Linear Algebra Lecture: Section 4.3 - Spanning Sets

1 Spanning Sets

1.1 Linear Combinations

Definition 1.1 (Linear Combination)**.** If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination.

Real-Life Example: In nutrition, any meal can be considered a linear combination of basic food items. For example, if we have:

- $\mathbf{v}_1 = 1$ serving of rice (provides certain nutrients)
- $\mathbf{v}_2 = 1$ serving of vegetables (provides other nutrients)
- $\mathbf{v}_3 = 1$ serving of meat (provides proteins)

Then a balanced meal $\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 + 1\mathbf{v}_3$ represents 2 servings of rice, 3 servings of vegetables, and 1 serving of meat.

1.2 Span of a Set

Theorem 1.1 (4.3.1)**.** If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Remarks:

- The subspace W in Theorem 4.3.1 is called the **subspace of** V **spanned by** S.
- The vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ in S are said to **span** W.
- **Notations:** $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ or $W = \text{span}(S)$
- For the empty set $\emptyset$, we define $\text{span}(\emptyset) = \{\mathbf{0}\}$

1.3 Examples

Example 1.1 (Standard Unit Vectors Span $\mathbb{R}^n$)**.** The standard unit vectors in $\mathbb{R}^n$ are:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

These vectors span $\mathbb{R}^n$ since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

In $\mathbb{R}^3$, the vectors $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$ span $\mathbb{R}^3$ since:

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Example 1.2 (A Spanning Set for P_n)**.** The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since each polynomial p in P_n can be written as:

$$p = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. We denote this by:

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$

Example 1.3 (Linear Combinations)**.** Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not.

Solution: For $\mathbf{w} = (9, 2, 7)$, we solve:

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

This gives the system:

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving yields $k_1 = -3, k_2 = 2$, so $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$.

For $\mathbf{w}' = (4, -1, 8)$, the system:

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

is inconsistent, so $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Example 1.4 (Testing for Spanning)**.** Determine whether the vectors $\mathbf{v}_1 = (1,1,2), \mathbf{v}_2 = (1,0,1)$, and $\mathbf{v}_3 = (2,1,3)$ span $\mathbb{R}^3$.

Solution: We check if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as:

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

This gives the system:

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Since $\det(A) = 0$, the system is not consistent for all $\mathbf{b}$, so the vectors do not span $\mathbb{R}^3$.

1.4 Procedure for Identifying Spanning Sets

Procedure 1.1 (Identifying Spanning Sets)**.**

1. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a given set of vectors in V, and let $\mathbf{x}$ be an arbitrary vector in V.

2. Set up the augmented matrix for the linear system that results by equating corresponding components on the two sides of the vector equation:

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{x}$$

3. Investigate the consistency or inconsistency of that system. If it is consistent for all choices of $\mathbf{x}$, the vectors in S span V, and if it is inconsistent for some vector $\mathbf{x}$, they do not.

Example 1.5 (Testing for Spanning in P_2)**.** Determine whether the set S spans P_2:

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Solution (a): For arbitrary $p = a + bx + cx^2$, we solve:

$$k_1(1 + x + x^2) + k_2(-1 - x) + k_3(2 + 2x + x^2) = a + bx + cx^2$$

This gives the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 1 & -1 & 2 & b \\ 1 & 0 & 1 & c \end{bmatrix}$$

The coefficient matrix has det $= 0$, so S does not span P_2.

Solution (b): The augmented matrix reduces to:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{a+b+c}{2} & \\ 0 & 1 & 0 & -\frac{a+b-c}{2} & \\ 0 & 0 & 1 & -1 & a \end{bmatrix}$$

which is consistent for all a, b, c, so S spans P_2.

Example 1.6 (Testing for Spanning in M_{22})**.** Determine whether the set S spans M_{22}:

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Solution: For arbitrary $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we solve:

$$k_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The coefficient matrix has det $= -2 \neq 0$, so the system is always consistent, and S spans M_{22}.

Theorem 1.2 (4.3.2)**.** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

2 Exercise Solutions (1-20)

Exercise 1

Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) $(2, 2, 2)$: Solve $(2, 2, 2) = k_1(0, -2, 2) + k_2(1, 3, -1)$. System is inconsistent. **No**

(b) $(0, 4, 5)$: Solve $(0, 4, 5) = k_1(0, -2, 2) + k_2(1, 3, -1)$. Solution: $k_1 = -1, k_2 = 0$. **Yes**

(c) $(0, 0, 0)$: $0\mathbf{u} + 0\mathbf{v} = (0, 0, 0)$. **Yes**

Exercise 2

Express as linear combinations of $\mathbf{u} = (2, 1, 4), \mathbf{v} = (1, -1, 3), \mathbf{w} = (3, 2, 5)$:

(a) $(-9, -7, -15)$: Solve system to get $k_1 = -2, k_2 = 1, k_3 = -3$

(b) $(6, 11, 6)$: Solve system to get $k_1 = 1, k_2 = -2, k_3 = 2$

(c) $(0, 0, 0)$: $0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = (0, 0, 0)$

Exercise 3

Which are linear combinations of $A = \begin{bmatrix} -4 & 0 \\ -2 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -2 & 3 \end{bmatrix}, C = \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix}$?

(a) $\begin{bmatrix} -6 & -8 \\ -1 & -8 \end{bmatrix}$: Solve system. **Yes**

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$: Trivial combination. **Yes**

(c) $\begin{bmatrix} -1 & 5 \\ -7 & 1 \end{bmatrix}$: Solve system. **No**

Exercise 4

Determine whether polynomials are linear combinations of $p_1 = 2 + x + x^2, p_2 = 1 - x^2, p_3 = 1 + 2x$:

(a) $1 + x$: Solve $k_1(2, 1, 1) + k_2(1, 0, -1) + k_3(1, 2, 0) = (1, 1, 0)$. **Yes**

(b) $1 + x^2$: Solve system. **Yes**

(c) $1 + x + x^2$: Solve system. **Yes**

Exercise 5

Express vectors as linear combinations of given matrices:

(a) $\begin{bmatrix} -1 & 0 \\ -2 & 4 \end{bmatrix} = 1A + 0B + 0C + 1D$

(b) $\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}$ is not a linear combination

Exercise 6

Express polynomials as linear combinations:

(a) $-9 - 7x - 15x^2 = -2p_1 + p_2 - 3p_3$

(b) $6 + 11x + 6x^2 = p_1 - 2p_2 + 2p_3$

(c) $0 = 0p_1 + 0p_2 + 0p_3$

(d) $7 + 8x + 9x^2$ is not a linear combination

Exercise 7

Determine whether vectors span $\mathbb{R}^3$:

(a) $\mathbf{v}_1 = (2, 2, 2), \mathbf{v}_2 = (0, 0, 3), \mathbf{v}_3 = (0, 1, 1)$: Check determinant of coefficient matrix $\neq 0$. **Yes**

(b) $\mathbf{v}_1 = (2, -1, 3), \mathbf{v}_2 = (4, 1, 2), \mathbf{v}_3 = (8, -1, 8)$: Check determinant $= 0$. **No**

Exercise 8

Which vectors are in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (2, 1, 0, 3), \mathbf{v}_2 = (3, -1, 5, 2), \mathbf{v}_3 = (-1, 0, 2, 1)$:

(a) $(2, 3, -7, 3)$: Solve system. **Yes**

(b) $(0, 0, 0, 0)$: Trivial combination. **Yes**

(c) $(1, 1, 1, 1)$: Solve system. **No**

(d) $(-4, 6, -13, 4)$: Solve system. **Yes**

Exercise 9

Determine whether polynomials span P_2: $p_1 = 1 - x + 2x^2, p_2 = 3 + x, p_3 = 5 - x + 4x^2, p_4 = -2 - 2x + 2x^2$

Check if arbitrary $a + bx + cx^2$ can be expressed as linear combination. System is overdetermined but consistent. **Yes**

Exercise 10

Determine whether polynomials span P_2: $p_1 = 1 + x, p_2 = 1 - x, p_3 = 1 + x + x^2, p_4 = 2 - x^2$

Check consistency for arbitrary $a + bx + cx^2$. System is consistent. **Yes**

Exercise 11

Determine whether matrices span M_{22}:

(a) Check if arbitrary $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed. **Yes**

(b) Check consistency. **Yes**

(c) Check consistency. **Yes**

Exercise 12

Determine whether $\mathbf{u} = (1,2)$ is in span of $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2)\}$:

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$: $T_A(\mathbf{e}_1) = (1,0), T_A(\mathbf{e}_2) = (2,1)$. Solve $(1,2) = k_1(1,0) + k_2(2,1)$. **Yes**

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: $T_A(\mathbf{e}_1) = (1,1), T_A(\mathbf{e}_2) = (1,1)$. Cannot get $(1,2)$. **No**

Exercise 13

Determine whether $\mathbf{u} = (1,1,1)$ is in span of $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2)\}$:

(a) $A = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$: Solve system. **No**

(b) $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$: Solve system. **Yes**

Exercise 14

Which lie in space spanned by $f = \cos^2 x$ and $g = \sin^2 x$:

(a) $\cos 2x = \cos^2 x - \sin^2 x$. **Yes**

(b) $3 + x^2$: Cannot be expressed. **No**

(c) $1 = \cos^2 x + \sin^2 x$. **Yes**

(d) $\sin x$: Cannot be expressed. **No**

(e) $0 = 0\cos^2 x + 0\sin^2 x$. **Yes**

Exercise 15

Determine whether $\{\mathbf{u}, \mathbf{v}\}$ spans W (solution space of $A\mathbf{x} = \mathbf{0}$):

(a) $\mathbf{u} = (1,0,-1,0), \mathbf{v} = (0,1,0,-1)$: Check if basis for nullspace. **Yes**

(b) $\mathbf{u} = (1,0,-1,0), \mathbf{v} = (1,1,-1,-1)$: Check dimension. **Yes**

Exercise 16

Determine whether $\{\mathbf{u}, \mathbf{v}\}$ spans W:

(a) $\mathbf{u} = (1,1,1,0), \mathbf{v} = (0,-1,0,1)$: Check rank. **Yes**

(b) $\mathbf{u} = (0,1,1,0), \mathbf{v} = (1,0,1,1)$: Check rank. **No**

Exercise 17

Determine whether $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ spans $\mathbb{R}^2$:

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$: Vectors are linearly independent. **Yes**

(b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$: Vectors are linearly dependent. **No**

Exercise 18

Determine whether $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans $\mathbb{R}^2$:

(a) Check if any two are linearly independent. **Yes**

(b) Check rank. **Yes**

Exercise 19

Show that $\text{span}(p_1, p_2) = \text{span}(q_1, q_2)$ where $p_1 = 1 + x^2, p_2 = 1 + x + x^2, q_1 = 2x, q_2 = 1 + x^2$

Express each vector as linear combination of the other set:

$$
\begin{aligned}
p_1 &= 1q_2 \\
p_2 &= 1q_2 + \frac{1}{2}q_1 \\
q_1 &= 2p_2 - 2p_1 \\
q_2 &= 1p_1
\end{aligned}
$$

Thus spans are equal.

Exercise 20

Show that $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$ where: $\mathbf{v}_1 = (1, 6, 4), \mathbf{v}_2 = (2, 4, -1), \mathbf{v}_3 = (-1, 2, 5), \mathbf{w}_1 = (1, -2, -5), \mathbf{w}_2 = (0, 8, 9)$

Express each vector as linear combination:

$$
\begin{aligned}
\mathbf{v}_1 &= 1\mathbf{w}_1 + 1\mathbf{w}_2 \\
\mathbf{v}_2 &= 2\mathbf{w}_1 + 1\mathbf{w}_2 \\
\mathbf{v}_3 &= -1\mathbf{w}_1 + 1\mathbf{w}_2 \\
\mathbf{w}_1 &= 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 \\
\mathbf{w}_2 &= -1\mathbf{v}_1 + 2\mathbf{v}_2 - 1\mathbf{v}_3
\end{aligned}
$$

Thus spans are equal.