TTK4115 Linear System Theory

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Solution to homework assignment 7

Problem 1: Linear quadratic regulation and minimum-energy estimation

a) From the equation of motion, we obtain

$$\ddot{z}(t) = 2\dot{z}(t) + 2u(t) + d(t).$$

This can be written as

$$\dot{x}(t) = Ax(t) + Bu(t) + \bar{B}d(t),$$

with

$$A = 2, \qquad B = 2 \qquad \text{and} \qquad \bar{B} = 1.$$

b) The controllability matrix is given by

$$\mathcal{C} = B = 2.$$

Because the controllability matrix has full row rank, i.e. $\operatorname{rank}(\mathcal{C}) = 1 = n$, we conclude that the system is controllable.

c) Substituting the control law (2) in the equation for the system (1), we obtain that the closed-loop system is given by

$$\dot{x}(t) = (A - BK)x(t) + \bar{B}d(t).$$

The eigenvalues of the closed-loop system can be calculated from the characteristic polynomial of $A - BK$, which is given by

$$\det(A - BK - \lambda) = 2 - 2K - \lambda.$$

The eigenvalue of $A - BK$ is equal to the root the characteristic polynomial of $A - BK$. Therefore, we obtain the eigenvalue $\lambda = 2 - 2K$.

The closed-loop system is marginally stable if the eigenvalue of the closed-loop system has a negative or zero real part. This implies that the closed-loop system is marginally stable if

$$\lambda = 2 - 2K \leq 0.$$

Hence, the closed-loop system is marginally stable for

$$K \geq 1.$$

d) The closed-loop system (1)-(2) is exponentially stable if the eigenvalue of the system has a negative real part. This implies that the closed-loop system is exponentially stable if

$$\lambda = 2 - 2K < 0.$$

Hence, the closed-loop system is marginally stable for

$$K > 1.$$

e) Substituting the values for A and B in the Riccati equation, we obtain

$$-\frac{4}{R}P^2 + 4P + Q = 0.$$

Solving for P yields

$$P = \frac{-4 \pm \sqrt{16 + 16\frac{Q}{R}}}{-\frac{8}{R}} = \frac{R}{2} \mp \frac{R}{2}\sqrt{1 + \frac{Q}{R}}.$$

The positive definite solution of Riccati equation is therefore given by

$$P = \frac{R}{2} + \frac{R}{2}\sqrt{1 + \frac{Q}{R}}.$$

Substituting P in the equation for K yields

$$K = \frac{BP}{R} = \frac{2\left(\frac{R}{2} + \frac{R}{2}\sqrt{1 + \frac{Q}{R}}\right)}{R} = 1 + \sqrt{1 + \frac{Q}{R}}.$$

f) Because $K > 1$ for all $Q, R > 0$, we conclude that the closed-loop system (1)-(2) is exponentially stable for all $Q, R > 0$.

g) Substituting $K = 3$ and $Q = 6$ in the equation for K gives

$$K = 1 + \sqrt{1 + \frac{Q}{R}} = 1 + \sqrt{1 + \frac{6}{R}} = 3.$$

From this, we obtain

$$1 + \frac{6}{R} = 4 \qquad \Rightarrow \qquad R = 2.$$

h) The equation (3) can be written as

$$y(t) = x(t) + n(t).$$

Therefore, we obtain

$$y(t) = Cx(t) + n(t),$$

with

$$C = 1.$$

i) Subsituting (4) and (5) in (1), we obtain

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BKy(t) + \bar{B}d(t)\\ &= Ax(t) - BK(Cx(t) + n(t)) + \bar{B}d(t)\\ &= (A - BKC)x(t) - BKn(t) + \bar{B}d(t).\end{aligned}$$

In Laplace domain, this equation is written as

$$sx(s) = (A - BKC)x(s) - BKn(s) + \bar{B}d(s),$$

from which it follows that

$$(s - A + BKC)x(s) = -BKn(s) + \bar{B}d(s).$$

Therefore, we have

$$x(s) = \frac{-BK}{s - A + BKC}n(s) + \frac{\bar{B}}{s - A + BKC}d(s).$$

Hence, the transfer function $\frac{x(s)}{d(s)}$ is given by

$$\frac{x(s)}{d(s)} = \frac{\bar{B}}{s - A + BKC}.$$

Substituting the values of A, B, $\bar{B}$ and C yields

$$\frac{x(s)}{d(s)} = \frac{1}{s - 2 + 2K}.$$

j) From the solution of i), we have

$$x(s) = \frac{-BK}{s - A + BKC}n(s) + \frac{\bar{B}}{s - A + BKC}d(s).$$

It follows that the transfer function $\frac{x(s)}{n(s)}$ is given by

$$\frac{x(s)}{n(s)} = \frac{-BK}{s - A + BKC}.$$

Substituting the values of A, B and C gives

$$\frac{x(s)}{n(s)} = \frac{-2K}{s - 2 + 2K}.$$

k) The limit for $s \to 0$ is given by

$$\lim_{s\to 0}\frac{x(s)}{d(s)} = \lim_{s\to 0}\frac{1}{s - 2 + 2K} = \frac{1}{-2 + 2K}.$$

l) The response of the state $x(t)$ to the disturbance $d(t)$ is given by the transfer function $\frac{x(s)}{d(s)}$. Substituting $s = j\omega$ (where j is the imaginary number) in the transfer function gives the frequency response function

$$\frac{x(j\omega)}{d(j\omega)} = \frac{1}{j\omega - 2 + 2K}.$$

Note that $d(t)$ is a low-frequency disturbance, which implies that the frequencies ω in the disturbance $d(t)$ are low. For sufficiently low frequencies ω, we have $|\omega| \ll |-2+2K|$. Therefore, the frequency response function can be approximated by

$$\frac{x(j\omega)}{d(j\omega)} \approx \frac{1}{-2 + 2K}.$$

Note that K should be larger than one for the system to be BIBO/exponentially stable. For $K > 1$, the magnitude of the frequency response function can be approximated by

$$\left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{1}{2(K-1)}.$$

From this, it follows that the effect of $d(t)$ on $x(t)$ is small if K is large.

m) The limit for $s \to \infty$ is given by

$$\lim_{s\to\infty} \frac{x(s)}{n(s)} = \lim_{s\to\infty} \frac{-2K}{s - 2 + 2K} = \lim_{s\to\infty} \frac{-2K}{s} = 0.$$

n) The response of the state $x(t)$ to the disturbance $n(t)$ is given by the transfer function $\frac{x(s)}{n(s)}$. Substituting $s = j\omega$ in the transfer function gives the frequency response function

$$\frac{x(j\omega)}{n(j\omega)} = \frac{-2K}{j\omega - 2 + 2K}.$$

Note that $n(t)$ is a high-frequency disturbance, which implies that the frequencies ω in the disturbance $n(t)$ are high. For sufficiently high frequencies ω, we have $|\omega| \gg |-2+2K|$. Therefore, the frequency response function can be approximated by

$$\frac{x(j\omega)}{n(j\omega)} \approx \frac{-2K}{j\omega}.$$

Note that K should be larger than one for the system to be BIBO/exponentially stable. For $K > 1$, the magnitude of the frequency response function can be approximated by

$$\left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{2K}{|\omega|}.$$

From this, it follows that the effect of $n(t)$ on $x(t)$ is small if K is small.

o) The block diagram of the system is given in Fig. 1.

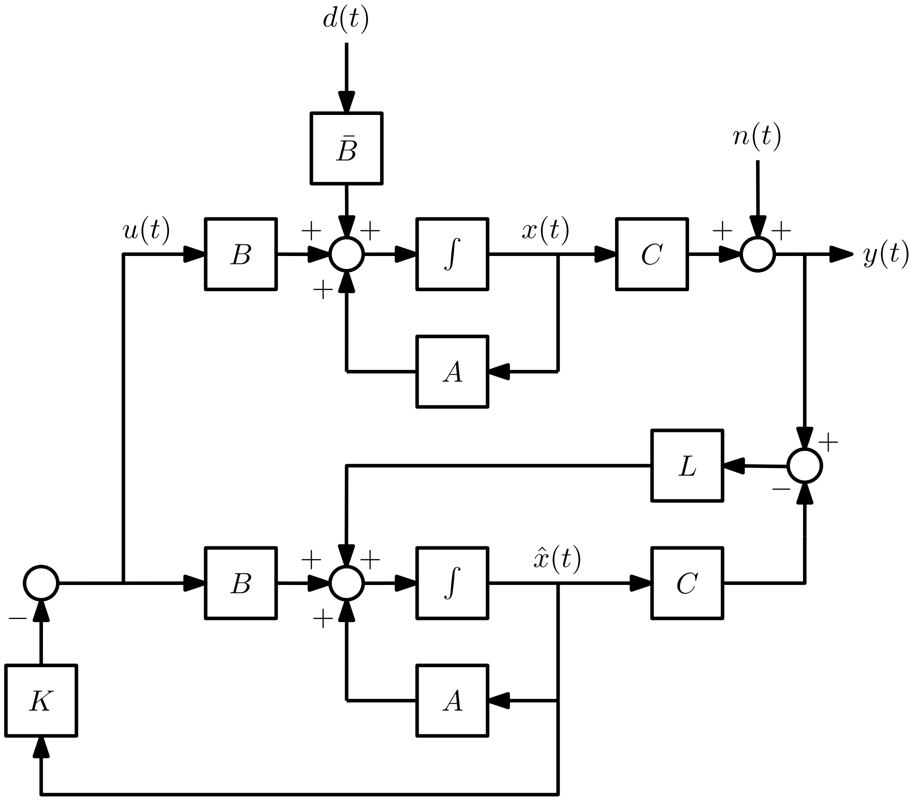

Fig. 1: System (1), (5), (6) and (7).

p) Using $e(t) = x(t) - \hat{x}(t)$ and the equations in (1), (5) and (6), we can write $\dot{e}(t)$ as

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t)\\ &= Ax(t) + Bu(t) + \bar{B}d(t) - (Ax(t) + Bu(t) + L(y(t) - C\hat{x}(t))\\ &= A(x(t) - \hat{x}(t)) + \bar{B}d(t) - L(Cx(t) + n(t) - C\hat{x}(t))\\ &= (A - LC)(x(t) - \hat{x}(t)) + \bar{B}d(t) - Ln(t)\\ &= (A - LC)e(t) + \bar{B}d(t) - Ln(t).\end{aligned}$$

Similarly, using $e(t) = x(t) - \hat{x}(t)$ and the equations in (1) and (7), $\dot{x}(t)$ can be written as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + \bar{B}d(t)\\ &= Ax(t) - BK\hat{x}(t) + \bar{B}d(t)\\ &= Ax(t) - BK(x(t) - e(t)) + \bar{B}d(t)\\ &= (A - BK)x(t) + BKe(t) + \bar{B}d(t).\end{aligned}$$

Combining the obtained expressions for $\dot{x}(t)$ and $\dot{e}(t)$ yields

$$\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} = \begin{bmatrix}A - BK & BK\\ 0 & A - LC\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} + \begin{bmatrix}\bar{B} & 0\\ \bar{B} & -L\end{bmatrix}\begin{bmatrix}d(t)\\ n(t)\end{bmatrix}.$$

Moreover, the out equation (7) can be written as

$$\begin{aligned} y(t) &= Cx(t) + n(t) \\ &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} d(t) \\ n(t) \end{bmatrix}. \end{aligned}$$

q) The eigenvalues of the system can be calculated from the characteristic polynomial of the system matrix, which is given by

$$\begin{aligned} \det\left(\begin{bmatrix} A - BK - \lambda & BK \\ 0 & A - LC - \lambda \end{bmatrix}\right) &= \det\left(\begin{bmatrix} -4 - \lambda & 6 \\ 0 & 2 - L - \lambda \end{bmatrix}\right) \\ &= (-4 - \lambda)(2 - L - \lambda) \\ &= (\lambda + 4)(\lambda + L - 2). \end{aligned}$$

The eigenvalues are equal to the roots the characteristic polynomial. Therefore, we obtain the eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -L + 2$. (Note that the eigenvalues are equal to the elements on the diagonal of the system matrix.)

The system is BIBO stable if the eigenvalue of the system have negative real parts. Hence, the system is BIBO stable for

$$L > 2.$$

r) Substituting the values for A $\bar{B}$ and C in the Riccati equation, we obtain

$$-SW^2 + 4W + \frac{1}{T} = 0$$

Solving for W yields

$$W = \frac{-4 \pm \sqrt{16 + 4\frac{S}{T}}}{-2S} = \frac{2}{S} \mp \frac{1}{S}\sqrt{4 + \frac{S}{T}}.$$

The positive definite solution of Riccati equation is therefore given by

$$W = \frac{2}{S} + \frac{1}{S}\sqrt{4 + \frac{S}{T}}.$$

Substituting W in the equation for L yields

$$L = WCS = \left(\frac{2}{S} + \frac{1}{S}\sqrt{4 + \frac{S}{T}}\right) S = 2 + \sqrt{4 + \frac{S}{T}}.$$

s) Because $L > 2$ for all $S, T > 0$, we conclude that the closed-loop system (1), (5), (6) and (7) is exponentially stable for all $S, T > 0$.

t) Taking the Laplace transform of the state equation in p), we obtain

$$\begin{bmatrix} sx(s) \\ se(s) \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix} + \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d(s) \\ n(s) \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} s-A+BK & -BK \\ 0 & s-A+LC \end{bmatrix} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix} = \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d(s) \\ n(s) \end{bmatrix}.$$

Subsequently, we obtain

$$\begin{aligned} \begin{bmatrix} x(s) \\ e(s) \end{bmatrix} &= \begin{bmatrix} s-A+BK & -BK \\ 0 & s-A+LC \end{bmatrix}^{-1} \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d(s) \\ n(s) \end{bmatrix} \\ &= \frac{1}{(s-A+BK)(s-A+LC)} \begin{bmatrix} s-A+LC & BK \\ 0 & s-A+BK \end{bmatrix} \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d(s) \\ n(s) \end{bmatrix} \\ &= \begin{bmatrix} \frac{(s-A+LC+BK)\bar{B}}{(s-A+BK)(s-A+LC)} & \frac{-BKL}{(s-A+BK)(s-A+LC)} \\ \frac{(s-A+BK)\bar{B}}{(s-A+BK)(s-A+LC)} & \frac{-(s-A+BK)L}{(s-A+BK)(s-A+LC)} \end{bmatrix} \begin{bmatrix} d(s) \\ n(s) \end{bmatrix}. \end{aligned}$$

Therefore, the transfer function $\frac{x(s)}{d(s)}$ is given by

$$\frac{x(s)}{d(s)} = \frac{(s-A+LC+BK)\bar{B}}{(s-A+BK)(s-A+LC)}.$$

Substituting the values of A, B, $\bar{B}$, C and K gives

$$\frac{x(s)}{d(s)} = \frac{s+4+L}{(s+4)(s-2+L)} = \frac{s+4+L}{s^2+(2+L)s-8+4L}.$$

u) From the solution of t), we have

$$\frac{x(s)}{n(s)} = \frac{-BKL}{(s-A+BK)(s-A+LC)}.$$

Substituting the values of A, B, C and K gives

$$\frac{x(s)}{n(s)} = \frac{-6L}{(s+4)(s-2+L)} = \frac{-6L}{s^2+(2+L)s-8+4L}.$$

v) The response of the state $x(t)$ to the disturbance $d(t)$ is given by the transfer function $\frac{x(s)}{d(s)}$. Substituting $s = j\omega$ in the transfer function gives the frequency response function

$$\frac{x(j\omega)}{d(j\omega)} = \frac{j\omega+4+L}{-\omega^2+j(2+L)\omega-8+4L}.$$

Note that $d(t)$ is a low-frequency disturbance, which implies that the frequencies ω in the disturbance $d(t)$ are low. For sufficiently low frequencies ω, we have

$|\omega| \ll |4+L|$, $|-\omega^2| \ll |-8+4L|$ and $|(2+L)\omega| \ll |-8+4L|$. Therefore, the frequency response function can be approximated by

$$\frac{x(j\omega)}{d(j\omega)} \approx \frac{4+L}{-8+4L}.$$

Note that L should be larger than two for the system to be BIBO/exponentially stable. For $L > 2$, the magnitude of the frequency response function can be approximated by

$$\left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{4+L}{4(L-2)}.$$

From this, we obtain that the magnitude of the frequency response function uniformly decreases as L increases (for $L > 2$). Therefore, we obtain that the effect of $d(t)$ on $x(t)$ is small if L is large.

w) The response of the state $x(t)$ to the disturbance $n(t)$ is given by the transfer function $\frac{x(s)}{n(s)}$. Substituting $s = j\omega$ in the transfer function gives the frequency response function

$$\frac{x(j\omega)}{n(j\omega)} = \frac{-6}{-\omega^2 + j(2+L)\omega - 8 + 4L}.$$

Note that $n(t)$ is a high-frequency disturbance, which implies that the frequencies ω in the disturbance $n(t)$ are high. For sufficiently high frequencies ω, we have $|-\omega^2| \gg |j(2+L)\omega|$ and $|-\omega^2| \gg |-8+4L|$. Therefore, the frequency response function can be approximated by

$$\frac{x(j\omega)}{n(j\omega)} \approx \frac{6L}{\omega^2}.$$

Note that L should be larger than two for the system to be BIBO/exponentially stable. For $L > 2$, the magnitude of the frequency response function can be approximated by

$$\left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{6L}{\omega^2}.$$

From this, it follows that the effect of $n(t)$ on $x(t)$ is small if L is small.

x) From i), we have that the transfer function $\frac{x(s)}{d(s)}$ for the system with output feedback is given by

$$\text{Output feedback:} \qquad \frac{x(s)}{d(s)} = \frac{1}{s-2+2K}.$$

Substituting $K = 3$ yields

$$\text{Output feedback:} \qquad \frac{x(s)}{d(s)} = \frac{1}{s+4}.$$

From t), we have that the transfer function $\frac{x(s)}{d(s)}$ for the system with state-estimated feedback is given by

$$\text{State-estimated feedback:} \qquad \frac{x(s)}{d(s)} = \frac{s+4+L}{s^2+(2+L)s-8+4L}.$$

Substituting $L = 12$ yields

$$\text{State-estimated feedback:} \qquad \frac{x(s)}{d(s)} = \frac{s+16}{s^2+14s+40}.$$

Substituting $s = j\omega$ gives

$$\text{Output feedback:} \qquad \frac{x(j\omega)}{d(j\omega)} = \frac{1}{j\omega+4}$$

and

$$\text{State-estimated feedback:} \qquad \frac{x(j\omega)}{d(j\omega)} = \frac{j\omega+16}{-\omega^2+j14\omega+40}.$$

Note that $d(t)$ is a low-frequency disturbance, which implies that the frequencies ω in the disturbance $d(t)$ are low. For low frequencies ω, these frequency response functions can be approximated by

$$\text{Output feedback:} \qquad \frac{x(j\omega)}{d(j\omega)} \approx \frac{1}{4}$$

and

$$\text{State-estimated feedback:} \qquad \frac{x(j\omega)}{d(j\omega)} \approx \frac{2}{5}.$$

The magnitude of the frequency response functions can be approximated by

$$\text{Output feedback:} \qquad \left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{1}{4}$$

and

$$\text{State-estimated feedback:} \qquad \left|\frac{x(j\omega)}{d(j\omega)}\right| \approx \frac{2}{5}.$$

From this, it follows that the effect of the low-frequency disturbance $d(t)$ on the state $x(t)$ is smaller with the output feedback than with the state-estimated feedback.

y) From j), we have that the transfer function $\frac{x(s)}{n(s)}$ for the system with output feedback is given by

$$\text{Output feedback:} \qquad \frac{x(s)}{n(s)} = \frac{-2K}{s-2+2K}.$$

Substituting $K = 3$ yields

$$\text{Output feedback:} \qquad \frac{x(s)}{n(s)} = \frac{-6}{s+4}.$$

From u), we have that the transfer function $\frac{x(s)}{n(s)}$ for the system with state-estimated feedback is given by

$$\text{State-estimated feedback:} \qquad \frac{x(s)}{n(s)} = \frac{-6L}{s^2 + (2+L)s - 8 + 4L}.$$

Substituting $L = 12$ yields

$$\text{State-estimated feedback:} \qquad \frac{x(s)}{n(s)} = \frac{-72}{s^2 + 14s + 40}.$$

Substituting $s = j\omega$ gives

$$\text{Output feedback:} \qquad \frac{x(j\omega)}{n(j\omega)} = \frac{-6}{j\omega + 4}$$

and

$$\text{State-estimated feedback:} \qquad \frac{x(j\omega)}{n(j\omega)} = \frac{-72}{-\omega^2 + j14\omega + 40}.$$

Note that $n(t)$ is a high-frequency disturbance, which implies that the frequencies ω in the disturbance $n(t)$ are high. For high frequencies ω, these frequency response functions can be approximated by

$$\text{Output feedback:} \qquad \frac{x(j\omega)}{n(j\omega)} \approx \frac{-6}{j\omega}$$

and

$$\text{State-estimated feedback:} \qquad \frac{x(j\omega)}{n(j\omega)} \approx \frac{72}{\omega^2}.$$

The magnitude of the frequency response functions can be approximated by

$$\text{Output feedback:} \qquad \left|\frac{x(j\omega)}{n(j\omega)}\right| \approx \frac{6}{|\omega|}$$

and

$$\text{State-estimated feedback:} \qquad \left|\frac{x(j\omega)}{n(j\omega)}\right| \approx \frac{72}{\omega^2}.$$

Note that for high frequencies ω, we have

$$\frac{6}{|\omega|} \gg \frac{72}{\omega^2}.$$

Therefore, it follows that the effect of the high-frequency disturbance $n(t)$ on the state $x(t)$ is larger with the output feedback than with the state-estimated feedback.