Linear Algebra Take-Home Quiz
Due: 48 hours from receipt
Student ID: [ABCD]
Registration Year: [YY]

Instructions:

- This quiz covers eigenvalues/eigenvectors, diagonalization, orthogonal sets, Gram-Schmidt process, QR decomposition, orthogonal matrices, and orthogonal diagonalization.
- Show all your work clearly and box your final answers.
- The quiz is designed to take approximately 2 hours to complete.
- Submit your solutions as a PDF file.

Problem 1: Eigenvalues and Eigenvectors (20 points)

Let

$$A = \begin{bmatrix} 3 + [[YY] mod 5 + 1] & 1 & 0 \\ 1 & 3 - [[ABCD]\ mod 4 + 1] & 0 \\ 0 & 0 & 2[[YY] mod 5 + 1] \end{bmatrix}$$

(a) Find all eigenvalues of A.
(b) For each eigenvalue, find a basis for the corresponding eigenspace.
(c) Determine whether A is diagonalizable. Justify your answer.

Problem 2: Orthogonal Sets and Projections (20 points)

Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ [[YY] mod 5 + 1] \end{bmatrix}$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.
(b) Normalize these vectors to obtain an orthonormal basis for $\mathbb{R}^3$.
(c) Let W be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Find the orthogonal projection of $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ [[ABCD]\ mod 4 + 1] \end{bmatrix}$ onto W.

Problem 3: Gram-Schmidt Process and QR Decomposition (20 points)

Apply the Gram-Schmidt process to the following vectors in $\mathbb{R}^3$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ [[YY] mod 5 + 1] \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ [[ABCD]\ mod 4 + 1] \\ 1 \end{bmatrix}$$

(a) Find an orthogonal basis for $\mathbb{R}^3$.
(b) Find an orthonormal basis for $\mathbb{R}^3$.
(c) Construct the QR decomposition of the matrix $B = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$.

Problem 4: Orthogonal Matrices (20 points)

Let

$$C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta = \frac{\pi}{[[YY] mod 5 + 1] + 2}$.

(a) Verify that C is an orthogonal matrix.
(b) Find C^{-1}.
(c) Compute $\det(C)$.
(d) Let $\mathbf{x} = \begin{bmatrix} 1 \\ [[ABCD]\ mod 4 + 1] \\ 2 \end{bmatrix}$. Find $\|C\mathbf{x}\|$ and verify that it equals $\|\mathbf{x}\|$.

Problem 5: Orthogonal Diagonalization (20 points)

Let

$$D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 + [[ABCD]\ mod 4 + 1] \end{bmatrix}$$

(a) Verify that D is symmetric.
(b) Find all eigenvalues of D and their multiplicities.
(c) Find an orthogonal matrix P that diagonalizes D.
(d) Write the spectral decomposition of D.