TTK4195
Modeling and Control of Robots
Assignment 4

Solution

February 14, 2014

Problem 4-3

$$R(\mathbf{a} \times \mathbf{b}) = R\mathbf{a} \times R\mathbf{b}$$

First, let us suppose that

$$\begin{aligned} R &= [\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}] \\ \mathbf{a} &= [a_1, a_2, a_3]^T \\ \mathbf{b} &= [b_1, b_2, b_3]^T \end{aligned}$$

$$R\mathbf{a} \times R\mathbf{b} = (a_1\mathbf{r_1} + a_2\mathbf{r_2} + a_3\mathbf{r_3}) \times (b_1\mathbf{r_1} + b_2\mathbf{r_2} + b_3\mathbf{r_3})$$

Now, since a rotation matrix is orthogonal and satisfies

$$\begin{aligned} \mathbf{r_1} \times \mathbf{r_2} &= \mathbf{r_3} \\ \mathbf{r_3} \times \mathbf{r_1} &= \mathbf{r_2} \\ \mathbf{r_2} \times \mathbf{r_3} &= \mathbf{r_1} \end{aligned}$$

which is a property it shares with the basic spatial vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we can therefore compute the cross product in the following manner

$$\begin{aligned} R\mathbf{a} \times R\mathbf{b} &= \begin{vmatrix} \mathbf{r_1} & \mathbf{r_2} & \mathbf{r_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{r_1} + (a_1b_3 - a_3b_1)\mathbf{r_2} + (a_1b_2 - a_2b_1)\mathbf{r_3} \end{aligned}$$

and since

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

we can write

$$
\begin{aligned}
R\mathbf{a} \times R\mathbf{b} &= [\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}] \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \\
&= R(\mathbf{a} \times \mathbf{b})
\end{aligned}
$$

The cross product computation uses the property of distributivity of the orthogonal column vectors $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$.

Problem 4-4

Verify Equation (4-10):
Set $Y = SX$. By commutativity of the inner product, $X^TY = Y^TX$, or $X^TSX = X^TS^TX$. Since S is skew-symmetric, $S^T + S = 0$. Thus, for any vector X, we have

$$0 = X^T(S + S^T)X = X^TSX + X^TS^TX = 2X^TSX$$

Therefore $X^TSX = 0$.

Problem 4-5

Verify Equation (4.17) by direct calculation:

$$
\begin{aligned}
\frac{dR_{y,\theta}}{d\theta} R_{y,\theta}^{-1} = \frac{dR_{y,\theta}}{d\theta} R_{y,\theta}^{T} &= \begin{bmatrix} -s_\theta & 0 & c_\theta \\ 0 & 0 & 0 \\ -c_\theta & 0 & -s_\theta \end{bmatrix} \begin{bmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = S(j) \\
\frac{dR_{z,\theta}}{d\theta} R_{z,\theta}^{-1} = \frac{dR_{z,\theta}}{d\theta} R_{z,\theta}^{T} &= \begin{bmatrix} -s_\theta & -c_\theta & 0 \\ c_\theta & -s_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S(k)
\end{aligned}
$$

Problem 4-6

Suppose that $a = [1, -1, 2]^T$ and that $R = R_{x,\frac{\pi}{2}}$. Show by direct calculation that $RS(a)R^T = S(Ra)$.

$$
R_{x,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S(a) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}
$$

$$
Ra = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad S(Ra) = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}
$$

Then

$$RS(a)R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = S(Ra)$$

Problem 4-8

Use Equation (2.43) to show that

$$R_{k,\theta} = I + S(k)sin(\theta) + S^2(k)vers(\theta)$$

k is a unit vector defining an axis: $k = [k_x, k_y, k_z]^T$. Then we get

$$S(k) = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$
$$S^2(k) = \begin{bmatrix} -k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & -k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & -k_x^2 - k_y^2 \end{bmatrix}$$

To solve the diagonal elements we use that k is a unit vector, such that

$$\|k\|^2 = k_x^2 + k_y^2 + k_z^2 = 1$$
$$k_x^2 - 1 = -(k_y^2 + k_z^2)$$
$$k_y^2 - 1 = -(k_x^2 + k_z^2)$$
$$k_z^2 - 1 = -(k_x^2 + k_y^2)$$

The first element of $R_{k,\theta}$ is therefore

$$r_{11} = 1 - (k_y^2 + k_z^2)(1 - c_\theta) = 1 + (k_x^2 - 1)(1 - c_\theta) = 1 + k_x^2(1 - c_\theta) - 1 + c_\theta = k_x^2 v_\theta + c_\theta$$

and we can show that

$$R_{k,\theta} = I + S(k)s_\theta + S^2(k)v_\theta$$

Problem 4-13

Given

$$R = R_{z,\psi} R_{y,\theta} R_{z,\phi}$$

we can compute the time derivative by noting that

$$\frac{d}{dx} R_{k',x} = S(k') R_{k',x}, \quad k' = [i, j, k]^T$$

from **Equation 4.18**. The derivative becomes

$$\dot{R} = \dot{R_{z,\psi}} R_{y,\theta} R_{z,\phi} + R_{z,\psi} \dot{R_{y,\theta}} R_{z,\phi} + R_{z,\psi} R_{y,\theta} \dot{R_{z,\phi}}$$

By using the chain rule, we can compute the individual derivatives as

$$\begin{aligned}
&\dot{R} = \frac{dR}{dx}\frac{dx}{dt}, \quad x = \psi, \theta, \phi \\
&\quad \Downarrow \\
&\dot{R} = \dot{\psi} S(k) R_{z,\psi} R_{y,\theta} R_{z,\phi} + R_{z,\psi} \dot{\theta} S(j) R_{y,\theta} R_{z,\phi} + R_{z,\psi} R_{y,\theta} \dot{\phi} S(k) R_{z,\phi}
\end{aligned}$$

By using **Equation 4.8**, we can rewrite this into

$$\begin{aligned}
\dot{R} &= S(\dot{\psi} k) R_z R_y R_z + S(R_z \dot{\theta} j) R_z R_y R_z + S(R_z R_y \dot{\phi} k) R_z R_y R_z \\
&= S(\omega) R
\end{aligned}$$

where

$$\begin{aligned}
\omega &= \dot{\psi} k + R_z \dot{\theta} j + R_z R_y \dot{\phi} k \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} -s_\psi \dot{\theta} \\ c_\psi \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} c_\psi s_\theta \dot{\phi} \\ s_\psi s_\theta \dot{\phi} \\ c_\theta \dot{\phi} \end{bmatrix} = \begin{bmatrix} c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta} \\ s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta} \\ c_\theta \dot{\phi} + \dot{\psi} \end{bmatrix} \\
&= (c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta}) \mathbf{i} + (s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta}) \mathbf{j} + (c_\theta \dot{\phi} + \dot{\psi}) \mathbf{k}
\end{aligned}$$

Problem 4-15

Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the homogeneous transformation

$$H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $v_1(t) = [3, 1, 0]^T$ relative to the frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

$$\begin{aligned}
P_1^0 &= H_1^0 P_1^1 = HP_1^1 \\
\begin{bmatrix} p_1^0 \\ 1 \end{bmatrix} &= \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1^1 \\ 1 \end{bmatrix} \\
p_1^0 &= Rp_1^1 + d \\
\frac{d}{dt}(p_1^0) &= \frac{d}{dt}(Rp_1^1 + d) \\
\dot{p}_1^0 &= \underbrace{\dot{R}p_1^1}_{0} + R\dot{p}_1^1 + \underbrace{\dot{d}}_{0} \\
v_1^0 &= Rv_1^1 \\
v_1^0 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}
\end{aligned}$$