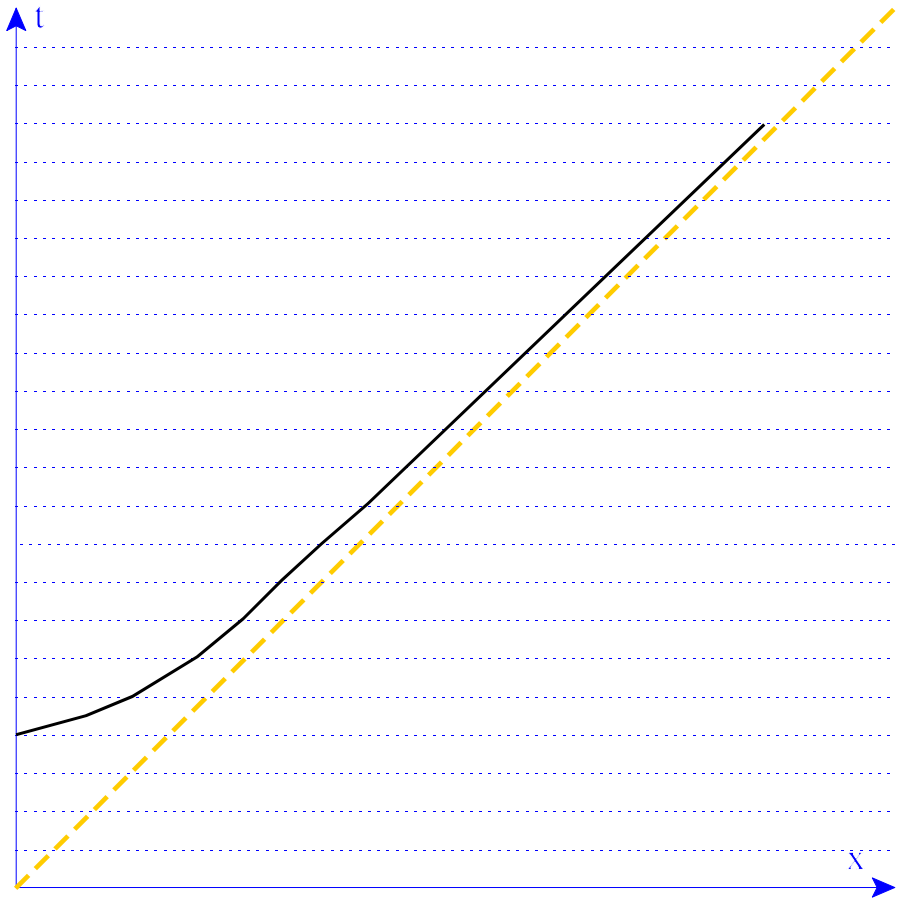
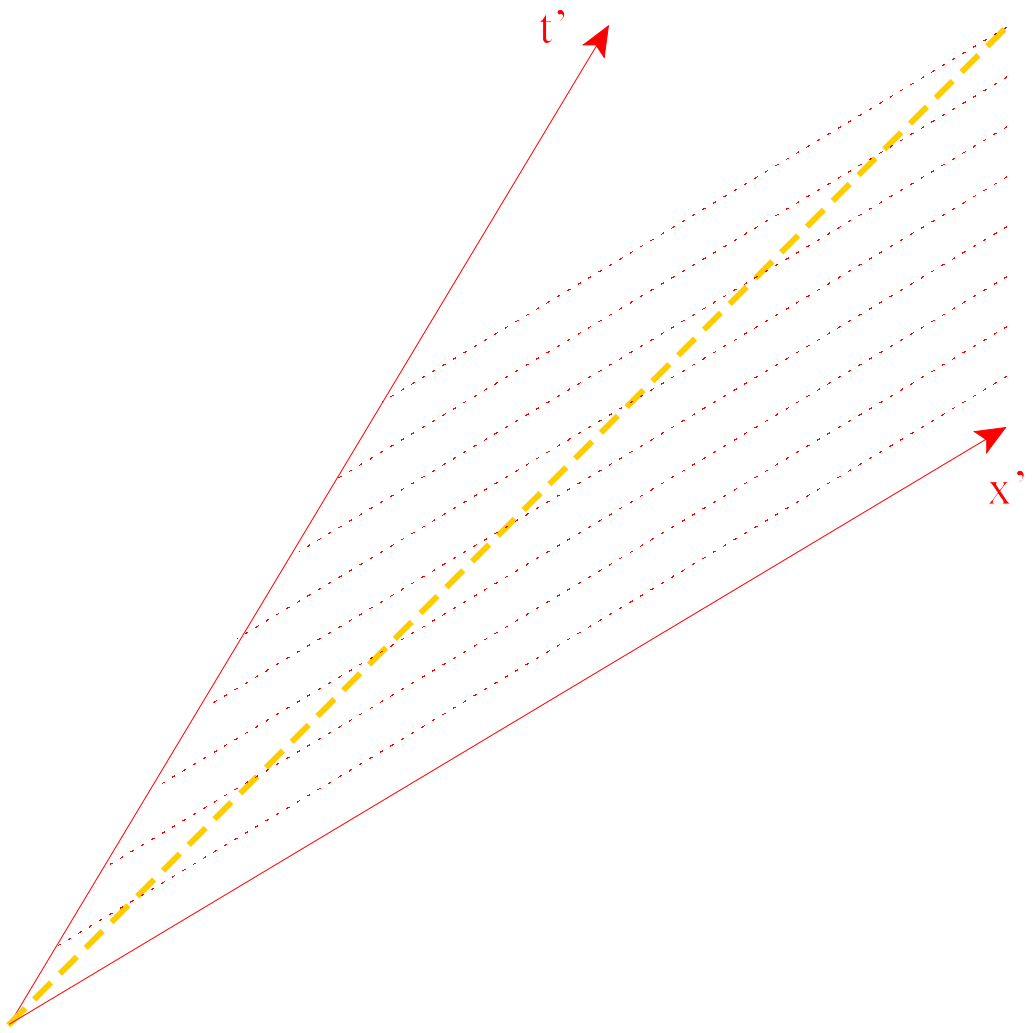
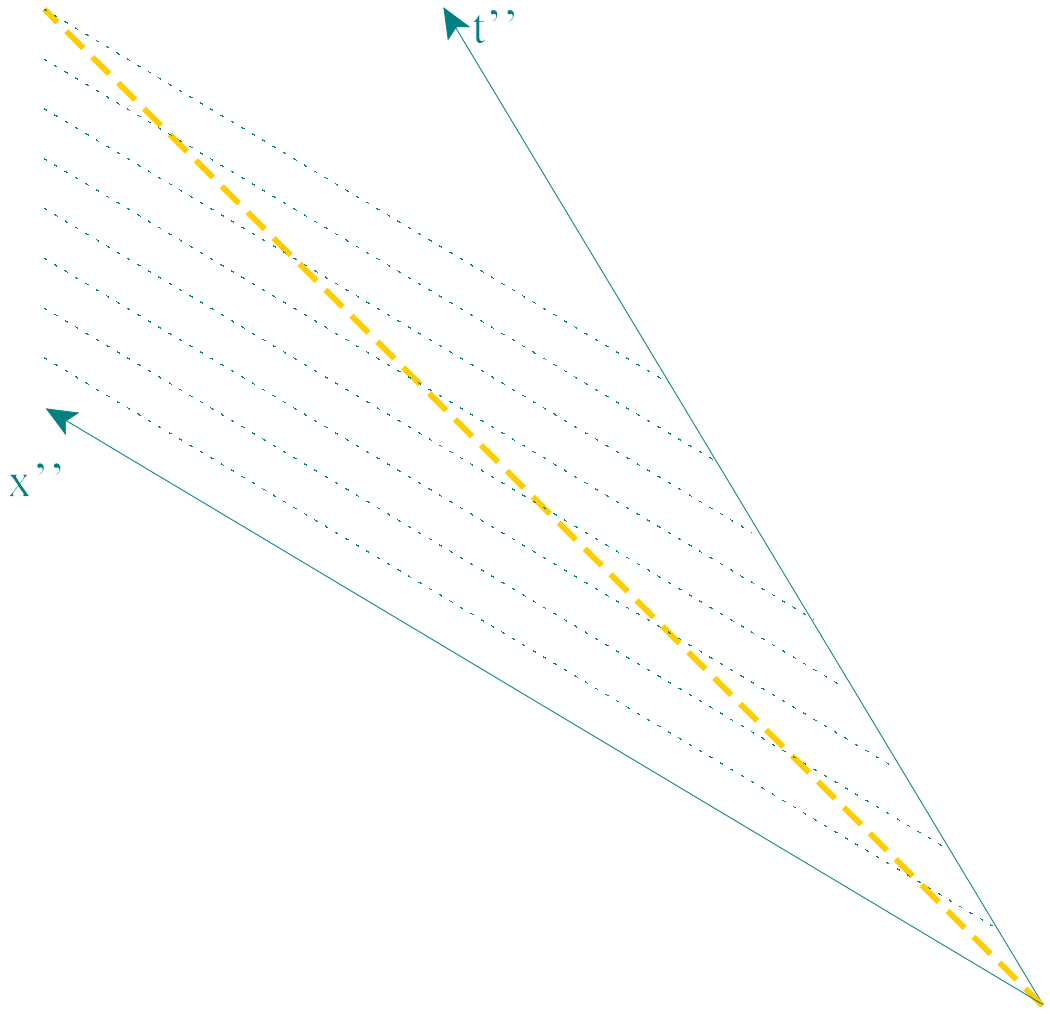
THE TWIN PARADOX



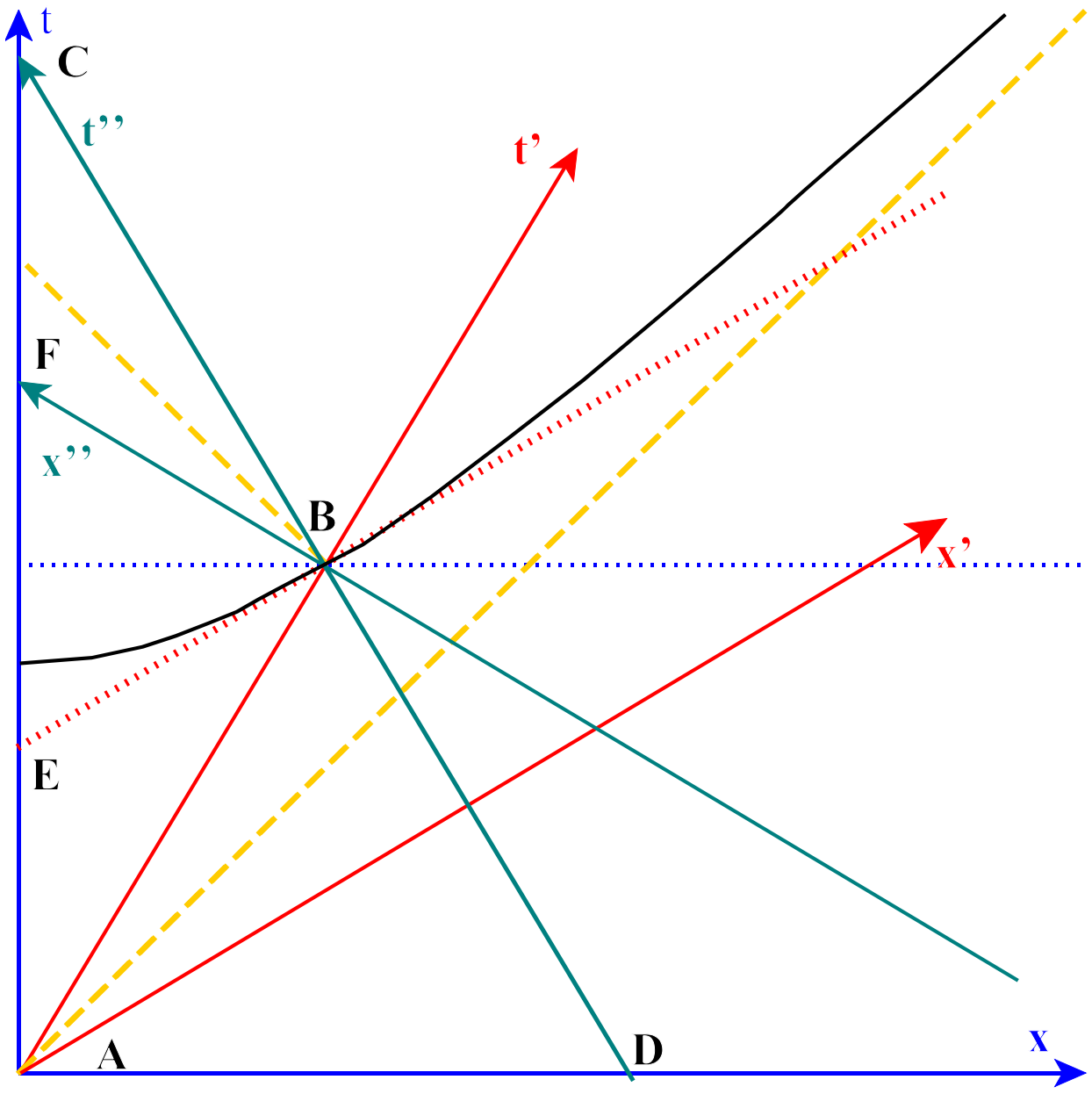
This is a basic spacetime diagram. We'll let this represent the inertial reference frame of Alphie, the "stay-at-home twin". Notice the black "unit hyperbola". Every event (point) on that curve has a spacetime interval of one unit to the origin. Where it intersects the t-axis, $t = 1$ and $x = 0$. At all other points on the curve, the unit interval will consist of some spatial interval and some temporal interval. (The unit hyperbola comes from the equation $t^2 - x^2 = 1$. In Euclidean space, we have unit circles, given by the equation $x^2 + y^2 = 1$.)



Let this represent the outgoing frame of Beth, the “adventurous twin”. Beth is at rest relative to this frame, standing at the origin of the x’-axis, so her world-line is just the t’-axis. She is traveling at a rate of .6 the velocity of light, relative to Alphie’s frame. Notice the planes of simultaneity parallel to the x’-axis.



Let this represent Beth's inbound frame. Notice the planes of simultaneity parallel to the x''-axis.



This is a composite of the three previous pictures. Beth leaves Alphie at event A, transfers to inbound frame at event B, and enjoys a reunion with Alphie at event C. Thus, Beth's world line is the line ABC. Beth heads to the right along Alphie's x-axis at velocity .6c. She has instructions to transfer to the inbound frame when her clock reads 100 meters of time, which she does at event B. Notice the dashed lines of simultaneity, and notice the black hyperbola. In this picture, it represents a 100-meter (of time) interval. Where it intersects the t'-axis, Beth's clocks read '100'. Where it intersects the t-axis, Alphie's clocks read '100'. According to Alphie, Beth's clocks are running slow, so she travels longer than 100 m in his frame. In fact, B occurs at 125 m in Alphie's frame. Of course, she makes the same claim about Alphie's clocks. She maintains that at event B, Alphie's clocks read 80 m, which is event E on the diagram.

It turns out that Beth will be younger than Alphie when they have their reunion. There are a number of ways of analyzing this, but the simplest is this. At event B, when Beth transfers to the inbound frame, she has aged 100 units, according to her clocks. However, that event B is simultaneous with t = 125 in Alphie's frame. So, Alphie judges her clocks to be running slow; he has aged 125 m while she has aged only 100 m. Now, just reverse the picture. It will take another 100 units of Beth's time to the reunion, and another 125 units of Alphie's time. Thus, Beth will have aged 200 units, while Alphie ages 250. You can complicate it by considering the events E and F. E occurs at t' = 100 and F occurs at t'' = 100, but E occurs at t = 80, and F at t = 170. But just as it took 80 of Alphie's units to get from A to E, it will take another 80 to get from F to C. But 170 + 80 = 250, as above.

Interestingly, after the transfer to the inbound frame, Beth will judge that Alphie is now older than her, whereas before the transfer, she would claim that he was younger. In fact, after the transfer, they would agree that Beth was younger than Alphie, and she remains so, right through the reunion.