Class Lecture: Section 4.8
Row Space, Column Space, and Null Space

Linear Algebra

Introduction to Matrix Spaces

In this section, we explore the fundamental vector spaces associated with matrices. These spaces provide deep insights into the structure of linear systems and their solutions.

Row and Column Vectors

Definition 1 (Row and Column Vectors of a Matrix)**.** *For an $m \times n$ matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the ***row vectors*** *are:*

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \ a_{12} \ \cdots \ a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \ a_{22} \ \cdots \ a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \ a_{m2} \ \cdots \ a_{mn}] \end{aligned}$$

and the ***column vectors*** *are:*

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \cdots, \ \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Example 1. *Let* $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

Row vectors:

- $\mathbf{r}_1 = [2 \ 1 \ 0]$

- $\mathbf{r}_2 = [3 \;\; -1 \;\; 4]$

Column vectors:

- $\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- $\mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Fundamental Matrix Spaces

Definition 2 (Row Space, Column Space, and Null Space)**.** *If A is an $m \times n$ matrix:*

- *The **row space** of A, denoted row(A), is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A*
- *The **column space** of A, denoted col(A), is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A*
- *The **null space** of A, denoted null(A), is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$*

Remark 1. *Throughout this section, we consider two fundamental questions:*

1. *What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A?*
2. *What relationships exist among the row space, column space, and null space of a matrix?*

These questions connect the algebraic properties of matrices with the geometric structure of their associated vector spaces.

Consistency and the Column Space

Theorem 1 (Theorem 4.8.1: Consistency Condition)**.** *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

Proof. The system $A\mathbf{x} = \mathbf{b}$ can be written as:

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of A. This equation holds if and only if $\mathbf{b}$ can be expressed as a linear combination of the column vectors, which is exactly the condition for $\mathbf{b}$ to be in the column space of A. □

Example 2. *Let $A\mathbf{x} = \mathbf{b}$ be the system:*

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Solving by Gaussian elimination yields: $x_1 = 2, x_2 = -1, x_3 = 3$
Therefore:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

This shows that $\mathbf{b}$ is indeed in the column space of A.

Homogeneous vs. Non-homogeneous Systems

The relationship between homogeneous and non-homogeneous systems is one of the most elegant and important concepts in linear algebra. Consider the systems:

$$A\mathbf{x} = \mathbf{0} \quad \text{(homogeneous)} \quad \text{and} \quad A\mathbf{x} = \mathbf{b} \quad \text{(non-homogeneous)}$$

If we have a particular solution $\mathbf{x}_p$ to the non-homogeneous system and any solution $\mathbf{x}_h$ to the homogeneous system, then:

$$A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So $\mathbf{x}_p + \mathbf{x}_h$ is also a solution to the non-homogeneous system. This reveals that the solution set of $A\mathbf{x} = \mathbf{b}$ is essentially a *translation* of the solution space of $A\mathbf{x} = \mathbf{0}$.

Theorem 2 (Theorem 4.8.2: Structure of Solutions)**.** *If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form:*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Remark 2. *The general solution of a consistent linear system can be expressed as:*

$$\textit{General Solution} = \textit{Particular Solution} + \textit{General Solution of Homogeneous System}$$

- $\mathbf{x}_0$ *is called a **particular solution** of $A\mathbf{x} = \mathbf{b}$*
- $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ *is called the **general solution** of $A\mathbf{x} = \mathbf{0}$*

Geometrically, the solution set of $A\mathbf{x} = \mathbf{b}$ is the translation by $\mathbf{x}_0$ of the solution space of $A\mathbf{x} = \mathbf{0}$.

Bases for Fundamental Spaces

Theorem 3 (Theorem 4.8.3: Invariance Under Row Operations)**.** *(a) Row equivalent matrices have the same row space*

(b) Row equivalent matrices have the same null space

Proof. (a) Elementary row operations involve only scalar multiplication and linear combinations of rows, so they preserve the row space

(b) Elementary row operations do not change the solution set of $A\mathbf{x} = \mathbf{0}$, so they preserve the null space

$\square$

Theorem 4 (Theorem 4.8.4: Bases from Row Echelon Form)**.** *If a matrix R is in row echelon form, then:*

- *The row vectors with the leading 1's form a basis for the row space of R*
- *The column vectors with the leading 1's of the row vectors form a basis for the column space of R*

Example 3. *Let* $R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row space basis:

- $[1 \ -2 \ 5 \ 0 \ 3]$
- $[0 \ 1 \ 3 \ 0 \ 0]$
- $[0 \ 0 \ 0 \ 1 \ 0]$

Column space basis:

- $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Example 4. *Find a basis for the row space of:*

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Reducing to row echelon form:

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space:

- $[1\ \ -3\ 4\ \ -2\ 5\ 4]$
- $[0\ 0\ 1\ 3\ \ -2\ \ -6]$
- $[0\ 0\ 0\ 0\ 1\ 5]$

Elementary Row Operations and Column Spaces

Theorem 5 (Theorem 4.8.5: Effect on Column Vectors)**.** *If A and B are row equivalent matrices, then:*

- *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent*
- *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B*

Remark 3. *Although elementary row operations can change the column space itself, they preserve the* dimension *of the column space and the linear dependence/independence relationships among column vectors.*

Example 5. *Find a basis for the column space of the matrix from Example 4 that consists of column vectors of A.*

From the row echelon form R, the pivot columns are columns 1, 3, and 5. Therefore, the corresponding columns of A:

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

Example 6. *Find a basis for the row space of*

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A.

Strategy: *Convert row space problem to column space problem via transposition.*

Transpose: $A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$

Reduce to row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns: 1, 2, 4. Corresponding rows of A:

- $[1\ \ -2\ 0\ 0\ 3]$ *(row 1)*
- $[2\ \ -5\ \ -3\ \ -2\ 6]$ *(row 2)*
- $[2\ 6\ 18\ 8\ 6]$ *(row 4)*

These form a basis for the row space of A.

Example 7. *Find a subset of* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ *that forms a basis for their span, where:*

$$\begin{aligned} \mathbf{v}_1 &= (1, 2, 2, -1) \\ \mathbf{v}_2 &= (-3, -6, -6, 3) \\ \mathbf{v}_3 &= (4, 9, 9, -4) \\ \mathbf{v}_4 &= (-2, -1, -1, 2) \\ \mathbf{v}_5 &= (5, 8, 9, -5) \\ \mathbf{v}_6 &= (4, 2, 7, -4) \end{aligned}$$

Form matrix with these vectors as columns and reduce to row echelon form. The pivot columns indicate that $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$ *form a basis.*

General Problem in $\mathbb{R}^n$

Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ in $\mathbb{R}^n$:

1. Find a subset of these vectors that forms a basis for span(S)

2. Express each vector that is not in the basis as a linear combination of the basis vectors

Example 8. *(a) Find a subset of the vectors that forms a basis for their span:*

$$\begin{aligned}\mathbf{v}_1 &= (1, -2, 0, 3)\\ \mathbf{v}_2 &= (2, -5, -3, 6)\\ \mathbf{v}_3 &= (0, 1, 3, 0)\\ \mathbf{v}_4 &= (2, -1, 4, -7)\\ \mathbf{v}_5 &= (5, -8, 1, 2)\end{aligned}$$

Form matrix with these as columns:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$
(b) Express other vectors as linear combinations:

$$\begin{aligned}\mathbf{v}_3 &= 2\mathbf{v}_1 - \mathbf{v}_2\\ \mathbf{v}_5 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4\end{aligned}$$

Procedure: Basis for Space Spanned by Vectors

1. **Form the matrix** A whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$

2. **Reduce the matrix** A to reduced row echelon form R

3. **Denote the column vectors** of R by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$

4. **Identify the pivot columns** of R (columns containing leading 1's). The corresponding column vectors of A form a basis for span(S)

5. **Obtain dependency equations** by expressing each $\mathbf{w}_i$ that does not contain a leading 1 as a linear combination of predecessors that do

6. **Replace vectors** $\mathbf{w}_i$ by $\mathbf{v}_i$ in each dependency equation to express non-basis vectors as linear combinations of basis vectors

This procedure efficiently solves both parts of the general problem: finding a basis and expressing all vectors in terms of that basis.

Exercise Solutions

Exercises 1-2: Express product as linear combination

Exercise 1a: $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

Exercise 1b: $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -13 \\ 22 \\ 17 \end{bmatrix}$

Exercise 2a: $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 25 \\ -13 \\ -1 \\ 48 \end{bmatrix}$

Exercise 2b: $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-5)\begin{bmatrix} 5 \\ -8 \end{bmatrix} = \begin{bmatrix} -19 \\ 58 \end{bmatrix}$

Exercises 3-4: Determine if b in col(A)

Exercise 3a: $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Augmented matrix: $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Last row: $0 = 1$, so system inconsistent. $\mathbf{b} \notin \text{col}(A)$

Exercise 3b: $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution: $x_1 = 1, x_2 = -2, x_3 = 0$. So $\mathbf{b} = 1\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \text{col}(A)$

Exercise 4a: $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 2 & 2 \end{bmatrix}$$

System inconsistent, so $\mathbf{b} \notin \text{col}(A)$

Exercise 4b: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 5 \\ 0 & 1 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Last row: $0 = 4$, so system inconsistent. $\mathbf{b} \notin \text{col}(A)$

Exercises 5-6: General solutions

Exercise 5: Particular: $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$
Homogeneous: $x_1 = 5r - 2s, x_2 = s, x_3 = s + t, x_4 = t$

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 5 \end{bmatrix} + r \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Exercise 6: Particular: $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$
Homogeneous: $x_1 = -3r + 4s, x_2 = r - s, x_3 = r, x_4 = s$

(a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Exercises 7-8: Vector form of general solution

Exercise 7a: $x_1 - 3x_2 = 1$, $2x_1 - 6x_2 = 2$

Augmented: $\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $x_2 = t$, then $x_1 = 1 + 3t$

General solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Homogeneous: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Exercise 7b: $x_1 + x_2 + 2x_3 = 5$, $x_1 + x_3 = -2$, $2x_1 + x_2 + 3x_3 = 3$

Augmented: $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $x_3 = t$, then $x_1 = -2 - t$, $x_2 = 7 - t$

General: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Homogeneous: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Exercise 8a: All equations are multiples of $x_1 - 2x_2 + x_3 + 2x_4 = -1$

Let $x_2 = r, x_3 = s, x_4 = t$, then $x_1 = -1 + 2r - s - 2t$

General: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Homogeneous: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Exercise 8b: System:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 4 \\ -2x_1 + x_2 + 2x_3 + x_4 &= -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 3 \\ 4x_1 - 7x_2 - 5x_4 &= -5 \end{aligned}$$

Augmented: $\begin{bmatrix} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Let $x_4 = t$, then $x_1 = 2 - t$, $x_2 = 1 - t$, $x_3 = -t$

General: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Homogeneous: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Exercises 9-10: Bases for null space and row space

Exercise 9a: $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 0, -16], [0, 1, -19]$

Null space: $x_1 = 16t, x_2 = 19t, x_3 = t$, so basis: $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

Exercise 9b: $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 0, -1/2]$

Null space: $x_1 = \frac{1}{2}t, x_2 = s, x_3 = t$, so basis: $\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Exercise 10a: $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row space basis: $[1, 0, 1, 0], [0, 1, 1, 0], [0, 0, 0, 1]$

Null space: $x_1 = -t, x_2 = -t, x_3 = t, x_4 = 0$, so basis: $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Exercise 10b: $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 0, 1, 2, 1], [0, 1, 1, 1, 2]$

Null space: Let $x_3 = r, x_4 = s, x_5 = t$, then $x_1 = -r - 2s - t$, $x_2 = -r - s - 2t$

Basis: $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Exercises 11-12: Bases from row echelon form

Exercise 11a: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 0, 2], [0, 0, 1]$

Column space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Exercise 11b: $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, -3, 0, 0], [0, 1, 0, 0]$

Column space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Exercise 12a: $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 2, 4, 5], [0, 1, -3, 0], [0, 0, 1, -3], [0, 0, 0, 1]$

Column space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Exercise 12b: $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row space basis: $[1, 2, -1, 5], [0, 1, 4, 3], [0, 0, 1, -7], [0, 0, 0, 1]$

Column space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix}$

Exercise 13: Methods for finding bases

13a: $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row space basis: $[1, 0, 1, 0, 1], [0, 1, -1, 0, -2], [0, 0, 0, 1, -2]$

Column space basis (from pivot columns): $\begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

13b: Using method from Example 6:

$A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Pivot columns 1,2,3 correspond to rows 1,2,3 of A

Basis from rows of A: $[1, -2, 5, 0, 3], [-2, 5, -7, 0, -6], [-1, 3, -2, 1, -3]$

Exercises 14-15: Basis for subspace spanned by vectors

Exercise 14: $(1, 1, -4, -3), (2, 0, 2, -2), (2, -1, 3, 2)$

Matrix: $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ -4 & 2 & 3 \\ -3 & -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: $(1, 1, -4, -3), (2, 0, 2, -2)$

Exercise 15: $(1, 1, 0, 0), (0, 0, 1, 1), (-2, 0, 2, 2), (0, -3, 0, 3)$

Matrix: $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: $(1, 1, 0, 0), (0, 0, 1, 1), (-2, 0, 2, 2)$

Exercises 16-17: Basis and linear combinations

Exercise 16: $v_1 = (1,0,1,1), v_2 = (-3,3,7,1), v_3 = (-1,3,9,3), v_4 = (-5,3,5,-1)$

Matrix: $\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: v_1, v_2

Dependencies: $v_3 = 2v_1 + v_2$, $v_4 = -2v_1 + v_2$

Exercise 17: $v_1 = (1,-1,5,2), v_2 = (-2,3,1,0), v_3 = (4,-5,9,4), v_4 = (0,4,2,-3), v_5 = (-7,18,2,-8)$

Matrix: $\begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: v_1, v_2, v_4

Dependencies: $v_3 = 2v_1 - v_2$, $v_5 = -4v_1 + 3v_2$

Exercises 18-19: Row space basis from rows of A

Exercise 18: From Exercise 10a: $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Pivot columns 1,2 correspond to rows 1,2 of A

Basis: $[1,4,5,2], [2,1,3,0]$

Exercise 19: From Exercise 10b: $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$$

RREF: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Pivot columns 1,2,3 correspond to rows 1,2,3 of A

Basis: $[1, 4, 5, 6, 9], [3, -2, 1, 4, -1], [-1, 0, -1, -2, -1]$

Exercise 20: Construct matrix with given null space

Null space basis: $v_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}$

We want A such that $Av_1 = 0$ and $Av_2 = 0$

Let $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$

From $Av_1 = 0$: $a - b + 3c + 2d = 0$, $e - f + 3g + 2h = 0$

From $Av_2 = 0$: $2a + 0b - 2c + 4d = 0$, $2e + 0f - 2g + 4h = 0$

Choose simple values: Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

Check: $Av_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Av_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (not zero)

Better: $A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Check: $Av_1 = \begin{bmatrix} 1 - 3 + 3 + 0 \\ 2 - 1 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (not zero)

Solution: Use rows orthogonal to both vectors. Dot products:

Row 1: $r \cdot v_1 = r_1 - r_2 + 3r_3 + 2r_4 = 0$

$r \cdot v_2 = 2r_1 + 0r_2 - 2r_3 + 4r_4 = 0$

Solve: From second: $r_1 = r_3 - 2r_4$

Substitute in first: $(r_3 - 2r_4) - r_2 + 3r_3 + 2r_4 = -r_2 + 4r_3 = 0$, so $r_2 = 4r_3$

Let $r_3 = 1, r_4 = 0$: $r_1 = 1, r_2 = 4, r_3 = 1, r_4 = 0 \rightarrow [1, 4, 1, 0]$

Let $r_3 = 0, r_4 = 1$: $r_1 = -2, r_2 = 0, r_3 = 0, r_4 = 1 \rightarrow [-2, 0, 0, 1]$

So $A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

Check: $Av_1 = \begin{bmatrix} 1 - 4 + 3 + 0 \\ -2 + 0 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$Av_2 = \begin{bmatrix} 2 + 0 - 2 + 0 \\ -4 + 0 + 0 + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Exercise 21: Linear transformations

$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix}$

(a) $b = (0, 0)$: Solve $Ax = 0$

Augmented: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8/3 & 0 \\ 0 & 1 & -4/3 & 0 \end{bmatrix}$

$x_1 = -\frac{8}{3}t, x_2 = \frac{4}{3}t, x_3 = t$

General: $x = t \begin{bmatrix} -8/3 \\ 4/3 \\ 1 \end{bmatrix}$

(b) $b = (1, 3)$: $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8/3 & 7/3 \\ 0 & 1 & -4/3 & -2/3 \end{bmatrix}$

$x_1 = \frac{7}{3} - \frac{8}{3}t, x_2 = -\frac{2}{3} + \frac{4}{3}t, x_3 = t$

General: $x = \begin{bmatrix} 7/3 \\ -2/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/3 \\ 4/3 \\ 1 \end{bmatrix}$

(c) $b = (-1, 1)$: $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8/3 & 1/3 \\ 0 & 1 & -4/3 & -2/3 \end{bmatrix}$

$x_1 = \frac{1}{3} - \frac{8}{3}t, x_2 = -\frac{2}{3} + \frac{4}{3}t, x_3 = t$

General: $x = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8/3 \\ 4/3 \\ 1 \end{bmatrix}$

Exercise 22: More linear transformations

$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

(a) $b = (0, 0, 0, 0)$: $Ax = 0$

Only trivial solution since columns linearly independent. $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $b = (1, 1, -1, -1)$: Check if consistent

Augmented: $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

From first two: $x_1 = 1/2, x_2 = 1$

Check third: $1/2 + 1 = 3/2 \neq -1$, so inconsistent

(c) $b = (2, 0, 0, 2)$: $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

From first two: $x_1 = 1, x_2 = 0$

Check third: $1 + 0 = 1 \neq 0$, so inconsistent

Exercises 23-24: Geometric interpretation

Exercise 23a: $x + y + z = 1$

Particular solution: $(1, 0, 0)$

Homogeneous: $x + y + z = 0 \to x = -s - t, y = s, z = t$

General: $(1, 0, 0) + s(-1, 1, 0) + t(-1, 0, 1)$

Exercise 23b: Plane in 3D parallel to plane through origin $x + y + z = 0$

Exercise 24a: $x + y = 1$

Particular: $(1, 0)$

Homogeneous: $x + y = 0 \to x = -t, y = t$

General: $(1, 0) + t(-1, 1)$

Exercise 24b: Line in 2D parallel to line through origin $x + y = 0$

Exercises 25-26: Linear systems

Exercise 25: $A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$

(a) Homogeneous: RREF: $\begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $x_2 = s, x_3 = t$, then $x_1 = -\frac{2}{3}s + \frac{1}{3}t$

General: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$

(b) Check $(1, 0, 1)$: $3(1) + 2(0) - 1(1) = 2 \neq 2$, $6(1) + 4(0) - 2(1) = 4 \neq 4$, $-3(1) - 2(0) + 1(1) = -2 \neq -2$

Actually $(1, 0, 1)$ gives $(2, 4, -2)$ which matches! So it is a solution.

(c) General: $(1, 0, 1) + s(-2/3, 1, 0) + t(1/3, 0, 1)$

(d) Direct: Augmented $\begin{bmatrix} 3 & 2 & -1 & 2 \\ 6 & 4 & -2 & 4 \\ -3 & -2 & 1 & -2 \end{bmatrix} \to$ RREF same as homogeneous

Let $x_2 = s, x_3 = t$, then $x_1 = 1 - \frac{2}{3}s + \frac{1}{3}t$

Exercise 26: $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix}$

(a) Homogeneous: RREF: $\begin{bmatrix} 1 & 0 & 11/5 \\ 0 & 1 & -2/5 \\ 0 & 0 & 0 \end{bmatrix}$

Let $x_3 = t$, then $x_1 = -\frac{11}{5}t, x_2 = \frac{2}{5}t$

General: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -11/5 \\ 2/5 \\ 1 \end{bmatrix}$

(b) Check $(1, 1, 1)$: $1 - 2 + 3 = 2 \neq 2$, $2 + 1 + 4 = 7 \neq 7$, $1 - 7 + 5 = -1 \neq -1$

Wait, $(1, 1, 1)$ gives $(2, 7, -1)$ which matches! So it is a solution.

(c) General: $(1,1,1)+t(-11/5,2/5,1)$

(d) Direct: Augmented $\begin{bmatrix}1 & -2 & 3 & 2\\ 2 & 1 & 4 & 7\\ 1 & -7 & 5 & -1\end{bmatrix} \to$ RREF same

Let $x_3 = t$, then $x_1 = 1 - \frac{11}{5}t, x_2 = 1 + \frac{2}{5}t$

Exercises 27-28: General solutions

Exercise 27: $\begin{bmatrix}3 & 4 & 1 & 2\\ 6 & 8 & 2 & 5\\ 9 & 12 & 3 & 10\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}3\\ 7\\ 13\end{bmatrix}$

Augmented RREF: $\begin{bmatrix}1 & 4/3 & 1/3 & 2/3 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

Let $x_2 = r, x_3 = s$, then $x_1 = 1 - \frac{4}{3}r - \frac{1}{3}s - \frac{2}{3}(1) = \frac{1}{3} - \frac{4}{3}r - \frac{1}{3}s$, $x_4 = 1$

General: $\begin{bmatrix}1/3\\ 0\\ 0\\ 1\end{bmatrix} + r\begin{bmatrix}-4/3\\ 1\\ 0\\ 0\end{bmatrix} + s\begin{bmatrix}-1/3\\ 0\\ 1\\ 0\end{bmatrix}$

Homogeneous: $r\begin{bmatrix}-4/3\\ 1\\ 0\\ 0\end{bmatrix} + s\begin{bmatrix}-1/3\\ 0\\ 1\\ 0\end{bmatrix}$

Particular: $(1/3, 0, 0, 1)$

Exercise 28: $\begin{bmatrix}9 & -3 & 5 & 6\\ 6 & -2 & 3 & 1\\ 3 & -1 & 3 & 14\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}4\\ 5\\ -8\end{bmatrix}$

Augmented RREF: $\begin{bmatrix}1 & -1/3 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

Let $x_2 = t$, then $x_1 = 1 + \frac{1}{3}t, x_3 = 2, x_4 = -1$

General: $\begin{bmatrix}1\\ 0\\ 2\\ -1\end{bmatrix} + t\begin{bmatrix}1/3\\ 1\\ 0\\ 0\end{bmatrix}$

Homogeneous: $t\begin{bmatrix}1/3\\ 1\\ 0\\ 0\end{bmatrix}$

Particular: $(1, 0, 2, -1)$

Exercises 29-30: Geometric interpretations

Exercise 29a: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Null space: $Ax = 0 \to x_2 = 0, x_1 = 0$, so x_3 free $\to$ z-axis

Column space: All vectors of form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \to$ xy-plane

Exercise 29b: Want null space = x-axis, column space = yz-plane

x-axis: vectors $(t, 0, 0)$

yz-plane: vectors $(0, a, b)$

Matrix where $(t, 0, 0)$ in null space: first column must be zero

Matrix where column space in yz-plane: all columns have first entry 0

So $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Check: Null space: $x_2 = 0, x_3 = 0 \to$ x-axis

Column space: span of $(0, 1, 0)$ and $(0, 0, 1) \to$ yz-plane

Exercise 30: Find 3×3 matrices with given null spaces

(a) Point (only zero vector): $A = I_3$ (null space $= \{0\}$)

(b) Line: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (null space = z-axis)

(c) Plane: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (null space = yz-plane)