Class Lecture: Section 4.1 - Real Vector Spaces

Linear Algebra

Vector Space Axioms

Definition 1 (Vector Space)**.** *Let V be an arbitrary nonempty set of objects for which two operations are defined:*

- ***Addition**: A rule for associating with each pair of objects u and v in V an object $u+v$, called the **sum** of u and v*
- ***Scalar multiplication**: A rule for associating with each scalar k and each object u in V an object ku, called the **scalar multiple** of u by k*

*If the following axioms are satisfied by all objects u, v, w in V and all scalars k and m, then we call V a **vector space** and the objects in V **vectors**:*

1. *If u and v are objects in V, then $u+v$ is in V.* *(Closure under addition)*
2. $u+v=v+u$ *(Commutativity)*
3. $u+(v+w)=(u+v)+w$ *(Associativity)*
4. *There exists an object in V, called the **zero vector**, denoted by 0, such that $0+u=u+0=u$ for all u in V*
5. *For each u in V, there is an object $-u$ in V, called a **negative** of u, such that $u+(-u)=(-u)+u=0$*
6. *If k is any scalar and u is any object in V, then ku is in V.* *(Closure under scalar multiplication)*
7. $k(u+v)=ku+kv$ *(Distributivity)*
8. $(k+m)u=ku+mu$ *(Distributivity)*
9. $k(mu)=(km)(u)$ *(Associativity of scalar multiplication)*
10. $1u=u$ *(Identity element)*

Steps to Verify a Vector Space

To show that a set with two operations is a vector space:

1. Identify the set V of objects that will become vectors
2. Identify the addition and scalar multiplication operations on V
3. Verify Axioms 1 and 6 (closure under addition and scalar multiplication)
4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold

Examples of Vector Spaces

Example 1 (The Zero Vector Space)**.** *Let V consist of a single object, denoted by 0, and define:*

$$0 + 0 = 0 \quad \text{and} \quad k0 = 0 \quad \text{for all scalars } k$$

All vector space axioms are satisfied. This is the ***zero vector space****.*

Example 2 ($\mathbb{R}^n$ is a Vector Space)**.** *Let $V = \mathbb{R}^n$ with the usual operations:*

$$u + v = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

$$ku = (ku_1, ku_2, \ldots, ku_n)$$

$\mathbb{R}^n$ is closed under addition and scalar multiplication, and satisfies all vector space axioms.

Example 3 (Vector Space of Infinite Sequences)**.** *Let V consist of all infinite sequences of real numbers:*

$$u = (u_1, u_2, \ldots, u_n, \ldots)$$

with componentwise operations:

$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots)$$

$$ku = (ku_1, ku_2, \ldots, ku_n, \ldots)$$

This vector space is denoted by $\mathbb{R}^\infty$.

Example 4 (Vector Space of 2×2 Matrices)**.** *Let V be the set of all 2×2 matrices with real entries, with matrix addition and scalar multiplication:*

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$ku = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

The zero vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the negative of u is $\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$.

Example 5 (Vector Space of Real-Valued Functions)**.** *Let V be the set of all real-valued functions defined on $(-\infty, \infty)$, with operations:*

$$(f+g)(x) = f(x) + g(x), \quad (kf)(x) = kf(x)$$

The zero vector is the function $f(x) = 0$ for all x, and the negative of f is $-f$ where $(-f)(x) = -f(x)$. This space is denoted by $F(-\infty, \infty)$.

Example 6 (A Set That Is Not a Vector Space)**.** *Let $V = \mathbb{R}^2$ with standard addition but non-standard scalar multiplication:*

$$u + v = (u_1 + v_1, u_2 + v_2), \quad ku = (ku_1, 0)$$

For $u = (u_1, u_2)$ with $u_2 \neq 0$, we have:

$$1u = 1(u_1, u_2) = (u_1, 0) \neq u$$

So Axiom 10 fails, and V is not a vector space.

Theorem 4.1.1

Theorem 1 (Properties of Vector Spaces)**.** *Let V be a vector space, u a vector in V, and k a scalar. Then:*

(a) $0u = 0$

(b) $k0 = 0$

(c) $(-1)u = -u$

(d) If $ku = 0$, then $k = 0$ or $u = 0$

Proof of (a).

$$\begin{aligned} 0u + 0u &= (0+0)u \quad \text{[Axiom 8]} \\ &= 0u \quad \text{[Property of the number 0]} \end{aligned}$$

Add $-(0u)$ to both sides:

$$\begin{aligned} [0u + 0u] + (-0u) &= 0u + (-0u) \\ 0u + [0u + (-0u)] &= 0u + (-0u) \quad \text{[Axiom 3]} \\ 0u + 0 &= 0 \quad \text{[Axiom 5]} \\ 0u &= 0 \quad \text{[Axiom 4]} \end{aligned}$$

□

Proof of (b).

$$\begin{aligned} k0 + k0 &= k(0+0) \quad \text{[Axiom 7]} \\ &= k0 \quad \text{[Axiom 4]} \end{aligned}$$

Add $-(k0)$ to both sides:

$$\begin{aligned} [k0 + k0] + (-k0) &= k0 + (-k0) \\ k0 + [k0 + (-k0)] &= k0 + (-k0) \quad \text{[Axiom 3]} \\ k0 + 0 &= 0 \quad \text{[Axiom 5]} \\ k0 &= 0 \quad \text{[Axiom 4]} \end{aligned}$$

□

Proof of (c). We show that $u + (-1)u = 0$:

$$\begin{aligned} u + (-1)u &= 1u + (-1)u \quad \text{[Axiom 10]} \\ &= (1 + (-1))u \quad \text{[Axiom 8]} \\ &= 0u \quad \text{[Property of numbers]} \\ &= 0 \quad \text{[Part (a)]} \end{aligned}$$

Therefore, $(-1)u = -u$ by the uniqueness of additive inverses. □

Proof of (d). Suppose $ku = 0$ and $k \neq 0$. Then:

$$\begin{aligned} \frac{1}{k}(ku) &= \frac{1}{k}0 \quad \text{[Multiply both sides by } \frac{1}{k}\text{]} \\ \left(\frac{1}{k}k\right) u &= 0 \quad \text{[Axiom 9 and part (b)]} \\ 1u &= 0 \quad \text{[Property of numbers]} \\ u &= 0 \quad \text{[Axiom 10]} \end{aligned}$$

So if $ku = 0$ and $k \neq 0$, then $u = 0$. □

Exercises

1. Let V be the set of all ordered pairs of real numbers, with operations:

$$u + v = (u_1 + v_1, u_2 + v_2), \quad ku = (0, ku_2)$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$.

Solution:

(a) For $u = (-1, 2)$, $v = (3, 4)$, $k = 3$:

$$u + v = (-1 + 3, 2 + 4) = (2, 6), \quad ku = (0, 3 \cdot 2) = (0, 6)$$

(b) V is closed under addition because the sum of two ordered pairs is another ordered pair. Closed under scalar multiplication because $(0, ku_2)$ is an ordered pair.

(c) Axioms 2, 3, 7, 8, 9 hold because addition is standard.

(d) Axiom 7: $k(u+v) = (0, k(u_2+v_2)) = (0, ku_2) + (0, kv_2) = ku + kv$
Axiom 8: $(k+m)u = (0, (k+m)u_2) = (0, ku_2) + (0, mu_2) = ku + mu$
Axiom 9: $k(mu) = k(0, mu_2) = (0, kmu_2) = (km)(0, u_2) = (km)u$

(e) Axiom 10 fails: $1u = (0, u_2) \neq u$ when $u_1 \neq 0$. So V is not a vector space.

2. Let V be the set of all ordered pairs of real numbers, with operations:

$$u + v = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad ku = (ku_1, ku_2)$$

Solution:

(a) For $u = (0, 4)$, $v = (1, -3)$, $k = 2$:

$$u + v = (0 + 1 + 1, 4 + (-3) + 1) = (2, 2), \quad ku = (2 \cdot 0, 2 \cdot 4) = (0, 8)$$

(b) $(0, 0)$ is not the zero vector because $(0, 0) + u = (0 + u_1 + 1, 0 + u_2 + 1) = (u_1 + 1, u_2 + 1) \neq u$

(c) $(-1, -1)$ is the zero vector because $(-1, -1) + u = (-1 + u_1 + 1, -1 + u_2 + 1) = (u_1, u_2) = u$

(d) For $u = (u_1, u_2)$, the negative is $-u = (-u_1 - 2, -u_2 - 2)$ because:

$$u + (-u) = (u_1 + (-u_1 - 2) + 1, u_2 + (-u_2 - 2) + 1) = (-1, -1) = 0$$

(e) Axiom 8 fails: $(k+m)u = ((k+m)u_1, (k+m)u_2)$ but $ku + mu = (ku_1, ku_2) + (mu_1, mu_2) = (ku_1 + mu_1 + 1, ku_2 + mu_2 + 1) \neq (k+m)u$

9. Determine whether the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with standard operations is a vector space.

Solution: Yes, this is a vector space (subspace of M_{22}). It is closed under addition and scalar multiplication:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$$

$$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}$$

All vector space axioms are satisfied.

11. Determine whether the set of all pairs of real numbers of the form $(1, x)$ with operations:

$$(1, y) + (1, y') = (1, y + y'), \quad k(1, y) = (1, ky)$$

is a vector space.

Solution: This is a vector space. The zero vector is $(1, 0)$, the negative of $(1, y)$ is $(1, -y)$. All axioms can be verified.

12. Determine whether the set of polynomials of the form $a_0 + a_1x$ with operations:

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x, \quad k(a_0 + a_1x) = (ka_0) + (ka_1)x$$

is a vector space.

Solution: Yes, this is the vector space P_1 (polynomials of degree 1). It satisfies all vector space axioms.