Assignment 2

1. a) $\bar{0} = (0,0,0)$

$\bar{0} + v = v$ Axiom 4

$(0,0,0) + (r,g,b) = (0+r,\ 0+g,\ 0+b)$
$= (r,g,b) = v$

$v + (-v) = 0$ Axiom 5

$(r,g,b) + (-r,-g,-b) = (r-r,\ g-g,\ b-b)$
$= (0,0,0) = 0$

b) 1) <u>Additive Identity:-</u>

$\bar{0} = (0,0,0)$, $v = (2,0,0)$

$v \oplus \bar{0} = (2,0,0) \oplus (0,0,0)$
$= (\min(2+0, 1), 0, 0)$
$= (1,0,0)$ Additive identity fails

$(1,0,0) \neq (2,0,0)$

This fails for values > 1.

2) <u>Axiom 5 : Scaler identity</u>:- $1 \odot v = v$

$1 \odot (2,0,0) = (\min(2,1), 0, 0)$
$= (1,0,0)$

$(1,0,0) \neq (2,0,0)$

Axiom 5 fails.

3) <u>Axiom 7; Distibutivity</u>:- $k(u+v) = ku + kv$

$u = (1,1,1)$ and $v = (1,2,3)$, $k = 0.5$

Taking L.H.S:

$u \oplus v = (\min(1+1, 1), \min(1+2), 1), \min(1+3, 1))$
$= (1,1,1)$

$k \odot (u \oplus v) = 0.5(1,1,1) = (0.5, 0.5, 0.5)$

Taking R.H.S:

$Ku = 0.5(1,1,1) = (0.5, 0.5, 0.5)$

$Kv = 0.5(1,2,3) = (0.5, 1, 1.5)$

$Ku + Kv = (\min(0.5+0.5), 1), \min(0.5+1, 1)$
$, \min(0.5+1.5, 1))$

$= (\min(1,1), \min(1.5,1), \min(2,1))$

$Ku + Kv = (1,1,1)$

$(0.5, 0.5, 0.5) \neq (1,1,1)$ Axiom 7 fails

4) Axiom 8: $(K+m)U = KU + mU$

$U = (1,2,3), K = 0.5, m = 0.8$

$K+m = 0.5 + 0.8 = 1.3$

$(1.3) \odot U = 1.3(1,2,3)$

$= (1.3, 2.6, 3.9)$

<u>Taking R.H.S:</u>

$KU = 0.5(1,2,3) = (0.5, 1, 1.5)$

$mU = 0.8(1,2,3) = (0.8, 1.6, 2.4)$

$KU + mU = (\min(0.5+0.8, 1), \min(1+1.6, 1),$
$\min(1.5+2.4, 1))$

$= \min(1.3, 1), \min(2.6, 1), \min(3.9, 1)$

$= (1,1,1)$

$(1.3, 2.6, 3.9) \neq (1,1,1)$ Axiom 8 fails.

2) $[w]_{B'} = P_{B \to B'} \cdot [w]_B$

$$[w]_{B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$[w]_{B'} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$$

5) <u>Axiom 9:</u> $K \odot (u+v) = K \odot u + K \odot v$

$K = -2,\ u = (-1,0,0),\ v = (0.3,0,0)$

$u \oplus v = (\min(-1+0.3, 1), 0, 0) = (-0.7, 0, 0)$

$K \odot (u \oplus v) = \min(-2(-0.7), 1) = \min(1.4, 1) = 1$

$K \odot u = \min(-2 \cdot (-1), 1) = \min(2,1) = 1$

$K \odot v = \min(-2 \cdot 0.3, 1) = \min(-0.6, 1) = -0.6$

$K \odot u \oplus K \odot v = \min(1 + (-0.6), 1) = 0.4$

$1 \neq 0.4$ Axiom 9 fails.

6) <u>Axiom 10:</u> $(a+b) \odot u = a \odot u \oplus b \odot u$

$a = -2,\ b = 0.2,\ u = (-1,0,0)$

$a+b = -1.8$

$(a+b) \odot u = \min(-1.8 \cdot (-1), 1) = \min(1.8, 1)$

$= 1$

$a \odot u = \min(-2 \cdot (-1), 1) = 1$

$b \odot u = \min(0.2 \cdot (-1), 1) = -0.2$

$a \odot u \oplus b \odot u = \min(1 + (-0.2), 1) = 0.8$

$1 \neq 0.8$ Axiom 10 fails.

2. a) $W_2 = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}$

1) zero vector $= (0,0,0,0)$

$0+0+0+0 \neq 1$

Zero vector is not in W_2 so W_2 is not a subspace.

b) $W_3 = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 = 2x_3 \text{ and } x_2 = x_4\}$

1) Zero vector $= (0,0,0,0)$

$0 = 2.0 \quad , \quad 0 = 0$

$0 = 0 \quad , \quad 0 = 0$

Zero vector is in W_3.

2) Closure under addition :

Let $U = (2a, b, a, b)$ and $V = (2c, d, c, d)$

$U + V = (2a+2c, b+d, a+c, b+d)$

$U + V = (2(a+c), b+d, a+c, b+d)$

$x_1 = 2(a+c) = 2x_3$

$x_2 = b+d = x_4$

$U+V$ is in W_3.

3) Closure under scaler multiplication:

Let $U = (2a, b, a, b)$

$KU = (2Ka, Kb, Ka, Kb)$

$x_1 = 2Ka = 2x_3$

$x_2 = Kb = x_4$

ku is in W_3.

Since all these requirements are fulfilled, W_3 is a subspace of R^4.

3. 1) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Let a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= aA_1 + bA_2 + cA_3 + dA_4$

$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since we can express any matrix in M_{22} as a linear combination of matrices in S, the set S spans M_{22}.

2) $B = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$

Let scalers K_1, K_2, K_3, K_4

$= K_1 A_1 + K_2 A_2 + K_3 A_3 + K_4 A_4 = B$

$= K_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + K_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$

$= \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$

$K_1 = 3, \quad K_2 = -2, \quad K_3 = 5, \quad K_4 = 4$

$B = 3A_1 - 2A_2 + 5A_3 + 4A_4$

4. $V_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, V_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, V_4 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

1) $K_1 V_1 + K_2 V_2 + K_3 V_3 + K_4 V_4 = 0$

$K_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + K_2 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + K_3 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + K_4 \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$K_1 + 3K_2 + 2K_3 = 0$

$2K_1 + K_2 - K_3 + 5K_4 = 0$ — ②

$-K_1 + 4K_2 + 5K_3 + 2K_4 = 0$ — ③

$K_1 = -2K_3 - 3K_2$ — ①

Substituting eq ① in eq ②:

$2(-2K_3 - 3K_2) + K_2 - K_3 + 5K_4 = 0$

$-4K_3 - 6K_2 + K_2 - K_3 + 5K_4 = 0$

$-5K_3 - 5K_2 + 5K_4 = 0$

$$-k_3 - k_2 + k_4 = 0$$
$$k_4 = k_2 + k_3$$

Substituting in eq ③:

$$-(-2k_3 - 3k_2) + 4k_2 + 5k_3 + 2(k_2 + k_3) = 0$$
$$2k_3 + 3k_2 + 4k_2 + 5k_3 + 2k_2 + 2k_3 = 0$$
$$9k_3 + 9k_2 = 0$$
$$k_3 + k_2 = 0$$
$$k_3 = -k_2$$

$$k_4 = k_2 - k_2 = 0$$
$$k_1 = 2k_2 - 3k_2 = -k_2$$

Let $k_2 = t$: (for any $t \in \mathbb{R}$)

$k_1 = -t$, $k_3 = -t$

If $t = 1$:

$k_2 = 1$, $k_1 = -1$, $k_3 = -1$, $k_4 = 0$

$$= -V_1 + V_2 - V_3$$

$$= -\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1+3-2 \\ -2+1+1 \\ 1+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is linearly dependent

2) $-V_1 + V_2 - V_3 + 0V_4 = 0$

$V_2 = V_3 + V_1$

V_2 is a linear combination of V_1 and V_3.

$+k_3) = 0$

$k_3 = 0$

3) $$M = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix}$$

$$\det|M| = 1\begin{vmatrix} 1 & b \\ 0 & c \end{vmatrix} + 0\begin{vmatrix} 0 & b \\ 0 & c \end{vmatrix} + a\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$\det = 1(c-0) + 0(0+0) + a(0-0)$

$\det = c$

For W to be linearly independent,
$c \neq 0$, a and b can be any value.

c)

5. $$B = \left\{ T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

0

$$B' = \left\{ S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, S_4 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

1) = $[A]_B = 3T_1 + T_2 + 2T_3 + 4T_4$

= $$[A]_B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

2) $P_{B \to B'} = \left[[T_1]_{B'} \; [T_2]_{B'} \; [T_3]_{B'} \; [T_4]_{B'} \right]$

- $[T_1]_{B'} \Rightarrow T_1 = K_1 S_1 + K_2 S_2 + K_3 S_3 + K_4 S_4$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = K_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + K_4 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} K_1 + K_2 &= 1 \\ K_1 - K_2 &= 0 \\ K_3 + K_4 &= 0 \\ K_3 - K_4 &= 0 \end{aligned}\right\} \begin{aligned} K_1 = K_2 &= 1/2, \\ K_3 = K_4 &= 0 \end{aligned}$$

$$[T_1]_{B'} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

- $[T_2]_{B'} \Rightarrow T_2 = K_1 S_1 + K_2 S_2 + K_3 S_3 + K_4 S_4$

$$\left.\begin{aligned} K_1 + K_2 &= 0 \\ K_1 - K_2 &= 1 \\ K_3 + K_4 &= 0 \\ K_3 - K_4 &= 0 \end{aligned}\right\} \begin{aligned} K_1 = 1/2, \; K_2 = -1/2, \\ K_3 = 0, \; K_4 = 0 \end{aligned}$$

$$[T_2]_{B'} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$$

• $[T_3]_{B'} \Rightarrow T_3 = K_1 S_1 + K_2 S_2 + K_3 S_3 + K_4 S_4$

$$\left.\begin{array}{l} K_1 + K_2 = 0 \\ K_1 - K_2 = 0 \\ K_3 + K_4 = 1 \\ K_3 - K_4 = 0 \end{array}\right\} K_1 = 0,\ K_2 = 0,\ K_3 = 1/2,\ K_4 = 1/2$$

$$= [T_3]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

• $[T_4]_{B'} \Rightarrow T_4 = K_1 S_1 + K_2 S_2 + K_3 S_3 + K_4 S_4$

$$\left.\begin{array}{l} K_1 + K_2 = 0 \\ K_1 - K_2 = 0 \\ K_3 + K_4 = 0 \\ K_3 - K_4 = 0 \end{array}\right\} K_1 = 0,\ K_2 = 0,\ K_3 = 1/2,\ K_4 = -1/2$$

$$= [T_4]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\star \quad P_{B \to B'} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

3) $[A]_{B'} = P_{B \to B'} \cdot [A]_B$

$$[A]_{B'} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$[A]_{B'} = \begin{bmatrix} 3/2 + 1/2 \\ 3/2 - 1/2 \\ 1 + 2 \\ 1 - 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

6. Part a :-

$$x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 = 0$$
$$2x_1 + 4x_2 - 2x_3 + 7x_4 - 3x_5 = 0$$
$$3x_1 + 6x_2 - 3x_3 + 10x_4 - 5x_5 = 0$$

1) $$A = \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & -2 & 0 \\ 2 & 4 & -2 & 7 & -3 & 0 \\ 3 & 6 & -3 & 10 & -5 & 0 \end{array}\right]$$

1. $R_2 - 2R_1$, $R_3 - 3R_1$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

2. $R_3 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- let $x_5 = t$, $x_3 = s$, $x_2 = r$:

$x_4 = -t$, $x_1 = 2t + 3t + s - 2r$

$x_1 = 5t + s - 2r$

$(x_1, x_2, x_3, x_4, x_5) = (5t + s - 2r, r, s, -t, t)$

$= t(5, 0, 0, -1, 1) + s(1, 0, 1, 0, 0) +$
$r(-2, 1, 0, 0, 0)$

Basis :- $(5,0,0,-1,1)$, $(1,0,1,0,0)$, $(-2,1,0,0,0)$.

2) Dimension $= 3$

3) $x_1 + x_2 + x_3 = 0$

$(5t + s - 2r) + r + s = 0$

$5t + 2s - r = 0$

$r = 5t + 2s$

- $x_1 = 5t + s - 2(5t + 2s)$

$x_1 = -5t - 3s$

- $x_2 = 5t + 2s$, $x_3 = s$, $x_4 = -t$, $x_5 = t$

$= (x_1, x_2, x_3, x_4, x_5) = (-5t - 3s, 5t + 2s, s, -t, t)$

$= t(-5, 5, 0, -1, 1) + s(-3, 2, 1, 0, 0)$

- Basis :- $(-5, 5, 0, -1, 1)$, $(-3, 2, 1, 0, 0)$
- Dimension $= 2$

Part b:-

1) 1. Zero vector:-

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$0^T = 0$, so $0 \in W$

2. Closed under addition:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

$A^T = A$, $B^T = B$

$$A+B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 6 & 5 \\ 2 & 5 & 9 \end{bmatrix}$$

$(A+B)^T = A^T + B^T = A+B$

so $A+B \in W$

3. Closed under scaler multipliation:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad K = 2$$

$$KA = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 8 \\ 6 & 8 & 10 \end{bmatrix}$$

$(KA)^T = KA$, so $KA \in W$

- Therefore, W is a subspace of M_{33}.

Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} +$$
$$d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Dimension = 6

2) 1. Zero vector:

$0(1) = 0$, $0(-1) = 0$

So $0 \in V$

2. Closed under addition:

$p(1) = 0$, $q(1) = 0$, $p(-1) = 0$, $q(-1) = 0$

- $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$
- $(p+q)(-1) = p(-1) + q(-1) = 0 + 0 = 0$

So $p+q \in V$

3. Closed under scaler multiplication:-

let $K \in R$

$(Kp)(1) = K.p(1) = K.0 = 0$

$(Kp)(-1) = K.p(-1) = K.0 = 0$

So $Kp \in V$

- Therefore, V is a subspace of P_4

Standard Basis :- $\{1, x, x^2, x^3, x^4\}$

$\dim(P_4) = 5$

Suppose $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ in P_4

1) $p(1) = a_0 + a_1 + a_2 + a_3 + a_4$

2) $p(-1) = a_0 - a_1 + a_2 - a_3 + a_4$

They are independent.

- $\dim(V) = 5 - 2 = 3$

$p(1) = 0, \; p(-1) = 0 \Rightarrow x = 1, \; x = -1$

$p(x) = (x^2 - 1) \cdot q(x)$

$\deg(q) = 2.$

Standard Basis = $\{1, x, x^2\}$.

1) $q(x) = p(x) \cdot 1 = (x^2 - 1) \cdot 1 = x^2 - 1$

2) $q(x) = p(x) \cdot x = (x^2 - 1) \cdot x = x^3 - x$

3) $q(x) = p(x) \cdot x^2 = (x^2 - 1) \cdot x^2 = x^4 - x^2$

$B = \{x^2 - 1, x^3 - x, x^4 - x^2\}$

7. $B = \left\{ u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\}$,

$B' = \left\{ u_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, u_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

$w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $[w]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

1) 1. $[u_1]_{B'} = k_1 u_1' + k_2 u_2'$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$k_1 - k_2 = 2$$
$$3k_1 - k_2 = 2$$

$k_1 = 0$, $k_2 = -2$

$$[u_1]_{B'} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

2. $[u_2]_{B'} = k_1 u_1' + k_2 u_2'$

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$k_1 - k_2 = 4$$
$$3k_1 - k_2 = -1$$

$k_1 = -5/2$, $k_2 = -13/2$

$$[u_2]_{B'} = \begin{bmatrix} -5/2 \\ -13/2 \end{bmatrix}$$

$$P_{B \to B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$$

8. $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$

$R = \begin{bmatrix} 1 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

1) row(A) = $\overset{r_1}{(1,-3,0,0,3,0)}$, $\overset{r_2}{(0,0,1,0,-2,0)}$, $\overset{r_3}{(0,0,0,1,1,0)}$,
$\overset{r_4}{(0,0,0,0,0,1)}$

2) col(A) = $\overset{c_1}{(1,2,2,-1)}$, $\overset{c_3}{(4,9,9,-4)}$, $\overset{c_4}{(-2,-1,-1,2)}$,
$\overset{c_6}{(4,2,7,-4)}$

3) free variables : x_2 , x_5

$x_1 = -3x_5 + 3x_2$

$x_3 = 2x_5$

$x_4 = -x_5$

$x_6 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

null(A) = {(3,1,0,0,0,0), (-3,0,2,-1,1,0)

4) $AX = b$

$$[A|b] = \left[\begin{array}{cccccc|c} 1 & -3 & 4 & 2 & 5 & 4 & 1 \\ 2 & -6 & 9 & -1 & 8 & 2 & -9 \\ 2 & -6 & 9 & -1 & 9 & 7 & -5 \\ -1 & 3 & -4 & 2 & -5 & -4 & 4 \end{array}\right]$$

1) $R_2 - 2R_1$, $R_3 - 2R_1$, $R_4 + R_1$

$$\left[\begin{array}{cccccc|c} 1 & -3 & 4 & 2 & 5 & 4 & 1 \\ 0 & 0 & 1 & 3 & -2 & -6 & -11 \\ 0 & 0 & 1 & 3 & -1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

- By looking at R_4, system is inconsistent. So b is not in column space of A.

9. A is a 5×7 matrix

1) • Largest rank = $\min(5, 7)$ = 5

• Nullity = number of columns − rank = $7 - 5 = 2$

2) a) Free variables = $7 - 4 = 3$

b) No unique solution because free variable exists so there would be infinite solutions.

3) a) rank = −nullity + number of columns = $-2 + 7 = -5$

b) rank = number of rows = 5

Since rank and number of rows are equal, column vectors of A will span $\mathbb{R}^5$.