Assignment 2

Q1. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $f(x) = Ax^2 + Bx + C$

- $f(b) = Ab^2 + Bb + C$
- $f(a) = Aa^2 + Ba + C$

$= \dfrac{f(b) - f(a)}{b - a} = \dfrac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$

$= \dfrac{Ab^2 + Bb + \cancel{C} - Aa^2 - Ba - \cancel{C}}{b - a}$

$= \dfrac{Ab^2 - Aa^2 + Bb - Ba}{b - a}$

$= \dfrac{A(b^2 - a^2) + B(b - a)}{b - a}$

$= \dfrac{(\cancel{b - a})(A(b + a) + B)}{\cancel{b - a}}$

$= \dfrac{f(b) - f(a)}{b - a} = A(b + a) + B$

- $f'(x) = 2Ax + B$
- $f'(c) = 2Ac + B$

$= f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$= 2\cancel{A}c + \cancel{B} = \cancel{A}(b + a) + \cancel{B}$

$= c = \dfrac{b + a}{2}$ = midpoint of a and b. Proved!

Q2. This function satisfies the hypothesis of mean value theorem as it is continous on closed interval $[a, b]$ and differentiable on open interval (a, b).

- $f(a) = f(-2) = 2(-2) + e^{-2(-2)} = -4 + e^{4}$
- $f(b) = f(3) = 2(3) + e^{-2(3)} = 6 + e^{-6}$

$\Rightarrow \dfrac{f(b) - f(a)}{b - a} = \dfrac{6 + e^{-6} - (-4 + e^{4})}{3 - (-2)}$

$= \dfrac{6 + e^{-6} + 4 - e^{4}}{3 + 2}$

$\Rightarrow \dfrac{f(b) - f(a)}{b - a} = \dfrac{10 + e^{-6} - e^{4}}{5}$

- $f'(x) = 2 - 2e^{-2t}$
- $f'(c) = 2 - 2e^{-2c}$

$\Rightarrow \dfrac{f(b) - f(a)}{b - a} = f'(c)$

$\Rightarrow \dfrac{10 + e^{-6} - e^{4}}{5} = 2 - 2e^{-2c}$

$\Rightarrow \cancel{10} + e^{-6} - e^{4} = \cancel{10} - 10e^{-2c}$

$\Rightarrow e^{-2c} = -\left(\dfrac{e^{-6} - e^{4}}{10}\right)$

$\Rightarrow -2c = \ln\left[-\left(\dfrac{e^{-6} - e^{4}}{10}\right)\right]$

$\Rightarrow c = \dfrac{-\ln\left[-\left(\dfrac{e^{-6} - e^{4}}{10}\right)\right]}{2}$

- Since ln of negative number is undefined, there are no values of c that satisfy Mean Value Theorem.

Q3.

- $R'(x) = u'v + uv'$ $\qquad u' = 2(x-1)$
$R'(x) = (2(x-1))(e^{3x}) + (x-1)^2)(3e^{3x})$ $\qquad v' = 3e^{3x}$
$R'(x) = e^{3x}[2(x-1) + 3(x-1)^2]$
$R'(x) = e^{3x}[2x-2+3x^2-6x+3]$
$R'(x) = e^{3x}(3x^2-4x+1)$

= $R'(x) = 0$
= $e^{3x}(3x^2-4x+1) = 0$
$e^{3x} = 0$, $\quad 3x^2-4x+1 = 0$ $\quad (-\infty, 1/3) \cup (1/3, 1)$
$x = 1, 1/3$ $\quad \cup (1, \infty)$

- $R'(1) = e^{3(1)}(3(1)^2 - 4(1)+1)$
$R'(1) = 0$
- $R'(1/3) = e^{3(1/3)}(3(1/3)^2 - 4(1/3)+1)$
$R'(1/3) = 0$
- $R'(-1) = e^{3(-1)}(3(-1)^2 - 4(-1)+1)$
- $R'(-1) = 0.39 > 0$
- $R'(0.5) = e^{3(0.5)}(3(0.5)^2 - 4(0.5)+1)$
$R'(0.5) = -1.12 < 0$
- $R'(2) = e^{3(2)}(3(2)^2 - 4(2)+1)$
$R'(2) = 2017.14 > 0$

= Increasing = $(-\infty, 1/3) \cup (1, \infty)$

= Decreasing = $(1/3, 1)$

- $R''(x) = u'v + uv'$ $\qquad u' = 3e^{3x}$
$R''(x) = (3e^{3x})(3x^2-4x+1) + (e^{3x})(6x-4)$ $\qquad v' = 6x-4$
$R''(x) = e^{3x}[9x^2-12x+3+6x-4]$
$R''(x) = e^{3x}(9x^2-6x-1)$

• $R''(x) = 0$

$e^{3x}(9x^2 - 6x - 1) = 0$

$e^{3x} = 0$, $9x^2 - 6x - 1 = 0$

$x = 0.8$, -0.13

$(-\infty, -0.13) \cup (-0.13, 0.8) \cup (0.8, \infty)$

• $R''(-0.13) = -0.04 < 0$
• $R''(0.8) = -0.44 < 0$
• $R''(-1) = 0.69 > 0$
• $R''(0) = -1 < 0$
• $R''(1) = 40.17 > 0$

= Concave up = $(-\infty, -0.13) \cup (0.8, \infty)$

= Concave down = $(-0.13, 0.8)$.

Q4. $P'(x) = -3x^2 + 18x + 120$

$P'(x) = 0$

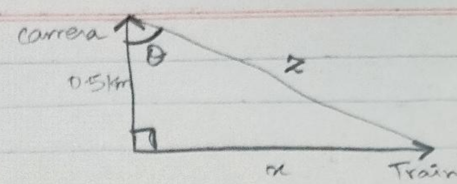
$-3x^2 + 18x + 120 = 0$

$x = 10, -4$

• Since $x \geq 5$, we will consider $x = 10$.

= $P(10) = -(10)^3 + 9(10)^2 + 120(10) - 400$

$P(10) = 700$ (Maximum profit).



Q5 i) $z^2 = x^2 + 0.5^2$

$z^2 = x^2 + 0.25$

$z = \sqrt{x^2 + 0.25}$

ii) $z' = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+0.25}}$

$x = \sqrt{z^2 - 0.25}$

$x = \sqrt{1 - 0.25}$

$x = \sqrt{0.75} = 0.86$

$z' = \frac{x}{\sqrt{x^2+0.25}}$

$z' = \frac{0.86}{\sqrt{0.86^2+0.25}}$

$z' = 0.86$ Km/min

iii) $\tan\theta = x/0.5$

$\theta = \tan^{-1}(2x)$

$\theta' = \frac{1}{1+(2x)^2} \cdot 2\frac{dx}{dt}$

$\theta' = \frac{1}{1+4x^2} \cdot 2\frac{dx}{dt}$

$\theta' = \frac{1}{1+4(0.86)} \cdot 2(0.86)$

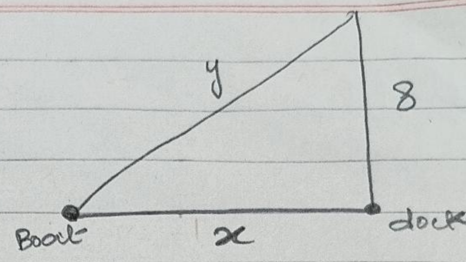
$\theta' = 0.38$ radians/min.

Q6

$$x^2 + 8^2 = y^2$$

$$x^2 + 64 = y^2 \quad , \; y = 11.31$$

$$2x \cdot \frac{dx}{dt} = 2y \frac{dy}{dt}$$



$$2(8) \cdot \frac{dx}{dt} = 2(11.31) \cdot -1$$

$$\frac{dx}{dt} = -1.41 \text{ ft/s.}$$

Boat is approaching the dock at the rate of -1.41 ft/s if boat is 8 ft away from dock.

Ex. 7.1

20. $\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta \, d\theta$

$= \int \sec x \tan x \, dx$

$= \sec x + c$

$= \sec(\sin\theta) + c$

$\sin\theta = x$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

21. $\int \frac{\operatorname{cosech}^2(2/x)}{x^2} dx$

$= \int \frac{-\operatorname{cosech}^2(u)}{2} du$

$= \frac{-1}{2} \int \operatorname{cosech}^2(u) \, du$

$= \frac{-1}{2}(-\coth(u)) + c$

$= \frac{1}{2}\coth(u) + c$

$= \frac{1}{2}\coth\left(\frac{2}{x}\right) + c.$

$\frac{2}{x} = u$

$\frac{du}{dx} = \frac{-2}{x^2}$

$du = \frac{-2}{x^2} dx$

$-\frac{du}{2} = \frac{dx}{x^2}$

22. $\int \frac{dx}{\sqrt{x^2-4}}$

$= \int \frac{2\sec\theta \tan\theta \, d\theta}{\sqrt{(2\sec\theta)^2 - 4}}$

$= \int \frac{2\sec\theta \tan\theta \, d\theta}{\sqrt{4\sec^2\theta - 4}}$

$= \int \frac{2\sec\theta \tan\theta \, d\theta}{\sqrt{4(\sec^2\theta - 1)}}$

$= \int \frac{\cancel{2} \sec\theta \cancel{\tan\theta} \, d\theta}{\cancel{2} \cancel{\tan\theta}}$

$= \int \sec\theta \, d\theta$

$= \ln(\tan\theta + \sec\theta) + c$

$= \ln\left(\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right) + c$

$= \ln(\sqrt{x^2-4} + x) - \ln 2$

$= \ln(\sqrt{x^2-4} + x) + c$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\frac{dx}{d\theta} = 2\sec\theta\tan\theta$

$dx = 2\sec\theta\tan\theta \, d\theta$

$x = a\sec\theta$

$x = 2\sec\theta$

$\sec\theta = \frac{x}{2}$

$\tan^2\theta = \sec^2\theta - 1$

$\tan^2\theta = \frac{x^2}{4} - 1$

$\tan^2\theta = \frac{x^2-4}{4}$

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

23. $\int \frac{e^{-x} \, dx}{4 - e^{-2x}}$

$= \int \frac{-du}{4 - u^2}$

$= -\int \frac{du}{(2+u)(2-u)}$

$= \frac{1}{(2+u)(2-u)} = \frac{A}{2+u} + \frac{B}{2-u}$

$= \frac{1}{(2+u)(2-u)} = \frac{A(2-u) + B(2+u)}{(2+u)(2-u)}$

$= 1 + 0u = 2A - Au + 2B + uB$

$= 2A + 2B = 1$

$= 2A + 2A = 1$

$= A = 1/4$

$(u)^2 = e^{-x} = e^{-2x}$

$\frac{du}{dx} = -e^{-x}$

$du = -e^{-x} \, dx$

$uB - uA = 0u$

$\cancel{u}(B - A) = 0\cancel{u}$

$B - A = 0$

$B = A = 1/4.$

$$= -\int \frac{du}{4-u^2} = -\frac{1}{4}\left(\int \frac{1}{2+u}\,du + \int \frac{1}{2-u}\,du\right)$$

$$= -\int \frac{du}{4-u^2} = -\frac{1}{4}\left(\ln(2+u) - \ln(2-u)\right) + C.$$

$$= -\int \frac{du}{4-u^2} = -\frac{1}{4}\ln\left(\frac{2+u}{2-u}\right) + C$$

$$= -\int \frac{du}{4-u^2} = -\frac{1}{4}\ln\left(\frac{2+e^{-x}}{2-e^{-x}}\right) + C$$

24. $\int \frac{\cos(\ln x)}{x}\,dx.$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

$= \int \cos u \,.\, du$

$= \sin u + C$

$= \sin(\ln x) + C$

25. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$ $\int \frac{du}{\sqrt{1-u^2}}$

$(u)^2 = e^x = e^{2x}$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$= \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u + C$

$= \sin^{-1}(e^x) + C.$

26. $\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$

$u = x^{-1/2-1}$

$\frac{du}{dx} = \frac{-1}{2} \,.\, x^{-3/2}$

$2\,du = \frac{1}{x^{3/2}}\,dx.$

$= \int \sinh(u) \,.\, 2\,du$

$= -2\cosh(u) + C$

$= -2\cosh(x^{-1/2}) + C$

27. $\int \frac{x\,dx}{\operatorname{cosec}(x^2)}$

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2} = x\,dx$$

$$\frac{1}{1/\sin x} = \sin x$$

$= \int \frac{1}{2} \frac{du}{\operatorname{cosec} u}$

$= \frac{1}{2} \int \frac{1\,du}{\operatorname{cosec} u}$

$= \frac{1}{2} \int \sin u\,du$

$= -\frac{1}{2} \cos u + c$

$= \frac{-1}{2} \cos x^2 + c$

28. $\int \frac{e^x\,dx}{\sqrt{4-e^{2x}}}$

$$u = e^x$$

$$\frac{du}{dx} = e^x$$

$$du = e^x\,dx$$

$= \int \frac{du}{\sqrt{4-u^2}}$

$= \sin^{-1}\left(\frac{u}{2}\right) + c$

$= \sin^{-1}\left(\frac{e^x}{2}\right) + c.$

29. $\int x\, 4^{-x^2}\, dx$

$= \int -4^{u}\, \frac{du}{2}$

$= \frac{-1}{2} \int 4^{u}\, du$

$= \frac{-1}{2} \frac{4^{u}}{\ln 4} + c$

$= \frac{-1}{2} \frac{4^{-x^2}}{\ln 4} + c$

$u = -x^2$

$\frac{du}{dx} = -2x$

$-\frac{du}{2} = x\, dx$

30. $\int 2^{\pi x}\, dx$

$= \frac{2^{\pi x}}{\pi \ln 2} + c.$

$a^{2x} = \frac{a^{2x}}{\ln a} \cdot \frac{1}{2}$

$2^{\pi x} = \frac{2^{\pi x}}{\pi \ln 2} + c.$