Solutions for Exam Problems in the Course TTK4195: Robots Modeling and Control

Monday, May 28, 2013

1. **Problem:** Consider the following sequence of rotations

   (a) Rotate by ϕ about the world axis x;

   (b) Rotate by θ about the world axis y;

   (c) Rotate by ψ about the current axis z;

   (d) Rotate by α about the world axis y

   Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication)

   Solution: The total rotation is

   $$R = R_{y,\alpha} \cdot R_{y,\theta} \cdot R_{x,\phi} \cdot R_{z,\psi}$$

   where

   $$R_{x,q}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q & -\sin q \\ 0 & \sin q & \cos q \end{bmatrix}, R_{y,q}=\begin{bmatrix} \cos q & 0 & \sin q \\ 0 & 1 & 0 \\ -\sin q & 0 & \cos q \end{bmatrix}, R_{z,q}=\begin{bmatrix} \cos q & -\sin q & 0 \\ \sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. **Problem:** Given three matrices

   $$T_1 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad T_2 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad T_3 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

   (a) Which of these matrices are orthogonal? **(5)**

   (b) Which of these matrices do represent a rotation? **(5)**

   Solution: (a) By definition, a matrix O is orthogonal if $OO^T = I$.

Computing the matrices $T_iT_i^T$ for $i = 1, 2, 3$ you can find that

$$T_1T_1^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{7}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{7}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$T_2T_2^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3T_3^T = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the matrices T_2 and T_3 are orthogonal, but T_1 is not.

(b) By definition, an orthogonal matrix T is a rotation if $\det T = 1$. Computing $\det T_2$ and $\det T_3$, we find that

$$\det T_2 = 1, \qquad \det T_3 = -1$$

Hence, T_2 is a rotation, but T_3 is not.

3. **Problem:** A motion of the center of mass of a rigid body with respect to an inertia frame is a rotation

$$R_1(t) = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}$$

Find an angular velocity of the motion **(5)**. Compute the vector of angular velocity when the rotation is **(10)**

$$R_2(t) = \begin{bmatrix} \cos t & -(\sin t)\cdot(\cos 2t) & (\sin t)\cdot(\sin 2t) \\ \sin t & (\cos t)\cdot(\cos 2t) & -(\cos t)\cdot(\sin 2t) \\ 0 & \sin 2t & \cos 2t \end{bmatrix}$$

Solution: (a) $R_1(t)$ is the basic rotation $R_{y,\phi(t)}$ with $\phi(t) = -t$. Hence

$$\vec{\omega}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{d}{dt}\phi(t) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{1}$$

An alternative way to compute $\vec{\omega}_1(t)$ is to use the formula

$$\left[\tfrac{d}{dt}R(t)\right] R(t)^T = S(\vec{\omega}(t)) = \begin{bmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{bmatrix}, \; \vec{\omega}(t) = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} \tag{2}$$

For $R_1(t)$ we obtain

$$\begin{aligned} \left[\tfrac{d}{dt}R_1(t)\right] R_1(t)^T &= \begin{bmatrix} -\sin t & 0 & -\cos t \\ 0 & 0 & 0 \\ \cos t & 0 & -\sin t \end{bmatrix} \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

which gives the same results as in (1).

To compute the angular velocity $\vec{\omega}_2(t)$ for the rotation $R_2(t)$ you can again use the formula (2). However, the easier way the solve the problem is to observe that

$$R_2(t) = \underbrace{\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_1^0(t)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{bmatrix}}_{R_2^1(t)} \tag{3}$$

Then the vector of angular velocity can be computed from the relation

$$\vec{\omega}_2(t) = \vec{\omega}_1^0(t) + R_2^1(t)\vec{\omega}_2^1(t)$$

where the vectors $\vec{\omega}_1^0(t)$ and $\vec{\omega}_2^1(t)$ are angular velocities of the transforms $R_1^0(t)$ and $R_2^1(t)$ respectively. $R_1^0(t)$ and $R_2^1(t)$ are basic rotations around z and x axes, hence

$$\vec{\omega}_1^0(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{\omega}_2^1(t) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the vector of angular velocity $\vec{\omega}_2(t)$ is

$$\vec{\omega}_2(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 1 \end{bmatrix}$$

4. **Problem:** Consider a robot depicted on Fig. 1. Derive the forward kinematic equations for this manipulator. In particular, introduce a DH-frame for each of the links of the robots, compute DH-parameters of homogeneous transforms between consecutive frames, presenting the solution of the problem do not perform multiplication of such transforms. **(20)**

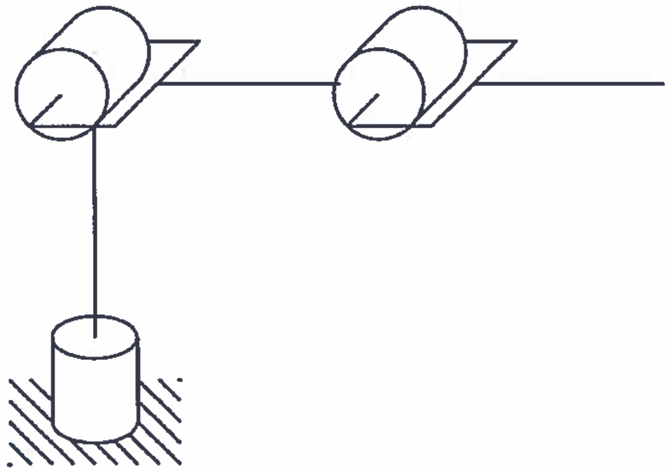

Figure 1: A robot from the problem 4: elbow manipulator

Solution: The parameters of sequence of homogeneous transforms written (as functions of angles q_1, q_2, q_3) in the DH-convention, i.e.

$$A = R_{z,\theta} \cdot T_{z,d} \cdot T_{x,a} \cdot R_{x,\alpha}$$

have the form

$$\begin{array}{lllll} A_1^0: & \theta = q_1, & d = d_1, & a = 0, & \alpha = \frac{\pi}{2} \\ A_2^1: & \theta = q_2, & d = 0, & a = a_2, & \alpha = 0 \\ A_3^2: & \theta = q_3, & d = 0, & a = a_3, & \alpha = 0 \end{array}$$

5. **Problem:** The pendulum on the cart system is depicted on Fig. 2. It has two degrees of freedom: the position x of center of mass of the cart on the horizontal and the angle θ of the pendulum with vertical counted anticlockwise. It is assumed that the rod of the pendulum is massless and the mass of the pendulum is all concentrated in the bob. While the

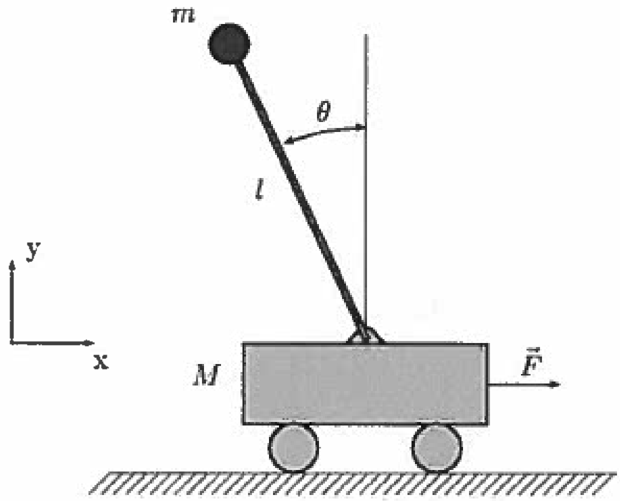

Figure 2: A robot from the problem 3: the cart-pendulum system

only one control variable in the system is the force applied to the cart along the horizontal direction.

Given physical parameters of the system (mass of the cart M, the mass of the bob m, the distance l from the suspension to the position of the bob), you are requested to implement the following tasks:

(a) Find the potential energy $\mathcal{P}(x, \theta)$ of the robot **(5)**

(b) Find the kinetic energy $\mathcal{K}(x, \theta, \dot{x}, \dot{\theta})$ of the robot **(10)**

(c) Obtain the Euler-Lagrange equations of the system dynamics **(15)**

(d) Assume that $M = m = 1$ [kg], $l = 1$ [m]. Compute the linearization of the system dynamics around the upright equilibrium $\{x_e = 0,\, \theta_e = 0\}$. Check, if the resulted linear control system is controllable. **(20)**

Solution: Before computing the kinetic and potential energies of the cart-pendulum system, let us observe that coordinates of centers of masses of the cart and the pendulum are $x_c = x$, $y_c = 0$ and

$$x_p = x_c - l \cdot \sin(\theta), \qquad y_p = l \cdot \cos(\theta)$$

and, therefore, their velocities are $\dot{x}_c = \dot{x}$, $\dot{y}_c = 0$ and

$$\dot{x}_p = \dot{x}_c - l \cdot \cos(\theta) \cdot \dot{\theta}, \qquad \dot{y}_p = -l \cdot \sin(\theta) \cdot \dot{\theta}$$

(a) The potential energy $\mathcal{P}(\cdot)$ of the robot is the potential energy of the pendulum

$$\mathcal{P} \;=\; mgy_p = mgl\cos(\theta)$$

(b) The kinetic energy of the robot is equal to the sum kinetic energies of the cart and the pendulum

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \tfrac{1}{2}M\left(\dot{x}_c^2 + \dot{y}_c^2\right) = \tfrac{1}{2}M\dot{x}^2 \\
\mathcal{K}_2 &= \tfrac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right) = \tfrac{1}{2}m\left(\dot{x} - l\cdot\cos(\theta)\cdot\dot{\theta}\right)^2 + \tfrac{1}{2}m\left(l\cdot\sin(\theta)\cdot\dot{\theta}\right)^2
\end{aligned}$$

Summing the terms we obtain

$$\mathcal{K} = \tfrac{1}{2}\cdot[M+m]\cdot\dot{x}^2 + \tfrac{1}{2}\cdot m\cdot l^2\cdot\dot{\theta}^2 - m\cdot l\cdot\cos(\theta)\cdot\dot{x}\cdot\dot{\theta}$$

(c) To derive the Euler-Lagrange equations, introduce the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

and compute partial derivatives of $\mathcal{L}$ with respect to variables x, θ, $\dot{x}$ and $\dot{\theta}$, i.e.

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = (M+m)\dot{x} - ml\cos(\theta)\dot{\theta}, \qquad \frac{\partial\mathcal{L}}{\partial x} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ml^2\dot{\theta} - ml\cos(\theta)\dot{x} \qquad \frac{\partial\mathcal{L}}{\partial\theta} = ml\sin(\theta)\dot{\theta}\dot{x} + mlg\sin(\theta)$$

The Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] - \frac{\partial\mathcal{L}}{\partial x} &= F \\
\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] - \frac{\partial\mathcal{L}}{\partial\theta} &= 0
\end{aligned}$$

are then

$$\begin{aligned} (M+m)\ddot{x} - ml\cos(\theta)\,\ddot{\theta} + ml\sin(\theta)\,\dot{\theta}^2 &= F \\ -ml\cos(\theta)\,\ddot{x} + ml^2\,\ddot{\theta} - mgl\sin(\theta) &= 0 \end{aligned}$$

or the same in the matrix form

$$\underbrace{\begin{bmatrix} M+m & -ml\cos(\theta) \\ -ml\cos(\theta) & ml^2 \end{bmatrix}}_{:=D} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F - ml\sin(\theta)\,\dot{\theta}^2 \\ mgl\sin(\theta) \end{bmatrix}$$

Multiplying both sides of the equation with D^{-1}, which is

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} = \frac{1}{\det D} \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}, \quad \det D = d_{11}d_{22} - d_{12}d_{21},$$

we resolve the equations with respect to the 2nd time derivatives

$$\begin{aligned} \ddot{x} &= \frac{d_{22}}{\det D}\left[F - ml\sin(\theta)\,\dot{\theta}^2\right] + \frac{-d_{12}}{\det D}\,mgl\sin(\theta) \\ \ddot{\theta} &= \frac{-d_{21}}{\det D}\left[F - ml\sin(\theta)\,\dot{\theta}^2\right] + \frac{d_{11}}{\det D}\,mgl\sin(\theta) \end{aligned} \tag{4}$$

(d) With new variables

$$z_1 = x, \quad z_2 = \theta, \quad z_3 = \dot{x}, \quad z_4 = \dot{\theta}$$

and assuming that parameters are $M = m = 1$ [kg], $l = 1$ [m], the dynamics (4), that is now

$$\begin{aligned} \ddot{x} &= \frac{1}{1+\sin^2\theta}\left[F - \sin(\theta)\,\dot{\theta}^2\right] + \frac{\cos\theta}{1+\sin^2\theta}\,g\sin(\theta) \\ \ddot{\theta} &= \frac{\cos\theta}{1+\sin^2\theta}\left[F - \sin(\theta)\,\dot{\theta}^2\right] + \frac{2}{1+\sin^2\theta}\,g\sin(\theta) \end{aligned}$$

can be rewritten in state-space form as

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ \frac{\sin(z_2)\cdot(g\cdot\cos(z_2) - z_4^2)}{1+\sin^2(z_2)} \\ \frac{\sin(z_2)\cdot(2g - \cos(z_2)\cdot z_4^2)}{1+\sin^2(z_2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{1+\sin^2(z_2)} \\ \frac{\cos(z_2)}{1+\sin^2(z_2)} \end{bmatrix} F \tag{5}$$

Linearization of the system (5) around the equilibrium

$$z_1 = z_2 = z_3 = z_4 = 0$$

has the form

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{=A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix}}_{=B} v \qquad (6)$$

Here the coefficients a_{3i} and a_{4i} with $i = 1, \ldots, 4$ are defined as

$$\begin{aligned} a_{3i} &= \frac{\partial}{\partial z_i}\left[\frac{\sin(z_2)\cdot(g\cdot\cos(z_2) - z_4^2)}{1+\sin^2(z_2)}\right]\Bigg|_{z_1=\cdots=z_4=0} && (7) \\ a_{4i} &= \frac{\partial}{\partial z_i}\left[\frac{\sin(z_2)\cdot(2g - \cos(z_2)\cdot z_4^2)}{1+\sin^2(z_2)}\right]\Bigg|_{z_1=\cdots=z_4=0} && (8) \end{aligned}$$

and

$$b_1 = \frac{1}{1+\sin^2(z_2)}\Bigg|_{z_1=\cdots=z_4=0} = 1, \quad b_2 = \frac{\cos(z_2)}{1+\sin^2(z_2)}\Bigg|_{z_1=\cdots=z_4=0} = 1 \quad (9)$$

Computing (7) and (8), we obtain

$$\begin{aligned} a_{31} = a_{33} = a_{34} = 0, \qquad a_{32} = g \qquad (10) \\ a_{41} = a_{43} = a_{44} = 0, \qquad a_{42} = 2g \qquad (11) \end{aligned}$$

As a result, matrices A and B take the form

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g & 0 & 0 \\ 0 & 2g & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad (12)$$

The linear system is controllable if the matrix

$$\Gamma = [B, AB, A^2B, A^3B] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ g \\ 2g \end{bmatrix}, \begin{bmatrix} g \\ 2g \\ 0 \\ 0 \end{bmatrix}\right]$$

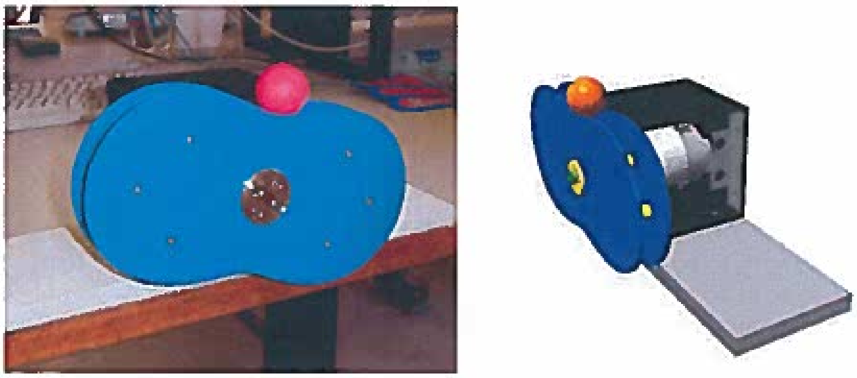

Figure 3: The photo and schematic view of the butterfly robot from the problem 6

is of full rank. Straightforward calculations show that its determinant is equal

$$\det \Gamma = -g^2 \neq 0$$

Hence, the linearization of the cart-pendulum system around its unstable upright equilibrium is controllable.

6. **Problem:** Consider the planar robot (the so-called butterfly robot built at Universita di Roma 'La Sapienza' by Prof. G. Oriolo), which photo and schematic view are depicted on Fig. 3. The robot consists of the following parts:

   - two identical wooden plates (blue on Fig.3), which are fixed in parallel to shape one rigid body and which can be rotated by a DC-motor about a center point
   - one ball (red/yellow on Fig.3), which can freely roll following the contour of the plates

   Suppose that the ball is modeled as a disk of a mass m rolling without slipping on the contour, see Fig. 4. Answer the following questions

   (a) Given a motion of the system for which the ball is always in contact with the contour, what is the velocity of the contact point between the disc and the contour in the inertia frame? Clarify the answer **(3)**

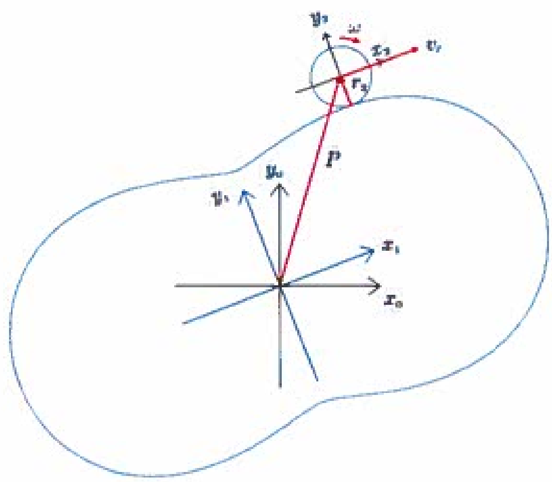

Figure 4: The schematic view of the butterfly robot from the problem 6, when the ball is modeled as a disc rolling on the contour.

(b) Is this mechanical system is holonomic or not? Clarify the answer **(4)**

(c) How many degrees of freedom does the system have? Clarify the answer **(4)**

(d) Under what conditions would the ball depart from the contour? Clarify the answer **(5)**

Solution:

(a) Since the ball is rolling on the frames' contour without slipping the velocities of the contact points at the ball and the contour are the same and equal to $[r \cdot \dot{\theta}]$, where r is the distance to the contact point from the axis of rotation of the frames.

(b) The dynamics of the system has a nontrivial velocity constraint at the contact point between the ball and the contour it rolls on. However this velocity constraint is holonomic and can be integrated. Indeed, since the ball rolls on a the contour without slipping the position of its center of mass and the angle of rotation can be uniquely computed from the path length it went along the contour.

(c) The system has two degrees of freedom. The position of the ball's center of mass and its angle of rotation can be computed from the path the ball went along the contour without slipping.

(d) In order to loose a contact with the surface the force (a vector with two nontrivial components in this example), which the ball applies to the frames at the contact point should have zero and then positive projection on a normal to the contour at the contact point drawn outside the frames. At this moment of time the ball will depart from the surface.