

Lecture Notes: Gaussian Elimination

Mathematics Department

1 Introduction to Systems of Linear Equations

A system of linear equations can be represented compactly by an **augmented matrix**. The goal of solving such a system is to transform this matrix into a simpler, equivalent form from which the solutions can be found through back-substitution. The tools we use for this transformation are called **elementary row operations**.

2 Elementary Row Operations

There are three types of elementary row operations that can be performed on a matrix without changing the solution set of the corresponding linear system.

Type I: Row Swapping

Interchange any two rows.

$$R_i \leftrightarrow R_j$$

(Swap row i with row j).

Type II: Row Scaling

Multiply all entries in a row by a nonzero constant.

$$R_i \rightarrow kR_i \quad (k \neq 0)$$

(Replace row i with k times itself).

Type III: Row Addition

Replace a row with the sum of itself and a multiple of another row.

$$R_i \rightarrow R_i + kR_j$$

(Replace row i with itself plus k times row j).

3 Echelon Form and Row-Reduced Echelon Form

3.1 Echelon Form (Row Echelon Form)

A matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All **nonzero rows** are above any rows of all zeros.
2. The **leading entry** (first nonzero number from the left, also called the **pivot**) of a nonzero row is always strictly to the right of the leading entry of the row above it.
3. All entries **in a column below a pivot** are zeros.

Examples of Matrices in Echelon Form

Example 1:
$$\begin{bmatrix} \mathbf{3} & 1 & 4 & -1 \\ 0 & \mathbf{2} & -5 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Example 2:
$$\begin{bmatrix} \mathbf{1} & -2 & 1 & 0 & 9 \\ 0 & 0 & \mathbf{7} & 3 & -4 \\ 0 & 0 & 0 & \mathbf{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3:
$$\begin{bmatrix} \mathbf{5} & 0 & 2 & 8 \\ 0 & \mathbf{1} & -3 & 0 \\ 0 & 0 & \mathbf{6} & 7 \end{bmatrix}$$

Example 4:
$$\begin{bmatrix} 0 & \mathbf{2} & 1 & -3 \\ 0 & 0 & 0 & \mathbf{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The **pivots** are in bold. Notice how each pivot is to the right of the one above it, and all entries below each pivot are zero. The pivots can be any nonzero number.

3.2 Row-Reduced Echelon Form (RREF)

A matrix is in **row-reduced echelon form (RREF)** if it satisfies all the conditions for echelon form **plus** two additional conditions:

4. The **leading entry in each nonzero row is 1** (called a **leading 1**).
5. Each **column containing a leading 1 has zeros in all its other entries**.

RREF is a *special type* of echelon form. Every matrix has a unique RREF.

Examples of Matrices in RREF

Example 1:
$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 3 \\ 0 & \mathbf{1} & 0 & -2 \\ 0 & 0 & \mathbf{1} & 5 \end{bmatrix}$$

Example 2:
$$\begin{bmatrix} \mathbf{1} & 4 & 0 & 0 & 7 \\ 0 & 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3:
$$\begin{bmatrix} \mathbf{1} & 0 & 5 \\ 0 & \mathbf{1} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 4:
$$\begin{bmatrix} 0 & \mathbf{1} & -2 & 0 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The **leading 1s** are in bold. Notice that each leading 1 is the only nonzero entry in its entire column. This is the defining feature of RREF.

Comparison: Echelon Form vs. RREF

Echelon Form:
$$\begin{bmatrix} \mathbf{2} & 4 & 6 \\ 0 & \mathbf{3} & 9 \\ 0 & 0 & \mathbf{5} \end{bmatrix}$$

RREF:
$$\begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

Echelon Form:
$$\begin{bmatrix} \mathbf{1} & -2 & 5 & 1 \\ 0 & \mathbf{3} & 2 & -4 \\ 0 & 0 & 0 & \mathbf{7} \end{bmatrix}$$

RREF:
$$\begin{bmatrix} \mathbf{1} & 0 & 19/3 & 0 \\ 0 & \mathbf{1} & 2/3 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

The echelon form is useful for Gaussian elimination and back-substitution. The RREF provides the solution(s) to the system directly, without the need for back-substitution.

4 The Gaussian Elimination Algorithm

The process of using elementary row operations to transform a matrix into echelon form is called **Gaussian elimination**. Once in echelon form, the system is solved using **back-substitution**.

Step-by-Step Procedure: Gaussian Elimination

Step 1: Start with the leftmost nonzero column. This is your **pivot column**.

Step 2: Select a pivot. Choose a nonzero entry in the pivot column (often the top entry, or the one with the largest absolute value for numerical stability) and swap rows to move it to the pivot position.

Step 3: Create zeros below the pivot. Use the row addition operation ($R_i \rightarrow R_i + kR_{\text{pivot}}$) to make all entries below the pivot zero.

Step 4: Repeat. Cover the row containing the current pivot and all rows above it. Apply Steps 1–3 to the remaining submatrix. Continue until the matrix is in **echelon form**.

Step 5: Back-Substitute. Starting from the last nonzero row, solve for the variable corresponding to the pivot. Substitute this value into the equation above to solve for the next variable, and repeat until all variables are found.

5 Interpreting Echelon Form: Types of Solutions

The final echelon form of an augmented matrix reveals the solution to the system.

- **No Solution:** If the echelon form has a row of the form $[0 \ 0 \ 0 \ | \ k]$ where $k \neq 0$.
- **Unique Solution:** If every column except the last is a pivot column.
- **Infinitely Many Solutions:** If not every column (excluding the last) is a pivot column. The variables corresponding to **non-pivot columns** are **free variables**.

6 Solutions to Problems 1-4

Problem 3

Suppose that the augmented matrix for a linear system has been reduced by row operations to the given row echelon form. Identify the pivot rows and columns and solve the system.

a.
$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 2, 3.

The system is consistent with a unique solution.

From Row 3: $x_3 = 5$

From Row 2: $x_2 + 2(5) = 2 \Rightarrow x_2 + 10 = 2 \Rightarrow x_2 = -8$

From Row 1: $x_1 - 3(-8) + 4(5) = 7 \Rightarrow x_1 + 24 + 20 = 7 \Rightarrow x_1 = 7 - 44 = -37$

Solution: $x_1 = -37$, $x_2 = -8$, $x_3 = 5$

b. $\left[\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 2, 3.

The system is consistent with a unique solution.

From Row 3: $x_3 + x_4 = 2$ (Equation A)

From Row 2: $x_2 + 4x_3 - 9x_4 = 3$ (Equation B)

From Row 1: $x_1 + 8x_3 - 5x_4 = 6$ (Equation C)

Since there is no pivot in column 4, x_4 is a free variable. Let $x_4 = t$.

From Eq A: $x_3 = 2 - t$

Substitute into Eq B: $x_2 + 4(2 - t) - 9t = 3 \Rightarrow x_2 + 8 - 4t - 9t = 3 \Rightarrow x_2 = 3 - 8 + 13t = 13t - 5$

Substitute into Eq C: $x_1 + 8(2 - t) - 5t = 6 \Rightarrow x_1 + 16 - 8t - 5t = 6 \Rightarrow x_1 = 6 - 16 + 13t = 13t - 10$

Solution: $x_1 = 13t - 10$, $x_2 = 13t - 5$, $x_3 = 2 - t$, $x_4 = t$

c. $\left[\begin{array}{ccccc|c} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 3, 4.

The system is consistent. Columns 2 and 5 are not pivot columns, so x_2 and x_5 are free variables.

Let $x_2 = s$ and $x_5 = t$.

From Row 3: $x_4 + 3x_5 = 9 \Rightarrow x_4 + 3t = 9 \Rightarrow x_4 = 9 - 3t$

From Row 2: $x_3 + x_4 + 6x_5 = 5$. Substitute x_4 : $x_3 + (9 - 3t) + 6t = 5 \Rightarrow x_3 + 9 + 3t = 5 \Rightarrow x_3 = -4 - 3t$

From Row 1: $x_1 + 7x_2 - 2x_3 + 0x_4 - 8x_5 = -3$. Substitute all:

$$x_1 + 7s - 2(-4 - 3t) - 8t = -3$$

$$x_1 + 7s + 8 + 6t - 8t = -3$$

$$x_1 + 7s - 2t + 8 = -3$$

$$x_1 = -11 - 7s + 2t$$

Solution: $x_1 = -11 - 7s + 2t$, $x_2 = s$, $x_3 = -4 - 3t$, $x_4 = 9 - 3t$, $x_5 = t$

d. $\left[\begin{array}{ccc|c} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$

Pivot Rows: 1, 2. **Pivot Columns:** 1, 2.

The last row corresponds to the equation $0 = 1$, which is a contradiction.

Solution: The system is inconsistent. There is no solution.

Problem 4

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 2, 3.

The system is consistent with a unique solution.

Solution: $x_1 = -3$, $x_2 = 0$, $x_3 = 7$

b. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 2, 3.

The system is consistent. Column 4 is not a pivot column, so x_4 is a free variable. Let $x_4 = t$.

From Row 3: $x_3 + x_4 = -5 \Rightarrow x_3 = -5 - t$

From Row 2: $x_2 + 3x_4 = 2 \Rightarrow x_2 = 2 - 3t$

From Row 1: $x_1 - 7x_4 = 8 \Rightarrow x_1 = 8 + 7t$

Solution: $x_1 = 8 + 7t$, $x_2 = 2 - 3t$, $x_3 = -5 - t$, $x_4 = t$

c. $\left[\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 3, 4.

The system is consistent. Columns 2 and 5 are not pivot columns, so x_2 and x_5 are free variables.

Let $x_2 = s$ and $x_5 = t$.

From Row 3: $x_4 + 5x_5 = 8 \Rightarrow x_4 + 5t = 8 \Rightarrow x_4 = 8 - 5t$

From Row 2: $x_3 + 4x_5 = 7 \Rightarrow x_3 + 4t = 7 \Rightarrow x_3 = 7 - 4t$

From Row 1: $x_1 - 6x_2 + 0x_3 + 0x_4 + 3x_5 = -2$

$x_1 - 6s + 3t = -2$

$x_1 = -2 + 6s - 3t$

Solution: $x_1 = -2 + 6s - 3t$, $x_2 = s$, $x_3 = 7 - 4t$, $x_4 = 8 - 5t$, $x_5 = t$

d. $\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$

Pivot Rows: 1, 2, 3. **Pivot Columns:** 1, 3.

The last row corresponds to the equation $0 = 1$, which is a contradiction.

Solution: The system is inconsistent. There is no solution.