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EXAM FOR COURSE TTK 4195
Robots Modeling and Control

Tuesday, May 21, 2010
Time: 09.00-13.00

Allowable aids: D - No printed or written material allowed. NTNU type approved calculator with an empty memory allowed. Language: English. Number of Pages: 4 (+ 4 formula sheet). This exam counts for 100% of the final grade.

This exam consists of 4 exercises each consisting of a number of questions. Each question gives a number of points and a sum of points is 100.

1. **Problem:** Given four orthogonal frames with the same origin $Ox_1y_1z_1$, $Ox_2y_2z_2$, $Ox_3y_3z_3$, $Ox_4y_4z_4$, suppose rotation matrices R_2^1, R_4^3 and R_4^1 that relate the orientations of the frames are

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_4^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_4^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the rotation matrix R_3^2. **(5)**

2. **Problem:** Consider the following sequence of rotations

(a) Rotate by ϕ about the world axis y;

(b) Rotate by θ about the world axis z;

(c) Rotate by ψ about the current axis x;

(d) Rotate by α about the world axis y

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication) **(5)**

3. **Problem:** Consider a robot depicted on Fig. 1

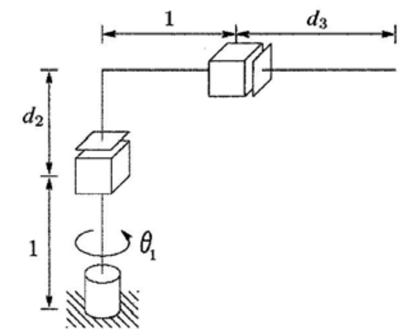

Figure 1: A robot of the RPP-type.

✓ (a) For each link introduce the frame following DH-convention and derive forward kinematics equations **(10)**

✓ (b) Solve the inverse position kinematics problem, i.e. given coordinates (position) of the origin of the tool frame of the cylindrical robot in the base frame $p_e = (x_e, y_e, z_e) \in \mathbb{R}^3$, find corresponding values for angle θ_1 and extensions d_2, d_3, with which the origin of the tool frame is in the requested position p_e. **(10)**

✓ (c) Compute the manipulator Jacobian for representation of linear and angular velocity of the origin of the tool frame, which is located at the end of the second prismatic link of the robot (see Fig. 1). **(10)**

✓ (d) Compute the total velocity of the origin of the tool frame when the variables θ_1, d_2 and d_3 are changing with time as follows

$$\theta_1(t) = \cos(3t), \quad d_2(t) = \sin(2t), \ . \ d_3(t) = \cos(t)$$

Computation can be based on the Jacobian computed on the previous step or the vector of velocity of this point can be computed directly. **(10)**

4. **Problem:** Consider the two link planar robot – the so-called inertia wheel pendulum – depicted on Fig. 2. It has two revolute joints, while the center of mass of the second link coincide with its suspension point.

   Given physical parameters of the system (masses of the links m_1, m_2; inertias I_1, I_2; the length of the first link l_1 and the distance to its center of mass l_{c1}), you are requested to implement the following tasks:

   ✓ (a) Find the potential energy $\mathcal{P}(q_1, q_2)$ of the robot. **(10)**

   ✓ (b) Find the kinetic energy $\mathcal{K}(q_1, q_2, \dot{q}_1, \dot{q}_2)$ of the robot. **(10)**

   ✓ (c) Obtain the Euler-Lagrange equations of the system dynamics. **(10)**

   ✓ (d) Compute the linearization of the system dynamics around the upright equilibrium $q_{1e} = q_{2e} = 0$ when both the first and the second joints of the robot are actuated. Is the resulted linear control system controllable? **(15)**

   (e) Assume that there is only one actuator that provides a control torque to change angular acceleration of the first link, while the second link is passive. Check that for this case the system dynamics is feedback linearizable. **(5)**

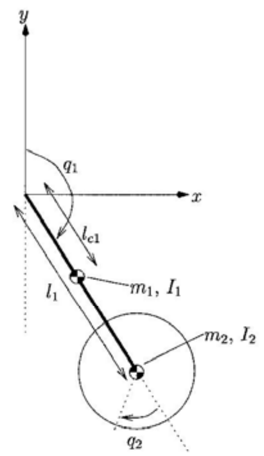

Figure 2: Robot from the problem 4: inertia wheel pendulum

List of Formulas for
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Robots Modeling and Control

1. If we are given $(N-1)$-rotation matrices

$$R_1^0, \quad R_2^1, \quad \ldots, \quad R_{(N-1)}^{(N-2)}$$

that represent consecutive rotations between the current frames

$$\left\{(x_0 y_0 z_0), (x_1 y_1 z_1)\right\}, \left\{(x_1 y_1 z_1), (x_2 y_2 z_2)\right\}, \ldots, \\ \left\{(x_{N-2} y_{N-2} z_{N-2}), (x_{N-1} y_{N-1} z_{N-1})\right\}$$

The formula to compute the position of the point in the 0-frame having known its position in the $(N-1)$-frame is

$$p^0 = R_1^0 R_2^1 R_3^2 \cdots R_{(N-1)}^{(N-2)} p^{(N-1)}$$

2. If there are 3 frames corresponding to 2 rigid motions

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \end{aligned}$$

then the overall motion is

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

3. Homogeneous transform A_i in DH-convention is a product

$$A_i = \text{Rot}_{z,\theta_i} \cdot \text{Trans}_{z,d_i} \cdot \text{Trans}_{x,a_i} \cdot \text{Rot}_{x,\alpha_i}$$

$$= \left[\begin{array}{ccc|c} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

with the following parameters: the link length a_i, the link twist α_i, the link offset d_i, the link angle θ_i.

4. If $R(t) \in \mathcal{SO}(3)$ is time-varying and has a time-derivative, then

$$\tfrac{d}{dt}R(t) = S(t)R(t) = S(\omega(t))R(t),$$

where $S(t)$ is skew symmetric matrix, $S(\cdot) \in so(3)$. The vector $\omega(t) \in \mathbb{R}^3$ is the angular velocity of rotating frame with respect to the fixed frame at time t.

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t)R_2^1(t)\cdots R_n^{n-1}(t) \;\Rightarrow\; \tfrac{d}{dt}R_n^0(t) = S(\omega_{0,n}^0(t))R_n^0(t)$$

Then the following rule is applied for computing angular velocity

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0(t) + R_1^0(t)\omega_{1,2}^1(t) + R_2^0(t)\omega_{2,3}^2(t) + \cdots + R_{n-1}^0(t)\omega_{n-1,n}^{n-1}(t) \end{aligned}$$

5. The velocity of any point p can be computed as

$$\begin{aligned} \tfrac{d}{dt}p^0(t) &= \tfrac{d}{dt}\left[R_n^0(q(t))\right]p^n + \tfrac{d}{dt}\left[o_n^0(q(t))\right] = S(\omega_{0,n}^0(t))R_n^0(q(t))p^n + v_n^0(t) \\ &= \omega_{0,n}^0(t) \times r(t) + v_n^0(t) \end{aligned}$$

where $\omega_{0,n}^0(t)$ is angular velocity of the n-th frame, $v_n^0(t)$ is the linear velocity of the origin of the n-th frame and $r(t) = p - o_n(t)$.

The manipulator Jacobian $J(q) = [J_v(q),\, J_\omega(q)]$ is the matrix function defined by the relations

$$v_n^0(t) = J_v(q(t))\,\tfrac{d}{dt}q(t) \qquad \omega_{0,n}^0(t) = J_\omega(q(t))\,\tfrac{d}{dt}q(t)$$

The function matrix $J_\omega(\cdot)$ can be expressed as

$$\omega_{0,n}^0(t) = \sum_{i=1}^{n}\rho_i\dot{q}_i z_{i-1}^0 = \underbrace{\left[\rho_1 z_0^0,\, \rho_2 z_1^0,\, \ldots,\, \rho_n z_{n-1}^0\right]}_{=J_\omega(q)}\begin{bmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{bmatrix}$$

where gains ρ_i are defined by the rule: If i^{th}-joint is prismatic then the gain ρ_i is 0, and it is 1 if the joint is revolute.

The matrix function $J_v(\cdot) = [J_{v_1}(\cdot), \ldots, J_{v_n}(\cdot)]$ is defined by

$$v_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n$$

where

$$J_{v_i} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_n^0 - o_{i-1}^0\right], & \text{for revolute joint} \end{cases}$$

6. Kinetic energy of a rigid body can be computed as

$$\mathcal{K} = \frac{1}{2}m|\vec{v}_c|^2 + \frac{1}{2}\vec{\omega}^T I \vec{\omega}$$

where $\vec{v}_c$ is the vector of velocity of the center of mass; $\vec{\omega}$ is the vector of angular velocity; I is the matrix of inertia.

7. The equation of motion of mechanical system with n-degrees of freedom are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, \quad 1 \le j \le n$$

where $\mathcal{L} = \mathcal{K} - \mathcal{P}$ is the Lagrangian of the system, $\tau = [\tau_1, \ldots, \tau_n]^T$ is the vector of generalized forces; $q = [q_1, \ldots, q_n]^T$ is the vector of generalized coordinates; $\dot{q} = [\dot{q}_1, \ldots, \dot{q}_n]^T$ is the vector of generalized velocities.