Section 1.7 Lecture Notes: Diagonal, Triangular, and Symmetric Matrices

1 Diagonal Matrices

1.1 Definition

A square matrix in which all entries off the main diagonal are zero is called a **diagonal matrix**. A general $n \times n$ diagonal matrix D is:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

1.2 Invertibility and Inverse

A diagonal matrix is invertible **if and only if all of its diagonal entries are nonzero**. Its inverse is another diagonal matrix:

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

1.3 Powers

Powers of diagonal matrices are straightforward to compute. For a positive integer k:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

1.4 Example 1

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & (-3)^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$A^{-5} = (A^{-1})^5 = \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & (-\frac{1}{3})^5 & 0 \\ 0 & 0 & (\frac{1}{2})^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

2 Triangular Matrices

2.1 Definitions

- A square matrix is **upper triangular** if all entries *below* the main diagonal are zero.
- A square matrix is **lower triangular** if all entries *above* the main diagonal are zero.
- A matrix is **triangular** if it is either upper or lower triangular.

2.2 General Examples

A general 4×4 upper triangular matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

A general 4×4 lower triangular matrix:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

2.3 Theorem 1.7.1: Properties of Triangular Matrices

(a) The transpose of a lower triangular matrix is upper triangular, and vice versa.

(b) The product of lower triangular matrices is lower triangular. The product of upper triangular matrices is upper triangular.

(c) A triangular matrix is invertible **if and only if** all its diagonal entries are nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular. The inverse of an invertible upper triangular matrix is upper triangular.

2.4 Example 3: Computations with Triangular Matrices

Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Invertibility:** By Theorem 1.7.1(c), A is invertible (all diagonal entries nonzero). B is *not* invertible (a diagonal entry is zero).
- **Inverse:** Theorem 1.7.1(d) tells us A^{-1} must be upper triangular.

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad \text{(Confirmed by calculation)}$$

- **Products:** Theorem 1.7.1(b) tells us AB and BA must be upper triangular.

$$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

3 Symmetric Matrices

3.1 Definition

A square matrix A is said to be **symmetric** if $A = A^T$.

3.2 Example 4

The following matrices are symmetric:

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

3.3 Theorems on Algebraic Properties

Theorem 1.7.2 If A and B are symmetric matrices of the same size, and k is any scalar, then:

(a) A^T is symmetric.

(b) $A+B$ and $A-B$ are symmetric.

(c) kA is symmetric.

Theorem 1.7.3 The product of two symmetric matrices A and B is symmetric **if and only if** A and B commute, i.e., $AB = BA$.

3.4 Example 5: Products of Symmetric Matrices

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$$

All three matrices are symmetric.

$$AB_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix} \quad \text{(Not symmetric} \Rightarrow A \text{ and } B_1 \text{ do not commute)}$$

$$AB_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{(Symmetric} \Rightarrow A \text{ and } B_2 \text{ commute)}$$

3.5 Theorem on the Inverse of a Symmetric Matrix

Theorem 1.7.4 If A is an invertible symmetric matrix, then A^{-1} is symmetric.
Proof: If A is symmetric and invertible, then:

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

Thus, A^{-1} is symmetric.

3.6 Example

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$. A is symmetric. Its inverse is:

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

We can confirm A^{-1} is symmetric.

3.7 Theorem on Products with the Transpose

Theorem 1.7.5 If A is an invertible matrix, then AA^T and A^TA are also invertible.
Furthermore, for *any* matrix A (not necessarily square or invertible), the products AA^T and A^TA are always **symmetric**.

$$(AA^T)^T = (A^T)^T A^T = AA^T \quad \text{and} \quad (A^TA)^T = A^T(A^T)^T = A^TA$$

3.8 Example 6: The Product of a Matrix and Its Transpose

Let A be the 2×3 matrix:

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then:

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Both $A^T A$ and AA^T are symmetric, as expected.