NTNU – Trondheim
Norwegian University of
Science and Technology

TTT4120 Digital Signal Processing
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Lecture: Inverse Z-transform and Residues

Prof. Stefan Werner
stefan.werner@ntnu.no
Office B329

Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 3.4.3 The inverse z-transform by partial-fraction expansion

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Inverse z-transform using partial fraction expansion
  - Mainly repetition of already covered or known topics
- Matlab implementation

Inverse z-transform

$$X(z) \overset{?}{\rightarrow} x[n]$$

- Three popular methods
  - Contour integration: $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$
  - Power series expansion: $X(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k}$
  - Partial fraction expansion and table lookup (rational functions):

$$X(z) = \sum_{k=1}^{N}\left(\frac{R_{k,1}}{(1-p_k z^{-1})} + \frac{R_{k,2}}{(1-p_k z^{-1})^2} + \cdots + \frac{R_{k,r}}{(1-p_k z^{-1})^r}\right)$$

Z-transform table

Sequence	Transform	ROC
$\delta[n]$	1	$\forall z$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u[-n-1]$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$(a^n \sin\omega_0 n)u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n \cos\omega_0 n)u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u[-n-1]$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$

5

Inverse z-transform by partial fractions

- Consider the rational expression

$$X(z) = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}} = \frac{B(z)}{A(z)}$$

- $B(z)$ and $A(z)$ are polynomials in variable z
- b_k and a_k are the coefficients of $B(z)$ and $A(z)$, respectively
- M is the degree of $B(z)$ and N is the degree of $A(z)$
- M roots of polynomial $B(z)$, satisfy $B(z_k) = 0$: called zeros of $H(z)$
- N roots of polynomial $A(z)$, satisfy $A(p_k) = 0$: called poles of $H(z)$

6

Inverse z-transform by partial fractions…

- Fundamental theorem of algebra:

  A polynomial of degree M' has exactly M' roots, counting multiplicities

- We may factor $X(z)$ as follows

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = b_0 z^{N-M} \frac{z^M + \tilde{b}_1 z^{M-1} + \cdots + \tilde{b}_M}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$= b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

- When coefficients a_k and b_k are real, complex poles or zeros occur in complex conjugate pairs

7

Inverse z-transform by partial fractions…

- Let us assume that
  - $M < N$, i.e., $H(z)$ is proper
  - Poles are distinct, i.e., all roots of $A(z)$ have multiplicity one
- Then we can perform a partial fraction of $X(z)$ to obtain

$$X(z) = \frac{R_1}{(1 - p_1 z^{-1})} + \frac{R_2}{(1 - p_2 z^{-1})} + \cdots + \frac{R_N}{(1 - p_N z^{-1})}$$

  where p_k is the kth pole of $X(z)$ and R_k is the residue at p_k
- When $p_k = p_l^*$ we have $R_k = R_l^*$

8

Inverse z-transform by partial fractions…

- Example: $X(z) = \frac{1}{1-\frac{1}{4}z^{-2}} = \frac{1/2}{1-\frac{1}{2}z^{-1}} + \frac{1/2}{1+\frac{1}{2}z^{-1}}$

- Let us verify:

$$\frac{1/2}{1-\frac{1}{2}z^{-1}} + \frac{1/2}{1+\frac{1}{2}z^{-1}} = \frac{\frac{1}{2}\left(1+\frac{1}{2}z^{-1}\right)+\frac{1}{2}\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

Inverse z-transform by partial fractions…

- Once in partial fraction form, inverse z-transform becomes simple:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$
$$= \mathcal{Z}^{-1}\left\{\frac{R_1}{(1-p_1z^{-1})} + \frac{R_2}{(1-p_2z^{-1})} + \cdots + \frac{R_N}{(1-p_Nz^{-1})}\right\}$$

- Finally to complete $x[n]$, we use the relation

$$\mathcal{Z}^{-1}\left\{\frac{1}{(1-p_kz^{-1})}\right\} = \begin{cases} p_k^n u[n], & \text{ROC: } |z| > |p_k| \\ -p_k^n u[-n-1], & \text{ROC: } |z| < |p_k| \end{cases}$$

- Depending on the ROCs, we may end up with causal, anti-causal and non-causal (stable or unstable) time-domain sequence $x[n]$

Inverse z-transform by partial fractions…

- Example: Causal system and stable system ($|z| > \max_k |p_k| < 1$)

$$x[n] = \sum_{k=1}^{N} R_k \mathcal{Z}^{-1}\left\{\frac{1}{(1-p_k z^{-1})}\right\} = \sum_{k=1}^{N} R_k p_k^n u[n]$$

- Example: Complex conjugated poles $R_1 = R_2^*$, with $|p_1| < 1$

$$\begin{aligned} x[n] &= R_1 p_1^n u[n] + R_1^*(p_1^*)^n u[n] \\ &= (R_1 p_1^n + R_1^*(p_1^*)^n) u[n] \\ &= |R_1||p_1|^n \left(e^{j(\angle R_1 + \angle p_1 n)} + e^{-j(\angle R_1 + \angle p_1 n)}\right) u[n] \\ &= 2|R_1||p_1|^n \cos(\angle R_1 + \angle p_1 n)\, u[n] \end{aligned}$$

11

Inverse z-transform by partial fractions…

$$X(z) \overset{?}{=} \frac{R_1}{(1-p_1 z^{-1})} + \frac{R_2}{(1-p_2 z^{-1})} + \cdots + \frac{R_N}{(1-p_N z^{-1})}$$

- Finding the partial fraction expansion:
  1. Factor $A(z)$, i.e., find poles $p_1, \dots, p_N$
  2. Find residues $R_1, \dots, R_N$

12

Finding residues

- Method 1: Solve linear equations (always works but can be tedious)
  1. Clear denominator terms

$$B(z) = \underbrace{\prod_{k=1}^{N}(1 - p_k z^{-1})}_{A(z)} X(z) = \sum_{k=1}^{N} R_k \prod_{j=1, j\neq k}^{N}(1 - p_j z^{-1})$$

  2. Equate coefficients on both sides

- Example: $\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{R_1}{1-\frac{1}{2}z^{-1}} + \frac{R_2}{1+\frac{1}{2}z^{-1}}$

  1. $1 = R_1\left(1 + \frac{1}{2}z^{-1}\right) + R_2\left(1 - \frac{1}{2}z^{-1}\right)$
  2. z^0: $1 = R_1 + R_2$

     z^{-1}: $0 = \frac{R_1}{2} - \frac{R_2}{2}$

13

Finding residues…

- Method 2: Multiply both sides by $1 - p_k z^{-1}$ to get
  1. $(1 - p_k z^{-1})X(z) = \frac{R_1(1-p_k z^{-1})}{(1-p_1 z^{-1})} + \cdots + R_k + \cdots + \frac{R_N(1-p_k z^{-1})}{(1-p_N z^{-1})}$
  2. and set $z = p_k$
- In general, we have the formula

$$R_k = (1 - p_k z^{-1})X(z)|_{z=p_k}$$

14

Finding residues…

- Example: $X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{R_1}{1-\frac{1}{2}z^{-1}} + \frac{R_2}{1+\frac{1}{2}z^{-1}}$

$$R_1 = \left(R_1 + \frac{R_2\left(1-\frac{1}{2}z^{-1}\right)}{1+\frac{1}{2}z^{-1}}\right)\Bigg|_{z=\frac{1}{2}} = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)}\Bigg|_{z=\frac{1}{2}} = \frac{1}{2}$$

$$R_2 = \left(\frac{R_2\left(1+\frac{1}{2}z^{-1}\right)}{1-\frac{1}{2}z^{-1}} + R_2\right)\Bigg|_{z=-\frac{1}{2}} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}\Bigg|_{z=-\frac{1}{2}} = \frac{1}{2}$$

15

Example: Inverse z-transform

- Find impulse response of system $H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$
- Solution:

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{3-4z^{-1}}{(1-0.5z^{-1})(1-3z^{-1})} = \frac{R_1}{1-0.5z^{-1}} + \frac{R_2}{1-3z^{-1}}$$

$$R_1 = (1-0.5z^{-1})H(z)|_{z=0.5} = \frac{3-4z^{-1}}{1-3z^{-1}}\Bigg|_{z=0.5} = \frac{3-8}{1-6} = 1$$

$$R_2 = (1-3z^{-1})H(z)|_{z=3} = \frac{3-4z^{-1}}{1-0.5z^{-1}}\Bigg|_{z=3} = \frac{3-\frac{4}{3}}{1-\frac{0.5}{3}} = 2$$

$$\Rightarrow h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.5z^{-1}}\right\} + \mathcal{Z}^{-1}\left\{\frac{2}{1-3z^{-1}}\right\}$$

16

Example: Inverse z-transform…

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.5z^{-1}}\right\} + \mathcal{Z}^{-1}\left\{\frac{2}{1-3z^{-1}}\right\}$$

- We may completely determine $h[n]$ after specifying ROC:
  - $h_1[n] = (0.5^n + 2 \cdot 3^n)u[n]$ with ROC $|z| > 3$
  - $h_2[n] = -(0.5^n + 2 \cdot 3^n)u[-n-1]$ with ROC $|z| < 0.5$
  - $h_3[n] = 0.5^n u[n] - 2 \cdot 3^n u[-n-1]$ with ROC $0.5 < |z| < 3$
- Summary:
  - $h_1[n]$ is causal and unstable
  - $h_2[n]$ is anti-causal and unstable
  - $h_3[n]$ is non-causal and stable

Final comments (optional)

- We assumed that $N > M$ so that $X(z)$ was proper
  - If $M \geq N$ can just express $X(z)$ as

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \underbrace{\frac{\tilde{b}_0 + \tilde{b}_1 z^{-1} + \cdots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}}_{\text{Proper rational part}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Polynomial part}}$$

  - Proper rational part handled as before while polynomial part trivial
- If a pole p_k has multiplicity r, the expansion has a more general form

$$\frac{R_{k,1}}{1-p_k z^{-1}} + \frac{R_{k,2}}{(1-p_k z^{-1})^2} + \cdots + \frac{R_{k,r}}{(1-p_k z^{-1})^r}$$

- Method 2 only provides $R_{k,r}$. Remaining residues using Method 1.

Matlab implementation

- Find impulse partial fraction representation of $H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$

```
Matlab
B = [3 -4];
A = [1 -3.5 1.5];

[R,P,C] = residuez(B,A);

R % Residues
P % Poles
C % Direct terms (if improper)
```

- Other useful Matlab functions:
  - `roots(a), poly([p1,p2]),impz(B,A)`

19

Summary

Today:

- Inverse z-transform
- Calculation of residues

Next:

- Sampling theorem

20

Illustrating example (optional)

A causal LTI system is described by the following difference equation:

$$y[n] = 0.81y[n-2] + x[n] - x[n-2]$$

- Determine:
  a) The system function $H(z)$
  b) The unit impulse response $h[n]$
  c) The frequency response function $H(\omega)$, and plot its magnitude and phase over $0 \leq \omega \leq \pi$

Illustrating example (optional)…

- The system function $H(z)$

$$Y(z) = 0.81z^{-2}Y(z) + X(z) - z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-2}}{1-0.81z^{-2}} = \frac{1-z^{-2}}{(1-0.9z^{-1})(1+0.9z^{-1})}$$

- Causality implies ROC: $|z| > 0.9$

Illustrating example (optional)…

- The unit impulse response $h[n]$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1-z^{-2}}{1-0.81z^{-2}}\right\}$$

$$= \mathcal{Z}^{-1}\left\{1 - 0.19z^{-2}\frac{1}{1-0.81z^{-2}}\right\} = \delta[n] - 0.19 \cdot h'[n-2]$$

where

$$h'[n] = \mathcal{Z}^{-1}\left\{\frac{1}{(1-0.9z^{-1})(1+0.9z^{-1})}\right\} = \mathcal{Z}^{-1}\left\{\frac{R_1}{1-0.9z^{-1}} + \frac{R_2}{1+0.9z^{-1}}\right\}$$

- $R_1 = R_2 = \frac{1}{2} \Longrightarrow$

$$h[n] = \delta[n] - \frac{1}{2}0.19 \cdot 0.9^{n-2} \cdot (1 + (-1)^{n-2})u[n-2]$$

$$= \delta[n] - 0.1173 \cdot 0.9^{n} \cdot (1 + (-1)^{n})u[n-2]$$

23

Illustrating example (optional)…

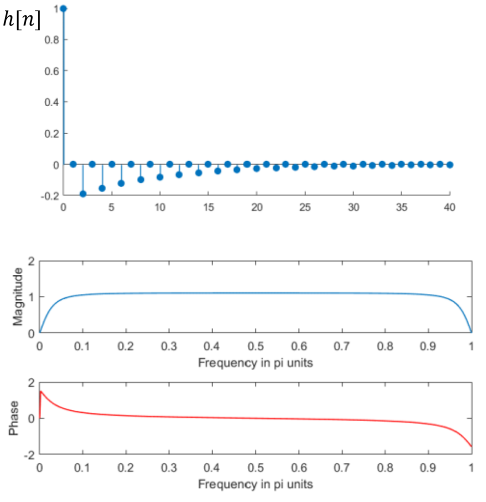
- Plot the frequency response: $H(\omega) = \frac{1-e^{-j2\omega}}{1-0.81e^{-j2\omega}}$

Matlab

```
B = [1 0 -1];
A = [1 0 -0.81];
W = [0:1:500]*pi/500;
H = freqz(B,A,W);

magH = abs(H); phaH = angle(H);

subplot(2,1,1); plot(W/pi,magH);
xlabel('Frequency in pi units')
ylabel('Magnitude')

subplot(2,1,2); plot(W/pi,phaH);
xlabel('Frequency in pi units')
ylabel('Phase')
```

24

Illustrating example (optional)…



25