



1.3 PREDICATES AND QUANTIFIERS

Agenda

- ◆ Predicates and Quantifiers
 - ◆ Existential Quantifier \exists
 - ◆ Universal Quantifier \forall



INTRO

Propositional logic studied in Chapter 1 cannot adequately express the meaning of all statements in mathematics and in natural language

e.g: suppose that we know that

1. “Every computer connected to the university network is functioning properly.”
2. “MATH3 is functioning properly,”
3. “CS2 is under attack by an intruder,”
4. “There is a computer on the university network that is under attack by an intruder.”

No rules of propositional logic allow us to conclude the truth of the statements above

1.3 PREDICATES



Statements involving variables, such as

“ $x > 3$,” “ $x = y + 3$,” “ $x + y = z$,”

“computer x is under attack by an intruder,”

“computer x is functioning properly,”

- ✓ Are often found in mathematical assertions, in computer programs, and in system specifications.
- ✓ These statements are neither true nor false when the values of the variables are not specified
- ✓ The statement “ x is greater than 3” has two parts. The first part, the variable x , is the subject of the statement.
- ✓ The second part—the **predicate**, “is greater than 3”—refers to a property that the subject of the statement can have
- ✓ We can denote the statement “ x is greater than 3” by $P(x)$, where P denotes the **predicate** “is greater than 3” and x is the variable
- ✓ The statement $P(x)$ is also said to be the value of the **propositional function** P at x
- ✓ Once a value has been assigned to the variable x , the statement $P(x)$ becomes a proposition and has a truth value

Motivating example



Consider the compound proposition:

If Jobs is an octopus then he has 8 limbs.

Q1: What are the atomic propositions and how do they form this proposition.

Q2: Is the proposition true or false?

Q3: Why?

Motivating example



A1: Let p = “Jobs is an octopus” and q = “Jobs has 8 limbs”.

The compound proposition is represented by $p \rightarrow q$.

A2: True!

A3: Conditional always true when p is false!

Q: Why is this not satisfying?

Motivating example



A: We wanted this to be true because of the fact that octopi have 8 limbs and not because of some (important) non-semantic technicality in the truth table of implication.

But recall that propositional calculus doesn't take semantics into account so there is no way that p could impact on q or affect the truth of $p \rightarrow q$.

Logical Quantifiers help to fix this problem. In our case the fix would look like:

For all x , if x is an octopus then x has 8 limbs.

Motivating example



Expressions such as the previous are built up from *propositional functions* —statements that have a variable or variables that represent various possible subjects. Then quantifiers are used to bind the variables and create a proposition with embedded semantics. For example:

“For all x , if x is an octopus then x has 8 limbs.”

there are two atomic propositional functions

- ◆ $P(x) = \text{“}x \text{ is an octopus”}$
- ◆ $Q(x) = \text{“}x \text{ has 8 limbs”}$

whose conditional $P(x) \rightarrow Q(x)$ is formed and is bound by “For all x ”.

Semantics



If logical propositions are to have meaning, they need to describe something. Up to now, propositions such as “Johnny is tall”, “Debbie is 5 years old”, and “Andre is immature” had no intrinsic meaning. Either they are true or false, but no more.

In order to endow such propositions with meaning, we need to have a *universe of discourse*, i.e. a collection of *subjects* (or *nouns*) about which the propositions relate.

Q: What is the universe of discourse for the three propositions above?

Semantics



A: There are many answers. Here are some:

- ◆ Johnny, Debbie and Andre (this is also the smallest correct answer)
- ◆ People in the world
- ◆ Animals

Java: The Java analog of a universe of discourse is a **type**. There are two categories of types in Java: *reference types* which consist of objects and arrays, and *primitive types* like **int**, **boolean**, **char**, etc. Examples of Java “universes” are:

int, **char**, **int[][]**, **Object**, **String**,
java.util.LinkedList, **Exception**, etc.

Predicates



A *predicate* is a property or description of subjects in the universe of discourse. The following predicates are all *italicized* :

- ◆ Johnny *is tall*.
- ◆ The bridge *is structurally sound*.
- ◆ 17 *is a prime number*.

Java: predicates are **boolean**-valued method calls-

- ◆ **someLinkedList**.*isEmpty()*
- ◆ *isPrime*(17)

Propositional Functions



By taking a variable subject denoted by symbols such as x , y , z , and applying a predicate one obtains a *propositional function* (or *formula*). When an object from the universe is plugged in for x , y , etc. a truth value results:

- ◆ x is tall. ...e.g. plug in $x = \text{Johnny}$
- ◆ y is structurally sound. ...e.g. plug in $y = \text{GWB}$
- ◆ n is a prime number. ...e.g. plug in $n = 111$

Java: propositional functions are **boolean** methods, rather than particular calls.

- ◆ `boolean isEmpty() {...} //in LinkedList`
- ◆ `boolean isPrime(int n) {...}`

Multivariable Predicates



Multivariable predicates generalize predicates to allow descriptions of relationships between subjects. These subjects may or may not even be in the same universe of discourse. For example:

- ◇ Johnny is taller than Debbie.
- ◇ 17 is greater than one of 12, 45.
- ◇ Johnny is at least 5 inches taller than Debbie.

Q: What universes of discourse are involved?

Multivariable Predicates



A: Again, many correct answers. The most obvious answers are:

- ◆ For “Johnny *is taller than* Debbie”
 - the universe of discourse of **both** variables is all people in the universe
- ◆ For “17 *is greater than one of* 12, 45”
 - the universe of discourse of all three variables is \mathbb{Z} (the set of integers)
- ◆ For “Johnny *is at least 5 inches taller than* Debbie”
 - the first and last variable have people as their universe of discourse, while the second variable has \mathbb{R} (the set of real numbers).

Multivariable Propositional Functions



The multivariable predicates, together with their variables create *multivariable propositional functions*. In the above examples, we have the following generalizations:

- ◇ x is taller than y
- ◇ a is greater than one of b, c
- ◇ x is at least n inches taller than y

In Java, a multivariable predicate is a **boolean** method with several arguments:

```
tallerByNumInches(Person x, double n, Person y)
{ ... }
```


Quantifiers



There are two quantifiers

- ◆ Existential Quantifier
“ \exists ” reads “there exists”
- ◆ Universal Quantifier
“ \forall ” reads “for all”

Each is placed in front of a propositional function and *binds* it to obtain a proposition with semantic value.

Existential Quantifier



- ◆ “ $\exists x P(x)$ ” is true when *an* instance can be found which when plugged in for x makes $P(x)$ true
- ◆ Like disjunctioning over entire universe $\exists x P(x)$
 $\Leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \dots$

Existential Quantifier. Example



Consider a universe consisting of

- ◊ Leo: a lion
- ◊ Jan: an octopus with all 8 tentacles
- ◊ Bill: an octopus with only 7 tentacles

And recall the propositional functions

- ◊ $P(x)$ = “ x is an octopus”
- ◊ $Q(x)$ = “ x has 8 limbs”

$$\exists x (P(x) \wedge Q(x))$$

Q: Is the proposition true or false?

Existential Quantifier. Example



A: True. Proposition is equivalent to
 $(P(\text{Leo}) \wedge Q(\text{Leo})) \vee (P(\text{Jan}) \wedge Q(\text{Jan})) \vee (P(\text{Bill}) \wedge Q(\text{Bill}))$

$P(\text{Jan})$ is true because Jan is an octopus with all 8 tentacles.

$P(\text{Jan}) \wedge Q(\text{Jan})$ is true, and the disjunction is true.

Quick survey



- ◆ I understand existential quantification
 - a) Very well
 - b) With some review, I'll be good
 - c) Not really
 - d) Not at all

The Universal Quantifier



- ◆ “ $\forall x P(x)$ ” true when *every* instance of x makes $P(x)$ true when plugged in
- ◆ Like conjunctioning over entire universe

$$\forall x P(x) \iff P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots$$

Universal Quantifier. Example



Consider the same universe and propositional functions as before.

$$\forall x (P(x) \rightarrow Q(x))$$

Q: Is the proposition true or false?

Universal Quantifier. Example



A: False. The proposition is equivalent to

$$(P(\text{Leo}) \rightarrow Q(\text{Leo})) \wedge (P(\text{Jan}) \rightarrow Q(\text{Jan})) \wedge (P(\text{Bill}) \rightarrow Q(\text{Bill}))$$

Bill is the *counter-example*, i.e. a value making an instance –and therefore the whole universal quantification– false.

$P(\text{Bill})$ is true because Bill is an octopus, while $Q(\text{Bill})$ is false because Bill only has 7 tentacles, not 8. Thus the conditional $P(\text{Bill}) \rightarrow Q(\text{Bill})$ is false since $T \rightarrow F$ gives F, and the conjunction is false.

Universal Quantification



- ◆ Think of \forall as a for loop:
- ◆ $\forall x P(x)$, where $1 \leq x \leq 10$
- ◆ ... can be translated as ...

```
for ( x = 1; x <= 10; x++ )  
    is P(x) true?
```

- ◆ If $P(x)$ is true for all parts of the for loop, then $\forall x P(x)$
 - ◆ Consequently, if $P(x)$ is false for any one value of the for loop, then $\forall x P(x)$ is false

Quick survey



- ◆ I understand universal quantification
 - a) Very well
 - b) With some review, I'll be good
 - c) Not really
 - d) Not at all

Illegal Quantifications



Once a variable has been bound, we cannot bind it again. For example the expression

$$\forall x (\forall x P (x))$$

is nonsensical. The interior expression $(\forall x P (x))$ bounded x already and therefore made it unobservable to the outside. Going back to our example, the English equivalent would be:

Everybody is an *everybody is an octopus*.

Multivariate Quantification



Quantification involving only one variable is fairly straightforward. Just a bunch of OR's or a bunch of AND's.

When two or more variables are involved each of which is bound by a quantifier, the order of the binding is important and the meaning often requires some thought.

Parsing Example



A: True.

For any “exists” we need to find a positive instance. Since x is the first variable in the expression and is “existential”, we need a number that works for all other y, z . Set $x = 0$ (want to ensure that $y - x$ is not too small).

Order matters



Set the universe of discourse to be all natural numbers $\{0, 1, 2, 3, \dots\}$.

Let $R(x, y) = "x < y"$.

Q1: What does $\forall x \exists y R(x, y)$ mean?

Q2: What does $\exists y \forall x R(x, y)$ mean?

Order matters



$R(x, y) = "x < y"$

A1: $\forall x \exists y R(x, y)$:

“All numbers x admit a bigger number y ”

A2: $\exists y \forall x R(x, y)$:

“Some number y is bigger than all x ”

Q: What's the true value of each expression?

Order matters



A: 1 is true and 2 is false.

$\forall x \exists y R(x, y)$: “All numbers x admit a bigger number y ”
--just set $y = x + 1$

$\exists y \forall x R(x, y)$: “Some number y is bigger than all numbers x ”
-- y is never bigger than itself, so setting $x = y$ is a counter example

Q: What if we have two quantifiers of the same kind?
Does order still matter?

Order matters –but not always



A: No! If we have two quantifiers of the same kind order is irrelevant.

$\forall x \forall y$ is the same as $\forall y \forall x$ because these are both interpreted as “for every combination of x and y ...”

$\exists x \exists y$ is the same as $\exists y \exists x$ because these are both interpreted as “there is a pair x, y ...”

Logical Equivalence with Formulas



DEF: Two logical expressions possibly involving propositional formulas and quantifiers are said to be *logically equivalent* if no-matter what universe and what particular propositional formulas are plugged in, the expressions always have the same truth value.

EG: $\forall x \exists y Q(x,y)$ and $\forall y \exists x Q(y,x)$ are equivalent
—names of variables don't matter.

EG: $\forall x \exists y Q(x,y)$ and $\exists y \forall x Q(x,y)$ are not!

DeMorgan Revisited



Recall DeMorgan's identities:

◆ Conjunctive negation:

$$\neg(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \dots \vee \neg p_n)$$

◆ Disjunctive negation:

$$\neg(p_1 \vee p_2 \vee \dots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n)$$

Since the quantifiers are the same as taking a bunch of AND's (\forall) or OR's (\exists) we have:

◆ Universal negation:

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

◆ Existential negation:

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

Negation Example



Compute: $\neg \forall x \exists y \ x^2 \leq y$

In English, we are trying to find the opposite of “every x admits a y greater or equal to x ’s square”. The opposite is that “some x does not admit a y greater or equal to x ’s square”

Algebraically, one just flips all quantifiers from \forall to \exists and vice versa, and negates the interior propositional function. In our case we get:

$$\exists x \forall y \neg (x^2 \leq y) \iff \exists x \forall y \ x^2 > y$$

Blackboard Exercises



Show that the following are logically equivalent:

- ◆ $(\forall x A(x)) \vee (\forall x B(x))$
- ◆ $\forall x, y A(x) \vee B(y)$

Blackboard Exercises



Need to show that

$$(\forall x A(x)) \vee (\forall x B(x)) \leftrightarrow \forall x, y A(x) \vee B(y)$$

is a tautology, so LHS and RHS must always have same truth values.

Blackboard Exercises



$$(\forall x A(x)) \vee (\forall x B(x)) \leftrightarrow \forall x, y A(x) \vee B(y)$$

CASE I) Assuming LHS true show RHS true.

Either $\forall x A(x)$ true OR $\forall x B(x)$ true

Case I.A) For all x , $A(x)$ true.

As $(T \vee \text{anything}) = T$, we can set

$\text{anything} = B(y)$ and obtain

For all x and y , $A(x) \vee B(y)$ –the RHS!

Case I.B) For all x , $B(x)$ true... similar to case (I.A)

Blackboard Exercises

$$(\forall x A(x)) \vee (\forall x B(x)) \leftrightarrow \forall x, y A(x) \vee B(y)$$



CASE II) Assume LHS false, show RHS false.

Both $\forall x A(x)$ false AND $\forall x B(x)$ false.

Thus $\exists x \neg A(x)$ true AND $\exists x \neg B(x)$ true.

Thus there is an example x_1 for which $A(x_1)$ is false and an example x_2 for which $B(x_2)$ is false.

As $F \vee F = F$, we have $A(x_1) \vee B(x_2)$ false.

Setting $x = x_1$ and $y = x_2$ gives a counterexample to $\forall x, y A(x) \vee B(y)$ showing the RHS false.

//QED

Translating from English



- ◆ Many ways to translate a given sentence
- ◆ Goal is to produce a logical expression that is simple and can be easily used in subsequent reasoning
- ◆ Steps:
 - ◆ Clearly identify the appropriate quantifier(s)
 - ◆ Introduce variable(s) and predicate(s)
 - ◆ Translate using quantifiers, predicates, and logical operators

Translating from English



- ◆ Consider “For every student in this class, that student has studied calculus”
- ◆ Rephrased: “For every student x in this class, x has studied calculus”
 - ◆ Let $C(x)$ be “ x has studied calculus”
 - ◆ Let $S(x)$ be “ x is a student”
- ◆ $\forall x C(x)$
 - ◆ True if the universe of discourse is all students in this class

Translating from English 2



- ◆ What about if the universe of discourse is all students (or all people?)
 - ◆ $\forall x (S(x) \wedge C(x))$
 - ◆ This is wrong! Why?
 - ◆ $\forall x (S(x) \rightarrow C(x))$
- ◆ Another option:
 - ◆ Let $Q(x,y)$ be “x has studied y”
 - ◆ $\forall x (S(x) \rightarrow Q(x, \text{calculus}))$

Translating from English 3



- ◆ Consider:
 - ◆ “Some students have visited Mexico”
 - ◆ “Every student in this class has visited Canada or Mexico”
- ◆ Let:
 - ◆ $S(x)$ be “ x is a student in this class”
 - ◆ $M(x)$ be “ x has visited Mexico”
 - ◆ $C(x)$ be “ x has visited Canada”

Translating from English 4



- ◆ Consider: “Some students have visited Mexico”
 - ◆ Rephrasing: “There exists a student who has visited Mexico”
- ◆ $\exists x M(x)$
 - ◆ True if the universe of discourse is all students
- ◆ What about if the universe of discourse is all people?
 - ◆ $\exists x (S(x) \rightarrow M(x))$
 - ◆ This is wrong! Why?
 - ◆ $\exists x (S(x) \wedge M(x))$

Translating from English 5



- ◆ Consider: “Every student in this class has visited Canada or Mexico”
- ◆ $\forall x (M(x) \vee C(x))$
 - ◆ When the universe of discourse is all students
- ◆ $\forall x (S(x) \rightarrow (M(x) \vee C(x)))$
 - ◆ When the universe of discourse is all people
- ◆ Why isn't $\forall x (S(x) \wedge (M(x) \vee C(x)))$ correct?

Translating from English 6



- ◆ Note that it would be easier to define $V(x, y)$ as “x has visited y”
 - ◆ $\forall x (S(x) \wedge V(x, \text{Mexico}))$
 - ◆ $\forall x (S(x) \rightarrow (V(x, \text{Mexico}) \vee V(x, \text{Canada})))$

Translating from English 7



- ◆ Translate the statements:
 - ◆ “All hummingbirds are richly colored”
 - ◆ “No large birds live on honey”
 - ◆ “Birds that do not live on honey are dull in color”
 - ◆ “Hummingbirds are small”
- ◆ Assign our propositional functions
 - ◆ Let $P(x)$ be “ x is a hummingbird”
 - ◆ Let $Q(x)$ be “ x is large”
 - ◆ Let $R(x)$ be “ x lives on honey”
 - ◆ Let $S(x)$ be “ x is richly colored”
- ◆ Let our universe of discourse be all birds

Translating from English 8



- ◆ Our propositional functions
 - ◆ Let $P(x)$ be “ x is a hummingbird”
 - ◆ Let $Q(x)$ be “ x is large”
 - ◆ Let $R(x)$ be “ x lives on honey”
 - ◆ Let $S(x)$ be “ x is richly colored”
- ◆ Translate the statements:
 - ◆ “All hummingbirds are richly colored”
 - ◆ $\forall x (P(x) \rightarrow S(x))$
 - ◆ “No large birds live on honey”
 - ◆ $\neg \exists x (Q(x) \wedge R(x))$
 - ◆ Alternatively: $\forall x (\neg Q(x) \vee \neg R(x))$
 - ◆ “Birds that do not live on honey are dull in color”
 - ◆ $\forall x (\neg R(x) \rightarrow \neg S(x))$
 - ◆ “Hummingbirds are small”
 - ◆ $\forall x (P(x) \rightarrow \neg Q(x))$

Quick survey



- ◆ I understand how to translate between English and quantified statements
 - a) Very well
 - b) With some review, I'll be good
 - c) Not really
 - d) Not at all