TTK4115 Linear System Theory

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Solution to homework assignment 5

Problem 1: Minimal realizations and state estimators

a) The transfer function $\hat{G}(s)$ is computed as follows:

$$\begin{aligned}
\hat{G}(s) &= \mathbf{C}(s\mathbf{I} + \mathbf{A})^{-1}\mathbf{B} + D \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s-4 & 0 \\ 3 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-4} & 0 \\ \frac{-3}{s^2-5s+4} & \frac{1}{s-1} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \\
&= -\frac{6}{s^2 - 5s + 4} + 1 \\
&= \frac{s^2 - 5s - 2}{s^2 - 5s + 4}.
\end{aligned}$$

b) Because the dimension of the system $n = 2$ is equal to the order of the denominator of of the transfer function $\hat{G}(s)$ in a) (the term in the denominator with the highest power is s^2), we conclude that the state-space equation in (1) is a minimal realization.

c) Because all minimal realizations are both controllable and observable and the state-space equation in (1) is a minimal realization, we conclude that the system is observable.

d) The observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}.$$

Because the observability matrix has full column rank, i.e. $\text{rank}(\mathcal{O}) = 2 = n$, we conclude that the system is observable. This matches our answer to c).

e) The block diagram of the system with state estimator is depicted in Fig. 1.

f) From the state-space equation of the system, the equation of the state estimator

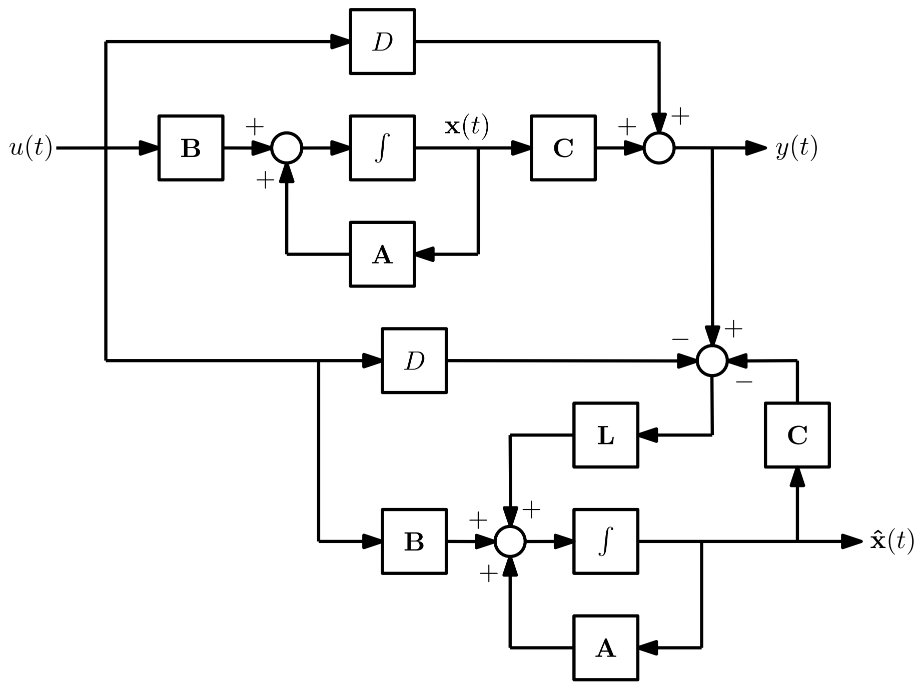

Fig. 1: Block diagram of system with state estimator.

and $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, it immediately follows that

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\
&= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{L}(y(t) - \hat{y}(t)) \\
&= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - \mathbf{L}(\mathbf{C}\mathbf{x}(t) + Du(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)) \\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\
&= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).
\end{aligned}$$

g) Let the estimator-gain matrix $\mathbf{L}$ be given by

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

where l_1 and l_2 are constants that have to be determined. The eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{L}\mathbf{C}$, which is given by

$$\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -l_1 \\ -3 & 1 - l_2 - \lambda \end{vmatrix} = \lambda^2 + (l_2 - 5)\lambda - 3l_1 - 4l_2 + 4.$$

For eigenvalues $\lambda_1 = -8$ and $\lambda_2 = -7$, the characteristic polynomial should be given by

$$\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = (-8 - \lambda)(-7 - \lambda) = \lambda^2 + 15\lambda + 56.$$

Comparing both expressions for the characteristic polynomial, we obtain the equations

$$l_2 - 5 = 15 \qquad \text{and} \qquad -3l_1 - 4l_2 + 4 = 56.$$

Solving for l_1 and l_2 yields $l_1 = -44$ and $l_2 = 20$. Hence, we obtain

$$\mathbf{L} = \begin{bmatrix} -44 \\ 20 \end{bmatrix}.$$

h) The state equation for $\mathbf{x}(t)$ can be written as

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\
&= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{B}Pr(t) \\
&= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{x}(t) - \mathbf{e}(t)) + \mathbf{B}Pr(t) \\
&= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t) + \mathbf{B}Pr(t).
\end{aligned}$$

For f), we already obtained that

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).$$

Therefore, we have

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}P \\ \mathbf{0} \end{bmatrix} r(t).$$

The output $y(t)$ is given by

$$\begin{aligned}
y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t) \\
&= \mathbf{C}\mathbf{x}(t) - D\mathbf{K}\hat{\mathbf{x}}(t) + DPr(t) \\
&= \mathbf{C}\mathbf{x}(t) - D\mathbf{K}(\mathbf{x}(t) - \mathbf{e}(t)) + DPr(t) \\
&= (\mathbf{C} - D\mathbf{K})\mathbf{x}(t) + D\mathbf{K}\mathbf{e}(t) + DPr(t),
\end{aligned}$$

which can be written as

$$y(t) = \begin{bmatrix} \mathbf{C} - D\mathbf{K} & D\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + DPr(t).$$

Hence, we obtain

$$\begin{aligned}
\dot{\mathbf{z}}(t) &= \mathbf{E}\mathbf{z}(t) + \mathbf{F}r(t) \\
y(t) &= \mathbf{G}\mathbf{z}(t) + Hr(t),
\end{aligned}$$

with matrices

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{B}P \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{C} - D\mathbf{K} & D\mathbf{K} \end{bmatrix}, \qquad H = DP.$$

Substituting the values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D, $\mathbf{K}$, $\mathbf{L}$ and P, we obtain

$$\mathbf{E} = \begin{bmatrix} -6 & 4 & 10 & -4 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 44 \\ 0 & 0 & -3 & -19 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} -5 & 3 & 5 & -2 \end{bmatrix}, \qquad H = -3.$$

i) The stability of the system in (2) is dependent on the eigenvalues of the system matrix $\mathbf{E}$. From the previous question, it follows that $\mathbf{E}$ is given by

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix}.$$

Note that $\mathbf{E}$ is a block-upper-triangular matrix, which implies that the eigenvalues of $\mathbf{E}$ are equal to the eigenvalues of the blocks on the diagonal of $\mathbf{E}$. The blocks on the diagonal of $\mathbf{E}$ are given by $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$.

The eigenvalues of $\mathbf{A} - \mathbf{BK}$ are given by the roots of its characteristic polynomial. The characteristic polynomial of $\mathbf{A} - \mathbf{BK}$ is given by

$$\det(\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I}) = \begin{vmatrix} -6-\lambda & 4 \\ -3 & 1-\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0.$$

The roots of its characteristic polynomial, and therefore the eigenvalues of $\mathbf{A} - \mathbf{BK}$, are given by $\lambda_1 = -3$ and $\lambda_2 = -2$.

In g), we have chosen the matrix $\mathbf{L}$ such that the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are given by $\lambda_3 = -8$ and $\lambda_4 = -7$.

Therefore, the eigenvalues of $\mathbf{E}$ are $\lambda_1 = -3$, $\lambda_2 = -2$, $\lambda_3 = -8$ and $\lambda_4 = -7$. Because all eigenvalues have negative real parts, the system is marginally stable, asymptotically stable and exponentially stable. The system is not unstable.

Problem 2: Process classification

a) The probability density function of the variable Φ is given by

$$f_\Phi(\phi) = \begin{cases} \dfrac{1}{2\pi}, & \text{if } -\pi \le \phi < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

The mean $\mu_X(t) = E[X(t)]$ is calculated as follows:

$$\begin{aligned}
\mu_X(t) &= E[X(t)] = E[a\sin(\omega t + \Phi)] = aE[\sin(\omega t + \Phi)] \\
&= a\int_{-\infty}^{\infty} \sin(\omega t + \phi) f_\Phi(\phi) d\phi = \frac{a}{2\pi}\int_{-\pi}^{\pi} \sin(\omega t + \phi) d\phi \\
&= \frac{a}{2\pi}\left[-\cos(\omega t + \phi)\right]_{-\pi}^{\pi} = \frac{a}{2\pi}\left[-\cos(\omega t + \pi) + \cos(\omega t + -\pi)\right] \\
&= \frac{a}{2\pi}\left[\cos(\omega t) - \cos(\omega t)\right] = 0.
\end{aligned}$$

b) The variance $\sigma_X^2(t) = E[X^2(t)]$ is given by

$$\begin{aligned}
\sigma_X^2(t) &= E[X^2(t)] = E[(a\sin(\omega t + \Phi))^2] = a^2E[\sin^2(\omega t + \Phi)] \\
&= a^2E\left[\frac{1-\cos(2\omega t + 2\Phi)}{2}\right] = \frac{a^2}{2}\left(1 - E\left[\cos(2\omega t + 2\Phi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \int_{-\infty}^{\infty} \cos(2\omega t + 2\phi) f_\Phi(\phi) d\phi\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega t + 2\phi) d\phi\right) = \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t + 2\pi) - \sin(2\omega t - 2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t) - \sin(2\omega t)\right]\right) = \frac{a^2}{2},
\end{aligned}$$

where we used the probability density function f_Φ in a).

c) Using the probability density function f_Φ in a), we obtain the following autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$:

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E[(a\sin(\omega t_1+\Phi))(a\sin(\omega t_2+\Phi))] \\
&= a^2 E[\sin(\omega t_1+\Phi)\sin(\omega t_2+\Phi)] \\
&= a^2 E\left[\frac{1}{2}\cos(\omega t_1+\Phi-(\omega t_2+\Phi)) - \frac{1}{2}\cos(\omega t_1+\Phi+(\omega t_2+\Phi))\right] \\
&= \frac{a^2}{2} E\left[\cos(\omega(t_1-t_2)) - \cos(\omega(t_1+t_2)+2\Phi)\right] \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - E[\cos(\omega(t_1+t_2)+2\Phi)]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \int_{-\infty}^{\infty}\cos(\omega(t_1+t_2)+2\phi)f_\Phi(\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega(t_1+t_2)+2\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\left[\frac{\sin(\omega(t_1+t_2)+2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)+2\pi) - \sin(\omega(t_1+t_2)-2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)) - \sin(\omega(t_1+t_2))\right]\right) \\
&= \frac{a^2}{2}\cos(\omega(t_1-t_2)).
\end{aligned}$$

Substituting $t_1 = t$ and $t_2 = t+\tau$, we get

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{a^2}{2}\cos(\omega(t-(t+\tau))) = \frac{a^2}{2}\cos(-\omega\tau) = \frac{a^2}{2}\cos(\omega\tau).$$

d) For every t the output $X(t)$ of the process lies in the interval $[-a, a]$. However, which value $X(t)$ takes in the interval $[-a, a]$ is dependent on the random variable Φ. Therefore, the output $X(t)$ of the process is not exactly predictable and, hence, the process is not deterministic.

e) Because the mean $\mu_X(t)$ is not dependent on the time origin (i.e. $\mu_X(t)$ is independent of t, see a)) and the autocorrelation function $R_X(t_1, t_2)$ in c) is only dependent on the time difference between sample points (i.e. $R_X(t_1, t_2)$ is dependent only on the time difference $t_2 - t_1$, since we can write $R_X(t_1, t_2) = R_X(\tau)$ for $t_1 = t$ and $t_2 = t + \tau$, see c)), the process is wide-sense stationary. In fact, it can be shown that all density functions associated with the process are independent of time, which implies that the process is stationary, which is a stronger property than wide-sense stationary.

f) While ergodicity applies to all density functions associated with the process, ergodicity in wide sense only applies to the mean and autocorrelation function of the process. For a process to be ergodic in wide sense, the time mean and the time autocorrelation function must be equivalent to the ensemble mean (i.e. μ_X) and the ensemble autocorrelation function (i.e. $R_X(\tau)$), respectively.

The time mean is given by

$$\begin{aligned}
\mathfrak{m}_X &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T a\sin(\omega t+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a}{T}\left[\frac{-\cos(\omega t+\Phi)}{\omega}\right]_0^T = \lim_{T\to\infty} \frac{a}{\omega T}\left[-\cos(\omega T+\Phi)+\cos(\Phi)\right] = 0.
\end{aligned}$$

The time autocorrelation function is given by

$$\begin{aligned}
\mathfrak{R}_X(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)X(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T (a\sin(\omega t+\Phi))(a\sin(\omega(t+\tau)+\Phi))dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \sin(\omega t+\Phi)\sin(\omega(t+\tau)+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \bigg(\frac{1}{2}\cos(\omega t+\Phi-(\omega(t+\tau)+\Phi)) \\
&\qquad -\frac{1}{2}\cos(\omega t+\Phi+\omega(t+\tau)+\Phi)\bigg)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\int_0^T \left(\cos(-\omega\tau)-\cos(2\omega t+\omega\tau+2\Phi)\right)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)t-\frac{\sin(2\omega t+\omega\tau+2\Phi)}{2\omega}\right]_0^T \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)T-\frac{\sin(2\omega T+\omega\tau+2\Phi)}{2\omega}+\frac{\sin(\omega\tau+2\Phi)}{2\omega}\right] \\
&= \frac{a^2}{2}\cos(\omega\tau).
\end{aligned}$$

Because the time mean $\mathfrak{m}_X$ and time autocorrelation function $\mathfrak{R}_X(\tau)$ are equal to the ensemble mean μ_X in a) and the ensemble autocorrelation function $R_X(\tau)$ in c), respectively, we conclude that the process is ergodic in wide sense. In fact, it can be shown that process is ergodic (not only in wide sense).

Problem 3: Linear system with white noise

a) White noise processes have a zero mean. Because the disturbance $w(t)$ is a white noise process, we have $\mu_w = 0$.

The reasoning behind this follows next. Let $v(t)$ be a white noise process. By definition, white noise has a flat spectrum. Therefore, the power spectrum density function associated with $v(t)$ is given by $S_v(j\omega) = \alpha_v$, where α_v is a nonnegative constant. Using the inverse Fourier transform, we obtain the corresponding autocorrelation function

$$R_v(\tau) = \mathcal{F}^{-1}\{S_v(j\omega)\} = \alpha_v \delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. We can define the zero-mean white-noise process $\bar{v}(t) = v(t) - \mu_v$, where $\mu_v = E[v(t)]$ is the mean of $v(t)$. Note that because $\bar{v}(t)$ is a white noise process, we have $S_{\bar{v}}(j\omega) = \alpha_{\bar{v}}$ for some nonnegative constant $\alpha_{\bar{v}}$. Similar as for $v(t)$, the autocorrelation function associated with $\bar{v}(t)$ is given by

$$R_{\bar{v}}(\tau) = \mathcal{F}^{-1}\{S_{\bar{v}}(j\omega)\} = \alpha_{\bar{v}} \delta(\tau).$$

Now, note that from the definition of the autocorrelation function, it follows that

$$\begin{aligned}
R_{\bar{v}}(\tau) &= E[\bar{v}(t)\bar{v}(t+\tau)] = E[(v(t) - \mu_v)(v(t+\tau) - \mu_v)] \\
&= E[v(t)v(t+\tau) - \mu_v v(t) - \mu_v v(t+\tau) + \mu_v^2] \\
&= E[v(t)v(t+\tau)] - \mu_v E[v(t)] - \mu_v E[v(t+\tau)] + \mu_v^2 \\
&= R_v(\tau) - \mu_v^2 - \mu_v^2 + \mu_v^2 = R_v(\tau) - \mu_v^2.
\end{aligned}$$

Substituting $R_v(\tau) = \alpha_v \delta(\tau)$ and $R_{\bar{v}}(\tau) = \alpha_{\bar{v}} \delta(\tau)$, we obtain

$$\alpha_{\bar{v}} \delta(\tau) = \alpha_v \delta(\tau) - \mu_v^2.$$

This is only valid for all τ if $\alpha_{\bar{v}} = \alpha_v$ and $\mu_v = 0$. Because the mean μ_v of $v(t)$ is equal to zero and $v(t)$ is an arbirtrary white noise process, we conclude that all white noise processes must have a zero mean.

b) The variance σ_w^2 can directly be obtained from the autocorrelation function $R_w(\tau)$:

$$\sigma_w^2 = E[w^2(t)] = R_w(0) = 4\delta(0) = \infty.$$

c) The power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$ is obtained by taking the Fourier transform of the autocorrelation function $R_w(\tau)$:

$$S_w(j\omega) = \mathcal{F}\{R_w(\tau)\} = \mathcal{F}\{4\delta(\tau)\} = 4\mathcal{F}\{\delta(\tau)\} = 4.$$

d) The transfer function $g(s) = \frac{y(s)}{w(s)}$ can be obtained from $g(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where $\mathbf{I}$ is the identity matrix. Hence, we get

$$\begin{aligned}
g(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 6s + 8} \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{s+8}{s^2 + 6s + 8}.
\end{aligned}$$

e) The poles of the system are equal to the roots of the denominator polynomial of the transfer function $g(s)$ (i.e. the roots of $s^2 + 6s + 8$) and are given by $\lambda_1 = -4$ and $\lambda_2 = -2$. Given that $g(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, we obtain

$$\begin{aligned} g(s) &= \frac{\alpha_1}{s+4} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{(s+2)(s+4)} + \frac{\alpha_2(s+4)}{(s+2)(s+4)} \\ &= \frac{(\alpha_1+\alpha_2)s + 2\alpha_1 + 4\alpha_2}{s^2+6s+8} = \frac{s+8}{s^2+6s+8}. \end{aligned}$$

From this, we conclude that

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 + 4\alpha_2 = 8.$$

Solving for α_1 and α_2 yields $\alpha_1 = -2$ and $\alpha_2 = 3$. Hence, the transfer function $g(s)$ can be written as

$$g(s) = \frac{-2}{s+4} + \frac{3}{s+2}.$$

By taking the inverse Laplace transform of the transfer function $g(s)$, we obtain the impulse response $g(t)$, which is given by

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+4} + \frac{3}{s+2}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -2e^{-4t} + 3e^{-2t}. \end{aligned}$$

It should be noted that $g(t) = -2e^{-4t} + 3e^{-2t}$ for $t \geq 0$. For $t < 0$, we have $g(t) = 0$.

f) Using $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$, the mean $\mu_y(t)$ is calculated as follows:

$$\begin{aligned} \mu_y(t) = E[y(t)] &= E\left[\int_0^t g(\tau)w(t-\tau)d\tau\right] = \int_0^t g(\tau)E[w(t-\tau)]d\tau \\ &= \int_0^t g(\tau)\mu_w d\tau = \mu_w \int_0^t g(\tau)d\tau = \mu_w \int_0^t (-2e^{-4\tau} + 3e^{-2\tau})d\tau \\ &= \mu_w \left[\frac{1}{2}e^{-4\tau} - \frac{3}{2}e^{-2\tau}\right]_0^t = \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} - \frac{1}{2} + \frac{3}{2}\right) \\ &= \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right). \end{aligned}$$

The stationary mean $\bar{\mu}_y$ is given by

$$\bar{\mu}_y = \lim_{t\to\infty} \mu_y(t) = \lim_{t\to\infty} \mu_w \left(\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right) = \mu_w.$$

From a), we have $\mu_w = 0$. Hence, we obtain $\mu_y = \mu_w = 0$.

g) Note that the variance $\sigma_y^2(t)$ is equal to the mean-square value of $y(t)$, i.e. $\sigma_y^2(t) = E[y^2(t)]$. It follows that

$$\begin{aligned}
\sigma_y^2(t) = E[y^2(t)] &= E\left[\int_0^t g(\tau_1)w(t-\tau_1)d\tau_1 \int_0^t g(\tau_2)w(t-\tau_2)d\tau_2\right] \\
&= E\left[\int_0^t g(\tau_2)\int_0^t g(\tau_1)w(t-\tau_1)w(t-\tau_2)d\tau_1 d\tau_2\right] \\
&= \int_0^t g(\tau_2)\int_0^t g(\tau_1)E\left[w(t-\tau_1)w(t-\tau_2)\right]d\tau_1 d\tau_2 \\
&= \int_0^t g(\tau_2)\int_0^t g(\tau_1)R_w(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2)\int_0^t g(\tau_1)\delta(\tau_2-\tau_1)d\tau_1 d\tau_2 \\
&= 4\int_0^t g(\tau_2)g(\tau_2)d\tau_2 = 4\int_0^t g^2(\tau_2)d\tau_2 \\
&= 4\int_0^t (-2e^{-4\tau_2}+3e^{-2\tau_2})^2 d\tau_2 = 4\int_0^t (4e^{-8\tau_2}-12e^{-6\tau_2}+9e^{-4\tau_2})d\tau_2 \\
&= 4\left[-\frac{1}{2}e^{-8\tau_2}+2e^{-6\tau_2}-\frac{9}{4}e^{-4\tau_2}\right]_0^t \\
&= 4\left(-\frac{1}{2}e^{-8t}+2e^{-6t}-\frac{9}{4}e^{-4t}+\frac{1}{2}-2+\frac{9}{4}\right) \\
&= -2e^{-8t}+8e^{-6t}-9e^{-4t}+3.
\end{aligned}$$

The stationary variance $\bar{\sigma}_y^2$ is given by

$$\bar{\sigma}_y^2 = \lim_{t\to\infty}\sigma_y^2(t) = \lim_{t\to\infty}\left(-2e^{-8t}+8e^{-6t}-9e^{-4t}+3\right) = 3.$$

Alternatively, we can first calculate the power spectral density function

$$S_y(s) = |g(s)|^2 S_w(s) = g(s)g(-s)S_w(s).$$

From c) and d), we have that $S_w(s) = 4$ and $g(s) = \frac{s+8}{s^2+6s+8}$, respectively. Therefore, the power spectral density function $S_y(s)$ can be written as

$$\begin{aligned}
S_y(s) &= \frac{s+8}{s^2+6s+8}\cdot\frac{-s+8}{(-s)^2+6(-s)+8}\cdot 4 \\
&= \frac{2s+16}{s^2+6s+8}\cdot\frac{2(-s)+16}{(-s)^2+6(-s)+8} \\
&= \frac{c(s)}{d(s)}\cdot\frac{c(-s)}{d(-s)},
\end{aligned}$$

where

$$c(s) = 2s+16 \qquad \text{and} \qquad d(s) = s^2+6s+8,$$

with $c_0 = 16$, $c_1 = 2$, $d_0 = 8$, $d_1 = 6$ and $d_2 = 1$. Therefore, we obtain

$$\bar{\sigma}_y^2 = \frac{1}{2\pi j}\int_{-j\omega}^{j\omega} \frac{c(s)c(-s)}{d(s)d(-s)} ds = \frac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2} = \frac{32 + 256}{96} = 3;$$

see Section 3.3 in Brown & Hwang for more details.

h) The power spectral density function $S_y(j\omega)$ of the output $y(t)$ is given by

$$S_y(j\omega) = |g(j\omega)|^2 S_w(j\omega) = g(j\omega)g(-j\omega)S_w(j\omega).$$

From c), we have that $S_w(j\omega) = 4$. In addition, using the transfer function $g(s) = \frac{s+8}{s^2+6s+8}$ in d), we obtain

$$\begin{aligned} S_y(j\omega) &= \frac{j\omega + 8}{(j\omega)^2 + 6(j\omega) + 8} \cdot \frac{(-j\omega) + 8}{(-j\omega)^2 + 6(-j\omega) + 8} \cdot 4 \\ &= \frac{j\omega + 8}{-\omega^2 + 6j\omega + 8} \cdot \frac{-j\omega + 8}{-\omega^2 - 6j\omega + 8} \cdot 4 \\ &= \frac{4\omega^2 + 256}{\omega^4 + 20\omega^2 + 64} = \frac{20}{\omega^2 + 4} - \frac{16}{\omega^2 + 16}. \end{aligned}$$