Linear Algebra Lecture: Section 4.4 - Linear Independence

1 Linear Independence

1.1 Introduction

In this section, we study when vectors in a set are "redundant" - meaning some vectors can be expressed in terms of others. This concept is crucial for understanding the dimension of vector spaces.

1.2 Linear Independence and Dependence

Definition 1.1 (Linear Independence/Dependence)**.** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then:

- S is **linearly independent** if no vector in S can be expressed as a linear combination of the others.
- S is **linearly dependent** if it is not linearly independent.
- For a single vector: $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

1.3 The Fundamental Test for Linear Independence

Theorem 1.1 (4.4.1 - Fundamental Test)**.** A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only solution to:

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

is the **trivial solution**: $k_1 = k_2 = \cdots = k_r = 0$.

Proof Sketch. If a nontrivial solution exists, say $k_1 \neq 0$, then:

$$\mathbf{v}_1 = -\frac{k_2}{k_1}\mathbf{v}_2 - \cdots - \frac{k_r}{k_1}\mathbf{v}_r$$

So $\mathbf{v}_1$ can be expressed in terms of the others, making the set linearly dependent. □

1.4 Examples

Example 1.1 (1 - Standard Unit Vectors in $\mathbb{R}^n$)**.** The standard unit vectors in $\mathbb{R}^n$:

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 1)$$

are linearly independent.

Proof: Consider $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n = \mathbf{0}$.
This gives $(k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$, so $k_1 = k_2 = \cdots = k_n = 0$.

Example 1.2 (2 - Testing Vectors in $\mathbb{R}^3$)**.** Determine whether $\mathbf{v}_1 = (1, -2, 3), \mathbf{v}_2 = (5, 6, -1), \mathbf{v}_3 = (3, 2, 1)$ are linearly independent.

Solution: Solve $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$:

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

This gives the system:

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

Solving yields: $k_1 = -\frac{1}{2}t, k_2 = -\frac{1}{2}t, k_3 = t$ for any t.
Since nontrivial solutions exist (e.g., $t = 1$), the vectors are **linearly dependent**.

Example 1.3 (3 - Testing Vectors in $\mathbb{R}^4$)**.** Determine whether $\mathbf{v}_1 = (1, 2, 2, -1), \mathbf{v}_2 = (4, 9, 9, -4), \mathbf{v}_3 = (5, 8, 9, -5)$ are linearly independent.

Solution: Solve $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$:
System:

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

Solving shows only trivial solution: $k_1 = k_2 = k_3 = 0$.
Thus, vectors are **linearly independent**.

Example 1.4 (4 - Polynomials $1, x, x^2, \ldots, x^n$)**.** Show that $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in P_n.

Solution: Consider $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ for all x.

A nonzero polynomial of degree n has at most n roots. Since this holds for infinitely many x, all coefficients must be zero: $a_0 = a_1 = \cdots = a_n = 0$.

Thus, the set is linearly independent.

Example 1.5 (5 - Testing Polynomials)**.** Determine whether $p_1 = 1 - x, p_2 = 5 + 3x - 2x^2, p_3 = 1 + 3x - x^2$ are linearly independent.

Solution: Solve $k_1p_1 + k_2p_2 + k_3p_3 = 0$:

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0$$

Equating coefficients:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned}$$

This system has nontrivial solutions, so the polynomials are **linearly dependent**.

1.5 Special Cases Theorems

Theorem 1.2 (4.4.2 - Special Cases)**.**

(a) Any finite set containing the zero vector is linearly dependent.

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Proof. (a) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{0}\}$, then:

$$0\mathbf{v}_1 + \cdots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

is a nontrivial linear combination giving zero.

(b) If one vector is a scalar multiple of the other, they are linearly dependent. If not, the only solution to $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ is trivial. □

Theorem 1.3 (4.4.3 - Maximum Size in $\mathbb{R}^n$)**.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

- n = dimension of the space $\mathbb{R}^n$
- r = number of vectors in the set

Proof. The equation $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ gives a homogeneous system of n equations in r unknowns. If $r > n$, there are more unknowns than equations, so nontrivial solutions exist. □

Example 1.6 (6 - Geometric Interpretation)**. In $\mathbb{R}^2$:**

- Two vectors: linearly independent if they don't lie on the same line through origin
- Three vectors: always linearly dependent (by Theorem 4.4.3)

In $\mathbb{R}^3$:

- Two vectors: linearly independent if they don't lie on the same line
- Three vectors: linearly independent if they don't lie in the same plane
- Four vectors: always linearly dependent

1.6 Linear Independence of Functions

For functions, we need special tools since the "coefficient" approach doesn't always work directly.

Definition 1.2 (2 - Wronskian)**.** If $f_1, f_2, \ldots, f_n$ are $n-1$ times differentiable on $(-\infty, \infty)$, then the **Wronskian** is:

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem 1.4 (4.4.4 - Wronskian Test)**.** If $f_1, f_2, \ldots, f_n$ have $n-1$ continuous derivatives and $W(x)$ is **not identically zero** on $(-\infty, \infty)$, then the functions are linearly independent.

Warning: The converse is false! If $W(x) \equiv 0$, the functions might still be linearly independent.

Example 1.7 (8 - Using Wronskian)**.** Show that $f_1 = x$ and $f_2 = \sin x$ are linearly independent.

Solution: Compute Wronskian:

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

Since $W(\pi/2) = (\pi/2)(0) - 1 = -1 \neq 0$, the Wronskian is not identically zero.
Thus, f_1 and f_2 are linearly independent.

Example 1.8 (9 - Exponential Functions)**.** Show that $f_1 = 1, f_2 = e^x, f_3 = e^{2x}$ are linearly independent.

Solution: Compute Wronskian:

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

Since $W(x) = 2e^{3x} \neq 0$ for all x, the functions are linearly independent.

2 Exercise Solutions

Exercise 1

Explain why the following form linearly dependent sets:

(a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in $\mathbb{R}^3$

Solution: $\mathbf{u}_2 = -5\mathbf{u}_1$, so they are scalar multiples.

(b) $\mathbf{u}_1 = (3, -1), \mathbf{u}_2 = (4, 5), \mathbf{u}_3 = (-4, 7)$ in $\mathbb{R}^2$

Solution: Three vectors in $\mathbb{R}^2$ are always linearly dependent (Theorem 4.4.3).

(c) $p_1 = 3 - 2x + x^2$ and $p_2 = 6 - 4x + 2x^2$ in P_2
Solution: $p_2 = 2p_1$, so they are scalar multiples.
(d) $A = \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}
Solution: Neither is a scalar multiple of the other, but check: $k_1A + k_2B = 0$ gives:

$$\begin{cases} 3k_1 + 3k_2 = 0 \\ 4k_1 - 4k_2 = 0 \\ 2k_1 - 2k_2 = 0 \\ 0k_1 + 0k_2 = 0 \end{cases}$$

Solution: $k_1 = t, k_2 = -t$, so linearly dependent.

Exercise 2

Determine whether vectors are linearly independent in $\mathbb{R}^3$:
(a) $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$
Solution: Solve $k_1(-3, 0, 4) + k_2(5, -1, 2) + k_3(1, 1, 3) = (0, 0, 0)$
System:

$$\begin{aligned} -3k_1 + 5k_2 + k_3 &= 0 \\ 0k_1 - k_2 + k_3 &= 0 \\ 4k_1 + 2k_2 + 3k_3 &= 0 \end{aligned}$$

Solving gives only trivial solution, so **linearly independent**.
(b) $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$
Solution: Four vectors in $\mathbb{R}^3$ are always linearly dependent (Theorem 4.4.3).

Exercise 3

Determine whether vectors are linearly independent in $\mathbb{R}^4$:
(a) $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (4, 2, 6, 4)$
Solution: Form matrix with vectors as columns and compute rank. Rank ¡ 4, so **linearly dependent**.
(b) $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$
Solution: Form matrix and compute rank. Rank = 4, so **linearly independent**.

Exercise 4

Determine whether polynomials are linearly independent in P_2:
(a) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$
Solution: Solve $k_1(2, -1, 4) + k_2(3, 6, 2) + k_3(2, 10, -4) = (0, 0, 0)$

System:

$$\begin{aligned} 2k_1 + 3k_2 + 2k_3 &= 0 \\ -k_1 + 6k_2 + 10k_3 &= 0 \\ 4k_1 + 2k_2 - 4k_3 &= 0 \end{aligned}$$

Nontrivial solutions exist, so **linearly dependent**.
(b) $1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$
Solution: Four vectors in P_2 (dimension 3) are always linearly dependent.

Exercise 5

Determine whether matrices are linearly independent:
(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}
Solution: Solve $k_1A + k_2B + k_3C = 0$:
System:

$$\begin{aligned} k_1 + k_2 &= 0 \\ 0k_1 + 2k_2 + k_3 &= 0 \\ k_1 + 2k_2 + 2k_3 &= 0 \\ 2k_1 + k_2 + k_3 &= 0 \end{aligned}$$

Only trivial solution, so **linearly independent**.
(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}
Solution: These are standard basis vectors for M_{23}, so **linearly independent**.

Exercise 6

Determine all values of k for which matrices are linearly independent in M_{22}:
$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$
Solution: Set up determinant condition. The matrices are linearly independent when the determinant of the coefficient matrix is nonzero. Solving gives specific k values where they become dependent.

Exercise 7

Determine whether three vectors lie in a plane in $\mathbb{R}^3$:
(a) $\mathbf{v}_1 = (2, -2, 0), \mathbf{v}_2 = (6, 1, 4), \mathbf{v}_3 = (2, 0, -4)$
Solution: Vectors lie in a plane if they are linearly dependent. Check determinant:

$$\begin{vmatrix} 2 & 6 & 2 \\ -2 & 1 & 0 \\ 0 & 4 & -4 \end{vmatrix} \neq 0$$

So they are linearly independent and do NOT lie in a plane.

(b) $\mathbf{v}_1 = (-6, 7, 2), \mathbf{v}_2 = (3, 2, 4), \mathbf{v}_3 = (4, -1, 2)$

Solution: Check determinant $= 0$, so linearly dependent and lie in a plane.

Exercise 8

Determine whether three vectors lie on the same line:

(a) $\mathbf{v}_1 = (-1, 2, 3), \mathbf{v}_2 = (2, -4, -6), \mathbf{v}_3 = (-3, 6, 0)$

Solution: $\mathbf{v}_2 = -2\mathbf{v}_1$, but $\mathbf{v}_3$ is not a multiple, so not all on same line.

(b) $\mathbf{v}_1 = (2, -1, 4), \mathbf{v}_2 = (4, 2, 3), \mathbf{v}_3 = (2, 7, -6)$

Solution: No vector is a scalar multiple of another, so not on same line.

(c) $\mathbf{v}_1 = (4, 6, 8), \mathbf{v}_2 = (2, 3, 4), \mathbf{v}_3 = (-2, -3, -4)$

Solution: $\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1$ and $\mathbf{v}_3 = -\frac{1}{2}\mathbf{v}_1$, so all on same line.

Exercise 9

(a) Show that $\mathbf{v}_1 = (0, 3, 1, -1), \mathbf{v}_2 = (6, 0, 5, 1), \mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set.

Solution: Solve $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$:

Find nontrivial solution, e.g., $k_1 = 1, k_2 = -1, k_3 = 1$ works.

(b) Express each vector as linear combination of others:

From the dependence relation: $\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$.

Exercise 10

(a) Show that $\mathbf{v}_1 = (1, 2, 3, 4), \mathbf{v}_2 = (0, 1, 0, -1), \mathbf{v}_3 = (1, 3, 3, 3)$ form a linearly dependent set.

Solution: Notice $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, so they are linearly dependent.

(b) Express: $\mathbf{v}_1 = \mathbf{v}_3 - \mathbf{v}_2$, $\mathbf{v}_2 = \mathbf{v}_3 - \mathbf{v}_1$, $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Exercise 11

Find real values of λ for which vectors form a linearly dependent set in $\mathbb{R}^3$:

$\mathbf{v}_1 = (\lambda, -\frac{1}{2}, -\frac{1}{2}), \mathbf{v}_2 = (-\frac{1}{2}, \lambda, -\frac{1}{2}), \mathbf{v}_3 = (-\frac{1}{2}, -\frac{1}{2}, \lambda)$

Solution: Vectors are linearly dependent when determinant of matrix with these vectors as columns is zero:

$$\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = 0$$

Solving gives $\lambda = 1$ or $\lambda = -\frac{1}{2}$.

Exercise 12

Under what conditions is a set with one vector linearly independent?

Solution: A single vector $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

Exercise 13

Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be multiplication by A, and $\mathbf{u}_1 = (1, 2), \mathbf{u}_2 = (-1, 1)$. Determine whether $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ is linearly independent:

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

Solution: $T_A(\mathbf{u}_1) = (-1, 4), T_A(\mathbf{u}_2) = (-2, -2)$. These are not scalar multiples, so linearly independent.

(b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

Solution: $T_A(\mathbf{u}_1) = (-1, -2), T_A(\mathbf{u}_2) = (-2, 4)$. Check determinant:

$$\begin{vmatrix} -1 & -2 \\ -2 & 4 \end{vmatrix} = -4 - 4 = -8 \neq 0$$

So linearly independent.

Exercise 14

Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A, and $\mathbf{u}_1 = (1, 0, 0), \mathbf{u}_2 = (2, -1, 1), \mathbf{u}_3 = (0, 1, 1)$. Determine whether $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent:

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

Solution: Compute images and check if they span $\mathbb{R}^3$. They do, so linearly independent.

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

Solution: Compute images - they don't span $\mathbb{R}^3$, so linearly dependent.