NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Filter Structures

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 9.1 Structures for the realization of discrete-time systems
  - 9.2 Structures for FIR Systems
  - 9.3 Structures for IIR Systems

- For a compressed overview of topics treated in the lecture, see "Filter implementation" on Blackboard

*Level of detail is defined by lectures and problem sets

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Contents and learning outcomes

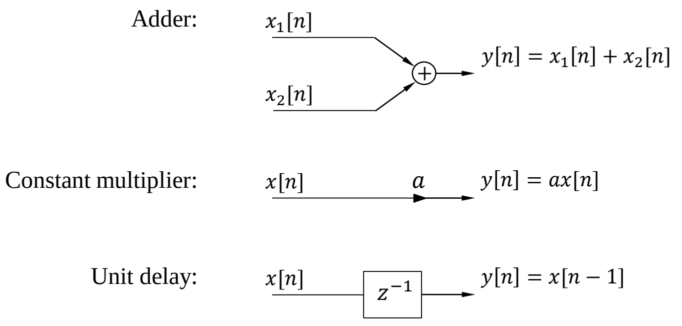
- Filter structures
  - Direct-form
  - Cascade form
  - Parallel form
  - Transposed form

Background

- So far, we have expressed a digital filter (system) using
  - System function
  - Frequency response
  - Impulse response
- How shall the filter be realized in practice?
  - Different filter structures dictate different design strategies
- Need to consider problems associated with quantization effects when finite-precision arithmetic is used in the implementation
  - Rounding errors in coefficients
  - Rounding errors in calculations

Basic elements

- Three elements to describe digital filter structure (Lecture 2):



Filter structures

- Rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}, a_0 = 1$$

- Order of system is called N if $a_N \neq 0$
- Poles and zeros of $H(z)$ determine frequency response
- Difference equation associated with an IIR filter is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

- Arranging equations in different ways ⇒ different implementations
  – Computational complexity, memory requirements, finite-precision effects

Filter structures…

- Computational complexity
  - Number of operations required to compute the output $y[n]$
- Memory requirements
  - Number of memory locations to store system parameters, inputs and outputs (past and present), intermediate values
- Finite-word-length effects (finite-precision)
  - Various structures are equivalent for infinite precision but behave differently with finite precision

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Filter structures...

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}},$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

- Three different implementation structures
  - Direct form: implement difference equation (two versions)
  - Cascade form: factor $H(z)$ into products of 2nd-order sections
  - Parallel form: partial fraction of $H(z)$ into 2nd-order sections

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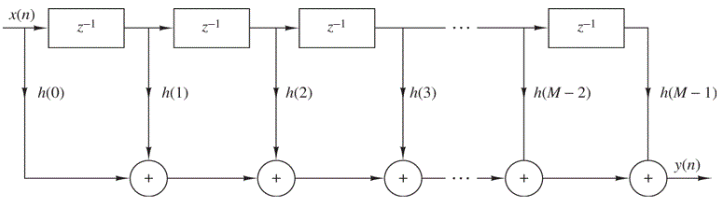
FIR filter

- Impulse response and system function

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

- Direct-form structure follows immediately from the convolution sum



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Linear-phase FIR filter

- Linear-phase filter satisfies symmetry or asymmetry condition

$$h[n] = \pm h[M-1-n]$$

- Example: Consider a length-7 Type 1 FIR transfer function with symmetric impulse response

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[2]z^{-4}$$
$$+h[1]z^{-5} + h[0]z^{-6}$$
$$= h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5})$$
$$+h[2](z^{-2} + z^{-4}) + h[3]z^{-3}$$

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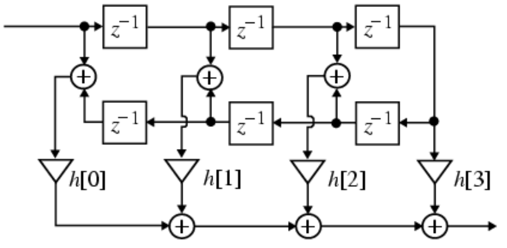
Linear-phase FIR filter…

- Example (cont.): $H(z) = h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5})$

$$+h[2](z^{-2} + z^{-4}) + h[3]z^{-3}$$



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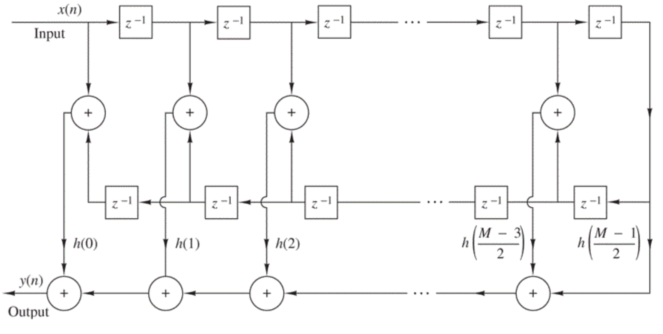
Linear-phase FIR filter…

- Linear-phase filter satisfies symmetry or asymmetry condition

$$h[n] = \pm h[M - 1 - n]$$

- Reduce the number of multiplications by a factor of two (2)



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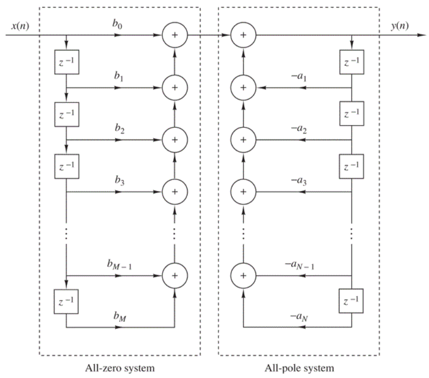
IIR filter

- Impulse response and system function obtained from

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$= B(z)\frac{1}{A(z)}$$



- Direct-form structure I
  - $M+N+1$ multiplications
  - $M+N+1$ memory locations

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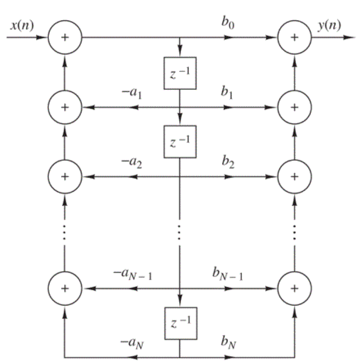
IIR filter…

- Can reverse the order without changing the system response

$$B(z)\frac{1}{A(z)} = \frac{1}{A(z)}B(z)$$



- Direct-form structure II
  - Only need to store past values of a single variable
  - Canonic ⇒ $\max(N, M)$ delays

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Cascade structure

- Consider a high-order IIR system with system function ($N \geq M$)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 \prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

- Filter coefficients are real-valued ⇒ poles and zeros are real-valued or come in complex-conjugate pairs
- Complex-conjugate poles/zeros together in one section, real-valued poles/zeros paired arbitrarily
- Write the system function as a cascade of second-order systems

$$H(z) = \prod_{k=1}^{K} H_k(z) = \prod_{k=1}^{K} \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{a_{k0} + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

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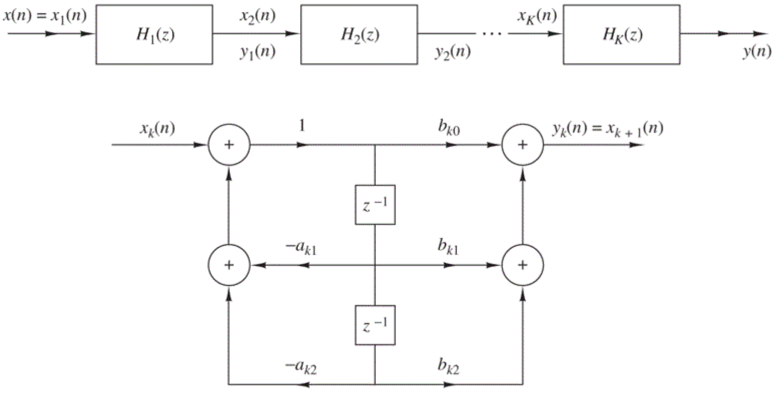
Cascade structure…



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Cascade structure…

- Example (from Mitra):

$$H(z) = \frac{0.44z^{-1}+0.362z^{-2}+0.02z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

$$= \left(\frac{0.44+0.362z^{-1}+0.02z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}\right)\left(\frac{z^{-1}}{1-0.4z^{-1}}\right)$$

Parallel-form structure

- Consider a high-order IIR system with system function

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

$$= \underbrace{\frac{\tilde{b}_0+\tilde{b}_1z^{-1}+\cdots+\tilde{b}_{N-1}z^{-(N-1)}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}}_{\text{Proper rational part}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M\geq N}$$

- Assume distinct poles and make a partial fraction expansion

$$H(z) = \sum_{k=1}^{N} \frac{R_k}{(1-p_kz^{-1})} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M\geq N}$$

- Rewrite first sum using 2nd-order sections
  - complex-cojugated poles, real-valued poles paired arbitrarily

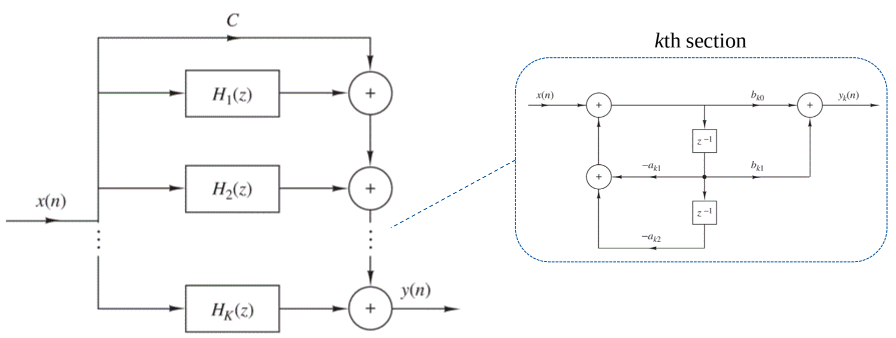
Parallel-form structure…

- Final form:

$$H(z) = \sum_{k=1}^{K} \frac{b_{k0}+b_{k1}z^{-1}}{a_{k0}+a_{k1}z^{-1}+a_{k2}z^{-2}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M \geq N}$$

- Filter input available to all biquad sections and polynomial section
- Output from all sections summed to form filter output
  - A parallel structure can be built to realize $H(z)$
  - Biquad section can be implemented using, e.g., direct form II

Parallel-form structure…



Parallel structure…

- Example (from Mitra):

$$H(z) = \frac{0.44z^{-1}+0.362z^{-2}+0.02z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

$$= -0.1 + \frac{0.6}{1-0.4z^{-1}} + \frac{-0.25}{1-(-0.4-0.5831j)z^{-1}} + \frac{-0.25}{1-(-0.4+0.5831j)z^{-1}}$$

$$= -0.1 + \frac{0.6}{1-0.4z^{-1}} + \frac{-0.5-0.2z^{-1}}{1+0.8z^{-1}+0.5z^{-2}}$$

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Example

- Determine the cascade and parallel realizations for system

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}\right)\left(1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}\right)}$$

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Example…

- Cascade form: Group pairwise, e.g.,

$$H(z)=\frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}\right)\left(1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}\right)}$$

$$H_1(z)=\frac{1-\frac{2}{3}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}=\frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}$$

$$H_2(z)=\frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}\right)\left(1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}\right)}=\frac{1+\frac{3}{2}z^{-1}-z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

$$\Rightarrow H(z)=10H_1(z)H_2(z)$$

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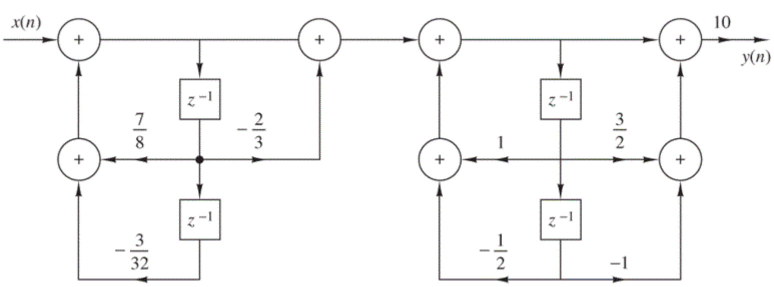
Example…

- Cascade form: $H(z)=10H_1(z)H_2(z)$



$$H_1(z)=\frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}\qquad H_2(z)=\frac{1+\frac{3}{2}z^{-1}-z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

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Example…

- Parallel form: Expand $H(z)$ in partial fraction expansion

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}\right)\left(1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}\right)}$$

$$= \frac{R_1}{1-\frac{3}{4}z^{-1}} + \frac{R_2}{1-\frac{1}{8}z^{-1}} + \frac{R_3}{1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}} + \frac{R_3^*}{1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}}$$

- Residue calculus

$$R_1 = 2.93,\ R_2 = -17.68,\ R_3 = 12.25 + j14.57$$

- Group terms into biquads

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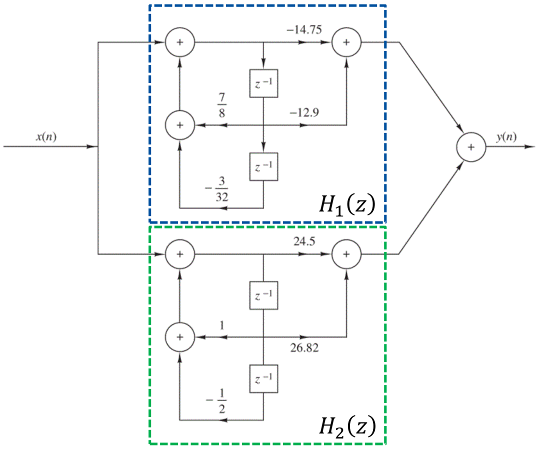
Example…

- Parallel form: $H(z) = \frac{-14.75-12.90z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{24.50+26.82z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$



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Summary

- Today we discussed:
  - Filter implementations
- Next:
  - Finite-precision and roundoff effects