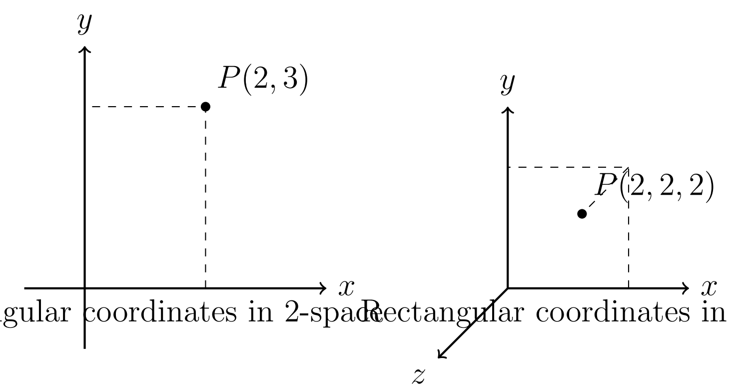
Section 4.5: Coordinates and Basis

Linear Algebra Lecture

Coordinate Systems in Linear Algebra

In analytic geometry, we use rectangular coordinate systems to establish one-to-one correspondences:

- Points in 2-space $\leftrightarrow$ Ordered pairs of real numbers
- Points in 3-space $\leftrightarrow$ Ordered triples of real numbers



Rectangular coordinates in 2-space Rectangular coordinates in 3-space

In linear algebra, we specify coordinate systems using vectors rather than coordinate axes. The key requirements are:

- **Directions** of basis vectors establish positive directions
- **Lengths** of basis vectors establish spacing between integer points
- Vectors must be **linearly independent**

Definition 0.1 (Basis for a Vector Space)**.** If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a **basis** for V if:

1. S **spans** V
2. S is **linearly independent**

Remark 0.1. Vector spaces fall into two categories:

- **Finite-dimensional**: There exists a finite set of vectors that spans V

- **Infinite-dimensional**: No finite set of vectors spans V

Example 0.1 (Standard Basis for $\mathbb{R}^n$)**.** The standard unit vectors:

$$e_1 = (1, 0, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad e_n = (0, 0, 0, \dots, 1)$$

form a basis for $\mathbb{R}^n$ called the **standard basis**. In particular:

- $\mathbb{R}^2$: $i = (1, 0)$, $j = (0, 1)$
- $\mathbb{R}^3$: $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

Example 0.2 (Standard Basis for P_n)**.** The set $S = \{1, x, x^2, \dots, x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less. This is called the **standard basis for** P_n.

Example 0.3 (Another Basis for $\mathbb{R}^3$)**.** Show that $v_1 = (1, 2, 1)$, $v_2 = (2, 9, 0)$, and $v_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

Solution: We must show these vectors are linearly independent and span $\mathbb{R}^3$. The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

Since $\det(A) = -1 \neq 0$, the vectors are linearly independent and span $\mathbb{R}^3$.

Example 0.4 (Standard Basis for M_{22})**.** The matrices:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for M_{22}. More generally, the mn different matrices with a single entry of 1 and zeros elsewhere form the standard basis for M_{mn}.

Example 0.5 (An Infinite-Dimensional Vector Space)**.** The vector space P_∞ of all polynomials with real coefficients is infinite-dimensional. If there were a finite spanning set $S = \{p_1, p_2, \dots, p_r\}$ with maximum degree n, then we couldn't express x^{n+1} as a linear combination of polynomials in S.

Theorem 0.1 (Uniqueness of Basis Representation)**.** If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ in exactly one way.

Remark 0.2 (Ordered Basis)**.** When the order of vectors in a basis matters, we call it an **ordered basis**. For example, $S = \{v_1, v_2, v_3\}$ and $S' = \{v_2, v_1, v_3\}$ are different ordered bases.

Definition 0.2 (Coordinates and Coordinate Vector)**.** If $S = \{v_1, v_2, \dots, v_n\}$ is an ordered basis for a vector space V, and

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

is the expression for a vector v in terms of the basis S, then:

- The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of v relative to the basis S**
- The vector $(c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ is called the **coordinate vector of v relative to S**
- Notation: $(v)_S = (c_1, c_2, \ldots, c_n)$ or $[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Example 0.6 (Coordinates Relative to Standard Basis for $\mathbb{R}^n$)**.** In the special case where $V = \mathbb{R}^n$ and S is the standard basis, the coordinate vector $(v)_S$ and the vector v are the same. For example, in $\mathbb{R}^3$:

$$v = (a, b, c) = ai + bj + ck \Rightarrow (v)_S = (a, b, c)$$

Example 0.7 (Coordinate Vectors Relative to Standard Bases)**.** (a) For $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ relative to standard basis for P_n: $(p)_S = (c_0, c_1, c_2, \ldots, c_n)$

(b) For $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to standard basis for M_{22}: $(B)_S = (a, b, c, d)$

Example 0.8 (Coordinates in $\mathbb{R}^3$)**.** Let $S = \{v_1, v_2, v_3\}$ where $v_1 = (1, 2, 1)$, $v_2 = (2, 9, 0)$, $v_3 = (3, 3, 4)$.

(a) Find $(v)_S$ for $v = (5, -1, 9)$

(b) Find v if $(v)_S = (-1, 3, 2)$

Solution:

(a) Solve $c_1v_1 + c_2v_2 + c_3v_3 = (5, -1, 9)$ to get $c_1 = 1$, $c_2 = -1$, $c_3 = 2$, so $(v)_S = (1, -1, 2)$

(b) $v = (-1)v_1 + 3v_2 + 2v_3 = (11, 31, 7)$

Exercise Solutions

1. **Show that $\{(2, 1), (3, 0)\}$ forms a basis for $\mathbb{R}^2$.**

   Solution: Let $v_1 = (2, 1)$, $v_2 = (3, 0)$. The coefficient matrix is:

   $$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad \det(A) = -3 \neq 0$$

   So the vectors are linearly independent. Since $\dim(\mathbb{R}^2) = 2$, they form a basis.

2. **Show that $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$ forms a basis for $\mathbb{R}^3$.**

   Solution: Let $v_1 = (3, 1, -4)$, $v_2 = (2, 5, 6)$, $v_3 = (1, 4, 8)$. The coefficient matrix is:

   $$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix}$$

$\det(A) = 3(40-24) - 2(8+16) + 1(6+20) = 48 - 48 + 26 = 26 \neq 0$ So the vectors are linearly independent and span $\mathbb{R}^3$.

3. **Show that $\{x^2+1, x^2-1, 2x-1\}$ forms a basis for P_2.**

   Solution: We need to show these polynomials are linearly independent and span P_2. Consider:

   $$c_1(x^2+1) + c_2(x^2-1) + c_3(2x-1) = 0$$

   This gives: $(c_1+c_2)x^2 + 2c_3x + (c_1 - c_2 - c_3) = 0$ So we have the system:

   $$\begin{aligned} c_1 + c_2 &= 0 \\ 2c_3 &= 0 \\ c_1 - c_2 - c_3 &= 0 \end{aligned}$$

   Solving gives $c_1 = c_2 = c_3 = 0$, so they're linearly independent. Since $\dim(P_2) = 3$, they form a basis.

4. **Show that $\{1+x, 1-x, 1-x^2, 1-x^3\}$ forms a basis for P_3.**

   Solution: $\dim(P_3) = 4$, so we need 4 linearly independent vectors. Write them in terms of standard basis:

   $$\begin{aligned} p_1 &= 1 + x = 1\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ p_2 &= 1 - x = 1\cdot 1 + (-1)\cdot x + 0\cdot x^2 + 0\cdot x^3 \\ p_3 &= 1 - x^2 = 1\cdot 1 + 0\cdot x + (-1)\cdot x^2 + 0\cdot x^3 \\ p_4 &= 1 - x^3 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + (-1)\cdot x^3 \end{aligned}$$

   The coordinate matrix is:

   $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   This matrix has rank 4, so the polynomials are linearly independent and form a basis.

5. **Show that the given matrices form a basis for M_{22}.**

   Solution: Let the matrices be:

   $$A_1 = \begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

   We need to show they're linearly independent. Consider:

   $$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives the system:

$$\begin{aligned}3c_1 + 0c_2 + 0c_3 + c_4 &= 0\\ 6c_1 - c_2 - 8c_3 + 0c_4 &= 0\\ -3c_1 - c_2 - 12c_3 - c_4 &= 0\\ -6c_1 + 0c_2 - 4c_3 + 2c_4 &= 0\end{aligned}$$

Solving shows only trivial solution, so they're linearly independent. Since $\dim(M_{22}) = 4$, they form a basis.

6. **Show that the given matrices form a basis for M_{22}.**

   Solution: Similar approach as Exercise 5.

7. **Show that the sets are not bases for $\mathbb{R}^3$:**

   (a) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

   (b) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

   Solution:

   (a) Check linear independence:

   $$\det \begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{bmatrix} = 2(1+7) - 4(-3+7) + 0 = 16 - 16 = 0$$

   So they're linearly dependent.

   (b) Check linear independence:

   $$\det \begin{bmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{bmatrix} = 1(20+2) - 2(30-8) - 1(-6-16) = 22 - 44 + 22 = 0$$

   So they're linearly dependent.

8. **Show that the vectors do not form a basis for P_2:** $1 - 3x + 2x^2$**,** $1 + x + 4x^2$**,** $1 - 7x$

   Solution: Check if they're linearly independent:

   $$c_1(1 - 3x + 2x^2) + c_2(1 + x + 4x^2) + c_3(1 - 7x) = 0$$

   This gives:

   $$\begin{aligned}c_1 + c_2 + c_3 &= 0\\ -3c_1 + c_2 - 7c_3 &= 0\\ 2c_1 + 4c_2 &= 0\end{aligned}$$

   Solving this system shows nontrivial solutions exist, so they're linearly dependent.

9. **Show that the matrices do not form a basis for M_{22}.**

   Solution: There are 4 matrices but $\dim(M_{22}) = 4$, so check if they're linearly independent. Setting up the linear combination equation and solving shows they're linearly dependent.

10. **Let V be spanned by $v_1 = \cos^2 x$, $v_2 = \sin^2 x$, $v_3 = \cos 2x$.**

    (a) Show $S = \{v_1, v_2, v_3\}$ is not a basis for V

    (b) Find a basis for V

    Solution:

    (a) Using trigonometric identity: $\cos 2x = \cos^2 x - \sin^2 x = v_1 - v_2$ So v_3 is a linear combination of v_1 and v_2, making S linearly dependent.

    (b) A basis for V is $\{\cos^2 x, \sin^2 x\}$ or $\{\cos^2 x, \cos 2x\}$, etc.

11. **Find coordinate vector of w relative to basis $S = \{u_1, u_2\}$ for $\mathbb{R}^2$:**

    (a) $u_1 = (2, -4)$, $u_2 = (3, 8)$; $w = (1, 1)$

    (b) $u_1 = (1, 1)$, $u_2 = (0, 2)$; $w = (a, b)$

    Solution:

    (a) Solve $c_1(2, -4) + c_2(3, 8) = (1, 1)$ System: $2c_1 + 3c_2 = 1$, $-4c_1 + 8c_2 = 1$ Solving: $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, so $(w)_S = (\frac{5}{28}, \frac{3}{14})$

    (b) Solve $c_1(1, 1) + c_2(0, 2) = (a, b)$ System: $c_1 = a$, $c_1 + 2c_2 = b \Rightarrow c_2 = \frac{b-a}{2}$ So $(w)_S = (a, \frac{b-a}{2})$