7.2 Orthogonal Diagonalization

In this section we will be concerned with the problem of diagonalizing a symmetric matrix A. As we will see, this problem is closely related to that of finding an orthonormal basis for R^n that consists of eigenvectors of A. Problems of this type are important because many of the matrices that arise in applications are symmetric.

The Orthogonal Diagonalization Problem

In Section 5.2 we defined two square matrices, A and B, to be *similar* if there is an *invertible* matrix P such that $P^{-1}AP = B$. In this section we will be concerned with the special case in which it is possible to find an *orthogonal* matrix P for which this relationship holds.

We begin with the following definition.

Orthogonal Diagonalization:

Definition 1

If A and B are square matrices, then we say that B is ***orthogonally similar*** to A if there is an orthogonal matrix P such that $B = P^T AP$.

If A is orthogonally similar to some diagonal matrix, say

$$P^T AP = D$$

then we say A is ***orthogonally diagonalizable*** and P ***orthogonally diagonalizes*** A.

Theorem 7.2.1

If A is an $n \times n$ matrix with real entries, then the following are equivalent.

(a) A is orthogonally diagonalizable.

(b) A has an orthonormal set of n eigenvectors.

(c) A is symmetric.

Properties of Symmetric Matrices

Our next goal is to devise a procedure for orthogonally diagonalizing a symmetric matrix, but before we can do so, we need the following critical theorem about eigenvalues and eigenvectors of symmetric matrices.

Theorem 7.2.2

If A is a symmetric matrix with real entries, then:

(a) The eigenvalues of A are all real numbers.

(b) Eigenvectors from different eigenspaces are orthogonal.

Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step 3. Form the matrix P whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

EXAMPLE 1 | Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

In Exercises **1–6**, *find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces.*

1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Sol:

1. $$\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda$$

The characteristic equation is $\lambda^2 - 5\lambda = 0$ and the eigenvalues are $\lambda = 0$ and $\lambda = 5$. Both eigenspaces are one-dimensional.

2. $$\begin{vmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{vmatrix} = \lambda^3 - 27\lambda - 54 = (\lambda-6)(\lambda+3)^2$$

The characteristic equation is $\lambda^3 - 27\lambda - 54 = 0$ and the eigenvalues are $\lambda = 6$ and $\lambda = -3$. The eigenspace for $\lambda = 6$ is one-dimensional; the eigenspace for $\lambda = -3$ is two-dimensional.

3. $$\begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda-3)$$

The characteristic equation is $\lambda^3 - 3\lambda^2 = 0$ and the eigenvalues are $\lambda = 3$ and $\lambda = 0$. The eigenspace for $\lambda = 3$ is one-dimensional; the eigenspace for $\lambda = 0$ is two-dimensional.

4. $$\begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = (\lambda-8)(\lambda-2)^2$$

The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ and the eigenvalues are $\lambda = 8$ and $\lambda = 2$. The eigenspace for $\lambda = 8$ is one-dimensional; the eigenspace for $\lambda = 2$ is two-dimensional.

5. $$\begin{vmatrix} \lambda-4 & -4 & 0 & 0 \\ -4 & \lambda-4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 - 8\lambda^3 = \lambda^3(\lambda-8)$$

The characteristic equation is $\lambda^4 - 8\lambda^3 = 0$ and the eigenvalues are $\lambda = 0$ and $\lambda = 8$. The eigenspace for $\lambda = 0$ is three-dimensional; the eigenspace for $\lambda = 8$ is one-dimensional.

6. $$\begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-2 & 1 \\ 0 & 0 & 1 & \lambda-2 \end{vmatrix} = \lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9 = (\lambda-1)^2(\lambda-3)^2$$

The characteristic equation is $\lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9 = 0$ and the eigenvalues are $\lambda = 1$ and $\lambda = 3$. Both eigenspaces are two-dimensional.

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

10. $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

Sol:

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda-6 & -2\sqrt{3} \\ -2\sqrt{3} & \lambda-7 \end{vmatrix} = \lambda^2 - 13\lambda + 30 = (\lambda-3)(\lambda-10)$ therefore A has eigenvalues 3 and 10.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{2}{\sqrt{3}}t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ \sqrt{3} \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} \end{bmatrix}$ so that the eigenspace

corresponding to $\lambda_2 = 10$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{\sqrt{3}}{2}t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix}$. We have $P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$.

9. $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

Sol:

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 0 & 36 \\ 0 & \lambda+3 & 0 \\ 36 & 0 & \lambda+23 \end{vmatrix}$

$= (\lambda+3)\begin{vmatrix} \lambda+2 & 36 \\ 36 & \lambda+23 \end{vmatrix} = (\lambda-25)(\lambda+3)(\lambda+50)$ therefore A has eigenvalues 25, -3, and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this

eigenspace.

The reduced row echelon form of $-3I-A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_2 = -3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$, $x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this

eigenspace.

The reduced row echelon form of $-50I-A$ is $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3 = -50$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{3}{4}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ forms a basis for this

eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal. This yields the columns of a matrix P that orthogonally

diagonalizes A: $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$.

We have $P^{-1}AP = P^{T}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$.

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Sol:

11. $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2$ therefore A has eigenvalues 3 and 0.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A resulting in

$$P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sol: Q8 to 14

8. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ therefore A has eigenvalues 2 and 4.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. We have $P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 0 & 36 \\ 0 & \lambda + 3 & 0 \\ 36 & 0 & \lambda + 23 \end{vmatrix}$

$= (\lambda + 3)\begin{vmatrix} \lambda + 2 & 36 \\ 36 & \lambda + 23 \end{vmatrix} = (\lambda - 25)(\lambda + 3)(\lambda + 50)$ therefore A has eigenvalues 25, -3, and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I-A$ is $\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_2=-3$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=0$, $x_2=t$, $x_3=0$. A vector $\mathbf{p}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$ forms a basis for this

eigenspace.

The reduced row echelon form of $-50I-A$ is $\begin{bmatrix}1&0&-\frac{3}{4}\\0&1&0\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3=-50$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=\frac{3}{4}t$, $x_2=0$, $x_3=t$. A vector $\mathbf{p}_3=\begin{bmatrix}3\\0\\4\end{bmatrix}$ forms a basis for this

eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P=\begin{bmatrix}-\frac{4}{5}&0&\frac{3}{5}\\0&1&0\\\frac{3}{5}&0&\frac{4}{5}\end{bmatrix}$.

We have $P^{-1}AP=P^TAP=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}25&0&0\\0&-3&0\\0&0&-50\end{bmatrix}$.

10. $\det(\lambda I-A)=\begin{vmatrix}\lambda-6&2\\2&\lambda-3\end{vmatrix}=(\lambda-2)(\lambda-7)$ therefore A has eigenvalues 2 and 7.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1&-\frac{1}{2}\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=2$

consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=\frac{1}{2}t$, $x_2=t$. A vector $\mathbf{p}_1=\begin{bmatrix}1\\2\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $7I-A$ is $\begin{bmatrix}1&2\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2=7$ consists

of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=-2t$, $x_2=t$. A vector $\mathbf{p}_2=\begin{bmatrix}-2\\1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A: $P=\begin{bmatrix}\frac{1}{\sqrt5} & -\frac{2}{\sqrt5}\\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5}\end{bmatrix}$.

We have $P^{-1}AP=P^TAP=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix}$.

11. $\det(\lambda I-A)=\begin{vmatrix}\lambda-2 & 1 & 1\\ 1 & \lambda-2 & 1\\ 1 & 1 & \lambda-2\end{vmatrix}=\lambda^3-6\lambda^2+9\lambda=\lambda(\lambda-3)^2$ therefore A has eigenvalues 3 and 0.

The reduced row echelon form of $3I-A$ is $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=\lambda_2=3$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1=-s-t$, $x_2=s$, $x_3=t$. Vectors $\mathbf{p}_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ and $\mathbf{p}_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1=\mathbf{p}_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ and $\mathbf{v}_2=\mathbf{p}_2-\frac{\langle\mathbf{p}_2,\mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1=\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}=\begin{bmatrix}-\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\\ 0\end{bmatrix}$ and $\mathbf{q}_2=\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}=\begin{bmatrix}-\frac{1}{\sqrt6}\\ -\frac{1}{\sqrt6}\\ \frac{2}{\sqrt6}\end{bmatrix}$.

The reduced row echelon form of $0I-A$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3=0$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1=t$, $x_2=t$, $x_3=t$. A vector $\mathbf{p}_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A resulting in

$$P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

12. $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda-2)$ therefore A has eigenvalues 0 and 2.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = 0$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors since the three vectors are already orthogonal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$.

We have $P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix} = (\lambda+25)^2(\lambda-25)^2$ therefore A has eigenvalues -25 and 25.

The reduced row echelon form of $-25I - A$ is $\begin{bmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = \lambda_2 = -25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = -\frac{4}{3}s$, $x_2 = s$, $x_3 = -\frac{4}{3}t$, $x_4 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ and

$\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3 = \lambda_4 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = \frac{3}{4}s$, $x_2 = s$, $x_3 = \frac{3}{4}t$, $x_4 = t$. Vectors $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

form a basis for this eigenspace.

Applying the Gram-Schmidt process to the two bases $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3, \mathbf{p}_4\}$ amounts to simply normalizing the vectors since the four vectors are already orthogonal. This yields the columns of a matrix P that

orthogonally diagonalizes A: $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -25 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$.

14. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & 0 & 0 \\ -1 & \lambda-3 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda-2)(\lambda-4)$ therefore A has eigenvalues 0, 2, and 4.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = 0$, $x_2 = 0$, $x_3 = s$, $x_4 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and

$\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 2$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$, $x_3 = 0$, $x_4 = 0$. A vector $\mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this

eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_4 = 4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = 0$, $x_4 = 0$. A vector $\mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for

this eigenspace.

Applying the Gram-Schmidt process to the three bases $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3\}$, and $\{\mathbf{p}_4\}$ amounts to simply normalizing the vectors since the four vectors are already orthogonal. This yields the columns of a

Spectral Decomposition

If A is a symmetric matrix with real entries that is orthogonally diagonalized by

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

$$A = PDP^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= [\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_n\mathbf{u}_n]\begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

Multiplying out, we obtain the formula

$$\boxed{A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T} \qquad (7)$$

which is called a ***spectral decomposition of A***.*

EXAMPLE 2 | A Geometric Interpretation of a Spectral Decomposition

The matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(verify). Normalizing these basis vectors yields

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

so a spectral decomposition of A is

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T = (-3)\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} + (2)\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \tag{8}$$

where, as noted above, the 2×2 matrices on the right side of (8) are the standard matrices for the orthogonal projections onto the eigenspaces corresponding to the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$, respectively.

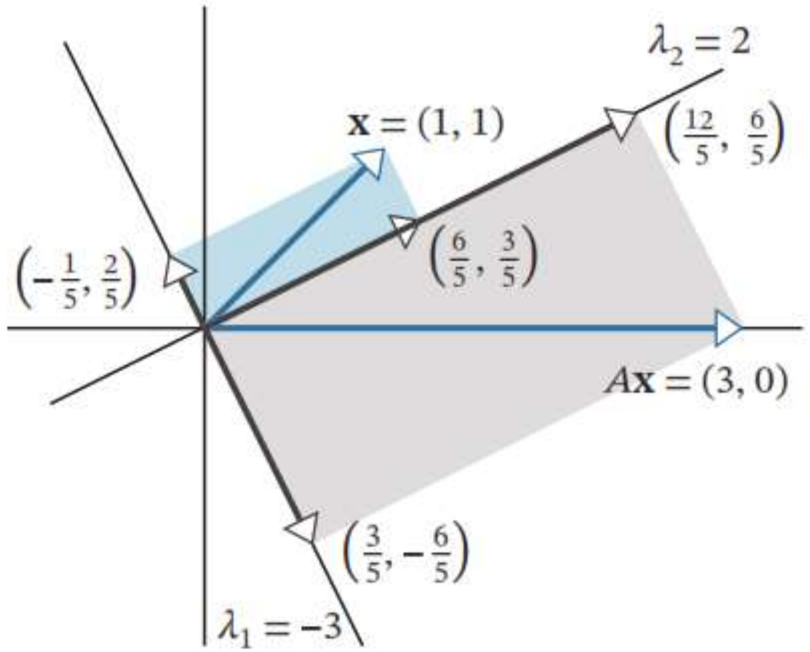
Now let us see what this spectral decomposition tells us about the image of the vector $\mathbf{x} = (1, 1)$ under multiplication by A. Writing $\mathbf{x}$ in column form, it follows that

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{9}$$

and from (8) that

$$\begin{aligned}
A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= (-3)\begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix} + (2)\begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}
\end{aligned} \tag{10}$$

Formulas (9) and (10) provide two different ways of viewing the image of the vector $(1, 1)$ under multiplication by A: Formula (9) tells us directly that the image of this vector is $(3, 0)$, whereas Formula (10) tells us that this image can also be obtained by projecting $(1, 1)$ onto the eigenspaces corresponding to $\lambda_1 = -3$ and $\lambda_2 = 2$ to obtain the vectors $\left(-\frac{1}{5}, \frac{2}{5}\right)$ and $\left(\frac{6}{5}, \frac{3}{5}\right)$, then scaling by the eigenvalues to obtain $\left(\frac{3}{5}, -\frac{6}{5}\right)$ and $\left(\frac{12}{5}, \frac{6}{5}\right)$, and then adding these vectors (see **Figure 7.2.1**).



$\lambda_2 = 2$
$\mathbf{x} = (1, 1)$
$\left(\frac{12}{5}, \frac{6}{5}\right)$
$\left(\frac{6}{5}, \frac{3}{5}\right)$
$\left(-\frac{1}{5}, \frac{2}{5}\right)$
$A\mathbf{x} = (3, 0)$
$\left(\frac{3}{5}, -\frac{6}{5}\right)$
$\lambda_1 = -3$

FIGURE 7.2.1

In Exercises 15–18, find the spectral decomposition of the matrix.

15. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

17. $\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

Sol: Q15 to 18

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-4)$ therefore A has eigenvalues 2 and 4.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = (2)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + (4)\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = (2)\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + (4)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

16. In the solution of Exercise 10, we have shown that $P=\begin{bmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$ orthogonally diagonalizes A:

$P^T AP=\begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix}$. Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix}6 & -2\\ -2 & 3\end{bmatrix}=(2)\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}+(7)\begin{bmatrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}=(2)\begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{4}{5}\end{bmatrix}+(7)\begin{bmatrix}\frac{4}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{1}{5}\end{bmatrix}.$$

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & -1 & -2 \\ -1 & \lambda+3 & -2 \\ -2 & -2 & \lambda \end{vmatrix} = (\lambda+4)^2(\lambda-2)$ therefore A has eigenvalues -4 and 2.

The reduced row echelon form of $-4I - A$ is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -4$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - 2t$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis

for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace:

$\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to

yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{2}t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ orthogonally diagonalizes A resulting in $P^T AP = D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} = (-4)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + (-4)\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= (-4)\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-4)\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

18. $\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 0 & 36 \\ 0 & \lambda+3 & 0 \\ 36 & 0 & \lambda+23 \end{vmatrix} = (\lambda-25)(\lambda+3)(\lambda+50)$ therefore A has eigenvalues 25, -3, and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 25$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$, $x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-50I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -50$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{3}{4}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal.

A matrix $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$ orthogonally diagonalizes A resulting in $P^T A P = D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} = (25)\begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + (-50)\begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$$= (25)\begin{bmatrix} \frac{16}{25} & 0 & -\frac{12}{25} \\ 0 & 0 & 0 \\ -\frac{12}{25} & 0 & \frac{9}{25} \end{bmatrix} + (-3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-50)\begin{bmatrix} \frac{9}{25} & 0 & \frac{12}{25} \\ 0 & 0 & 0 \\ \frac{12}{25} & 0 & \frac{16}{25} \end{bmatrix}.$$