Section 1.1: Introduction to Linear Systems

Lecture Notes and Problem Solutions

NUCES

Key Concepts

- **Linear Equation:** $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where a_i, b are constants.
- **System of Linear Equations:** A set of linear equations with the same variables.
  - **Solution:** Values for variables that satisfy all equations.
  - **Consistent:** Has at least one solution (unique or infinitely many).
  - **Inconsistent:** No solution exists.

Examples

Example 1 (Consistent System)

Solve:

$$\begin{cases} x + y = 4 \\ 2x - y = 5 \end{cases}$$

Solution:

$$\begin{aligned} \text{Add equations: } & (x+y) + (2x-y) = 4 + 5 \Rightarrow 3x = 9 \Rightarrow x = 3. \\ \text{Substitute } x = 3 \text{ into first equation: } & 3 + y = 4 \Rightarrow y = 1. \\ \textbf{Unique solution: } & (3, 1). \end{aligned}$$

Example 2 (Inconsistent System)

Solve:

$$\begin{cases} x + y = 3 \\ x + y = 5 \end{cases}$$

Solution: No solution exists (parallel lines).

Problems and Detailed Solutions

Part A: Identify Consistency (Problems 1–5)

Determine if each system is **consistent** or **inconsistent**. If consistent, state whether it has a **unique solution** or **infinitely many solutions**.

1. $\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$ **Step 1:** Add the two equations to eliminate y:

$$(2x - y) + (x + y) = 1 + 5 \Rightarrow 3x = 6 \Rightarrow x = 2.$$

Step 2: Substitute $x = 2$ into the second equation:

$$2 + y = 5 \Rightarrow y = 3.$$

Conclusion: Consistent with $\boxed{(2,3)}$ as the unique solution.

2. $\begin{cases} x + 2y = 4 \\ 3x + 6y = 12 \end{cases}$ **Step 1:** Divide the second equation by 3:

$$x + 2y = 4 \text{ (same as first equation)}.$$

Conclusion: Consistent with $\boxed{\text{infinitely many solutions}}$ (both equations represent the same line).

3. $\begin{cases} x - y = 0 \\ 2x - 2y = 3 \end{cases}$ **Step 1:** Multiply the first equation by 2:

$$2x - 2y = 0 \text{ vs. } 2x - 2y = 3.$$

Conclusion: Inconsistent ($\boxed{\text{no solution}}$; parallel lines).

4. $\begin{cases} 3x + y = 2 \\ 6x + 2y = 4 \end{cases}$ **Step 1:** Divide the second equation by 2:

$$3x + y = 2 \text{ (same as first equation)}.$$

Conclusion: Consistent with $\boxed{\text{infinitely many solutions}}$.

5. $\begin{cases} 4x - 2y = 6 \\ 6x - 3y = 9 \end{cases}$ **Step 1:** Divide the first equation by 2 and the second by 3:

$$2x - y = 3 \text{ (both equations simplify to this)}.$$

Conclusion: Consistent with $\boxed{\text{infinitely many solutions}}$.

Part B: Solve Systems Algebraically (Problems 6–10)

Solve using **substitution** or **elimination**.

6. $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$ **Step 1:** Add the equations:

$$(x + y) + (x - y) = 7 + 3 \Rightarrow 2x = 10 \Rightarrow x = 5.$$

Step 2: Substitute $x = 5$ into the first equation:

$$5 + y = 7 \Rightarrow y = 2.$$

Solution: $\boxed{(5,2)}$.

7. $\begin{cases} 2x + 3y = 8 \\ 4x - y = 6 \end{cases}$ **Step 1:** Multiply the second equation by 3:

$$12x - 3y = 18.$$

Step 2: Add to the first equation:

$$(2x + 3y) + (12x - 3y) = 8 + 18 \Rightarrow 14x = 26 \Rightarrow x = \frac{13}{7}.$$

Step 3: Substitute $x = \frac{13}{7}$ into the second original equation:

$$4\left(\frac{13}{7}\right) - y = 6 \Rightarrow y = \frac{52}{7} - 6 = \frac{10}{7}.$$

Solution: $\boxed{\left(\frac{13}{7}, \frac{10}{7}\right)}$.

Part C: Applications (Problems 11–12)

11. **Geometry:** Find the intersection of $3x + y = 5$ and $x - 2y = 4$. **Step 1:** Solve the first equation for y:

$$y = 5 - 3x.$$

Step 2: Substitute into the second equation:

$$x - 2(5 - 3x) = 4 \Rightarrow x - 10 + 6x = 4 \Rightarrow 7x = 14 \Rightarrow x = 2.$$

Step 3: Substitute $x = 2$ back into $y = 5 - 3x$:

$$y = 5 - 6 = -1.$$

Solution: $\boxed{(2, -1)}$.

12. **Mixture Problem:** A chemist mixes 10% and 30% acid solutions to make 20 L of 20% acid. How much of each is used? **Step 1:** Let x = volume of 10% solution, y = volume of 30% solution.

$$\begin{cases} x + y = 20 \quad \text{(total volume)} \\ 0.1x + 0.3y = 0.2 \times 20 \quad \text{(acid content)} \end{cases}$$

Step 2: Simplify the second equation:

$$0.1x + 0.3y = 4 \Rightarrow x + 3y = 40.$$

Step 3: Subtract the first equation ($x + y = 20$) from this:

$$(x + 3y) - (x + y) = 40 - 20 \Rightarrow 2y = 20 \Rightarrow y = 10.$$

Step 4: Substitute $y = 10$ into $x + y = 20$:

$$x = 10.$$

Solution: $\boxed{\text{10 L of 10\% solution and 10 L of 30\% solution}}$.

Part D: Augmented Matrices (Problems 13–15)

Convert systems to augmented matrices.

13. $\begin{cases} 2x - y + z = 3 \\ x + 2y - z = 1 \end{cases}$

$$\boxed{\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \end{array}\right]}$$

14. $\begin{cases} x + y = 0 \\ y - z = 5 \\ 2x + z = 3 \end{cases}$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 5 \\ 2 & 0 & 1 & 3 \end{array}\right]}$$

Part E: Word Problems (Problems 16–17)

16. **Economics:** A theater sells tickets (\$10 adult, \$6 child). Total sales are \$3200 for 400 tickets. How many of each were sold? **Step 1:** Let A = number of adult tickets, C = number of child tickets.

$$\begin{cases} A + C = 400 \\ 10A + 6C = 3200 \end{cases}$$

Step 2: Solve the first equation for A:

$$A = 400 - C.$$

Step 3: Substitute into the second equation:

$$10(400 - C) + 6C = 3200 \Rightarrow 4000 - 4C = 3200 \Rightarrow C = 200.$$

Step 4: Substitute $C = 200$ into $A = 400 - C$:

$$A = 200.$$

Solution: $\boxed{\text{200 adult tickets and 200 child tickets}}$.

Part F: True/False (Problems 18–20)

18. Every linear system has at least one solution. **False.** Inconsistent systems (e.g., parallel lines) have no solution.

19. A system with more equations than variables is always inconsistent. **False.** Example:

$$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \\ 3x + 3y = 6 \end{cases}$$

is consistent (infinitely many solutions).

20. The system $\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$ has infinitely many solutions. **True.** The second equation is a multiple of the first.