

# Character Sheaves and Trace of Hecke Category

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# Main Theorem

## Theorem

The category  $\mathcal{C}h_G$  of *unipotent character sheaves* is equivalent to both the *trace* and *center* of the Hecke category  $\mathcal{H}_G$ .

- This is an example of the *trace principle*, a unifying theme in geometric representation theory.
- We will be working in the sheaf-theoretic context of *D-modules*.

# Outline

- 1 Conventions
- 2 TFT, with a Toy Model
- 3 Hecke Category
- 4 Character Sheaves
- 5 Integral Transforms

# Notations & Conventions

- The term “linear category” means “stable  $\mathbb{C}$ -linear  $\infty$ -category”, or “pre-triangulated  $\mathbb{C}$ -linear dg category”.
- A “scheme”  $X$  means a quasicompact, separated derived scheme of finite type over  $\mathbb{C}$ .
- $\mathcal{D}(X)$  is the linear category of (complexes of) D-modules on  $X$ .
- All functors are derived.
- Fix a Borel subgroup  $B$  of a complex reductive group  $G$ .
- Horizontal bars (e.g.  $\overline{\frac{G}{B}}$ ) symbolize quotients by *conjugation* actions.
- Slashes  $B \backslash G$  ( $G/B$ ) denote quotients by left (right) actions.

# Preliminaries: Double Quotients

## Observation

A *double quotient* can be written as a fiber product of delooping stacks:

$$K \backslash G / H \simeq \mathbf{B}K \times_{\mathbf{B}G} \mathbf{B}H,$$

where  $K, H$  are subgroups of  $G$  (or just groups with a homomorphism to  $G$ ) and  $\mathbf{B}K \rightarrow \mathbf{B}G$ ,  $\mathbf{B}H \rightarrow \mathbf{B}G$  are from functoriality of  $\mathbf{B}$ .

- This is a quotient of a  $K \times H$ -action on  $G$ .

# Preliminaries: Conjugation Actions

## Observation

A quotient by a *conjugation action* can also be written as a fiber product of delooping stacks:

$$\frac{G}{H} \simeq \mathbf{B}H \times_{(\mathbf{B}G)^2} \mathbf{B}G,$$

where the morphisms come from  $\mathbf{B}H \rightarrow \mathbf{B}G$  and the diagonal  $\Delta_{\mathbf{B}G}$ .

- In particular, a quotient by conjugation is a special case of a double quotient:

$$\frac{G}{H} \simeq H \backslash (G \times G) / G.$$

- The quotient of  $G$  by the conjugation action on itself is

$$\frac{G}{G} \simeq \mathbf{B}G \times_{\mathbf{B}G \times \mathbf{B}G} \mathbf{B}G = \text{Hom}(S^1, \mathbf{B}G).$$

# TFT

## Definition (TFT)

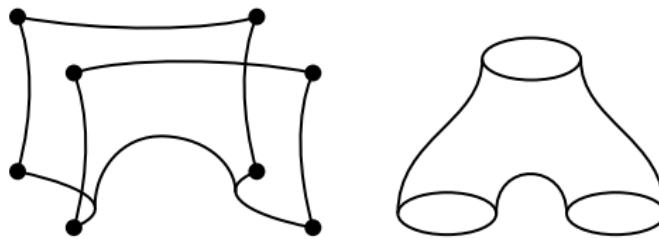
Let  $\mathcal{C}$  be a symmetric monoidal  $(\infty, 2)$ -category. An oriented  $2$ -dimensional TFT (*topological field theory*) valued in  $\mathcal{C}$  is a symmetric monoidal functor

$$Z: \text{Bord}_2 \rightarrow \mathcal{C},$$

where  $\text{Bord}_2$  is the *oriented bordism*  $(\infty, 2)$ -category, whose

- objects are oriented 0-manifolds,
- 1-morphisms are oriented 1-dimensional bordisms,
- 2-morphisms are oriented 2-dimensional bordisms between 1-bordisms.

# TFT



Examples of 2-morphisms in  $\text{Bord}_2$

# Preliminaries: Center of Group Algebra

Let  $G$  be a finite group.

- The *group algebra*  $\mathbb{C}[G]$  is the space of functions on  $G$ , equipped with convolution.
- The space of *class functions*  $\mathbb{C}[G]^G \simeq \mathbb{C}[\frac{G}{G}] \simeq Z(\mathbb{C}[G])$  is also the *center* of the group algebra.

# Preliminaries: Morita Category

- There is a 2-category  $\text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}})$  called the *Morita 2-category*, whose
  - objects are  $\mathbb{C}$ -algebras,
  - morphisms  $A \rightarrow B$  are  $(A, B)$ -bimodules,
  - 2-morphisms are module homomorphisms.
- The Morita category embeds into the 2-category of linear categories:

$$\text{Mod}: \text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}}) \hookrightarrow \text{Cat}_{\mathbb{C}}.$$

# 2D Dijkgraaf–Witten: Rep. Theory of Finite Groups

A toy model of TFT called *Dijkgraaf–Witten theory* encodes all familiar structures in  $\mathbb{C}$ -representation theory of a finite group  $G$ . It is a functor

$$Z_G: \text{Bord}_2 \rightarrow \text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}}),$$

where

- $Z_G(*) = \mathbb{C}[G]$ , or the category  $\text{Rep}(G)^{\text{fd}}$  of fin. dim. representations of  $G$ ;
- $Z_G(S^1) = \mathbb{C}[G]^G \simeq \mathbb{C}[\frac{G}{G}] \simeq Z(\mathbb{C}[G])$ ;
- for a closed oriented surface  $\Sigma$ ,  $Z_G(\Sigma) = \# \frac{\text{Hom}(\pi_1(\Sigma), G)}{G} \in \mathbb{C}$  is the “number” of principal  $G$ -bundles on  $\Sigma$ , where a bundle  $P$  counts as  $1/\#\text{Aut}(P)$ .

# Preliminaries: Hecke Algebras

Let  $H$  be a subgroup of  $G$  and let  $V = \text{Ind}_H^G(\mathbb{C}) = \mathbb{C}[G/H]$ .

- The *Hecke algebra* of  $(G, H)$  is

$$\text{End}_G(V) \simeq \mathbb{C}[(G/H) \times (G/H)]^G \simeq \mathbb{C}[H \backslash G / H].$$

- Consider the maps

$$\frac{G}{G} \xleftarrow{q} \frac{G}{H} \xrightarrow{\epsilon} H \backslash G / H.$$

The pull-push  $q_*\epsilon^*: \mathbb{C}[H \backslash G / H] \rightarrow \mathbb{C}[\frac{G}{G}] = Z(\mathbb{C}[G])$  equals the generalized trace map

$$(\varphi \in \text{End}_G(V)) \mapsto (g \mapsto \text{tr}(g \circ \varphi)).$$

# Hecke Category

## Definition (Hecke category)

The *Hecke category* is the linear category

$$\mathcal{H} = \mathcal{D}(B \backslash G / B) = \mathcal{D}(\mathbf{B}B \times_{\mathbf{B}G} \mathbf{B}B)$$

of “ $B$ -biequivariant  $\mathcal{D}$ -modules on  $G$ ”.

# Hecke Category (generalized)

## Definition' (Hecke category)

For a general morphism  $p: X \rightarrow Y$ , the *Hecke category* is defined as

$$\mathcal{H} = \mathcal{D}(X \times_Y X).$$

- In our special case,  $X = \mathbf{B}B$  and  $Y = \mathbf{B}G$ .
- Assumptions on the morphism  $p: X \rightarrow Y$ :
  - The morphism  $p$  is proper;
  - The diagonal  $\Delta_X: X \rightarrow X \times X$  is smooth.

These assumptions guarantee the adjunctions and dualities we need.

# Hecke Category and TFT

## Proposition

There exists a unique TFT

$$Z: \text{Bord}_2 \rightarrow \text{Alg}_{(1)}(\text{St}_{\mathbb{C}})$$

such that  $Z(*) = \mathcal{H}$ .

# Horocycle Correspondence

## Definition (horocycle correspondence)

The *horocycle correspondence* is the correspondence given by the following diagram.

$$\begin{array}{ccc} \frac{G}{G} & \xleftarrow{q} & \frac{G}{B} & \xrightarrow{\epsilon} & B \backslash G / B \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathbf{B}G \times \mathbf{B}G & \xleftarrow[p]{(BG)^2} & \mathbf{B}B \times \mathbf{B}G & \xrightarrow{\Delta_{\mathbf{B}B}} & (\mathbf{B}B)^2 \times \mathbf{B}G \\ & & (BG)^2 & & (BG)^2 \end{array}$$

# Horocycle Correspondence (generalized)

## Definition' (horocycle correspondence)

More generally we consider the following correspondence,

$$Y \times_{Y^2} Y \xleftarrow{q} X \times_{Y^2} Y \xrightarrow{\epsilon} X^2 \times_{Y^2} Y = X \times_Y X$$

which is a base-change of

$$Y \xleftarrow{p} X \xrightarrow{\Delta_X} X^2.$$

- Recall our assumptions:  $p$  is proper;  $\Delta_X$  is smooth.

# Some Useful Identities

We will make use of the following basic identities.

- $\mathcal{L}Y = Y \times_{Y \times Y} Y$ .
- $X \times_Y X = (X \times X) \times_{Y \times Y} Y$ .
- $X \times_{X \times X} (X \times_Y X) = X \times_{Y \times Y} Y = X \times_Y \mathcal{L}Y$ .
- More generally,

$$\begin{aligned} X \times_{X \times X} (X \times_Y X)^{\times_X n} &\simeq X \times_{X \times X} (X)^{\times_Y (n+1)} \\ &\simeq (X)^{\times_Y n} \times_Y \mathcal{L}Y. \end{aligned}$$

# Harish-Chandra Transform

## Definition (Harish-Chandra transform)

The *Harish-Chandra transform*  $F$  is the pull-push along the horocycle correspondence:

$$F: \mathcal{D}(X^2 \times_{Y^2} Y) \xrightarrow{\epsilon^!} \mathcal{D}(X \times_{Y^2} Y) \xrightarrow{q_*} \mathcal{D}(Y \times_{Y^2} Y).$$

- $q$  is a base change of  $p$  and hence proper;  $\epsilon$  is a base change of  $\Delta_X$  and hence smooth.

## Proposition (right adjoint to Harish-Chandra transform)

The functor  $F$  has a right adjoint

$$F^r: \mathcal{D}(Y \times_{Y^2} Y) \xrightarrow{q^!} \mathcal{D}(X \times_{Y^2} Y) \xrightarrow{\epsilon_*[-2 \dim X]} \mathcal{D}(X^2 \times_{Y^2} Y).$$

# Character Sheaves

## Definition (unipotent character sheaves)

The linear category of *unipotent character sheaves* is the full subcategory

$$\mathcal{C}h_G \subset \mathcal{D}\left(\frac{G}{G}\right)$$

generated under colimits by the image of  $F$ .

- Traditional unipotent character sheaves are simple objects of the heart of  $\mathcal{C}h_G$ .
- Geometric characterization: adjoint-equivariant D-modules on  $G$  with singular support in the nilpotent cone and unipotent central character.

# Integral Transforms

## Proposition (integral transform)

Let  $Y$  be a stack with smooth diagonal. The *integral transform construction* for D-modules on  $X_1, X_2$  is

$$\mathcal{D}(X_1 \times X_2) \rightarrow \text{Fun}^L(\mathcal{D}(X_1), \mathcal{D}(X_2)), M \mapsto (\text{pr}_2)_*(\text{pr}_1^! - \otimes M).$$