

Character Sheaves and Trace of Hecke Category

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Main Theorem

Theorem

The category \mathcal{Ch}_G of *unipotent character sheaves* is equivalent to both the *trace* and *center* of the Hecke category \mathcal{H}_G .

- This is an example of the *trace principle*, a unifying theme in geometric representation theory.
- We will be working in the sheaf-theoretic context of *D-modules*.

Outline

- 1 Conventions
- 2 TFT, with a Toy Model
- 3 Hecke Category
- 4 Character Sheaves
- 5 Integral Transforms

Notations & Conventions

- The term “linear category” means “stable \mathbb{C} -linear ∞ -category”, or “pre-triangulated \mathbb{C} -linear dg category”.
- A “scheme” X means a quasicompact, separated derived scheme of finite type over \mathbb{C} .
- $\mathcal{D}(X)$ is the linear category of (complexes of) D-modules on X .
- All functors are derived.
- Fix a Borel subgroup B of a complex reductive group G .
- Horizontal bars (e.g. $\frac{G}{B}$) symbolize quotients by *conjugation* actions.
- Slashes $B \backslash G$ (G/B) denote quotients by left (right) actions.

Preliminaries: Double Quotients

Observation

A *double quotient* can be written as a fiber product of delooping stacks:

$$K \backslash G / H \simeq \mathbf{B}K \times_{\mathbf{B}G} \mathbf{B}H,$$

where K, H are subgroups of G (or just groups with a homomorphism to G) and $\mathbf{B}K \rightarrow \mathbf{B}G, \mathbf{B}H \rightarrow \mathbf{B}G$ are from functoriality of \mathbf{B} .

- This is a quotient of a $K \times H$ -action on G .

Preliminaries: Conjugation Actions

Observation

A quotient by a *conjugation action* can also be written as a fiber product of delooping stacks:

$$\frac{G}{H} \simeq \mathbf{B}H \times_{(\mathbf{B}G)^2} \mathbf{B}G,$$

where the morphisms come from $\mathbf{B}H \rightarrow \mathbf{B}G$ and the diagonal $\Delta_{\mathbf{B}G}$.

- In particular, a quotient by conjugation is a special case of a double quotient:

$$\frac{G}{H} \simeq H \backslash (G \times G) / G.$$

- The quotient of G by the conjugation action on itself is

$$\frac{G}{G} \simeq \mathbf{B}G \times_{\mathbf{B}G \times \mathbf{B}G} \mathbf{B}G = \mathrm{Hom}(S^1, \mathbf{B}G).$$

TFT

Definition (TFT)

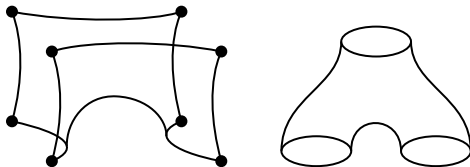
Let \mathcal{C} be a symmetric monoidal $(\infty, 2)$ -category. An oriented *2-dimensional TFT* (*topological field theory*) valued in \mathcal{C} is a symmetric monoidal functor

$$Z: \text{Bord}_2 \rightarrow \mathcal{C},$$

where Bord_2 is the *oriented bordism* $(\infty, 2)$ -category, whose

- objects are oriented 0-manifolds,
- 1-morphisms are oriented 1-dimensional bordisms,
- 2-morphisms are oriented 2-dimensional bordisms between 1-bordisms.

TFT

Examples of 2-morphisms in Bord_2

Preliminaries: Center of Group Algebra

Let G be a finite group.

- The *group algebra* $\mathbb{C}[G]$ is the space of functions on G , equipped with convolution.
- The space of *class functions* $\mathbb{C}[G]^G \simeq \mathbb{C}[\frac{G}{G}] \simeq Z(\mathbb{C}[G])$ is also the *center* of the group algebra.

Preliminaries: Morita Category

- There is a 2-category $\text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}})$ called the *Morita 2-category*, whose
 - objects are \mathbb{C} -algebras,
 - morphisms $A \rightarrow B$ are (A, B) -bimodules,
 - 2-morphisms are module homomorphisms.
- The Morita category embeds into the 2-category of linear categories:

$$\text{Mod}: \text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}}) \hookrightarrow \text{Cat}_{\mathbb{C}}.$$

2D Dijkgraaf–Witten: Rep. Theory of Finite Groups

A toy model of TFT called *Dijkgraaf–Witten theory* encodes all familiar structures in \mathbb{C} -representation theory of a finite group G . It is a functor

$$Z_G: \text{Bord}_2 \rightarrow \text{Alg}_{(1)}(\text{Vect}_{\mathbb{C}}),$$

where

- $Z_G(*) = \mathbb{C}[G]$, or the category $\text{Rep}(G)^{\text{fd}}$ of fin. dim. representations of G ;
- $Z_G(S^1) = \mathbb{C}[G]^G \simeq \mathbb{C}[\frac{G}{G}] \simeq Z(\mathbb{C}[G])$;
- for a closed oriented surface Σ , $Z_G(\Sigma) = \# \frac{\text{Hom}(\pi_1(\Sigma), G)}{G} \in \mathbb{C}$ is the “number” of principal G -bundles on Σ , where a bundle P counts as $1/\#\text{Aut}(P)$.

Preliminaries: Hecke Algebras

Let H be a subgroup of G and let $V = \text{Ind}_H^G(\mathbb{C}) = \mathbb{C}[G/H]$.

- The *Hecke algebra* of (G, H) is

$$\text{End}_G(V) \simeq \mathbb{C}[(G/H) \times (G/H)]^G \simeq \mathbb{C}[H \backslash G/H].$$

- Consider the maps

$$\frac{G}{G} \xleftarrow{q} \frac{G}{H} \xrightarrow{\epsilon} H \backslash G/H.$$

The pull-push $q_*\epsilon^*: \mathbb{C}[H \backslash G/H] \rightarrow \mathbb{C}[\frac{G}{G}] = Z(\mathbb{C}[G])$ equals the generalized trace map

$$(\varphi \in \text{End}_G(V)) \mapsto (g \mapsto \text{tr}(g \circ \varphi)).$$

Hecke Category

Definition (Hecke category)

The *Hecke category* is the linear category

$$\mathcal{H} = \mathcal{D}(B \backslash G / B) = \mathcal{D}(\mathbf{B}B \times_{\mathbf{B}G} \mathbf{B}B)$$

of “ B -biequivariant \mathcal{D} -modules on G ”.

Hecke Category (generalized)

Definition' (Hecke category)

For a general morphism $p: X \rightarrow Y$, the *Hecke category* is defined as

$$\mathcal{H} = \mathcal{D}(X \times_Y X).$$

- In our special case, $X = \mathbf{B}B$ and $Y = \mathbf{B}G$.
- Assumptions on the morphism $p: X \rightarrow Y$:
 - The morphism p is proper;
 - The diagonal $\Delta_X: X \rightarrow X \times X$ is smooth.

These assumptions guarantee the adjunctions and dualities we need.

Hecke Category and TFT

Proposition

There exists a unique TFT

$$Z: \text{Bord}_2 \rightarrow \text{Alg}_{(1)}(\text{St}_{\mathbb{C}})$$

such that $Z(*) = \mathcal{H}$.

Horocycle Correspondence

Definition (horocycle correspondence)

The *horocycle correspondence* is the correspondence given by the following diagram.

$$\begin{array}{ccccc}
 \frac{G}{G} & \xleftarrow{q} & \frac{G}{B} & \xrightarrow{\epsilon} & B \backslash G / B \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 \mathbf{B}G \times_{(\mathbf{B}G)^2} \mathbf{B}G & \xleftarrow{p} & \mathbf{B}B \times_{(\mathbf{B}G)^2} \mathbf{B}G & \xrightarrow{\Delta_{\mathbf{B}B}} & (\mathbf{B}B)^2 \times_{(\mathbf{B}G)^2} \mathbf{B}G
 \end{array}$$

Horocycle Correspondence (generalized)

Definition' (horocycle correspondence)

More generally we consider the following correspondence,

$$Y \times_{Y^2} Y \xleftarrow{q} X \times_{Y^2} Y \xrightarrow{\epsilon} X^2 \times_{Y^2} Y = X \times_Y X$$

which is a base-change of

$$Y \xleftarrow{p} X \xrightarrow{\Delta_X} X^2.$$

- Recall our assumptions: p is proper; Δ_X is smooth.

Some Useful Identities

We will make use of the following basic identities.

- $\mathcal{L}Y = Y \times_{Y \times Y} Y$.
- $X \times_Y X = (X \times X) \times_{Y \times Y} Y$.
- $X \times_{X \times X} (X \times_Y X) = X \times_{Y \times Y} Y = X \times_Y \mathcal{L}Y$.
- More generally,

$$\begin{aligned} X \times_{X \times X} (X \times_Y X)^{\times_{X^n}} &\simeq X \times_{X \times X} (X)^{\times_Y (n+1)} \\ &\simeq (X)^{\times_Y n} \times_Y \mathcal{L}Y. \end{aligned}$$

Harish-Chandra Transform

Definition (Harish-Chandra transform)

The *Harish-Chandra transform* F is the pull-push along the horocycle correspondence:

$$F: \mathcal{D}(X^2 \times_{Y^2} Y) \xrightarrow{\epsilon^!} \mathcal{D}(X \times_{Y^2} Y) \xrightarrow{q_*} \mathcal{D}(Y \times_{Y^2} Y).$$

- q is a base change of p and hence proper; ϵ is a base change of Δ_X and hence smooth.

Proposition (right adjoint to Harish-Chandra transform)

The functor F has a right adjoint

$$F^r: \mathcal{D}(Y \times_{Y^2} Y) \xrightarrow{q^!} \mathcal{D}(X \times_{Y^2} Y) \xrightarrow{\epsilon_*[-2 \dim X]} \mathcal{D}(X^2 \times_{Y^2} Y).$$

Character Sheaves

Definition (unipotent character sheaves)

The linear category of *unipotent character sheaves* is the full subcategory

$$\mathcal{Ch}_G \subset \mathcal{D}\left(\frac{G}{G}\right)$$

generated under colimits by the image of F .

- Traditional unipotent character sheaves are simple objects of the heart of \mathcal{Ch}_G .
- Geometric characterization: adjoint-equivariant D-modules on G with singular support in the nilpotent cone and unipotent central character.

Integral Transforms

Proposition (integral transform)

Let Y be a stack with smooth diagonal. The *integral transform construction* for D-modules on X_1, X_2 is

$$\mathcal{D}(X_1 \times X_2) \rightarrow \mathrm{Fun}^L(\mathcal{D}(X_1), \mathcal{D}(X_2)), \quad M \mapsto (\mathrm{pr}_2)_*(\mathrm{pr}_1^! - \otimes M).$$