1.1 Library

- Declare a `Library` class to support the following functions.

```
Output binom(final Derivative deriv, final MarketData mkt, int n)

int impvol(final Derivative deriv, final MarketData mkt, int n,
           int max_iter, double tol, Output out)
```

- The `Library` class is a `final` class.

  1. The function `binom` implements the binomial model to compute the fair value and fugit of a derivative.
  2. The function `impvol` executes a loop of iterations to calculate the implied volatility of a derivative.

- Additional details, including the various input and output classes mentioned above, will be given below.

1.2 Class MarketData

- A MarketData object contains market data values to be used for valuation of a derivative.

```
final class MarketData
{
    public double Price;            // market price of security
    public double S;                // stock price
    public double r;                // interest rate
    public double sigma;            // volatility
    public double t0;               // current time
...
}
```

- The values of the MarketData class members are set in a calling application (for example the main() program), and the MarketData object is passed as an input to the above library functions.
- It is your responsibility to write any class methods, etc. that you consider appropriate.

1.3 Class `Output`

- An `Output` object contains the results of calculations by the library functions.

  ```
  final class Output
  {
      public double FV;       // fair value
      public double fugit;    // fugit
      public double impvol;   // implied volatility
      public int num_iter;    // number of iterations to compute implied volatility
  ...
  }
  ```

- The values of the `Output` class members are set in the library functions and returned to the calling application.

  1. Note that not all output fields may be populated by a given fumction.
  2. For example the function `binom` populates the values of `FV` and `fugit`, but not `impvol` and `num_iter`.

- It is your responsibility to write any class methods, etc. that you consider appropriate.

1.4 Class `Node`

- To calculate the fair value and fugit of a derivative, the binomial model should allocate `Node` objects.

```
final class Node
{
    ...
}
```

- It is your responsibility to decide what data members and methods the `Node` class should contain.
- **It is your responsibility to decide how the binomial model allocates the `Node` objects, e.g. array or ArrayList or HashSet, etc.**
- In other words, *you formulate the software design.*

1.5 Class `Derivative`

- The class `Derivative` is an **abstract base class** which supports virtual functions for use in the valuation of derivatives.

```
abstract class Derivative
{
    public double T;                            // data member
    public void terminalCondition(Node n)       // virtual function
    public void valuationTest(Node n)           // virtual function
}
```

- All the derivatives we shall treat in this course have an expiration time `T`.
- Hence the `Derivative` class contains a data member `double T`.
- The virtual functions must be overridden by non-abstract derived classes.
  1. The virtual function `terminalCondition` sets the payoff value and the fugit value on the expiration date.
  2. The virtual function `valuationTest` is called when traversing the tree in the binomial model, to make decisions about early exercise and to set the fair value and fugit to appropriate values.
  3. The `Node` object must therefore contain suitable data members and methods for the above functions to perform their tasks.

1.6 Class `VanillaOption`

- The class `VanillaOption` is a non-abstract class which extends `Derivative`, to support the valuation of ordinary (vanilla) options.
- It must support put and call options, also American and European exercise policies.
- Therefore the `VanillaOption` must contain suitable indicative data members (primary key).
- The `VanillaOption` class must also override the virtual functions.

```
class VanillaOption extends Derivative
{
    ...                                       // data members
    public void terminalCondition(Node n)     // override
    public void valuationTest(Node n)         // override
...
}
```

1.7 Class BermudanOption

- The class BermudanOption is a non-abstract class which is an American-style vanilla option, but it allows early exercise only in the time window $w_{\text{begin}} \leq t \leq w_{\text{end}}$.

```
class BermudanOption extends ...
{
    public double window_begin;
    public double window_end;
...
}
```

- A real Bermudan option can have many time windows, but we shall support only one time window.
- The BermudanOption class must contain suitable indicative data members and override the virtual functions in Derivative.

1.8 Classes

- The overall set of classes is therefore as follows.

```
class Library
class MarketData
class Output
class Node
abstract class Derivative & non-abstract classes
```

- It must be possible for me to declare a non-abstract class (new financial derivative) and override the virtual functions in `Derivative`.
- The functions in your library must be able to calculate the fair value, fugit and implied volatility of an object of that class I declare.
- *Polymorphism:* it must be possible for me to declare such a class and your program code must support it **without modifying any of your program code.**

Queens College, CUNY, Department of Computer Science
Computational Finance
CSCI 365 / 765
Fall 2017
Instructor: Dr. Sateesh Mane

November 15, 2017

17 Lecture 17a

Binomial model: worked examples

- We display worked examples to calculate option fair values using the **binomial model.**
- We calculate American options and demonstrate how to incorporste early exercise.
- **There is no explicit mathematical probability theory in this lecture.**

17.6 Binomial model: summary of tree

- We make a very simple model of the stock price movements.
- We discretize the time to expiration $T - t_0$ into n equal steps of size

$$\Delta t = \frac{T - t_0}{n}\,. \tag{17.6.1}$$

- At each step, we approximate that the stock price can go to only one of two future values at the next step.
- If the stock price is S at a node at the timestep i, then the stock price either goes up by a factor u to Su or down by a factor d to Sd at the next timestep $i + 1$.
- A sketch is shown in Fig. 1 for three timesteps.
- The binomial tree **recombines.** Hence it has $O(n^2)$ nodes, not $O(2^n)$ nodes.

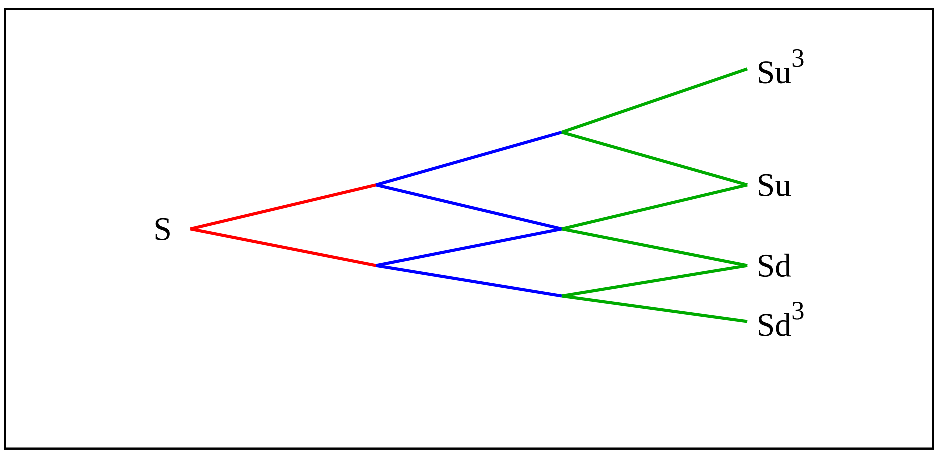

Figure 1: Sketch of binomial stock price movements for three timesteps.

17.7 Binomial model: parameters

- The probability of taking an up step is p and the probability of taking a down step is $q = 1-p$.
- Let the risk–free interest rate be r (a constant).
- Let the volatility of the stock be σ (a constant).
- Suppose the stock pays continuous dividends at a rate q.
- Then the values of u, d, p and q are given as follows:

$$u = e^{\sigma\sqrt{\Delta t}}\,, \tag{17.7.1a}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u\,, \tag{17.7.1b}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d}\,, \tag{17.7.1c}$$

$$q = \frac{u - e^{(r-q)\Delta t}}{u-d}\,. \tag{17.7.1d}$$

- Comments:
  1. There is no flexilibity in the locations of the nodes in the binomial tree.
  2. There are hidden assumptions in the above derivation. We require both p and q to be positive (or at least zero). Hence to obtain meaningful values for the probabilities we must have
  $$d \le e^{(r-q)\Delta t} \le u\,. \tag{17.7.2}$$
  3. The inequalities in eq. (17.7.2) then yield
  $$e^{-\sigma\sqrt{\Delta t}} \le e^{(r-q)\Delta t} \le e^{\sigma\sqrt{\Delta t}}\,. \tag{17.7.3}$$
  4. The inequalities in eq. (17.7.3) are usually satisfied in practice, but can fail is the value of σ is very small, or if the value of Δt is not small enough.

17.8 Binomial model: valuation

- We formulate the valuation procedure for any derivative on a stock.
- We value the derivative by working ***backwards*** from the final timestep to the initial timestep.
- Consider a node at the timestep i and let the stock price at that node be S.
  Let the derivative value at that node be V.
- The node at the timestep i is connected to two nodes at the timestep $i+1$, with the values Su and Sd, respectively. Let the derivative fair values at those nodes be V_u and V_d, respectively.
  1. For a European style derivative, the fair value V at the timestep i is calculated as follows:
  $$V_{\rm Eur} = e^{-r\Delta t}\,(pV_u + qV_d)\,. \tag{17.8.1}$$
  2. For an American style derivative, **we compare the value from eq. (17.8.1) to the derivative intrinsic value.** If the value from eq. (17.8.1) is less than the derivative's intrinsic value, we set the fair value V at that node to the intrinsic value $V_{\rm intrinsic}$ instead. Hence for an American option
  $$V_{\rm Am} = \max\Big\{\, e^{-r\Delta t}\,(pV_u + qV_d)\,, V_{\rm intrinsic} \Big\}\,. \tag{17.8.2}$$
- A sketch is shown in Fig. 2.
- Notice the arrows in Fig. 2 point backwards: we work backwards through the tree.

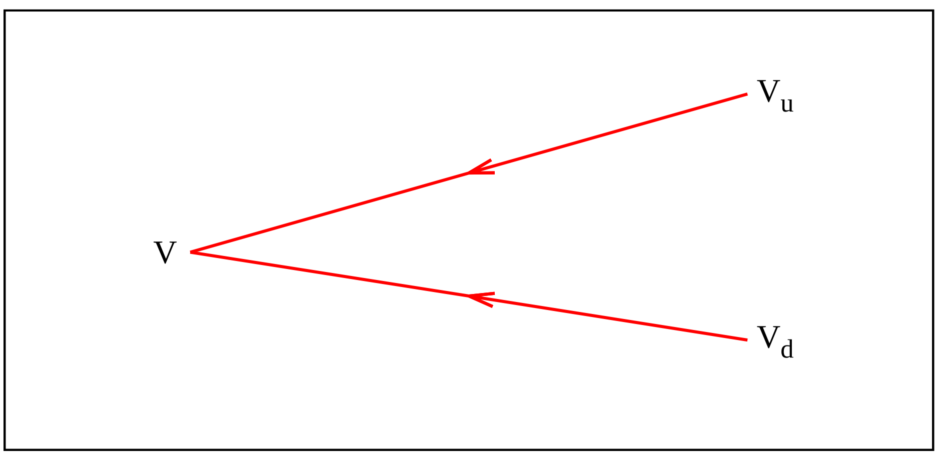

Figure 2: Sketch of binomial tree and derivative fair values at nodes at timesteps i and $i+1$.

17.9 Worked example: Put option

17.9.1 Parameter values

- We value a European put option using a binomial tree.
- The current time is $t_0 = 0$.
- The current stock price is $S_0 = 100$.
- The stock does not pay dividends.
- The stock volatility is $\sigma = 0.5$.
- The risk-free interest rate is $r = 0.1$.
- The option strike is $K = 100$ and the expiration time is $T = 0.3$.
- We make a binomial tree with three timesteps $n = 3$ so $\Delta t = 0.3/3 = 0.1$.
- Then the values of the relevant parameters are as follows:

$$e^{r\Delta t} \simeq 1.01005\,, \tag{17.9.1a}$$
$$e^{-r\Delta t} \simeq 0.99005\,, \tag{17.9.1b}$$
$$u \simeq 1.1713\,, \tag{17.9.1c}$$
$$d \simeq 0.8538\,, \tag{17.9.1d}$$
$$p \simeq 0.4922\,, \tag{17.9.1e}$$
$$q \simeq 0.5078\,. \tag{17.9.1f}$$

17.9.2 Stock price nodes

The stock price values at the nodes of the binomial tree are shown in Fig. 3.

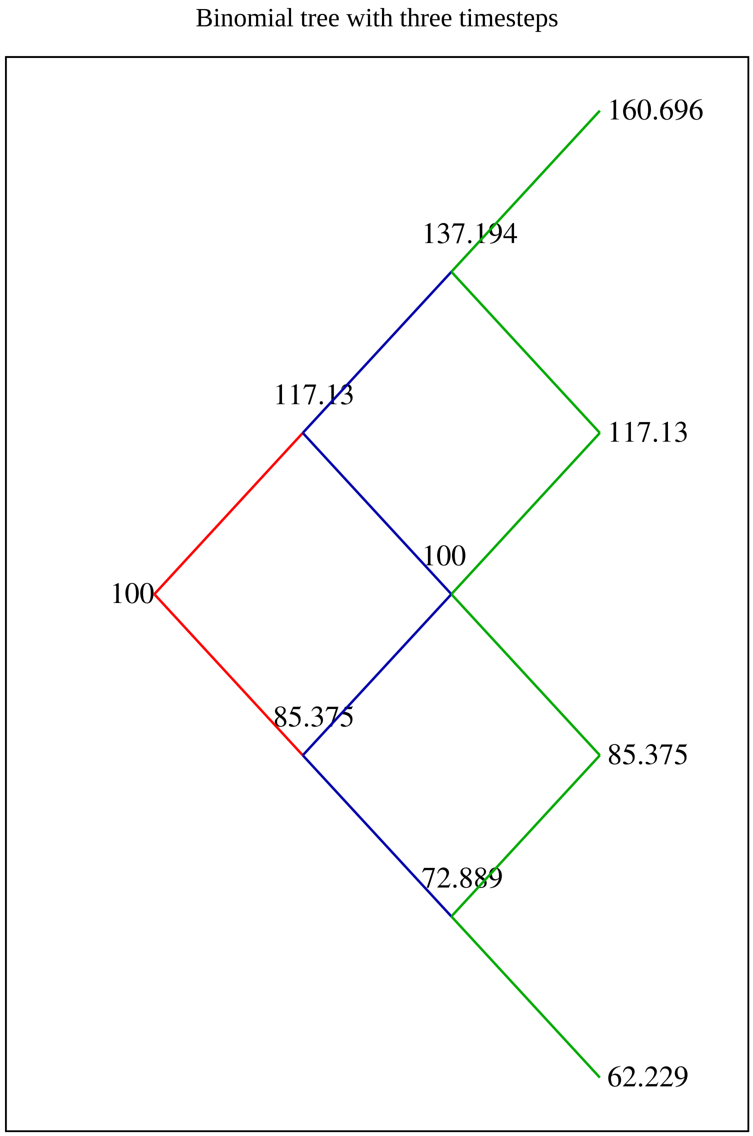

Figure 3: Example binomial tree with three timesteps. The stock prices at each node are listed.

17.9.3 Valuation of European put

- We calculate the fair value of a **European put.**
- The option valuation tree is shown in Fig. 4.
- The option fair values at expiration ($i = 3$) are filled in first.

$$
\begin{aligned}
V(Su^3) &= \max(100 - 160.696, 0) && = 0\,, && (17.9.2a)\\
V(Su) &= \max(100 - 117.130, 0) && = 0\,, && (17.9.2b)\\
V(Sd) &= \max(100 - 85.375, 0) && \simeq 14.625\,, && (17.9.2c)\\
V(Sd^3) &= \max(100 - 62.229, 0) && \simeq 37.771\,. && (17.9.2d)
\end{aligned}
$$

- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:

$$
\begin{aligned}
V(Su^2) &= e^{-r\Delta t}(p \times 0 + q \times 0) && = 0\,, && (17.9.3a)\\
V(S) &= e^{-r\Delta t}(p \times 0 + q \times 14.625) && \simeq 7.353\,, && (17.9.3b)\\
V(Sd^2) &= e^{-r\Delta t}(p \times 14.625 + q \times 37.771) && \simeq 26.116\,. && (17.9.3c)
\end{aligned}
$$

- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:

$$
\begin{aligned}
V(Su) &= e^{-r\Delta t}(p \times 0 + q \times 7.353) && \simeq 3.696\,, && (17.9.4a)\\
V(Sd) &= e^{-r\Delta t}(p \times 7.353 + q \times 26.116) && \simeq 16.712\,. && (17.9.4b)
\end{aligned}
$$

- Finally, the European put option fair value using a three step binomial tree is

$$
p_{\text{binom}} = e^{-r\Delta t}(p \times 3.696 + q \times 16.712) \simeq 10.203\,. \qquad (17.9.5)
$$

- The European put option fair value using the Black–Scholes formula is

$$
p_{\text{BS}} = Ke^{-r(T-t_0)}N(-d_2) - SN(-d_1) \simeq 9.317\,. \qquad (17.9.6)
$$

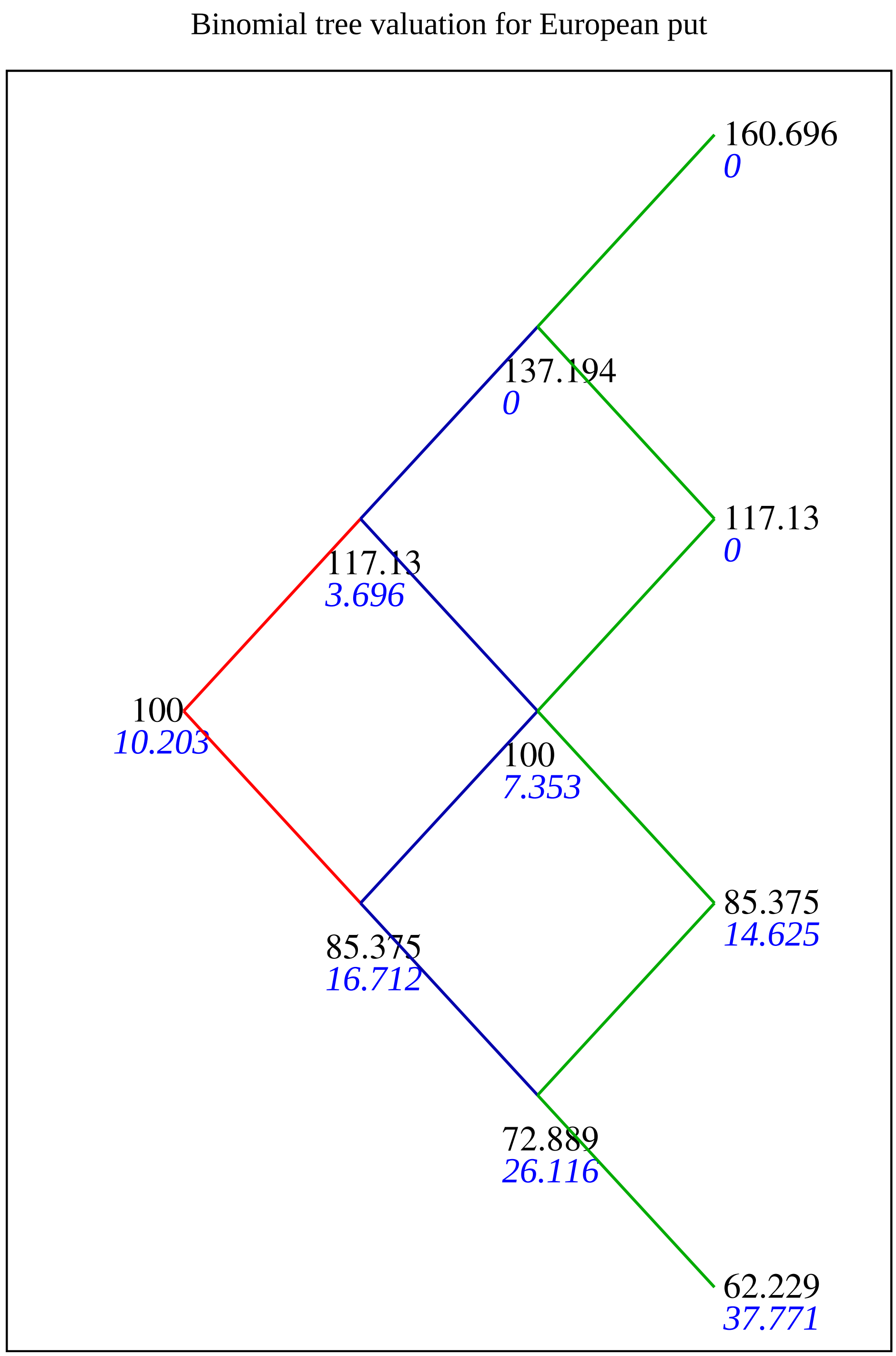

Figure 4: Valuation of European put option using the binomial tree in Fig. 3.

17.9.4 Valuation of American put

- We calculate the fair value of an **American put.**
- The option valuation tree is shown in Fig. 5.
- The option fair values at expiration ($i = 3$) are filled in first.
- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:
  1. We calculate the discounted expectations:

$$V(Su^2) = e^{-r\Delta t}(p \times 0 + q \times 0) \qquad = 0\,, \tag{17.9.7a}$$
$$V(S) = e^{-r\Delta t}(p \times 0 + q \times 14.625) \qquad \simeq 7.353\,, \tag{17.9.7b}$$
$$\boldsymbol{X(Sd^2)} = e^{-r\Delta t}(p \times 14.625 + q \times 37.771) \qquad \simeq 26.116\,. \tag{17.9.7c}$$

  2. At the node Sd^2, the value is called "$\boldsymbol{X}$" because it is **less than the intrinsic value.**
  3. The intrinsic value of the American put at this node is higher:

$$\max(K - S, 0) \simeq 100 - 72.889 = 27.111\,. \tag{17.9.8}$$

  4. Hence the fair value at this node is set to the intrinsic value

$$V_{\rm node} = 27.111\,. \tag{17.9.9}$$

- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:
  1. We calculate the discounted expectations:

$$V(Su) = e^{-r\Delta t}(p \times 0 + q \times 7.353) \qquad \simeq 3.696\,, \tag{17.9.10a}$$
$$V(Sd) = e^{-r\Delta t}(p \times 7.353 + q \times 27.111) \qquad \simeq 17.213\,. \tag{17.9.10b}$$

  2. In each case, the result is higher than the intrinsic value at that node.
  3. Hence early exercise is not optimal at either node.
- Finally, the American put option fair value using a three step binomial tree is

$$P_{\rm binom} = e^{-r\Delta t}(p \times 3.696 + q \times 17.213) \simeq 10.455\,. \tag{17.9.11}$$

- This is higher than the intrinsic value at that node, hence early exercise is not optimal.
- The fair value of the American put $P_{\rm binom}$ in eq. (17.9.11) is higher than the fair value of a European put $p_{\rm binom}$ with the same parameters (see eq. (17.9.5)).
- The Black–Scholes formula cannot calculate the fair value of an American put option.
- The *Black–Scholes equation* (also the Black–Scholes–Merton equation) can treat American options, but the equations must be solved numerically.
- The binomial model is one of the simplest numerical algorithms for valuing derivatives.

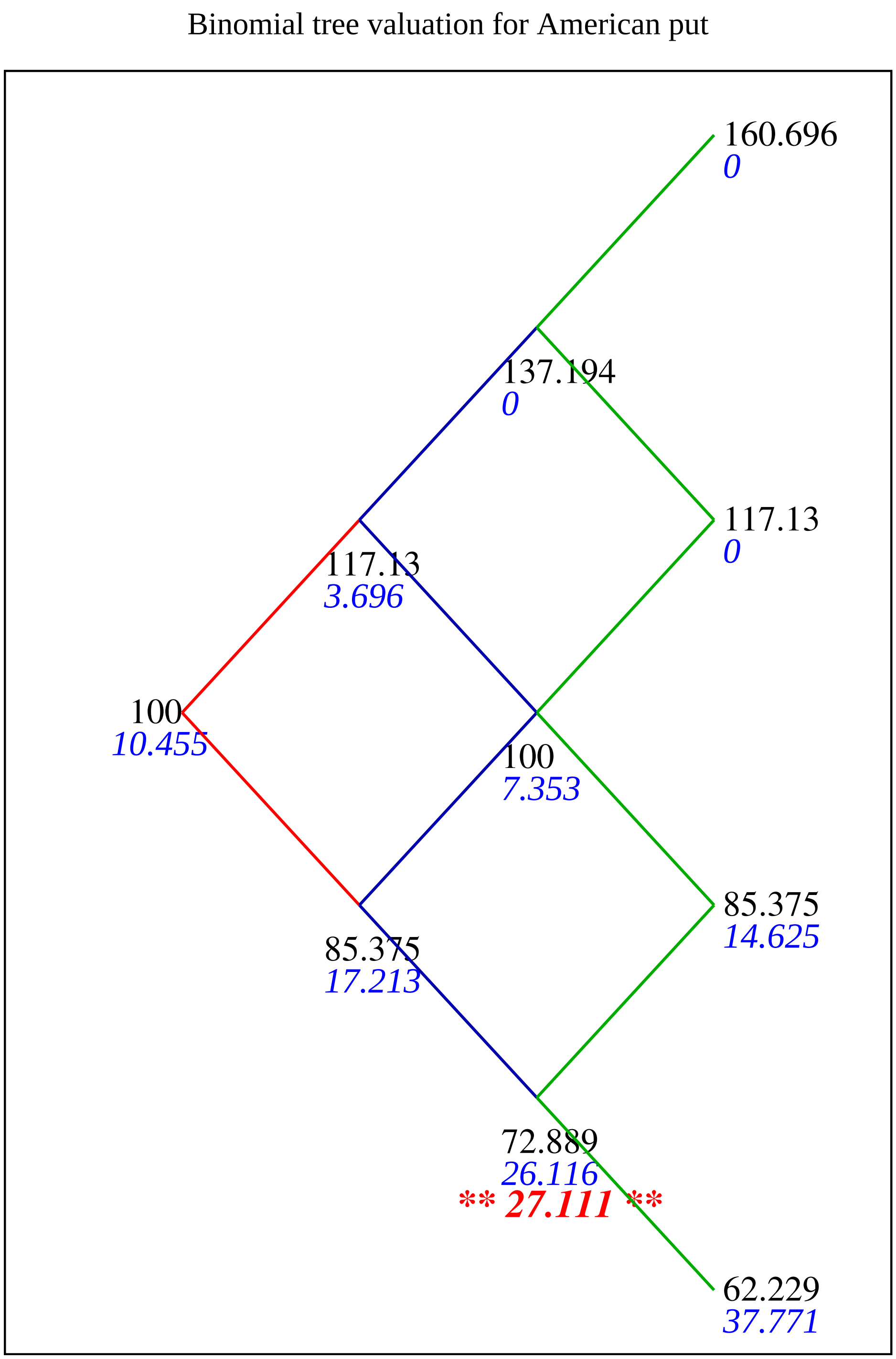

Figure 5: Valuation of American put option using the binomial tree in Fig. 3.

17.9.5 Valuation of European call

- We calculate the fair value of a **European call.**
- The option valuation tree is shown in Fig. 6.
- The option fair values at expiration ($i = 3$) are filled in first.

$$\begin{aligned}
V(Su^3) &= \max(160.696 - 100, 0) && \simeq 60.696\,, && (17.9.12a)\\
V(Su) &= \max(117.130 - 100, 0) && \simeq 17.130\,, && (17.9.12b)\\
V(Sd) &= \max(85.375 - 100, 0) && = 0\,, && (17.9.12c)\\
V(Sd^3) &= \max(62.229 - 100, 0) && = 0\,. && (17.9.12d)
\end{aligned}$$

- The option fair values at the remaining nodes are calculated using eq. (17.8.1).
- The fair values at the nodes for the step $i = 2$ are calculated as follows:

$$\begin{aligned}
V(Su^2) &= e^{-r\Delta t}(p \times 60.696 + q \times 17.130) && \simeq 38.190\,, && (17.9.13a)\\
V(S) &= e^{-r\Delta t}(p \times 17.130 + q \times 0) && \simeq 8.348\,, && (17.9.13b)\\
V(Sd^2) &= e^{-r\Delta t}(p \times 0 + q \times 0) && = 0\,. && (17.9.13c)
\end{aligned}$$

- The fair values at the step $i = 1$ are calculated using the values at the step $i = 2$:

$$\begin{aligned}
V(Su) &= e^{-r\Delta t}(p \times 38.190 + q \times 8.348) && \simeq 22.807\,, && (17.9.14a)\\
V(Sd) &= e^{-r\Delta t}(p \times 8.348 + q \times 0) && \simeq 4.068\,. && (17.9.14b)
\end{aligned}$$

- Finally, the European call option fair value using a three step binomial tree is

$$c_{\text{binom}} = e^{-r\Delta t}(p \times 22.807 + q \times 4.068) \simeq 13.159\,. \qquad (17.9.15)$$

- The European call option fair value using the Black–Scholes formula is

$$c_{\text{BS}} = SN(d_1) - Ke^{-r(T-t_0)}N(d_2) \simeq 12.272\,. \qquad (17.9.16)$$

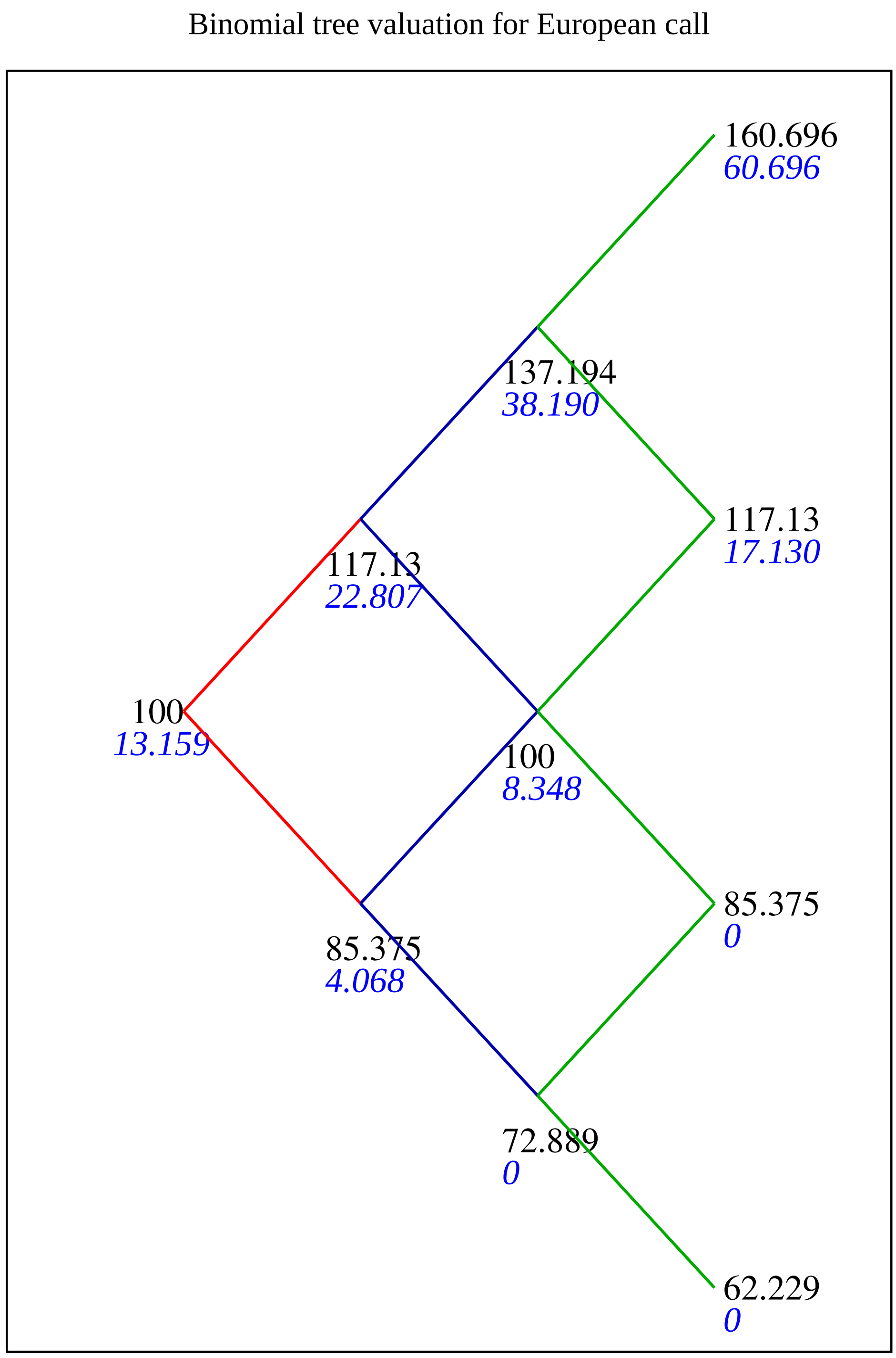

Figure 6: Valuation of European call option using the binomial tree in Fig. 3.

17.9.6 Valuation of American call

- If the stock does not pay dividends, then the value of an American call equals the value of a European call on the same stock, with the same strike and expiration.
- The binomial model confirms this behavior.
- The valuation of an American call using the binomial model yields the same results, at all nodes, as the valuation of the European call.
- Hence the American call option fair value using a three step binomial tree is

$$C_{\text{binom}} \simeq 13.159\,. \tag{17.9.17}$$

17.10 Tests: put–call parity

- If the underlying stock does not pay dividends, the put–call parity formula is

$$c - p = S - \mathrm{PV}(K) = S - Ke^{-r(T-t_0)}\,. \tag{17.10.1}$$

- For the given parameter values, we obtain

$$S - Ke^{-r(T-t_0)} = 100 - 100\,e^{-0.1\times 0.3} = 100 - 100\,e^{-0.03} \simeq 2.955\,. \tag{17.10.2}$$

- The fair values using the Black–Scholes formula agree with eq. (17.10.1):

$$c_{\mathrm{BS}} - p_{\mathrm{BS}} \simeq 12.272 - 9.317 = 2.955\,. \tag{17.10.3}$$

- Using eqs. (17.9.5) and (17.9.15), the fair values using the binomial model yield

$$c_{\mathrm{binom}} - p_{\mathrm{binom}} \simeq 13.159 - 10.203 = 2.956\,. \tag{17.10.4}$$

- **The fair values using the binomial model also agrees with eq.** (17.10.1)**.**
- Remember that put–call parity **does not depend on a probability model** for the stock price movements.
- **Therefore all option valuation models, including in particular the binomial model, must satisfy put–call parity.**

17.11 Tests: inequalities for American options

- If the underlying stock does not pay dividends, the fair values of American calls and puts satisfy the following inequalities (derived from rational option pricing theory)

$$S - K \quad \leq \quad C - P \quad \leq \quad S - \mathrm{PV}(K)\,. \tag{17.11.1}$$

- From the data, $S = K = 100$, hence $S - K = 0$.

- Also from eq. (17.10.2), $S - \mathrm{PV}(K) \simeq 2.955$.

- Hence using these parameter values in eq. (17.11.1), we must have

$$0 \quad \leq \quad C - P \quad \leq \quad 2.955\,. \tag{17.11.2}$$

- Using eqs. (17.9.11) and (17.9.17), the fair values using the binomial model yield

$$C_{\rm binom} - P_{\rm binom} \simeq 13.159 - 10.455 = 2.704\,. \tag{17.11.3}$$

- Hence eq. (17.11.3) satisfies eq. (17.11.2), and therefore (for these parameter values), the rational option pricing inequalities in eq. (17.11.1).

Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Fall 2017

Instructor: Dr. Sateesh Mane

November 17, 2017

18 Lecture 18

Fugit

- In this lecture we study the concept of **fugit**.
- The **fugit** is the mean or expectation of the life of a derivative.
- The fugit is therefore measured in units of time, i.e. years.
  1. The term **fugit** is frequently called the "median life" of a derivative.
  2. This is technically incorrect, because the fugit is really an expectation, not a median.
  3. However, people in the financial industry are not big on mathematical rigor.
  4. Hence "median life" is the common description.
- The fugit of a European option is obviously the time to expiration of the option.
- The fugit of an American option can be less than the time to expiration of the option, because of the possibility of early exercise.
- The term fugit is Latin, basically "time passes" or "time flies" to indicate the passage of time.
- **There is no explicit mathematical probability theory in this lecture.**

18.1 Fugit calculation using binomial model

- The concept and definition of fugit is independent of any probability model for the stock price movements, but we shall restrict our analysis to Geometric Brownian Motion.
- We can calculate the fugit of a derivative easily using a binomial model.
- First we calculate the fair value of the derivative using a binomial model.
- In the process of calculating the fair value, we tag each node in the binomial tree:
  1. Node = "dead" if the derivative is exercised (or terminated) at that node.
  2. Node = "alive" if the derivative is not terminated at that node (therefore still alive).
- There is no standard notation for fugit. Let us denote the fugit by τ.
- We initialize the fugit calculation by setting $\tau = T - t_0$ at all the nodes at the final (terminal) timestep $i = n$.
- We then calculate the fugit by working backwards through the binomial tree.
- The fugit calculation is similar to that for the fair value.
- Recall that at a node with stock price S at the timestep i, connected to nodes Su and Sd at the timestep $i+1$, the formula to calculate the fair value V is

$$V = e^{-r\Delta t}\left(pV_u + qV_d\right). \tag{18.1.1}$$

- To calculate the fugit, we employ the same formula but without the discount factor

$$\tau = p\tau_u + q\tau_d\,. \tag{18.1.2}$$

- If the derivative is "dead" at a node, we set $\tau = t - t_0$, where t is the time at that node.
- The value of fugit is given by the value of τ at the node at the initial timestep $i = 0$.
- By construction, the answer lies in the interval $0 \le \tau \le T - t_0$.

18.2 Fugit calculation using Black–Scholes–Merton equation

- Recall the Black–Scholes–Merton equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \,\frac{\partial^2 V}{\partial S^2} + (r-q)S\,\frac{\partial V}{\partial S} - rV = 0\,. \tag{18.2.1}$$

- The partial differential equation for the fugit is the same, but without the interest rate compounding:

$$\frac{\partial \tau}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \,\frac{\partial^2 \tau}{\partial S^2} + (r-q)S\,\frac{\partial \tau}{\partial S} = 0\,. \tag{18.2.2}$$

- We solve eq. (18.2.2) with the terminal condition $\tau = T - t_0$ at $t = T$.
- At every point in the (S,t) plane, we set $\tau = t - t_0$ if $V(S,t)$ is dead at that value of S and t.
- The fugit of the derivative is the value of τ at the current time and stock price (S_0, t_0).
- By construction, the answer lies in the interval $0 \le \tau \le T - t_0$.

18.3 Interpretation of fugit

- The concept of fugit is similar to that of the **Macaulay duration** of a bond.
- Basically, if all the cashflows of a derivative were paid at one time t_* (and discounted to the present time t_0), what would that time be? We wish to solve for t_* such that

$$e^{-r(t_*-t_0)}(\text{All cashflows paid at time } t_*) = \text{Fair value of derivative}\,. \tag{18.3.1}$$

- The answer is the fugit $t_* = \tau$.
- There is of coourse the complication that the casjflows of a derivative are weighted by probabilities, hence the value of the fugit will change if, for example, the volatility changes.
- Some people use the fugit to estimate which of the cashflows of a derivative are the "most important" in their overall contribution to the fair value, although this is not a rigorous mathematical description.
- For example, if an American option has an expiration time T and the fugit is (say) much less than $T - t_0$, it is an indication that the option will probably be exercised early and the terminal payoff of the option is not important. One should search to check what events are occurring around the fugit time. Perhaps the underlying stock pays a big dividend on a particular date, and it may be optimal to exercise early, to own the stock and collect that dividend.

18.4 Worked example: American put

- Recall the worked example of an American put in Lecture 17a.
- The input parameters were

$$S_0 = 100\,, \tag{18.4.1a}$$
$$K = 100\,, \tag{18.4.1b}$$
$$r = 0.1\,, \tag{18.4.1c}$$
$$q_{\rm div} = 0\,, \tag{18.4.1d}$$
$$\sigma = 0.5\,, \tag{18.4.1e}$$
$$T = 0.3\,, \tag{18.4.1f}$$
$$t_0 = 0\,. \tag{18.4.1g}$$

- The binomial tree had three timesteps ($n = 3$).
- From the above data we calculated the following parameter values:

$$e^{r\Delta t} \simeq 1.01005\,, \tag{18.4.2a}$$
$$e^{-r\Delta t} \simeq 0.99005\,, \tag{18.4.2b}$$
$$u \simeq 1.1713\,, \tag{18.4.2c}$$
$$d \simeq 0.8538\,, \tag{18.4.2d}$$
$$p \simeq 0.4922\,, \tag{18.4.2e}$$
$$q \simeq 0.5078\,. \tag{18.4.2f}$$

- The option valuation tree of the American put is shown in Fig. 1.

Binomial tree valuation for American put

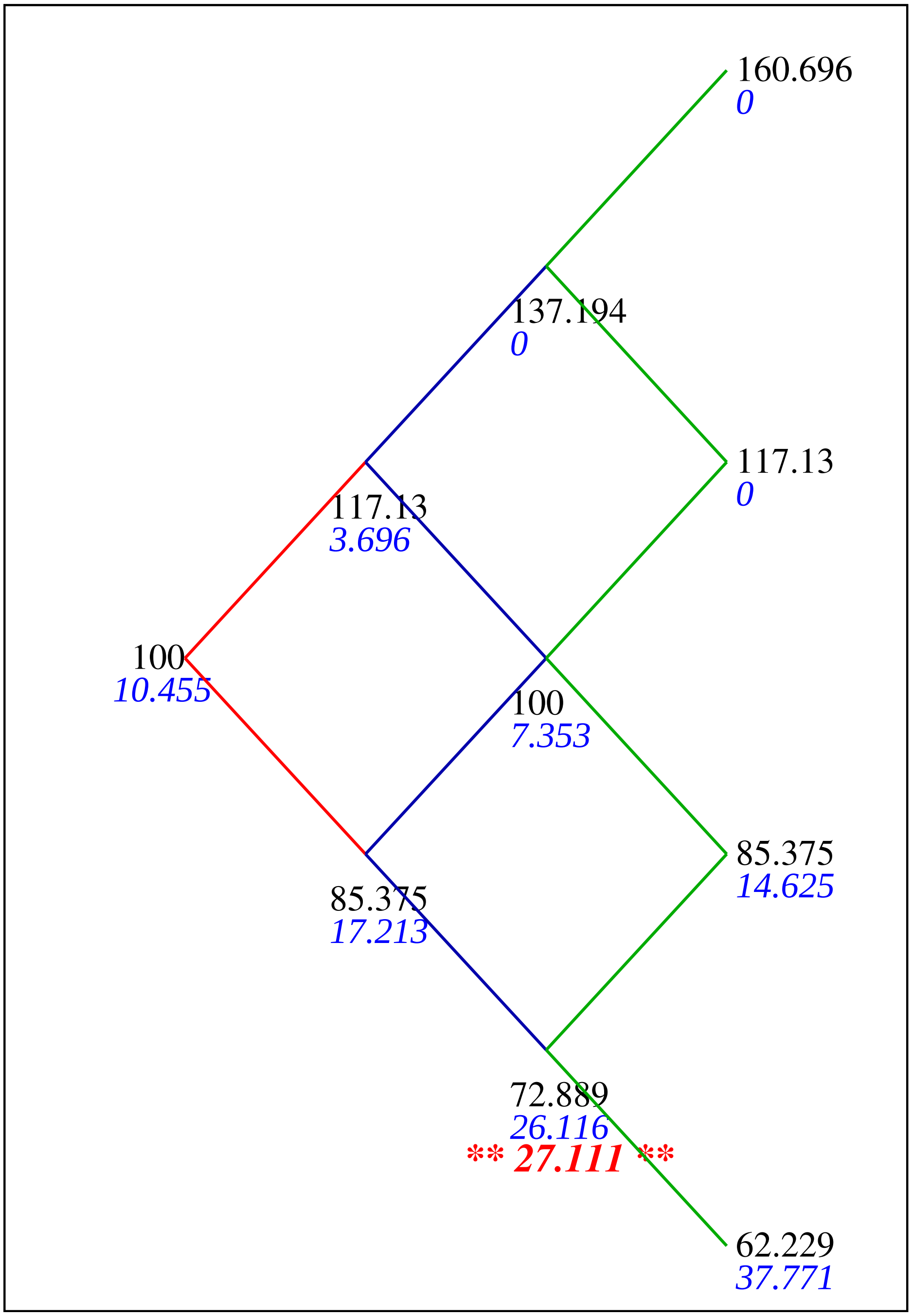

Figure 1: Valuation of American put option using a binomial tree with three timesteps, using the parameter values in eq. (18.4.2).

- We now calculate the fugit for the above American put.
- The option valuation tree of the American put is shown in Fig. 2.
- We begin with the nodes at the expiration time, hence $i = 3$.
  1. We set $\tau = T - t_0 = 0.3$ at every node for $i = 3$.
  2. It does not matter if the option is in or out of the money.
- Next we calculate the fugit values at the timestep $i = 2$, i.e. $t - t_0 = 0.2$.
  1. The values at the top two nodes are calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.3 = (p + q) \times 0.3 = 0.3 \,. \qquad (18.4.3)$$

  2. The American put was exercised early at the bottom node, hence we set $\tau = t - t_0 = 0.2$.
- Next we calculate the fugit values at the timestep $i = 1$, i.e. $t - t_0 = 0.1$.
  1. The value at the top node is calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.3 = (p + q) \times 0.3 = 0.3 \,. \qquad (18.4.4)$$

  2. The value at the bottom node is calculated as follows:

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.2 \simeq 0.249 \,. \qquad (18.4.5)$$

  3. This is a weighted average of the fugit values at the nodes for $i = 2$, where there was early exercise at one of those nodes. Hence the fugit calculation now yields a more complicated result.
- Finally we calculate the fugit value at the initial timestep $i = 0$, i.e. $t = t_0 = 0$.
- The fugit of the American put, using a binomial tree with three timesteps, is given by

$$\tau = p\tau_u + q\tau_d = p \times 0.3 + q \times 0.249 \simeq 0.274 \,. \qquad (18.4.6)$$

- The fugit value is less than the time to expiration $T - t_0 = 0.3$ because there was early exercise at one node in the binomial tree.
- A more accurate value using $n = 1000$ timesteps yields $\tau \simeq 0.259$.

Binomial tree fugit calculation for American put

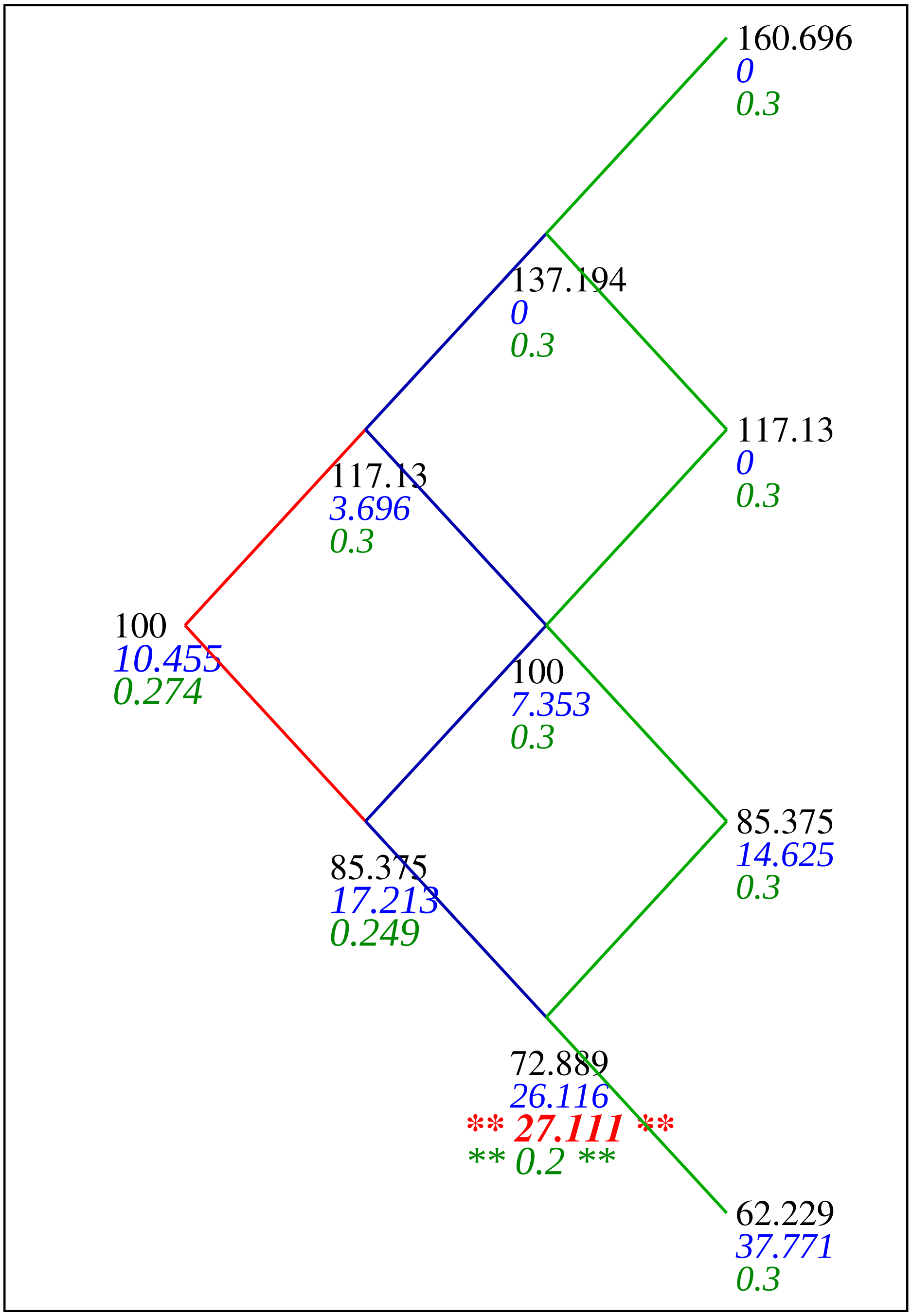

Figure 2: Calculation of fugit of American put option using the binomial tree shown in Fig. 1.

Queens College, CUNY, Department of Computer Science
Computational Finance
CSCI 365 / 765
Fall 2017
Instructor: Dr. Sateesh Mane

November 15, 2017

17 Lecture 17

Option pricing: binomial model

- In this lecture we study the **binomial model** to calculate the fair values of options.
- In fact, the binomial model can be used for any derivative on a stock, not just options.
- The binomial model can be employed to calculate the fair values of **American options.**
- To date, we have no algorithm to calculate the fair values of American options.
- The binomial model is not necessarily the most accurate option pricing model.
- However, it is simple to understand, easy to implement and computationally fast.
- For these reaons, the binomial model is important and is popular in practice.
- The binomial model assumes the stock price obeys Geometric Brownian Motion.
- **However, there is no explicit mathematical probability theory in this lecture.**

17.1 Binomial model: constructing the tree

17.1.1 One timestep

- Let us make a very simple model of the stock price movements.
- Let us discretize the time to expiration $T - t_0$ into n equal steps of size

$$\Delta t = \frac{T - t_0}{n} \,. \tag{17.1.1}$$

- At each step, we approximate that the stock price can go to only one of two future values at the next step.
- In more detail, suppose the stock price at the initial timestep $i = 1$ is S.
- At the next timestep $i = 2$, we say the stock price either goes up by a factor u to Su or down by a factor d to Sd.
- A sketch is shown in Fig. 1

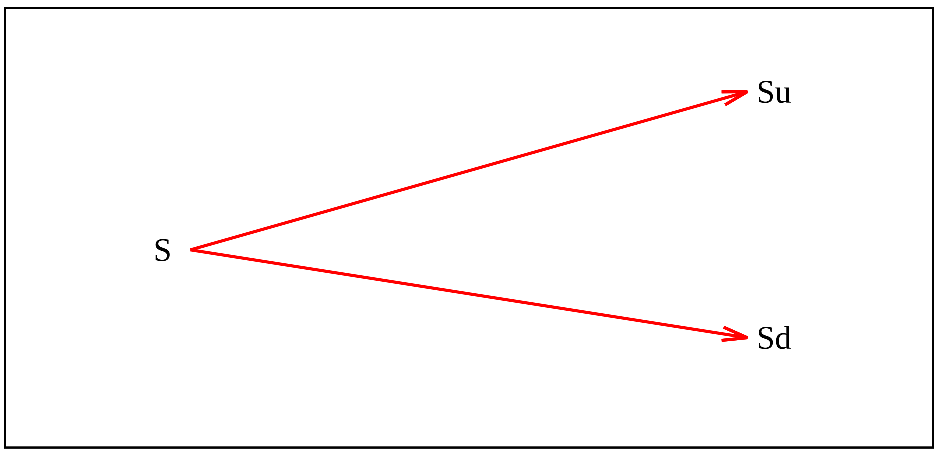

Figure 1: Sketch of binomial stock price movements for one timestep.

- The binomial model assumes the stock price obeys Geometric Brownian Motion.
- For this reason we express the stock price movements as ratios Su and Sd, as opposed to equal arithmetic steps $S \pm \delta S$.

17.1.2 Two timesteps

- After two timesteps, the final stock price levels are Su^2 (up-up), Sud (up-down), Sdu (down-up) and Sd^2 (down-down).
- Note that $Sud = Sdu$ so the up-down and down-up paths both lead to the same final node.
- We say that the paths **recombine.**
- Hence there are only three (not four) distinct "nodes" at the timestep $i + 2$.
- It is conventional to set $d = 1/u$ so $ud = 1$.
- Then $Sud = Sdu = S$ and the value equals the original stock price level S.
- A sketch is shown in Fig. 2

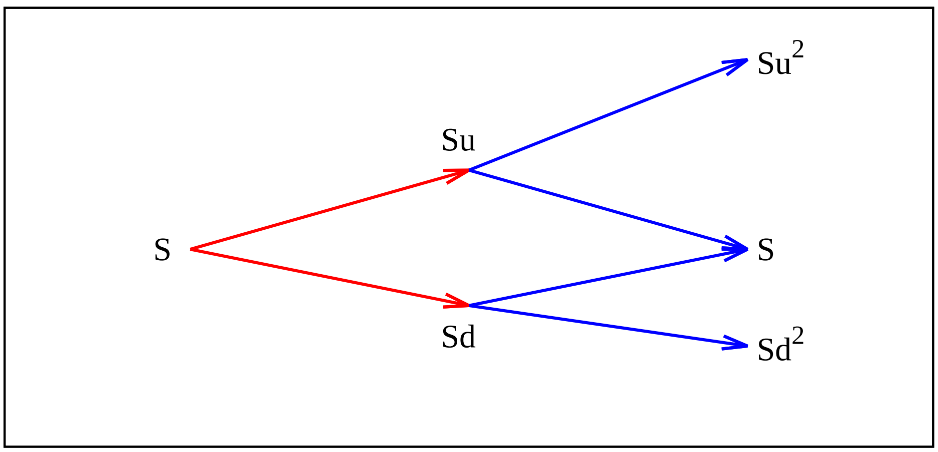

Figure 2: Sketch of binomial stock price movements for two timesteps.

17.1.3 Multiple timesteps

- We obviously generate more "nodes" at each timestep, creating a **tree** of nodes.
- A sketch is shown in Fig. 3 for three timesteps.

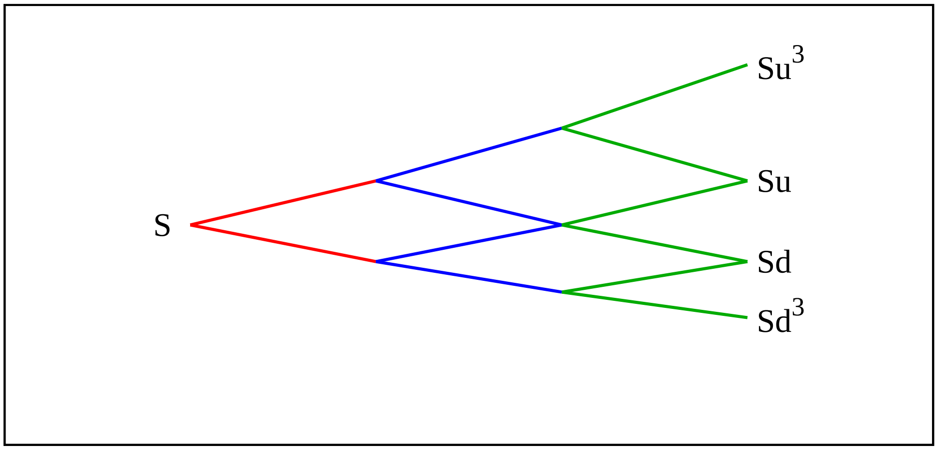

Figure 3: Sketch of binomial stock price movements for three timesteps.

- We call it a **binomial tree** because the stock price can go to only one of two values at each timestep.
- Also, because the paths recombine, there are totally i nodes at the timestep i.
- The recombination property of the binomial tree is very important.
  1. The recombination property means that a tree with n timesteps has only $\frac{1}{2}n(n+1)$ nodes.
  2. Hence the computation time for a recombining tree is of **polynomial complexity** $O(n^2)$**.**
  3. If the tree did not recombine, then a tree with n timesteps would have $2^n - 1$ nodes.
  4. **The computation time of a non-recombining tree would have exponential complexity, which is very bad.**

17.2 Binomial model: parameter values

- Let us say the probability of taking an up step is p and the probability of taking a down step is $q = 1 - p$.
- Let the risk-free interest rate be r (a constant).
- Let the volatility of the stock be σ (a constant).
- Suppose the stock pays continuous dividends at a rate q.
- Then the values of u, d, p and q are given as follows:

$$u = e^{\sigma\sqrt{\Delta t}}\,, \tag{17.2.1a}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u\,, \tag{17.2.1b}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}\,, \tag{17.2.1c}$$

$$q = \frac{u - e^{(r-q)\Delta t}}{u - d}\,. \tag{17.2.1d}$$

- The above expressions will be derived later, in Sec. 17.5.

17.3 Binomial model: valuation

- We value an option by working ***backwards*** from the final timestep to the initial timestep.
- This is because we know the value of the option at the expiration time, at all the stock price levels (nodes) in the binomial tree. We use that information to systematically work backwards to determine the option's fair value at the nodes at earlier times.
- Let us begin with a European call option.
- Consider a node at the timestep i and let the stock price at that node be S. Let the option value be C. We wish to calculate the value of C.
- The node at the timestep i is connected to two nodes at the timestep $i+1$, with the values Su and Sd, respectively.
- Let the option fair values at those nodes be C_u and C_d, respectively.
- A sketch is shown in Fig. 4.

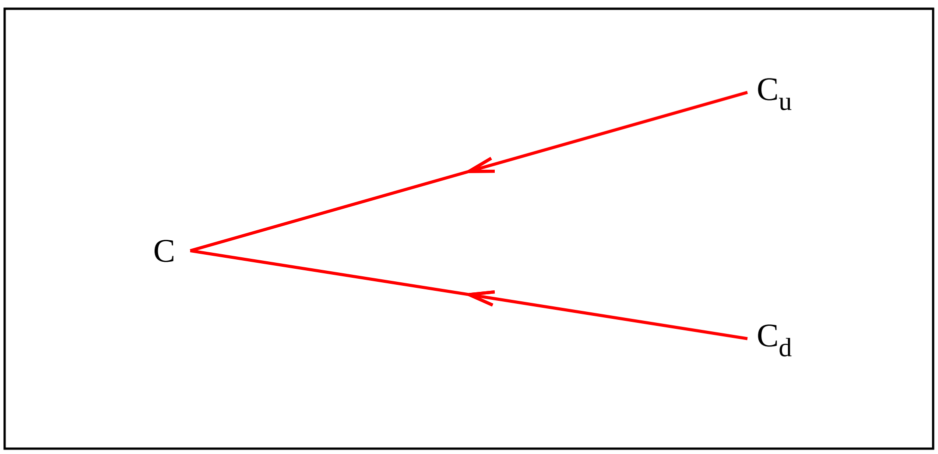

Figure 4: Sketch of binomial tree and option fair values at nodes at timesteps i and $i+1$.

- By hypothesis, we have worked backwards through the binomial tree so we know the values of C_u and C_d. We shall use that information to calculate the value of C.
- Notice the arrows in Fig. 4 point backwards: we are working backwards through the tree.

- We calculate the value of the option fair value C at the timestep i as follows:

$$C = e^{-r\Delta t} \left(pC_u + qC_d \right) . \tag{17.3.1}$$

  1. Essentially, eq. (17.3.1) says that the value of C is given by calculating the expectation value (probability weighted average) of C_u and C_d, and discounting the result at the risk-free rate r.
  2. **The formal statement is that value of the option fair value at the timestep i is the discounted expected option fair value at the timestep $i + 1$.**

- Hence this is the valuation procedure:

  1. We compute the option value at all the nodes at the final timestep (expiration time $t = T$), say $i = n$.
  2. We employ eq. (17.3.1) to calculate the option fair values at all the nodes at the timestep $i = n - 1$.
  3. We repeat the above procedure, looping backwards through the binomial tree to calculate the option fair values at all the nodes at the earlier timesteps $i = n - 2, n - 3$, etc.
  4. We loop backwards through the binomial tree until we obtain the option fair value at the initial node, say $i = 0$, where $t = t_0$ and the stock price is S_0.

- The exact same procedure also works for a European put option. The only difference is the value of the option at the terminal timestep (expiration). The formula corresponding to eq. (17.3.1) is, with an obvious notation

$$P = e^{-r\Delta t} \left(pP_u + qP_d \right) . \tag{17.3.2}$$

- The same procedure also works for **American** options, with one important modification, because of the possibility of early exercise.

  1. For an American option, **we compare the value from eq. (17.3.1) to the option's intrinsic value.**
  2. If the value from eq. (17.3.1) is less than the option's intrinsic value, we set the option fair value at that node to the intrinsic value instead.
  3. In other words, it is optimal to exercise the American option early at that value of the stock price and time.
  4. This is because the value of an American option cannot be less than its intrinsic value. Else we exercise the option immediately and make an arbitrage profit.
  5. Hence the formula for the value of an American call option C is modified as follows:

$$C_{\text{American}} = \max\Big\{ e^{-r\Delta t} \left(pC_u + qC_d \right), \max(S - K, 0) \Big\} . \tag{17.3.3}$$

  6. For an American put option, the corresponding formula is as follows:

$$P_{\text{American}} = \max\Big\{ e^{-r\Delta t} \left(pP_u + qP_d \right), \max(K - S, 0) \Big\} . \tag{17.3.4}$$

17.4 Comments on the binomial model

- The binomial model is simple to understand and easy to code (implement in software).
- It runs quickly (computation time) and is popular and is widely used in the financial industry.
- Nevertheless, the binomial model does have various limitations.
- There is no flexilibity in the locations of the nodes in the binomial tree.
- For example, we might wish to have a closer spacing of nodes at values of the stock price close to the option strike. The binomial model does not allow this. If we wish to have closely spaced nodes near the strike price, we must increase the value of n, which decreases the spacing of the nodes throughout the tree.
- Let us consider an option with one year to expiration. Suppose $S_0 = K = 100$. Let the volatility be 50%, i.e. $\sigma = 0.5$. Suppose we employ $n = 1000$, which is a value used by academics in options research.
  1. Then the highest and lowest stock price levels in the binomial tree are
  $$S_0(e^{\sigma\sqrt{\Delta t}})^n = 100\left(e^{0.5\sqrt{0.001}}\right)^{1000} \simeq 7.4 \times 10^8 \,. \tag{17.4.1a}$$
  $$S_0(e^{-\sigma\sqrt{\Delta t}})^n = 100\left(e^{-0.5\sqrt{0.001}}\right)^{1000} \simeq 1.4 \times 10^{-5} \,. \tag{17.4.1b}$$
  2. These are absurdly large and small values, but there is nothing we can do about this.
  3. We cannot arbitrarily truncate the binomial tree without upsetting the option valuation.

17.5 Binomial model: derivation of parameters

- This section presents the derivation of the parameter values in Sec. 17.2.
- This section contains a small amount of mathematical probability theory.
- Let us say the probability of taking an up step is p and the probability of taking a down step is $q = 1 - p$.
- We determine the values of p and q by equating the expressions for mean and variance of S after one timestep, calculated using the binomial model, to the known expressions in terms of the risk-free interest rate and volatility.
- We assume that the stock pays continuous dividends at a rate q.
- Let the risk-free interest rate be r (a constant).
  1. Then if the stock price is S at the timestep i, after one timestep the expectation value of the stock price is
  $$\mathbb{E}[S]_{i+1} = Se^{(r-q)\Delta t} . \tag{17.5.1}$$
  2. The above expression does **not** require Geometric Brownian Motion.
  3. Using the binomial tree, the expectation value of S at the timestep $i+1$ is
  $$\mathbb{E}[S]_{i+1} = pSu + qSd = S(pu + qd) . \tag{17.5.2}$$
  4. Equating the two expressions for the expectations yields
  $$pu + qd = e^{(r-q)\Delta t} . \tag{17.5.3}$$
  5. This can be solved for p and q by using the additional relation $p + q = 1$.
  6. The answer is
  $$p = \frac{e^{(r-q)\Delta t} - d}{u - d} , \qquad q = \frac{u - e^{(r-q)\Delta t}}{u - d} . \tag{17.5.4}$$
- However, this is not a complete solution because we have not specified the values of u and d.
- There are of course hidden assumptions in the above derivation. We require both p and q to be positive (or at least zero). Hence to obtain meaningful values for the probabilities we must have
$$d \le e^{(r-q)\Delta t} \le u . \tag{17.5.5}$$
- **Technical mathematical note about eq. (17.5.1).**
  1. It was stated in Lecture 14 that the growth of the stock price random variable dS^r is
  $$dS^r = \mu S^r \, dt + \sigma S^r \, dW_t . \tag{17.5.6}$$
  2. However, eq. (17.5.1) makes no reference to μ and contains the risk-free rate r instead.
  3. Technically, we have performed a mathematical transformation, as in the derivation of the Black-Scholes equation.
  4. The values of p and q are the called **risk-neutral probabilities.**

- We determine the values of u and d by fitting to the variance of S in one timestep, which is
- Let the volatility of the stock be σ (a constant).

1. Then if the stock price is S at the timestep i, after one timestep the variance of the stock price is
$$\mathrm{Var}(S)_{i+1} = \sigma^2 S^2 \Delta t \,. \tag{17.5.7}$$
2. **The above expression is derived using Geometric Brownian Motion.**
3. Using the binomial tree, the variance of S at the timestep $i+1$ is
$$\mathrm{Var}(S)_{i+1} = pS^2u^2 + qS^2d^2 - S^2e^{2(r-q)\Delta t} \tag{17.5.8}$$
4. Equating the two expressions for the variance (and retaining terms to $O(\Delta t)$ only) yields
$$pu^2 + qd^2 - (1 + 2(r-q)\Delta t) = \sigma^2 \Delta t \,. \tag{17.5.9}$$
5. The solution is
$$u = e^{\sigma\sqrt{\Delta t}} \,, \qquad d = e^{-\sigma\sqrt{\Delta t}} \,. \tag{17.5.10}$$

- Hence the probabilities p and q are given by eq. (17.5.4) where the values of u and d are given by eq. (17.5.10). Writing it out explicitly, the expressions are
$$p = \frac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \,, \qquad q = \frac{e^{\sigma\sqrt{\Delta t}} - e^{(r-q)\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \,. \tag{17.5.11}$$
- The inequalities in eq. (17.5.5) then yield
$$e^{-\sigma\sqrt{\Delta t}} \le e^{(r-q)\Delta t} \le e^{\sigma\sqrt{\Delta t}} \,. \tag{17.5.12}$$
- The inequalities in eq. (17.5.12) are usually satisfied in practice, but can fail is the value of σ is very small, or if the value of Δt is not small enough.