X[n] = -\sum_{k=1}^{p} a_k X[n-k] + \sum_{k=0}^{q} b_k W[n-k]$
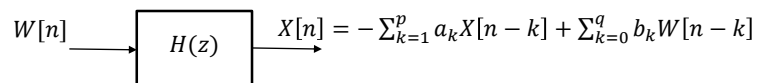
- For an autoregressive moving average (ARMA) process, filter $H(z)$ has both zeros and poles,

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

and is described in time domain as

$$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + \sum_{k=0}^{q} b_k W[n-k]$$

14

14

Parameter estimation

$$W[n] \longrightarrow \boxed{H(z)} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + \sum_{k=0}^{q} b_k W[n-k]$$

- The models are described by a few parameters: $\{a_k\}$, $\{b_k\}$, and σ_W^2
- Parameter values are unknown $\Rightarrow$ need to estimate them from $X[n]$
- We need to find the parameters such that the model "resembles," or is "close" to, the true process
- Restrict our study to AR models

15

15

Parameter estimation…

$$W[n] \longrightarrow \boxed{H(z)} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

- Restrict our study to AR models
  - can model processes characterized by sharp peaks in the spectrum
  - other types of spectra can be modeled by increasing the model order, i.e., the number of filter coefficients
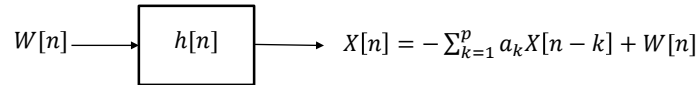 - suitable for many practical physical processes, e.g., speech

16

16

Statistical description of AR(*p*) process

- Model the problem with linear system

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

- WGN process $W[n]$: $E\{W[n]W[n-l]\} = \sigma_W^2 \delta[l]$
- The AR(p) process has p filter coefficients $\{a_k\}_{k=1}^{p}$
- Find a relation between $\gamma_{XX}[l]$ and coefficients a_k and σ_W^2

17

17

Statistical description of AR(*p*) process

- Autocorrelation function $\gamma_{XX}[l]$:

$$\gamma_{XX}[l] = E\{X[n]X[n-l]\}$$

$$= E\left\{\left(-\sum_{k=1}^{p} a_k X[n-k] + W[n]\right)X[n-l]\right\}$$

$$= E\left\{-\sum_{k=1}^{p} a_k X[n-k]X[n-l] + W[n]X[n-l]\right\}$$

$$= -\sum_{k=1}^{p} a_k E\{X[n-k]X[n-l]\} + E\{W[n]X[n-l]\}$$

$$= -\sum_{k=1}^{p} a_k \gamma_{XX}[l-k] + \gamma_{WX}[l]$$

- Take a closer look at the crosscorrelation term $\gamma_{WX}[l]$

18

18

9

Statistical description of AR(*p*) process…

- Crosscorrelation term $\gamma_{WX}[l]$:

$$\gamma_{WX}[l] = E\{W[n]X[n-l]\} = E\{W[n+l]X[n]\}$$

$$= E\{W[n+l]\left(-\sum_{k=1}^{p} a_k X[n-k] + W[n]\right)\}$$

$$= E\{-\sum_{k=1}^{p} a_k W[n+l]X[n-k] + W[n+l]W[n]\}$$

$$= -\sum_{k=1}^{p} a_k E\{W[n+l]X[n-k]\} + E\{W[n+l]W[n]\}$$

$$= 0 + \sigma_W^2 \delta[l]$$

- Crosscorrelation term only take non-zero value at lag $l = 0$

19

19

Statistical description of AR(*p*) process…

- Autocorrelation function $\gamma_{XX}[l]$ for an AR(*p*) process:

$$\boxed{\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l| - k] + \sigma_W^2 \delta[l]}$$

- Discussion: How can we use the above equation to find
$\gamma_{XX}[l]$ for all l, and what knowledge is required?

20

20

10

Statistical description of AR(*p*) process…

- Autocorrelation function $\gamma_{XX}[l]$ for an AR(*p*) process:

$$\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l| - k] + \sigma_W^2 \delta[l]$$

- Discussion: How can we use the above equation to find…
  - Crosscorrelation function $\gamma_{XX}[l]$ is specified for all l when
    1) the *p* filter coefficients $\{a_k\}_{k=1}^{p}$, and;
    2) the *p*+1 first values of $\gamma_{XX}[l]$, i.e., $\gamma_{XX}[0], \gamma_{XX}[1], \ldots, \gamma_{XX}[p]$
- How to find model parameters to the model $\{a_k\}_{k=1}^{p}$ given $\gamma_{XX}[l]$

21

21

Statistical description of AR(*p*) process…

$$\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l| - k] + \sigma_W^2 \delta[l]$$

- Linear equation system ($\gamma_{XX}[l] = \gamma_{XX}[-l]$):

$$l = 1: -\gamma_{XX}[1] = a_1 \gamma_{XX}[0] + a_2 \gamma_{XX}[1] + \cdots + a_p \gamma_{XX}[p - 1]$$

$$l = 2: -\gamma_{XX}[2] = a_1 \gamma_{XX}[1] + a_2 \gamma_{XX}[0] + \cdots + a_p \gamma_{XX}[p - 2]$$

$$l = 3: -\gamma_{XX}[3] = a_1 \gamma_{XX}[2] + a_2 \gamma_{XX}[1] + \cdots + a_p \gamma_{XX}[p - 3]$$

$$\vdots$$

$$l = p: -\gamma_{XX}[p] = a_1 \gamma_{XX}[p - 1] + a_2 \gamma_{XX}[p - 2] + \cdots + a_p \gamma_{XX}[0]$$

22

22

Statistical description of AR(*p*) process…

- Linear equation system in matrix form (Yule-Walker equations):

$$\underbrace{\begin{bmatrix} \gamma_{XX}[0] & \gamma_{XX}[1] & \cdots & \gamma_{XX}[p-1] \\ \gamma_{XX}[1] & \gamma_{XX}[0] & \cdots & \gamma_{XX}[p-2] \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{XX}[p-1] & \gamma_{XX}[p-2] & \cdots & \gamma_{XX}[0] \end{bmatrix}}_{\Gamma_{XX}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}_{a} = - \underbrace{\begin{bmatrix} \gamma_{XX}[1] \\ \gamma_{XX}[2] \\ \vdots \\ \gamma_{XX}[p] \end{bmatrix}}_{\gamma_{XX}}$$

and

$$\sigma_W^2 = \gamma_{XX}[0] + \sum_{k=1}^{p} a_k \gamma_{XX}[k]$$

- Solve for coefficients

$$\boldsymbol{a} = -\boldsymbol{\Gamma}_{XX}^{-1} \boldsymbol{\gamma}_{XX}$$

23

23

Statistical description of AR(*p*) process…

- Sanity check: Model process $X[n] = -aX[n-1] + W[n]$ using an AR(2) process, and find parameters a_1 and a_2

  AR(1) process: $\gamma_{XX}[l] = \sigma_W^2 \frac{(-a)^{|l|}}{1-a^2}, |l| \geq 0$

  Yule-Walker equations:

  $$\underbrace{\begin{bmatrix} \gamma_{XX}[0] & \gamma_{XX}[1] \\ \gamma_{XX}[1] & \gamma_{XX}[0] \end{bmatrix}}_{\Gamma_{XX}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{a} = - \underbrace{\begin{bmatrix} \gamma_{XX}[1] \\ \gamma_{XX}[2] \end{bmatrix}}_{\gamma_{XX}}$$

  $$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} -a \\ a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}^{-1}\begin{bmatrix} -a \\ a^2 \end{bmatrix}$$

  Solve for coefficients: $a_1 =?, a_2 =?$

24

24

PSD of an AR(*p*) process

- Once we have filter coefficients $\{a_k\}_{k=1}^{p}$ we can compute the PSD

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

$$\gamma_{WW}[k] = \sigma_W^2 \delta[k], \qquad\qquad \Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f) = |H(f)|^2 \sigma_W^2$$
$$\Gamma_{WW}(f) = \sigma_W^2$$

- Frequency response of filter

$$H(f) = \frac{1}{1+\tilde{A}(f)}, \text{ with } \tilde{A}(z) = \sum_{k=1}^{p} a_k z^{-k}$$

- Finally we obtain the PSD as

$$\Gamma_{XX}(f) = \frac{\sigma_W^2}{\left|1 + \tilde{A}(f)\right|^2}$$

25

25

PSD of an AR(*p*) process…

- Example: Model $X[n] = W[n] - bW[n-1]$, $W[n] \sim N(0,1)$ using an AR(2) process, and find parameters a_1 and a_2

- Autocorrelation: $\gamma_{XX}[l] = (1+b^2)\delta[l] - b\delta[l-1] - b\delta[l+1]$
- Yule-Walker equations:

$$\begin{bmatrix} 1+b^2 & -b \\ -b & 1+b^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} -b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1+b^2 & -b \\ -b & 1+b^2 \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Solve for coefficients: $a_1 = \frac{b(1+b^2)}{b^2(1+b^2)+1}$, $a_2 = \frac{b^2}{b^2(1+b^2)+1}$

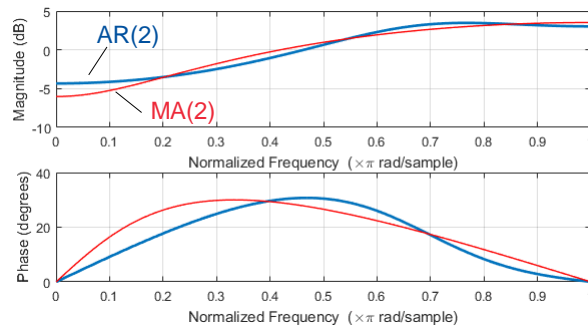$$\sigma_W^2 = \gamma_{XX}[0] + a_1 \gamma_{XX}[1] + a_2 \gamma_{XX}[2]$$

26

26

13

PSD of an AR(p) process…

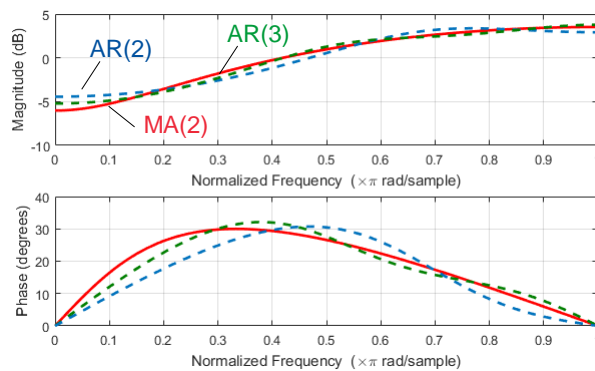- Example: Model $X[n] = W[n] - 0.5W[n-1]$, $W[n] \sim N(0,1)$

27

27

PSD of an AR(p) process…

- Example: Model $X[n] = W[n] - 0.5W[n-1]$, $W[n] \sim N(0,1)$

28

28

14

PSD of an AR(*p*) process…

- Example: Power spectrum of an AR process is given by

$$\Gamma_{XX}(f) = \frac{\sigma_W^2}{|A(f)|^2} = \frac{25}{\left|1 - e^{-j2\pi f} + \frac{1}{2}e^{-j4\pi f}\right|^2}$$

  where σ_W^2 is the variance of the input sequence.

  a) Determine the difference equation that generates the AR process when the excitation is white noise
  b) Determine the system function for the whitening filter

29

29

PSD of an AR(*p*) process…

- Only access to finite-length realization, $x[n]$, of process $X[n]$
  - True $\gamma_{XX}[l]$ must be estimated from $x[n] \Rightarrow \hat{\gamma}_{XX}[l]$
  - Parameter values computed using $\hat{\gamma}_{XX}[l]$ becomes parameter estimates $\{\hat{a}_k\} \Rightarrow$ Power spectrum estimate

$$\hat{\Gamma}_{XX}(f) = \frac{\hat{\sigma}_W^2}{\left|\hat{A}(f)\right|^2} = \frac{\hat{\sigma}_W^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_k e^{-j2\pi fk}\right|^2}$$

$$\gamma_{XX}[l] \longrightarrow \{a_k\} \longrightarrow \Gamma_{XX}(f)$$
$$\downarrow \text{(estimation)}$$
$$\hat{\gamma}_{XX}[l] \longrightarrow \{\hat{a}_k\} \longrightarrow \hat{\Gamma}_{XX}(f)$$

30

30

PSD of an AR(*p*) process…

- Example: Estimate $\Gamma_{XX}(f)$ from an N-point realization of
$$X[n] = aX[n-1] + W[n], W[n] \sim N(0, \sigma_w^2)$$

1. Compute estimates of $\hat{\gamma}_{XX}[0]$ and $\hat{\gamma}_{XX}[1]$:

$$\hat{\gamma}_{XX}[l] = \frac{1}{N}\sum_{n=0}^{N-|l|-1} x[n]x[n+|l|], l = 0,1$$

2. Estimate AR(1) parameter and noise variance (Yule-Walker):

$$\hat{a} = -\frac{\hat{\gamma}_{XX}[1]}{\hat{\gamma}_{XX}[0]}, \sigma_W^2 = \hat{\gamma}_{XX}[0] + \hat{a}\hat{\gamma}_{XX}[1]$$
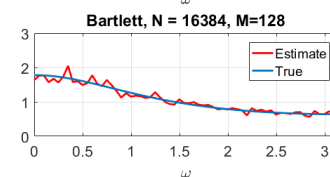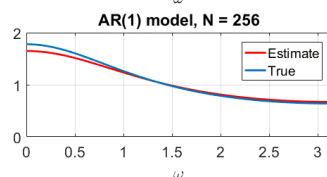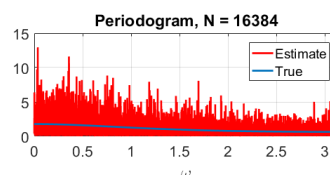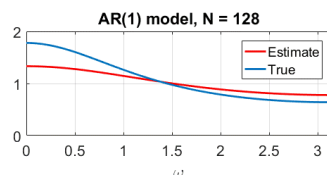
3. Power spectrum estimate:

$$\hat{\Gamma}_{XX}(f) = \frac{\hat{\sigma}_W^2}{\left|\hat{A}(f)\right|^2} = \frac{\hat{\sigma}_W^2}{\left|1+\hat{a}e^{-j2\pi fk}\right|^2}$$

31

31

PSD of an AR(1) process…
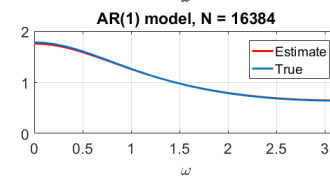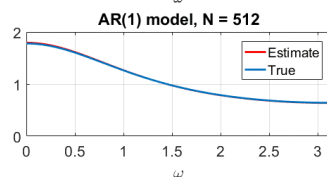
32

32

PSD of an AR(1) process…

```matlab
Matlab
% AR prosess
a = [1 -0.25];
[H,Omega] = freqz(1,a,1024); % True spectrum

% WGN
N = 2^14;
W = randn(N,1);

% Observed AR prosess
X = filter(1,a,W);

% Estimate AR process
[a_e, sigmaW2_e] = aryule(X,1); % Estimate
[He,W]=freqz(sigmaW2_e, a_e, 1024);

plot(W/pi,10*log10(abs(H))), hold on
plot(W/pi,10*log10(abs(He)))
```

33

33

Summary

- Today we discussed:
  – Parametric spectral estimation

- Next:
  – Linear prediction

34

34