Inner Product Spaces
Ex# 6.1 and 6.2

Inner Product Spaces

Definition 1

An ***inner product*** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

Definition 2

If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

Theorem 6.1.1

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then:

(*a*) $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$.

(*b*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

(*c*) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(*d*) $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{v}$.

- Ex 1: (The Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u} = (u_1, u_2, \cdots, u_n) \quad , \quad \mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Def., this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

Weighted Euclidean Inner Product

The Euclidean inner product is the most important inner product on R^n. However, there are various applications in which it is desirable to modify the Euclidean inner product by *weighting its terms differently. More precisely, if*

$$w_1, w_2, \ldots, w_n$$

are *positive real numbers, which we shall call* ***weights, and if*** $\mathbf{u} = (u_1, u_2, \ldots, u_n)$

and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots w_n u_n v_n$$

defines an inner product on R^n; it is called the ***weighted Euclidean inner product with weights*** $w_1, w_2, \ldots, w_n$

Ex 6.2: (A different inner product for R^n)

Show that the function defines an inner product on R^2, where

$$\mathbf{u} = (u_1, u_2) \qquad \mathbf{v} = (v_1, v_2)$$

and $$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Sol:

$$(a)\ \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$(b)\ \mathbf{w} = (w_1, w_2)$$

$$\begin{aligned}
\Rightarrow \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) \\
&= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2 \\
&= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2) \\
&= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle
\end{aligned}$$

$(c)\quad c\langle \mathbf{u}, \mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v}\rangle$

$(d)\quad \langle \mathbf{v}, \mathbf{v}\rangle = v_1^{\,2} + 2v_2^{\,2} \geq 0$

$\langle \mathbf{v}, \mathbf{v}\rangle = 0 \Rightarrow v_1^{\,2} + 2v_2^{\,2} = 0 \qquad \Rightarrow \; v_1 = v_2 = 0 \;\; (\mathbf{v} = 0)$

Since all four axioms of inner product are satisfied, therefore, the given function is an inner product space for R^2.

Ex 6.3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let

$$\mathbf{v} = (1, 2, 1)$$

Then $< \mathbf{v}, \mathbf{v} >= (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Theorem : (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space V, and let c be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

  $$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:

  $$||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

- Distance between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \; 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

(2) $\|\mathbf{v}\| \neq 1 \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of $\mathbf{v}$)

$\mathbf{v} \neq 0$

not a unit vector

- Properties of norm:

  (1) $\|\mathbf{u}\| \geq 0$

  (2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

  (3) $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$

- Properties of distance:

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Theorem 5.2：

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

(2) Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

(3) Pythagorean theorem :

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

1. Let R^2 have the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

and let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$. Compute the stated quantities.

a. $\langle \mathbf{u}, \mathbf{v} \rangle$

b. $\langle k\mathbf{v}, \mathbf{w} \rangle$

c. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$

d. $\|\mathbf{v}\|$

e. $d(\mathbf{u}, \mathbf{v})$

f. $\|\mathbf{u} - k\mathbf{v}\|$

1. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(3) + 3(1)(2) = 12$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = 2((3)(3))(0) + 3((3)(2))(-1) = -18$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 2(1+3)(0) + 3(1+2)(-1) = -9$

(d) $\| \mathbf{v} \| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = [2(3)(3) + 3(2)(2)]^{1/2} = \sqrt{30}$

(e) $d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \langle (-2,-1), (-2,-1) \rangle^{1/2} = [2(-2)(-2) + 3(-1)(-1)]^{1/2} = \sqrt{11}$

(f) $\| \mathbf{u} - k\mathbf{v} \| = \langle (-8,-5), (-8,-5) \rangle^{1/2} = [2(-8)(-8) + 3(-5)(-5)]^{1/2} = \sqrt{203}$

2. Follow the directions of Exercise 1 using the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \tfrac{1}{2}u_1v_1 + 5u_2v_2$$

2. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}(1)(3) + 5(1)(2) = \frac{23}{2}$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}((3)(3))(0) + 5((3)(2))(-1) = -30$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}(1+3)(0) + 5(1+2)(-1) = -15$

(d) $\| \mathbf{v} \| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\frac{1}{2}(3)(3) + 5(2)(2)\right]^{1/2} = \sqrt{\frac{49}{2}} = \frac{7}{\sqrt{2}}$

(e) $d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \langle (-2,-1), (-2,-1) \rangle^{1/2} = \left[\frac{1}{2}(-2)(-2) + 5(-1)(-1)\right]^{1/2} = \sqrt{7}$

(f) $\| \mathbf{u} - k\mathbf{v} \| = \langle (-8,-5), (-8,-5)^{1/2} = \left[\frac{1}{2}(-8)(-8) + 5(-5)(-5)\right]^{1/2} = \sqrt{157}$

Inner Products Generated by Matrices

The Euclidean inner product and the weighted Euclidean inner products are special cases of a general class of inner products on R^n called ***matrix inner products***. To define this class of inner products, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n that are expressed in *column form*, and let A be an *invertible* $n \times n$ matrix. It can be shown (Exercise 47) that if $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} \tag{5}$$

also defines an inner product; it is called the ***inner product on R^n generated by A***.

Recall from Table 1 of Section 3.2 that if $\mathbf{u}$ and $\mathbf{v}$ are in column form, then $\mathbf{u} \cdot \mathbf{v}$ can be written as $\mathbf{v}^T\mathbf{u}$ from which it follows that (5) can be expressed as

$$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u}$$

or equivalently as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A\mathbf{u} \tag{6}$$

In Exercises 3–4, compute the quantities in parts (a)–(f) of Exercise 1 using the inner product on R^2 generated by A.

3. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Exercise 1

1. Let R^2 have the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

and let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$. Compute the stated quantities.

a. $\langle \mathbf{u}, \mathbf{v} \rangle$

b. $\langle k\mathbf{v}, \mathbf{w} \rangle$

c. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$

d. $\|\mathbf{v}\|$

e. $d(\mathbf{u}, \mathbf{v})$

f. $\|\mathbf{u} - k\mathbf{v}\|$

3. (a) $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}\right) = \begin{bmatrix}3\\ 2\end{bmatrix}\cdot\begin{bmatrix}8\\ 5\end{bmatrix} = 34$

(b) $\langle k\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}9\\ 6\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ -1\end{bmatrix}\right) = \begin{bmatrix}24\\ 15\end{bmatrix}\cdot\begin{bmatrix}-1\\ -1\end{bmatrix} = -39$

(c) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}4\\ 3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ -1\end{bmatrix}\right) = \begin{bmatrix}11\\ 7\end{bmatrix}\cdot\begin{bmatrix}-1\\ -1\end{bmatrix} = -18$

(d) $\|\mathbf{v}\| = \langle \mathbf{v},\mathbf{v}\rangle^{1/2} = \left[\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}8\\ 5\end{bmatrix}\cdot\begin{bmatrix}8\\ 5\end{bmatrix}\right)^{1/2} = \sqrt{89}$

(e) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left[\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2\\ -1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2\\ -1\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-5\\ -3\end{bmatrix}\cdot\begin{bmatrix}-5\\ -3\end{bmatrix}\right)^{1/2} = \sqrt{34}$

(f) $\|\mathbf{u}-k\mathbf{v}\| = \left[\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-8\\ -5\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-8\\ -5\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-21\\ -13\end{bmatrix}\cdot\begin{bmatrix}-21\\ -13\end{bmatrix}\right)^{1/2} = \sqrt{610}$

4. **(a)** $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} = 7$

(b) $\langle k\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}9\\6\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}9\\12\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = 12$

(c) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = 5$

(d) $\|\mathbf{v}\| = \langle \mathbf{v},\mathbf{v}\rangle^{1/2} = \left[\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}\right)^{1/2} = \sqrt{25} = 5$

(e) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left[\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-2\\-3\end{bmatrix}\cdot\begin{bmatrix}-2\\-3\end{bmatrix}\right)^{1/2} = \sqrt{13}$

(f) $\|\mathbf{u}-k\mathbf{v}\| = \left[\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-8\\-11\end{bmatrix}\cdot\begin{bmatrix}-8\\-11\end{bmatrix}\right)^{1/2} = \sqrt{185}$

In Exercises 7–8, use the inner product on R^2 generated by the matrix A to find $\langle \mathbf{u}, \mathbf{v} \rangle$ for the vectors $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

7. $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

$\begin{bmatrix} 0 & \sqrt{5} \end{bmatrix}$

7. $$\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 26 \\ 6 \end{bmatrix} = -24$$

8. $$\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 0 \end{bmatrix} = -42$$

9. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}\left(U^T V\right) = \operatorname{tr}\left(\begin{bmatrix} 1 & 13 \\ 10 & 2 \end{bmatrix} \right) = 3$.

In Exercises 5–6, find a matrix that generates the stated weighted inner product on R^2.

5. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ **6.** $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

5. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

6. $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$

EXAMPLE 4 | Matrices Generating Weighted Euclidean Inner Products

The standard Euclidean and weighted Euclidean inner products are special cases of matrix inner products. The standard Euclidean inner product on R^n is generated by the $n \times n$ identity matrix, since setting $A = I$ in Formula (5) yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$$

and the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n \tag{7}$$

is generated by the matrix

$$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{w_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix}$$

This can be seen by observing that A^TA is the $n \times n$ diagonal matrix whose diagonal entries are the weights $w_1, w_2, \ldots, w_n$.

Other Examples of Inner Products

So far, we have only considered examples of inner products on R^n. We will now consider examples of inner products on some of the other kinds of vector spaces that we discussed earlier.

▶ EXAMPLE 6 The Standard Inner Product on M_{nn}

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) \tag{8}$$

defines an inner product on M_{nn} called the ***standard inner product*** on that space (see Definition 8 of Section 1.3 for a definition of trace). This can be proved by confirming that the four inner product space axioms are satisfied, but we can see why this is so by computing (8) for the 2×2 matrices

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

This yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

which is just the dot product of the corresponding entries in the two matrices. And it follows from this that

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}\langle U^T U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)} = \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2} = \sqrt{14}$$

In Exercises 9–10, compute the standard inner product on M_{22} of the given matrices.

9. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

10. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

In Exercises 21–22, find $\|U\|$ and $d(U, V)$ relative to the standard inner product on M_{22}.

21. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

22. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

9. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}\left(U^T V\right) = \operatorname{tr}\left(\begin{bmatrix} 1 & 13 \\ 10 & 2 \end{bmatrix}\right) = 3$.

10. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}\left(U^T V\right) = \operatorname{tr}\left(\begin{bmatrix} 4 & -18 \\ 8 & 52 \end{bmatrix}\right) = 56$.

POLYNOMIALS

If

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n \quad \text{and} \quad \mathbf{q} = b_0 + b_1x + \cdots + b_nx^n$$

are polynomials in P_n, then the following formula defines an inner product on P_n (verify) that we will call the ***standard inner product*** on this space:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n \qquad (9)$$

The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

In Exercises 11–12, find the standard inner product on P_2 of the given polynomials.

11. $\mathbf{p} = -2 + x + 3x^2, \ \mathbf{q} = 4 - 7x^2$

12. $\mathbf{p} = -5 + 2x + x^2, \ \mathbf{q} = 3 + 2x - 4x^2$

11. $\langle \mathbf{p}, \mathbf{q} \rangle = (-2)(4) + (1)(0) + (3)(-7) = -29$

12. $\langle \mathbf{p}, \mathbf{q} \rangle = (-5)(3) + (2)(2) + (1)(-4) = -15$

EXAMPLE 9 | Working with the Evaluation Inner Product

Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad \text{and} \quad x_2 = 2$$

Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ and $\|\mathbf{p}\|$ for the polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$.

Solution It follows from (10) and (11) that

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (4)(-1) + (0)(1) + (4)(3) = 8$$

$$\|\mathbf{p}\| = \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + [p(x_2)]^2} = \sqrt{[p(-2)]^2 + [p(0)]^2 + [p(2)]^2}$$

$$= \sqrt{4^2 + 0^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

In Exercises **15–16**, *a sequence of sample points is given. Use the evaluation inner product on P_3 at those sample points to find $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials*

$$\mathbf{p} = x + x^3 \quad \textit{and} \quad \mathbf{q} = 1 + x^2$$

15. $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1$

16. $x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$

15. $\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$= (-10)(5) + (-2)(2) + (0)(1) + (2)(2) = -50$

16. $\langle \mathbf{p}, \mathbf{q} \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2)$

$= (-2)(2) + (0)(1) + (2)(2) + (10)(5) = 50$

In Exercises 17–18, find $\|\mathbf{u}\|$ *and* $d(\mathbf{u}, \mathbf{v})$ *relative to the weighted Euclidean inner product* $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ *on* R^2.

17. $\mathbf{u} = (-3, 2)$ and $\mathbf{v} = (1, 7)$

18. $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$

In Exercises 19–20, find $\|\mathbf{p}\|$ *and* $d(\mathbf{p}, \mathbf{q})$ *relative to the standard inner product on* P_2.

19. $\mathbf{p} = -2 + x + 3x^2$, $\mathbf{q} = 4 - 7x^2$

20. $\mathbf{p} = -5 + 2x + x^2$, $\mathbf{q} = 3 + 2x - 4x^2$

In Exercises 21–22, find $\|U\|$ and $d(U,V)$ relative to the standard inner product on M_{22}.

21. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

22. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

In Exercises 23–24, let

$$\mathbf{p} = x + x^3 \quad \text{and} \quad \mathbf{q} = 1 + x^2$$

Find $\|\mathbf{p}\|$ *and* $d(\mathbf{p}, \mathbf{q})$ *relative to the evaluation inner product on* P_3 *at the stated sample points.*

23. $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1$

24. $x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$

In Exercises 25–26, find $\|\mathbf{u}\|$ *and* $d(\mathbf{u}, \mathbf{v})$ *for the vectors* $\mathbf{u} = (-1, 2)$ *and* $\mathbf{v} = (2, 5)$ *relative to the inner product on* R^2 *generated by the matrix* A.

25. $A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

25. $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -4 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 7 \end{bmatrix}\right)^{1/2} = \sqrt{65}$

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -12 \\ -24 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ -24 \end{bmatrix}\right)^{1/2} = 12\sqrt{5}$

26. $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right)^{1/2} = \sqrt{58}$

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -9 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ -6 \end{bmatrix}\right)^{1/2} = 3\sqrt{13}$

Ex# 6.1 (1-26)

Cauchy–Schwarz Inequality

Recall from Formula (20) of Section 3.2 that the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \tag{1}$$

We were assured that this formula was valid because it followed from the Cauchy–Schwarz inequality (Theorem 3.2.4) that

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1 \tag{2}$$

as required for the inverse cosine to be defined. The following generalization of the Cauchy–Schwarz inequality will enable us to define the angle between two vectors in *any* real inner product space.

Theorem 6.2.1

Cauchy–Schwarz Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{3}$$

Theorem 6.2.2

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a real inner product space V, and if k is any scalar, then:

(a) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(b) $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ **[Triangle inequality for distances]**

Definition 1

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V are called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Definition 1

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

In Exercises 1–2, find the cosine of the angle between the vectors with respect to the Euclidean inner product.

1. a. $\mathbf{u} = (1, -3)$, $\mathbf{v} = (2, 4)$

   b. $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = (2, 4, -9)$

   c. $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{v} = (-3, -3, -3, -3)$

2. a. $\mathbf{u} = (-1, 0)$, $\mathbf{v} = (3, 8)$

   b. $\mathbf{u} = (4, 1, 8)$, $\mathbf{v} = (1, 0, -3)$

   c. $\mathbf{u} = (2, 1, 7, -1)$, $\mathbf{v} = (4, 0, 0, 0)$

In Exercises 3–4, find the cosine of the angle between the vectors with respect to the standard inner product on P_2.

3. $\mathbf{p} = -1 + 5x + 2x^2, \ \mathbf{q} = 2 + 4x - 9x^2$

4. $\mathbf{p} = x - x^2, \ \mathbf{q} = 7 + 3x + 3x^2$

In Exercises 5–6, find the cosine of the angle between A and B with respect to the standard inner product on M_{22}.

5. $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

In Exercises 7–8, determine whether the vectors are orthogonal with respect to the Euclidean inner product.

7. a. $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$

   b. $\mathbf{u} = (-2, -2, -2)$, $\mathbf{v} = (1, 1, 1)$

   c. $\mathbf{u} = (a, b)$, $\mathbf{v} = (-b, a)$

8. a. $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (0, 0, 0)$

   b. $\mathbf{u} = (-4, 6, -10, 1)$, $\mathbf{v} = (2, 1, -2, 9)$

   c. $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (-c, 0, a)$

In Exercises 9–10, show that the vectors are orthogonal with respect to the standard inner product on P_2.

9. $\mathbf{p} = -1 - x + 2x^2$, $\mathbf{q} = 2x + x^2$

10. $\mathbf{p} = 2 - 3x + x^2$, $\mathbf{q} = 4 + 2x - 2x^2$

In Exercises 11–12, show that the matrices are orthogonal with respect to the standard inner product on M_{22}.

11. $U = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $V = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

12. $U = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

17. Do there exist scalars k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

Orthogonality of $\mathbf{p}_1$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle = (2)(1) + (k)(2) + (6)(3) = 2k + 20 = 0$ so $k = -10$. Likewise, orthogonality of $\mathbf{p}_2$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = (l)(1) + (5)(2) + (3)(3) = l + 19$ so $l = -19$.

Substituting the values of k and l obtained above yields the polynomials $\mathbf{p}_1 = 2 - 10x + 6x^2$ and $\mathbf{p}_2 = -19 + 5x + 3x^2$ which are not orthogonal since $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = (2)(-19) + (-10)(5) + (6)(3) = -70 \neq 0$. We conclude that no scalars k and l exist that make the three vectors mutually orthogonal.

18. Show that the vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$$

are orthogonal with respect to the inner product on R^2 that is generated by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

[See Formulas (5) and (6) of Section 6.1.]

or equivalently as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A \mathbf{u} \tag{6}$$

19. Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad x_2 = 2$$

Show that the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to this inner product.

32.

6.2 (1-12, 17-19)