TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 3

Hand-out time:	Monday, September 16, 2013, at 12:00
Hand-in deadline:	Friday, September 27, 2013, at 12:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results.

Problem 1: Controllability tests

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

with state $\mathbf{x}(t)$, input $\mathbf{u}(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system.

b) Use the controllability matrix to determine whether the system is controllable or not.

c) Calculate the eigenvalues of $\mathbf{A}$.

d) Use the Popov-Belevitch-Hautus test for controllability to determine whether the system is controllable or not.

We want to use the Lyapunov test for controllability to determine if the system is controllable. The corresponding Lyapunov equation is given by

$$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T,$$

where $\mathbf{W}$ is a symmetric matrix.

e) State the conditions that $\mathbf{A}$ must satisfy in order to be able to use the Lyapunov test to check if the system is controllable.

f) Does $\mathbf{A}$ satisfy these conditions? Motivate your answer.

g) Calculate the matrix $\mathbf{W}$ from the Lyapunov equation.

h) Conclude from your answer in g) whether the system is controllable or not. Motivate your answer.

Problem 2: Controllable decompositions and stabilizability

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{1}$$

with state $\mathbf{x}(t)$ and input $\mathbf{u}(t)$, where $\mathbf{A}$ is a matrix of size $n \times n$. Let λ^* be an eigenvalue of $\mathbf{A}$. We will refer to λ^* as a *controllable eigenvalue* of system (1) if the condition

$$\operatorname{rank}\begin{bmatrix}\mathbf{A} - \lambda^*\mathbf{I} & \mathbf{B}\end{bmatrix} = n$$

holds. Similarly, if this condition does not hold, λ^* will be referred to as an *uncontrollable eigenvalue* of system (1).

a) Keeping the Popov-Belevitch-Hautus test for controllability in mind, how many controllable eigenvalues and how many uncontrollable eigenvalues does system (1) have if system (1) is controllable? Motivate your answer.

b) Keeping the Popov-Belevitch-Hautus test for stabilizability in mind, which property do the uncontrollable eigenvalues of system (1) need to have if system (1) is stabilizable? Motivate your answer.

Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \tag{2}$$

with state $\mathbf{x}(t)$, input $u(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}.$$

c) Calculate the eigenvalues of $\mathbf{A}$.

d) Show that the system (2) has two controllable eigenvalues and one uncontrollable eigenvalue.

e) Conclude from your answers in a) and d) whether the system is controllable or not. Motivate your answer.

f) Conclude from your answers in b), c) and d) whether the system is stabilizable or not. Motivate your answer.

g) Calculate the eigenvectors of $\mathbf{A}$.

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the two eigenvectors corresponding to the controllable eigenvalues of system (2). Moreover, let $\mathbf{q}_3$ be the eigenvector corresponding to the uncontrollable eigenvalue of system (2).

h) Calculate the matrix $\mathbf{Q} = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix}$.

i) Use the similarity transform $\mathbf{x}(t) = \mathbf{Q}\hat{\mathbf{x}}(t)$ to transform system (2) to the following form:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t). \tag{3}$$

j) Show that system (3) is a controllable decomposition of system (2), i.e., show that the matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ have the form:

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}_c & \hat{\mathbf{A}}_{12} \\ \mathbf{0} & \hat{\mathbf{A}}_u \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \hat{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix},$$

where $\hat{\mathbf{A}}_c$ and $\hat{\mathbf{B}}_c$ are the controllable parts of the matrices $\mathbf{A}$ and $\mathbf{B}$, and $\hat{\mathbf{A}}_u$ is the uncontrollable part of the matrix $\mathbf{A}$.

Problem 3: State feedback

Consider the following system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

with state $\mathbf{x}(t)$, input $u(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system.

b) Use the controllability matrix to determine whether the system is controllable or not.

Consider a state-feedback controller of the following form:

$$u(t) = -\mathbf{K}\mathbf{x}(t),$$

where $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is a feedback matrix. The closed-loop system (i.e. the system with controller) can be written as

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t).$$

c) Compute the matrix $\bar{\mathbf{A}}$ as a function of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{K}$.

d) Calculate the characteristic polynomial of the closed-loop system matrix $\bar{\mathbf{A}}$ as a function of k_1 and k_2.

We aim to find the values of k_1 and k_2 such that the poles of the closed-loop system, which are equal to the eigenvalues of $\bar{\mathbf{A}}$, are given by

$$\bar{\lambda}_1 = -2 \qquad \text{and} \qquad \bar{\lambda}_2 = -4.$$

Note that if the eigenvalues of $\bar{\mathbf{A}}$ are given by $\bar{\lambda}_1$ and $\bar{\lambda}_2$, the characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\det(\bar{\mathbf{A}} - \lambda\mathbf{I}) = (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda) = 0. \tag{4}$$

e) Calculate the values of k_1 and k_2 such that the characteristic polynomial of $\bar{\mathbf{A}}$ is equal to (4).

f) Calculate the matrix $\bar{\mathbf{A}}$ with the obtained values of k_1 and k_2 in e) and check if the eigenvalues of $\bar{\mathbf{A}}$ are equal to $\bar{\lambda}_1$ and $\bar{\lambda}_2$.