National University
of Computer & Emerging Sciences

Serial No. 389581

Course Code & Title: Linear Algebra Quiz 3

Roll No: 24K-0746 Section: BCS-3K Student's Signature: ______ Date: ______

Tick (✓) Extra Sheet No: 1 2 3 4 5 Invigilator's Signature: ______

$YY = 24, \quad ABCD = 0746$

1. Problem 1:-

- $[YY] \bmod 5 + 1 = 24 \bmod 5 + 1$
  $= 4 + 1 = 5$
- $[ABCD] \bmod 4 + 1 = 746 \bmod 4 + 1$
  $= 2 + 1 = 3$

$$A = \begin{bmatrix} 3+5 & 1 & 0 \\ 1 & 3-3 & 0 \\ 0 & 0 & 2(5) \end{bmatrix} = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

a) $A - \lambda I = \begin{bmatrix} \lambda-8 & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-10 \end{bmatrix}$

$= (\lambda-8)(\lambda(\lambda-10)) + 1(-1(\lambda-10)) + 0$

$= (\lambda-10)(\lambda(\lambda-8)-1)$

$= (\lambda-10)(\lambda^2 - 8\lambda - 1)$

$\lambda = 10, \quad 4+\sqrt{17}, \quad 4-\sqrt{17}$

b) ① For $\lambda = 10$:

$$\begin{bmatrix} 10-8 & -1 & 0 \\ -1 & 10 & 0 \\ 0 & 0 & 10-10 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1) $R_1 + 2R_2$

$$\begin{bmatrix} 0 & 19 & 0 \\ -1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) $R_1 / 19$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3) $R_2 - 10R_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4) $-R_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5) $R_1 \Leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$ : $x_1 = 0$, $x_2 = 0$

$(x_1, x_2, x_3) = t(0, 0, 1)$.

② For $\lambda = 4+\sqrt{17}$ :

$$A - \lambda I = \begin{bmatrix} -4+\sqrt{17} & -1 & 0 \\ -1 & 4+\sqrt{17} & 0 \\ 0 & 0 & -6+\sqrt{17} \end{bmatrix}$$

1) $R_3 / -6+\sqrt{17}$

$$\begin{bmatrix} -4+\sqrt{17} & -1 & 0 \\ -1 & 4+\sqrt{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $R_1 + (-4+\sqrt{17})R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 4+\sqrt{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) $-R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -4-\sqrt{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4) $R_1 \longleftrightarrow R_2$ , $R_2 \longleftrightarrow R_3$

$$\begin{bmatrix} 1 & -4-\sqrt{17} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_2 = t$ , $x_1 = 4+\sqrt{17}\, t$ , $x_3 = 0$

$(x_1, x_2, x_3) = t(4+\sqrt{17}, 1, 0)$

③ $\lambda = 4 - \sqrt{17}$

$$A - \lambda I = \begin{bmatrix} 4+\sqrt{17} & -1 & 0 \\ -1 & 4-\sqrt{17} & 0 \\ 0 & 0 & -6-\sqrt{17} \end{bmatrix}$$

1) $R_3 / -6-\sqrt{17}$

$$\begin{bmatrix} 4+\sqrt{17} & -1 & 0 \\ -1 & 4-\sqrt{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $R_1 + (-4-\sqrt{17})R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 4-\sqrt{17} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) $R_1 \leftrightarrow R_2$, $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & 4-\sqrt{17} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_2 = t$, $x_1 = 4-\sqrt{17}\,t$, $x_3 = 0$

$(x_1, x_2, x_3) = t(4-\sqrt{17}, 1, 0)$

C) According to Theorem 5.2.2, an $n \times n$ matrix with n distinct eigenvalues is diagonalizab
Since it is a 3x3 matrix with 3 distinct eigen values, it is diagonalizable.

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$YX \bmod 5 + 1 = 24 \bmod 5 + 1 = 4 + 1 = 5$

2. Problem 2:-

a) $\langle V_1 . V_2 \rangle = (1)(1) + (1)(-1) + (0)(0) = 0$

$\langle V_1 . V_3 \rangle = (0)(0) + (1)(0) + (0)(5) = 0$

$\langle V_2 . V_3 \rangle = (1)(0) + (-1)(0) + (0)(5) = 0$

All three dot products are zero so $\{V_1, V_2, V_3\}$ is an orthogonal set.

b) $\|V_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

$\|V_2\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$

$\|V_3\| = \sqrt{0^2 + 0^2 + 5^2} = 5$

$$q_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,0)}{\sqrt{2}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$$

$$q_2 = \frac{V_2}{\|V_2\|} = \frac{(1,-1,0)}{\sqrt{2}} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$$

$$q_3 = \frac{V_3}{\|V_3\|} = \frac{(0,0,5)}{5} = (0,0,1)$$

c) ABCD mod 4 + 1 = 746 mod 4 + 1 = 2 + 1 = 3

$$\text{proj}_W u = \langle u, v_1 \rangle v_1 / \|v_1\| + \langle u, v_2 \rangle v_2 / \|v_2\|$$

$$\langle u,v_1\rangle v_1 + \langle u,v_2\rangle v_2 = ((2)(1)+(0)(1)+(3)(0))(1,1,0) + ((2)(1)+(0)(-1)+(3)(0))(1,-1,0)$$

$$= (1,1,0)(2) + (1,-1,0)\,2$$

$$= (2,2,0) + (2,-2,0)$$

$$\text{proj}_W u = \frac{(2,2,0)}{(\sqrt{2})^2} + \frac{(2,-2,0)}{(\sqrt{2})^2}$$

$$= \frac{(2,2,0)}{2} + \frac{(2,-2,0)}{2}$$

$$= (1,1,0) + (1,-1,0)$$

$$= (2,0,0)$$

3. Problem 3:-

YY mod 5 + 1 = 5

ABCD mod 4 + 1 = 3

a)

$$v_1 = u_1 = (1,1,0)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2}$$

$$= (1,0,5) - \frac{((1)(1)+(0)(1)+(5)(0))(1,1,0)}{1^2+1^2+0^2}$$

$$= (1,0,5) - \frac{(1,1,0)}{2}$$

$$= (1,0,5) - (1/2, 1/2, 0)$$

$$v_2 = (1/2, -1/2, 5)$$

Q/Part No.



Rough Work

$$V_3 = U_3 - \frac{\langle U_3, V_1\rangle V_1}{\|V_1\|^2} - \frac{\langle U_3, V_2\rangle V_2}{\|V_2\|^2}$$

$$= (0,3,1) - \frac{(0+3+0)(1,1,0)}{2} - \frac{(0 - 3/2 + 5)(1/2, -1/2, 5)}{51/2}$$

$$= (0,3,1) - \frac{3(1,1,0)}{2} - \frac{7}{51}(1/2, -1/2, 5)$$

$$= (0,3,1) - (3/2, 3/2, 0) - (7/102, -7/102, 35/51)$$

$$V_3 = (-80/51, 80/51, 16/51).$$

orthogonal basis: $\{V_1, V_2, V_3\}$.

b) $$q_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,0)}{\sqrt{2}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$$

$$q_2 = \frac{V_2}{\|V_2\|} = \frac{(1/2, -1/2, 5)}{\sqrt{51/2}} = (\sqrt{2}/2\sqrt{51}, -\sqrt{2}/2\sqrt{51}, 5\sqrt{2}/\sqrt{51})$$

$$q_3 = \frac{V_3}{\|V_3\|} = \frac{(-80/51, 80/51, 16/51)}{16\sqrt{51}/51} = (-5/\sqrt{51}, 5/\sqrt{51}, 1/\sqrt{51})$$

c) $$R = \begin{bmatrix} \langle U_1, q_1\rangle & \langle U_2, q_1\rangle & \langle U_3, q_1\rangle \\ 0 & \langle U_2, q_2\rangle & \langle U_3, q_2\rangle \\ 0 & 0 & \langle U_3, q_3\rangle \end{bmatrix}$$

$$R = \begin{bmatrix} (1/\sqrt{2} + 1/\sqrt{2} + 0) & (\sqrt{2}/2\sqrt{51} + 0 + 25\sqrt{2}/\sqrt{51}) & (0 + 3/\sqrt{2} + 0) \\ 0 & (1/\sqrt{2} + 0 + 0) & (0 - 3\sqrt{2}/2\sqrt{51} + 5\sqrt{2}/\sqrt{51}) \\ 0 & 0 & (0 + 15/\sqrt{51} + 1/\sqrt{51}) \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 3/\sqrt{2} \\ 0 & \sqrt{\frac{51}{2}} & 7/\sqrt{102} \\ 0 & 0 & 16\sqrt{51}/51 \end{bmatrix}$$

Q/Part No.



Rough Work

4. Problem 4:-

$\theta = \pi/7$

a) $C^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C^T.C = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Performing matrix multiplication directly

$C^T.C = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C is orthogonal matrix.

b) Since $C \cdot C^T = I$, $C^{-1} = C^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\det(C) = \cos\theta(\cos\theta) + \sin\theta(\sin\theta) + 0$

$= \cos^2\theta + \sin^2\theta = 1$

d) $||CX||$:

$CX = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$CX = \begin{bmatrix} (1)(\cos\theta) + (3)(-\sin\theta) + (0)(2) \\ (1)(\sin\theta) + (3)(\cos\theta) + (0)(2) \\ (0)(1) + (0)(3) + (1)(2) \end{bmatrix} = \begin{bmatrix} \cos\theta - 3\sin\theta \\ \sin\theta + 3\cos\theta \\ 2 \end{bmatrix}$

- $||CX|| = \sqrt{(\cos\theta - 3\sin\theta)^2 + (\sin\theta + 3\cos\theta)^2 + 2^2}$
$||CX|| = \sqrt{\cos^2\theta - 6\cos\theta\sin\theta + 9\sin^2\theta + \sin^2\theta + 6\cos\theta\sin\theta + 9\cos^2\theta + 4}$
$||CX|| = \sqrt{10\cos^2\theta + 10\sin^2\theta + 4}$
$||CX|| = \sqrt{10(\cos^2\theta + \sin^2\theta) + 4}$
$||CX|| = \sqrt{10(1) + 4} = \sqrt{14}$

- $||X|| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{1+9+4} = \sqrt{14}$
$||CX|| = ||X||$

5. Problem 5:-

a)
$$D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Since $D = D^T$, D is symmetric.

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b) $D - \lambda I = \begin{bmatrix} \lambda-2 & -1 & 0 \\ -1 & \lambda-2 & 0 \\ 0 & 0 & \lambda-6 \end{bmatrix}$

$= (\lambda-2)(\lambda-2)(\lambda-6) + 1(-1(\lambda-6)) + 0$

$= (\lambda-6)((\lambda-2)^2 - 1)$

$= (\lambda-6)(\lambda^2 - 4\lambda + 4 - 1)$

$= (\lambda-6)(\lambda^2 - 4\lambda + 3)$

$= \lambda = 6, \quad \lambda = 3, \quad \lambda = 1$

- Multiplicity of $\lambda = 6 = 1$
- Multiplicity of $\lambda = 3 = 1$
- Multiplicity of $\lambda = 1 = 1$

c) ① For $\lambda = 6$:

$D - 6I = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1) $R_1 + 4R_2$

$\begin{bmatrix} 0 & 15 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2) $R_2/15$

$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3) $R_2 - 4R_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4) $-R_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5) $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$, $x_1 = 0$, $x_2 = 0$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2 For $\lambda = 3$:

$$D - 3I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

1) $R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

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2) $-R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

let $x_2 = t$, $x_1 = t$, $x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow p_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(3) For $\lambda = 2$:

$$D - \underline{I} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

1) $R_2 - R_1$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

2) $-R_2/5$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

let $x_2 = t$, $x_1 = -t$, $x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow p_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Applying Gram - Schmidt process on $p_1, p_2, p_3$:

- $v_1 = p_1 = (0,0,1)$
- $v_2 = p_2 - \frac{\langle p_2, v_1 \rangle v_1}{\|v_1\|^2}$

$= (1,1,0) - \frac{(0+0+0)(0,0,1)}{1}$

$= (1,1,0) - (0,0,0) = (1,1,0)$

- $v_3 = p_3 - \frac{\langle p_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle p_3, v_2 \rangle v_2}{\|v_2\|^2}$

$= (-1,1,0) - \frac{(0+0+0)(0,0,1)}{1} - \frac{(\cancel{-1}+\cancel{1}+0)(1,1,0)}{2}$

$= (-1,1,0) - (0,0,0) - (0,0,0) = (-1,1,0)$

- $q_1 = \frac{v_1}{\|v_1\|} = \frac{(0,0,1)}{1} = (0,0,1)$

- $q_2 = \frac{v_2}{\|v_2\|} = \frac{(1,1,0)}{\sqrt{2}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$

- $q_3 = \frac{v_3}{\|v_3\|} = \frac{(-1,1,0)}{\sqrt{2}} = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

d) $D = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T$

$$D = 6\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0,0,1\end{bmatrix} + 3\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix} +$$

$$1\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}$$

$$D = 6\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + 3\begin{bmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&0\end{bmatrix} + 1\begin{bmatrix}1/2&-1/2&0\\-1/2&1/2&0\\0&0&0\end{bmatrix}$$

$$\cancel{D = \begin{bmatrix}0&0&0\\0&0&0\\0&0&6\end{bmatrix}}$$