National University of Computer and Emerging Sciences

Final Exam Solutions - Fall 2025

Q1: Linear Transformations and Subspaces

Part (a): Linear Transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$

Given: $T(x_1, x_2, x_3) = (2x_1 + x_2 - 3x_3, -x_1 + 4x_2 + x_3)$.

i. Find the standard matrix A of T.

To find the standard matrix A, we apply T to the standard basis vectors e_1, e_2, e_3:

$$T(e_1) = T(1,0,0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$T(e_2) = T(0,1,0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$T(e_3) = T(0,0,1) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The standard matrix is:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 4 & 1 \end{bmatrix}$$

ii. Range Check and Pre-image

Determine if $b = (7, 2)$ is in the range. Solve $Ax = b$:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 7 \\ -1 & 4 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 2 \\ 2 & 1 & -3 & 7 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 2 \\ 0 & 9 & -1 & 11 \end{array}\right]$$

The system is consistent. Let $x_3 = -2$. From R_2: $9x_2 - (-2) = 11 \implies 9x_2 = 9 \implies x_2 = 1$.
From R_1: $-x_1 + 4(1) - 2 = 2 \implies -x_1 = 0 \implies x_1 = 0$.
Specific pre-image: $x = (0, 1, -2)$.

Grading Criteria for Q1(a)	Marks
i. Correct calculation of basis transformations	1
i. Correct Matrix A	1
ii. Setup of Augmented Matrix	1
ii. Correct Row Reduction/Consistency Check	2
ii. Specific pre-image vector found	1

Part (b): Subspace W

Equation: $x_1 + 2x_2 - x_3 + 3x_4 = 0$.

i. Show that W is a subspace.

1. **Zero Vector:** $0 + 2(0) - 0 + 3(0) = 0$. $\mathbf{0} \in W$.
2. **Additivity:** Let $u, v \in W$. Since equations are linear/homogeneous, $(u + v)$ satisfies the equation.
3. **Homogeneity:** Let $c \in \mathbb{R}$. $c(x_1 + \dots) = c(0) = 0$. Thus $cu \in W$.

Conclusion: W is a subspace.

ii. Check if $v = (1, -1, 3, 0) \in W$.

Substitute into equation:

$$(1) + 2(-1) - (3) + 3(0) = 1 - 2 - 3 = -4 \neq 0$$

Conclusion: $v \notin W$.

Grading Criteria for Q1(b)	Marks
i. Verifying Zero Vector, Additivity, and Homogeneity (1+2+2)	5
ii. Correct substitution and conclusion	2

Part (c): Null Space

Check if $u = (5, 3, -2)$ is in $Nul(A)$ for $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$.

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Conclusion: Yes, $u \in Nul(A)$.

Grading Criteria for Q1(c)	Marks
Matrix-Vector multiplication setup	2
Correct arithmetic	1
Correct Conclusion	1

Q2: Eigenvalues, QR, and Quadratic Forms

Part (a): Diagonalization

Given: $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

i. **Eigenvalues:** $\det(A - \lambda I) = (2-\lambda)[(-\lambda)(3-\lambda) - (-2)(1)] = (2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$. Roots: $\lambda = 1, 2, 2$.

ii. **Rank of $\lambda I - A$:** - For $\lambda = 1:$

$$I - A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two pivots. Rank = 2.
For $\lambda = 2:$

$$2I - A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One pivot. Rank = 1.

iii. **Diagonalizable?** For $\lambda = 2$, Alg. Mult (2) = Geo. Mult (3 − Rank(1) = 2). Yes, diagonalizable.

iv. **P and P^{-1}:**

- Eigenvector for $\lambda = 1:$ Solve $(I - A)x = 0$. From RREF in (ii): $x_1 = -2x_3, x_2 = x_3$. Let $x_3 = 1 \implies v_1 = (-2, 1, 1)$. - Eigenvectors for $\lambda = 2:$ Solve $(2I - A)x = 0$. From RREF in (ii): $x_1 = -x_3$. x_2 is free. Let $x_3 = 1, x_2 = 0 \implies v_2 = (-1, 0, 1)$. Let $x_3 = 0, x_2 = 1 \implies v_3 = (0, 1, 0)$.

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To find P^{-1}, use row reduction $[P \mid I]$ or cofactor method.

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

v. **Verify:** Calculate $P^{-1}AP$ yields $D = \text{diag}(1, 2, 2)$.

vi. **Find A^6:**

$$A^6 = PD^6P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix} P^{-1} = \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$

Grading Criteria for Q2(a)	Marks
i. Finding Eigenvalues (1+1+1)	3
ii. Rank calculation (3+3)	6
iii. Justification	2
iv. P and P^{-1} matrices (4+2)	6
v. Verification	2
vi. A^6 calculation	3

Part (b): QR Decomposition

Matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Using Gram-Schmidt:

$$\begin{aligned} &u_1 = (1,2,1) \quad (\text{norm } \sqrt{6}) \\ &u_2 = a_2 - \text{proj}_{u_1} a_2 = (7/6, -4/6, 1/6) \to \text{normalize } v_2 = (7,-4,1) \\ &u_3 = a_3 - \text{proj}_{u_1} a_3 - \text{proj}_{u_2} a_3 \propto (-1,-1,3) \end{aligned}$$

Resulting Matrices:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} & \frac{-1}{\sqrt{11}} \\ \frac{2}{\sqrt{6}} & \frac{-4}{\sqrt{66}} & \frac{-1}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{66}} & \frac{3}{\sqrt{11}} \end{bmatrix},$$

Matrix R: $R = Q^T A$ (Upper triangular). $r_{11} = \|u_1\| = \sqrt{6}.r_{12} = \langle q_1, a_2 \rangle = 5/\sqrt{6}.r_{13} = \langle q_1, a_3 \rangle = 5/\sqrt{6}.r_{22} = \|u_2\| = \sqrt{66}/6$. (Using original u_2) $r_{23} = \langle q_2, a_3 \rangle = 23/\sqrt{66}$. $r_{33} = \|u_3\| = 18\sqrt{11}/66 = 3/\sqrt{11}$.

$$R = \begin{bmatrix} \sqrt{6} & \frac{5}{\sqrt{6}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{\sqrt{66}}{6} & \frac{23}{\sqrt{66}} \\ 0 & 0 & \frac{3}{\sqrt{11}} \end{bmatrix}$$

Grading Criteria for Q2(b)	Marks
Gram-Schmidt Process (1+2+2)	5
Matrix Q construction	2
Matrix R calculation (6 ×0.5)	3

Part (c): Basis and Inner Product

1. **Basis Condition:** $\det([u_1, u_2, u_3]) \neq 0$.

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 4 & t \end{bmatrix} = -(t-6) \neq 0 \implies t \neq 6$$

2. **Distance and Product:** Let $t = 0$. $d(u_1, u_2) = \|(-1,-1,-2)\| = \sqrt{6}$. $\langle u_1, u_2 + u_3 \rangle = 29 + 2t$.

Grading Criteria for Q2(c)	**Marks**
Determinant and condition on t (2+2)	4
Distance and Inner Product (2+2)	4

Part (d): Quadratic Form

$Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$.

- Symmetric Matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$.
- Eigenvalues: $\lambda = 1, 4, 7$.
- New Form: $Q' = y_1^2 + 4y_2^2 + 7y_3^2$.
- **Nature:** Positive Definite (all $\lambda > 0$).

Grading Criteria for Q2(d)	**Marks**
Matrix Construction diagonal and off-diagonal (1+1)	2
Eigenvalues (1+1+1)	3
New Form	2
Definiteness Check	1

Q3: SVD and Matrix Factorization

Part (a): Singular Value Decomposition (SVD)

Matrix: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$.

1. **Compute $A^T A$:**

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

2. **Singular Values:** $\sigma_1 = \sqrt{3}, \sigma_2 = \sqrt{2}$.

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

3. **Matrix V :** Eigenvectors of $A^T A$ are standard basis vectors.

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. **Matrix U:** $u_i = \frac{1}{\sigma_i} A v_i$.

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find u_3 via Gram-Schmidt or Cross Product (must be orthogonal to u_1, u_2 and norm 1).

Cross product of $(1, 1, -1)$ and $(0, 1, 1)$ is $(2, -1, 1)$. Normalize: $u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

$$U = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Grading Criteria for Q3(a)	**Marks**
Computation of $A^T A$ and Eigenvalues	5
Singular Values (Σ)	3
Matrix V	4
Matrix U (including finding orthogonal u_3)	8

Part (b): Low-Rank Factorization $\hat{A} = CR$

Matrix: $A = \begin{bmatrix} 5 & 4 & 13 \\ 3 & 0 & 3 \\ 4 & 4 & 12 \end{bmatrix}$.

1. **Identify Basis (Matrix C):** Observe that $Col_3 = 1(Col_1) + 2(Col_2)$. Col_1 and Col_2 are linearly independent.
$$C = \begin{bmatrix} 5 & 4 \\ 3 & 0 \\ 4 & 4 \end{bmatrix}$$

2. **Coefficients (Matrix R):** Express columns of A via columns of C.
$Col_1 = 1 \cdot c_1 + 0 \cdot c_2$. $Col_2 = 0 \cdot c_1 + 1 \cdot c_2$. $Col_3 = 1 \cdot c_1 + 2 \cdot c_2$.
$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

3. **Result:** $\hat{A} = CR$ reconstructs A exactly.

Grading Criteria for Q3(b)	**Marks**
Identification of independent columns	5
Construction of C	3
Construction of R	5
Final Product Verification	2