Assignment 2

MT-1004: Linear Algebra

Due: October 18, 2025

Instructions

- Submit your solutions electronically as a PDF file
- Show all your work and reasoning

Problem 1 (Vector Space - Application in Computer Science)**.** *In computer graphics and game development, colors are often represented as vectors. Consider the set V of all RGB color vectors of the form (r, g, b) where $r, g, b \in \mathbb{R}$ represent red, green, and blue components respectively.*

Define two operations on V:

- ***Color Blending (Addition)**: $(r_1, g_1, b_1) \oplus (r_2, g_2, b_2) = (r_1 + r_2, g_1 + g_2, b_1 + b_2)$*
- ***Color Scaling (Scalar Multiplication)**: $k \odot (r, g, b) = (kr, kg, kb)$ for $k \in \mathbb{R}$*

However, in practical applications, we need to ensure colors don't exceed display capabilities. Consider the modified operations:

- ***Bounded Addition**: $(r_1, g_1, b_1) \oplus (r_2, g_2, b_2) = (\min(r_1+r_2, 1), \min(g_1+g_2, 1), \min(b_1+b_2, 1))$*
- ***Bounded Scaling**: $k \odot (r, g, b) = (\min(kr, 1), \min(kg, 1), \min(kb, 1))$ for $k \in \mathbb{R}$*

(a) *Show that V with the standard operations (first definition) forms a vector space. Identify the zero vector and additive inverses.*

(b) *Determine whether V with the bounded operations (second definition) forms a vector space. If not, identify which vector space axioms fail and provide counterexamples.*

Problem 2 (Subspace - Application in Data Science)**.** *In machine learning, datasets are often represented as vectors in $\mathbb{R}^n$. Consider a dataset where each data point is a vector (x_1, x_2, x_3, x_4) representing four features of an object.*

Let $V = \mathbb{R}^4$ be the vector space of all possible data points. Consider the following subsets:

(b) $W_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}$

(c) $W_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 2x_3 \text{ and } x_2 = x_4\}$

For each subset:

(i) Determine whether it is a subspace of $\mathbb{R}^4$ *using the Subspace Test.*

(ii) If it is not a subspace, explain which condition fails and provide a specific counterexample.

Problem 3. *In computer graphics, transformations are often represented using matrices. Consider a 2D graphics system where we want to represent all possible linear transformations that can be applied to points in the plane.*

Let $S = \{A_1, A_2, A_3, A_4\}$ *be a set of* 2×2 *matrices where:*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1. *Show that* S *spans the vector space* M_{22} *of all* 2×2 *matrices.*

2. *A computer graphics programmer wants to create a new transformation matrix:*

$$B = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$$

   Express B *as a linear combination of the matrices in* S*.*

Problem 4. *In machine learning and data science, feature vectors often represent different attributes of data points. Consider a dataset where each data point is represented by a 3-dimensional feature vector.*

Let $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ *be a set of feature vectors where:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

1. *Determine whether the set* V *is linearly independent or linearly dependent. Show your work using the definition of linear independence.*

2. *If the set is linearly dependent, identify the dependency relationship and express one vector as a linear combination of the others.*

3. *In data preprocessing, we often encounter the problem of multicollinearity where features are highly correlated. Consider a modified set:*

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$

   Find the conditions on a, b, c *such that* W *is linearly independent.*

Problem 5. *In computer graphics and data science, we often work with different coordinate systems to represent the same data. Consider a vector space V of 2D geometric transformations with basis:*

$$B = \left\{ T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ T_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ T_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This is the standard basis for M_{22} representing basic transformation matrices.

Now consider an alternative basis B' that represents shear transformations and scaling:

$$B' = \left\{ S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\ S_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ S_4 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

A transformation matrix used in computer vision is given by:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

1. *Find the coordinate vector $[A]_B$ of matrix A relative to the standard basis B.*
2. *Find the transition matrix $P_{B \to B'}$ from basis B to basis B'.*
3. *Using the transition matrix from part (1), find the coordinate vector $[A]_{B'}$ of matrix A relative to basis B'.*

Problem 6. *In machine learning and data compression, understanding the dimension of vector spaces is crucial for feature selection and dimensionality reduction. Consider the following problems:*

Part a: Solution Spaces in Linear Systems

A machine learning model uses a linear system to extract features from image data. The system is represented by:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 7x_4 - 3x_5 &= 0 \\ 3x_1 + 6x_2 - 3x_3 + 10x_4 - 5x_5 &= 0 \end{aligned}$$

1. *Find a basis for the solution space of this homogeneous system.*
2. *Determine the dimension of the solution space.*
3. *If we add the constraint $x_1 + x_2 + x_3 = 0$ to the system, how does this affect the dimension of the solution space? Find the new basis and dimension.*

Part b: Subspaces in Data Representation

In data science, we often work with structured data that forms subspaces of larger vector spaces.

1. *Consider the set W of all 3×3 symmetric matrices (matrices where $A = A^T$). Prove that W is a subspace of M_{33} and find its dimension.*

2. *A data compression algorithm works with polynomials of degree at most 4 that satisfy $p(1) = 0$ and $p(-1) = 0$. Show that these polynomials form a subspace of P_4 and find its dimension. Provide an explicit basis for this subspace.*

Problem 7. *Consider the vector space $\mathbb{R}^2$ with two bases:*

$$B = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\} \quad \textit{and} \quad B' = \left\{ \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

Let $\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ *and suppose its coordinate vector relative to basis B is* $[\mathbf{w}]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

1. *Find the transition matrix $P_{B \to B'}$ from basis B to basis B'.*

2. *Use the transition matrix $P_{B \to B'}$ to find the coordinate vector $[\mathbf{w}]_{B'}$.*

Problem 8. *Let A be the matrix given by:*

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Its reduced row echelon form (RREF) is:

$$R = \begin{bmatrix} 1 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1. *Find a basis for the **row space** of A, row(A).*

2. *Find a basis for the **column space** of A, col(A). (Your basis must consist of column vectors from the original matrix A).*

3. *Find a basis for the **null space** of A, null(A).*

4. *Is the vector $\mathbf{b} = \begin{bmatrix} 1 & -9 & -5 & 4 \end{bmatrix}^T$ in the column space of A? Justify your answer.*

Problem 9. *Let A be a 5×7 matrix.*

1. *What is the largest possible value for the* ***rank*** *of A? What is the smallest possible value for the* ***nullity*** *of A? Explain your reasoning.*

2. *Suppose the linear system $A\mathbf{x} = \mathbf{b}$ is consistent and rank$(A) = 4$.*

   (a) *How many parameters (free variables) are in the general solution of $A\mathbf{x} = \mathbf{b}$?*

   (b) *Is the solution unique? Why or why not?*

3. *Now, suppose A is the coefficient matrix of a homogeneous system, $A\mathbf{x} = \mathbf{0}$, and its null space is 2-dimensional.*

   (a) *What is the rank of A?*

   (b) *Do the* ***column vectors*** *of A span $\mathbb{R}^5$? Justify your answer using the concept of rank.*

Grading Rubric:

- ***Correctness (60%):*** *Accurate mathematical solutions*
- ***Reasoning (25%):*** *Clear explanation of steps and methods*
- ***Application (15%):*** *Meaningful interpretation in computer science context*