# Goodwillie Calculus

Jinyi Wang

Tsinghua University

Autumn 2024

# Outline

# Conventions and notations

| Whenever I say $\cdots$ | I mean $\cdots$ |
|---|---|
| category | category with equivalences, or $\infty$-category |
| functor | homotopy functor, or $\infty$-functor |
| (co)limit | homotopy (co)limit, or $\infty$-(co)limit |
| (co)fiber | homotopy (co)fiber |

# Conventions and notations

| Notation | Meaning |
| --- | --- |
| Top | the category of (unbased) spaces, or any topos |
| $\mathsf{Top}_{/X}$ | the category of spaces over $X$, or the slice topos over $X$ |
| $\mathsf{Top}_*$ | the category of based spaces |
| $\mathcal{S}p$ | the category of spectra |

# Conventions and notations

We use the following notion of connectedness, which is off by one from tradition.

**Definition (connectedness of a map)**

A map $f\colon Y \to X$ between spaces is $n$-*connected* if the following equivalent condition holds:

- For all $y \in Y$ the map $\pi_*(Y, y) \to \pi_*(X, f(y))$ is an isomorphism on degrees $\leq n$ and an epimorphism on degree $n+1$;
- Every fiber of $f$ is $n$-connected.

This notion has the following merits:

- A space $Y$ is $n$-connected iff $Y \to *$ is $n$-connected.
- A map $f\colon Y \to X$ is $n$-connected iff it is $n$-connected as an object of $\mathrm{Top}_{/X}$. Connectedness of an object is defined by connectedness of mapping spaces into it.

# Section 1

# An analogy to differential calculus

# Notion of smallness

Goodwillie describes his theory of functor calculus as

**Goodwillie, *The Differential Calculus of Homotopy Functors* (1991)**

a kind of deformation theory, ... to describe the *infinitesimal change* in $A(X)$ produced by an *infinitesimal* change in $X$. (A small change in $X$ is a highly connected map $Y \to X$.)

- Fundamental notion: small $=$ highly connected.
- A map $Y \to X$ is a small change iff its fibers are small.

# Notion of smallness

- Basic fact: suspension increases connectivity of spaces.
- If we keep suspending a space, it gets smaller and smaller.
- Sanity check: suspension also increases connectivity of maps, i.e. for a map $f\colon A \to B$, the map $Sf\colon SA \to SB$ is more connected than $f$. (Gooswillie's article uses this fact without proof. Proof by passing to slice topos?)
- Sanity check: if $f\colon A \to B$ is $n$-connected and $g\colon B \to C$ is $m$-connected, the composition $gf$ is $\min(m, n)$-connected.

# Notion of smallness

- We will also consider the relative (fiberwise) version of suspension $S_X\colon \mathsf{Top}_{/X} \to \mathsf{Top}_{/X}$ defined by the following pushout.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & S_X Y \end{tikzcd}$$

  Note that pushouts in $\mathsf{Top}_{/X}$ are just pushouts in $\mathsf{Top}$, and $X$ is the terminal object of $\mathsf{Top}_{/X}$.
- Fact: fiberwise suspension increases connectivity of objects of $\mathsf{Top}_{/X}$ (i.e. connectivity of maps $Y \to X$).

# Notion of smallness

- Trivial observation: the loop-space functor decreases connectivity of spaces by $1$.
- If suspension is like multiplying by a small quantity, then taking the loop space is like dividing by this small quantity.

| Homotopy theory | calculus |
|---:|---|
| {pointed spaces} | ring of formal power series $k[[x]]$ |
| circle | $x$ |
| suspension | multiplication by $x$ |
| loopspace | division by $x$ |
| {pointed connected spaces} | ideal $J = xk[[x]]$ |
| {pointed $n$-connected spaces} | ideal $J^{n+1} = x^{n+1}k[[x]]$ |

The analogy goes on.

# Blakers–Massey Theorem

**Theorem (Blakers–Massey)**

Given a pushout diagram in Top

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

where $f$ is $m$-connected and $g$ is $n$-connected, the map $A \to B \times_D C$ is $(m+n)$-connected.

- Taking $B = C = *$, we have $D \simeq SA$, and we get Freudenthal Suspension Theorem. This implies that suspension increases connectivity of spaces.
- Taking $B = C$ and $f = g$, we have $D \simeq S_B A$. This implies that fiberwise suspension increases connectivity of maps.
- "In a small neighborhood a pushout diagram is nearly a pullback." $\to$ notion of *linear functors*

# An analogy to differential calculus

- The differential of a function $f\colon \mathbb{R} \to \mathbb{R}$ is a linear function that describes
$$f(y) - f(x)$$
up $2k$-th order when $y - x$ is $k$-th order infinitesimal.
- The differential of a functor $F\colon \mathsf{Top} \to \mathsf{Top}_*$ is a linear functor that describes
$$\mathrm{fiber}(F(Y) \to F(X))$$
up to $2k$-homotopy type when $Y \to X$ is $k$-connected.

Section 2

# Linear functors and the first derivative

# Linear functors

## Definition (linear functors)

A functor $L\colon \mathsf{Top} \to \mathsf{Top}_*$ is *linear* if

- $L$ is *excisive*, i.e. $L$ takes coCartesian (pushout) squares to Cartesian (pullback) squares;
- $L$ is *reduced*, i.e. $L(*) \simeq *$.

- Every functor $F$ gives a reduced functor $\overline{F} := \mathrm{fiber}(F(-) \to F(*))$.
- If $F$ is excisive, then $\overline{F}$ is linear. This is analogous to the fact that for a first-degree function $f$, $f(-) - f(0)$ is a linear function.
- The term *excisive* comes from the *axiom of excision* in homology theory.

# Linear functors

We will also consider the "relative" (fiberwise) notion of linear functors.

**Definition (linear functors out of $\mathsf{Top}_{/X}$)**

A functor $L\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ is *linear* if

- $L$ is *excisive*, i.e. $L$ takes coCartesian (pushout) squares to Cartesian (pullback) squares;
- $L$ is *reduced*, i.e. $L(\mathrm{id}_X) \simeq *$.

- Every functor $F$ gives a reduced functor $\overline{F} := \mathrm{fiber}(F(-) \to F(X))$.
- If $L\colon \mathsf{Top} \to \mathsf{Top}_*$ is linear, then for any space $X$, the functor

  $$\mathrm{fiber}(L(-) \to L(X))\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$$

  is also linear. This is analogous to the following property of linear functions: if $\ell\colon \mathbb{R} \to \mathbb{R}$ is linear, then $y \mapsto \ell(x+y) - \ell(x)$ is also linear. The proof uses the fact that pullbacks in $(\mathsf{Top}_*)_{/L(X)}$ are just pullbacks in $\mathsf{Top}_*$, and taking fiber preserves pullbacks.

# Linear functors and $\Omega$-spectra

**Proposition (linear functors induce $\Omega$-spectra)**

For a linear functor $L$ and a space $Y$, the square (1) is a pushout, so the square (2) is a pullback, giving $L(Y) \simeq \Omega L(SY)$.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & SY \end{tikzcd} \qquad \begin{tikzcd} L(Y) \arrow[r] \arrow[d] & L(*) \arrow[d] \\ L(*) \arrow[r] & L(SY) \end{tikzcd}$$

(1) (2)

In particular, $\{L(S^i)\}$ have the structure of an $\Omega$-spectrum, called the *coefficient spectrum* of $L$.

- Fact: this gives the equivalence between linear functors and spectra.

# Nonlinear functors and sequential spectra

## Remark (nonlinear functors induce prespectra)

For a reduced functor $F\colon \mathsf{Top} \to \mathsf{Top}_*$ and a space $Y$, the diagram

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(SY) \end{tikzcd}$$

gives a map $F(Y) \to \Omega F(SY)$. In particular we have a prespectrum (also called a *sequential spectrum*) $\{F(S^i)\}$ with structure maps $F(S^i) \to \Omega F(S^{i+1})$, which we call the *coefficient prespectrum* of $F$. Also note that, if $F$ is linear, this prespectrum is an $\Omega$-spectrum.

# Differential at $*$

**Definition (differential of a reduced functor at $*$)**

For a reduced functor $F\colon \mathsf{Top} \to \mathsf{Top}_*$, define the *differential* of $F$ at $*$ to be

$$D_*F\colon \mathsf{Top} \to \mathsf{Top}_*, \quad Y \mapsto \operatorname{colim}_{k\to+\infty} \Omega^k F(S^k Y).$$

- If $F$ is linear, then $\Omega^k F(S^k Y)$ is independent of $k$, and $D_*F \simeq F$.
- For a reduced functor $F$, the coefficient spectrum of $D_*F$ is $\{D_*F(S^i)\} = \{\operatorname{colim}_{k\to+\infty} \Omega^k F(S^{k+i})\}$, which is the $\Omega$*-spectrification* of the coefficient prespectrum of $F$. In analogy to differential calculus, this spectrum is sometimes called the *derivative* of $F$ at $*$.
- Example. The differential of $\mathrm{id}_{\mathsf{Top}_*}$ is $Y \mapsto \Omega^\infty\Sigma^\infty Y = \operatorname{colim}_{k\to+\infty}$.

# 1-jet at $*$

**Definition (1-jet of a functor at $*$)**

For a functor $F\colon \mathsf{Top} \to \mathsf{Top}_*$, define its *1-jet* at $*$ to be

$$P_*F\colon \mathsf{Top} \to \mathsf{Top}_*, \quad Y \mapsto \operatorname{colim}_{k\to+\infty} \Omega^k_{F(*)} F(S^k Y),$$

where $\Omega_{F(*)}$ denotes the "loop-space functor"[1] on $(\mathsf{Top}_*)_{/F(*)}$, i.e. $\Omega_{F(*)}F(SY)$ is the following pullback.

$$\begin{array}{ccc} \Omega_{F(*)}F(SY) & \longrightarrow & F(*) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & F(SY) \end{array}$$

Denote the functor $Y \mapsto \Omega_{F(*)}F(SY)$ by $TF$, then the definition of $P_*F$ can be written as $Y \mapsto \operatorname{colim}_k T^k F(Y)$.

[1] It might not be a functor, but rather a syntactic sugar designed for better analogy to the reduced case.

# Differential at $*$

**Proposition-definition (differential of a general functor at $*$)**

For a functor $F\colon \mathsf{Top} \to \mathsf{Top}_*$, there is an equivalence of functors

$$P_*\overline{F} \to \overline{P_*F}$$

and we define it to be the differential $D_*F$.

- There is a pullback diagram

$$\begin{array}{ccc} P_*\overline{F}(Y) & \longrightarrow & P_*F(Y) \\ \downarrow & & \downarrow \\ P_*\overline{F}(*) & \longrightarrow & P_*F(*) \end{array}$$

  where $P_*\overline{F}(*) \simeq *$.

# 1-jet and differential at $X$

**Definition (1-jet and differential of a functor at $X$)**

For a functor $F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$, define the $1$-*jet* of $F$ at $X$ to be

$$P_X F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*, \quad Y \mapsto \operatorname{colim}_{k\to+\infty} \Omega^k_{F(X)} F(S^k_X Y).$$

Define the *differential* $D_X F$ of $F$ at $X$ to be $P_X \overline{F} \simeq \overline{P_X F}$.

- We always have $P_X F(X) \simeq F(X)$.
- If $F$ is excisive, then $F \simeq P_X F$.
- If $F$ is linear, then $F \simeq D_X F$.

# First order approximation

**Definition (two functors agreeing to first order)**

Two functors $F, G\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ are said to *agree to first order* via a natural transformation $\alpha\colon F \to G$ if the following condition holds for some constants $c, \kappa$:

- Whenever $Y \to X$ is $k$-connected and $k \geq \kappa$, the map $\alpha\colon F(Y) \to G(Y)$ is $(2k - c)$-connected.

- "Whenever $Y \to X$ is sufficiently connected, $\alpha\colon F(Y) \to G(Y)$ is about twice as connected."
- This is analogous to functions agreeing to first order: whenever $y - x$ is a $k$-th order infinitesimal, $f(y) - g(y)$ is of $2k$-th order.
- Here $\mathsf{Top}_{/X}$ is analogous to a neighborhood centered at $X$.

# First order approximation

- We want a functor $F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ to agree to first order with its 1-jet $P_X F$. Specifically, if $Y \to X$ is sufficiently connected, then we want the natural map $F(Y) \to P_X F(Y)$ to be about twice as connected.
- This is implied by a condition called *stable excision*, which is satisfied by most functors in practice.

# Stable excision

## Definition (stable excision)

A functor $F$ is said to be *stably excisive* if the following condition holds for some constants $c, \kappa$:

- If $A \to B$ and $A \to C$ are respectively $k_1$-, $k_2$-connected and $k_1, k_2 \geq \kappa$, and $D$ is the pushout on the left,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(D) \end{array}$$

  then the square on the right is $(k_1 + k_2 - c)$-Cartesian, meaning that $F(A) \to F(B) \times_{F(D)} F(C)$ is $(k_1 + k_2 - c)$-connected.

- Blakers–Massey Theorem tells us that $\mathrm{id}_{\mathsf{Top}}$ is stably excisive.
- The smaller the constants $c, \kappa$, the closer $F$ is to being excisive.

# First order approximation

The key lemma to first order approximation is the following.

**Lemma**

Stable excision for $F$ implies stable excision for the new functor $Y \mapsto \Omega_{F(X)} F(S_X Y)$, with improved (smaller) constants $c, \kappa$.

The proof uses the following facts.

- $S_X$ preserves cocartesian squares.
- $S_X$ increases connectivity of maps, i.e. for a $k_1$-connected map $Y \to Y_1$, the map $S_X Y \to S_X Y_1$ is $(k_1 + 1)$-connected.
- $\Omega_X$ preserves connectivity of maps.

# First order approximation

## Proposition

If $F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ is stably excisive, then

- $P_X F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ is excisive;
- $D_X F\colon \mathsf{Top}_{/X} \to \mathsf{Top}_*$ is linear;
- $F$ agrees to first order with $P_X F$;
- $\overline{F}$ agrees to first order with $D_X F$.

# Section 3

# The Taylor tower

# Cubes

## Definition (cube)

Let $P(S)$ denote the poset of subsets of a finite set $S$. An $S$-*cube* in a category $\mathcal{C}$ is a functor

$$X\colon P(S) \to \mathcal{C}.$$

A *face* of an $S$-cube is its restriction to a sub-poset $\{V \mid U \subset V \subset W\}$ for some $U \subset W \subset S$, which can be regarded as a $(W \setminus U)$-cube.

$$X$$

$\varnothing$-cube

$$X \longrightarrow X_0$$

$\{0\}$-cube

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X_{01}
\end{tikzcd}$$

$\{0,1\}$-cube

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & X_0 \arrow[dr] \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dd] & & X_{02} \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr] & & X_{01} \arrow[dr] & \\
& X_{12} \arrow[rr] & & X_{012}
\end{tikzcd}$$

$\{0,1,2\}$-cube

# Cubes

- Point of view: an $(n+1)$-dimensional cube is a map between $n$-dimensional cubes.
- The notion of $k$*-connectivity of a map* can be generalized to cubes in two ways: $k$*-Cartesian* and $k$*-coCartesian* cubes.

# (co)Cartesian cubes

**Definition (Cartesian cube)**

An $S$-cube $X\colon P(S) \to \mathcal{C}$ is called *Cartesian* if $X$ is a limit diagram, where $X(\varnothing)$ is a limit of the rest of the cube, $X|_{P(S)\setminus\{\varnothing\}}$.The cube is called $k$-Cartesian if the gap map $X(\varnothing) \to \lim X|_{P(S)\setminus\{\varnothing\}}$ is $k$-connected.

**Definition (coCartesian cube)**

An $S$-cube $X\colon P(S) \to \mathcal{C}$ is called *coCartesian* if $X$ is a colimit diagram, where $X(S)$ is a colimit of the rest of the cube, $X|_{P(S)\setminus\{S\}}$. The cube is called $k$-coCartesian if the cogap map $\operatorname{colim} X|_{P(S)\setminus\{S\}} \to X(S)$ is $k$-connected.

- A $\varnothing$-cube (an object) is (co)Cartesian iff it is a final (initial) object.
- A $\{0\}$-cube (a morphism) is (co)Cartesian iff it is an equivalence.

# Strongly coCartesian cubes

**Definition (strongly coCartesian cube)**

An $n$-cube $X\colon P(\{1,\cdots,n\}) \to \mathcal{C}$ is called *strongly coCartesian* if the following equivalent conditions hold:

- every $2$-dimensional face of $X$ is coCartesian;
- every face of dimension $\geq 2$ is coCartesian;
- $X$ is a left Kan extension of $X|_{\{\varnothing,\{1\},\cdots,\{n\}\}}$.

- The Kan extension condition translates to: a map from $X$ to another cube is completely determined by its restriction on $X|_{\{\varnothing,\{1\},\cdots,\{n\}\}}$.
- Particularly, $X$ can be recovered from $X|_{\{\varnothing,\{1\},\cdots,\{n\}\}}$ by repeated pushouts.
- Example. Every $\{0\}$-cube is strongly coCartesian (because it has no $2$-dim. faces). A $\{0,1\}$-cube is strongly coCartesian iff it is coCartesian.

# Excisive functors

**Definion ($n$-excisive functor)**

A functor $F$ is called *$n$-excisive* if it sends *strongly* coCartesian $\{0, 1, \cdots, n\}$-cubes to Cartesian cubes.

- A $(-1)$-excisive functor sends every object to the final object.
- A $0$-excisive functor is essentially a constant functor.
- A $1$-excisive functor is an excisive functor defined above.
- $\cdots$
- An $n$-excisive functor is analogous to a polynomial of order $\leq n$.

# Taylor approximation

## Proposition (Taylor approximation of functors)

Denote by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ the category of $n$-excisive functors $\mathcal{C} \to \mathcal{D}$. The inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has a left adjoint

$$P_n \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D}).$$

In other words, a functor $F \colon \mathcal{C} \to \mathcal{D}$ has a "best approximation" $F \to P_n(F)$ to some $n$-excisive functor $P_n(F)$.

- The construction of $P_n(F)$ is by a colimit very similar to $P_1(F)$ which we described.
- Moreover, the functor $P_n$ preserves finite limits. This makes $P_n$ examples of *left exact localizations*. In the case where $\mathcal{D}$ is a topos, we see that $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ is again a topos.

# Taylor approximation

### $(-1)$-th order approximation

For a functor $F\colon \mathcal{C} \to \mathcal{D}$, the functor $P_{-1}(F)$ sends everything to the final object of $\mathcal{D}$; it is a $(-1)$-excisive functor.

### $0$-th order approximation

For a functor $F\colon \mathcal{C} \to \mathcal{D}$, the functor $P_0(F)$ sends everything to $F(*)$, where $*$ is the final object of $\mathcal{C}$; it is a $0$-excisive functor.

### first-order approximation

For a functor $F\colon \mathcal{C} \to \mathcal{D}$, the functor $P_1(X)$ is the $1$-jet, or the first-order approximation we have constructed; it is a $1$-excisive functor.

# The Taylor tower

## Observation (Taylor tower)

The excisive approximations for a functor $F$ form a tower

$$F \to \cdots \to P_2(F) \to P_1(F) \to P_0(F) \to P_{-1}(F).$$

The "difference" between successive terms

$$D_n(F) = \mathrm{fiber}(P_n(F) \to P_{n-1}(F))$$

may be thought of as the "$n$-th term of the Taylor series".

# The Taylor Tower

## A factorization system

A morphism $F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is called

- a $P_n$*-local morphism*, if $P_{>n}(F) \to P_{>n}(G)$ is an equivalence, where $P_{>n}(F)$ denotes the fiber of $F \to P_n(F)$;
- an $P_n$*-equivalence*, if $P_n(F) \to P_n(G)$ is an equivalence.

## Theorem

The pair ($P_n$-equivalence, $P_n$-local morphisms) is a factorization system, i.e. every morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ can be factored as a $P_n$-equivalence followed by a $P_n$-local morphism.

- This situation is similar to the ($n$-connected, $n$-truncated) factorization for spaces. This is the theory of *modalities in topoi*.

# References


- Goodwillie, T. (2003). *Calculus III: Taylor Series.* arXiv:math/0310481 [math.AT]
- nLab authors. (2024). *Goodwillie calculus.* https://ncatlab.org/nlab/show/Goodwillie+calculus
- Haine, P. (2017). *An Introduction to Goodwillie Calculus.*
- Anel, M., Biedermann, G., Finster, E., and Joyal, A. *Goodwillie's Calculus of Functors and Higher Topos Theory.* https://arxiv.org/abs/1703.09632v2