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EXAM FOR COURSE TTK 4195
Robots Modeling and Control

Tuesday, May 20, 2011
Time: 09.00-13.00

Allowable aids: D - No printed or written material allowed. NTNU type approved calculator with an empty memory allowed. Language: English. Number of Pages: 6 (+ 4 formula sheet). This exam counts for 100% of the final grade.

This exam consists of 4 exercises containing a number of questions. Each question gives a number of points with a total sum of 100 points.

1. **Problem:** Given four orthogonal frames with the same origin $Ox_1y_1z_1$, $Ox_2y_2z_2$, $Ox_3y_3z_3$, $Ox_4y_4z_4$, suppose rotation matrices R_2^1, R_4^3 and R_4^1 that relate the orientations of the frames are

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_4^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_4^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Find the rotation matrix R_3^2. **(5)**

2. **Problem:** Consider the following sequence of rotations

   (a) Rotate by ϕ about the world axis z;

   (b) Rotate by θ about the world axis x;

   (c) Rotate by ψ about the current axis z;

   (d) Rotate by α about the world axis y

   Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication). **(5)**

3. **Problem:** Consider a robot depicted in Fig. 1

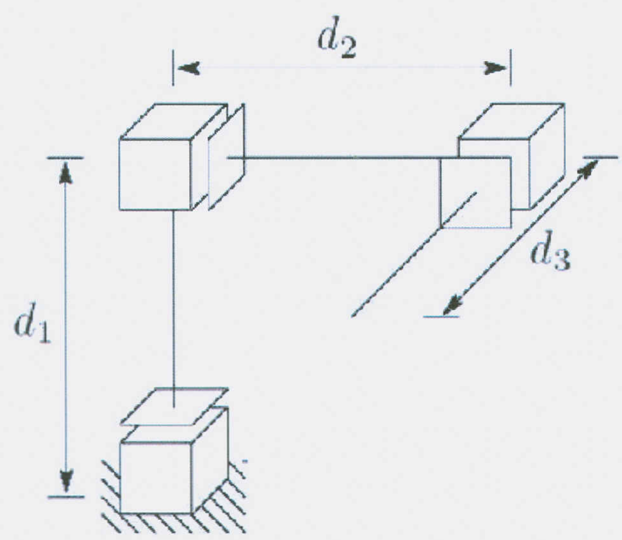

Figure 1: A robot of the PPP-type.

(a) For each link introduce the frame following DH-convention and derive forward kinematics equations. **(10)**

(b) Solve the inverse position kinematics problem, i.e. given coordinates (position) of the origin of the tool frame of the robot in the base frame $p_e = (x_e, y_e, z_e) \in \mathbb{R}^3$, find corresponding values for extensions d_1, d_2 and d_3, with which the origin of the tool frame is in the requested position p_e. **(5)**

(c) Compute the manipulator Jacobian for representation of linear and angular velocity of the origin of the tool frame, which is located at the end of the last prismatic link of the robot (see Fig. 1). **(10)**

(d) Compute the total velocity of the origin of the tool frame when the variables d_1, d_2 and d_3 are changing with time as follows

$$d_1(t) = \cos(t), \quad d_2(t) = \sin(2t), \quad d_3(t) = \cos(3t)$$

Computation can be based on the Jacobian computed on the previous step or the vector of velocity of this point can be computed directly. **(5)**

4. **Problem:** Consider the planar robot (the so-called butterfly robot built at Universita di Roma 'La Sapienza' by Prof. Guiseppe Oriolo) for which photo and schematic views are depicted in Figs. 2 and 3.

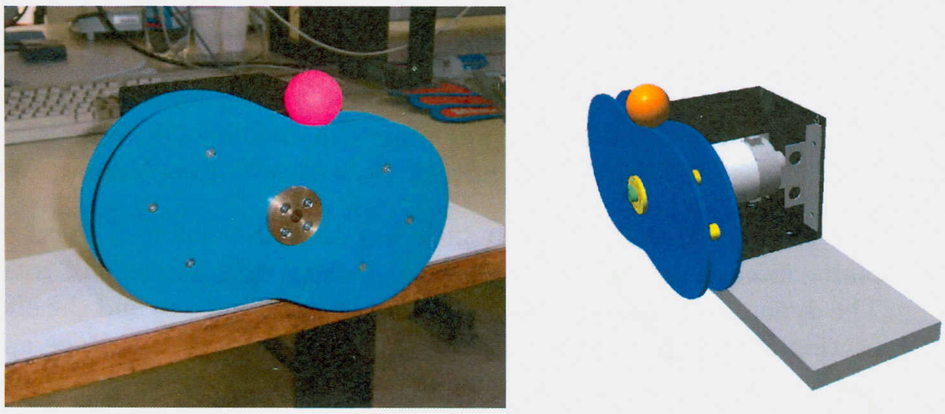

Figure 2: The photo and the schematic side view of the Butterfly robot

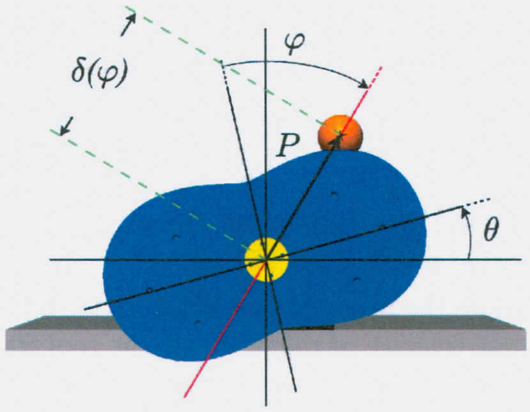

Figure 3: The schematic front view of the Butterfly robot: the angle θ represents the rotation of the two rigidly connected identical plates with respect to horizontal; the angle ϕ characterizes the relative position P of the center of mass of the ball; $\delta(\phi)$ is the distance to the ball from the axis of rotation of the plates, it is a known smooth function of ϕ that characterizes the geometry of the plates' contour.

The robot consists of the following parts:

- two identical symmetric wooden plates (*blue* in Figs. 2 and 3), which are fixed in parallel to shape one rigid body and which can be rotated by a DC-motor about the center point (*yellow* in Fig. 3);
- one ball (*red* in Figs. 2 and 3), which can freely roll following the contour of the plates. Its geometry is parameterized by the given smooth function $\delta(\phi)$ representing the distance between the center of mass of the ball and the axis of rotation of the plates, see Fig. 3.

The robot is manufactured in such way that the center of mass of the plates (*blue* parts in Figs. 2 and 3) coincides with the axis of rotation. The dynamics of the electrical components is fast enough and the effect of friction is pre-compensated so that the control signal (a voltage applied to the DC-motor) is equal to an external torque applied to the axis of rotation of the plates.

You are requested to complete the following tasks:

(a) Suppose that the ball is modeled as a point mass m and the moment of inertia of the plates around the axis of rotation equals to J. Find the potential energy $\mathcal{P}(\theta, \phi)$ of the robot. **(10)**

(b) Suppose that the ball is modeled as a point mass m and the moment of inertia of the plates around the axis of rotation equals to J. Find the kinetic energy $\mathcal{K}(\theta, \phi)$ of the robot. **(10)**

(c) Suppose that the ball is modeled as a point mass m and the moment of inertia of the plates around the axis of rotation equals to J. Derive the Euler-Lagrange equations of the system dynamics. **(15)**

(d) Suppose that the ball is modeled as a point mass m and the moment of inertia of the plates around the axis of rotation equals to J. Deduce the equations that can be solved to find the value of the angle ϕ_{des} for the unstable equilibriums $\theta_e = 0$, $\phi_e = \pm\phi_{des}$ depicted on Fig 4. **(10)**

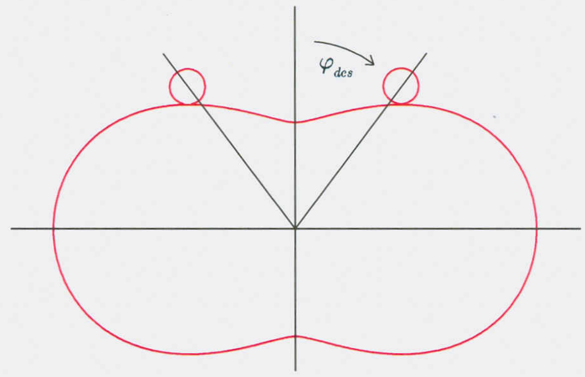

Figure 4: The configuration of the Butterfly robot, for which the system has one stable equilibrium at $\theta_e = \phi_e = 0$ and two unstable ones at $\theta_e = 0$, $\phi_e = \pm\phi_{des}$.

(e) Suppose that the ball is modeled as a disc of a mass m rolling without slipping on the contour, see Fig. 5.

i. Is this mechanical system holonomic or not? Clarify the answer. **(5)**

ii. How many degrees of freedom will the system have? Clarify the answer. **(5)**

iii. Under what condition would the ball depart from the contour? Clarify the answer. **(5)**

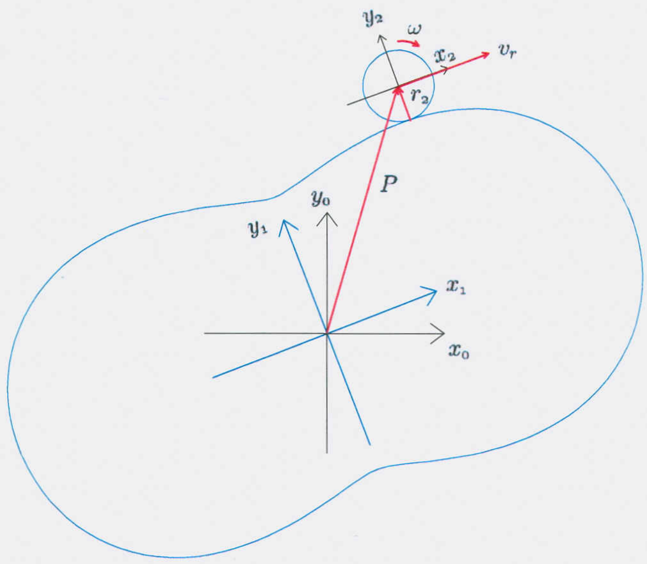

Figure 5: The schematic view of the Butterfly robot, when the ball is modeled as a disc rolling without slipping on the contour.

List of Formulas for
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Robots Modeling and Control

1. If we are given $(N-1)$-rotation matrices

$$R_1^0, \quad R_2^1, \quad \ldots, \quad R_{(N-1)}^{(N-2)}$$

that represent consecutive rotations between the current frames

$$\{(x_0 y_0 z_0), (x_1 y_1 z_1)\}, \{(x_1 y_1 z_1), (x_2 y_2 z_2)\}, \ldots, \{(x_{N-2} y_{N-2} z_{N-2}), (x_{N-1} y_{N-1} z_{N-1})\}$$

The formula to compute the position of the point in the 0-frame having known its position in the $(N-1)$-frame is

$$p^0 = R_1^0 \, R_2^1 \, R_3^2 \cdots R_{(N-1)}^{(N-2)} \, p^{(N-1)}$$

2. If there are 3 frames corresponding to 2 rigid motions

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \end{aligned}$$

then the overall motion is

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

3. Homogeneous transform A_i in DH-convention is a product

$$A_i = \mathrm{Rot}_{z,\theta_i} \cdot \mathrm{Trans}_{z,d_i} \cdot \mathrm{Trans}_{x,a_i} \cdot \mathrm{Rot}_{x,\alpha_i}$$

$$= \left[\begin{array}{ccc|c} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

with the following parameters: the link length a_i, the link twist α_i, the link offset d_i, the link angle θ_i.

4. If $R(t) \in \mathcal{SO}(3)$ is time-varying and has a time-derivative, then

$$\tfrac{d}{dt}R(t) = S(t)R(t) = S(\omega(t))R(t),$$

where $S(t)$ is skew symmetric matrix, $S(\cdot) \in so(3)$. The vector $\omega(t) \in \mathbb{R}^3$ is the angular velocity of rotating frame with respect to the fixed frame at time t.

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t)R_2^1(t)\cdots R_n^{n-1}(t) \;\Rightarrow\; \tfrac{d}{dt}R_n^0(t) = S(\omega_{0,n}^0(t))R_n^0(t)$$

Then the following rule is applied for computing angular velocity

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0(t) + R_1^0(t)\omega_{1,2}^1(t) + R_2^0(t)\omega_{2,3}^2(t) + \cdots + R_{n-1}^0(t)\omega_{n-1,n}^{n-1}(t) \end{aligned}$$

5. The velocity of any point p can be computed as

$$\begin{aligned} \tfrac{d}{dt}p^0(t) &= \tfrac{d}{dt}\left[R_n^0(q(t))\right]p^n + \tfrac{d}{dt}\left[o_n^0(q(t))\right] = S(\omega_{0,n}^0(t))R_n^0(q(t))p^n + v_n^0(t) \\ &= \omega_{0,n}^0(t) \times r(t) + v_n^0(t) \end{aligned}$$

where $\omega_{0,n}^0(t)$ is angular velocity of the n-th frame, $v_n^0(t)$ is the linear velocity of the origin of the n-th frame and $r(t) = p - o_n(t)$.

The manipulator Jacobian $J(q) = [J_v(q),\, J_\omega(q)]$ is the matrix function defined by the relations

$$v_n^0(t) = J_v(q(t))\,\tfrac{d}{dt}q(t) \qquad \omega_{0,n}^0(t) = J_\omega(q(t))\,\tfrac{d}{dt}q(t)$$

The function matrix $J_\omega(\cdot)$ can be expressed as

$$\omega_{0,n}^0(t) = \sum_{i=1}^{n} \rho_i \dot q_i z_{i-1}^0 = \underbrace{\left[\rho_1 z_0^0,\ \rho_2 z_1^0,\ \ldots,\ \rho_n z_{n-1}^0\right]}_{=J_\omega(q)} \begin{bmatrix} \dot q_1 \\ \vdots \\ \dot q_n \end{bmatrix}$$

where gains ρ_i are defined by the rule: If i^{th}-joint is prismatic then the gain ρ_i is 0, and it is 1 if the joint is revolute.

The matrix function $J_v(\cdot) = [J_{v_1}(\cdot), \ldots, J_{v_n}(\cdot)]$ is defined by

$$v_n^0(t) = J_{v_1}(\cdot)\dot q_1 + J_{v_2}(\cdot)\dot q_2 + \cdots + J_{v_n}(\cdot)\dot q_n$$

where

$$J_{v_i} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_n^0 - o_{i-1}^0\right], & \text{for revolute joint} \end{cases}$$

6. Kinetic energy of a rigid body can be computed as

$$\mathcal{K} = \frac{1}{2} m |\vec{v}_c|^2 + \frac{1}{2} \vec{\omega}^T I \vec{\omega}$$

where $\vec{v}_c$ is the vector of velocity of the center of mass; $\vec{\omega}$ is the vector of angular velocity; I is the matrix of inertia.

7. The equation of motion of mechanical system with n-degrees of freedom are

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, \quad 1 \leq j \leq n$$

where $\mathcal{L} = \mathcal{K} - \mathcal{P}$ is the Lagrangian of the system, $\tau = [\tau_1, \ldots, \tau_n]^T$ is the vector of generalized forces; $q = [q_1, \ldots, q_n]^T$ is the vector of generalized coordinates; $\dot{q} = [\dot{q}_1, \ldots, \dot{q}_n]^T$ is the vector of generalized velocities.