Section 4.6: Dimension

Linear Algebra Lecture

Introduction

In the previous section, we saw that the standard basis for $\mathbb{R}^n$ has n vectors. This suggests a link between the number of vectors in a basis and the "dimension" of a vector space. In this section, we make this idea precise.

Fundamental Theorems

Theorem 0.1 (4.6.1)**.** All bases for a finite-dimensional vector space have the same number of vectors.

Theorem 0.2 (4.6.2)**.** Let V be a finite-dimensional vector space, and let $\{v_1, v_2, \ldots, v_n\}$ be any basis for V.

(a) If a set in V has more than n vectors, then it is linearly dependent.

(b) If a set in V has fewer than n vectors, then it does not span V.

These theorems tell us that in an n-dimensional space:

- You cannot have more than n linearly independent vectors
- You need at least n vectors to span the space
- A basis has exactly the right number of vectors: enough to span but not so many as to be dependent

Definition of Dimension

Definition 0.1 (Dimension)**.** The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Remark 0.1 (Degree of Freedom as Dimension)**.** The dimension of a vector space can be thought of as the number of "degrees of freedom" or the number of independent parameters needed to specify any vector in the space.

- A line has 1 degree of freedom $\Rightarrow$ dimension 1

- A plane has 2 degrees of freedom $\Rightarrow$ dimension 2
- Space around us has 3 degrees of freedom $\Rightarrow$ dimension 3

Examples

Example 0.1 (Dimensions of Familiar Vector Spaces)**.**

$$\begin{aligned}
\dim(\mathbb{R}^n) &= n \quad \text{[The standard basis has } n \text{ vectors]} \\
\dim(P_n) &= n+1 \quad \text{[The standard basis has } n+1 \text{ vectors]} \\
\dim(M_{mn}) &= mn \quad \text{[The standard basis has } mn \text{ vectors]}
\end{aligned}$$

Example 0.2 (Dimension of Span(S))**.** If $S = \{v_1, v_2, \ldots, v_r\}$ is a linearly independent set, then they automatically form a basis for span(S), so:

$$\dim\left[\text{span}\{v_1, v_2, \ldots, v_r\}\right] = r$$

In words: the dimension of the space spanned by a linearly independent set of vectors equals the number of vectors in that set.

Example 0.3 (Dimension of a Solution Space)**.** Find a basis for and the dimension of the solution space of:

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 + 15x_6 &= 0 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution: The general solution is:

$$\begin{aligned}
x_1 &= -3r - 4s - 2t \\
x_2 &= r \\
x_3 &= -2s \\
x_4 &= s \\
x_5 &= t \\
x_6 &= 0
\end{aligned}$$

In vector form:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

So the basis is:

$$\{(-3, 1, 0, 0, 0, 0),\ (-4, 0, -2, 1, 0, 0),\ (-2, 0, 0, 0, 1, 0)\}$$

and dim(solution space) = 3.

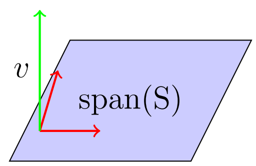
Plus/Minus Theorem

Theorem 0.3 (4.6.3 - Plus/Minus Theorem)**.** Let S be a nonempty set of vectors in a vector space V.

(a) If S is linearly independent, and if v is a vector in V that is outside of span(S), then the set $S \cup \{v\}$ is still linearly independent.

(b) If v is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{v\}$ denotes the set obtained by removing v from S, then:

$$\text{span}(S) = \text{span}(S - \{v\})$$



Adding v outside span(S) maintains linear independence

Example 0.4 (Applying the Plus/Minus Theorem)**.** Show that $p_1 = 1 - x^2$, $p_2 = 2 - x^2$, and $p_3 = x^3$ are linearly independent.

Solution: The set $S = \{p_1, p_2\}$ is linearly independent since neither is a scalar multiple of the other. Since p_3 cannot be expressed as a linear combination of p_1 and p_2 (it has x^3 term while they don't), we can adjoin it to S to get the linearly independent set $\{p_1, p_2, p_3\}$.

Basis Characterization Theorem

Theorem 0.4 (4.6.4)**.** Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

This is a very useful theorem! In an n-dimensional space, if you have exactly n vectors, you only need to check one condition:

- Either check they span V, OR
- Check they are linearly independent

The other condition will follow automatically.

Example 0.5 (Bases by Inspection)**.** (a) Explain why $v_1 = (-3, 7)$ and $v_2 = (5, 5)$ form a basis for $\mathbb{R}^2$.

Solution: Since neither vector is a scalar multiple of the other, they are linearly independent. Since $\dim(\mathbb{R}^2) = 2$ and we have 2 linearly independent vectors, they form a basis by Theorem 4.6.4.

(b) Explain why $v_1 = (2, 0, -1)$, $v_2 = (4, 0, 7)$, and $v_3 = (-1, 1, 4)$ form a basis for $\mathbb{R}^3$.

Solution: Vectors v_1 and v_2 are linearly independent and lie in the xz-plane. Vector v_3 has a y-component, so it's outside this plane. Thus $\{v_1, v_2, v_3\}$ is linearly independent. Since $\dim(\mathbb{R}^3) = 3$ and we have 3 linearly independent vectors, they form a basis.

Extension and Reduction Theorems

Theorem 0.5 (4.6.5)**.** Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

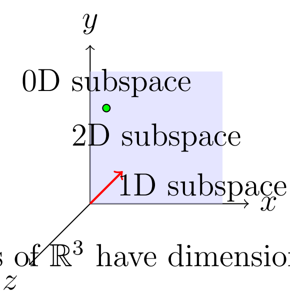
(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Theorem 0.6 (4.6.6)**.** If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite-dimensional

(b) $\dim(W) \leq \dim(V)$

(c) $W = V$ if and only if $\dim(W) = \dim(V)$



Subspaces of $\mathbb{R}^3$ have dimension 0, 1, 2, or 3

Exercise Solutions

Exercises 1-6: Solution Spaces

1. **Find a basis for and dimension of:**

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

Solution: Write augmented matrix and reduce:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ -2 & -1 & 2 & | & 0 \\ -1 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So $x_1 = x_3$, $x_2 = 0$, x_3 free. Solution: $(x_1, x_2, x_3) = t(1, 0, 1)$.

Basis: $\{(1, 0, 1)\}$, Dimension: 1

2. **Find a basis for and dimension of:**

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

Solution: Reduce the system:

$$\begin{bmatrix} 3 & 1 & 1 & 1 & | & 0 \\ 5 & -1 & 1 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & | & 0 \\ 0 & 1 & \frac{1}{4} & 1 & | & 0 \end{bmatrix}$$

So $x_1 = -\frac{1}{4}x_3$, $x_2 = -\frac{1}{4}x_3 - x_4$, with x_3, x_4 free.

Solution: $(x_1, x_2, x_3, x_4) = s(-\frac{1}{4}, -\frac{1}{4}, 1, 0) + t(0, -1, 0, 1)$

Basis: $\{(-\frac{1}{4}, -\frac{1}{4}, 1, 0),\ (0, -1, 0, 1)\}$, Dimension: 2

3. **Find a basis for and dimension of:**

$$2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 5x_3 = 0$$
$$x_2 + x_3 = 0$$

Solution: From third equation: $x_2 = -x_3$. From second: $x_1 = -5x_3$. Check first: $2(-5x_3) + (-x_3) + 3x_3 = -8x_3 = 0 \Rightarrow x_3 = 0$.

So only solution is trivial: $(0, 0, 0)$. Basis: $\{\}$ (empty set), Dimension: 0

4. **Find a basis for and dimension of:**

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0$$

Solution: The second equation is just 2 times the first, so we have only one independent equation:

$$x_1 - 4x_2 + 3x_3 - x_4 = 0 \Rightarrow x_1 = 4x_2 - 3x_3 + x_4$$

With x_2, x_3, x_4 free.

Solution: $(x_1, x_2, x_3, x_4) = s(4, 1, 0, 0) + t(-3, 0, 1, 0) + u(1, 0, 0, 1)$

Basis: $\{(4, 1, 0, 0),\ (-3, 0, 1, 0),\ (1, 0, 0, 1)\}$, Dimension: 3

5. **Find a basis for and dimension of:**

$$x_1 - 3x_2 + x_3 = 0$$
$$2x_1 - 6x_2 + 2x_3 = 0$$
$$3x_1 - 9x_2 + 3x_3 = 0$$

Solution: All equations are multiples of the first: $x_1 - 3x_2 + x_3 = 0$

So $x_1 = 3x_2 - x_3$, with x_2, x_3 free.

Solution: $(x_1, x_2, x_3) = s(3, 1, 0) + t(-1, 0, 1)$

Basis: $\{(3, 1, 0),\ (-1, 0, 1)\}$, Dimension: 2

6. **Find a basis for and dimension of:**

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

Solution: Solve the system:

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 3 & 2 & -2 & | & 0 \\ 4 & 3 & -1 & | & 0 \\ 6 & 5 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Only solution is trivial: $(0, 0, 0)$. Basis: $\{\}$, Dimension: 0

Exercise 7: Subspaces of $\mathbb{R}^3$

(a) **The plane** $3x - 2y + 5z = 0$

Solution: Solve for one variable: $3x = 2y - 5z \Rightarrow x = \frac{2}{3}y - \frac{5}{3}z$

So $(x, y, z) = s(\frac{2}{3}, 1, 0) + t(-\frac{5}{3}, 0, 1)$

Basis: $\{(\frac{2}{3}, 1, 0),\ (-\frac{5}{3}, 0, 1)\}$ or clearer: $\{(2, 3, 0),\ (-5, 0, 3)\}$

Dimension: 2

(b) **The plane** $x - y = 0$

Solution: $x = y$, z free.

So $(x, y, z) = s(1, 1, 0) + t(0, 0, 1)$

Basis: $\{(1, 1, 0),\ (0, 0, 1)\}$, Dimension: 2

(c) **The line** $x = 2t, y = -t, z = 4t$

Solution: $(x, y, z) = t(2, -1, 4)$

Basis: $\{(2, -1, 4)\}$, Dimension: 1

(d) **All vectors of form** (a, b, c) **where** $b = a + c$

Solution: $(a, b, c) = (a, a + c, c) = a(1, 1, 0) + c(0, 1, 1)$

Basis: $\{(1, 1, 0),\ (0, 1, 1)\}$, Dimension: 2

Exercise 8: Subspaces of $\mathbb{R}^4$

(a) **All vectors of form** $(a, b, c, 0)$

Solution: $(a, b, c, 0) = a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0)$

Basis: $\{(1, 0, 0, 0),\ (0, 1, 0, 0),\ (0, 0, 1, 0)\}$, Dimension: 3

(b) **All vectors where** $d = a + b$ **and** $c = a - b$

Solution: $(a, b, c, d) = (a, b, a - b, a + b) = a(1, 0, 1, 1) + b(0, 1, -1, 1)$

Basis: $\{(1, 0, 1, 1),\ (0, 1, -1, 1)\}$, Dimension: 2

(c) **All vectors where** $a = b = c = d$

Solution: $(a, b, c, d) = (a, a, a, a) = a(1, 1, 1, 1)$

Basis: $\{(1, 1, 1, 1)\}$, Dimension: 1

Exercise 10: Subspace of P_3

Find dimension of subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ with $a_0 = 0$.

Solution: These polynomials have form $a_1x + a_2x^2 + a_3x^3 = a_1(x) + a_2(x^2) + a_3(x^3)$
Basis: $\{x, x^2, x^3\}$, Dimension: 3

Exercise 12: Extending to Basis for $\mathbb{R}^3$

Find standard basis vector that can be added to $\{v_1, v_2\}$ to produce basis for $\mathbb{R}^3$:

(a) $v_1 = (-1, 2, 3)$, $v_2 = (1, -2, -2)$

Solution: Check if v_1, v_2 are linearly independent:

$$\det \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} = 2 - 2 = 0 \quad \text{(dependent in x-y plane)}$$

But in 3D, they might be independent. Check if one is scalar multiple of the other: No.

So they span a 2D plane. We need a vector not in this plane. Try $e_1 = (1, 0, 0)$.

Check: $\det \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix} = -1(0 - 0) - 1(0 - 0) + 1(-4 + 6) = 2 \neq 0$

So $\{v_1, v_2, e_1\}$ is a basis.

(b) $v_1 = (1, -1, 0)$, $v_2 = (3, 1, -2)$

Solution: Check linear independence: Not scalar multiples, so independent.

They span a plane. Try $e_3 = (0, 0, 1)$.

Check: $\det \begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = 1(1-0) - 3(-1-0) + 0 = 1 + 3 = 4 \neq 0$

So $\{v_1, v_2, e_3\}$ is a basis.

Exercise 13: Extending to Basis for $\mathbb{R}^4$

Find standard basis vectors for $\mathbb{R}^4$ that can be added to $\{v_1, v_2\}$:

$$v_1 = (1, -4, 2, -3), \quad v_2 = (-3, 8, -4, 6)$$

Solution: Check if v_1, v_2 are linearly independent: Not scalar multiples, so independent.
We need 2 more vectors. Try $e_1 = (1, 0, 0, 0)$ and $e_3 = (0, 0, 1, 0)$.
Check the 4×4 determinant or row reduce:

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ -4 & 8 & 0 & 0 \\ 2 & -4 & 0 & 1 \\ -3 & 6 & 0 & 0 \end{bmatrix} \to \text{Row reduction shows rank 4}$$

So $\{v_1, v_2, e_1, e_3\}$ is a basis.

Exercise 15: Extending to Basis for $\mathbb{R}^3$

The vectors $v_1 = (1, -2, 3)$ and $v_2 = (0, 5, -3)$ are linearly independent. Enlarge to basis for $\mathbb{R}^3$.

Solution: We need one more vector not in $\text{span}\{v_1, v_2\}$. Try $e_1 = (1, 0, 0)$.

Check: $\det \begin{bmatrix} 1 & 0 & 1 \\ -2 & 5 & 0 \\ 3 & -3 & 0 \end{bmatrix} = 1(0-0) - 0(0-0) + 1(6-15) = -9 \neq 0$

So $\{v_1, v_2, e_1\}$ is a basis.

Exercise 16: Extending to Basis for $\mathbb{R}^4$

The vectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge to basis for $\mathbb{R}^4$.

Solution: We need 2 more vectors. Try $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$.

The set $\{v_1, v_2, e_3, e_4\}$ is clearly linearly independent and has 4 vectors, so it's a basis for $\mathbb{R}^4$.

Exercise 17: Basis for Subspace of $\mathbb{R}^3$

Find basis for subspace of $\mathbb{R}^3$ spanned by:

$$v_1 = (1, 0, 0), \; v_2 = (1, 0, 1), \; v_3 = (2, 0, 1), \; v_4 = (0, 0, -1)$$

Solution: Notice all vectors have y-coordinate 0. So they all lie in the xz-plane.
We can find a basis by removing dependent vectors:

- $v_3 = v_1 + v_2$ (since $(2,0,1) = (1,0,0) + (1,0,1)$)
- v_4 is independent of v_1, v_2

So basis: $\{v_1, v_2, v_4\} = \{(1,0,0),\ (1,0,1),\ (0,0,-1)\}$
Dimension: 3 (they span the entire xz-plane? Wait, xz-plane is 2D! Let's check:)
Actually, all vectors have form $(a, 0, b)$, which is 2D. So maximum dimension is 2.
Let's find the actual basis:

- $v_1 = (1,0,0)$
- $v_2 = (1,0,1)$
- $v_2 - v_1 = (0,0,1)$

So the span is all vectors of form $(a, 0, b)$, which has basis $\{(1,0,0),\ (0,0,1)\}$.
Dimension: 2

Exercise 18: Basis for Subspace of $\mathbb{R}^4$

Find basis for subspace of $\mathbb{R}^4$ spanned by:

$$v_1 = (1,1,1,1),\ v_2 = (2,2,2,0),\ v_3 = (0,0,0,3),\ v_4 = (3,3,3,4)$$

Solution: Check dependencies:

- $v_4 = v_1 + v_3$ (since $(3,3,3,4) = (1,1,1,1) + (0,0,0,3)$)
- v_2 is not a multiple of v_1
- v_3 is independent

So basis: $\{v_1, v_2, v_3\} = \{(1,1,1,1),\ (2,2,2,0),\ (0,0,0,3)\}$
Check linear independence: The matrix has rank 3, so dimension is 3.

Exercise 19: Kernel Dimension

Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A. Find dimension of subspace where $T_A(x) = 0$.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Solution: Solve $Ax = 0$. Row reduce:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2, so nullity = 3 - 2 = 1. Dimension: 1

(b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

Solution: All rows are multiples of $(1, 2, 0)$. Rank = 1, so nullity = 3 - 1 = 2. Dimension: 2

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Solution: Matrix is invertible (determinant = 1(1-0) - 0 + 0 = 1 0). Rank = 3, nullity = 0. Dimension: 0

Exercise 20: Kernel Dimension in $\mathbb{R}^4$

Let T_A be multiplication by A. Find dimension of subspace where $T_A(x) = 0$.

(a) $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

Solution: A is 2×4, rank 2. By row reduction, rank = 2, so nullity = 4 - 2 = 2. Dimension: 2

(b) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Solution: A is 3×4, rank 3. By row reduction, rank = 3, so nullity = 4 - 3 = 1. Dimension: 1