Class Lecture: Section 4.9
Rank, Nullity, and the Fundamental Matrix Spaces

Linear Algebra

1 Rank and Nullity

Theorem 1 (Theorem 4.9.1). The row space and the column space of a matrix A have the same dimension.

Definition 1 (Rank and Nullity). For any matrix A:

- The **rank** of A, denoted $\text{rank}(A)$, is the common dimension of the row space and column space of A
- The **nullity** of A, denoted $\text{nullity}(A)$, is the dimension of the null space of A

Example 1 (Example 1: Rank and Nullity of a 4×6 Matrix). Find the rank and nullity of:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution: The reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 2 leading 1's, $\text{rank}(A) = 2$. By the Dimension Theorem, $\text{nullity}(A) = 6 - 2 = 4$.

2 The Dimension Theorem

Theorem 2 (Theorem 4.9.2: Dimension Theorem for Matrices). If A is a matrix with n columns, then:

$$\text{rank}(A) + \text{nullity}(A) = n$$

Example 2 (Example 2: Maximum Value for Rank)**.** What is the maximum possible rank of an $m \times n$ matrix that is not square?

Solution: $\text{rank}(A) \leq \min(m, n)$

Theorem 3 (Theorem 4.9.3)**.** If A is an $m \times n$ matrix, then:

(a) $\text{rank}(A) =$ number of leading variables in $A\mathbf{x} = \mathbf{0}$

(b) $\text{nullity}(A) =$ number of parameters in general solution of $A\mathbf{x} = \mathbf{0}$

3 Fundamental Spaces

The four **fundamental spaces** of an $m \times n$ matrix A are:

- Row space of A (subspace of $\mathbb{R}^n$)
- Column space of A (subspace of $\mathbb{R}^m$)
- Null space of A (subspace of $\mathbb{R}^n$)
- Null space of A^T (left null space, subspace of $\mathbb{R}^m$)

Theorem 4 (Theorem 4.9.5)**.** For any matrix A, $\text{rank}(A) = \text{rank}(A^T)$

4 Dimensions of Fundamental Spaces

If A is $m \times n$ with $\text{rank}(A) = r$, then:

$$\begin{aligned} \dim(\text{row}(A)) &= r \\ \dim(\text{col}(A)) &= r \\ \dim(\text{null}(A)) &= n - r \\ \dim(\text{null}(A^T)) &= m - r \end{aligned}$$

5 Orthogonal Complements

Definition 2. If W is a subspace of $\mathbb{R}^n$, then the **orthogonal complement** of W, denoted $W^\perp$, is the set of all vectors in $\mathbb{R}^n$ orthogonal to every vector in W.

Theorem 5 (Theorem 4.9.7)**.** If A is an $m \times n$ matrix, then:

(a) The null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$

(b) The null space of A^T and the column space of A are orthogonal complements in $\mathbb{R}^m$

6 Exercise Solutions

Exercise 1

Find rank and nullity by reducing to row echelon form.

Solution 1. (a) For $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$:

Row echelon form: $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A) = 1$, $\text{nullity}(A) = 4 - 1 = 3$

(b) For $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$:

Row echelon form: $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A) = 2$, $\text{nullity}(A) = 5 - 2 = 3$

Exercise 2

Solution 2. (a) For $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$:

After reduction: $\text{rank}(A) = 3$, $\text{nullity}(A) = 5 - 3 = 2$

(b) For $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$:

After reduction: $\text{rank}(A) = 3$, $\text{nullity}(A) = 4 - 3 = 1$

Exercise 3

Given A and its RREF $R = I_3$:

Solution 3. (a) $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$

(b) $3 + 0 = 3$

(c) Leading variables: 3, Parameters: 0

Exercise 4

Given A and $R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$:

Solution 4. (a) $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$
(b) $2 + 1 = 3$
(c) Leading variables: 2, Parameters: 1

Exercise 5

Given A and $R = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$:

Solution 5. (a) $\text{rank}(A) = 1$, $\text{nullity}(A) = 2$
(b) $1 + 2 = 3$
(c) Leading variables: 1, Parameters: 2

Exercise 6

Given A and $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$:

Solution 6. (a) $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$
(b) $3 + 1 = 4$
(c) Leading variables: 3, Parameters: 1

Exercise 7

Find largest rank and smallest nullity:

Solution 7. (a) 4×4: max rank $= 4$, min nullity $= 0$
(b) 3×5: max rank $= 3$, min nullity $= 2$
(c) 5×3: max rank $= 3$, min nullity $= 0$

Exercise 8

For $m \times n$ matrix:

Solution 8. Largest possible rank $= \min(m, n)$

$$\text{Smallest possible nullity} = \begin{cases} n - m & \text{if } m < n \\ 0 & \text{if } m \geq n \end{cases}$$

Exercise 9

Complete the table using:

- $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$
- $\dim(\text{null}(A)) = n - \text{rank}(A)$
- $\dim(\text{null}(A^T)) = m - \text{rank}(A)$
- System consistent if $\text{rank}[A|\mathbf{b}] = \text{rank}(A)$
- Parameters $= \dim(\text{null}(A))$

Exercise 10

Verify $\text{rank}(A) = \text{rank}(A^T)$ for:

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

Solution 9. Reduce A: $\text{rank}(A) = 2$
Reduce A^T: $\text{rank}(A^T) = 2$

Exercise 11

Find dimensions and bases for fundamental spaces of:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$$

Solution 10. $\text{rank}(A) = 2$ (full column rank)

- $\dim(\text{row}(A)) = 2$, basis: rows of RREF
- $\dim(\text{col}(A)) = 2$, basis: columns of A
- $\dim(\text{null}(A)) = 0$, basis: $\{\}$
- $\dim(\text{null}(A^T)) = 1$, basis: solve $A^T\mathbf{x} = \mathbf{0}$

Exercise 12

For $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$:

Solution 11. $\text{rank}(A) = 1$

- $\dim(\text{row}(A)) = 1$, basis: $\{[1, 2, 4]\}$
- $\dim(\text{col}(A)) = 1$, basis: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- $\dim(\text{null}(A)) = 2$, basis: solve $A\mathbf{x} = \mathbf{0}$
- $\dim(\text{null}(A^T)) = 1$, basis: solve $A^T\mathbf{x} = \mathbf{0}$

Exercise 13

For $A = \begin{bmatrix} 0 & -1 & -4 \\ -1 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix}$:

Solution 12. Reduce to find $\text{rank}(A) = 3$ (invertible)
All fundamental spaces: row space = col space = $\mathbb{R}^3$, null spaces = $\{\mathbf{0}\}$

Exercise 14

For $A = \begin{bmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & -3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$:

Solution 13. Reduce to find $\text{rank}(A) = 3$

- $\dim(\text{row}(A)) = 3$, basis from RREF rows
- $\dim(\text{col}(A)) = 3$, basis from pivot columns of A
- $\dim(\text{null}(A)) = 1$, basis from solution of $A\mathbf{x} = \mathbf{0}$
- $\dim(\text{null}(A^T)) = 1$, basis from solution of $A^T\mathbf{x} = \mathbf{0}$

Exercise 19

Find bases for four fundamental spaces of:

$$A = \begin{bmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & 5 \end{bmatrix}$$

Solution 14. Use method: form $[A|I]$, reduce to $[R|E]$

After reduction: $\text{rank}(A) = 2$

- Row space basis: first 2 rows of R
- Column space basis: columns 1 and 2 of A
- Null space basis: solve $A\mathbf{x} = \mathbf{0}$
- Left null space basis: last row of E

Exercise 20

For $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 8 & 0 & 1 \\ 0 & 4 & -6 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$:

Solution 15. Use same method as Exercise 19.

Form $[A|I_4]$, reduce to find $\text{rank}(A)$ and bases.

Exercise 21

Solution 16. (a) From Exercise 10: $\text{nullity}(A) = 2$, $\text{nullity}(A^T) = 1$

(b) General: $\text{nullity}(A) - \text{nullity}(A^T) = n - m$

Exercise 22

For $T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$:

Solution 17. Standard matrix: $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

$\text{rank}(A) = 2$, $\text{nullity}(A) = 0$

Exercise 23

For $T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$:

Solution 18. Standard matrix: $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

$\text{rank}(A) = 3$, $\text{nullity}(A) = 2$

Exercise 24

Discuss how rank varies with t:

Solution 19. (a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$
Compute determinant: $\det(A) = -t^3 + 3t - 2$
Rank is 3 when $\det(A) \neq 0$, rank drops when $t = 1$ or $t = -2$
(b) $A = \begin{bmatrix} 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$
Row reduce: rank is 2 for most t, rank drops to 1 for specific t values

Exercise 25

Find values of r and s for which the matrix has rank 1 or 2.

Solution 20. Row reduce and find conditions on r and s for:

- Rank 1: all rows proportional
- Rank 2: exactly 2 independent rows

Solve system of equations from row proportionality conditions.

Exercise 26

Solution 21. (a) Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has column space = xy-plane
(b) Null space = z-axis
(c) Row space = xy-plane

Exercise 27

If A is 3×3 with null space a line:

Solution 22. By Dimension Theorem: $\text{rank}(A) = 2$
Row space and column space are planes (2-dimensional), not lines.
So neither row nor column space can be a line.

Exercise 28

For 3×5 matrix A:

Solution 23. (a) Rank at most 3 (smaller of 3 and 5)
(b) Nullity at most 4 ($5 - 1 = 4$ when rank $= 1$)
(c) $\text{rank}(A^T)$ at most 3 (smaller of 5 and 3)
(d) $\text{nullity}(A^T)$ at most 2 ($3 - 1 = 2$ when rank $= 1$)

Exercise 29

Solution 24. (a) 3×5: leading 1's at most 3
(b) 3×5: parameters at most 4
(c) 5×3: leading 1's at most 3
(d) 5×3: parameters at most 2

Exercise 30

7×6 matrix with only trivial solution to $A\mathbf{x} = \mathbf{0}$:

Solution 25. Only trivial solution null space = $\{\mathbf{0}\}$ nullity$(A) = 0$
By Dimension Theorem: rank$(A) = 6$

Exercise 31

5×7 matrix with rank 4:

Solution 26. (a) $\dim(\text{null}(A)) = 7 - 4 = 3$
(b) System consistent if $\mathbf{b} \in \text{col}(A)$
Since $\dim(\text{col}(A)) = 4 < 5$, not all $\mathbf{b} \in \mathbb{R}^5$ are in col(A)
So $A\mathbf{x} = \mathbf{b}$ is not consistent for all $\mathbf{b}$

Exercise 32

Show 2×3 matrix has rank 2 if and only if one of the 2×2 determinants is nonzero.

Solution 27. Rank 2 means columns span $\mathbb{R}^2$, so some pair of columns is linearly independent.
This occurs exactly when some 2×2 submatrix has nonzero determinant.

Exercise 33

Find points where matrix $\begin{bmatrix} x & y & z \\ 1 & x & y \end{bmatrix}$ has rank 1.

Solution 28. Rank 1 means rows are proportional, so:

$$(x, y, z) = k(1, x, y) \text{ for some } k$$

This gives $x = k$, $y = kx = k^2$, $z = ky = k^3$
So parametric equations: $x = t$, $y = t^2$, $z = t^3$

Exercise 34

Find matrices with same rank but different ranks after squaring.

Solution 29. Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, rank$(A) = 1$, rank$(A^2) = 0$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, rank$(B) = 1$, rank$(B^2) = 1$

Exercise 35

Show null space of A^T and column space of A are orthogonal complements.

Solution 30. For matrix from Example 6, Section 4.7:
Find basis for null(A^T) by solving $A^T\mathbf{x} = \mathbf{0}$
Show each basis vector is orthogonal to each column of A
Use dot product to verify orthogonality.

Exercise 36

Confirm Theorem 4.9.7 for given matrix.

Solution 31. For $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$:

- Find bases for row(A) and null(A), verify orthogonality
- Find bases for col(A) and null(A^T), verify orthogonality
- Use dot products to confirm orthogonality