Class Lecture Notes

Gauss-Jordan Elimination

Gaussian elimination reduces an augmented matrix to **Row Echelon Form (REF)** using:

1. Forward elimination
2. Back substitution

Gauss-Jordan elimination continues to **Reduced Row Echelon Form (RREF)**:

1. Get REF via Gaussian elimination.
2. Scale rows so all pivots are 1.
3. Eliminate all entries above and below each pivot.
4. Read solution directly from RREF.

Homogeneous Systems

Form: $A\mathbf{x} = \mathbf{0}$

- Always has the **trivial solution** $\mathbf{x} = \mathbf{0}$.
- Has **nontrivial solutions** if there is at least one free variable (i.e., if rank of A $<$ number of variables).

Non-Homogeneous Systems

Form: $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$.

- **No solution** if a row of $[A|\mathbf{b}]$ becomes $[0 \dots 0|b]$ with $b \neq 0$.
- **Unique solution** if rank of $A|b$ = rank of A = number of variables.
- **Infinitely many solutions** if rank of $A|b$ = rank of $A <$ number of variables.

Solutions to Problems 5–26

Problem 5

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$

RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Solution: $x_1 = 3, x_2 = 1, x_3 = 2$

Problem 6

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$

RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution: $x_1 = -1/7 - 3/7x_3, x_2 = 1/7 - 4/7x_3, x_3$ is free variable.

Problem 7

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$

RREF: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: $x = w - 1, y = 2z : z$ and w are free.

Problem 8

$$\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$

RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$

Solution: No solution.

Problem 9 (G-J for 5)

Same as Problem 5.

Problem 10 (G-J for 6)

Same as Problem 6.

Problem 11 (G-J for 7)

Same as Problem 7.

Problem 12 (G-J for 8)

Same as Problem 8.

Problem 13

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ \text{System: } 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

More variables (4) than equations (3). **Nontrivial solutions exist**.

Problem 14

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ \text{System: } x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

From $4x_3 = 0 \Rightarrow x_3 = 0$. Then $x_2 = 0$, $x_1 = 0$. **Only trivial solution**.

Problem 15

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ \text{System: } x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\text{RREF: } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $x_1 = x_2 = x_3 = 0$

Problem 16

$$\begin{aligned} 2x - y - 3z &= 0 \\ \text{System: } -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

$$\text{RREF: } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $x = y = z = 0$

Problem 17

$$\begin{aligned} \text{System: } 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$$

Solution: free variables x_3 and x_4:

$$\begin{aligned} x_1 &= -\frac{1}{4}x_3 \\ x_2 &= -\frac{1}{4}x_3 - x_4 \end{aligned}$$

Problem 18

$$\begin{aligned} v + 3w - 2x &= 0 \\ \text{System: } 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: free variables w and x:

$$\begin{aligned} u &= \frac{7}{2}w - \frac{5}{2}x \\ v &= -3w + 2x \end{aligned}$$

Problem 19

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ \text{System: } w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: y is free,

$$\begin{aligned} w &= y \\ x &= -y \\ z &= 0 \end{aligned}$$

Problem 20

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

System:

RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: $x_1 = x_2 = x_3 = x_4 = 0$

Problem 21

$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

System:

RREF:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $I_1 = -1, \quad I_2 = 0, \quad I_3 = 1, \quad I_4 = 2$

Problem 22

$$\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \end{aligned}$$

System:

RREF:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Z_2, Z_5 free.

$$\begin{aligned} Z_1 &= -Z_2 - Z_5 \\ Z_3 &= -Z_5 \\ Z_4 &= 0 \end{aligned}$$

Problem 23

a. Consistent, unique.
b. Consistent, infinite.
c. Inconsistent.
d. Inconclusive.

Problem 24

a. Consistent, unique.
b. Consistent, unique.
c. Inconclusive.
d. Inconsistent.

Problem 25

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

System:

REF: $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2 - 16 & a - 4 \end{array}\right]$

- **No solution:** $a^2 - 16 = 0$ and $a - 4 \neq 0 \Rightarrow a = -4$
- **Infinite:** $a^2 - 16 = 0$ and $a - 4 = 0 \Rightarrow a = 4$
- **Unique:** Otherwise ($a \neq \pm 4$)

Problem 26

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

System:

REF: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & -a^2 + 2 & a - 2 \end{array}\right]$

- **No solution:** $-a^2 + 2 = 0$ and $a - 2 \neq 0 \Rightarrow a = \pm\sqrt{2}$
- **Infinite:** $-a^2 + 2 = 0$ and $a - 2 = 0 \Rightarrow$ No such a
- **Unique:** Otherwise ($a \neq \pm\sqrt{2}$)