



# Orthogonal Matrices

## Definition 1

A square matrix  $A$  is said to be **orthogonal** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I \quad (1)$$

A matrix transformation  $T_A: R^n \rightarrow R^n$  is said to be an **orthogonal transformation** or an **orthogonal operator** if  $A$  is an orthogonal matrix.

### EXAMPLE 1 | A $3 \times 3$ Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## **EXAMPLE 2** | Rotation and Reflection Matrices Are Orthogonal

Recall from Table 5 of Section 1.8 that the standard matrix for the counterclockwise rotation about the origin of  $R^2$  through an angle  $\theta$  is

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This matrix is orthogonal for all choices of  $\theta$  since

$$A^T A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We leave it for you to verify that the reflection matrices in Tables 1 and 2 of Section 1.8 are all orthogonal.

### Theorem 7.1.1

The following are equivalent for an  $n \times n$  matrix  $A$ .

- (a)  $A$  is orthogonal.
- (b) The row vectors of  $A$  form an orthonormal set in  $R^n$  with the Euclidean inner product.
- (c) The column vectors of  $A$  form an orthonormal set in  $R^n$  with the Euclidean inner product.

**Warning** Note that an orthogonal matrix has *orthonormal* rows and columns—not simply orthogonal rows and columns.

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \mathbf{c}_1^T & \mathbf{r}_1 \mathbf{c}_2^T & \cdots & \mathbf{r}_1 \mathbf{c}_n^T \\ \mathbf{r}_2 \mathbf{c}_1^T & \mathbf{r}_2 \mathbf{c}_2^T & \cdots & \mathbf{r}_2 \mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \mathbf{c}_1^T & \mathbf{r}_n \mathbf{c}_2^T & \cdots & \mathbf{r}_n \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_n \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \cdot \mathbf{r}_1 & \mathbf{r}_n \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{r}_n \end{bmatrix}$$

It is evident from this formula that  $AA^T = I$  if and only if

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}_2 \cdot \mathbf{r}_2 = \cdots = \mathbf{r}_n \cdot \mathbf{r}_n = 1$$

and

$$\mathbf{r}_i \cdot \mathbf{r}_j = 0 \quad \text{when } i \neq j$$

which are true if and only if  $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$  is an orthonormal set in  $R^n$ . ■



### Theorem 7.1.2

- (a) The transpose of an orthogonal matrix is orthogonal.
- (b) The inverse of an orthogonal matrix is orthogonal.
- (c) A product of orthogonal matrices is orthogonal.
- (d) If  $A$  is orthogonal, then  $\det(A) = 1$  or  $\det(A) = -1$ .

### EXAMPLE 3 | $\det(A) = \pm 1$ for an Orthogonal Matrix $A$

The matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal since its row (and column) vectors form orthonormal sets in  $\mathbb{R}^2$  with the Euclidean inner product. We leave it for you to verify that  $\det(A) = 1$  and that interchanging the rows produces an orthogonal matrix whose determinant is  $-1$ .



In each part of Exercises **1–4**, determine whether the matrix is orthogonal, and if so find its inverse.

**1. a.**  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

**b.**  $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

**2. a.**  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**b.**  $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

Sol:

1. (a)  $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$  and  $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$  therefore  $A$  is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b)  $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$  and  $A^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$  therefore

$$A \text{ is an orthogonal matrix; } A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

2. (a)  $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  and  $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  therefore  $A$  is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)  $AA^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{bmatrix} \neq I$  therefore  $A$  is not an orthogonal matrix.

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3. a. 
$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

b. 
$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

4. a. 
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

b. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$



Sol:

3. (a)  $\|\mathbf{r}_1\| = \sqrt{0^2 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1$  so the matrix is not orthogonal.

(b)  $AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$  and  $A^T A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$

therefore  $A$  is an orthogonal matrix;  $A^{-1} = A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4. (a)  $AA^T = A^T A = I$  therefore  $A$  is an orthogonal matrix;  $A^{-1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

(b)  $\|\mathbf{r}_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{7}{12}} \neq 1$ . The matrix is not orthogonal.

In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating  $A^T A$ , then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1.

$$5. \quad A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \quad 6. \quad A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

### Theorem 7.1.1

The following are equivalent for an  $n \times n$  matrix  $A$ .

- (a)  $A$  is orthogonal.
- (b) The row vectors of  $A$  form an orthonormal set in  $R^n$  with the Euclidean inner product.
- (c) The column vectors of  $A$  form an orthonormal set in  $R^n$  with the Euclidean inner product.

Sol:

$$5. \quad A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} = I;$$

row vectors of  $A$ ,  $\mathbf{r}_1 = [\frac{4}{5} \ 0 \ -\frac{3}{5}]$ ,  $\mathbf{r}_2 = [-\frac{9}{25} \ \frac{4}{5} \ -\frac{12}{25}]$ ,  $\mathbf{r}_3 = [\frac{12}{25} \ \frac{3}{5} \ \frac{16}{25}]$ , form an orthonormal set since  $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$  and  $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$ ;

column vectors of  $A$ ,  $\mathbf{c}_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{9}{25} \\ \frac{12}{25} \end{bmatrix}$ ,  $\mathbf{c}_2 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$ ,  $\mathbf{c}_3 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}$ , form an orthonormal set since  $\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$  and  $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$ .

$$6. \quad A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = I$$

row vectors of  $A$ ,  $\mathbf{r}_1 = [\frac{1}{3} \ \frac{2}{3} \ \frac{2}{3}]$ ,  $\mathbf{r}_2 = [\frac{2}{3} \ -\frac{2}{3} \ \frac{1}{3}]$ ,  $\mathbf{r}_3 = [-\frac{2}{3} \ -\frac{1}{3} \ \frac{2}{3}]$ , form an orthonormal set since  $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$  and  $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$ ;

column vectors of  $A$ ,  $\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$ ,  $\mathbf{c}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$ ,  $\mathbf{c}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ , form an orthonormal set since  $\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$  and  $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$ .