Section 1.5: Elementary Matrices and a Method for Finding A^{-1}

Definitions

1. **Definition 1 (Row Equivalence)**
   Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

2. **Definition 2 (Elementary Matrix)**
   A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Examples of Elementary Matrices

1. $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$
   Obtained by multiplying the second row of I_2 by -3.

2. $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
   Obtained by interchanging the second and fourth rows of I_4.

3. $E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   Obtained by adding 3 times the third row of I_3 to the first row.

4. $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   The identity matrix (obtained by multiplying the first row of I_3 by 1).

Detailed Solutions to Problems 1 and 2

Exercise 1: Determine whether the given matrix is elementary.

(a) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$
Answer: Yes.
Reason: This matrix is obtained from $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by adding -5 times the first row to the second row.

(b) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$
Answer: No.
Reason: This matrix cannot be obtained from I_2 by a single elementary row operation. For example, interchanging rows of I_2 gives $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is not equal to the given matrix.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
Answer: No.
Reason: This matrix has a row of zeros, which cannot result from a single elementary row operation on I_3. Additionally, elementary matrices are always invertible, but this matrix is not invertible.

(d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
Answer: No.
Reason: This matrix is not square (4×3), and elementary matrices must be square. It also contains a row of zeros.

Exercise 2: Determine whether the given matrix is elementary.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$
Answer: Yes.
Reason: This matrix is obtained from I_2 by multiplying the second row by $\sqrt{3}$.

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
Answer: Yes.
Reason: This matrix is obtained from I_3 by interchanging the first and third rows.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$
Answer: Yes.
Reason: This matrix is obtained from I_3 by adding 9 times the third row to the second row.

(d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Answer: No.
Reason: This matrix requires two elementary row operations:

- Multiply the first row by -1 to get $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- Interchange the second and third rows.

Since it requires two operations, it is not an elementary matrix.

Exercise 3:

Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $E = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by adding -3 times row 2 to row 1 of I_2
- **Inverse operation:** Add 3 times row 2 to row 1

(b) $E = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by multiplying row 1 by -7
- **Inverse operation:** Multiply row 1 by $-\frac{1}{7}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by adding -5 times row 1 to row 3
- **Inverse operation:** Add 5 times row 1 to row 3

(d) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

- This matrix is not square (4×3), so it cannot be an elementary matrix
- Elementary matrices must be square since they are obtained from identity matrices
- Therefore, no inverse operation exists to restore it to an identity matrix

Exercise 4:

Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by adding -3 times row 1 to row 2
- **Inverse operation:** Add 3 times row 1 to row 2

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by multiplying row 3 by 3
- **Inverse operation:** Multiply row 3 by $\frac{1}{3}$

(c) $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:

- This matrix is not square (4×3), so it cannot be an elementary matrix
- Elementary matrices must be square since they are obtained from identity matrices
- Therefore, no inverse operation exists to restore it to an identity matrix

(d) $E = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

- This elementary matrix was obtained by adding $-\frac{1}{2}$ times row 3 to row 1
- **Inverse operation:** Add $\frac{1}{2}$ times row 3 to row 1

Theorem 1.5.1 and Examples

Theorem 1.5.1 (Row Operations by Matrix Multiplication)

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Example 2: Using Elementary Matrices

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

The elementary matrix E is obtained by adding 3 times the first row of I_3 to the third row. Then:

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

This is exactly the result of adding 3 times the first row of A to the third row.

Example 3: Row Operations and Inverse Row Operations

1. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   Multiply the second row by 7, then multiply the second row by $\frac{1}{7}$.

2. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   Interchange the first and second rows, then interchange them again.

3. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   Add 5 times the second row to the first, then add -5 times the second row to the first.

Exercise Solutions

Exercise 5:

An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 \\ 3 & -6 & -6 \end{bmatrix}$

Solution:

- **Row operation:** Interchange rows 1 and 2.
- **Verification:**
$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 \\ 3 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -6 \\ -1 & -2 & 5 \end{bmatrix}$$
This is indeed the result of interchanging rows 1 and 2 of A.

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

Solution:

- **Row operation:** Add -3 times row 2 to row 3.
- **Verification:**
$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ -1 & 9 & 4 & -12 \end{bmatrix}$$
This matches the result of adding -3 times row 2 to row 3 of A.

(c) $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Solution:

- **Row operation:** Add 4 times row 3 to row 1.
- **Verification:**
$$EA = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$
This matches the result of adding 4 times row 3 to row 1 of A.

Exercise 6:

An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 \\ 3 & -6 & -6 \end{bmatrix}$

Solution:

- **Row operation:** Multiply row 1 by -6.
- **Verification:**
$$EA = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 \\ 3 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 12 & -30 \\ 3 & -6 & -6 \end{bmatrix}$$
This matches the result of multiplying row 1 of A by -6.

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

Solution:

- **Row operation:** Add -4 times row 1 to row 2.
- **Verification:**
$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 \\ -7 & 1 & -1 & 21 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$
This matches the result of adding -4 times row 1 to row 2 of A.

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

Solution:

- **Row operation:** Identity operation (no change).
- **Verification:**

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 \\ 1 & -3 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

The product is identical to A, confirming that no row operation was applied.

Inversion Algorithm and Examples

Inversion Algorithm

To find the inverse of an invertible matrix A:

1. Form the augmented matrix $[A \mid I]$
2. Perform elementary row operations to reduce A to the identity matrix I
3. The same operations will transform I to A^{-1}, yielding $[I \mid A^{-1}]$
4. If a row of zeros appears on the left side during the process, A is not invertible

Example 4: Using Row Operations to Find A^{-1}

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example 5: Showing That a Matrix Is Not Invertible

Consider $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Since we obtained a row of zeros on the left side, A is not invertible.

Example 6: Analyzing Homogeneous Systems

Use Theorem 1.5.3 to determine whether the given homogeneous system has nontrivial solutions.

(a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned}$$

Coefficient matrix is invertible (from Example 4), so only trivial solution.

(b)

$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Coefficient matrix is not invertible (from Example 5), so nontrivial solutions exist.

Exercise Solutions 11-18

Exercise 11:

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution: (Same as Example 4)

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

Solution:

$$\begin{bmatrix} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix}$$

Row of zeros on left A is not invertible.

Exercise 12:

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

(a) $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

Solution: Multiply all rows by 30 to eliminate fractions:

$$\begin{bmatrix} 6 & 6 & -20 & 30 & 0 & 0 \\ 6 & 6 & 3 & 0 & 30 & 0 \\ 6 & -24 & 3 & 0 & 0 & 30 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -\frac{10}{3} & 5 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 5 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{bmatrix}$$

Continue with row operations... (solution would show full reduction)

(b) $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{3} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

Solution: Similar approach as (a), multiply to eliminate fractions and perform row operations.

Exercise 13:

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Exercise 15:

$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

Solution: Use inversion algorithm:

$$\begin{bmatrix} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{bmatrix} \to \cdots \to A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Exercise 16:

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

Solution: Lower triangular matrix. Inverse is:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$$