TTK4115 Linear System Theory

Department of Engineering Cybernetics

NTNU

Solution to homework assignment 6

Problem 1: Discrete-time Kalman filter

a) Fig. 1 shows a block diagram of the system.

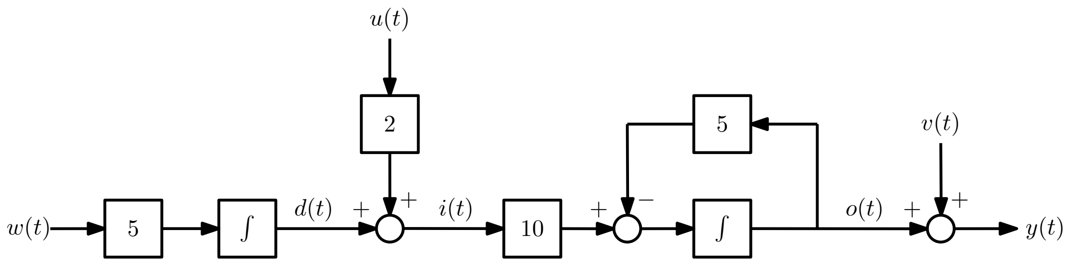

Fig. 1: Block diagram of the system.

b) From the transfer function $g(s) = \frac{o(s)}{i(s)} = \frac{10}{s+5}$, it follows that

$$(s+5)o(s) = 10i(s).$$

By taking the inverse Laplace transform, the following dynamics are obtained:

$$\dot{o}(t) + 5o(t) = 10i(t).$$

This can be written as

$$\dot{o}(t) = -5o(t) + 10i(t).$$

Substituting $i(t) = 2u(t) + d(t)$, we get

$$\dot{o}(t) = -5o(t) + 10d(t) + 20u(t).$$

From $d(t) = 5\int_0^t w(\tau)d\tau$, it follows that

$$\dot{d}(t) = 5w(t).$$

Combining these two differential equations and the output equation $y(t) = o(t) + v(t)$, we obtain the following system

$$\begin{bmatrix}\dot{o}(t)\\ \dot{d}(t)\end{bmatrix} = \begin{bmatrix}-5 & 10\\ 0 & 0\end{bmatrix}\begin{bmatrix}o(t)\\ d(t)\end{bmatrix} + \begin{bmatrix}20\\ 0\end{bmatrix}u(t) + \begin{bmatrix}0\\ 5\end{bmatrix}w(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}o(t)\\ d(t)\end{bmatrix} + \begin{bmatrix}1\end{bmatrix}v(t).$$

This can be written as

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t), \\
y(t) &= \mathbf{C}\mathbf{x}(t) + Hv(t),
\end{aligned}
$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$ and matrices

$$
\mathbf{A} = \begin{bmatrix} -5 & 10 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 \end{bmatrix}.
$$

c) The discrete-time system matrices are given by

$$
\begin{aligned}
\mathbf{A}_d &= \mathbf{I} + \Delta t \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -5 & 10 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \\
\mathbf{B}_d &= \Delta t \mathbf{B} = \frac{1}{5} \begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \\
\mathbf{G}_d &= \Delta t \mathbf{G} = \frac{1}{5} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
\mathbf{C}_d &= \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
H_d &= H = \begin{bmatrix} 1 \end{bmatrix}.
\end{aligned}
$$

d) The observability matrix is given by

$$
\mathcal{O} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d \mathbf{A}_d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.
$$

Because the observability matrix has full column rank, i.e. $\text{rank}(\mathcal{O}) = 2 = n$, we conclude from [C: Theorem 6.O1] that the system is observable.

e) Substituting the update law $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-)$ in the definition of $\mathbf{P}_k$ yields

$$
\begin{aligned}
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T] \\
&= E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))(\mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))^T] \\
&= E[(\mathbf{e}_k^- - \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))(\mathbf{e}_k^- - \mathbf{K}_k(y_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))^T].
\end{aligned}
$$

Now, by substituting the output equation $y_k = \mathbf{C}_d \mathbf{x}_k + H_d v_k$ in the obtained expression for $\mathbf{P}_k$, we get

$$
\begin{aligned}
\mathbf{P}_k &= E[(\mathbf{e}_k^- - \mathbf{K}_k(\mathbf{C}_d \mathbf{x}_k + H_d v_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))(\mathbf{e}_k^- - \mathbf{K}_k(\mathbf{C}_d \mathbf{x}_k + H_d v_k - \mathbf{C}_d \hat{\mathbf{x}}_k^-))^T] \\
&= E[(\mathbf{e}_k^- - \mathbf{K}_k(\mathbf{C}_d \mathbf{e}_k^- + H_d v_k))(\mathbf{e}_k^- - \mathbf{K}_k(\mathbf{C}_d \mathbf{e}_k^- + H_d v_k))^T] \\
&= E[((\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)\mathbf{e}_k^- - \mathbf{K}_k H_d v_k)((\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)\mathbf{e}_k^- - \mathbf{K}_k H_d v_k)^T] \\
&= E[(\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)\mathbf{e}_k^- \mathbf{e}_k^{-T}(\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)^T - (\mathbf{I} + \mathbf{K}_k \mathbf{C}_d)\mathbf{e}_k^- v_k H_d \mathbf{K}_k^T \\
&\qquad - \mathbf{K}_k H_d v_k \mathbf{e}_k^{-T}(\mathbf{I} + \mathbf{K}_k \mathbf{C}_d)^T + \mathbf{K}_k H_d v_k v_k H_d \mathbf{K}_k^T] \\
&= (\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)E[\mathbf{e}_k^- \mathbf{e}_k^{-T}](\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)^T - (\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)E[\mathbf{e}_k^- v_k]H_d \mathbf{K}_k^T \\
&\qquad - \mathbf{K}_k H_d E[v_k \mathbf{e}_k^{-T}](\mathbf{I} - \mathbf{K}_k \mathbf{C}_d)^T + \mathbf{K}_k H_d E[v_k^2] H_d \mathbf{K}_k^T.
\end{aligned}
$$

Note that $\mathbf{e}_k^-$ is uncorrelated with v_k, i.e. $E[\mathbf{e}_k^- v_k] = \mathbf{0}$. Substituting $E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] = \mathbf{P}_k^-$ and $E[v_k^2] = R$ gives

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{C}_d)\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C}_d)^T + RH_d^2\mathbf{K}_k\mathbf{K}_k^T.$$

f) The expression for $\mathbf{P}_k$ obtained in e) can be rewritten as

$$\begin{aligned}\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}_d\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}_d^T\mathbf{K}_k^T + \mathbf{K}_k\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T\mathbf{K}_k^T + RH_d^2\mathbf{K}_k\mathbf{K}_k^T \\ &= \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}_d\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}_d^T\mathbf{K}_k^T + \mathbf{K}_k(\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2)\mathbf{K}_k^T.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d(\text{trace } \mathbf{P}_k)}{d\mathbf{K}_k} &= -\mathbf{C}_d\mathbf{P}_k^- - (\mathbf{P}_k^-\mathbf{C}_d^T)^T + (\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2)\mathbf{K}_k^T \\ &\quad + (\mathbf{K}_k(\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2))^T \\ &= -2\mathbf{C}_d\mathbf{P}_k^- + 2(\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2)\mathbf{K}_k^T.\end{aligned}$$

Note that we used that $\mathbf{P}_k^-$ is symmetric, i.e. $\mathbf{P}_k^- = \mathbf{P}_k^{-T}$. Now, since the Kalman gain is optimal, we have

$$\frac{d(\text{trace } \mathbf{P}_k)}{d\mathbf{K}_k} = \mathbf{0}^T.$$

Combining the last two equations and taking the transpose yields

$$-\mathbf{P}_k^-\mathbf{C}_d^T + \mathbf{K}_k(\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2) = \mathbf{0}.$$

Rewriting this equation, we obtain that the Kalman gain is given by

$$\mathbf{K}_k = \frac{\mathbf{P}_k^-\mathbf{C}_d^T}{\mathbf{C}_d\mathbf{P}_k^-\mathbf{C}_d^T + RH_d^2}.$$

g) From the expression $\hat{\mathbf{x}}_{k+1}^- = E[\mathbf{x}_{k+1}] = E[\mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k]$, it follows that

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{A}_d E[\mathbf{x}_k] + \mathbf{B}_d u_k + \mathbf{G}_d E[w_k].$$

Substituting $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k$ and $E[w_k] = 0$ yields

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{A}_d\hat{\mathbf{x}}_k + \mathbf{B}_d u_k.$$

h) Substituting $\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k$ and $\hat{\mathbf{x}}_{k+1}^- = \mathbf{A}_d\hat{\mathbf{x}}_k + \mathbf{B}_d u_k$ in the expression for $\mathbf{P}_{k+1}^-$, we obtain

$$\begin{aligned}\mathbf{P}_{k+1}^- &= E[\mathbf{e}_{k+1}^-\mathbf{e}_{k+1}^{-T}] = E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T] \\ &= E[(\mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k - \mathbf{A}_d\hat{\mathbf{x}}_k - \mathbf{B}_d u_k) \\ &\qquad \cdot (\mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d u_k + \mathbf{G}_d w_k - \mathbf{A}_d\hat{\mathbf{x}}_k - \mathbf{B}_d u_k)^T] \\ &= E[(\mathbf{A}_d(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}_d w_k)(\mathbf{A}_d(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{G}_d w_k)^T] \\ &= E[(\mathbf{A}_d\mathbf{e}_k + \mathbf{G}_d w_k)(\mathbf{A}_d\mathbf{e}_k + \mathbf{G}_d w_k)^T] \\ &= E[\mathbf{A}_d\mathbf{e}_k\mathbf{e}_k^T\mathbf{A}_d^T + \mathbf{A}_d\mathbf{e}_k w_k\mathbf{G}_d^T + \mathbf{G}_d w_k\mathbf{e}_k^T\mathbf{A}_d^T + \mathbf{G}_d w_k w_k\mathbf{G}_d^T] \\ &= \mathbf{A}_d E[\mathbf{e}_k\mathbf{e}_k^T]\mathbf{A}_d^T + \mathbf{A}_d E[\mathbf{e}_k w_k]\mathbf{G}_d^T + \mathbf{G}_d E[w_k\mathbf{e}_k^T]\mathbf{A}_d^T + \mathbf{G}_d E[w_k^2]\mathbf{G}_d^T.\end{aligned}$$

Note that $\mathbf{e}_k$ is uncorrelated with w_k, i.e. $E[\mathbf{e}_k w_k] = \mathbf{0}$. Substituting $E[\mathbf{e}_k \mathbf{e}_k^T] = \mathbf{P}_k$ and $E[w_k^2] = Q$ gives

$$\mathbf{P}_{k+1}^- = \mathbf{A}_d \mathbf{P}_k \mathbf{A}_d^T + Q \mathbf{G}_d \mathbf{G}_d^T.$$

i) For $k = 0$, we first calculate the Kalman gain $\mathbf{K}_0$ as follows

$$\begin{aligned}\mathbf{K}_0 &= \frac{\mathbf{P}_0^- \mathbf{C}_d^T}{\mathbf{C}_d \mathbf{P}_0^- \mathbf{C}_d^T + RH_d^2} \\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + 1\right)^{-1} = \begin{bmatrix}1\\0\end{bmatrix}\frac{1}{2} = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}.\end{aligned}$$

Next, we update the state estimate

$$\begin{aligned}\hat{\mathbf{x}}_0 &= \hat{\mathbf{x}}_0^- + \mathbf{K}_0(y_0 - \mathbf{C}_d \hat{\mathbf{x}}_0^-) \\ &= \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\left(2 - \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}2 = \begin{bmatrix}1\\0\end{bmatrix}\end{aligned}$$

and covariance matrix

$$\begin{aligned}\mathbf{P}_0 &= (\mathbf{I} - \mathbf{K}_0\mathbf{C}_d)\mathbf{P}_0^-(\mathbf{I} - \mathbf{K}_0\mathbf{C}_d)^T + RH_d^2\mathbf{K}_0\mathbf{K}_0^T \\ &= \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)^T + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}\frac{1}{4} & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}.\end{aligned}$$

The projection ahead of the state estimate and the covariance matrix to $k = 1$ is given by

$$\begin{aligned}\hat{\mathbf{x}}_1^- &= \mathbf{A}_d\hat{\mathbf{x}}_0 + \mathbf{B}_d u_0 \\ &= \begin{bmatrix}0 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}4\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}_1^- &= \mathbf{A}_d\mathbf{P}_0\mathbf{A}_d^T + Q\mathbf{G}_d\mathbf{G}_d^T \\ &= \begin{bmatrix}0 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\2 & 1\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \\ &= \begin{bmatrix}4 & 2\\2 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 3\end{bmatrix} = \begin{bmatrix}4 & 2\\2 & 4\end{bmatrix}.\end{aligned}$$

Now, for $k = 1$, we do the same steps. The Kalman gain $\mathbf{K}_1$ is given by

$$\begin{aligned}\mathbf{K}_1 &= \frac{\mathbf{P}_1^- \mathbf{C}_d^T}{\mathbf{C}_d \mathbf{P}_1^- \mathbf{C}_d^T + RH_d^2} \\ &= \begin{bmatrix}4 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + 1\right)^{-1} = \begin{bmatrix}4\\2\end{bmatrix}\frac{1}{5} = \begin{bmatrix}\frac{4}{5}\\\frac{2}{5}\end{bmatrix} = \begin{bmatrix}0.8\\0.4\end{bmatrix}.\end{aligned}$$

The updated state estimate and covariance matrix are obtained as follows:

$$
\begin{aligned}
\hat{\mathbf{x}}_1 &= \hat{\mathbf{x}}_1^- + \mathbf{K}_1(y_1 - \mathbf{C}_d\hat{\mathbf{x}}_1^-) \\
&= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \left(-1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} 5 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{P}_1 &= (\mathbf{I} - \mathbf{K}_1\mathbf{C}_d)\mathbf{P}_1^-(\mathbf{I} - \mathbf{K}_1\mathbf{C}_d)^T + RH_d^2\mathbf{K}_1\mathbf{K}_1^T \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^T + \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 \end{bmatrix} + \frac{1}{25} \begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix} \\
&= \frac{1}{25} \begin{bmatrix} 4 & 2 \\ 2 & 76 \end{bmatrix} + \frac{1}{25} \begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 20 & 10 \\ 10 & 80 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 3.2 \end{bmatrix}.
\end{aligned}
$$

From the projection ahead to $k = 2$, it follows that the state estimate $\hat{\mathbf{x}}_2^-$ and the associated error covariance matrix $\mathbf{P}_2^-$ are respectively given by

$$
\begin{aligned}
\hat{\mathbf{x}}_2^- &= \mathbf{A}_d\hat{\mathbf{x}}_1 + \mathbf{B}_d u_1 \\
&= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{P}_2^- &= \mathbf{A}_d\mathbf{P}_1\mathbf{A}_d^T + Q\mathbf{G}_d\mathbf{G}_d^T \\
&= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 16 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 64 & 32 \\ 32 & 16 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 64 & 32 \\ 32 & 31 \end{bmatrix} = \begin{bmatrix} 12.8 & 6.4 \\ 6.4 & 6.2 \end{bmatrix}.
\end{aligned}
$$