Lecture Notes: Section 4.2 - Subspaces

Linear Algebra

1 Subspaces

1.1 Definition and the Subspace Test

Definition 1.1. A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 1.2 (Subspace Test)**.** If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied:

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W. (Closure under addition)

(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W. (Closure under scalar multiplication)

1.2 Examples and Non-Examples

Example 1.3 (The Zero Subspace)**.** If V is any vector space, and $W = \{\mathbf{0}\}$, then W is closed under addition and scalar multiplication since:

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for any scalar k. Thus, W is a subspace of V, called the **zero subspace**.

Example 1.4 (Lines Through the Origin in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** If W is a line through the origin of $\mathbb{R}^2$ or $\mathbb{R}^3$, then adding two vectors on the line or multiplying by a scalar produces another vector on the line. Thus, W is closed under addition and scalar multiplication, making it a subspace.

Example 1.5 (Planes Through the Origin in $\mathbb{R}^3$)**.** If W is a plane through the origin in $\mathbb{R}^3$, then the sum of any two vectors in the plane, or any scalar multiple of a vector in the plane, remains in the plane. Hence, W is a subspace.

Example 1.6 (A Subset of $\mathbb{R}^2$ That Is Not a Subspace)**.** Let $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$. This set is **not** a subspace because it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1) \in W$, but $(-1)\mathbf{v} = (-1, -1) \notin W$.

Example 1.7 (Subspaces of M_{nn})**.** The following subsets of M_{nn} are subspaces:

- The set of symmetric $n \times n$ matrices.
- The set of upper triangular $n \times n$ matrices.
- The set of diagonal $n \times n$ matrices.

Each is closed under addition and scalar multiplication.

1.3 Intersections and Solution Spaces

Theorem 1.8. If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Theorem 1.9. The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

Remark 1.10. Because of this theorem, the solution set of a homogeneous system is often called the **solution space** of the system.

Example 1.11 (Solution Spaces of Homogeneous Systems)**.** Describe the solution space geometrically for each system:

(a)
$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: $x - 2y + 3z = 0$. This is a **plane through the origin**.

(b)
$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: Parametric equations $x = -5t, y = -t, z = t$. This is a **line through the origin**.

(c) The only solution is $\mathbf{x} = \mathbf{0}$. The solution space is $\{\mathbf{0}\}$.

(d) All $(x, y, z) \in \mathbb{R}^3$ are solutions. The solution space is $\mathbb{R}^3$.

Theorem 1.12. If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

2 Exercise Solutions

Exercise 1

Use the Subspace Test to determine which sets are subspaces of $\mathbb{R}^3$.

(a) All vectors of the form $(a, 0, 0)$.
Solution: Yes. Closed under addition and scalar multiplication.

(b) All vectors of the form $(a, 1, 1)$.
Solution: No. Does not contain the zero vector $(0, 0, 0)$.

(c) All vectors of the form (a, b, c), where $b = a + c$.
Solution: Yes. Let $\mathbf{u} = (u_1, u_1 + u_3, u_3)$, $\mathbf{v} = (v_1, v_1 + v_3, v_3)$. Then:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, (u_1 + v_1) + (u_3 + v_3), u_3 + v_3)$$

$$k\mathbf{u} = (ku_1, k(u_1 + u_3), ku_3) = (ku_1, ku_1 + ku_3, ku_3)$$

Both satisfy the condition $b = a + c$.

Exercise 2

(a) All vectors of the form (a, b, c), where $b = a + c + 1$.
Solution: No. Does not contain the zero vector.

(b) All vectors of the form $(a, b, 0)$.
Solution: Yes. This is the xy-plane in $\mathbb{R}^3$, a subspace.

(c) All vectors of the form (a, b, c) for which $a + b = 7$.
Solution: No. Not closed under addition. If $\mathbf{u} = (7, 0, 0)$, $\mathbf{v} = (0, 7, 0)$, then $\mathbf{u} + \mathbf{v} = (7, 7, 0)$, but $7 + 7 = 14 \neq 7$.

Exercise 3

Determine which sets are subspaces of M_{nn}.

(a) The set of all diagonal $n \times n$ matrices.
Solution: Yes. Closed under addition and scalar multiplication.

(b) The set of all $n \times n$ matrices A such that $\det(A) = 0$.
Solution: No. Not closed under addition. Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both have determinant 0, but $A + B = I$ has determinant 1.

(c) The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$.
Solution: Yes. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B) = 0$, and $\operatorname{tr}(kA) = k \operatorname{tr}(A) = 0$.

(d) The set of all symmetric $n \times n$ matrices.
Solution: Yes. $(A + B)^T = A^T + B^T = A + B$, and $(kA)^T = kA^T = kA$.

Exercise 4

(a) The set of all $n \times n$ matrices A such that $A^T = -A$ (skew-symmetric).
Solution: Yes. Closed under addition and scalar multiplication.

(b) The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
Solution: No. Does not contain the zero matrix.

(c) The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed B.
Solution: Yes. If $AB = BA$ and $CB = BC$, then $(A+C)B = AB+CB = BA+BC = B(A+C)$, and $(kA)B = k(AB) = k(BA) = B(kA)$.

(d) The set of all invertible $n \times n$ matrices.
Solution: No. Not closed under addition; does not contain the zero matrix.

Exercise 5

Determine which sets are subspaces of P_3.

(a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
Solution: Yes. The set of polynomials with zero constant term. Closed under addition and scalar multiplication.

(b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
Solution: Yes. If $p(1) = 0$ and $q(1) = 0$, then $(p+q)(1) = 0$ and $(kp)(1) = k \cdot 0 = 0$.

Exercise 19

Determine whether the solution space of $A\mathbf{x} = \mathbf{0}$ is a line, plane, or the origin.

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$
Solution: Row reduction shows 3 pivots. Only solution is $\mathbf{x} = \mathbf{0}$. **The origin only**.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$
Solution: Row reduction shows 3 pivots. Only solution is $\mathbf{x} = \mathbf{0}$. **The origin only**.

(c) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$
Solution: All rows are multiples of the first. One pivot, two free variables. Solution set: $x = 3s - t, y = s, z = t$, or $\mathbf{x} = s(3,1,0) + t(-1,0,1)$. This is a **plane through the origin**. Equation: $x - 3y + z = 0$.

(d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

Solution: Row reduction yields 2 pivots, one free variable. Solution set: $x = -3t, y = -2t, z = t$, or $\mathbf{x} = t(-3, -2, 1)$. This is a **line through the origin**. Parametric equations: $x = -3t, y = -2t, z = t$.