Solutions to Exam Problems in the Course TTK 4195
– Modeling and Control of Robots –
Tuesday, May 29, 2012

1. **Problem:** Suppose that four orthogonal coordinate frames $\{o_1x_1y_1z_1\}$, $\{o_2x_2y_2z_2\}$, $\{o_3x_3y_3z_3\}$ and $\{o_4x_4y_4z_4\}$ are given together with rotation matrices R_2^1, R_4^3 and R_4^1 describing the respective orientations as

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_4^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_4^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Compute the rotation matrix R_3^2. **(5)**

Solution: The rotation R_4^1 is defined by the consecutive rotations

$$R_4^1 = R_2^1 R_3^2 R_4^3 .$$

Therefore,

$$\begin{aligned} R_3^2 &= \left[R_2^1\right]^{-1} R_4^1 \left[R_4^3\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

2. **Problem:** Consider the following sequence of rotations:

   (a) Rotate by ϕ about the world z-axis;
   (b) Rotate by θ about the world x-axis;
   (c) Rotate by ψ about the current z-axis;
   (d) Rotate by α about the world y-axis.

Apply the composition law for rotations and simplify the resulting product (do not perform the matrix multiplication). **(5)**

Solution: The total rotation is

$$\begin{aligned} R &= \underbrace{R_{z,\phi}\left[R_{z,\phi}^{-1} R_{x,\theta} R_{z,\phi}\right]}_{R_{x,\theta}R_{z,\phi}} R_{z,\psi} \left[R_{x,\theta}R_{z,\phi}R_{z,\psi}\right]^{-1} R_{y,\alpha} \left[R_{x,\theta}R_{z,\phi}R_{z,\psi}\right] \\ &= R_{y,\alpha}R_{x,\theta}R_{z,\phi}R_{z,\psi} \end{aligned}$$

where

$$R_{x,q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q & -\sin q \\ 0 & \sin q & \cos q \end{bmatrix}, R_{y,q} = \begin{bmatrix} \cos q & 0 & \sin q \\ 0 & 1 & 0 \\ -\sin q & 0 & \cos q \end{bmatrix}, R_{z,q} = \begin{bmatrix} \cos q & -\sin q & 0 \\ \sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. **Problem:** Consider the RPP-robot depicted in Fig. 1

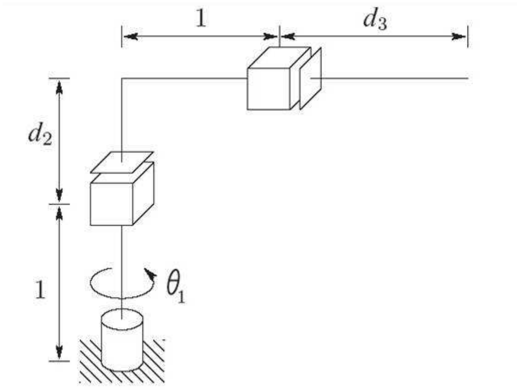


Figure 1: Robot of RPP-type.

(a) Sketch the workspace of the robot and classify the resulting geometric shape. **(2)**

(b) Introduce coordinate frames for each link according to the Denavit-Hartenberg (DH) convention and make a table of the DH parameters. Derive the forward kinematic equations using homogeneous transformation matrices. **(10)**

(c) Solve the inverse position kinematics problem for a given position of the tool frame $p_e^0 = (x_e^0, y_e^0, z_e^0) \in \mathbb{R}^3$ with respect to the base frame. That is, find the configuration of joint angle θ_1 and joint extensions d_2, d_3 such that the requested position p_e^0 is obtained for the origin of the tool frame, which is located at the end of the second prismatic link of the robot. **(9)**

(d) Compute the manipulator Jacobian for representation of linear and angular velocity of the origin of the tool frame, which is located at the end of the second prismatic link of the robot. **(9)**

(e) Compute the total velocity of the origin of the tool frame when the variables θ_1, d_2 and d_3 are changing with time as follows

$$\theta_1(t) = \cos(3t), \quad d_2(t) = \sin(2t), \quad d_3(t) = \cos(t)\,.$$

The computation can be based on the Jacobian from the previous step or based on the velocity vector of this point. **(5)**

Solution:

(a) The robot workspace has cylindrical shape, see Fig. 2.

(b) The parameters for the DH-frames in Fig. 3(a) are

Link	θ_i	d_i	a_i	α_i
1	θ_1	1	0	0
2	0	d_2	0	$-\frac{\pi}{2}$
3	0	$1 + d_3$	0	0

where the homogeneous transformation is defined as the product

$$A_i(\theta_i, d_i, a_i, \alpha_i) = \mathrm{Rot}_{z,\theta_i} \cdot \mathrm{Trans}_{z,d_i} \cdot \mathrm{Trans}_{x,a_i} \cdot \mathrm{Rot}_{x,\alpha_i}\ .$$

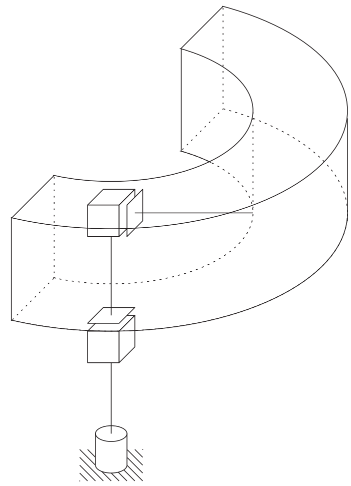


Figure 2: Cylindrical workspace of the robot in Problem 3.

With the above parameters we obtain the homogeneous transformations as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The transformation matrices to each link frame from the base frame can now be found as

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_2^0 = A_1A_2 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1(1+d_3) \\ s_1 & 0 & c_1 & c_1(1+d_3) \\ 0 & -1 & 0 & 1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Given $p_e = (x_e, y_e, z_e)$ shown in Fig. 3(b), we are requested to solve the equations

$$z_e = 1 + d_2, \quad x_e = (1+d_3)\cos(\theta_1 + \pi/2), \quad y_e = (1+d_3)\sin(\theta_1+\pi/2)$$

with respect to variables θ_1, d_2 and d_3. The solutions are

$$d_2 = z_e - 1, \quad d_3 = \sqrt{x_e^2 + y_e^2} - 1, \quad \theta_1 = \arctan2(y_e, x_e) - \pi/2$$

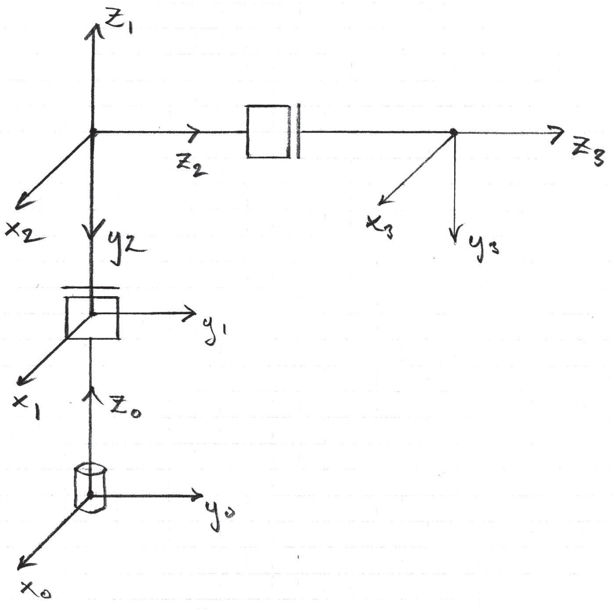
where $\arctan2(y_e, x_e)$ is used instead of $\arctan(y_e/x_e)$ so that solutions are found within a full revolution interval. Alternatively, with the vector to the origin of the tool frame as

$$o_3 = \begin{bmatrix} -s_1(1+d_3) \\ c_1(1+d_3) \\ 1+d_2 \end{bmatrix} = p_e$$

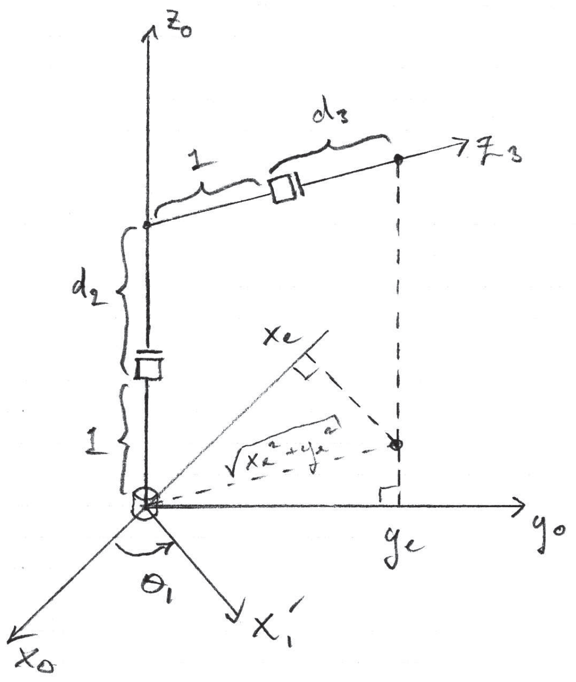
we obtain the equation set

$$x_e = -(1+d_3)\sin(\theta_1), \quad y_e = (1+d_3)\cos(\theta_1), \quad z_e = 1+d_2$$

giving the same solution.



(a) DH frames.



(b) Geometry for inverse kinematics.

Figure 3: Illustrative schematics of the robot in Problem 3.

(d) Given the DH-frames for each of three links of the robot, the manipulator Jacobian is computed as

$$J_\omega = \left[\rho_1 \vec{z}_0,\, \rho_2 \vec{z}_1,\, \rho_3 \vec{z}_2\right], \quad J_v = \left[\vec{z}_0 \times (\vec{p}_e - \vec{o}_0)\,,\, \vec{z}_1,\, \vec{z}_2\right]$$

The gain ρ_1 is equal to 1, because the first joint is revolute, while $\rho_2 = \rho_3 = 0$ because the second and third joints are prismatic. This implies that

$$J_\omega = \left[\vec{z}_0,\, 0 \cdot \vec{z}_1,\, 0 \cdot \vec{z}_2\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Computing of J_v is straightforward if one observes that

$$\vec{z}_0 = \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{z}_2 = \begin{bmatrix} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{bmatrix}, \ \vec{o}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\vec{p}_e = (1 + d_2)\vec{z}_0 + (1 + d_3)\vec{z}_2$. Then

$$\vec{z}_0 \times (\vec{p}_e - \vec{o}_0) = (1 + d_3) \begin{bmatrix} -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}$$

and the final form for J_v becomes

$$J_v = \begin{bmatrix} -(1 + d_3)\cos(\theta_1) & 0 & -\sin(\theta_1) \\ -(1 + d_3)\sin(\theta_1) & 0 & \cos(\theta_1) \\ 0 & 1 & 0 \end{bmatrix}$$

(e) The velocity of any point of tool frame can be computed by the formula

$$\vec{v} = \vec{v}_e + \vec{\omega} \times \vec{r}$$

where $\vec{r}$ is a vector from the origin of the tool frame to the point; $\vec{v}_e$ is the velocity of the origin of the tool frame and $\vec{\omega}$ is the angular velocity of the tool frame. We are asked to compute the velocity of the origin of the tool frame as a function of θ_1, d_2 and d_3 so that $\vec{r} = 0$ and

$$\vec{v} = J_v(\theta_1, d_2, d_3) \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

Substituting the functions $\theta_1(t)$, $d_2(t)$ and $d_3(t)$ are their time derivatives into the last formula, we obtain

$$\vec{v} = \begin{bmatrix} -\big(1+\cos(t)\big)\cos\big(\cos(3t)\big) & 0 & -\sin\big(\cos(3t)\big) \\ -\big(1+\cos(t)\big)\sin\big(\cos(3t)\big) & 0 & \cos\big(\cos(3t)\big) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3\sin(3t) \\ 2\cos(2t) \\ -\sin(t) \end{bmatrix}$$

Multiplication gives the answer.

4. **Problem:** Consider the planar elbow manipulator depicted in Fig. 5. Its motion is constrained by a slider for the end effector.

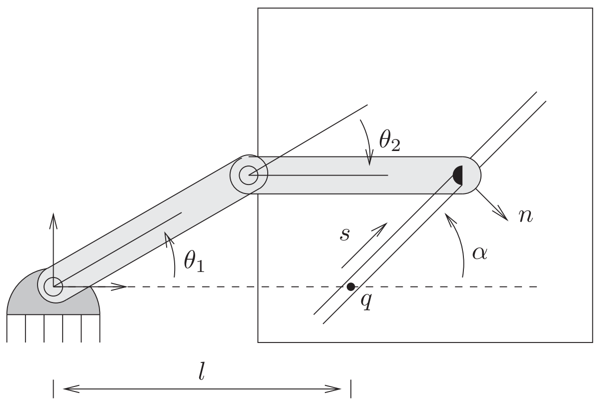


Figure 4: Planar elbow manipulator forced to slide along a slot, which resembles a simple crank mechanism.

Let the slot be a straight line passing through the point $q = (l, 0)$ by an angle α with respect to the x-axis of the base frame.

(a) Find the vector $\vec{n}$ normal to the direction of the slot. Denote the link lengths as l_1 and l_2: derive the end-effector position $p_e^0(\theta_1, \theta_2) \in \mathbb{R}^2$ from forward kinematics with respect to the base frame. **(5)**

(b) Compute the motion constraint on the robot as $h(\cdot) = 0$ requiring that the position of the end effector remains in the slot. **(5)**

(c) Explain whether the motion constraint is holonomic or nonholonomic and mention implications for the system dynamics. **(5)**

Solution:

(a) The vector normal to the direction of the slot is given by

$$\vec{n} = \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix} .$$

From forward kinematics we find the end-effector position as

$$p_e^0(\theta_1, \theta_2) = \begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} .$$

(b) The constraint that keeps the end effector moving in the slot is described by the set of all points $p_e^0(\theta_1, \theta_2) \in \mathbb{R}^2$ for which the vector $(p_e^0(\theta_1, \theta_2) - q)$ is orthogonal to the normal vector on the slot $\vec{n}$. Hence, the constraint becomes

$$h(\theta_1, \theta_2) = \left(\begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} - \begin{bmatrix} l \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix} = 0 .$$

(c) The motion constraint is holonomic as it is expressed with respect to configuration variables only. Here we have two configuration variables and one holonomic constraint so that the motion of the system lies on a $(2-1)$-dimensional hypersurface defined by the level set of $h(\theta(t)) = 0$ for all $t > 0$. Differentiating $h(\theta)$ along a motion we obtain the differential relations $< dh(\theta), \dot{\theta} >= 0$ where $dh(\theta)$ is the normal vector on the hypersurface and $\dot{\theta}$ are the directions of feasible motion. Forces applied in the direction of $dh(\theta)$ do no work and therefore cause no motion normal to the slot. It is possible to parameterize a motion on the hypersurface by recuded dynamics for only one configuration variable, for instance, by resolving the constraint force after substituting a function $\theta_2 = f(\theta_1)$ from a smooth map $f : h(\theta_1, \theta_2) \to \theta_2$ or directly from inverse kinematics $\theta = f(s)$ with s being the position along the slot into the system dynamics together with corresponding derivatives.

5. **Problem:** Given a mechanical system in standard matrix form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u \ , \quad q \in \mathbb{R}^n$$

and the desired trajectory $q_d = q_d(t)$, introduce an inverse dynamics control law for u that stabilizes the error dynamics. **(5)**

Solution: The joint space inverse dynamics control law has the form

$$\begin{aligned} u &= M(q)a_q + C(q, \dot{q})\dot{q} + G(q) \\ a_q &= \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \end{aligned}$$

where $K_p > 0$ and $K_d > 0$ are feedback gains on the position and velocity errors, respectively. Then the closed loop system is

$$\ddot{q} = a_q = \ddot{q}_d(t) + K_d\big(\dot{q}_d(t) - \dot{q}\big) + K_p\big(q_d(t) - q\big) \ ,$$

which can be rewritten in error variables as

$$\ddot{e} + K_d\dot{e} + K_p e = 0 \ , \quad e = q_d(t) - q$$

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -K_p & -K_d \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} .$$

It can be shown that the error dynamics vanish exponentially.

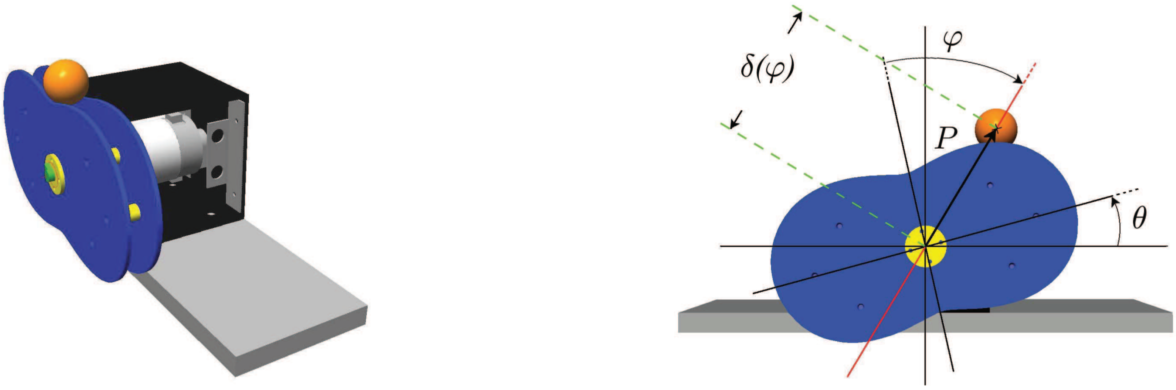


Figure 5: Schematic views of the Butterfly robot built at Universita di Roma 'La Sapienza' by Prof. Guiseppe Oriolo. The angle θ represents the rotation of the two rigidly connected identical plates with respect to horizontal; the angle φ characterizes the relative position P of the center of mass of the ball; $\delta(\varphi)$ is a known smooth function describing the distance to the ball from the axis of rotation according to the geometric contour of the plates.

6. **Problem:** Consider the two-degree-of-freedom planar robot in Fig. 5.

   The so-called butterfly robot consists of the following parts:

   - two identical symmetric plates (*blue* in Fig. 5), which are fixed in parallel to shape one rigid body and which can be rotated by a DC-motor about the center point (*yellow* in Fig. 5);
   - one ball (*orange* in Fig. 5), which can freely roll along the contour of the plates. Its position is parameterized by a given smooth function $\delta(\varphi)$ representing the distance between the center of mass of the ball and the axis of rotation of the plates.

   The robot is manufactured in such a way that the center of mass of the combined plates coincides with the axis of rotation. The dynamics of the electric motor is fast enough and the effect of friction is pre-compensated so that the control signal is equal to an external torque applied to the axis of rotation of the plates.

   <u>You are requested to complete the following tasks:</u>

   Suppose that the ball is modeled as a point mass m and the moment of inertia of the combined plates around the axis of rotation equals to J.

   (a) Derive the potential energy $\mathcal{P}(\theta, \varphi)$ of the robot. **(5)**

   (b) Derive the kinetic energy $\mathcal{K}(\theta, \varphi)$ of the robot. **(10)**

   (c) Derive the Euler-Lagrange equations of the system dynamics. **(15)**

   (d) Derive the equations that must be solved to find the value of the angle φ_{des} describing the unstable equilibria ($\theta_e = 0,\ \varphi_e = \pm\varphi_{des}$) depicted in Fig 6. **(5)**

   Solution:

   (a) The potential energy $\mathcal{P}(\cdot)$ of the robot is the sum of potential energies of the links

$$\mathcal{P} \quad = \quad mg\delta(\phi)\sin(\theta + \pi/2 - \phi) = mg\delta(\phi)\cos(\theta - \phi)$$

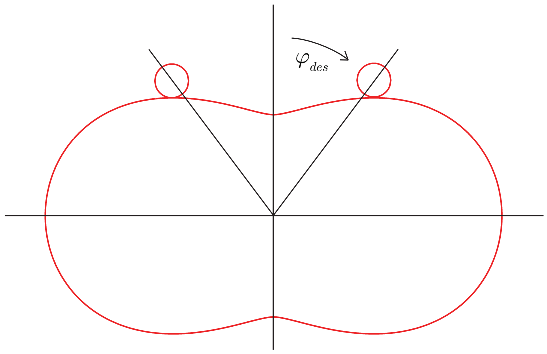


Figure 6: The configuration of the Butterfly robot for which the system has one stable equilibrium at $\theta_e = \varphi_e = 0$ and two unstable ones at $\theta_e = 0$, $\varphi_e = \pm\varphi_{des}$.

(b) The kinetic energy of the robot is equal to the sum kinetic energies of links

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$$

where

$$\begin{aligned} \mathcal{K}_1 &= \frac{1}{2}J\dot{\theta}^2 \\ \mathcal{K}_2 &= \frac{1}{2}m\delta^2(\phi)(\dot{\theta}-\dot{\phi})^2 + \frac{1}{2}m(\delta_\phi'(\phi))^2\dot{\phi}^2 \end{aligned}$$

Summing the terms we obtain

$$\mathcal{K} = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\delta^2(\phi)(\dot{\theta}-\dot{\phi})^2 + \frac{1}{2}m(\delta_\phi'(\phi))^2\dot{\phi}^2$$

(c) To derive the Euler-Lagrange equations, introduce the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

and compute partial derivatives of $\mathcal{L}$ with respect to variables θ, ϕ, i.e.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= J\dot{\theta} + m\delta^2(\phi)(\dot{\theta}-\dot{\phi}) \\ \frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= -m\delta^2(\phi)(\dot{\theta}-\dot{\phi}) + m(\delta_\phi'(\phi))^2\dot{\phi} \\ \frac{\partial\mathcal{L}}{\partial\theta} &= mg\delta(\phi)\sin(\theta-\phi) \\ \frac{\partial\mathcal{L}}{\partial\phi} &= m\delta(\phi)\delta_\phi'(\phi)(\dot{\theta}-\dot{\phi})^2 + m\delta_\phi'(\phi)\delta_\phi''(\phi)\dot{\phi}^2 - \\ & \quad mg\delta_\phi'(\phi)\cos(\theta-\phi) - mg\delta(\phi)\sin(\theta-\phi) \end{aligned}$$

The Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] - \frac{\partial\mathcal{L}}{\partial\theta} &= \tau_1 \\ \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right] - \frac{\partial\mathcal{L}}{\partial\phi} &= 0 \end{aligned}$$

where

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] &= J\ddot{\theta} + 2m\delta(\phi)\delta_{\phi}^{'}(\phi)\dot{\phi}(\dot{\theta} - \dot{\phi}) + m\delta^2(\phi)(\ddot{\theta} - \ddot{\phi}) \\
\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] &= -2m\delta(\phi)\delta_{\phi}^{'}(\phi)\dot{\phi}(\dot{\theta} - \dot{\phi}) - m\delta^2(\phi)(\ddot{\theta} - \ddot{\phi}) + \\
& \quad m(\delta_{\phi}^{'}(\phi))^2\ddot{\phi} + 2m\delta_{\phi}^{'}(\phi))\delta_{\phi}^{''}(\phi))\ddot{\phi}^2
\end{aligned}
$$

(d) In order to find a point of unstable equilibrium, we assume that $\ddot{\phi} = 0$, $\dot{\phi} = 0$, $\ddot{\theta} = 0$, $\dot{\theta} = 0$ in the second Euler-Lagrange equation. We obtain:

$$
\begin{aligned}
-mg\delta_{\phi}^{'}(\phi)\cos(-\phi) + mg\delta(\phi)\sin(-\phi) = 0 \\
-\delta_{\phi}^{'}(\phi)\cos(-\phi) + \delta(\phi)\sin(-\phi) = 0
\end{aligned}
$$

This equation can be solved to find the value of the angle ϕ_{des} ($\phi_{des} \neq 0$) for the unstable equilibriums.