Class Lectures: Inner Product Spaces
Sections 6.1 and 6.2

Linear Algebra

Lecture 1: Section 6.1 - Inner Products

Introduction to Inner Product Spaces

In Chapter 3, we studied the dot product in $\mathbb{R}^n$ and used it to define length, angle, distance, and orthogonality. Now we generalize these concepts to arbitrary vector spaces.

Definition 1 (Inner Product)**.** An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V such that for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars k:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (Symmetry)
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (Additivity)
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ (Homogeneity)
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$ (Positivity)

A real vector space with an inner product is called a **real inner product space**.

Examples of Inner Products

Example 1 (Euclidean Inner Product)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the **Euclidean inner product** is:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Example 2 (Weighted Euclidean Inner Product)**.** For $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ with weights $w_1, w_2 > 0$:

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2$$

For example, $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ defines an inner product.

Norm and Distance in Inner Product Spaces

Definition 2 (Norm and Distance)**.** If V is a real inner product space, then:

- The **norm** of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$
- The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$
- A vector of norm 1 is called a **unit vector**

Theorem 1 (Properties of Norm and Distance)**.** For vectors $\mathbf{u}, \mathbf{v}$ in a real inner product space and scalar k:

1. $\|\mathbf{v}\| \geq 0$ with equality iff $\mathbf{v} = \mathbf{0}$
2. $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality iff $\mathbf{u} = \mathbf{v}$

Matrix Inner Products

Definition 3 (Matrix Inner Product)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ (in column form) and A be an invertible $n \times n$ matrix. The **inner product generated by** A is:

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = \mathbf{v}^T A^T A \mathbf{u}$$

Example 3. The weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ is generated by:

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Other Important Inner Products

Example 4 (Standard Inner Product on M_{nn})**.** For matrices $U, V \in M_{nn}$:

$$\langle U, V \rangle = \mathrm{tr}(U^T V) = \sum_{i,j} u_{ij} v_{ij}$$

Example 5 (Standard Inner Product on P_n)**.** For polynomials $p = a_0 + a_1x + \cdots + a_nx^n$, $q = b_0 + b_1x + \cdots + b_nx^n$:

$$\langle p, q \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$

Example 6 (Evaluation Inner Product on P_n)**.** For distinct sample points $x_0, x_1, \ldots, x_n$:

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n)$$

Example 7 (Integral Inner Product on $C[a, b)$. $]For continuous functionsf, gon$[a,b]: $\langle f, g \rangle = \int_a^b f(x)g(x)dx$

Algebraic Properties

Theorem 2 (Algebraic Properties of Inner Products)**.** For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a real inner product space and scalar k:

1. $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$
4. $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$
5. $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$

Lecture 2: Section 6.2 - Angle and Orthogonality

Cauchy-Schwarz Inequality

Theorem 3 (Cauchy-Schwarz Inequality)**.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, then:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

Proof Sketch: Consider $\langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v} \rangle \geq 0$ for all $t \in \mathbb{R}$. This gives a quadratic in t that must have non-positive discriminant, leading to the inequality.

Angle Between Vectors

From the Cauchy-Schwarz inequality, we have:

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1$$

Definition 4 (Angle Between Vectors)**.** The **angle** θ between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by:

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right), \quad 0 \leq \theta \leq \pi$$

Example 8. In M_{22} with standard inner product, for:

$$U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

We have $\langle U, V \rangle = 16$, $\|U\| = \sqrt{30}$, $\|V\| = \sqrt{14}$, so:

$$\cos\theta = \frac{16}{\sqrt{30}\sqrt{14}} \approx 0.78$$

Triangle Inequalities

Theorem 4 (Triangle Inequalities)**.** For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a real inner product space:

1. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (Triangle inequality for vectors)
2. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (Triangle inequality for distances)

Orthogonality

Definition 5 (Orthogonal Vectors)**.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example 9 (Orthogonality Depends on Inner Product)**.** Vectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$ are:

- Orthogonal with Euclidean inner product: $\mathbf{u} \cdot \mathbf{v} = 0$
- Not orthogonal with weighted inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$: $\langle \mathbf{u}, \mathbf{v} \rangle = 1$

Theorem 5 (Generalized Theorem of Pythagoras)**.** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a real inner product space, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Orthogonal Complements

Definition 6 (Orthogonal Complement)**.** If W is a subspace of a real inner product space V, then the **orthogonal complement** of W is:

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$$

Theorem 6 (Properties of Orthogonal Complements)**.** If W is a subspace of a real inner product space V, then:

1. $W^\perp$ is a subspace of V
2. $W \cap W^\perp = \{\mathbf{0}\}$
3. If V is finite-dimensional, then $(W^\perp)^\perp = W$

Example 10 (Finding an Orthogonal Complement)**.** For $W = \text{span}\{(1, 3, -2, 0, 2, 0), (2, 6, -5, -2, 4, -3), .$ in $\mathbb{R}^6$, the orthogonal complement $W^\perp$ is the null space of the matrix whose rows are the basis vectors of W.

Key Points to Remember

- Inner products generalize the dot product to arbitrary vector spaces
- Different inner products give different notions of length, distance, and orthogonality
- The Cauchy-Schwarz inequality is fundamental for defining angles
- Orthogonality depends on the choice of inner product
- Orthogonal complements generalize the concept of perpendicular subspaces