TTK4195
Modeling and Control of Robots
Assignment 9

Solution

April 8, 2014

Problem 1

We start by defining the state vector, this vector can be comprised by four states that contain information about joint variables and their derivatives. By simplicity and compactness let us combine the angles into one state and the derivatives into another so that

$$x_1 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad \Rightarrow \quad x = \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix}$$

Moreover, if we rewrite the EoM, we can get the joint accelerations on the left hand side.

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \dot{x}_2 = -M(x_1)^{-1}\left[C(x_1, x_2)x_2 + G(x_1)\right] + M(x_1)^{-1}B(x_1)u$$

And since we are only interested in measuring the position variables q, the output function becomes

$$y = x_1$$

The state space model is now given in standard form by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 \\ -M(x_1)^{-1}\left[C(x_1, x_2)x_2 + G(x_1)\right] \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ M(x_1)^{-1}B(x_1) \end{bmatrix} \\ h(x) &= x_1 \end{aligned}$$

Problem 2

We find all the equilibrium points by setting the differential equations to zero and solve for the states. From the first state equation, and from a physical point of view, it is easy to see that

$$\dot{x}_1 = x_2^* = 0$$

Furthermore, we have that

$$\dot{x}_2 = -M(x_1^*)^{-1}\left[C(x_1^*, x_2^*)x_2^* + G(x_1^*)\right] = 0$$

$$M(x_1)^{-1} \neq 0, \text{ or equivalently } M(x_1) \neq 0$$
$$\Downarrow$$
$$C(x_1^*, x_2^*)x_2^* + G(x_1^*) = 0$$

$$x_2^* = 0, \quad \Rightarrow \quad G(x_1^*) = 0$$

So, not surprisingly, we find that the equilibriums of the pendubot is only dependant on the gravity torque vector that tells us something about the potential energy of the robot. There are only certain configurations where the robot remains still. Let us find these configurations mathematically

$$\begin{bmatrix} p_4\cos(q_1^*) + p_5\cos(q_1^* + q_2^*) \\ p_5\cos(q_1^* + q_2^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since all the p's are positive definite we get that

$$\begin{bmatrix} \cos(q_1^*) \\ \cos(q_1^* + q_2^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Several equilibria exists. First, we note that two configurations of q_1 is possible (not counting multiple revolutions), these are

$$q_1^* = \frac{\pi}{2}, \quad \text{and} \quad q_1^* = \frac{3\pi}{2}$$

From these two we can find the remaining configurations of q_2

$$\underline{q_1^* = \frac{\pi}{2}}$$

$$q_2^* = \frac{\pi}{2} - \frac{\pi}{2} = 0$$
$$q_2^* = \frac{3\pi}{2} - \frac{\pi}{2} = \pi$$

$$\underline{q_1^* = \frac{3\pi}{2}}$$

$$q_2^* = \frac{\pi}{2} - \frac{3\pi}{2} = -\pi$$
$$q_2^* = \frac{3\pi}{2} - \frac{3\pi}{2} = 0$$

We end up with four equilibrium points for the pendubot. In physical terms, they describe the robots first limb as standing up or hanging down with the second limb either standing or hanging in both those cases. In summary, we have:

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} \frac{\pi}{2} \\ \pi \end{bmatrix} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} \frac{3\pi}{2} \\ -\pi \end{bmatrix} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} \frac{3\pi}{2} \\ 0 \end{bmatrix} \\ 0 \end{bmatrix}$$

Problem 3

First, we define deviation variables Δx, Δu and Δy.

$$\begin{aligned} \Delta x &= x - x^* \\ \Delta u &= u - u^* \\ \Delta y &= y - h(x^*) \end{aligned}$$

A Taylor expansion about the equilibrium $(x^*\ u^*)$ yields

$$\dot{x} \approx f(x^*) + g(x^*)u^* + \left[\frac{\partial}{\partial x}(f(x) + g(x)u)\right]_{x^*,u^*} \Delta x + \left[\frac{\partial}{\partial u} g(x)u\right]_{x^*,u^*} \Delta u$$

$f(x^*) + g(x^*)u^* = 0$ and renaming the deviation variables to z, ω and v, we get the linear state space system

$$\begin{aligned} \dot{z} &= Az + H\omega \\ v &= Cz \end{aligned}$$

Where the system matrix A and input matrix H can be computed as

$$A = \begin{bmatrix} \left[\frac{\partial}{\partial x_1}(f_1(x) + g_1(x)u)\right]_{x^*,u^*} & \left[\frac{\partial}{\partial x_2}(f_1(x) + g_1(x)u)\right]_{x^*,u^*} \\ \left[\frac{\partial}{\partial x_1}(f_2(x) + g_2(x)u)\right]_{x^*,u^*} & \left[\frac{\partial}{\partial x_2}(f_2(x) + g_2(x)u)\right]_{x^*,u^*} \end{bmatrix}$$

$$H = \begin{bmatrix} \left[\frac{\partial}{\partial u} g_1(x)u\right]_{x^*,u^*} \\ \left[\frac{\partial}{\partial u} g_2(x)u\right]_{x^*,u^*} \end{bmatrix}$$

Here, we must expand the state vector to obtain all four states.

$$
\begin{aligned}
\frac{\partial}{\partial x_{11}}(f_1(x) + g_1(x)u) = 0 \\
\frac{\partial}{\partial x_{12}}(f_1(x) + g_1(x)u) = 0 \\
\frac{\partial}{\partial x_{21}}(f_1(x) + g_1(x)u) = 1 \\
\frac{\partial}{\partial x_{22}}(f_1(x) + g_1(x)u) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_{11}}(f_2(x) + g_2(x)u) = 0 \\
\frac{\partial}{\partial x_{12}}(f_2(x) + g_2(x)u) = 0 \\
\frac{\partial}{\partial x_{21}}(f_2(x) + g_2(x)u) = 0 \\
\frac{\partial}{\partial x_{22}}(f_2(x) + g_2(x)u) = 1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_1}(f_{3,4}(x) + g_{3,4}(x)u) &= \frac{\partial}{\partial x_1}\left(-M(x_1)^{-1}\left[C(x_1, x_2)x_2 + G(x_1)\right] + M(x_1)^{-1}B(x_1)u\right) \\
\frac{\partial}{\partial x_2}(f_{3,4}(x) + g_{3,4}(x)u) &= \frac{\partial}{\partial x_2}\left(-M(x_1)^{-1}\left[C(x_1, x_2)x_2 + G(x_1)\right] + M(x_1)^{-1}B(x_1)u\right)
\end{aligned}
$$

For convenience sake we use MATLAB to differentiate and obtain the linearized system.

$$
\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{p_2p_4 - p_3p_5}{p_1p_2 - p_3^2} & -\frac{p_3p_5}{p_1p_2 - p_3^2} & 0 & 0 \\ \frac{p_1p_5 - p_2p_4 - p_3p_4 + p_3p_5}{p_1p_2 - p_3^2} & \frac{p_5(p_1 + p_3)}{p_1p_2 - p_3^2} & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \frac{p_2}{p_1p_2 - p_3^2} \\ -\frac{p_2 + p_3}{p_1p_2 - p_3^2} \end{bmatrix} \omega \tag{2}
$$

$$
v = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} z \tag{3}
$$

Problem 4

The linearized system (2) is controllable if the rank of the controllability matrix is full (i.e. equals the order of the system). We compute the controllability matrix with the help of the `ctrb`-command in MATLAB. The system is controllable since the rank turned out to be 4, which is the same as the order of the system.

Asymptotic (in fact exponential) stability of the linearized closed loop system depends on the poles of the closed loop system. All poles must lie strictly in the left half plane of the complex plane. Let us start by defining a general full-state feedback control law

$$\omega = -Kz$$

Closing the loop we get

$$\dot{z} = Az + H(-Kz) = (A - HK)z$$

The $A - HK$-matrix must be Hurwitz in order to ensure local stability of the non-linear system, this can be done with the help of the `place`-command in MATLAB. Furthermore, this control law assumes that all states can be fed back, one must therefore also implement an observer to obtain all states.

$$K = 1 \cdot 10^3[-6.135 \ \ -3.048 \ \ -0.368 \ \ -0.178]$$

Using this feedback matrix will stabilize the upright-upright equilibrium. The poles were chosen to be uniformly distributed on a half circle in the left half plane with a radius of 50.

Problem 5

Full feedback linearization is not possible for this system because the evolution of one of the joint variables remains unaffected by the linearizing control law from input to output. Consider the second time derivatives of the output

$$\ddot{y}_1 = f_3(x) + g_3(x)u$$
$$\ddot{y}_2 = f_4(x) + g_4(x)u$$

and the linearizing control for y_1 with the auxiliary control variable v

$$u = g_3^{-1}(x)[v - f_3(x)]$$

then the dynamics of y_2 become

$$\ddot{y}_2 = f_4(x) + g_4(x)g_3^{-1}(x)[\ddot{y}_1 - f_3(x)]$$

It means that the partially linearized system has some internal dynamics that is uncontrollable.

Problem 6

Substitute the following functions

$$q_1 = kq_2$$
$$\dot{q}_1 = k\dot{q}_2$$
$$\ddot{q}_1 = k\ddot{q}_2$$

to equation second of the EoM and the dynamics can be reduced to only one differential equation of second order the following form:

$$\alpha(q_2)\ddot{q}_2 + \beta(q_2)\dot{q}_2^2 + \gamma(q_2) = 0$$

Analysing obtained above differential equation we can see that the motion of q_2, and inherently q_1, is constrained.