Solutions for Exam Problems in the Course TTK4195: Robots Modeling and Control

Monday, May 25, 2009

1. **Problem:** Consider the following sequence of rotations

   (a) Rotate by ϕ about the world axis x;

   (b) Rotate by θ about the world axis y;

   (c) Rotate by ψ about the current axis z;

   (d) Rotate by α about the world axis y

   Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication)

   Solution: The total rotation is

$$R = R_{y,\alpha} \cdot R_{y,\theta} \cdot R_{x,\phi} \cdot R_{z,\psi}$$

   where

$$R_{x,q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q & -\sin q \\ 0 & \sin q & \cos q \end{bmatrix}, R_{y,q} = \begin{bmatrix} \cos q & 0 & \sin q \\ 0 & 1 & 0 \\ -\sin q & 0 & \cos q \end{bmatrix}, R_{z,q} = \begin{bmatrix} \cos q & -\sin q & 0 \\ \sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. **Problem:** Given three matrices

$$T_1 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad T_2 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad T_3 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

   (a) Which of these matrices are orthogonal? **(5)**

   (b) Which of these matrices do represent a rotation? **(5)**

   Solution: (a) By definition, a matrix O is orthogonal if $OO^T = I$.

Computing the matrices $T_i T_i^T$ for $i = 1, 2, 3$ you can find that

$$T_1 T_1^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$T_2 T_2^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3 T_3^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the matrices T_2 and T_3 are orthogonal, but T_1 is not.

(b) By definition, an orthogonal matrix T is a rotation if $\det T = 1$. Computing $\det T_2$ and $\det T_3$, we find that

$$\det T_2 = 1, \qquad \det T_3 = -1$$

Hence, T_2 is a rotation, but T_3 is not.

3. **Problem:** A motion of the center of mass of a rigid body with respect to an inertia frame is a rotation

$$R_1(t) = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}$$

Find an angular velocity of the motion **(5)**. Compute the vector of angular velocity when the rotation is **(10)**

$$R_2(t) = \begin{bmatrix} \cos t & -(\sin t) \cdot (\cos 2t) & (\sin t) \cdot (\sin 2t) \\ \sin t & (\cos t) \cdot (\cos 2t) & -(\cos t) \cdot (\sin 2t) \\ 0 & \sin 2t & \cos 2t \end{bmatrix}$$

Solution: (a) $R_1(t)$ is the basic rotation $R_{y,\phi(t)}$ with $\phi(t) = -t$. Hence

$$\vec{\omega}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{d}{dt}\phi(t) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad (1)$$

An alternative way to compute $\vec{\omega}_1(t)$ is to use the formula

$$\left[\tfrac{d}{dt}R(t)\right]R(t)^T = S(\vec{\omega}(t)) = \begin{bmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{bmatrix}, \ \vec{\omega}(t) = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} \tag{2}$$

For $R_1(t)$ we obtain

$$\begin{aligned} \left[\tfrac{d}{dt}R(t)\right]R(t)^T &= \begin{bmatrix} -\sin t & 0 & -\cos t \\ 0 & 0 & 0 \\ \cos t & 0 & -\sin t \end{bmatrix} \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

which gives the same results as in (1).

To compute the angular velocity $\vec{\omega}_2(t)$ for the rotation $R_2(t)$ you can again use the formula (2). However, the easier way the solve the problem is to observe that

$$R_2(t) = \underbrace{\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_1^0(t)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{bmatrix}}_{R_2^1(t)} \tag{3}$$

Then the vector of angular velocity can be computed from the relation

$$\vec{\omega}_2(t) = \vec{\omega}_1^0(t) + R_2^1(t)\vec{\omega}_2^1(t)$$

where the vectors $\vec{\omega}_1^0(t)$ and $\vec{\omega}_2^1(t)$ are angular velocities of the transforms $R_1^0(t)$ and $R_2^1(t)$ respectively. $R_1^0(t)$ and $R_2^1(t)$ are basic rotations around z and x axes, hence

$$\vec{\omega}_1^0(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{\omega}_2^1(t) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the vector of angular velocity $\vec{\omega}_2(t)$ is

$$\vec{\omega}_2(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 1 \end{bmatrix}$$

4. **Problem:** Consider a robot depicted on Fig. 1. Derive the forward kinematic equations for this manipulator. In particular, introduce a DH-frame for each of the links of the robots, compute DH-parameters of homogeneous transforms between consecutive frames, presenting the solution of the problem do not perform multiplication of such transforms. **(20)**

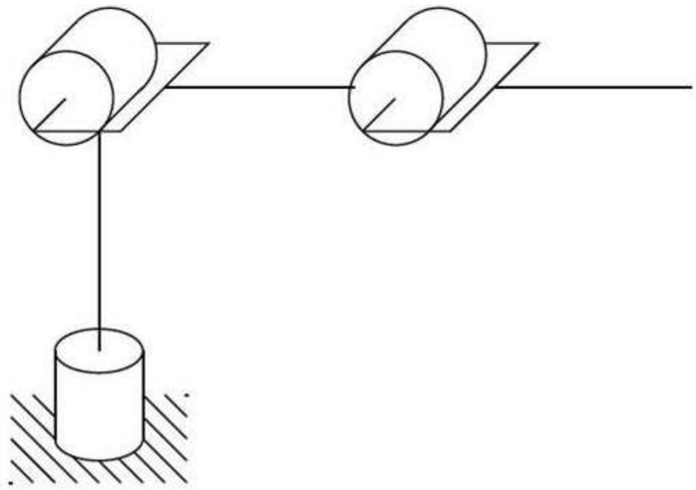

Figure 1: A robot from the problem 4: elbow manipulator

Solution: The parameters of sequence of homogeneous transforms written (as functions of angles q_1, q_2, q_3) in the DH-convention, i.e.

$$A = R_{z,\theta} \cdot T_{z,d} \cdot T_{x,a} \cdot R_{x,\alpha}$$

have the form

$$\begin{array}{lllll} A_1^0 : & \theta = q_1, & d = d_1, & a = 0, & \alpha = \frac{\pi}{2} \\ A_2^1 : & \theta = q_2, & d = 0, & a = a_2, & \alpha = 0 \\ A_3^2 : & \theta = q_3, & d = 0, & a = a_3, & \alpha = 0 \end{array}$$

5. **Problem:** The pendulum on the cart system is depicted on Fig. 2. It has two degrees of freedom: the position x of center of mass of the cart on the horizontal and the angle θ of the pendulum with vertical counted anticlockwise. It is assumed that the rod of the pendulum is massless and the mass of the pendulum is all concentrated in the bob. While the

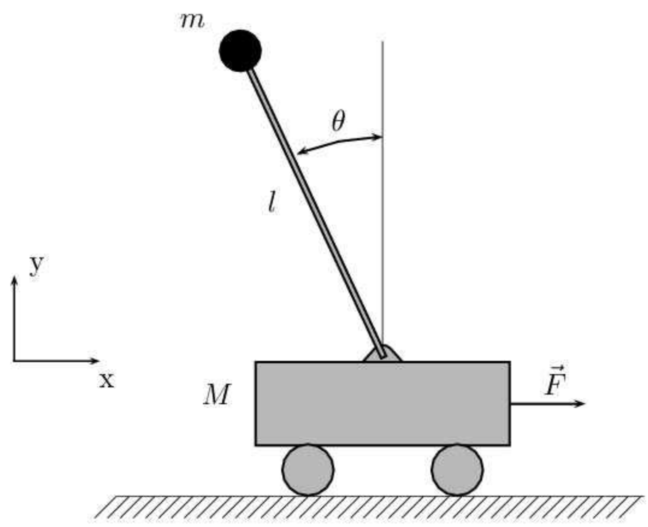

Figure 2: A robot from the problem 3: the cart-pendulum system

only one control variable in the system is the force applied to the cart along the horizontal direction.

Given physical parameters of the system (mass of the cart M, the mass of the bob m, the distance l from the suspension to the position of the bob), you are requested to implement the following tasks:

(a) Find the potential energy $\mathcal{P}(x, \theta)$ of the robot **(5)**

(b) Find the kinetic energy $\mathcal{K}(x, \theta, \dot{x}, \dot{\theta})$ of the robot **(10)**

(c) Obtain the Euler-Lagrange equations of the system dynamics **(15)**

(d) Assume that $M = m = 1$ [kg], $l = 1$ [m]. Compute the linearization of the system dynamics around the upright equilibrium $\{x_e = 0,\ \theta_e = 0\}$. Check, if the resulted linear control system is controllable. **(20)**

Solution: Before computing the kinetic and potential energies of the cart-pendulum system, let us observe that coordinates of centers of masses of the cart and the pendulum are $x_c = x$, $y_c = 0$ and

$$x_p = x_c - l \cdot \sin(\theta), \qquad y_p = l \cdot \cos(\theta)$$

and, therefore, their velocities are $\dot{x}_c = \dot{x}$, $\dot{y}_c = 0$ and

$$\dot{x}_p = \dot{x}_c - l \cdot \cos(\theta) \cdot \dot{\theta}, \qquad \dot{y}_p = -l \cdot \sin(\theta) \cdot \dot{\theta}$$

(a) The potential energy $\mathcal{P}(\cdot)$ of the robot is the potential energy of the pendulum

$$\mathcal{P} \;=\; mgy_p = mgl\cos(\theta)$$

(b) The kinetic energy of the robot is equal to the sum kinetic energies of the cart and the pendulum

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \tfrac{1}{2}M\left(\dot{x}_c^2 + \dot{y}_c^2\right) = \tfrac{1}{2}M\dot{x}^2 \\
\mathcal{K}_2 &= \tfrac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right) = \tfrac{1}{2}m\left(\dot{x} - l \cdot \cos(\theta) \cdot \dot{\theta}\right)^2 + \tfrac{1}{2}m\left(l \cdot \sin(\theta) \cdot \dot{\theta}\right)^2
\end{aligned}$$

Summing the terms we obtain

$$\mathcal{K} = \tfrac{1}{2} \cdot [M + m] \cdot \dot{x}^2 + \tfrac{1}{2} \cdot m \cdot l^2 \cdot \dot{\theta}^2 - m \cdot l \cdot \cos(\theta) \cdot \dot{x} \cdot \dot{\theta}$$

(c) To derive the Euler-Lagrange equations, introduce the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

and compute partial derivatives of $\mathcal{L}$ with respect to variables x, θ, $\dot{x}$ and $\dot{\theta}$, i.e.

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M + m)\dot{x} - ml\cos(\theta)\dot{\theta}, \qquad \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\cos(\theta)\dot{x} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = ml\sin(\theta)\dot{\theta}\dot{x} + mlg\sin(\theta)$$

The Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] - \frac{\partial \mathcal{L}}{\partial x} &= F \\
\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} &= 0
\end{aligned}$$

are then

$$
\begin{aligned}
(M+m)\ddot{x} - ml\cos(\theta)\,\ddot{\theta} + ml\sin(\theta)\,\dot{\theta}^2 &= F \\
-ml\cos(\theta)\,\ddot{x} + ml^2\,\ddot{\theta} - mgl\sin(\theta) &= 0
\end{aligned}
$$

or the same in the matrix form

$$
\underbrace{\begin{bmatrix} M+m & -ml\cos(\theta) \\ -ml\cos(\theta) & ml^2 \end{bmatrix}}_{:=D} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F - ml\sin(\theta)\,\dot{\theta}^2 \\ mgl\sin(\theta) \end{bmatrix}
$$

Multiplying both sides of the equation with D^{-1}, which is

$$
\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} = \frac{1}{\det D} \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}, \qquad \det D = d_{11}d_{22} - d_{12}d_{21},
$$

we resolve the equations with respect to the 2nd time derivatives

$$
\begin{aligned}
\ddot{x} &= \frac{d_{22}}{\det D}\left[F - ml\sin(\theta)\,\dot{\theta}^2\right] + \frac{-d_{12}}{\det D}\,mgl\sin(\theta) \\
\ddot{\theta} &= \frac{-d_{21}}{\det D}\left[F - ml\sin(\theta)\,\dot{\theta}^2\right] + \frac{d_{11}}{\det D}\,mgl\sin(\theta)
\end{aligned}
\tag{4}
$$

(d) With new variables

$$
z_1 = x, \quad z_2 = \theta, \quad z_3 = \dot{x}, \quad z_4 = \dot{\theta}
$$

and assuming that parameters are $M = m = 1$ [kg], $l = 1$ [m], the dynamics (4), that is now

$$
\begin{aligned}
\ddot{x} &= \frac{1}{1+\sin^2\theta}\left[F - \sin(\theta)\,\dot{\theta}^2\right] + \frac{\cos\theta}{1+\sin^2\theta}\,g\sin(\theta) \\
\ddot{\theta} &= \frac{\cos\theta}{1+\sin^2\theta}\left[F - \sin(\theta)\,\dot{\theta}^2\right] + \frac{2}{1+\sin^2\theta}\,g\sin(\theta)
\end{aligned}
$$

can be rewritten in state-space form as

$$
\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \frac{\sin(z_2)\cdot\left(g\cdot\cos(z_2) - z_4^2\right)}{1+\sin^2(z_2)} \\ \frac{\sin(z_2)\cdot\left(2g - \cos(z_2)\cdot z_4^2\right)}{1+\sin^2(z_2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{1+\sin^2(z_2)} \\ \frac{\cos(z_2)}{1+\sin^2(z_2)} \end{bmatrix} F \tag{5}
$$

Linearization of the system (5) around the equilibrium

$$z_1 = z_2 = z_3 = z_4 = 0$$

has the form

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{=A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix}}_{=B} v \tag{6}$$

Here the coefficients a_{3i} and a_{4i} with $i = 1, \ldots, 4$ are defined as

$$a_{3i} = \frac{\partial}{\partial z_i}\left[\frac{\sin(z_2)\cdot(g\cdot\cos(z_2) - z_4^2)}{1+\sin^2(z_2)}\right]\Bigg|_{z_1=\cdots=z_4=0} \tag{7}$$

$$a_{4i} = \frac{\partial}{\partial z_i}\left[\frac{\sin(z_2)\cdot(2g - \cos(z_2)\cdot z_4^2)}{1+\sin^2(z_2)}\right]\Bigg|_{z_1=\cdots=z_4=0} \tag{8}$$

and

$$b_1 = \frac{1}{1+\sin^2(z_2)}\Bigg|_{z_1=\cdots=z_4=0} = 1, \quad b_2 = \frac{\cos(z_2)}{1+\sin^2(z_2)}\Bigg|_{z_1=\cdots=z_4=0} = 1 \tag{9}$$

Computing (7) and (8), we obtain

$$a_{31} = a_{33} = a_{34} = 0, \qquad a_{32} = g \tag{10}$$

$$a_{41} = a_{43} = a_{44} = 0, \qquad a_{42} = 2g \tag{11}$$

As a result, matrices A and B take the form

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g & 0 & 0 \\ 0 & 2g & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{12}$$

The linear system is controllable if the matrix

$$\Gamma = [B, AB, A^2B, A^3B] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ g \\ 2g \end{bmatrix}, \begin{bmatrix} g \\ 2g \\ 0 \\ 0 \end{bmatrix}\right]$$

is of full rank. Straightforward calculations show that its determinant is equal

$$\det \Gamma = -g^2 \neq 0$$

Hence, the linearization of the cart-pendulum system around its unstable upright equilibrium is controllable.