Lecture Notes: Section 1.8
Introduction to Linear Transformations

Mathematics Department

1 Foundational Concepts

1.1 Ordered n-Tuples and $\mathbb{R}^n$

A solution to a system of n linear equations can be expressed as an **ordered n-tuple**:

$$(s_1, s_2, \ldots, s_n)$$

For $n = 2$ and $n = 3$, these are called **ordered pairs** and **ordered triples**, respectively. Order matters: $(1, 2) \neq (2, 1)$.

The set of all ordered n-tuples of real numbers is denoted by $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are called **vectors**.

1.2 Column-Vector Form

Vectors can be represented in **column-vector form**, which is often more convenient for matrix operations:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

1.3 Standard Basis Vectors

For each $i = 1, 2, \ldots, n$, the **standard basis vector** $\mathbf{e}_i$ has a 1 in the ith position and 0 elsewhere:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

These vectors form a **basis** for $\mathbb{R}^n$, meaning any vector $\mathbf{x} \in \mathbb{R}^n$ can be written uniquely as a linear combination of them:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

2 Transformations

Definition 2.1 (Transformation)**.** A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted $T : \mathbb{R}^n \to \mathbb{R}^m$, is a rule that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a unique vector $T(\mathbf{x}) \in \mathbb{R}^m$.

- $\mathbb{R}^n$ is the **domain**.

- $\mathbb{R}^m$ is the **codomain**.
- The vector $T(\mathbf{x})$ is the **image** of $\mathbf{x}$ under T.
- The set of all images is the **range**.
- If $m = n$, T may be called an **operator** on $\mathbb{R}^n$.

3 Matrix Transformations

A system of linear equations,

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n, \end{aligned}$$

can be written in matrix form as $\mathbf{w} = A\mathbf{x}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

This defines a **matrix transformation** $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by:

$$T_A(\mathbf{x}) = A\mathbf{x}$$

We say T_A is **multiplication by** A.

Example 3.1. The transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned}$$

is a matrix transformation with standard matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}.$$

For $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}.$$

Example 3.2 (Zero Transformation)**.** If $\mathbf{0}$ is the $m \times n$ zero matrix, then $T_{\mathbf{0}}(\mathbf{x}) = \mathbf{0}\mathbf{x} = \mathbf{0}$ maps every vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

Example 3.3 (Identity Operator)**.** If I is the $n \times n$ identity matrix, then $T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$ maps every vector in $\mathbb{R}^n$ to itself.

3.1 Properties of Matrix Transformations

Theorem 3.1. For any matrix A and transformation $T_A(\mathbf{x}) = A\mathbf{x}$:

1. $T_A(\mathbf{0}) = \mathbf{0}$
2. $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (Homogeneity)
3. $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (Additivity)
4. $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

These properties imply that matrix transformations preserve linear combinations:

$$T_A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T_A(\mathbf{v}_1) + c_2T_A(\mathbf{v}_2) + \cdots + c_kT_A(\mathbf{v}_k)$$

4 Linearity and Standard Matrices

Theorem 4.1 (1.8.2)**.** A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **matrix transformation** (i.e., $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A) **if and only if** it satisfies the following **linearity conditions** for all vectors $\mathbf{u}, \mathbf{v}$ and scalar k:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (Additivity)
2. $T(k\mathbf{u}) = kT(\mathbf{u})$ (Homogeneity)

A transformation satisfying these conditions is called a **linear transformation**.

Theorem 4.2 (1.8.3)**.** Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation, and conversely. The terms *linear transformation* and *matrix transformation* are synonymous.

Theorem 4.3 (1.8.4)**.** If $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$, then $A = B$. There is a one-to-one correspondence between $m \times n$ matrices and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ is called the **standard matrix** for T.

4.1 Procedure for Finding the Standard Matrix

The standard matrix A for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by:

$$A = [\,T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)\,]$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{R}^n$.

Procedure:

1. Find the images of the standard basis vectors, $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$.
2. Construct the matrix A using these images as the columns.

Example 4.1 (Example 4)**.** Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}.$$

Compute:

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

Thus, the standard matrix is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}.$$

Example 4.2 (Example 5)**.** For the transformation in Example 4, compute $T\left(\begin{bmatrix}1\\4\end{bmatrix}\right)$ using the standard matrix.

$$T\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = A\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & -3\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}6\\-11\\3\end{bmatrix}.$$

Example 4.3 (Example 6)**.** Find the standard matrix for $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}3x_1 + x_2\\2x_1 - 4x_2\end{bmatrix} = \begin{bmatrix}3 & 1\\2 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

The standard matrix is $\begin{bmatrix}3 & 1\\2 & -4\end{bmatrix}$.

Example 4.4 (Example 7)**.** Find the standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ given that

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}-5\\5\end{bmatrix}, \quad T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}7\\-6\end{bmatrix}.$$

First, express the standard basis vectors as linear combinations of the given vectors. Solve:

$$\begin{bmatrix}1\\0\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\-1\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} = k_1\begin{bmatrix}-1\\1\end{bmatrix} + k_2\begin{bmatrix}2\\-1\end{bmatrix}.$$

The solutions are $c_1 = 1, c_2 = 1$ and $k_1 = 2, k_2 = 1$. Use linearity:

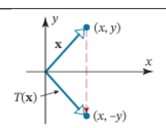
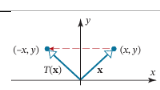
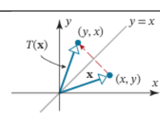
$$T(\mathbf{e}_1) = T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}-5\\5\end{bmatrix} + \begin{bmatrix}7\\-6\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix},$$
$$T(\mathbf{e}_2) = 2T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}-10\\10\end{bmatrix} + \begin{bmatrix}7\\-6\end{bmatrix} = \begin{bmatrix}-3\\4\end{bmatrix}.$$

Thus, the standard matrix is $A = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix}$.

5 Geometric Linear Operators

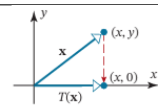
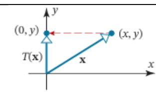
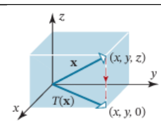
5.1 Reflection Operators

Reflections map a point to its mirror image across a line or plane through the origin.

Operator	**Illustration**	**Images of Basis Vectors**	**Standard Matrix**
Reflection about x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = (1, 0)$ $T(\mathbf{e}_2) = (0, -1)$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection about y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = (-1, 0)$ $T(\mathbf{e}_2) = (0, 1)$	$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection about $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = (0, 1)$ $T(\mathbf{e}_2) = (1, 0)$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

5.2 Projection Operators

Projections map a point onto a line or plane through the origin orthogonally.

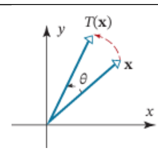
Operator	Illustration	Images of Basis Vectors	Standard Matrix
Projection onto x-axis $T(x,y)=(x,0)$		$T(\mathbf{e}_1)=(1,0)$ $T(\mathbf{e}_2)=(0,0)$	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto y-axis $T(x,y)=(0,y)$		$T(\mathbf{e}_1)=(0,0)$ $T(\mathbf{e}_2)=(0,1)$	$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$
Projection onto xy-plane $T(x,y,z)=(x,y,0)$		$T(\mathbf{e}_1)=(1,0,0)$ $T(\mathbf{e}_2)=(0,1,0)$ $T(\mathbf{e}_3)=(0,0,0)$	$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$

5.3 Rotation Operators

Rotations in $\mathbb{R}^2$ move points along circular arcs centered at the origin.

Theorem 5.1. The standard matrix for the counterclockwise rotation about the origin through an angle θ is

$$R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Operator	Illustration	Images of Basis Vectors	Standard Matrix
Counterclockwise Rotation by θ		$T(\mathbf{e}_1)=(\cos\theta,\sin\theta)$ $T(\mathbf{e}_2)=(-\sin\theta,\cos\theta)$	$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Example 5.1 (Example 8)**.** Find the image of $\mathbf{x}=(1,1)$ under a rotation of $\pi/6$ radians (30°).

$$R_{\pi/6} = \begin{bmatrix}\cos(\pi/6) & -\sin(\pi/6)\\ \sin(\pi/6) & \cos(\pi/6)\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$$

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}-1}{2}\\ \frac{1+\sqrt{3}}{2}\end{bmatrix} \approx \begin{bmatrix}0.37\\1.37\end{bmatrix}$$

So, $T(1,1)\approx(0.37,1.37)$.