

Orthogonal Matrices

Definition 1

A square matrix A is said to be ***orthogonal*** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^TA = I \tag{1}$$

A matrix transformation $T_A : R^n \rightarrow R^n$ is said to be an ***orthogonal transformation*** or an ***orthogonal operator*** if A is an orthogonal matrix.

EXAMPLE 1 | A 3×3 Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



EXAMPLE 2 | Rotation and Reflection Matrices Are Orthogonal

Recall from Table 5 of Section 1.8 that the standard matrix for the counterclockwise rotation about the origin of R^2 through an angle θ is

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

This matrix is orthogonal for all choices of θ since

$$A^T A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We leave it for you to verify that the reflection matrices in Tables 1 and 2 of Section 1.8 are all orthogonal.

Theorem 7.1.1

The following are equivalent for an $n \times n$ matrix A .

- (a) A is orthogonal.
- (b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
- (c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Warning Note that an orthogonal matrix has *orthonormal* rows and columns—not simply orthogonal rows and columns.

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \mathbf{c}_1^T & \mathbf{r}_1 \mathbf{c}_2^T & \cdots & \mathbf{r}_1 \mathbf{c}_n^T \\ \mathbf{r}_2 \mathbf{c}_1^T & \mathbf{r}_2 \mathbf{c}_2^T & \cdots & \mathbf{r}_2 \mathbf{c}_n^T \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \mathbf{c}_1^T & \mathbf{r}_n \mathbf{c}_2^T & \cdots & \mathbf{r}_n \mathbf{c}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_n \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_n \cdot \mathbf{r}_1 & \mathbf{r}_n \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{r}_n \end{bmatrix}$$

It is evident from this formula that $AA^T = I$ if and only if

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}_2 \cdot \mathbf{r}_2 = \cdots = \mathbf{r}_n \cdot \mathbf{r}_n = 1$$

and

$$\mathbf{r}_i \cdot \mathbf{r}_j = 0 \quad \text{when } i \neq j$$

which are true if and only if $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$ is an orthonormal set in R^n . ■

Theorem 7.1.2

- (a) The transpose of an orthogonal matrix is orthogonal.
- (b) The inverse of an orthogonal matrix is orthogonal.
- (c) A product of orthogonal matrices is orthogonal.
- (d) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

EXAMPLE 3 | $\det(A) = \pm 1$ for an Orthogonal Matrix A

The matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal since its row (and column) vectors form orthonormal sets in R^2 with the Euclidean inner product. We leave it for you to verify that $\det(A) = 1$ and that interchanging the rows produces an orthogonal matrix whose determinant is -1 .

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find its inverse.

1. a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

Sol:

1. (a) $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$ and $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$ therefore A is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$ and $A^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$ therefore

A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

2. (a) $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ therefore A is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $AA^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{bmatrix} \neq I$ therefore A is not an orthogonal matrix.

3. a.
$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

b.
$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

4. a.
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Sol:

3. (a) $\| \mathbf{r}_1 \| = \sqrt{0^2 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1$ so the matrix is not orthogonal.

(b) $AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$ and $A^T A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$

therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4. (a) $AA^T = A^T A = I$ therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

(b) $\| \mathbf{r}_2 \| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{7}{12}} \neq 1$. The matrix is not orthogonal.

In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating $A^T A$, then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1.

$$5. \quad A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$6. \quad A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Theorem 7.1.1

The following are equivalent for an $n \times n$ matrix A .

- (a) A is orthogonal.
- (b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
- (c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Sol:

$$5. \quad A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} = I;$$

row vectors of A , $\mathbf{r}_1 = \left[\frac{4}{5} \ 0 \ -\frac{3}{5} \right]$, $\mathbf{r}_2 = \left[-\frac{9}{25} \ \frac{4}{5} \ -\frac{12}{25} \right]$, $\mathbf{r}_3 = \left[\frac{12}{25} \ \frac{3}{5} \ \frac{16}{25} \right]$, form an orthonormal set since

$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$ and $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$;

$$\text{column vectors of } A, \quad \mathbf{c}_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{9}{25} \\ \frac{12}{25} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}, \text{ form an orthonormal set since}$$

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$ and $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$.

$$6. \quad A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = I$$

row vectors of A , $\mathbf{r}_1 = \left[\frac{1}{3} \ \frac{2}{3} \ -\frac{2}{3} \right]$, $\mathbf{r}_2 = \left[\frac{2}{3} \ -\frac{2}{3} \ -\frac{1}{3} \right]$, $\mathbf{r}_3 = \left[-\frac{2}{3} \ -\frac{1}{3} \ \frac{2}{3} \right]$, form an orthonormal set since

$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$ and $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$;

$$\text{column vectors of } A, \quad \mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \text{ form an orthonormal set since}$$

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$ and $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$.