Solutions for Exam Problems in the Course Modeling and Control of Robots

Tuesday May 20th 2008

1. **Problem:** Given three orthogonal frames with the same origin $Ox_1y_1z_1$, $Ox_2y_2z_2$, $Ox_3y_3z_3$, suppose rotation matrices R_2^1 and R_3^1 are

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad R_3^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the rotation matrix R_3^2. **(10)**

Solution: The rotation R_3^1 is defined as the product of two consecutive rotations

$$R_3^1 = R_2^1 R_3^2$$

Therefore,

$$\begin{aligned} R_3^2 &= \left[R_2^1\right]^{-1} R_3^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Here we have used the fact that the inverse of any rotation matrix is equal to its transpose

$$R^{-1} = R^T.$$

2. **Problem:** Consider a robot depicted on Fig. 1

   (a) For each link introduce the frame following DH-convention and derive forward kinematics equations. **(10)**

   (b) Solve the inverse position kinematics problem, i.e. given coordinates (position) of the tool frame of the cylindrical robot in the base frame $p_e = (x_e, y_e, z_e) \in \mathbb{R}^3$ find corresponding values for the angle θ_1 and the extensions d_2, d_3, with which the tool frame is in the requested position p_e. **(10)**

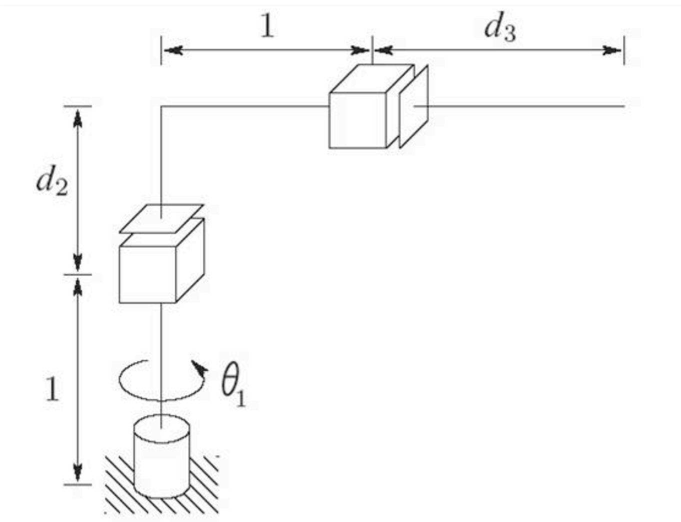

Figure 1: A robot from the problem 2.

(c) Compute the manipulator Jacobian for representation of linear and angular velocity of the origin of the tool frame, which is located at the end of the second prismatic link of the robot, see Fig. 1. **(10)**

(d) Compute the total velocity of the origin of the tool frame when the variables θ_1, d_2 and d_3 are changing with time as follows:

$$\theta_1(t) = \sin(t), \quad d_2(t) = \cos(2t), \quad d_3(t) = \sin(3t)$$

Computation can be based on the Jacobian computed in the previous step, or the vector of velocity of this point can be computed directly. **(10)**

Solution:

(a) The parameters for the DH-frames in Fig. 2 are

Link	a_i	α_i	d_i	θ_i
1	0	0	1	θ_1
2	0	$-\frac{\pi}{2}$	d_2	0
3	0	0	$1 + d_3$	0

where the homogeneous transform $A_i = A_i(\theta_i, d_i, a_i, \alpha_i)$ is defined as a product

$$A_i(\theta_i, d_i, a_i, \alpha_i) = \mathrm{Rot}_{z,\theta_i} \cdot \mathrm{Trans}_{z,d_i} \cdot \mathrm{Trans}_{x,a_i} \cdot \mathrm{Rot}_{x,\alpha_i}$$

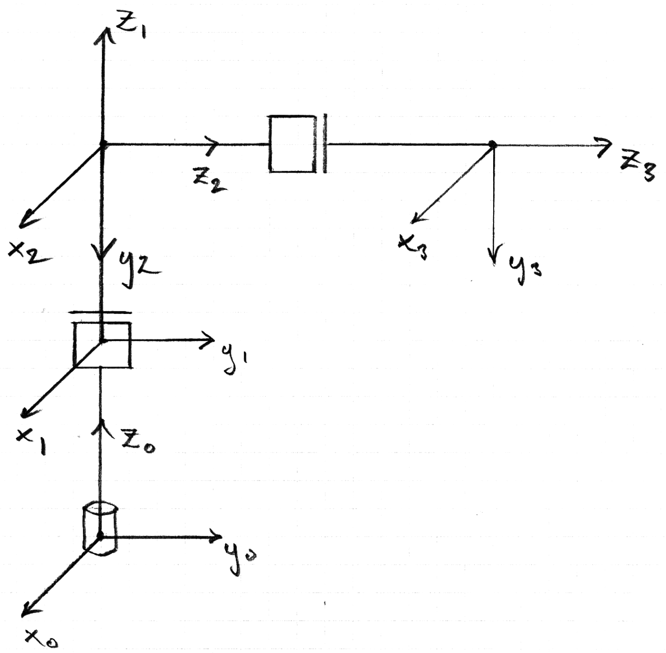

Figure 2: Frames for the robot in problem 2.

resulting in the matrix

$$A_i = A_i(\theta_i, d_i, a_i, \alpha_i) = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With the above parameters we obtain the homogeneous transforms as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The transformation matrices to each link frame from the base frame can now be found as

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1(1+d_3) \\ s_1 & 0 & c_1 & c_1(1+d_3) \\ 0 & -1 & 0 & 1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Given $p_e = (x_e, y_e, z_e)$, we are requested to solve the equations

$$x_e = (1+d_3)\cos(\theta_1+\pi/2), \quad y_e = (1+d_3)\sin(\theta_1+\pi/2), \quad z_e = 1+d_2$$

with respect to variables θ_1, d_2 and d_3. The solutions are

$$d_2 = z_e - 1, \quad d_3 = \sqrt{x_e^2 + y_e^2} - 1, \quad \theta_1 = \mathrm{Atan2}(x_e, y_e) - \pi/2.$$

Alternatively, with the vector to the origin of the tool frame as

$$o_3 = \begin{bmatrix} -s_1(1+d_3) \\ c_1(1+d_3) \\ 1+d_2 \end{bmatrix} = p_e$$

we have the equation set

$$x_e = -(1+d_3)\sin(\theta_1), \quad y_e = (1+d_3)\cos(\theta_1), \quad z_e = 1+d_2.$$

and the solutions become

$$d_2 = z_e - 1, \quad d_3 = \sqrt{x_e^2 + y_e^2} - 1, \quad \theta_1 = \mathrm{Atan2}(y_e, -x_e).$$

(Here we have used $\mathrm{Atan2}(x, y)$ as defined in the textbook.)

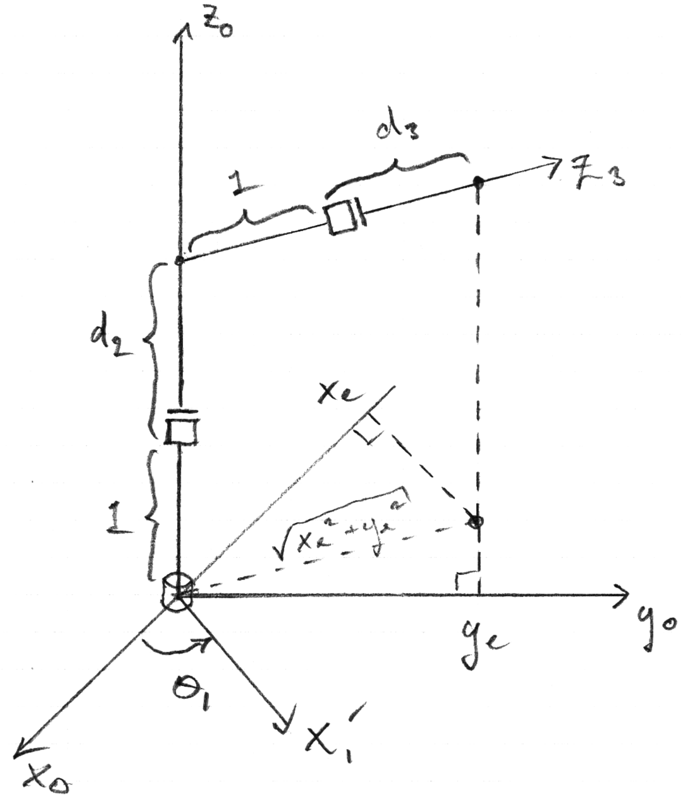

Figure 3: Geometry for inverse kinematics in problem 2.

(c) Given the DH-frames for each of three links of the robot, the manipulator Jacobian is computed as

$$J_\omega = \left[\rho_1 \vec{z}_0,\, \rho_2 \vec{z}_1,\, \rho_3 \vec{z}_2\right], \quad J_v = \left[\vec{z}_0 \times (\vec{p}_e - \vec{o}_0),\, \vec{z}_1,\, \vec{z}_2\right]$$

The gain ρ_1 is equal to 1, because the first joint is revolute, while $\rho_2 = \rho_3 = 0$ because the second and third joints are prismatic. This implies that

$$J_\omega = \left[\vec{z}_0,\, 0 \cdot \vec{z}_1,\, 0 \cdot \vec{z}_2\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Computing of J_v is straightforward if one observes that

$$\vec{z}_0 = \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; \vec{z}_2 = \begin{bmatrix} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{bmatrix}, \; \vec{o}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\vec{p}_e = (1+d_2)\vec{z}_0 + (1+d_3)\vec{z}_2$. Then

$$\vec{z}_0 \times (\vec{p}_e - \vec{o}_0) = (1+d_3)\begin{bmatrix} -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}$$

and the final form for J_v becomes

$$J_v = \begin{bmatrix} -(1+d_3)\cos(\theta_1) & 0 & -\sin(\theta_1) \\ -(1+d_3)\sin(\theta_1) & 0 & \cos(\theta_1) \\ 0 & 1 & 0 \end{bmatrix}.$$

(d) The velocity of any point of tool frame can be computed by the formula

$$\vec{v} = \vec{v}_e + \vec{\omega} \times \vec{r}$$

where $\vec{r}$ is a vector from the origin of the tool frame to the point; $\vec{v}_e$ is the velocity of the origin of the origin of the tool frame and $\vec{\omega}$ is the angular velocity of the tool frame. We are asked to compute the velocity of the origin of the tool frame as a function of θ_1, d_2 and d_3 so that $\vec{r} = 0$ and

$$\vec{v} = J_v(\theta_1, d_2, d_3)\begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

Substituting the functions $\theta_1(t)$, $d_2(t)$ and $d_3(t)$ are their time derivatives into the last formula, we obtain

$$\vec{v} = \begin{bmatrix} -\big(1+\sin(3t)\big)\cos\big(\sin(t)\big) & 0 & -\sin\big(\sin(t)\big) \\ -\big(1+\sin(3t)\big)\sin\big(\sin(t)\big) & 0 & \cos\big(\sin(t)\big) \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \cos(t) \\ -2\sin(2t) \\ 3\cos(3t) \end{bmatrix}$$

multiplication gives the answer as

$$\vec{v} = \begin{bmatrix} -\big(1+\sin(3t)\big)\cos\big(\sin(t)\big)\cos(t) - 3\sin\big(\sin(t)\big)\cos(3t) \\ -\big(1+\sin(3t)\big)\sin\big(\sin(t)\big)\cos(t) + 3\cos\big(\sin(t)\big)\cos(3t) \\ -2\sin(2t) \end{bmatrix}.$$

Additionally, the angular velocity for the tool frame will be

$$\vec{\omega} = J_\omega(\theta_1, d_2, d_3)\begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}.$$

We can immediately see that the answer will be

$$\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix},$$

but since $\vec{r} = 0$ it will not contribute to the final answer.

3. **Problem:** The two link planar robot – the so-called inertia wheel pendulum – depicted on Fig. 4. It has two revolute joints, while the center

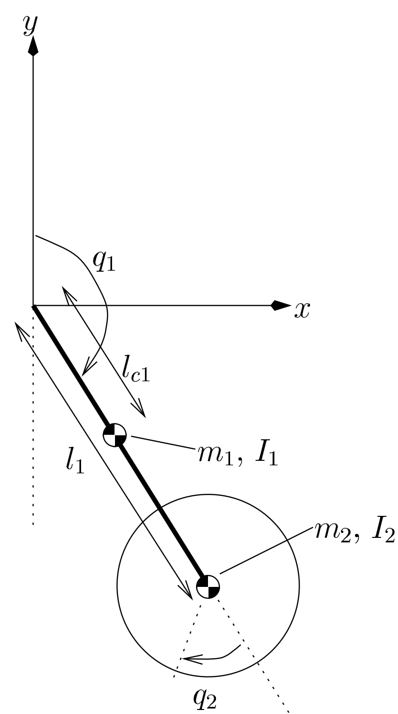

Figure 4: A robot from the problem 3: inertia wheel pendulum

of mass of the second link coincide with its suspension point.

Given physical parameters of the system (masses of the links m_1, m_2; inertias I_1, I_2; the length of the first link l_1 and the distance to its center of mass l_{c1}), you are requested to implement the following tasks:

(a) Find the potential energy $\mathcal{P}(q_1, q_2)$ of the robot **(10)**

(b) Find the kinetic energy $\mathcal{K}(q_1, q_2, \dot{q}_1, \dot{q}_2)$ of the robot **(10)**

(c) Obtain the Euler-Lagrange equations of the system dynamics **(15)**

(d) Assume that both links are actuated and compute the linearization of the system dynamics around the upright equilibrium $q_{1e} = q_{2e} = 0$. Is the resulted linear control system controllable? **(15)**

Solution: Before computing kinetic and potential energies of the inertia wheel pendulum, let us observe that coordinates of centers of masses of its 1st and 2nd links are

$$\left\{ \begin{array}{rcl} x_{c1} & = & l_{c1}\sin(q_1) \\ y_{c1} & = & l_{c1}\cos(q_1) \end{array} \right. \qquad \left\{ \begin{array}{rcl} x_{c2} & = & l_1\sin(q_1) \\ y_{c2} & = & l_1\cos(q_1) \end{array} \right.$$

and, therefore, their velocities are

$$\left\{ \begin{array}{rcl} \dot{x}_{c1} & = & l_{c1}\cos(q_1)\cdot\dot{q}_1 \\ \dot{y}_{c1} & = & -l_{c1}\sin(q_1)\cdot\dot{q}_1 \end{array} \right. \qquad \left\{ \begin{array}{rcl} \dot{x}_{c2} & = & l_1\cos(q_1)\cdot\dot{q}_1 \\ \dot{y}_{c2} & = & -l_1\sin(q_1)\cdot\dot{q}_1 \end{array} \right.$$

(a) The potential energy $\mathcal{P}(\cdot)$ of the robot is the sum of potential energies of the links

$$\begin{array}{rcl} \mathcal{P} & = & m_1 g y_{c1} + m_2 g y_{c2} = m_1 g l_{c1}\cos(q_1) + m_2 g l_1 \cos(q_1) \\ & = & \big(m_1 l_{c1} + m_2 l_1\big)\, g\cos(q_1). \end{array}$$

(b) The kinetic energy of the robot is equal to the sum kinetic energies of links

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$$

where

$$\begin{array}{rcl} \mathcal{K}_1 & = & \dfrac{1}{2}m_1\left(\dot{x}_{c1}^2 + \dot{y}_{c1}^2\right) + \dfrac{1}{2}I_1\omega_1^2 \\ & = & \dfrac{1}{2}m_1\left([l_{c1}\cos(q_1)\cdot\dot{q}_1]^2 + [-l_{c1}\sin(q_1)\cdot\dot{q}_1]^2\right) + \dfrac{1}{2}I_1\dot{q}_1^2 \\ & = & \dfrac{1}{2}\left[m_1 l_{c1}^2 + I_1\right]\dot{q}_1^2 \\ \mathcal{K}_2 & = & \dfrac{1}{2}m_2\left(\dot{x}_{c2}^2 + \dot{y}_{c2}^2\right) + \dfrac{1}{2}I_2\omega_2^2 \\ & = & \dfrac{1}{2}m_1\left([l_1\cos(q_1)\cdot\dot{q}_1]^2 + [-l_1\sin(q_1)\cdot\dot{q}_1]^2\right) + \dfrac{1}{2}I_2\left[\dot{q}_1 + \dot{q}_2\right]^2 \\ & = & \dfrac{1}{2}m_2 l_1^2\dot{q}_1^2 + \dfrac{1}{2}I_2\left[\dot{q}_1 + \dot{q}_2\right]^2 \end{array}$$

Summing the terms we obtain

$$\mathcal{K} = \frac{1}{2}\left[m_1 l_{c1}^2 + I_1 + m_2 l_1^2\right]\dot{q}_1^2 + \frac{1}{2}I_2\left[\dot{q}_1 + \dot{q}_2\right]^2.$$

Alternatively, we can find the kinetic energy using

$$\mathcal{K} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{2}\left\{m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I_i R_i^T J_{\omega_i}\right\}\right]\dot{q} = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

where we can start finding the needed rotation matrices by finding the DH-parameters for the manipulator

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	q_1'
2	0	0	0	q_2'

where $q_i' = \frac{\pi}{2} - q_i$. If we compute the transformation matrices, we obtain

$$T_1^0 = \begin{bmatrix} s_1 & -c_1 & 0 & l_1 s_1 \\ c_1 & s_1 & 0 & l_1 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_2^0 = \begin{bmatrix} -c_{12} & -s_{12} & 0 & l_1 s_1 \\ s_{12} & -c_{12} & 0 & l_1 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The geometric Jacobians for the mass centers will be

$$J_{v_1} = \begin{bmatrix} z_0 \times (o_{c_1} - o_0) & 0 \end{bmatrix} = \begin{bmatrix} -l_{c1}s_1 & 0 \\ l_{c1}c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_1} = \begin{bmatrix} z_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} z_0 \times (o_{c_2} - o_0) & z_1 \times (o_{c_2} - o_1) \end{bmatrix} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_1} = \begin{bmatrix} z_0 & z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

where we note that $o_{c_2} = o_1$.
We now obtain

$$m_1 J_{v_1}^T J_{v_1} = \begin{bmatrix} m_1 \left({l_{c1}}^2 {s_1}^2 + {l_{c1}}^2 {c_1}^2 \right) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 {l_{c1}}^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_2 J_{v_2}^T J_{v_2} = \begin{bmatrix} m_2 {l_1}^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_1}^T R_1 I_1 R_1^T J_{\omega_1} = \begin{bmatrix} I_{1zz} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_2}^T R_2 I_2 R_2^T J_{\omega_2} = \begin{bmatrix} I_{2zz} & I_{2zz} \\ I_{2zz} & I_{2zz} \end{bmatrix} \Rightarrow \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

and

$$D(q) = \begin{bmatrix} m_1 {l_{c1}}^2 + m_2 {l_1}^2 + I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

which when inserted in the expression for the kinetic energy above, yields the same result as already given.

(c) To derive the Euler-Lagrange equations, introduce the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

and compute partial derivatives of $\mathcal{L}$ with respect to variables q_1, q_2, $\dot{q}_1$ and $\dot{q}_2$, i.e.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q_1} &= -\frac{\partial \mathcal{P}}{\partial q_1} = \big(m_1 l_{c1} + m_2 l_1 \big)\, g \sin(q_1) \\ \frac{\partial \mathcal{L}}{\partial q_2} &= -\frac{\partial \mathcal{P}}{\partial q_2} = 0 \\ \frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= \frac{\partial \mathcal{K}}{\partial \dot{q}_1} = \left[m_1 l_{c1}^2 + I_1 + m_2 l_1^2 \right] \dot{q}_1 + I_2 \left[\dot{q}_1 + \dot{q}_2 \right] \\ \frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= \frac{\partial \mathcal{K}}{\partial \dot{q}_2} = I_2 \left[\dot{q}_1 + \dot{q}_2 \right] \end{aligned}$$

The Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right] - \frac{\partial \mathcal{L}}{\partial q_1} &= u_1 \\ \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right] - \frac{\partial \mathcal{L}}{\partial q_2} &= u_2 \end{aligned}$$

are then

$$\begin{aligned} \left[m_1 l_{c1}^2 + I_1 + m_2 l_1^2 + I_2\right]\ddot{q}_1 + I_2\ddot{q}_2 - \left(m_1 l_{c1} + m_2 l_1\right) g \sin(q_1) &= u_1 \\ I_2\ddot{q}_1 + I_2\ddot{q}_2 &= u_2 \end{aligned}$$

or the same in the matrix form

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \phi(q_1) + u_1 \\ u_2 \end{bmatrix}$$

with $d_{11} = m_1 l_{c1}^2 + I_1 + m_2 l_1^2 + I_2$, $d_{12} = d_{21} = d_{22} = I_2$ and $\phi(q_1) = -\left(m_1 l_{c1} + m_2 l_1\right) g \sin(q_1)$. Multiplying both sides of the equation with

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} = \frac{1}{\det D}\begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}, \quad \det D = d_{11}d_{22} - d_{12}d_{21}$$

we resolve the equations with respect to the 2nd time derivatives

$$\begin{aligned} \ddot{q}_1 &= \frac{d_{22}}{\det D}\left[\phi(q_1) + u_1\right] - \frac{d_{12}}{\det D} u_2 \\ \ddot{q}_2 &= -\frac{d_{21}}{\det D}\left[\phi(q_1) + u_1\right] + \frac{d_{11}}{\det D} u_2 \end{aligned} \tag{1}$$

(d) With new variables

$$x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{q}_1, \quad x_4 = \dot{q}_2$$

the dynamics can be rewritten in state-space form as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ \dfrac{d_{22}}{\det D}\phi(x_1) \\ -\dfrac{d_{21}}{\det D}\phi(x_1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{d_{22}}{\det D} & -\dfrac{d_{12}}{\det D} \\ -\dfrac{d_{21}}{\det D} & \dfrac{d_{11}}{\det D} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2}$$

Linearization of the system (2) around the equilibrium

$$x_1 = x_2 = x_3 = x_4 = 0$$

is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \end{bmatrix}}_{=A} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{=B} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

Here $a_1 = -\frac{d_{22}}{\det D}\left(m_1 l_{c1} + m_2 l_1\right) g$, $a_2 = \frac{d_{21}}{\det D}\left(m_1 l_{c1} + m_2 l_1\right) g$, and

$$b_{11} = \frac{d_{22}}{\det D}, \quad b_{12} - \frac{d_{12}}{\det D}, \quad b_{21} = -\frac{d_{21}}{\det D}, \quad b_{22} = \frac{d_{11}}{\det D}$$

The linear system is controllable if the matrix

$$\Gamma = [B, AB] = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right]$$

is of full rank. It will be the case if

$$\det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \neq 0$$

Indeed, this determinant is different from zero because

$$\begin{aligned} \det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \frac{1}{D^2} \det \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix} = \frac{1}{D^2} \cdot D \\ &= \frac{1}{(m_1 l_{c1}^2 + I_1 + m_2 l_1^2) I_2} > 0 \end{aligned}$$