Eigen Value and Eigen Vector

Eigenvalues and Eigenvectors

- Linear equations Ax = b come from steady state problems. Eigen values have their greatest importance in *dynamic problems. The solution of du/dt= Au is changing with time*—growing or decaying or oscillating.

- Almost all vectors change direction, when they are multiplied by square matrix A.

- *Certain exceptional vectors "x" are in the same direction as Ax. Those are the "Eigen vectors".*

- The basic equation is Ax = λx. The number λ is an "Eigen value" of A.

- The eigen value tells whether the special vector "x" is stretched or shrunk or reversed or left unchanged—when it is multiplied by A.

Note:

The prefix *eigen-* is adopted from the German word "eigen" for "own" in the sense of a characteristic description (that is why the eigenvectors are sometimes also called characteristic vectors, and, similarly, the eigenvalues are also known as characteristic values).

Definition 1

If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** λ.

The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case $A\mathbf{0} = \lambda\mathbf{0}$, which holds for every A and λ.

Eigenvalues and Eigenvectors

- Eigenvalue problem:

  If A is an $n\times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$？

- Eigenvalue and eigenvector:

  A ： an $n\times n$ matrix

  λ ： a scalar

  $\boldsymbol{x}$ ： a nonzero vector in R^n

Eigenvalue

$$A\underset{\uparrow}{x} = \overset{\downarrow}{\lambda}\underset{\uparrow}{x}$$

Eigenvector

- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad x_1=\begin{bmatrix}1\\ 0\end{bmatrix} \quad x_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Eigenvalue

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=2\begin{bmatrix}1\\ 0\end{bmatrix}=2x_1$$

Eigenvector

Eigenvalue

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=-1\begin{bmatrix}0\\ 1\end{bmatrix}=(-1)x_2$$

Eigenvector

- Theorem 2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

Let A is an $n\times n$ matrix.

(1) An eigenvalue of A is a scalar λ such that $\det(\lambda \mathrm{I} - A) = 0$

(2) The eigenvectors of A corresponding to λ are the nonzero

solutions of $(\lambda \mathrm{I} - A)x = 0$

- Note:

$$Ax = \lambda x \quad \Rightarrow \quad (\lambda \mathrm{I} - A)x = 0$$ (homogeneous system)

$(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$

- Characteristic equation of A:

$$\det(\lambda \mathrm{I} - A) = 0$$

- Ex 3: (Finding eigenvalues and eigenvectors)

Find the eigenvalues and eigenvectors of matrix A.

$$A=\begin{bmatrix}2 & -12\\ 1 & -5\end{bmatrix}$$

Sol: Characteristic equation:

$$(\lambda \mathrm{I}-A)=\begin{vmatrix}\lambda-2 & 12\\ -1 & \lambda+5\end{vmatrix}$$

$$=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

$$\Rightarrow \lambda=-1,-2$$

Algebraic multiplicity :
-1 is 1 and -2 is 1

Eigenvalue: $\lambda_1=-1, \lambda_2=-2$

$$(1)\lambda_1 = -1 \qquad \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$(2)\lambda_2 = -2 \qquad \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ t \neq 0$$

Geometric multiplicity :
-(4,1) is 1 and (3,1) is 1

- Ex 4: (Finding eigenvalues and eigenvectors)

Find the eigenvalues and corresponding eigenvectors for the matrix A.

What is the dimension of the eigenspace of each eigenvalue?

$$A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Sol: Characteristic equation:

$$\left|\lambda \mathrm{I}-A\right|=\begin{vmatrix}\lambda-2 & -1 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda-2\end{vmatrix}=(\lambda-2)^3=0$$

Eigenvalue: $\lambda=2$

Algebraic multiplicity : 2 is 3:

The eigenspace of A corresponding to : $\lambda = 2$

$$(\lambda \mathrm{I} - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

Geometric multiplicity : 2

- Ex 5：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

Eigenvalue: $\lambda_1=1, \lambda_2=2, \lambda_3=3$

Algebraic multiplicity : 1 is 2, 2 is 1 and 3 is 1

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace of A corresponding to $\lambda = 1$

Geometric multiplicity : 2

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

Geometric multiplicity : 1

$$(3)\lambda_3 = 3 \qquad \Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

Geometric multiplicity : 1

- Theorem 3: (Eigenvalues for triangular matrices)

  If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 6: (Finding eigenvalues for diagonal and triangular matrices)

$$(a) A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b) A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b)\ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Ex 7: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

Sol:

$$|\lambda I-A|=\begin{bmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{bmatrix}=(\lambda+2)^2(\lambda-4)$$

eigenvalues $\lambda_1=4,\ \lambda_2=-2$

The eigenspaces for these two eigenvalues are as follows.

$B_1=\{(1,1,0)\}$ Basis for $\lambda_1=4$

$B_2=\{(1,-1,0),(0,0,1)\}$ Basis for $\lambda_2=-2$

More on the Equivalence Theorem

As our final result in this section, we will use Theorem 5.1.4 to add one additional part to Theorem 4.9.8.

Theorem 5.1.5

Equivalent Statements

If A is an $n \times n$ matrix in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(*a*) A is invertible.
(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(*c*) The reduced row echelon form of A is I_n.
(*d*) A is expressible as a product of elementary matrices.
(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(*g*) $\det(A) \neq 0$.
(*h*) The column vectors of A are linearly independent.
(*i*) The row vectors of A are linearly independent.
(*j*) The column vectors of A span R^n.
(*k*) The row vectors of A span R^n.
(*l*) The column vectors of A form a basis for R^n.
(*m*) The row vectors of A form a basis for R^n.
(*n*) A has rank n.
(*o*) A has nullity 0.
(*p*) The orthogonal complement of the null space of A is R^n.
(*q*) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
(*r*) $\lambda = 0$ is not an eigenvalue of A.

In Exercises 1–4, confirm by multiplication that $\mathbf{x}$ *is an eigenvector of* A*, and find the corresponding eigenvalue.*

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In each part of Exercises 5–6, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

5. **a.** $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

6. **a.** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

In Exercises 7–12, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

7. $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

In Exercises **13–14**, *find the characteristic equation of the matrix by inspection.*

13. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

In Exercises 15–16, find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula. [Suggestion: *Work with the standard matrix for the operator.*]

15. $T(x, y) = (x + 4y, 2x + 3y)$

16. $T(x, y, z) = (2x - y - z,\ x - z,\ -x + y + 2z)$