A=CR decomposition

CR Decomposition using RREF Method

A = **CR** where:

- **C** = columns of A that form a basis for its column space
- **R** = row reduced echelon form of A

- **A = C × R**
- **C** (m×r): Contains the *independent columns* of A (basis for Col(A))
- **R** (r×n): Row Reduced Echelon Form of A (without zero rows)
- **r** = rank of A
- Every column of A = linear combination of columns of C
- Coefficients given by R

1. Compute **RREF(A)**
2. Identify **pivot columns** (1st nonzero in each row = leading 1's)
3. **C** = original A's columns corresponding to pivot columns
4. **R** = RREF(A) with *zero rows removed*
5. Verify: **A = CR**

Let's decompose: **Into CR**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

Compute RREF(A)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

- R2 ← R2 − 2R1:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

- R3 ← R3 − R1:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

- Swap R2 ↔ R3:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- R3 is zero row; R2 ← (−1)×R2:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- R1 ← R1 − 3R2:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF(A) =

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From RREF(A):

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Leading 1's in **column 1** and **column 3**
- **Pivot columns:** 1 and 3
- **Rank r = 2**

C = original columns of A corresponding to pivot columns 1 and 3:

From

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

Column 1: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Column 3: $\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 1 & 2 \end{bmatrix}$$

R = RREF(A) with zero rows removed:

RREF(A) was

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Remove zero row (row 3):

$$R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Check:

$$CR = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Question 1 — Image Compression

A grayscale image patch is represented by the matrix

$$A = \begin{bmatrix} 12 & 24 & 36 \\ 10 & 20 & 30 \\ 8 & 16 & 24 \end{bmatrix}$$

In image compression, low-rank approximations reduce size.

Perform the CR decomposition of A to identify its **rank-1 representation.**

Question 2 — Database Query Optimization

A database engine stores join-costs between three tables as:

$$A = \begin{bmatrix} 4 & 8 & 12 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Query optimizers use factorization to reduce computation.

Find the CR decomposition of A so the system can express A as a product of column and row bases.

Question 3 — Machine Learning (Feature Reduction)

A machine-learning model stores the correlation matrix of 3 features as:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The model wants to reduce redundant features.

Use CR decomposition to extract the independent feature set.