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NTNU – Trondheim
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Exercise 11
TTK4130 Modeling and Simulation

Problem 1 (Tank with liquid)

A tank with area A is filled with an incompressible liquid with (constant) density ρ and level h. The liquid volume is then $V = Ah$ and the mass of the liquid in the tank is $m = V\rho$. Liquid enters the tank through a pipe with mass flow $w_i = \rho A_i v_i$, where A_i is the pipe cross section, and v_i is the velocity (constant over the cross section). Liquid leaves the tank through a second pipe with mass flow $w_u = \rho A_u v_u$ where A_u is the cross section of the pipe and v_u is the velocity.

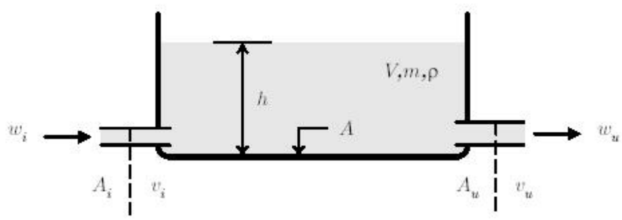

Figure 1: Tank with liquid

Use a mass balance for the tank to set up a differential equation for the level h.

Solution: The principle of mass conservation is

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_V \rho dV = 0,$$

that is, the mass is constant in a material volume. Using eq. (10.90) in the book, the liquid mass balance for a fixed volume V_f (the total volume of the tank) becomes (eq. (11.8) in the book):

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho \mathrm{d}V}_{\substack{\text{rate of change}\\ \text{of liquid mass}\\ \text{in } V_f}} = - \underbrace{\iint_{\partial V_f} \rho \mathbf{v}^{\mathsf{T}}\mathbf{n}\mathrm{d}A}_{\substack{\text{net increase of mass}\\ \text{by flow in and out}\\ \text{of } V_f}}$$

(Alternatively, one could assume the liquid volume as "control volume" and use eq. (11.10).)

We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\rho V = \rho A\dot{h},$$

(that is, the total mass of liquid in the tank), and

$$-\iint_{\partial V_f} \rho \mathbf{v}^{\mathsf{T}}\mathbf{n}\mathrm{d}A = \rho v_i A_i - \rho v_u A_u$$

(flow in and out of the tank).

Inserted into the mass balance above, this gives

$$A\rho\dot{h} = \rho v_i A_i - \rho v_u A_u$$
$$\dot{h} = \frac{A_i}{A}v_i - \frac{A_u}{A}v_u.$$

Problem 2 (Mixing, reactions (Exam 2010))

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then consists of a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

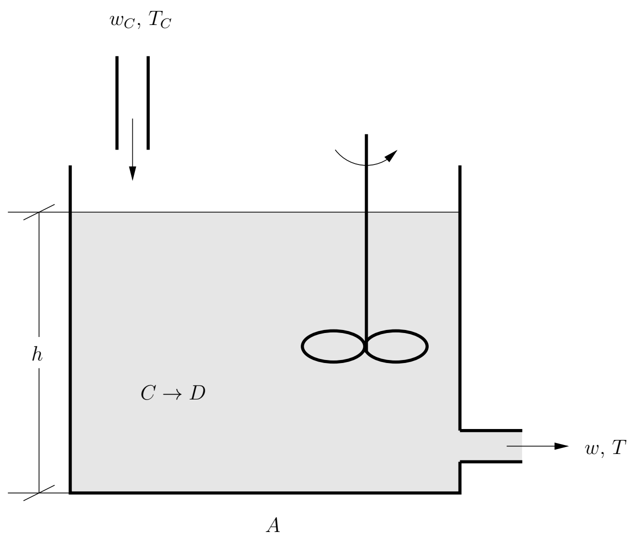

Figure 2: Tank reactor

(a) Set up a differential equation for the level of the tank. (Hint: Use the ordinary overall mass balance. Assume that the average density ρ is constant.)

Solution: The mass balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho \mathrm{d}V = -\iint_{\partial V_f} \rho \vec{v} \cdot \vec{n} \mathrm{d}A$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (\rho A h) = w_C - w$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h = \frac{w_C - w}{\rho A}$$

(b) In a material volume V_m, the following holds:

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_{V_m} \rho_C \mathrm{d}V = -\iiint_{V_m} J \mathrm{d}V.$$

Use this together with the appropriate form of the transport theorem to explain that the mass balance for substance C on integral form in a fixed control volume V_f is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho_C \mathrm{d}V = -\iiint_{V_f} J \mathrm{d}V - \iint_{\partial V_f} \rho_C \vec{v} \cdot \vec{n} \mathrm{d}A.$$

(In this particular case, the natural control volume, the volume of liquid in the tank, is not fixed, but this can be ignored since $\rho_C \vec{v}_c \cdot \vec{n} = 0$ – expansion of the volume does not accumulate more of substance C.)

Solution: Take eq. (10.90) in the book, set $\phi = \rho_C$ and insert the first equation to obtain the result.

(c) Use this to write up the mass balance for the mass of substance C in the tank ($\frac{\mathrm{d}}{\mathrm{d}t} m_C = \ldots$). Assume here, and for the rest of the problem, that J is proportional to the density of substance C, $J = k\frac{m_C}{V}$, and that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow, $w_{C,out} = \frac{m_C}{m_C + m_D} w$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_C A h) = w_C - JV - w_{C,out}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} m_C = w_C - k m_C - \frac{m_C}{m_C + m_D} w$$

(d) What is the mass balance equation on integral form for substance D (in a fixed volume)? Use this to write up the mass balance of substance D.

Solution: For substance D, we have

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_{V_m} \rho_D \mathrm{d}V = \iiint_{V_m} J \mathrm{d}V.$$

Insertion into (10.90) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho_D \mathrm{d}V = \iiint_{V_f} J \mathrm{d}V - \iint_{\partial V_f} \rho_D \vec{v} \cdot \vec{n} \mathrm{d}A.$$

Solving the integrals, give

$$\frac{\mathrm{d}}{\mathrm{d}t} m_D = JV - w_D = k m_C - \frac{m_D}{m_C + m_D} w.$$

(e) Check that the solution in (c) and (d) agrees with the answer in (a).

Solution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} m &= \frac{\mathrm{d}}{\mathrm{d}t} m_C + \frac{\mathrm{d}}{\mathrm{d}t} m_D \\
&= w_C - k m_C - \frac{m_C}{m_C + m_D} w + k m_C - \frac{m_D}{m_C + m_D} w \\
&= w_C - w.
\end{aligned}$$

This agrees with the solution to (a).

(f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Assume no 'heat flux' (the tank is well insulated). Assume the internal energy is $u = c_p T$.

Solution: The book does not treat energy balances with "internally generated" energy. We must therefore derive the energy balance on integral form for this (as we did for the mass balance above).

Under the assumptions made, (11.164) takes the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{V_m}\rho u \mathrm{d}V = \iiint\limits_{V_m} cJ\mathrm{d}V$$

($e = u$, pressure work and heat flux ignored, but heat from reaction added.) Insertion into (11.169) (for a fixed volume) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho u \mathrm{d}V = \iiint\limits_{V_f} cJ\mathrm{d}V - \iint\limits_{\partial V_f}\rho u\vec{v}\cdot\vec{n}\mathrm{d}A.$$

Inserting $u = c_pT$ and resolving the integrals, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p TV\right) &= cJV + w_C c_p T_C - w c_p T\\ \rho c_p V\frac{\mathrm{d}}{\mathrm{d}t}T + \rho c_p TA\frac{\mathrm{d}}{\mathrm{d}t}h &= cJV + w_C c_p T_C - w c_p T\end{aligned}$$

Insertion of the result in (a), and using $JV = km_C$,

$$\begin{aligned}\rho c_p Ah\frac{\mathrm{d}}{\mathrm{d}t}T + c_pT(w_C - w) &= ckm_C + w_C c_p T_C - w c_p T\\ \frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{ckm_C + c_p w_C(T_C - T)}{\rho c_p Ah}\end{aligned}$$

Correct result (without derivation of the energy balance) will give full score.

Problem 3 (Compressor, momentum balance, Bernoulli's equation)

A compressor takes in air with pressure p_0 and velocity $v_0 = 0$ from the surroundings. The air flows through a duct into the compressor. For control, it would be beneficial to have a measurement of the mass flow into the compressor. However, this measurement is not available.

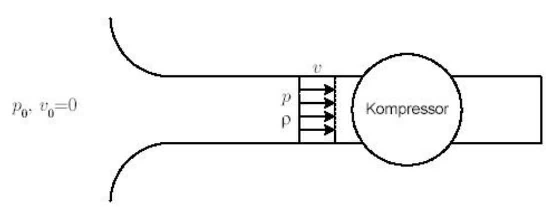

Figure 3: Compressor

Instead, there is a pressure measurement in the duct, giving a measurement p. How can the mass flow w and velocity v be found from this measurement? Assume that the density ρ in the duct is constant and known, there is no friction, and that the velocity is uniform over the cross-section where the pressure transmitter is located.

Hint: Use (the stationary) Bernoulli's equation.

Solution: Bernoulli's equation for frictionless, incompressible flow along a streamline relates pressure, velocity and elevation in two points:

$$\frac{p_1 - p_0}{\rho} + \frac{1}{2}\left(v_1^2 - v_0^2\right) + (z_1 - z_0)g = 0.$$

Choosing point 1 to be the location of the pressure transmitter ($p_1 = p$, $v_1 = v$ and $z_1 = 0$) and point 0 to be the duct inlet ($p_0 = p_0$, $v_0 = 0$ and $z_0 = 0$), we get

$$p = p_0 - \frac{1}{2}v^2\rho$$

that can be solved to

$$v = \sqrt{\frac{2\left(p_0 - p\right)}{\rho}}$$

(note that p must be smaller than p_0). This gives mass flow

$$w = A\rho v = A\sqrt{2\rho\left(p_0 - p\right)}$$