TTK4195
Modeling and Control of Robots
Assignment 3

Solution

February 8, 2014

Problem 3-10

First, the coordinate frames need to be placed according to the DH convention. This is done as shown by Figure 1.

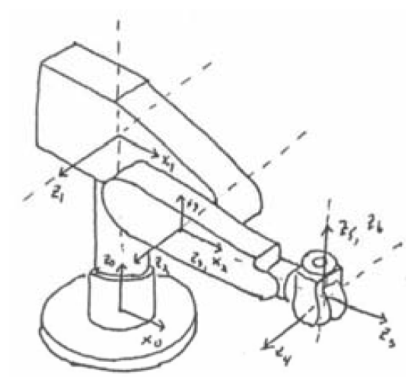

Figure 1: Coordinate frames placed according to the DH convention.

Second, the DH parameters must be found. These are listed in Table 1.

Link	θ	d	a	α
1	θ_1	13	0	$\frac{\pi}{2}$
2	θ_2	d_s	8	0
3	θ_3	0	0	$\frac{\pi}{2}$
4	θ_4	8	0	$-\frac{\pi}{2}$
5	θ_5	0	0	$\frac{\pi}{2}$
6	θ_6	d_e	0	0

Table 1: DH parameters for the coordinate frames in Figure 1. The distances are given in inches.

Third, the homogeneous transformation matrices between coordinate frame $i-1$ and i need to be found. These are named A_i, and are found to be

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & 8c_2 \\ s_2 & c_2 & 0 & 8s_2 \\ 0 & 0 & 1 & d_s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & 0 & s_3 & 8c_3 \\ s_3 & 0 & -c_3 & 8s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3-15

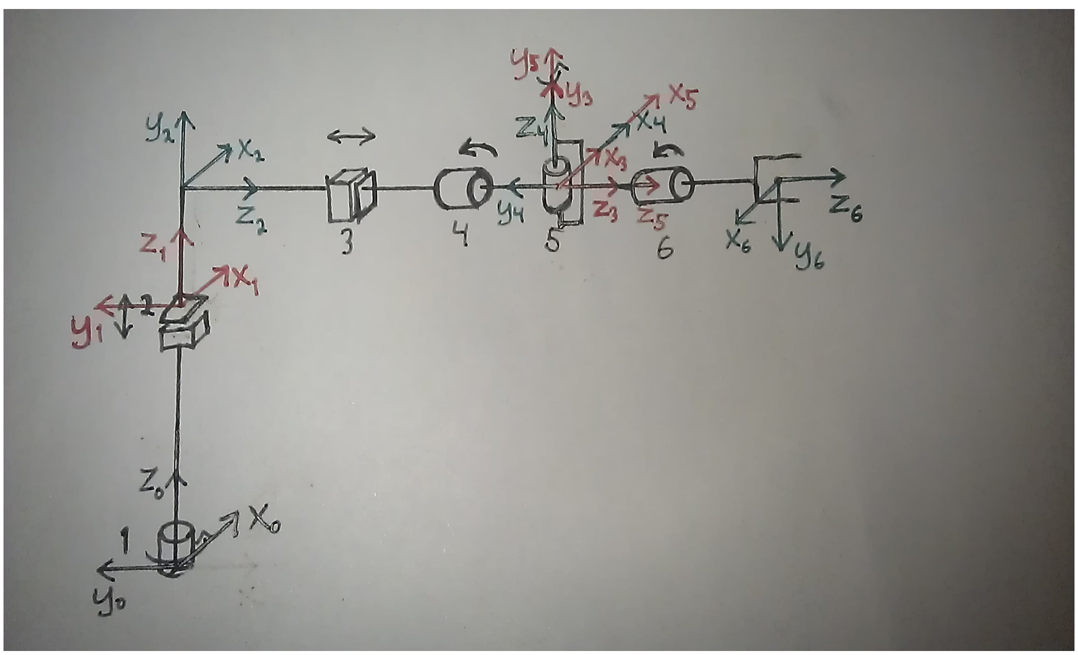

Figure 2: Sketch of problem 3-15

	a	α	d	θ
1	0	0	d_1	θ_1^*
2	0	$\pi/2$	d_2^*	0
3	0	0	$1+d_3^*$	0
4	0	$-\pi/2$	0	θ_4^*
5	0	$\pi/2$	0	θ_5^*
6	0	0	d_6	θ_6^*

Table 2: DH-parameters, Problem 3-15.

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = o_3^0$$

We obtain o_3^0 from the T_3^0 matrix that we can calculate from the forward kinematics with the DH parameters specified in Table 2.

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & s_1 & s_1(d_3^*+1) \\ s_1 & 0 & -c_1 & -c_1(d_3^*+1) \\ 0 & 1 & 0 & d_2^*+1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} s_1(d_3^*+1) \\ -c_1(d_3^*+1) \\ d_2^*+1 \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

From these equations, we can derive the first three joint variables.

$$\begin{aligned} \theta_1 &= \text{Atan2}((o_y - d_6 r_{23}), (o_x - d_6 r_{13})) \\ d_2 &= o_z - d_6 r_{33} - 1 \\ d_3 &= \sqrt{(o_x - d_6 r_{13})^2 + (o_y - d_6 r_{23})^2} - 1 \end{aligned}$$

To find expressions for the remaining three joint variables, we try to find euler angles that correspond to

$$R_6^3 = (R_3^0)^T R$$

R_3^0 is given by the homogeneous transformation T_3^0.

$$R_3^0 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \ (R_3^0)^{-1} = (R_3^0)^T = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix}$$

$$\begin{aligned} R_6^3 = A &= \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\ &= \begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} \\ r_{31} & r_{32} & r_{33} \\ s_1 r_{11} - c_1 r_{21} & s_1 r_{12} - c_1 r_{22} & s_1 r_{13} - c_1 r_{23} \end{bmatrix} \end{aligned}$$

To give the end-effector the wanted orientation, we find the Euler angles that satisfy the ZYZ**-Euler angle transformation**.

$$A = R_{ZYZ}$$

$$\begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} \\ r_{31} & r_{32} & r_{33} \\ s_1 r_{11} - c_1 r_{21} & s_1 r_{12} - c_1 r_{22} & s_1 r_{13} - c_1 r_{23} \end{bmatrix} = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 c_6 - s_4 c_6 & c_4 s_5 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 \\ -s_5 c_6 & s_5 s_6 & c_5 \end{bmatrix}$$

There are several solutions to this problem

1. Not both of a_{13} and a_{23} are zero.

   (a)

$$\begin{aligned} \theta_5 &= \text{Atan2}((s_1 r_{13} - c_1 r_{23}), \sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}) \\ \Rightarrow \ \theta_4 &= \text{Atan2}((c_1 r_{13} + s_1 r_{23}), r_{33}) \\ \theta_6 &= \text{Atan2}((-s_1 r_{11} + c_1 r_{21}), (s_1 r_{12} - c_1 r_{22})) \end{aligned}$$

   (b)

$$\begin{aligned} \theta_5 &= \text{Atan2}((s_1 r_{13} - c_1 r_{23}), -\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}) \\ \Rightarrow \ \theta_4 &= \text{Atan2}((-c_1 r_{13} - s_1 r_{23}), -r_{33}) \\ \theta_6 &= \text{Atan2}((s_1 r_{11} - c_1 r_{21}), (-s_1 r_{12} + c_1 r_{22})) \end{aligned}$$

2. $a_{13} = a_{23} = 0$.

   (a) $a_{33} = 1$

$$R* = \begin{bmatrix} c_{\theta_4+\theta_6} & -s_{\theta_4+\theta_6} & 0 \\ s_{\theta_4+\theta_6} & c_{\theta_4+\theta_6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The sum is determined to be

$$\theta_4 + \theta_6 = \text{Atan2}((c_1 r_{11} + s_1 r_{21}), (r_{31}))$$

There are therefore infinitely many solutions because only the sum can be determined, θ_4 is usually chosen to be 0 by convention. The total solution set is

$$\begin{aligned}
\theta_4 &= 0 \\
\theta_5 &= 0 \\
\theta_6 &= \text{Atan2}((c_1 r_{11} + s_1 r_{21}), (r_{31}))
\end{aligned}$$

(b) $r_{33} = -1$

$$R* = \begin{bmatrix} -c_{\theta_4-\theta_6} & -s_{\theta_4-\theta_6} & 0 \\ s_{\theta_4-\theta_6} & c_{\theta_4-\theta_6} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The sum is determined to be

$$\theta_4 - \theta_6 = \text{Atan2}((-c_1 r_{11} - s_1 r_{21}), (-r_{31}))$$

There are therefore infinitely many solutions because only the sum can be determined, θ_4 is usually chosen to be 0 by convention. The total solution set is

$$\begin{aligned}
\theta_4 &= 0 \\
\theta_5 &= 0 \\
\theta_6 &= -\text{Atan2}((-c_1 r_{11} - s_1 r_{21}), (-r_{31}))
\end{aligned}$$

Problem 3-16

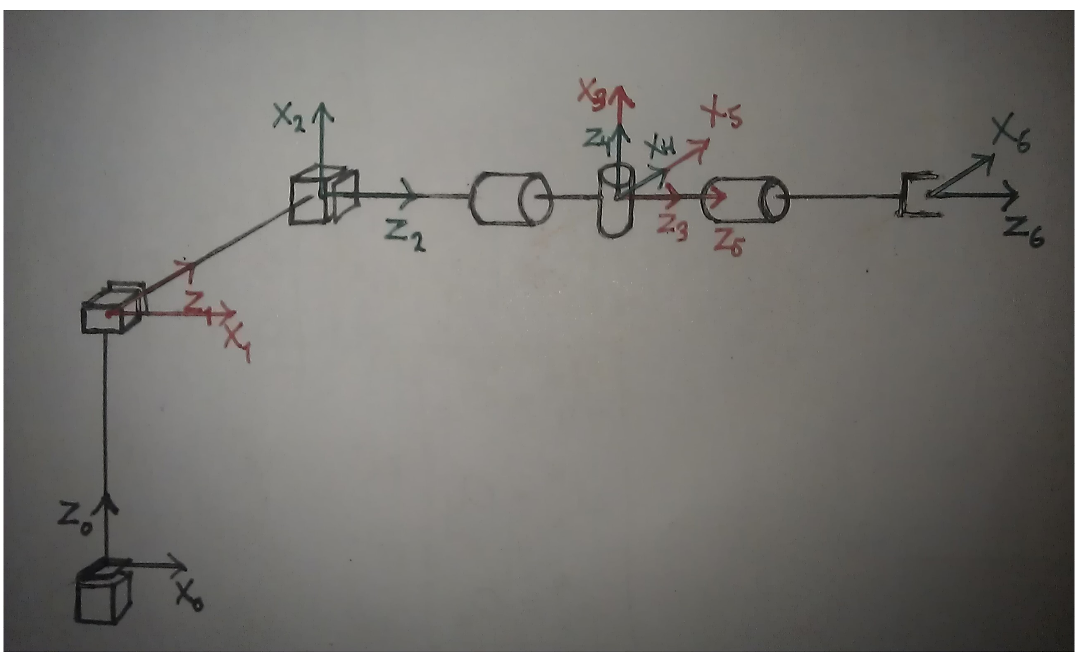

Figure 3: Sketch of problem 3-16

	a	α	d	θ
1	0	$-\pi/2$	d_1^*	0
2	0	$-\pi/2$	d_2^*	$-\pi/2$
3	0	0	d_3^*	0
4	0	$-\pi/2$	0	θ_4^*
5	0	$\pi/2$	0	θ_5^*
6	0	0	d_6	θ_6^*

Table 3: DH-parameters, Problem 3-16.

The first three joint variables are found by

$$T_3^0 = \begin{bmatrix} 0 & 0 & 1 & d_3^* \\ 0 & -1 & 0 & d_2^* \\ 1 & 0 & 0 & d_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} d_3^* \\ d_2^* \\ d_1^* \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

The remaining three are found in a similar fashion as in **Problem 3-15**.

$$\begin{aligned} R_6^3 &= (R_3^0)^T R = A \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\ &= \begin{bmatrix} r_{31} & r_{32} & r_{33} \\ -r_{21} & -r_{22} & -r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix} \end{aligned}$$

Solving for the Euler angles to satisfy the Euler transformation gives

1. Not both of a_{13} and a_{23} are zero.

   (a)

$$\begin{aligned} & \theta_5 = \text{Atan2}(r_{13}, \sqrt{1 - r_{13}^2}) \\ \Rightarrow\ & \theta_4 = \text{Atan2}(r_{33}, -r_{23}) \\ & \theta_6 = \text{Atan2}(-r_{11}, r_{12}) \end{aligned}$$

   (b)

$$\begin{aligned} & \theta_5 = \text{Atan2}(r_{13}, -\sqrt{1 - r_{13}^2}) \\ \Rightarrow\ & \theta_4 = \text{Atan2}(-r_{33}, r_{23}) \\ & \theta_6 = \text{Atan2}(r_{11}, -r_{12}) \end{aligned}$$

2. $a_{13} = a_{23} = 0$.

   (a) $a_{33} = 1$

$$R* = \begin{bmatrix} c_{\theta_4+\theta_6} & -s_{\theta_4+\theta_6} & 0 \\ s_{\theta_4+\theta_6} & c_{\theta_4+\theta_6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The sum is determined to be

$$\theta_4 + \theta_6 = \text{Atan2}(r_{31}, -r_{21})$$

   There are therefore infinitely many solutions because only the sum can be determined, θ_4 is usually chosen to be 0 by convention. The total solution set is

$$\begin{aligned} \theta_4 &= 0 \\ \theta_5 &= 0 \\ \theta_6 &= \text{Atan2}(r_{31}, -r_{21}) \end{aligned}$$

   (b) $a_{33} = -1$

$$R* = \begin{bmatrix} -c_{\theta_4-\theta_6} & -s_{\theta_4-\theta_6} & 0 \\ s_{\theta_4-\theta_6} & c_{\theta_4-\theta_6} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   The sum is determined to be

$$\theta_4 - \theta_6 = \text{Atan2}(-r_{31}, -r_{32})$$

   There are therefore infinitely many solutions because only the sum can be determined, θ_4 is usually chosen to be 0 by convention. The total solution set is

$$\begin{aligned} \theta_4 &= 0 \\ \theta_5 &= 0 \\ \theta_6 &= -\text{Atan2}(-r_{31}, -r_{32}) \end{aligned}$$

Problem 4-7

$$R = R_{x,\theta} R_{y,\phi}$$

$$\frac{\partial R}{\partial \phi} = R_{x,\theta} \frac{\partial R_{y,\phi}}{\partial \phi}$$

Differentiating rotation matrices is equivalent to multiplying the rotation matrix with a skew symmetric matrix. From **Equation 4.17** we see that

$$\frac{d}{d\phi}R_{y,\phi} = S(j)R_{y,\phi}$$
$$\Downarrow$$
$$\begin{aligned}
\frac{\partial R}{\partial \phi} &= R_{x,\theta}S(j)R_{y,\phi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} -s_\phi & 0 & c_\phi \\ s_\theta c_\phi & 0 & s_\theta s_\phi \\ -c_\theta c_\phi & 0 & -c_\theta s_\phi \end{bmatrix}
\end{aligned}$$

Evaluating the partial derivative at $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$ results in

$$\frac{\partial R}{\partial \phi}\Big|_{\theta=\phi=\frac{\pi}{2}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$