Ex#6.3

6.3 (1-14, 27-31)

(44-49)

Orthogonal and Orthonormal Sets

Recall from Section 6.2 that two vectors in an inner product space are said to be *orthogonal* if their inner product is zero. The following definition extends the notion of orthogonality to *sets* of vectors in an inner product space.

Definition 1

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

NORMALIZING:

- Notes:

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector.

(2) $\|\mathbf{v}\| \neq 1 \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of $\mathbf{v}$)

$\mathbf{v} \neq 0$

not a unit vector

Theorem 6.3.1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Since an orthonormal set is orthogonal, and since its vectors are nonzero (norm 1), it follows from Theorem 6.3.1 that every *orthonormal* set is linearly independent.

Orthonormal Bases

- Orthogonal:

  A set S of vectors in an inner product space V is called an **orthogonal set** if every pair of vectors in the set is orthogonal.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$
$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \quad i \neq j$$

- Orthonormal:

  An orthogonal set in which each vector is a unit vector is called **orthonormal**.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$
$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Note:

  If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

In an inner product space, a basis consisting of orthonormal vectors is called an ***orthonormal basis***, and a basis of orthogonal vectors is called an ***orthogonal basis***. A familiar example of an orthonormal basis is the standard basis for R^n with the Euclidean inner product:

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

Show that the following set is an orthonormal basis.

$$S = \left\{ \underset{\mathbf{v}_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)}, \quad \underset{\mathbf{v}_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right)}, \quad \underset{\mathbf{v}_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right)} \right\}$$

Sol:

Show that the three vectors are mutually orthogonal.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Show that each vector is of length 1.

$$\| \mathbf{v}_1 \| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\| \mathbf{v}_2 \| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\| \mathbf{v}_3 \| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus *S* is an orthonormal set.

1. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^2.

   a. $(0,1),\ (2,0)$

   b. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

   c. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

   d. $(0,0),\ (0,1)$

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.

   a. $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

   b. $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

   c. $(1,0,0),\ \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ (0,0,1)$

   d. $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

Sol:

1. **(a)** $\langle(0,1),\ (2,0)\rangle = 0+0=0$;

$\|(0,1)\|=1$; $\quad\|(2,0)\|=2\neq 1$;

The set is orthogonal, but is not orthonormal.

(b) $\left\langle\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2}+\frac{1}{2}=0$;

$\left\|\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}}=1$; $\quad\left\|\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}}=1$

The set is orthogonal and orthonormal.

(c) $\left\langle\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2}-\frac{1}{2}=-1\neq 0$;

The set is not orthogonal (therefore, it is not orthonormal either).

(d) $\langle(0,0),\ (0,1)\rangle = 0+0=0$;

$\|(0,0)\|=0\neq 1$; $\quad\|(0,1)\|=1$;

The set is orthogonal, but is not orthonormal.

Sol:

2. **(a)** $\left\langle\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\rangle = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$; $\left\langle\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2} + 0 + \frac{1}{2} = 0$;

$\left\langle\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}} \neq 0$;

The set is not orthogonal (therefore, it is not orthonormal either).

(b) $\left\langle\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$; $\left\langle\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$;

$\left\langle\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$;

$\left\|\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$; $\left\|\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$; $\left\|\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$;

The set is orthogonal and orthonormal.

(c) $\left\langle(1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle = 0 + 0 + 0 = 0$; $\langle(1, 0, 0), (0, 0, 1)\rangle = 0 + 0 + 0 = 0$;

$\left\langle\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (0, 0, 1)\right\rangle = 0 + 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0$;

The set is not orthogonal (therefore, it is not orthonormal either).

3. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on P_2 (see Example 7 of Section 6.1).

a. $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2,\ p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2,$

$p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

b. $p_1(x) = 1,\ p_2(x) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ p_3(x) = x^2$

Sol: **3.** **(a)** $\langle p_1(x),\ p_2(x)\rangle = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0;$

$\langle p_1(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0;$

$\langle p_2(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0;$

The set is orthogonal.

(b) $\langle p_1(x),\ p_2(x)\rangle = 1(0) + 0\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0;\quad \langle p_1(x),\ p_3(x)\rangle = 1(0) + 0(0) + 0(1) = 0;$

$\langle p_2(x),\ p_3(x)\rangle = 0(0) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \neq 0;$

The set is not orthogonal.

4. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on M_{22} (see Example 6 of Section 6.1).

a. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

Sol:

4. **(a)** Denoting the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$, $C = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, and $D = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ we calculate

$$\langle A,B\rangle = (1)(0)+(0)\left(\tfrac{2}{3}\right)+(0)\left(\tfrac{1}{3}\right)+(0)\left(-\tfrac{2}{3}\right)=0;$$

$$\langle A,C\rangle = (1)(0)+(0)\left(\tfrac{2}{3}\right)+(0)\left(-\tfrac{2}{3}\right)+(0)\left(\tfrac{1}{3}\right)=0;$$

$$\langle A,D\rangle = (1)(0)+(0)\left(\tfrac{1}{3}\right)+(0)\left(\tfrac{2}{3}\right)+(0)\left(\tfrac{2}{3}\right)=0;$$

$$\langle B,C\rangle = (0)(0)+\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)+\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)+\left(-\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)=0;$$

$$\langle B,D\rangle = (0)(0)+\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)+\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)+\left(-\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)=0;$$

$$\langle C,D\rangle = (0)(0)+\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)+\left(-\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)+\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)=0;$$

In Exercises 5–6, show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & 0 & -\frac{2}{3} \end{bmatrix}$

sol:

5. Let us denote the column vectors $\mathbf{u}_1 = (1,0,-1)$, $\mathbf{u}_2 = (2,0,2)$, and $\mathbf{u}_3 = (0,5,0)$. These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 2+0-2=0$, $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 0+0+0=0$, and $\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0+0+0=0$.

It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an orthogonal basis for the column space of A. We proceed to normalize each column vector:

$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{1+0+1}}(1,0,-1) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$; $\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{4+0+4}}(2,0,2) = \left(\frac{2}{2\sqrt{2}}, 0, \frac{2}{2\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$;

$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{0+25+0}}(0,5,0) = (0, 1, 0)$. A resulting orthonormal basis for the column space is

$\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0, 1, 0)\right\}$.

Coordinates Relative to Orthonormal Bases

One way to express a vector $\mathbf{u}$ as a linear combination of basis vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

is to convert the vector equation

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

to a linear system and solve for the coefficients $c_1, c_2, \dots, c_n$. However, if the basis happens to be orthogonal or orthonormal, then the following theorem shows that the coefficients can be obtained more simply by computing appropriate inner products.

Theorem 6.3.2

(*a*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle\mathbf{v}_n \tag{4}$$

Using the terminology and notation from Definition 2 of Section 4.5, it follows from Theorem 6.3.2 that the coordinate vector of a vector $\mathbf{u}$ in V relative to an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$\boxed{(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}, \ldots, \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\right)} \tag{6}$$

and relative to an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$\boxed{(\mathbf{u})_S = \left(\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n\rangle\right)} \tag{7}$$

EXAMPLE 5 | A Coordinate Vector Relative to an Orthonormal Basis

Let

$$\mathbf{v}_1 = (0,1,0), \quad \mathbf{v}_2 = \left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right), \quad \mathbf{v}_3 = \left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

It is easy to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for R^3 with the Euclidean inner product. Express the vector $\mathbf{u} = (1,1,1)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_S$.

Solution We leave it for you to verify that

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1, \quad \langle \mathbf{u}, \mathbf{v}_2 \rangle = -\tfrac{1}{5}, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v}_3 \rangle = \tfrac{7}{5}$$

Therefore, by Theorem 6.3.2 we have

$$\mathbf{u} = \mathbf{v}_1 - \tfrac{1}{5}\mathbf{v}_2 + \tfrac{7}{5}\mathbf{v}_3$$

that is,

$$(1,1,1) = (0,1,0) - \tfrac{1}{5}\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) + \tfrac{7}{5}\left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

Thus, the coordinate vector of $\mathbf{u}$ relative to S is

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle) = \left(1, -\tfrac{1}{5}, \tfrac{7}{5}\right)$$

EXAMPLE 6 | An Orthonormal Basis from an Orthogonal Basis

(*a*) Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \quad \mathbf{w}_2 = (3, 0, 3), \quad \mathbf{w}_3 = (-4, 0, 4)$$

form an orthogonal basis for R^3 with the Euclidean inner product, and use that basis to find an orthonormal basis by normalizing each vector.

(*b*) Express the vector $\mathbf{u} = (1, 2, 4)$ as a linear combination of the orthonormal basis vectors obtained in part (*a*).

Solution (*a*) The given vectors form an orthogonal set since

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0, \quad \langle \mathbf{w}_1, \mathbf{w}_3 \rangle = 0, \quad \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0$$

It follows from Theorem 6.3.1 that these vectors are linearly independent and hence form a basis for R^3 by Theorem 4.6.4. We leave it for you to calculate the norms of $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$ and then obtain the orthonormal basis

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0, 1, 0), \quad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Solution (*b*) It follows from Formula (4) that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3$$

We leave it for you to confirm that

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_1 \rangle &= (1, 2, 4) \cdot (0, 1, 0) = 2 \\
\langle \mathbf{u}, \mathbf{v}_2 \rangle &= (1, 2, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{5}{\sqrt{2}} \\
\langle \mathbf{u}, \mathbf{v}_3 \rangle &= (1, 2, 4) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}}
\end{aligned}$$

and hence that

$$(1, 2, 4) = 2(0, 1, 0) + \frac{5}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) + \frac{3}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

7. Verify that the vectors

$$\mathbf{v}_1 = \left(-\tfrac{3}{5}, \tfrac{4}{5}, 0\right),\ \mathbf{v}_2 = \left(\tfrac{4}{5}, \tfrac{3}{5}, 0\right),\ \mathbf{v}_3 = (0, 0, 1)$$

form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(b) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Sol:

7. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -\frac{12}{25} + \frac{12}{25} + 0 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$;

$\|\mathbf{v}_1\| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1$; $\|\mathbf{v}_2\| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1$; $\|\mathbf{v}_3\| = \sqrt{0 + 0 + 1} = 1$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim\left(R^3\right) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. This basis is orthonormal, so by Theorem 6.3.2(b),

$$\begin{aligned}
\mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\
&= \left(-\tfrac{3}{5} - \tfrac{8}{5} + 0\right)\mathbf{v}_1 + \left(\tfrac{4}{5} - \tfrac{6}{5} + 0\right)\mathbf{v}_2 + (0 + 0 + 2)\mathbf{v}_3 \\
&= -\tfrac{11}{5}\mathbf{v}_1 - \tfrac{2}{5}\mathbf{v}_2 + 2\mathbf{v}_3.
\end{aligned}$$

8. Use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in Exercise 7.

Sol:

8. It was shown in Exercise 7 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form an orthonormal basis for R^3. By Theorem 6.3.2(b),

$$\begin{aligned}\mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\ &= \left(-\tfrac{9}{5} - \tfrac{28}{5} + 0\right)\mathbf{v}_1 + \left(\tfrac{12}{5} - \tfrac{21}{5} + 0\right)\mathbf{v}_2 + (0 + 0 + 4)\mathbf{v}_3 \\ &= -\tfrac{37}{5}\mathbf{v}_1 - \tfrac{9}{5}\mathbf{v}_2 + 4\mathbf{v}_3.\end{aligned}$$

9. Verify that the vectors

$$\mathbf{v}_1 = (2, -2, 1), \quad \mathbf{v}_2 = (2, 1, -2), \quad \mathbf{v}_3 = (1, 2, 2)$$

form an orthogonal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(a) to express the vector $\mathbf{u} = (-1, 0, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Sol:

9. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 4 - 2 - 2 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 2 - 4 + 2 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 2 + 2 - 4 = 0$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim\left(R^3\right) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. By Theorem 6.3.2(a),

$$\begin{aligned}
\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 \\
&= \frac{-2+0+2}{4+4+1}\mathbf{v}_1 + \frac{-2+0-4}{4+1+4}\mathbf{v}_2 + \infty\mathbf{v}_3 \\
&= 0\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3.
\end{aligned}$$

10. Verify that the vectors

$$\mathbf{v}_1 = (1, -1, 2, -1), \quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1), \quad \mathbf{v}_4 = (1, 0, 0, 1)$$

form an orthogonal basis for R^4 with respect to the Euclidean inner product, and then use Theorem 6.3.2(a) to express the vector $\mathbf{u} = (1, 1, 1, 1)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$. Sol:

10. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -2-2+6-2=0$ $\quad$ $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 1-2+0+1=0$ $\quad$ $\langle \mathbf{v}_1, \mathbf{v}_4 \rangle = 1+0+0-1=0$

$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -2+4+0-2=0$ $\quad$ $\langle \mathbf{v}_2, \mathbf{v}_4 \rangle = -2+0+0+2=0$ $\quad$ $\langle \mathbf{v}_3, \mathbf{v}_4 \rangle = 1+0+0-1=0$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim\left(R^4\right)=4$, this set forms a basis for R^4 by Theorem 4.6.4. We use Theorem 6.3.2(a):

$$\begin{aligned}
\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 + \frac{\langle \mathbf{u}, \mathbf{v}_4 \rangle}{\|\mathbf{v}_4\|^2}\mathbf{v}_4 \\
&= \frac{1-1+2-1}{1+1+4+1}\mathbf{v}_1 + \frac{-2+2+3+2}{4+4+9+4}\mathbf{v}_2 + \frac{1+2+0-1}{1+4+0+1}\mathbf{v}_3 + \frac{1+0+0+1}{1+0+0+1}\mathbf{v}_4 \\
&= \frac{1}{7}\mathbf{v}_1 + \frac{5}{21}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3 + 1\mathbf{v}_4.
\end{aligned}$$

*In Exercises **11–14**, find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in the stated exercise.*

11. Exercise 7

12. Exercise 8

13. Exercise 9

14. Exercise 10

Sol:

11. $(\mathbf{u})_S = \left(-\frac{11}{5}, -\frac{2}{5}, 2\right)$

12. $(\mathbf{u})_S = \left(-\frac{37}{5}, -\frac{9}{5}, 4\right)$

13. $(\mathbf{u})_S = \left(0, -\frac{2}{3}, \frac{1}{3}\right)$

14. $(\mathbf{u})_S = \left(\frac{1}{7}, \frac{5}{21}, \frac{1}{3}, 1\right)$

Theorem 6.3.3

Projection Theorem

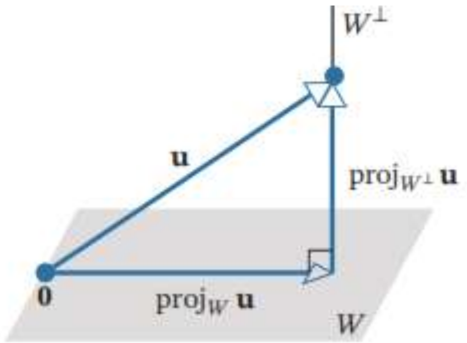
If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \tag{8}$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$.

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in Formula (8) are commonly denoted by

$$\mathbf{w}_1 = \text{proj}_W \mathbf{u} \quad \text{and} \quad \mathbf{w}_2 = \text{proj}_{W^\perp} \mathbf{u} \tag{9}$$



These are called the ***orthogonal projection of* u *on W*** and the ***orthogonal projection of* u *on* $W^\perp$**, respectively. The vector $\mathbf{w}_2$ is also called the ***component of* u *orthogonal to W***. Using the notation in (9), Formula (8) can be expressed as

$$\boxed{\mathbf{u} = \text{proj}_W \mathbf{u} + \text{proj}_{W^\perp} \mathbf{u}} \tag{10}$$

$$\boxed{\mathbf{u} = \text{proj}_W \mathbf{u} + (\mathbf{u} - \text{proj}_W \mathbf{u})}$$

Theorem 6.3.4

Let W be a finite-dimensional subspace of an inner product space V.

(*a*) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{12}$$

(*b*) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r \tag{13}$$

Although Formulas (12) and (13) are expressed in terms of orthogonal and orthonormal basis vectors, the resulting vector $\text{proj}_W \mathbf{u}$ does not depend on the basis vectors that are used.

EXAMPLE 7 | Calculating Projections

Let R^3 have the Euclidean inner product, and let W be the subspace spanned by the orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$. From Formula (13) the orthogonal projection of $\mathbf{u} = (1, 1, 1)$ on W is

$$\begin{aligned}\text{proj}_W \mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 \\ &= (1)(0, 1, 0) + \left(-\tfrac{1}{5}\right)\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) \\ &= \left(\tfrac{4}{25}, 1, -\tfrac{3}{25}\right)\end{aligned}$$

The component of $\mathbf{u}$ orthogonal to W is

$$\text{proj}_{W^\perp} \mathbf{u} = \mathbf{u} - \text{proj}_W \mathbf{u} = (1, 1, 1) - \left(\tfrac{4}{25}, 1, -\tfrac{3}{25}\right) = \left(\tfrac{21}{25}, 0, \tfrac{28}{25}\right)$$

Observe that $\text{proj}_{W^\perp} \mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so this vector is orthogonal to each vector in the space W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, as it should be.

The Gram–Schmidt Process

We have seen that orthonormal bases exhibit a variety of useful properties. Our next theorem, which is the main result in this section, shows that every nonzero finite-dimensional vector space has an orthonormal basis. The proof of this result is extremely important since it provides an algorithm, or method, for converting an arbitrary basis into an orthonormal basis.

Theorem 6.3.5

Every nonzero finite-dimensional inner product space has an orthonormal basis.

The Gram–Schmidt Process

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for r steps)

Optional Step. To convert the orthogonal basis into an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$, normalize the orthogonal basis vectors.

EXAMPLE 8 | Using the Gram–Schmidt Process

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1, 1, 1), \quad \mathbf{u}_2 = (0, 1, 1), \quad \mathbf{u}_3 = (0, 0, 1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

Solution

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

Step 2. $$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1} \mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$= (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3. $$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2} \mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Thus,

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

form an orthogonal basis for R^3. The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for R^3 is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

27. $\mathbf{u}_1 = (1, -3),\ \mathbf{u}_2 = (2, 2)$ **28.** $\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (3, -5)$

In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

29. $\mathbf{u}_1 = (1, 1, 1),\ \mathbf{u}_2 = (-1, 1, 0),\ \mathbf{u}_3 = (1, 2, 1)$

30. $\mathbf{u}_1 = (1, 0, 0),\ \mathbf{u}_2 = (3, 7, -2),\ \mathbf{u}_3 = (0, 4, 1)$

31. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, 2, 1, 0), \quad \mathbf{u}_2 = (1, -1, 0, 0),$$
$$\mathbf{u}_3 = (1, 2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, 1)$$

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

Sol:

The reduced row echelon form of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the three column vectors of the original matrix, $\mathbf{u}_1 = (6, 2, -2, 6)$, $\mathbf{u}_2 = (1, 1, -2, 8)$, and $\mathbf{u}_3 = (-5, 1, 5, -7)$ form a basis for the column space. Applying the Gram-Schmidt process yields an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$\mathbf{v}_1 = \mathbf{u}_1 = (6,2,-2,6)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \tfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$(1,1,-2,8) - \tfrac{6+2+4+48}{36+4+4+36}(6,2,-2,6) = (1,1,-2,8) - \tfrac{3}{4}(6,2,-2,6)$$

$$= \left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (-5,1,5,-7) - \tfrac{-30+2-10-42}{36+4+4+36}(6,2,-2,6) - \tfrac{\frac{35}{2}-\frac{1}{2}-\frac{5}{2}-\frac{49}{2}}{\frac{49}{4}+\frac{1}{4}+\frac{1}{4}+\frac{49}{4}}\left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$= (-5,1,5,-7) + (6,2,-2,6) + \tfrac{2}{5}\left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right) = \left(-\tfrac{2}{5}, \tfrac{14}{5}, \tfrac{14}{5}, \tfrac{2}{5}\right)$$

Theorem 6.3.7

QR-Decomposition

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}$$

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a QR-decomposition of A is

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

EXAMPLE 10 | *QR*-Decomposition of a 3×3 Matrix

Find a *QR*-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors (see Example 8)

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a *QR*-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

In Exercises 45–48, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A.

45. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

46. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$

49. Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

In partitioned form, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$. By inspection, $\mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$, so the column vectors of A are not linearly independent and A does not have a QR-decomposition.