Section 1.6: More on Linear Systems and Invertible Matrices

1 Theorems and Examples

Theorem 1.6.1

A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Theorem 1.6.2

If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix b, the system of equations $Ax = b$ has exactly one solution, namely, $x = A^{-1}b$.

Example 1: Solution Using A^{-1}

Consider the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form: $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

From previous section:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Solution:

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Example 2: Solving Multiple Systems

Solve simultaneously:

(a) $x_1 + 2x_2 + 3x_3 = 4$, $2x_1 + 5x_2 + 3x_3 = 5$, $x_1 + 8x_3 = 9$

(b) $x_1 + 2x_2 + 3x_3 = 1$, $2x_1 + 5x_2 + 3x_3 = 6$, $x_1 + 8x_3 = -6$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Solutions:

- System (a): $x_1 = 1, x_2 = 0, x_3 = 1$
- System (b): $x_1 = 2, x_2 = 1, x_3 = -1$

2 Exercise Solutions 1-8

Exercise 1

$x_1 + x_2 = 2$, $5x_1 + 6x_2 = 9$

Coefficient matrix: $A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$

$A^{-1} = \frac{1}{1} \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$

Solution: $x = A^{-1}b = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Exercise 2

$4x_1 - 3x_2 = -3$, $2x_1 - 5x_2 = 9$

$A = \begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix}$, $A^{-1} = \frac{1}{-14} \begin{bmatrix} -5 & 3 \\ -2 & 4 \end{bmatrix}$

$x = A^{-1}b = \frac{1}{-14} \begin{bmatrix} -5 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

Exercise 3

System:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

Find A^{-1} using inversion algorithm, then $x = A^{-1}b$

Exercise 4

System:

$$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$$

Solve using matrix inversion method.

Exercise 5

System:

$$\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$$

Use $x = A^{-1}b$ approach.

Exercise 6

System:

$$\begin{aligned} -w - 2y - 3z &= 0 \\ w + x + 4y + 4z &= 7 \\ w + 3x + 7y + 9z &= 4 \\ -w - 2x - 4y - 6z &= 6 \end{aligned}$$

4×4 system, solve using matrix inversion.

Exercise 7

$3x_1 + 5x_2 = b_1$, $x_1 + 2x_2 = b_2$

$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

Solution: $x = A^{-1}b = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Exercise 8

System:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 5x_3 &= b_2 \\ 3x_1 + 5x_2 + 8x_3 &= b_3 \end{aligned}$$

Find A^{-1} and express solution in terms of b_1, b_2, b_3.

3 Consistency Conditions

Example 3

Determine conditions for consistency:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

Consistency condition: $b_3 - b_1 - b_2 = 0$ or $b_3 = b_1 + b_2$

Example 4

System:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$$

Reduced form:

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix}$$

No restrictions on b_1, b_2, b_3. Unique solution:

$$\begin{aligned} x_1 &= -40b_1 + 16b_2 + 9b_3 \\ x_2 &= 13b_1 - 5b_2 - 3b_3 \\ x_3 &= 5b_1 - 2b_2 - b_3 \end{aligned}$$

4 Exercise Solutions 9-12

Exercise 9

System: $x_1 - 5x_2 = b_1$, $3x_1 + 2x_2 = b_2$

Augmented matrix for both cases:

$$\begin{bmatrix} 1 & -5 & b_1 \\ 3 & 2 & b_2 \end{bmatrix}$$

(i) $b_1 = 1, b_2 = 4$: Solve to get $x_1 = 2, x_2 = 0.2$
(ii) $b_1 = -2, b_2 = 5$: Solve to get $x_1 = 1, x_2 = 0.6$

Exercise 10

System:

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= b_1 \\ x_1 + 9x_2 - 2x_3 &= b_2 \\ 6x_1 + 4x_2 - 8x_3 &= b_3 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} -1 & 4 & 1 & b_1 \\ 1 & 9 & -2 & b_2 \\ 6 & 4 & -8 & b_3 \end{bmatrix}$$

Solve for both cases using elimination.

Exercise 11

$4x_1 - 7x_2 = b_1$, $x_1 + 2x_2 = b_2$

Augmented matrix:

$$\begin{bmatrix} 4 & -7 & b_1 \\ 1 & 2 & b_2 \end{bmatrix}$$

Solve for all four cases:

- (i) $b_1 = 0, b_2 = 1$
- (ii) $b_1 = -4, b_2 = 6$
- (iii) $b_1 = -1, b_2 = 3$
- (iv) $b_1 = -5, b_2 = 1$

Exercise 12

System:

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 3 & 5 & b_1 \\ -1 & -2 & 0 & b_2 \\ 2 & 5 & 4 & b_3 \end{bmatrix}$$

Solve for all three cases using elimination.

5 Exercise Solutions 13-17: Consistency Conditions

Exercise 13

Determine conditions on b_1, b_2 for consistency of:

$$\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 7 & b_2 + 2b_1 \end{bmatrix} \quad (R_2 \leftarrow R_2 + 2R_1)$$

No row of zeros appears, so the system is consistent for all b_1, b_2. No conditions needed.

Exercise 14

Determine conditions on b_1, b_2 for consistency of:

$$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & b_2 \\ 6 & -4 & b_1 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & b_2 \\ 0 & 0 & b_1 - 2b_2 \end{bmatrix} \quad (R_2 \leftarrow R_2 - 2R_1)$$

For consistency: $b_1 - 2b_2 = 0$ or $b_1 = 2b_2$

Exercise 15

Determine conditions on b_1, b_2, b_3 for consistency of:

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & -3 & 12 & b_3 + 3b_1 \end{bmatrix} \quad \begin{matrix} (R_2 \leftarrow R_2 - 4R_1) \\ (R_3 \leftarrow R_3 + 3R_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{bmatrix} \quad (R_3 \leftarrow R_3 + R_2)$$

For consistency: $b_3 + b_2 - b_1 = 0$ or $b_3 = b_1 - b_2$

Exercise 16

Determine conditions on b_1, b_2, b_3 for consistency of:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2 + 4b_1 \\ 0 & -1 & 0 & b_3 + 4b_1 \end{bmatrix} \quad \begin{matrix} (R_2 \leftarrow R_2 + 4R_1) \\ (R_3 \leftarrow R_3 + 4R_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & 0 & b_3 + 4b_1 \\ 0 & -3 & -2 & b_2 + 4b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -b_3 - 4b_1 \\ 0 & 0 & -2 & b_2 + 4b_1 + 3b_3 + 12b_1 \end{bmatrix} \quad (R_3 \leftarrow R_3 - 3R_2)$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -b_3 - 4b_1 \\ 0 & 0 & 1 & -\frac{1}{2}(b_2 + 3b_3 + 16b_1) \end{bmatrix}$$

No row of zeros appears, so the system is consistent for all b_1, b_2, b_3. No conditions needed.

Exercise 17

Determine conditions on b_1, b_2, b_3, b_4 for consistency of:

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{bmatrix}$$

Perform row operations:

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & b_2 + 2b_1 \\ 0 & -1 & 11 & 5 & b_3 + 3b_1 \\ 0 & 1 & -11 & -5 & b_4 - 4b_1 \end{bmatrix} \quad \begin{matrix} (R_2 \leftarrow R_2 + 2R_1) \\ (R_3 \leftarrow R_3 + 3R_1) \\ (R_4 \leftarrow R_4 - 4R_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 + b_1 \\ 0 & 0 & 0 & 0 & b_4 + b_2 - 2b_1 \end{bmatrix} \quad \begin{matrix} (R_3 \leftarrow R_3 - R_2) \\ (R_4 \leftarrow R_4 + R_2) \end{matrix}$$

For consistency, both conditions must be satisfied:

$$\begin{aligned} b_3 - b_2 + b_1 = 0 \quad &\text{or} \quad b_3 = b_2 - b_1 \\ b_4 + b_2 - 2b_1 = 0 \quad &\text{or} \quad b_4 = 2b_1 - b_2 \end{aligned}$$