Section 4.7: Change of Basis

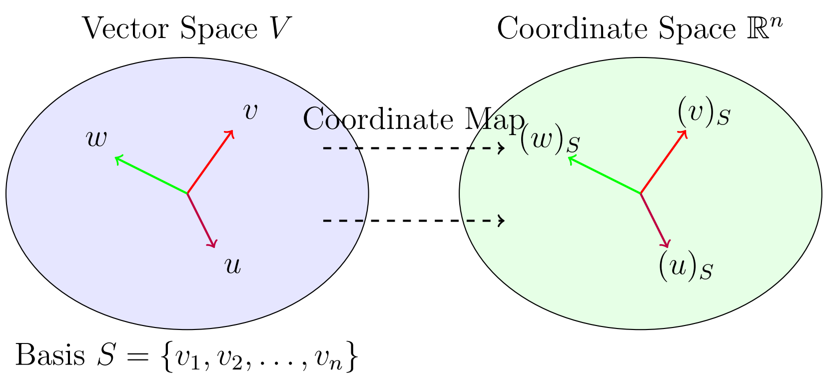
Linear Algebra Lecture

Coordinate Maps

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a finite-dimensional vector space V, and $(v)_S = (c_1, c_2, \ldots, c_n)$ is the coordinate vector of v relative to S, then the mapping:

$$v \to (v)_S$$

creates a one-to-one correspondence between vectors in the general vector space V and vectors in the Euclidean vector space $\mathbb{R}^n$. We call this the **coordinate map** relative to S from V to $\mathbb{R}^n$.



We often express coordinate vectors in matrix form:

$$[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The Change-of-Basis Problem

In many applications, we need to work with more than one coordinate system. This leads to the fundamental problem:

Definition 0.1 (Change-of-Basis Problem)**.** If v is a vector in a finite-dimensional vector space V, and we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[v]_B$ and $[v]_{B'}$ related?

Remark 0.1. We refer to:

- B as the "**old basis**"
- B' as the "**new basis**"

Our objective is to find the relationship between the old and new coordinates of a fixed vector v.

Solution to the Change-of-Basis Problem

Theorem 0.1 (Change-of-Basis Solution)**.** If we change the basis for a vector space V from an old basis $B = \{u_1, u_2, \ldots, u_n\}$ to a new basis $B' = \{u'_1, u'_2, \ldots, u'_n\}$, then for each vector v in V, the new coordinate vector $[v]_{B'}$ is related to the old coordinate vector $[v]_B$ by:

$$[v]_{B'} = P[v]_B$$

where the columns of P are the coordinate vectors of the old basis vectors relative to the new basis:

$$P = \big[[u_1]_{B'} \quad [u_2]_{B'} \quad \cdots \quad [u_n]_{B'}\big]$$

Transition Matrices

Definition 0.2 (Transition Matrix)**.** The matrix P in the change-of-basis formula is called the **transition matrix** from B to B' and is denoted by:

$$P_{B\to B'} = \big[[u_1]_{B'} \quad [u_2]_{B'} \quad \cdots \quad [u_n]_{B'}\big]$$

Similarly, the transition matrix from B' to B is:

$$P_{B'\to B} = \big[[u'_1]_B \quad [u'_2]_B \quad \cdots \quad [u'_n]_B\big]$$

Remark 0.2. Memory Aid: The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

Example 0.1 (Finding Transition Matrices)**.** Consider bases $B = \{u_1, u_2\}$ and $B' = \{u'_1, u'_2\}$ for $\mathbb{R}^2$, where:

$$u_1 = (1, 0), \quad u_2 = (0, 1), \quad u'_1 = (1, 1), \quad u'_2 = (2, 1)$$

(a) Find $P_{B\to B'}$ (from B to B')

(b) Find $P_{B'\to B}$ (from B' to B)

Solution (a): We need $[u_1]_{B'}$ and $[u_2]_{B'}$. Solve:

$$u_1 = au'_1 + bu'_2 \Rightarrow (1, 0) = a(1, 1) + b(2, 1)$$
$$u_2 = cu'_1 + du'_2 \Rightarrow (0, 1) = c(1, 1) + d(2, 1)$$

Solving these systems:

$$(1,0) = -1(1,1) + 1(2,1) \Rightarrow [u_1]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$(0,1) = 2(1,1) + (-1)(2,1) \Rightarrow [u_2]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So:

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Solution (b): We need $[u_1']_B$ and $[u_2']_B$. Since B is standard basis:

$$u_1' = (1,1) = 1(1,0) + 1(0,1) \Rightarrow [u_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u_2' = (2,1) = 2(1,0) + 1(0,1) \Rightarrow [u_2']_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So:

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Transforming Coordinates

The transition matrices allow us to convert between coordinate systems:

Theorem 0.2 (Coordinate Transformation)**.** If B and B' are bases for a finite-dimensional vector space V, then for every vector v in V:

$$[v]_{B'} = P_{B \to B'}[v]_B$$
$$[v]_B = P_{B' \to B}[v]_{B'}$$

Example 0.2 (Change of Coordinates)**.** Let B and B' be the bases from Example 1. Given:

$$[v]_B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

find $[v]_{B'}$.

Solution: Using the transition matrix from Example 1(a):

$$[v]_{B'} = P_{B \to B'}[v]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+10 \\ -3-5 \end{bmatrix} = \begin{bmatrix} 13 \\ -8 \end{bmatrix}$$

Invertibility of Transition Matrices

Theorem 0.3 (4.7.1)**.** If P is the transition matrix from a basis B to a basis B' for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B' to B.

Proof. For any vector v:

$$[v]_{B'} = P_{B\to B'}[v]_B \quad \text{and} \quad [v]_B = P_{B'\to B}[v]_{B'}$$

Substituting:

$$[v]_{B'} = P_{B\to B'}P_{B'\to B}[v]_{B'} \Rightarrow P_{B\to B'}P_{B'\to B} = I$$

Similarly, $P_{B'\to B}P_{B\to B'} = I$. Thus, $P_{B'\to B} = (P_{B\to B'})^{-1}$. □

Example 0.3. For the matrices from Example 1:

$$P_{B\to B'}P_{B'\to B} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Confirming they are inverses.

Efficient Method for Computing Transition Matrices

For $\mathbb{R}^n$, we have an efficient procedure:

Theorem 0.4 (Transition Matrix Computation)**.** To compute $P_{B\to B'}$ for bases in $\mathbb{R}^n$:

1. Form the partitioned matrix [new basis | old basis] (vectors as columns)
2. Reduce to reduced row echelon form: $[I \mid P]$
3. The matrix P on the right is $P_{B\to B'}$

$$\boxed{\text{[New Basis | Old Basis]}} \xrightarrow{\text{Row Reduction}} \boxed{[I \mid P_{B\to B'}]}$$

Example 0.4 (Example 1 Revisited)**.** Use the efficient method to find:

(a) $P_{B\to B'}$ from $B = \{(1,0),(0,1)\}$ to $B' = \{(1,1),(2,1)\}$

(b) $P_{B'\to B}$ from B' to B

Solution (a): B is old, B' is new:

$$[\text{New} \mid \text{Old}] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1 \end{bmatrix}$$

Row reduce:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

So $P_{B\to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ (matches Example 1)

Solution (b): B' is old, B is new:

$$[\text{New} \mid \text{Old}] = \begin{bmatrix} 1 & 0 & | & 1 & 2 \\ 0 & 1 & | & 1 & 1 \end{bmatrix}$$

Already in RREF, so $P_{B'\to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Transition to Standard Basis

Theorem 0.5 (4.7.2)**.** Let $B = \{u_1, u_2, \ldots, u_n\}$ be any basis for $\mathbb{R}^n$ and let $S = \{e_1, e_2, \ldots, e_n\}$ be the standard basis. If vectors are in column form, then:

$$P_{B\to S} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

That is, the transition matrix from any basis to the standard basis is simply the matrix whose columns are the basis vectors.

Exercise Solutions

1. **Consider bases** $B = \{u_1, u_2\}$ **and** $B' = \{u_1', u_2'\}$ **for** $\mathbb{R}^2$**:**

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \; u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \; u_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; u_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Solution:

(a) **Find** $P_{B'\to B}$**:** We need $[u_1']_B$ and $[u_2']_B$.
Solve $u_1' = au_1 + bu_2$:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = a\begin{bmatrix} 2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} 2a + 4b = 1 \\ 2a - b = 3 \end{cases}$$

Solving: $a = \frac{13}{10}$, $b = -\frac{2}{5}$, so $[u_1']_B = \begin{bmatrix} \frac{13}{10} \\ -\frac{2}{5} \end{bmatrix}$
Solve $u_2' = cu_1 + du_2$:

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = c\begin{bmatrix} 2 \\ 2 \end{bmatrix} + d\begin{bmatrix} 4 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} 2c + 4d = -1 \\ 2c - d = -1 \end{cases}$$

Solving: $c = -\frac{1}{2}$, $d = 0$, so $[u_2']_B = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$
Thus: $P_{B'\to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$

(b) **Find** $P_{B\to B'}$**:** Use efficient method:

$$[B' \mid B] = \begin{bmatrix} 1 & -1 & | & 2 & 4 \\ 3 & -1 & | & 2 & -1 \end{bmatrix}$$

Row reduce:

$$\to \begin{bmatrix} 1 & 0 & | & 0 & -\frac{5}{2} \\ 0 & 1 & | & -2 & -\frac{13}{2} \end{bmatrix}$$

So $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$

(c) **For $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, find $[w]_B$ and $[w]_{B'}$:**

Since B is not standard, solve $w = au_1 + bu_2$:

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = a\begin{bmatrix} 2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} 2a + 4b = 3 \\ 2a - b = -5 \end{cases}$$

Solving: $a = -\frac{17}{10}$, $b = \frac{8}{5}$, so $[w]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$

Then $[w]_{B'} = P_{B\to B'}[w]_B = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

(d) **Check:** Verify $w = -4u_1' - 4u_2' = -4(1,3) - 4(-1,-1) = (-4+4, -12+4) = (0,-8)$? Wait, this gives $(0,-8)$ but $w = (3,-5)$.

Let me recompute: Actually, $[w]_{B'} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$ means $w = -4u_1' - 4u_2' = -4(1,3) - 4(-1,-1) = (-4+4, -12+4) = (0,-8)$

But $w = (3,-5)$, so there's an error. Let me recompute $P_{B\to B'}$ carefully.

Using the efficient method properly:

$$[B' \mid B] = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$: $\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & -4 & -13 \end{bmatrix}$

$R_2 \to \frac{1}{2}R_2$: $\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -2 & -\frac{13}{2} \end{bmatrix}$

$R_1 \to R_1 + R_2$: $\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{bmatrix}$

So $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$ is correct.

Then $[w]_{B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{17}{5} - \frac{52}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

Check: $w = -4u_1' - 7u_2' = -4(1,3) - 7(-1,-1) = (-4+7, -12+7) = (3,-5)$

2. **Repeat with:**

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ u_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ u_2' = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Solution: Since B is standard basis:

(a) $P_{B'\to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ (columns are u_1', u_2')

(b) $P_{B\to B'} = (P_{B'\to B})^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$

(c) For $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $[w]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ (since B is standard)

$$[w]_{B'} = P_{B\to B'}[w]_B = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 \\ -13 \end{bmatrix}$$

(d) Check: $w = -\frac{3}{11}u'_1 - \frac{13}{11}u'_2 = -\frac{3}{11}(2,1) - \frac{13}{11}(-3,4) = (-\frac{6}{11}+\frac{39}{11}, -\frac{3}{11}-\frac{52}{11}) = (3,-5)$

3. **Consider bases for $\mathbb{R}^3$:**

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \ B' = \left\{ \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Solution:

(a) **Find $P_{B\to B'}$:** Use efficient method:

$$[B' \mid B] = \begin{bmatrix} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{bmatrix}$$

Row reduce to get $[I \mid P_{B\to B'}]$

(b) **For $w = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$, find $[w]_B$ and $[w]_{B'}$:**

Solve $w = au_1 + bu_2 + cu_3$:

$$\begin{cases} 2a + 2b + c = -5 \\ a - b + 2c = 8 \\ a + b + c = -5 \end{cases}$$

Solving: $a = -2, b = -1, c = -1$, so $[w]_B = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$

Then $[w]_{B'} = P_{B\to B'}[w]_B$

4. **Similar to Exercise 3 with different bases.**

5. **Let V be spanned by $f_1 = \sin x$, $f_2 = \cos x$:**

(a) **Show $g_1 = 2\sin x + \cos x$, $g_2 = 3\cos x$ form basis:**

Check linear independence: If $c_1g_1 + c_2g_2 = 0$, then:

$$c_1(2\sin x + \cos x) + c_2(3\cos x) = 2c_1 \sin x + (c_1 + 3c_2)\cos x = 0$$

Since $\{\sin x, \cos x\}$ is linearly independent, $2c_1 = 0$ and $c_1 + 3c_2 = 0$, so $c_1 = c_2 = 0$.

(b) **Find $P_{B'\to B}$ where $B' = \{g_1, g_2\}$, $B = \{f_1, f_2\}$:**
Express g_1, g_2 in terms of f_1, f_2:

$$g_1 = 2f_1 + f_2 \Rightarrow [g_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$g_2 = 0f_1 + 3f_2 \Rightarrow [g_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

So $P_{B'\to B} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

(c) **Find $P_{B\to B'}$:** Inverse of previous matrix:

$$P_{B\to B'} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

(d) **For $h = 2\sin x - 5\cos x$, find $[h]_B$ and $[h]_{B'}$:**
$[h]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ (since $h = 2f_1 - 5f_2$)
$[h]_{B'} = P_{B\to B'}[h]_B = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(e) **Check:** $h = 1 \cdot g_1 - 2 \cdot g_2 = (2\sin x + \cos x) - 2(3\cos x) = 2\sin x - 5\cos x$

6. **Consider bases for P_1:**

$$B = \{6 + 3x, 10 + 2x\}, \quad B' = \{2, 3 + 2x\}$$

Solution:

(a) **Find $P_{B'\to B}$:** Express B' vectors in B coordinates.
Solve $2 = a(6 + 3x) + b(10 + 2x)$:

$$2 = (6a + 10b) + (3a + 2b)x \Rightarrow \begin{cases} 6a + 10b = 2 \\ 3a + 2b = 0 \end{cases}$$

Solving: $a = -\frac{2}{9}, b = \frac{1}{3}$, so $[q_1]_B = \begin{bmatrix} -\frac{2}{9} \\ \frac{1}{3} \end{bmatrix}$
Solve $3 + 2x = c(6 + 3x) + d(10 + 2x)$:

$$3 + 2x = (6c + 10d) + (3c + 2d)x \Rightarrow \begin{cases} 6c + 10d = 3 \\ 3c + 2d = 2 \end{cases}$$

Solving: $c = \frac{7}{9}, d = -\frac{1}{6}$, so $[q_2]_B = \begin{bmatrix} \frac{7}{9} \\ -\frac{1}{6} \end{bmatrix}$
Thus $P_{B'\to B} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$

(b) **Find $P_{B\to B'}$:** Use $P_{B\to B'} = (P_{B'\to B})^{-1}$

(c) **For $p = -4 + x$, find coordinates:**
Solve $-4 + x = \alpha(6 + 3x) + \beta(10 + 2x)$:

$$-4 + x = (6\alpha + 10\beta) + (3\alpha + 2\beta)x \Rightarrow \begin{cases} 6\alpha + 10\beta = -4 \\ 3\alpha + 2\beta = 1 \end{cases}$$

Solving: $\alpha = 1, \beta = -1$, so $[p]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then $[p]_{B'} = P_{B\to B'}[p]_B$

7. **Let $B_1 = \{(1, 2), (2, 3)\}$, $B_2 = \{(1, 3), (1, 4)\}$:**

(a) $P_{B_2\to B_1}$**:** Use efficient method:

$$[B_1 \mid B_2] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{bmatrix}$$

Row reduce to get $P_{B_2\to B_1}$

(b) $P_{B_1\to B_2}$**:** Inverse of previous matrix

(c) **Confirm inverses:** Multiply to get identity

(d) **For $w = (0, 1)$:** Find coordinates and transform

(e) **For $w = (2, 5)$:** Find coordinates and transform

8. **Let S be standard basis, $B = \{(2, 1), (-3, 4)\}$:**

(a) $P_{B\to S}$**:** $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ (Theorem 4.7.2)

(b) $P_{S\to B}$**:** Inverse: $\frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$

(c) **Confirm inverses**

(d) **For $w = (5, -3)$:** $[w]_B = P_{S\to B}[w]_S$

(e) **For $w = (3, -5)$:** $[w]_S = P_{B\to S}[w]_B$

9. **Similar to Exercise 8 in $\mathbb{R}^3$**

10. **Reflection about $y = x$:**

(a) $P_{B\to S}$**:** Reflection sends $(1, 0)$ to $(0, 1)$ and $(0, 1)$ to $(1, 0)$, so:

$$P_{B\to S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) **Show $P^T = P_{S\to B}$:** Since reflection is its own inverse, $P_{S\to B} = P_{B\to S} = P$, and $P^T = P$.

11. **Reflection about line at angle θ:**
    (a) $P_{B\to S}$**:** Rotation matrix
    (b) **Show** $P^T = P_{S\to B}$**:** Orthogonal matrix property

12. **Given** $P_{B_1\to B_2}$ **and** $P_{B_2\to B_3}$**, find** $P_{B_3\to B_1}$**:**

$$P_{B_3\to B_1} = P^{-1}_{B_1\to B_3} = (P_{B_2\to B_3}P_{B_1\to B_2})^{-1} = P^{-1}_{B_1\to B_2}P^{-1}_{B_2\to B_3}$$

13. **Transition matrix composition:**
    - $P_{B'\to C} = P_{B\to C}P_{B'\to B}$
    - $P_{C\to B'} = P_{B\to B'}P_{C\to B}$

14. **Effect of reversing basis order:** Changes columns of transition matrix

15. **Given P, find what basis it represents:**
    (a) Columns are basis vectors relative to standard basis
    (b) Inverse gives basis from standard to B

16. **Given P, find basis B:** Solve $P = [B \mid S]$ interpretation

17. **Linear transformation on standard basis:**
    (a) Apply $T(x_1, x_2) = (2x_1 + 3x_2, 5x_1 - x_2)$ to standard basis
    $T(1,0) = (2,5)$, $T(0,1) = (3,-1)$, so $P_{B\to S} = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$

18. **Similar to Exercise 17 in $\mathbb{R}^3$**

19. **If $[w]_B = w$ for all w, then B is standard basis**