Assignment 1

1. $x - 4y = 1$, $2x - y = -3$, $-x - 3y = 4$

$$[A|b] = \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right]$$

1) $R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ -1 & -3 & 4 \end{array}\right]$$

2) $R_3 + R_1$

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right]$$

3) $R_3 + R_2$

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right]$$

4) $R_2/7$

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{array}\right]$$

5) $R_1 + 4R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & -13/7 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{array}\right]$$

- Point of Intersection:-

$x = -13/7$, $y = -5/7$

2. $\left[\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array}\right]$

1) $R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{array}\right]$$

2) $R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{array}\right]$$

* For consistent system, $k+2g+h=0$.

3. $w + x + y + z = 6$

$w + y + z = 4$

$w + y = 2$

a) Leading variables :- w, x, z

b) Free variable :- y

c) $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{array}\right]$$

1) $R_2 - R_1$, $R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -4 \end{array}\right]$$

2) $R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

3) $-R_2$, $-R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

4) $R_1 - R_2$, $R_1 - R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$x = 2$, $z = 2$, $y = t$, $w = 2 - t$

d)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ -2 & 0 & -2 & 0 & -3 \end{array}\right]$$

1) $R_2 - R_1$, $R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -4 \\ -2 & 0 & -2 & 0 & -3 \end{array}\right]$$

2) $R_4 + 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -4 \\ 0 & 2 & 0 & 2 & 9 \end{array}\right]$$

3) $R_1 + R_2$, $R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 2 & 9 \end{array}\right]$$

4) $-R_2, -R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 9 \end{array}\right]$$

5) $R_4 - 2R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right]$$

6) $R_1 - R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 5 \end{array}\right]$$

7) $R_4 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$0w + 0x + 0y + 0z = 1$$

System is Inconsistent.

4.

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 6 & 7 \\ 1 & 2 & 2 & 11 & 10 \\ 2 & 3 & -4 & 3 & 3 \end{array}\right]$$

a) 1) $R_2 - R_1$, $R_3 - 2R_1$

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 1 & -6 & -9 & -11 \end{array}\right]$$

2) $R_3 - R_2$

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 0 & -7 & -14 & -14 \end{array}\right]$$

3) $-R_3/7$

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 6 & 7 \\ 0 & 1 & 1 & 5 & 3 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

4) $R_2 - R_3$, $R_1 - R_3$,

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 4 & 5 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

5) $R_1 - R_2$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

- For b_1 :- $x_1 = 1$, $x_2 = 3$, $x_3 = 2$
- For b_2 :- $x_1 = 4$, $x_2 = 1$, $x_3 = 2$

b)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 0 & 1 \end{array}\right]$$

1) $R_2 - R_1$, $R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -6 & -2 & 0 & 1 \end{array}\right]$$

2) $R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -7 & -1 & -1 & 1 \end{array}\right]$$

3) $-R_3/7$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/7 & 1/7 & -1/7 \end{array}\right]$$

4) $R_2 - R_3$, $R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1/7 & 1/7 \\ 0 & 1 & 0 & -8/7 & 6/7 & 1/7 \\ 0 & 0 & 1 & 1/7 & 1/7 & -1/7 \end{array}\right]$$

5) $R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -8/7 & 6/7 & 1/7 \\ 0 & 0 & 1 & 1/7 & 1/7 & -1/7 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -8/7 & 6/7 & 1/7 \\ 1/7 & 1/7 & -1/7 \end{bmatrix}$$

1) $AX = b_1$

$X = A^{-1} b_1$

$$X = \begin{bmatrix} 2 & -1 & 0 \\ -8/7 & 6/7 & 1/7 \\ 1/7 & 1/7 & -1/7 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \\ 3 \end{bmatrix}$$

$$X = \begin{array}{r} \\ (2,-1,0) \\ (-8/7, 6/7, 1/7) \\ (1/7, 1/7, -1/7) \end{array} \begin{array}{c} (6,11,3) \\ \begin{bmatrix} 12 - 11 + 0 \\ -\frac{48}{7} + \frac{66}{7} + \frac{3}{7} \\ \frac{6}{7} + \frac{11}{7} - \frac{3}{7} \end{bmatrix} \end{array} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

2) $AX = b_2$

$X = A^{-1} b_2$

$$X = \begin{bmatrix} 2 & -1 & 0 \\ -8/7 & 6/7 & 1/7 \\ 1/7 & 1/7 & -1/7 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 3 \end{bmatrix}$$

$$X = \begin{array}{r} \\ (2,-1,0) \\ (-8/7, 6/7, 1/7) \\ (1/7, 1/7, -1/7) \end{array} \begin{array}{c} (7,10,3) \\ \begin{bmatrix} 14 - 10 + 0 \\ -8 + 60/7 + 3/7 \\ 1 + 10/7 - 3/7 \end{bmatrix} \end{array} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

5.

$$C = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{bmatrix}$$

a) $|C| = a\begin{vmatrix} a & 1 \\ 0 & a \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix} + 0\begin{vmatrix} 0 & a \\ 1 & 0 \end{vmatrix}$

$|C| = a(a^2-0) - 1(0-1) + 0(0-a)$

$|C| = a^3 + 1$

b) $|C| \neq 0$

$a^3+1 \neq 0$

$a \neq -1$ (C is invertible)

c) • If $a \neq -1$, the system will have Unique solution.

• If $a = -1$, the system will either has no solution or infinitely many solutions depending on the values of constants.

6. a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

b) This is a shear transformation (specifically a horizontal shear). The y-coordinate remains unchanged while the x-coordinate is shifted by an amount proportional to the y-coordinate. The matrix has the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ which is the standard form for a horizontal shear transformation.

c) $T(0,0) = (0+2,0,0) = (0,0)$
$T(0,1) = (0+2\cdot1, 1) = (2,1)$
$T(1,0) = (1+2\cdot0, 0) = (1,0)$
$T(1,1) = (1+2\cdot1, 1) = (3,1)$

- After one application, square becomes parallelogram
- After many applications, it will extend infinitely to right.

7.
$$ax + by - 3z = -3$$
$$-2x - by + cz = -1$$
$$ax + 3y - cz = -3$$

$x = 1, y = -1, z = 2$
$a = b = c = ?$

$$a(1) + b(-1) - 3(2) = -3$$
$$a - b = 3 \quad \text{— } 1$$

$$-2(1) - b(-1) + c(2) = -1$$
$$b + 2c = 1 \quad \text{— } 2$$

$$a(1) + 3(-1) - c(2) = -3$$
$$a - 2c = 0 \quad \text{— } 3$$

$a = 2$, $b = -1$, $c = 1$

8. A linear system is guaranteed to be consistent if its coefficient matrix has a pivot position in every row. This is because the presence of a pivot in each row ensures that when the augmented matrix is reduced to row echelon form, no row assumes the form $[0\ 0 \dots 0 | b]$

where $b \neq 0$, which would indicate an equation like $0 = b$ and thus consistency. Instead, each row contributes meaningfully to the system, allowing for either a unique solution or infinitely many solutions.

9. a) i) $A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

1) $R_1/5$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

2) $R_2 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 6/5 & 4/5 & -3/5 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

3) $5R_2/6$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

4) $R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 0 & 1/3 & 1/2 & -5/6 & 1 \end{array}\right]$$

5)

6)

7)

8)

5) $3R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

6) $R_2 - 2R_3/3$

$$\left[\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

7) $R_1 - 3R_2/5$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 11/10 & -3/2 & 6/5 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

8) $R_1 - 2R_3/5$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}$$

$AA^{-1} = I_3$ (Verified)

ii) $B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

$|B| = 0$

Since the determinant is 0, inverse is not possible.

b) i) $AX = 0$

$X = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ii) $BX = 0$

$\left[\begin{array}{ccc|c} -1 & 3 & -4 & 0 \\ 2 & 4 & 1 & 0 \\ -4 & 2 & -9 & 0 \end{array}\right]$

1) $R_2 + 2R_1$, $R_3 - 4R_1$

$\left[\begin{array}{ccc|c} -1 & 3 & -4 & 0 \\ 0 & 10 & -7 & 0 \\ 0 & -10 & 7 & 0 \end{array}\right]$

2) $-R_1$, $R_2/10$

$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -7/10 & 0 \\ 0 & -10 & 7 & 0 \end{array}\right]$

3)

4) R

L

y

c) i)

3) $R_3 + 10R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -7/10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4) $R_1 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/10 & 0 \\ 0 & 1 & -7/10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = t$

$y = \dfrac{7t}{10}$, $x = -\dfrac{19t}{10}$

$$X = \begin{bmatrix} -19t/10 \\ 7t/10 \\ t \end{bmatrix}$$

c) i) $Ax = b$

$$X = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 2-1+0 \\ -6+5-10 \\ 6-5+15 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

ii) $Bx = b$

$$\left[\begin{array}{ccc|c} -1 & 3 & -4 & 4 \\ 2 & 4 & 1 & 2 \\ -4 & 2 & -9 & 5 \end{array}\right]$$

1) $R_2 + 2R_1$, $R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} -1 & 3 & -4 & 4 \\ 0 & 10 & -7 & 10 \\ 0 & -10 & 7 & -11 \end{array}\right]$$

2) $R_3 + R_2$

$$\left[\begin{array}{ccc|c} -1 & 3 & -4 & 4 \\ 0 & 10 & -7 & 10 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

According to $0 = -1$, the system is inconsistent and has no solution.

11. a) $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$

b) $T(U) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

c) $AX = b$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]$$

1) $R_2 -$

2) $R_2/14$

3) $R_3 - 4$

4) $R_1 + 3$

$X =$

d) From th

solution.

e)

$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

1) $R_2 - 3R_1$, $R_3 + R_1$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right]$$

2) $R_2/14$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 4 & -2 \end{array}\right]$$

3) $R_3 - 4R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$$

4) $R_1 + 3R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$X = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

d) From the row reduction, we have a unique solution. Therefore, there is only one such x.

e)

$$AX = C$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right]$$

1) $R_2 - 3R_1$, $R_3 + R_1$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right]$$

2) $R_2/14$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 4 & 8 \end{array}\right]$$

3) $R_3 - 4R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 10 \end{array}\right]$$

The last row gives $0 = 10$ which shows that the system is inconsistent and c is not in the range of T.

12. $A = [T(e_1) \quad T(e_2)]$

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

13. a)

Node	Flow In	Flow Out
A	x_1	$x_3 + x_4 + 40$
B	200	$x_1 + x_2$
C	$x_2 + x_3$	$x_5 + 100$
D	$x_4 + x_5$	60

$$x_1 - x_3 - x_4 = 40$$
$$x_1 + x_2 = 200$$
$$x_2 + x_3 - x_5 = 100$$
$$x_4 + x_5 = 60$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

1) $R_2 - R_1$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

2) $R_3 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

3) $-R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right]$$

4) $R_4 - R_3$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_5 = t$, $x_3 = s$

$x_4 = 60 - t$

$x_2 = 160 - x_4 - x_3 \Rightarrow 100 + t - s$

$x_1 = 40 + x_3 + x_4 \Rightarrow 100 - t + s$

b) $x_5 = 60$

$x_2 = 160 - x_3 \Rightarrow 160 - s$

$x_1 = 40 + x_3 \Rightarrow 40 + s$

c) To minimize x_1, minimize x_3. The smallest allowed x_3 is 0. $x_1 = 40$

14.

Node	Flow In	Flow Out
A	x_1	$x_2 + 100$
B	$x_2 + 50$	x_3
C	x_3	$x_4 + 120$
D	$x_4 + 150$	x_5
E	x_5	$x_6 + 80$
F	$x_6 + 100$	x_1

$x_1 - x_2 = 100$

$x_3 - x_2 = 50$

$x_3 - x_4 = 120$

$x_5 - x_4 = 150$

$x_5 - x_6 = 80$

$x_1 - x_6 = 100$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & -1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 1 & 0 & 0 & 0 & 0 & -1 & 100 \end{array}\right]$$

1) $R_6 - R_1$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & -1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

2) $R_6 + R_2$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & -1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \end{array}\right]$$

3) $R_1 - R_2$

$$\left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \end{array}\right]$$

4) $-R_2$

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right.$$

5) $R_1 + R_3$,

6) $-R_4$

7) $R_1 + R_4$,

4) $-R_2$

$$\left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \end{array}\right]$$

5) $R_1 + R_3$, $R_2 + R_3$, $R_6 - R_3$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & 1 & 0 & -1 & 0 & 0 & 70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \end{array}\right]$$

6) $-R_4$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & 1 & 0 & -1 & 0 & 0 & 70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \end{array}\right]$$

7) $R_1 + R_4$, $R_2 + R_4$, $R_3 + R_4$, $R_6 - R_4$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 20 \\ 0 & 1 & 0 & 0 & -1 & 0 & -80 \\ 0 & 0 & 1 & 0 & -1 & 0 & -30 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{array}\right]$$

8) $R_1 + R_5,\ R_2 + R_5,\ R_3 + R_5,\ R_4 + R_5,\ R_6 - R_5$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_6 = t \Rightarrow t \geq 0$

$x_5 = 80 + t \Rightarrow t \geq -80$

$x_4 = -70 + t \Rightarrow \boxed{t \geq 70}$

$x_3 = 50 + t \Rightarrow t \geq -50$

$x_2 = t \Rightarrow t \geq 0$

$x_1 = 100 + t \Rightarrow t \geq -100$

The smallest possible value for x_6 is 70.

15.
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

1) $R_3 - aR_2$
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & b(b-a) & c(c-a) \end{vmatrix}$$

2) $R_2 - aR_1$
$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b(b-a) & c(c-a) \end{vmatrix}$$

3) $R_3 - bR_2$
$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{vmatrix}$$ (Upper Triangular)

$\text{Det} = 1 \times (b-a) \times (c-a)(c-b)$

$\text{Det} = (b-a)(c-a)(c-b)$ Proved!