# DISCRETE MATHEMATICS INTRO

**Discrete mathematics** – study of mathematical structures and objects that are fundamentally **discrete** rather than **continuous**.

- **Examples of objects** with discrete values are – **integers**, **graphs**, or **statements in logic**.
- Discrete mathematics and **computer science**. – Concepts from discrete mathematics are useful for describing **objects and problems in computer algorithms and programming languages**. These have applications in cryptography, automated theorem proving, and software development.

Chapter 1. The Foundations: Logic and Proofs

# 1.1 PROPOSITIONAL LOGIC

# 1.1 Propositional Logic……

## Introduction

- **Logic:**
  - defines a formal language for representing knowledge and for making logical inferences
  - It helps us to understand how to construct a valid argument

**Logic defines:**

- Syntax of statements
- The meaning of statements
- The rules of logical inference (manipulation)

- A **proposition** is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.

- Are the following sentences propositions?
  - Kigali is the capital of Rwanda. (Yes)
  - Read this carefully. (No)
  - 1+2=3 (Yes)
  - x+1=2 (No)
  - What time is it? (No)

# 1.1 Propositional Logic…..

- Propositional Logic – the area of logic that deals with propositions
- Propositional Variables – variables that represent propositions: $p$, $q$, $r$, $s$
  - E.g. Proposition $p$ – “Today is Friday.”
- Truth values – T, F

# 1.1 Propositional Logic……..

DEFINITION 1

Let $p$ be a proposition. The negation of $p$, denoted by $\neg p$, is the statement
"It is not the case that $p$."

The proposition $\neg p$ is read "not $p$." The truth value of the negation of $p$, $\neg p$ is the opposite of the truth value of $p$.

- Examples
  - Find the negation of the proposition "Today is Friday." and express this in simple English.
    Solution: The negation is "It is not the case that *today is Friday*."
    In simple English, "Today is not Friday." or "It is not Friday today."
  - Find the negation of the proposition "At least 10 inches of rain fell today in Kigali " and express this in simple English.
    Solution: The negation is "It is not the case that *at least 10 inches of rain fell today in Kigali*"

# 1.1 Propositional Logic……….

- Note: Always assume fixed times, fixed places, and particular people unless otherwise noted.
- Truth table:

The Truth Table for the Negation of a Proposition.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

- Logical operators are used to form new propositions from two or more existing propositions. The logical operators are also called connectives.

# 1.1 Propositional Logic….

DEFINITION 2

Let $p$ and $q$ be propositions. The *conjunction* of $p$ and $q$, denoted by $p \land q$, is the proposition "$p$ and $q$". The conjunction $p \land q$ is true when both $p$ and $q$ are true and is false otherwise.

- Examples
  - Find the conjunction of the propositions $p$ and $q$ where $p$ is the proposition "Today is Friday." and $q$ is the proposition "It is raining today.", and the truth value of the conjunction.

    Solution: The conjunction is the proposition "Today is Friday and it is raining today." The proposition is true on rainy Fridays.

# 1.1 Propositional Logic….

DEFINITION 3

Let $p$ and $q$ be propositions. The *disjunction* of $p$ and $q$, denoted by $p \vee q$, is the proposition "$p$ or $q$". The conjunction $p \vee q$ is false when both $p$ and $q$ are false and is true otherwise.

- Note:

  *inclusive or* : The disjunction is true when at least one of the two propositions is true.
  - E.g. "Students who have taken calculus or computer science can take this class." – those who take one or both classes.

  *exclusive or* : The disjunction is true only when one of the proposition is true.
  - E.g. "Students who have taken calculus or computer science, but not both, can take this class." – only those who take one of them.
- Definition 3 uses *inclusive or*.

# 1.1 Propositional Logic…

DEFINITION 4

Let $p$ and $q$ be propositions. The *exclusive or* of $p$ and $q$, denoted by $p$ $q$, is the $\oplus$ proposition that is true when exactly one of $p$ and $q$ is true and is false otherwise.

The Truth Table for the Conjunction of Two Propositions.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

The Truth Table for the Disjunction of Two Propositions.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

The Truth Table for the Exclusive Or (XOR) of Two Propositions.

| $p$ | $q$ | $p$ $q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# 1.1 Propositional Logic…….

## Conditional Statements

DEFINITION 5

Let $p$ and $q$ be propositions. The *conditional statement* $p \rightarrow q$, is the proposition "if $p$, then $q$." The conditional statement is false when $p$ is true and $q$ is false, and true otherwise. In the conditional statement $p \rightarrow q$, $p$ is called the *hypothesis* (or *antecedent* or *premise*) and $q$ is called the *conclusion* (or *consequence*).

- A conditional statement is also called an implication.
- Example: "If I am elected, then I will lower taxes." $p \rightarrow q$

implication:

| | | | |
|---|---|---|---|
| elected, lower taxes. | T | T | T |
| not elected, lower taxes. | F | T | T |
| not elected, not lower taxes. | F | F | T |
| elected, not lower taxes. | T | F | F |

# 1.1 Propositional Logic……

- Example:
  - Let $p$ be the statement "Maria learns discrete mathematics." and $q$ the statement "Maria will find a good job." Express the statement $p \longrightarrow q$ as a statement in English.

  Solution: Any of the following -

  "If Maria learns discrete mathematics, then she will find a good job.

  "Maria will find a good job when she learns discrete mathematics."

  "For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

  "Maria will find a good job unless she does not learn discrete mathematics."

# Different statement to express $p \rightarrow q$

- You will encounter most if not all of the following ways to express the above  conditional statement

"if $p$, then $q$"
"if $p$, $q$"
"$p$ is sufficient for $q$"
"$q$ if $p$"
"$q$ when $p$"
"a necessary condition for $p$ is $q$"
"$q$ unless $\neg p$"
"$p$ implies $q$"
"$p$ only if $q$"
"a sufficient condition for $q$ is $p$"
"$q$ whenever $p$"
"$q$ is necessary for $p$"
"$q$ follows from $p$"

# 1.1 Propositional Logic…..

- Other conditional statements:
  - Converse of $p \longrightarrow q$ : $q \longrightarrow p$
  - Contrapositive of $p \longrightarrow q$ : $\neg q \longrightarrow \neg p$
  - Inverse of $p \longrightarrow q$ : $\neg p \longrightarrow \neg q$

# 1.1 Propositional Logic……

DEFINITION 6

Let $p$ and $q$ be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "$p$ if and only if $q$." The biconditional statement $p \leftrightarrow q$ is true when $p$ and $q$ have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
- "*if and only if*" can be expressed by "*iff*"
- Example:
  - Let $p$ be the statement "You can take the bus" and let $q$ be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

    "You can take the bus if and only if you buy a ticket."

    Implication:

    If you buy a ticket you can take the bus.

    If you don't buy a ticket you cannot take the bus.

# 1.1 Propositional Logic…..

The Truth Table for the Biconditional $p \leftrightarrow q$.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# 1.1 Propositional Logic…….

## Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \lor \neg q) \longrightarrow (p \land q).$$

The Truth Table of $(p \lor \neg q) \rightarrow (p \land q)$.

| $p$ | $q$ | $\neg q$ | $p \lor \neg q$ | $p \land q$ | $(p \lor \neg q) \rightarrow (p \land q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| F | T | F | F | F | T |
| F | F | T | T | F | F |

# 1.1 Propositional Logic……

## Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.

| Operator | Precedence |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

# 1.1 Propositional Logic……

## Translating English Sentences

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let $q$, $r$, and $s$ represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old." The sentence can be translated into:

$$(r \land \neg s) \rightarrow \neg q.$$

# 1.1 Propositional Logic…….

- Example: How can this English sentence be translated into a logical expression?

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let $a$, $c$, and $f$ represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman." The sentence can be translated into:

$$a \rightarrow (c \vee \neg f).$$

# 1.1 Propositional Logic

## Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*.

| $x$ | $y$ | $x \vee y$ | $x \wedge y$ | $x \quad y$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 |

# 1.1 Propositional Logic

DEFINITION 7

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

- Example: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110

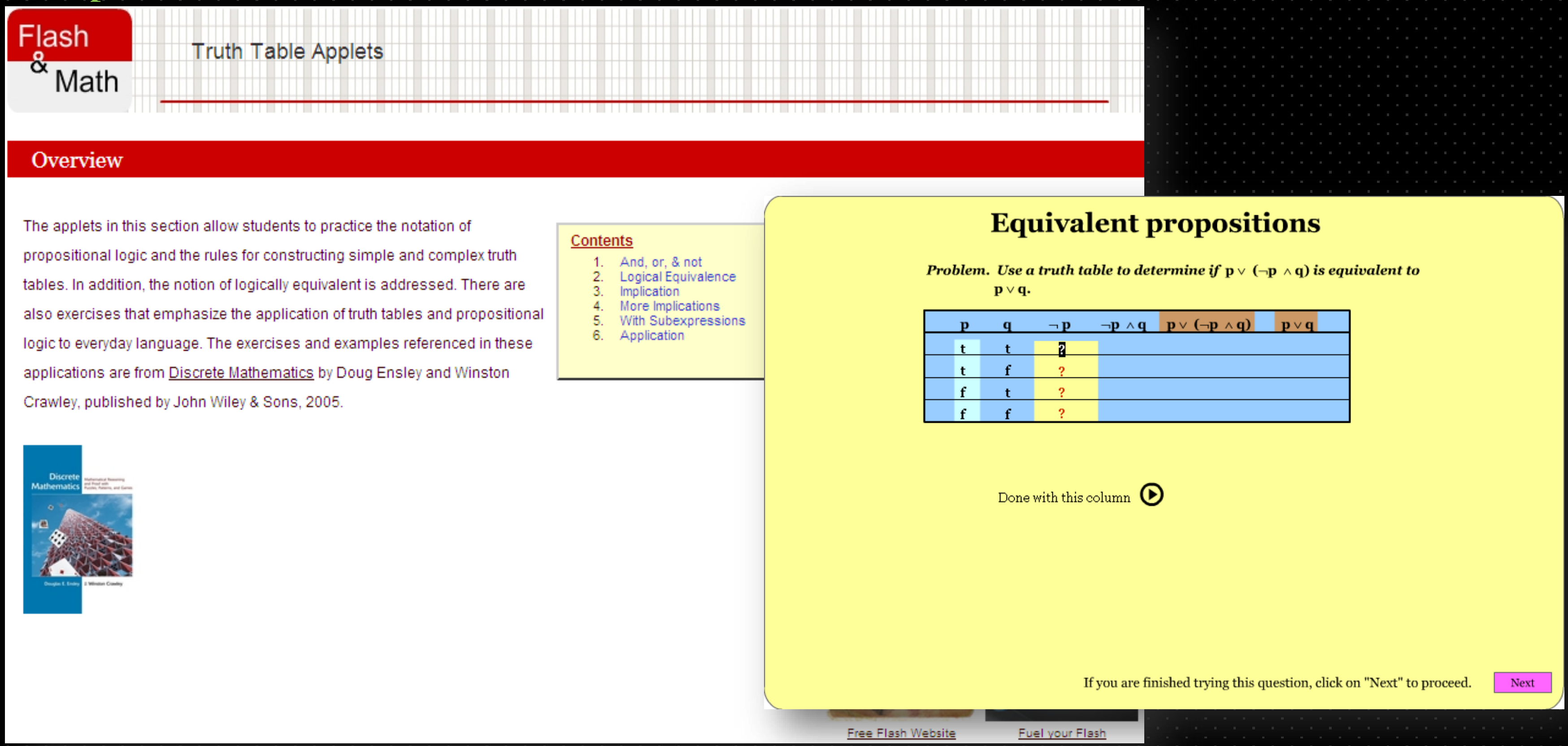
11 0001 1101

---------------

11 1011 1111 bitwise *OR*
01 0001 0100 bitwise *AND*
10 1010 1011 bitwise *XOR*

# Extra Practice

- For Extra Practice you can check out the following links:
  For practice with truth tables, here is a nice ones:
- http://www.flashandmath.com/mathlets/discrete/truthtables/



Chapter 1. The Foundations: Logic and Proofs

# 1.2 PROPOSITIONAL EQUIVALENCES

# 1.2 Propositional Equivalences

## Agenda

- Tautologies
- Logical Equivalences

# 1.2 Propositional Equivalences

## Introduction

DEFINITION 1.1

A compound proposition is called a *tautology* if no matter what truth values of propositions have, its own truth value is T.

E.g. $p \vee \neg p$ (Law of excluded middle)

DEFINITION 1.2

The opposite to a tautology, is a compound proposition that's always false – a *contradiction*.

E.g. $p \wedge \neg p$

DEFINITION 1.3

On the other hand, a compound proposition whose truth value isn't constant is called a *contingency*.

E.g. $p \rightarrow \neg p$

# 1.2 Propositional Equivalences

The easiest way to see if a compound proposition is a tautology/contradiction is to use a truth table.

Examples of a Tautology and a Contradiction.

| $p$ | $\neg p$ | $p \vee \neg p$ | $p \wedge \neg p$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

# 1.2 Propositional Equivalences

## Tautology example Part 1

Demonstrate that

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

is a tautology in two ways:

1. Using a truth table – show that

   $[\neg p \wedge (p \vee q)] \rightarrow q$ is always true

2. Using a proof (will get to this later).

# 1.2 Propositional Equivalences

## Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \rightarrow q$ |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

# 1.2 Propositional Equivalences

## Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \rightarrow q$ |
|---|---|---|---|---|---|
| T | T | F | | | |
| T | F | F | | | |
| F | T | T | | | |
| F | F | T | | | |

# 1.2 Propositional Equivalences

## Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \rightarrow q$ |
|---|---|---|---|---|---|
| T | T | F | T | | |
| T | F | F | T | | |
| F | T | T | T | | |
| F | F | T | F | | |

# 1.2 Propositional Equivalences

## Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \to q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | |
| T | F | F | T | F | |
| F | T | T | T | T | |
| F | F | T | F | F | |

# 1.2 Propositional Equivalences

## Tautology by truth table

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \wedge (p \vee q)$ | $[\neg p \wedge (p \vee q)] \to q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | F | T | F | T |
| F | T | T | T | T | T |
| F | F | T | F | F | T |

# 1.2 Propositional Equivalences

## Tautologies, contradictions and programming

Tautologies and contradictions in your code usually correspond to poor programming design. E.g.

- ```
  while(x <= 3 || x > 3)
        x++;
  ```
- ```
  if(x > y)
        if(x == y)
              return "never got here";
  ```

# 1.2 Propositional Equivalences

## Logical Equivalences

DEFINITION 2

Two compound propositions $p$, q are *logically equivalent* if their biconditional joining $p \leftrightarrow q$ is a *tautology*. *Logical equivalence* is denoted by $p \Leftrightarrow q$ ($p \equiv q$ ).

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- E.g. Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$ .

| $p$ | $q$ | $\neg p$ | $p \rightarrow q$ | $\neg p \vee q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

E.g. The *contrapositive* of a logical implication is the reversal of the implication, while negating both components. I.e. the *contrapositive* of $p \rightarrow q$ is $\neg q \rightarrow \neg p$ . As we'll see next: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| | | |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| | | | | |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| | | | | |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| T | T | F | | |
| T | F | T | | |
| F | T | F | | |
| F | F | T | | |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| T | T | F | F | |
| T | F | T | F | |
| F | T | F | T | |
| F | F | T | T | |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $p$ | $q$ | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | T | F | F |
| F | T | F | T | T |
| F | F | T | T | T |

Q: why does this work given definition of $\Leftrightarrow$ ?

# 1.2 Propositional Equivalences

## Logical Equivalences

A: $p \Leftrightarrow q$ by definition means that $p \leftrightarrow q$ is a tautology. Furthermore, the biconditional is true exactly when the truth values of $p$ and of $q$ are identical. So if the last column of truth tables of $p$ and of $q$ is identical, the biconditional join of both is a tautology.

# 1.2 Propositional Equivalences

## Logical Non-Equivalence of Conditional and Converse

The *converse* of a logical implication is the reversal of the implication. I.e. the converse of $p \rightarrow q$ is $q \rightarrow p$.

E.g. The converse of "If Donald is a duck then Donald is a bird." is "If Donald is a bird then Donald is a duck."

As we'll see next: $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent.

# 1.2 Propositional Equivalences

Logical Non-Equivalence of Conditional and Converse

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \leftrightarrow (q \to p)$ |
|---|---|---|---|---|
| | | | | |

# 1.2 Propositional Equivalences

## Logical Non-Equivalence of Conditional and Converse

| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

# 1.2 Propositional Equivalences

Logical Non-Equivalence of Conditional and Converse

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \leftrightarrow (q \to p)$ |
|---|---|---|---|---|
| T | T | T | | |
| T | F | F | | |
| F | T | T | | |
| F | F | T | | |

# 1.2 Propositional Equivalences

Logical Non-Equivalence of Conditional and Converse

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \leftrightarrow (q \to p)$ |
|---|---|---|---|---|
| T | T | T | T | |
| T | F | F | T | |
| F | T | T | F | |
| F | F | T | T | |

# 1.2 Propositional Equivalences

## Logical Non-Equivalence of Conditional and Converse

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \leftrightarrow (q \to p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

# 1.2 Propositional Equivalences

## Derivational Proof Techniques

When compound propositions involve more and more atomic components, the size of the truth table for the compound propositions increases

Q1: How many rows are required to construct the truth-table of:
$( (q \leftrightarrow (p \rightarrow r )) \wedge (\neg(s \wedge r) \vee \neg t) ) \rightarrow (\neg q \rightarrow r )$

Q2: How many rows are required to construct the truth-table of a proposition involving $n$ atomic components?

# 1.2 Propositional Equivalences

## Derivational Proof Techniques

A1: 32 rows, each additional variable doubles the number of rows

A2: In general, $2^n$ rows

Therefore, as compound propositions grow in complexity, truth tables become more and more unwieldy. Checking for tautologies/logical equivalences of complex propositions can become a chore, especially if the problem is obvious.

# 1.2 Propositional Equivalences

## Derivational Proof Techniques

E.g. consider the compound proposition

$$(p \rightarrow p) \vee (\neg(s \wedge r) \vee \neg t)) \vee (\neg q \rightarrow r)$$

Q: Why is this a tautology?

# 1.2 Propositional Equivalences

## Derivational Proof Techniques

A: Part of it is a tautology $(p \rightarrow p)$ and the disjunction of True with any other compound proposition is still True:

$(p \rightarrow p) \lor (\neg(s \land r) \lor \neg t)) \lor (\neg q \rightarrow r)$

$\Leftrightarrow \quad T \lor (\neg(s \land r) \lor \neg t)) \lor (\neg q \rightarrow r)$

$\Leftrightarrow \quad T$

Derivational techniques formalize the intuition of this example.

# Tables of Logical Equivalences

- Identity laws
  Like adding 0
- Domination laws
  Like multiplying by 0
- Idempotent laws
  Delete redundancies
- Double negation
  "I don't like you, not"
- Commutativity
  Like "$x+y = y+x$"
- Associativity
  Like "$(x+y)+z = y+(x+z)$"
- Distributivity
  Like "$(x+y)z = xz+yz$"
- De Morgan

**TABLE Logical Equivalences.**

| *Equivalence* | *Name* |
|---|---|
| $p \wedge \mathbf{T} \Leftrightarrow p$<br>$p \vee \mathbf{F} \Leftrightarrow p$ | Identity laws |
| $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$<br>$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$ | Domination laws |
| $p \vee p \Leftrightarrow p$<br>$p \wedge p \Leftrightarrow p$ | Idempotent laws |
| $\neg(\neg p) \Leftrightarrow p$ | Double negation law |
| $p \vee q \Leftrightarrow q \vee p$<br>$p \wedge q \Leftrightarrow q \wedge p$ | Commutative laws |
| $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ | Associative laws |
| $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$<br>$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ | De Morgan's laws |

# Tables of Logical Equivalences

- Excluded middle
- Negating created opposite
- Definition of implication in terms of Not and Or

**TABLE Some Useful Logical Equivalences.**

| | |
|---|---|
| $p \vee \neg p \iff \mathbf{T}$ | ULE 1 |
| $p \wedge \neg p \iff \mathbf{F}$ | ULE 2 |
| $(p \rightarrow q) \iff (\neg p \vee q)$ | ULE 3 |

# 1.2 Propositional Equivalences

## DeMorgan Identities

DeMorgan's identities allow for simplification of negations of complex expressions

- Conjunctional negation:

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$$

"It's not the case that all are true iff one is false."

- Disjunctional negation:

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$$

"It's not the case that one is true iff all are false."

# 1.2 Propositional Equivalences

## Tautology example Part 2

Demonstrate that

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

is a tautology in two ways:

1. Using a truth table (did above)
2. Using a proof relying on Tables 1 and 2 to derive **True** through a series of logical equivalences

# 1.2 Propositional Equivalences

## Constructing New Logical Equivalences

- E.g. Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent.

Solution:

$\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \lor q)$ *by example on slide 35*

$\Leftrightarrow \neg(\neg p) \land \neg q$ *by the De Morgan law*

$\Leftrightarrow p \land \neg q$ *by the double negation law*

- *E.g. Show that* $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \land q) \rightarrow (p \lor q) \Leftrightarrow \neg(p \land q) \lor (p \lor q)$ *by example on slide 35*

$\Leftrightarrow (\neg p \lor \neg q) \lor (p \lor q)$ *by the De Morgan law*

$\Leftrightarrow (\neg p \lor p) \lor (\neg q \lor q)$ *by the associative and communicative law for disjunction*

$\Leftrightarrow \mathrm{T} \lor \mathrm{T}$

$\Leftrightarrow \mathrm{T}$

- *Note: The above examples can also be done using truth tables.*

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\, F \vee (\neg p \wedge q)] \rightarrow q$ ULE

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\, F \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q\,] \rightarrow q$ Identity

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\, F \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q\,] \rightarrow q$ Identity

$\Leftrightarrow \neg\, [\neg p \wedge q\,] \vee q$ ULE

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\, F \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q\,] \rightarrow q$ Identity

$\Leftrightarrow \neg\, [\neg p \wedge q\,] \vee q$ ULE

$\Leftrightarrow [\neg(\neg p) \vee \neg q\,] \vee q$ DeMorgan

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\mathrm{F} \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q] \rightarrow q$ Identity

$\Leftrightarrow \neg [\neg p \wedge q] \vee q$ ULE

$\Leftrightarrow [\neg(\neg p) \vee \neg q] \vee q$ DeMorgan

$\Leftrightarrow [p \vee \neg q] \vee q$ Double Negation

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\mathrm{F} \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q] \rightarrow q$ Identity

$\Leftrightarrow \neg [\neg p \wedge q] \vee q$ ULE

$\Leftrightarrow [\neg(\neg p) \vee \neg q] \vee q$ DeMorgan

$\Leftrightarrow [p \vee \neg q] \vee q$ Double Negation

$\Leftrightarrow p \vee [\neg q \vee q]$ Associative

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ Distributive

$\Leftrightarrow [\mathrm{F} \vee (\neg p \wedge q)] \rightarrow q$ ULE

$\Leftrightarrow [\neg p \wedge q] \rightarrow q$ Identity

$\Leftrightarrow \neg [\neg p \wedge q] \vee q$ ULE

$\Leftrightarrow [\neg(\neg p) \vee \neg q] \vee q$ DeMorgan

$\Leftrightarrow [p \vee \neg q] \vee q$ Double Negation

$\Leftrightarrow p \vee [\neg q \vee q]$ Associative

$\Leftrightarrow p \vee [q \vee \neg q]$ Commutative

# Tautology by proof

$[\neg p \wedge (p \vee q)] \rightarrow q$

| | |
|---|---|
| $\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ | Distributive |
| $\Leftrightarrow [\, F \vee (\neg p \wedge q)] \rightarrow q$ | ULE |
| $\Leftrightarrow [\neg p \wedge q\,] \rightarrow q$ | Identity |
| $\Leftrightarrow \neg\, [\neg p \wedge q\,] \vee q$ | ULE |
| $\Leftrightarrow [\neg(\neg p) \vee \neg q\,] \vee q$ | DeMorgan |
| $\Leftrightarrow [p \vee \neg q\,] \vee q$ | Double Negation |
| $\Leftrightarrow p \vee [\neg q \vee q\,]$ | Associative |
| $\Leftrightarrow p \vee [q \vee \neg q\,]$ | Commutative |
| $\Leftrightarrow p \vee T$ | ULE |

# Tautology by proof

| | |
|---|---|
| $[\neg p \wedge (p \vee q)] \rightarrow q$ | |
| $\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$ | Distributive |
| $\Leftrightarrow [\mathrm{F} \vee (\neg p \wedge q)] \rightarrow q$ | ULE |
| $\Leftrightarrow [\neg p \wedge q] \rightarrow q$ | Identity |
| $\Leftrightarrow \neg [\neg p \wedge q] \vee q$ | ULE |
| $\Leftrightarrow [\neg(\neg p) \vee \neg q] \vee q$ | DeMorgan |
| $\Leftrightarrow [p \vee \neg q] \vee q$ | Double Negation |
| $\Leftrightarrow p \vee [\neg q \vee q]$ | Associative |
| $\Leftrightarrow p \vee [q \vee \neg q]$ | Commutative |
| $\Leftrightarrow p \vee \mathrm{T}$ | ULE |
| $\Leftrightarrow \mathrm{T}$ | Domination |

# How to prove two propositions are equivalent?

- Two methods:
  - Using truth tables
    - Not good for long formula
    - In this course, only allowed if specifically stated!
  - Using the logical equivalences
    - The preferred method

- Example:
  - Show that:

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

# Using Truth Tables

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

| p | q | r | $p \to r$ | $q \to r$ | $(p \to r) \vee (q \to r)$ | $p \wedge q$ | $(p \wedge q) \to r$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | F | T | F |
| T | F | T | T | T | T | F | T |
| T | F | F | F | T | T | F | T |
| F | T | T | T | T | T | F | T |
| F | T | F | T | F | T | F | T |
| F | F | T | T | T | T | F | T |
| F | F | F | T | T | T | F | T |

# Using Logical Equivalences

$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$ Original statement

$(\neg p \lor r) \lor (\neg q \lor r) \equiv \neg(p \land q) \lor r$

Definition of implication $p \to q \equiv \neg p \lor q$

$(\neg p \lor r) \lor (\neg q \lor r) \equiv (\neg p \lor \neg q) \lor r$

De Morgan's Law $\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg p \lor r \lor \neg q \lor r \equiv (\neg p \lor \neg q) \lor r$

Associativity of Or $(\neg p \lor r) \lor (\neg q \lor r) \equiv \neg p \lor r \lor \neg q \lor r$

$\neg p \lor \neg q \lor r \lor r \equiv \neg p \lor \neg q \lor r$

Rearranging

$\neg p \lor \neg q \lor r \equiv \neg p \lor \neg q \lor r$

Idempotent Law $r \lor r \equiv r$

# Quick survey

- I understood the logical equivalences on the last slide

a) Very well
b) Okay
c) Not really
d) Not at all

# Logical Thinking

- At a trial:
  - Bill says: "Sue is guilty and Fred is innocent."
  - Sue says: "If Bill is guilty, then so is Fred."
  - Fred says: "I am innocent, but at least one of the others is guilty."

- Let b = Bill is innocent, f = Fred is innocent, and s = Sue is innocent
- Statements are:
  - $\neg s \wedge f$
  - $\neg b \rightarrow \neg f$
  - $f \wedge (\neg b \vee \neg s)$

# Can all of their statements be true?

- Show: $(\neg s \wedge f) \wedge (\neg b \rightarrow \neg f) \wedge (f \wedge (\neg b \vee \neg s))$

| b | f | s | ¬b | ¬f | ¬s | ¬s∧f | ¬b→¬f | | f∧(¬b∨¬s) |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | T | | F |
| T | T | F | F | F | T | T | T | | T |
| T | F | T | F | T | F | F | T | | F |
| T | F | F | F | T | T | F | T | | F |
| F | T | T | T | F | F | F | F | | T |
| F | T | F | T | F | T | T | F | | T |
| F | F | T | T | T | F | F | T | | F |
| F | F | F | T | T | T | F | T | | F |

# Are all of their statements true? Show values for s, b, and f such that the equation is true

| | |
|---|---|
| $(\neg s \wedge f) \wedge (\neg b \rightarrow \neg f) \wedge (f \wedge (\neg b \vee \neg s)) \equiv T$ | Original statement |
| $(\neg s \wedge f) \wedge (b \vee \neg f) \wedge (f \wedge (\neg b \vee \neg s)) \equiv T$ | Definition of implication |
| $\neg s \wedge f \wedge (b \vee \neg f) \wedge f \wedge (\neg b \vee \neg s) \equiv T$ | Associativity of AND |
| $\neg s \wedge f \wedge f \wedge (b \vee \neg f) \wedge (\neg b \vee \neg s) \equiv T$ | Re-arranging |
| $\neg s \wedge f \wedge (b \vee \neg f) \wedge (\neg b \vee \neg s) \equiv T$ | Idempotent law |
| $f \wedge (b \vee \neg f) \wedge \neg s \wedge (\neg s \vee \neg b) \equiv T$ | Re-arranging |
| $f \wedge (b \vee \neg f) \wedge \neg s \equiv T$ | Absorption law |
| $(f \wedge (b \vee \neg f)) \wedge \neg s \equiv T$ | Re-arranging |
| $((f \wedge b) \vee (f \wedge \neg f)) \wedge \neg s \equiv T$ | Distributive law |
| $((f \wedge b) \vee F) \wedge \neg s \equiv T$ | Negation law |
| $(f \wedge b) \wedge \neg s \equiv T$ | Domination law |
| $f \wedge b \wedge \neg s \equiv T$ | Associativity of AND |

# What if it weren't possible to assign such values to s, b, and f?

| | |
|---|---|
| $(\neg s \wedge f) \wedge (\neg b \to \neg f) \wedge (f \wedge (\neg b \vee \neg s)) \wedge s = T$ | Original statement |
| $(\neg s \wedge f) \wedge (b \vee \neg f) \wedge (f \wedge (\neg b \vee \neg s)) \wedge s = T$ | Definition of implication |
| | … (same as previous slide) |
| $(f \wedge b) \wedge \neg s \wedge s = T$ | Domination law |
| $f \wedge b \wedge \neg s \wedge s = T$ | Re-arranging |
| $f \wedge b \wedge F = T$ | Negation law |
| $f \wedge F = T$ | Domination law |
| $F = T$ | Domination law |
| | Contradiction! |

# Quick survey

- I feel I can prove a logical equivalence myself

a) Absolutely
b) With a  bit more practice
c) Not really
d) Not at all