TTK4195 Exam May 2017 – Solution Manual

Problem 1.

a)

The relations between the frames are given by

$$T_{\mathcal{B}}^{\mathcal{N}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T_{\mathcal{F}}^{\mathcal{N}} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \end{bmatrix},$$

where one can note that $T_{\mathcal{N}}^{\mathcal{B}} = \left(T_{\mathcal{N}}^{\mathcal{B}}\right)^{-1} = T_{\mathcal{B}}^{\mathcal{N}}$. Hence

$$\begin{aligned} \vec{a}_n &= T_{\mathcal{F}}^{\mathcal{N}}\, \vec{a}_f = \Big(a_1/2 + a_3\sqrt{3}/2, a_1\sqrt{3}/2 - a_3/2, \;\; a_2\Big)_n, \\ \vec{a}_b &= T_{\mathcal{N}}^{\mathcal{B}}\, \vec{a}_n = \Big(a_1\sqrt{3}/2 - a_3/2, a_1/2 + a_3\sqrt{3}/2, -a_2\Big)_b. \end{aligned}$$

b)

Inserting $(a_1, a_2, a_3) = (0, 1, 0)$ into the expressions above, one obtains

$$\begin{aligned} \vec{a}_n &= (0, 0, \;\; 1)_n, \\ \vec{a}_b &= (0, 0, -1)_b. \end{aligned}$$

Problem 2.

a)

Let $\vec{e} = (e_x, e_y, e_z)^T$ and consider Figure 1. It is then clear that

$$\begin{aligned} \sin(-\phi) &= \frac{e_x}{\sqrt{e_x^2 + e_y^2}} &&\Rightarrow \quad \phi = -\arcsin\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}, \\ \cos(-\theta) &= \frac{e_z}{\|\vec{e}\|} &&\Rightarrow \quad \theta = -\arccos\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}, \end{aligned}$$

if we assume positive rotations counter-clockwise. Hence ψ is left to represent the rotation with constant angular velocity around $\vec{e}$, such that $(\phi(t), \theta(t), \psi(t)) = (-\pi/4, -\pi/4, ct)$ where $c \in \mathbb{R}$ is some constant.

b)

It is here enough to realize that due to the rotation $R(t)$, the z-axis of the body-fixed frame is aligned with the vector $\vec{e}$, such that its angular velocity in the body-fixed frame is simply $\vec{\omega}_b = (0, 0, c)^T$.

A more thorough approach is the following. As $R(t) = R_{\phi,z}R_{\theta,x}R_{\psi,z}$, i.e.,

$$R(t) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

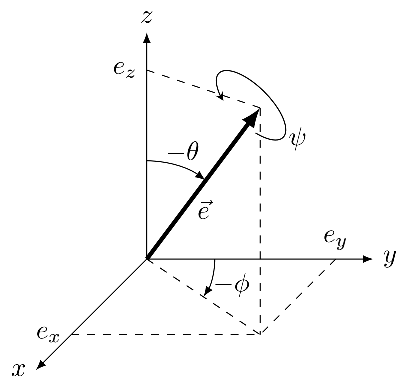

Figure 1: Euler angles corresponding to the ZXZ-parametrization from the rotation around the vector $\vec{e}$.

one obtains, by inserting the values of (ϕ, θ, ψ) from a), the rotation matrix

$$R(t) = \begin{bmatrix} \frac{\sqrt{2}}{2}\cos(ct) + \frac{1}{2}\sin(ct) & \frac{1}{2}\cos(ct) - \frac{\sqrt{2}}{2}\sin(ct) & \frac{1}{2} \\ \frac{1}{2}\sin(ct) - \frac{\sqrt{2}}{2}\cos(ct) & \frac{\sqrt{2}}{2}\sin(ct) + \frac{1}{2}\cos(ct) & \frac{1}{2} \\ -\frac{\sqrt{2}}{2}\sin(ct) & -\frac{\sqrt{2}}{2}\cos(ct) & \frac{\sqrt{2}}{2}, \end{bmatrix}$$

whose third column equals the vector $\vec{e}$. Its derivative with respect to time is

$$\dot{R}(t) = c\begin{bmatrix} \frac{-\sqrt{2}}{2}\sin(ct) + \frac{1}{2}\cos(ct) & -\frac{1}{2}\sin(ct) - \frac{\sqrt{2}}{2}\cos(ct) & 0 \\ \frac{1}{2}\cos(ct) + \frac{\sqrt{2}}{2}\sin(ct) & \frac{\sqrt{2}}{2}\cos(ct) - \frac{1}{2}\sin(ct) & 0 \\ -\frac{\sqrt{2}}{2}\cos(ct) & \frac{\sqrt{2}}{2}\sin(ct) & 0 \end{bmatrix}.$$

Given the angular velocity in the inertial frame $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$, we know that

$$\dot{R}(t) = S(\vec{\omega})R(t)$$

for the skew-symmetric matrix

$$S(\vec{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

As the third column of $\dot{R}(t)$ contains only zeros, one gets the relations

$$\omega_z = \sqrt{2}\omega_x = \sqrt{2}\omega_y,$$

while from element $(1, 1)$ one gets the relation

$$-\omega_z\left(\frac{1}{2}\sin(ct) - \frac{\sqrt{2}}{2}\cos(ct)\right) + \omega_y\left(-\frac{\sqrt{2}}{2}\sin(ct)\right) = -c\left(-\frac{\sqrt{2}}{2}\sin(ct) + \frac{1}{2}\cos(ct)\right)$$

which, by using the relation $\omega_z = \sqrt{2}\omega_y$, can be solved for ω_z, resulting in $\omega_z = c\sqrt{2}/2$; hence

$$\vec{\omega} = c(1/2, 1/2, \sqrt{2}/2)^T.$$

As the rotation matrix takes any vector measured in the body-fixed from to the inertial frame, the relation

$$\vec{\omega} = R(t)\vec{\omega}_b$$

must hold. As $\vec{\omega} = c\vec{e}$ and the third column of $R(t)$ equals $\vec{e}$, it thus follows that $\vec{\omega}_b = (0, 0, c)^T$ (one can of course also solve $\vec{\omega}_b = R(t)^T\vec{\omega}$).

Problem 3.

a)

There are a few things to realize in this exercise. Firstly, the manipulator is a Cartesian manipulator with a spherical wrist, thus one can have the same origin for all the frames in the wrist. Secondly, there is an offset a from the base to the first joint which must be taken into account.

A possible frame assignment for the manipulator can be seen in Figure 2. The corresponding DH parameters are given in Table 1. Note the addition of a fixed frame to account for the constant offset between frames o_0 and o_1 along z_0.

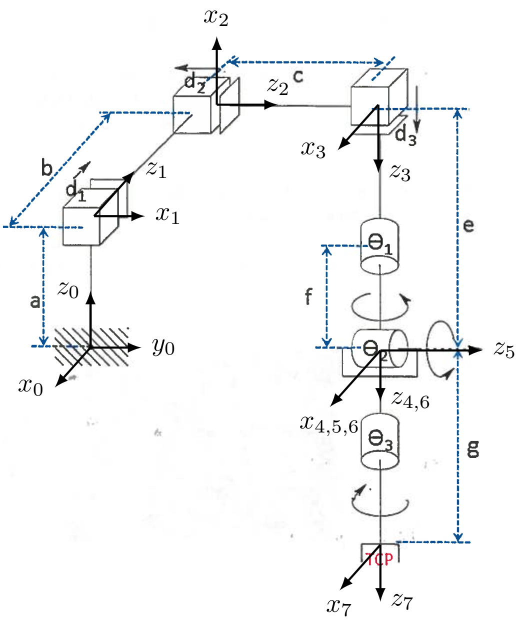

Figure 2: Frame assignment following the DH convention for the Cartesian manipulator with a spherical wrist.

Link	a_i	α_i	d_i	θ_i
1	0	$-\pi/2$	a	$\pi/2$
2	0	$-\pi/2$	$b+d_1$	$-\pi/2$
3	0	$-\pi/2$	$c-d_2$	$-\pi/2$
4	0	0	$e+d_3$	0
5	0	$\pi/2$	0	θ_1
6	0	$-\pi/2$	0	θ_2
7	0	0	g	θ_3

Table 1: DH parameters corresponding to the frame assignment in Figure 2.

As the first three joints correspond to a Cartesian manipulator, one can simply find, either by observations or by computing the homogeneous transformations matrices, that

$$T_4^0 = \begin{bmatrix} 1 & 0 & 0 & -b-d_1 \\ 0 & -1 & 0 & c-d_2 \\ 0 & 0 & -1 & a-e-d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the spherical wrist, we compute the homogeneous transformations matrices:

$$A_5 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_7 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

resulting in

$$T_7^4 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & -c_1s_2 & -gc_1s_2 \\ s_1c_2c_3 + c_1s_3 & -s_1c_2s_3 + c_1c_3 & -s_1s_2 & -gs_1s_2 \\ s_2c_3 & -s_2s_3 & c_2 & gc_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$. The forward kinematics of the manipulator are thus given by

$$T_7^0 = T_4^0 T_7^4 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & -c_1s_2 & -b - d_1 - gc_1s_2 \\ -s_1c_2c_3 - c_1s_3 & s_1c_2s_3 - c_1c_3 & -s_1s_2 & gs_1s_2 + c - d_2 \\ -s_2c_3 & s_2s_3 & -c_2 & a - e - d_3 - gc_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

b)

From (1), we get

$$p_e = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -b - d_1 - gc_1s_2 \\ gs_1s_2 + c - d_2 \\ a - e - d_3 - gc_2 \end{bmatrix}. \tag{2}$$

Since θ_1 and θ_2 appear in this equation, the inverse kinematics can have multiple solutions if p_e is within reach of the manipulator. However, given some desired orientation of the end effector, represented by the Euler-angles $(\phi_e, \theta_e, \psi_e)$, one can determine $(\theta_1, \theta_2, \theta_3)$ from the upper left 3×3 part of (1), such that one can consequently determine the values of (d_1, d_2, d_3) from (2).

Problem 4.

a)

Let the subscript w denote the wheel and p the pendulum. The positions of the centers of mass relative to the inertial frame (x is positive rightwards and y upwards) are thus

$$\begin{aligned} x_w &= r\theta_2 \quad \Rightarrow \dot{x}_w = r\dot{\theta}_2; \quad & x_p &= r\theta_2 + \sin(\theta_1) \quad \Rightarrow \quad \dot{x}_p = r\dot{\theta}_2 + \dot{\theta}_1\cos(\theta_1); \\ y_w &= 0 \quad \Rightarrow \dot{y}_w = 0; \quad & y_p &= \cos(\theta_1) \quad \Rightarrow \quad \dot{y}_p = -\dot{\theta}_1\sin(\theta_1). \end{aligned}$$

The potential energy of the system is therefore

$$\mathcal{P} = m_1 g y_p + m_2 g y_w = m_1 g \cos(\theta_1).$$

b)

Since both θ_1 and θ_2 are absolute angles measured relative to the inertial frame, the angular velocities are simply $\vec{\omega}_w = (0, 0, -\theta_2)^T$ and $\vec{\omega}_p = (0, 0, -\theta_1)^T$, while the linear velocities are $\vec{v}_w = (\dot{x}_w, \dot{y}_w, 0)^T$ and $\vec{v}_p = (\dot{x}_p, \dot{y}_p, 0)^T$. The kinetic energy of the system is thus

$$\begin{aligned} \mathcal{K} =& \frac{1}{2} m_1 \vec{v}_p \cdot \vec{v}_p + \frac{1}{2} m_2 \vec{v}_w \cdot \vec{v}_w + \frac{1}{2} J_1 \vec{\omega}_p \cdot \vec{\omega}_p + \frac{1}{2} J_2 \vec{\omega}_w \cdot \vec{\omega}_w \\ =& \frac{1}{2}\left(m_1 r^2 + m_2 r^2 + J_2\right)\dot{\theta}_2^2 + \frac{1}{2}\left(m_1 + J_1\right)\dot{\theta}_1^2 + m_1 r \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1). \end{aligned}$$

c)

Let us define $p_1 := m_1 r^2 + m_2 r^2 + J_2$, $p_2 := m_1 + J_1$, $p_3 := m_1 r$ and $p_4 := m_1 g$. The Lagrangian of the system can then be written as

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2} p_1 \dot{\theta}_2^2 + \frac{1}{2} p_2 \dot{\theta}_1 + p_3 \dot{\theta}_1 \dot{\theta} \cos(\theta_1) - p_4 \cos(\theta_1).$$

Computing its partial derivatives, we find:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -p_3 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1) + p_4 \sin(\theta_1), \quad \frac{\partial \mathcal{L}}{\partial \theta_2} = 0 \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = p_2 \dot{\theta}_1 + p_3 \dot{\theta}_2 \cos(\theta_1), \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = p_1 \dot{\theta}_2 + p_3 \dot{\theta}_1 \cos(\theta_1).$$

Thus the Euler-Lagrange equations for the system are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = p_2 \ddot{\theta}_1 + p_3 \ddot{\theta}_2 \cos(\theta_1) - p_4 \sin(\theta_1), \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = p_1 \ddot{\theta}_2 + p_3 \ddot{\theta}_1 \cos(\theta_1) - p_3 \dot{\theta}_1^2 \sin(\theta_1). \tag{4}$$

d)

For $q = [\theta_1, \theta_2]^T$, we can write the EL equations on the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u \tag{5}$$

with

$$M(q) = \begin{bmatrix} p_2 & p_3\cos(\theta_1) \\ p_3\cos(\theta_1) & p_1 \end{bmatrix}, \quad C(q,\dot{q}) = \begin{bmatrix} 0 & 0 \\ -p_3\dot{\theta}_1\sin(\theta_1) & 0 \end{bmatrix}, \quad G(q) = \begin{bmatrix} -p_4\sin(\theta_1) \\ 0 \end{bmatrix},$$

and where $B(q)$ needs to be determined. In order to find how $u = F$ acts on the model, i.e. finding $B(q)$, we use the method of Newton-Euler. Firstly, Newton's second law for the two bodies gives

$$\begin{aligned} m_1\ddot{x}_p &= -F_x, \\ m_2\ddot{x}_w &= F_x + F_s - F, \end{aligned} \tag{6}$$

where F_x is the constraint force acting between the wheel and the pendulum in the x-direction, while F_s is the tangential force between the ground and the wheel arising from the no slip constraint $x_w = r\theta_2$; see Figure 3. Secondly, from Euler's law, we get

$$J_2\ddot{\theta}_2 = -rF_s. \tag{7}$$

Combining (6) and (7) together with the fact that $\ddot{x}_w = r\ddot{\theta}_2$ and $\ddot{x}_p = \ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1) - \dot{\theta}_1^2\sin(\theta_1)$, we get

$$m_2r\ddot{\theta}_2 = -m_1r\ddot{\theta}_2 - m_1\ddot{\theta}_1\cos(\theta_1) + m_1\dot{\theta}_1^2\sin(\theta_1) - \frac{1}{r}J_2\ddot{\theta}_2 - F.$$

Multiplying this equation with r and rearranging, we get

$$m_2r^r\ddot{\theta}_2 + m_1r^2\ddot{\theta}_2 + rm_1\ddot{\theta}_1\cos(\theta_1) - rm_1\dot{\theta}_1^2\sin(\theta_1) + J_2\ddot{\theta}_2 = -rF,$$

whose left-hand side is equivalent to (4), meaning that

$$B = B(q) = \begin{bmatrix} 0 \\ -r \end{bmatrix}$$

as there are no forces acting upon the pendulum other than gravity and the constraints forces which are inherently embedded in the generalized coordinate θ_1.

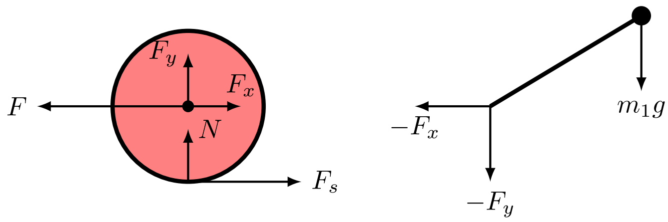

Figure 3: Forces acting on the wheel and the pendulum.

We can now solve (5) for $\ddot{q}$:

$$\ddot{q} = M(q)^{-1}\left(-C(q,\dot{q})\dot{q} - G(q) + Bu\right),$$

yielding

$$\begin{aligned} \ddot{\theta}_1 &= f_3(x,u) = \frac{1}{p_1p_2 - p_3^2\cos^2(\theta_1)}\left(-p_3^2\dot{\theta}_1^2\cos(\theta_1)\sin(\theta_1) + p_1p_4\sin(\theta_1) + rp_3\cos(\theta_1)u\right), \\ \ddot{\theta}_2 &= f_4(x,u) = \frac{1}{p_1p_2 - p_3^2\cos^2(\theta_1)}\left(p_2p_3\dot{\theta}_1^2\sin(\theta_1) - p_3p_4\cos(\theta_1)\sin(\theta_1) - p_2ru\right). \end{aligned}$$

for $x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)^T$ such that $\dot{x} = f(x,u) = [\dot{\theta}_1, \dot{\theta}_2, f_3(x,u), f_4(x,u)]^T$.

Before linearizing about the upright equilibrium $x_* = (0,0,0,0)^T$, one can see that since $f_3(x_*,0) = f_4(x_*,0) = 0$, one does not need to differentiate the denominator $p_1p_2 - p_3^2\cos^2(\theta_1)$ which simply becomes $p_1p_3 - p_3^2$ at x_*. Moreover, one does not need to consider terms in f of the form $g_1(x,u)g_2(x,u)$ if $g_1(x_*,0) = g_2(x_*,0) = 0$ since these disappear in the linearization.

The linearized system takes the form

$$\delta\dot{x} = \underbrace{\frac{\partial f}{\partial x}(x,u)\bigg|_{x=x_*,u=0}}_{=A}\delta x + \underbrace{\frac{\partial f}{\partial u}(x,u)\bigg|_{x=x_*,u=0}}_{=B} u$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \\ d \end{bmatrix}$$

where

$$a := \frac{p_1p_4}{p_1p_2 - p_3^2}, \quad b := \frac{rp_3}{p_1p_2 - p_3^2}, \quad c := \frac{-p_3p_4}{p_1p_2 - p_3^2} \quad \text{and} \quad d := \frac{-rp_2}{p_1p_2 - p_3^2}.$$

The controllability matrix is therefore given by

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & b & 0 & ab \\ 0 & d & 0 & bc \\ b & 0 & ab & 0 \\ d & 0 & bc & 0 \end{bmatrix}$$

which has full rank ($\text{rank}(\mathcal{C}) = 4$) such that the system is controllable about x_* if and only if $a \neq 1$ and $d \neq bc$.

e)

Slipping occurs when the size of the tangential force F_s is greater than the normal force between the wheel and the ground multiplied with the friction coefficient, i.e., $|F_s| > \mu N$. If x_c denotes the position of the point of contact on the wheel between it and the ground, then $\dot{x}_c \neq 0$ if the wheel slips while rolling. This means that the relation $x_w = r\theta_2$ is no longer valid; hence x_w must be added as a coordinate, such that 3 coordinates are needed to describe the configuration of the system: θ_1, θ_2 and x_w.