Class Lecture: Eigenvalues, Eigenvectors, and Diagonalization

Mathematics 301: Linear Algebra

Chapter 5, Sections 5.1-5.2

5.1 Eigenvalues and Eigenvectors

Introduction

Eigenvalues and eigenvectors are fundamental concepts in linear algebra with applications across mathematics, physics, engineering, and computer science. The term "eigen" comes from German meaning "own," "characteristic," or "individual."

Definition 1. If A is an $n \times n$ matrix, then a **nonzero** vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an eigenvector corresponding to λ.

Geometric Interpretation: When $\mathbf{x}$ is an eigenvector, multiplication by A preserves the direction of $\mathbf{x}$, only scaling it by the eigenvalue λ.

Computing Eigenvalues

Theorem 1 (Characteristic Equation)**.** If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies:

$$\det(\lambda I - A) = 0$$

This is called the **characteristic equation** of A.

The polynomial $p(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial**. An $n \times n$ matrix has at most n distinct eigenvalues.

Example 1. Find eigenvalues of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Solution:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 3)(\lambda + 1) = 0 \\ \Rightarrow \lambda &= 3, -1 \end{aligned}$$

Theorem 2. If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on the main diagonal.

Finding Eigenvectors and Eigenspaces

For each eigenvalue λ, the eigenvectors are the nonzero solutions of:

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

The solution space of this system is called the **eigenspace** corresponding to λ.

Example 2. Find bases for eigenspaces of $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

Solution: Characteristic equation: $\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0$
For $\lambda = 2$:

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -3$:

$$\mathbf{x} = t \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

Important Properties

Theorem 3. A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

5.2 Diagonalization

Similarity Transformations

Definition 2. If A and B are square matrices, then B is **similar** to A if there exists an invertible matrix P such that:

$$B = P^{-1}AP$$

Similar matrices share many properties:

- Same determinant
- Same invertibility status
- Same rank and nullity
- Same trace
- Same characteristic polynomial
- Same eigenvalues

Definition 3. A square matrix A is **diagonalizable** if it is similar to a diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

When is a Matrix Diagonalizable?

Theorem 4. If A is an $n \times n$ matrix, the following are equivalent:

1. A is diagonalizable
2. A has n linearly independent eigenvectors

Theorem 5.
1. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.
2. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Procedure for Diagonalization

1. Find all eigenvalues of A
2. For each eigenvalue λ, find a basis for the eigenspace
3. If total number of basis vectors $= n$, then A is diagonalizable
4. Form P with eigenvectors as columns
5. $P^{-1}AP = D$ will be diagonal with eigenvalues on diagonal

Example 3. Diagonalize $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Solution: Characteristic polynomial: $(\lambda - 1)(\lambda - 2)^2$
Eigenvectors:

$$\lambda = 2 : \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1 : \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Let $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, then:

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Computing Powers of Matrices

If $A = PDP^{-1}$, then:

$$A^k = PD^kP^{-1}$$

This provides an efficient way to compute high powers of matrices.

Geometric and Algebraic Multiplicity

Definition 4. For an eigenvalue λ_0 of A:

- **Geometric multiplicity**: dimension of the eigenspace
- **Algebraic multiplicity**: number of times $(\lambda - \lambda_0)$ appears as a factor in the characteristic polynomial

Theorem 6.
1. Geometric multiplicity $\leq$ Algebraic multiplicity
2. A is diagonalizable if and only if for every eigenvalue, geometric multiplicity = algebraic multiplicity

Key Results

Theorem 7. If λ is an eigenvalue of A with eigenvector $\mathbf{x}$, then for any positive integer k, λ^k is an eigenvalue of A^k with the same eigenvector $\mathbf{x}$.

Theorem 8. If A is diagonalizable and invertible, then A^{-1} is also diagonalizable.