Progsheet 2 - Introduction to the Numerics of partial differential equations

Author: Sven Ullmann and Jonathan Schnitzler

This is an exercise sheet for Linear FEM.

We create a Trianglulation `[p,e,t]` with `n:=|I|` according to Exercise Sheet 03 from Introduction to the numerics of ordinary differential equations. With the given two meshes we want to solve the poisson problem

$$-\Delta u = 1 \quad \text{on}\, \Omega$$
$$u = 0 \quad \text{on}\, \partial\Omega$$

We will consider **solve** and **adaptive solve**

a) Poisson initialization

Firstfully we consider what is p, e and t of a triangulation. We will use the `Parial Differential Equation Toolbox` from `Matlab` with important functions such as:

- `initmesh` - initializes a mesh
- `pdegplot(.)` - plots the geometry (g) of the pde

The Triplet `[p,e,t]` is of major importance. They represent the mesh data and the components stand for

p - node points:

Besteht aus x und y daten

Wird bei bisect beispielsweise um 1 erhöht

e - edge assiciativity - Boundary:

Rows:

1. & 2. indices of starting and ending point,

3. & 4. the starting and ending parameter values

5. the boundary edge segment number

6. & 7. left- and right-hand side subdomain numbers.

t - triangular elements:

Rows:

1. - 3. Corners in counter clockwise order

4. Number of domain

(I think we don't use Decomposed geometry matrix DECSG

and instead use a geometry representation matlab file like `squareg.m` or `sectorg.m`)

I - index set:

set of indices of the inner points of a triangulation

Calculation on reference Element and B

```
% function [B,shift] = get_B(p1,p2,p3)
% %Calculates the linear transformation B and shift t
% shift = p1;
% B = [p2-shift, p3-shift];
% end
```

Integration on reference Element

The reference Element is the Simplex with Edges (0,0),(1,0),(0,1)

Function

```
reference = @(x,y) [1-x-y, x, y];
```

Derivative

```
grad_reference = [-1, 1, 0;...
                  -1, 0, 1];
```

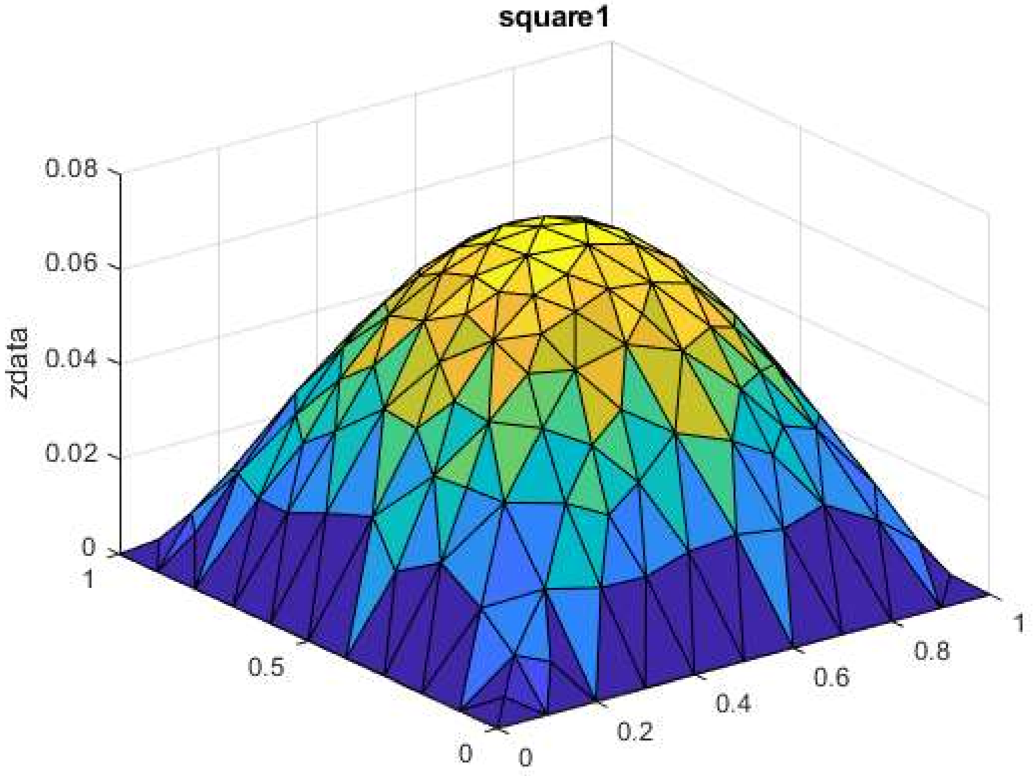
Index mapping and check if it is border

```
% NOTE:
% The task to check if on boundary can be done even more efficient
% following snipplet is one approach, but I forgot to keep track of
% the local coordinates ... since it is not so much more efficient
% I left it out and noted it at this point.
%             idx = zeros(1,3);
%             put = 1;
%             for i = 1:3
%                 ih = t(i,t_elem);
%                 if ~any(ih == bound_points)
%                     idx(put) = ih;
%                     put = put +1;
%                 end
%             end
%             idx = nonzeros(idx);
%             idx = perms(idx);
%
%             for i = 1:size(tuples,2)
%                 A(idx(i,1), idx(i,2)) = ...
%                     A(idx(i,1), idx(i,2)) + A_helper;
%             end
```

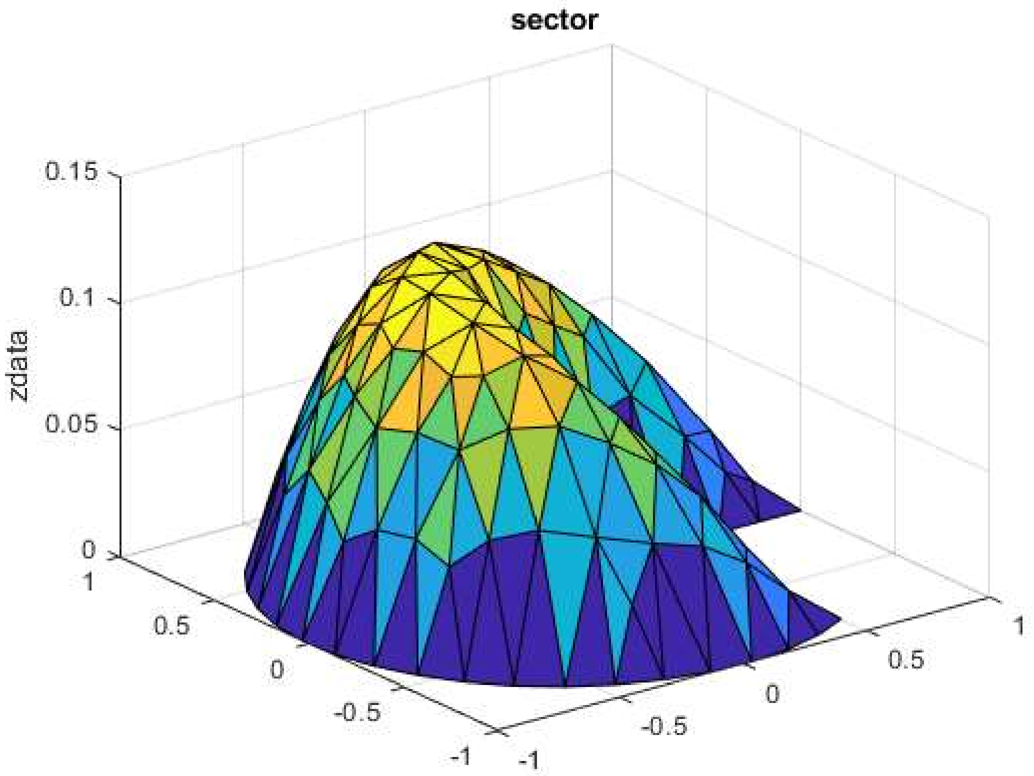
Solve Poisson

```
[pSq,eSq,tSq] = initmesh('squareg');
[pSec, eSec, tSec] = initmesh('sectorg');
[pSemiCirc, eSemiCirc, tSemiCirc] = initmesh('semicircleg');

% function u = poisson(p,e,t)
%     n = size(p,2);
%     interior = 1:n;
%     interior(union(e(1,:),e(2,:))) = [];
%     u = zeros(n,1);
%
%     [A,b] = poisson_init(p,e,t);
%     u(interior) = A\b;
```

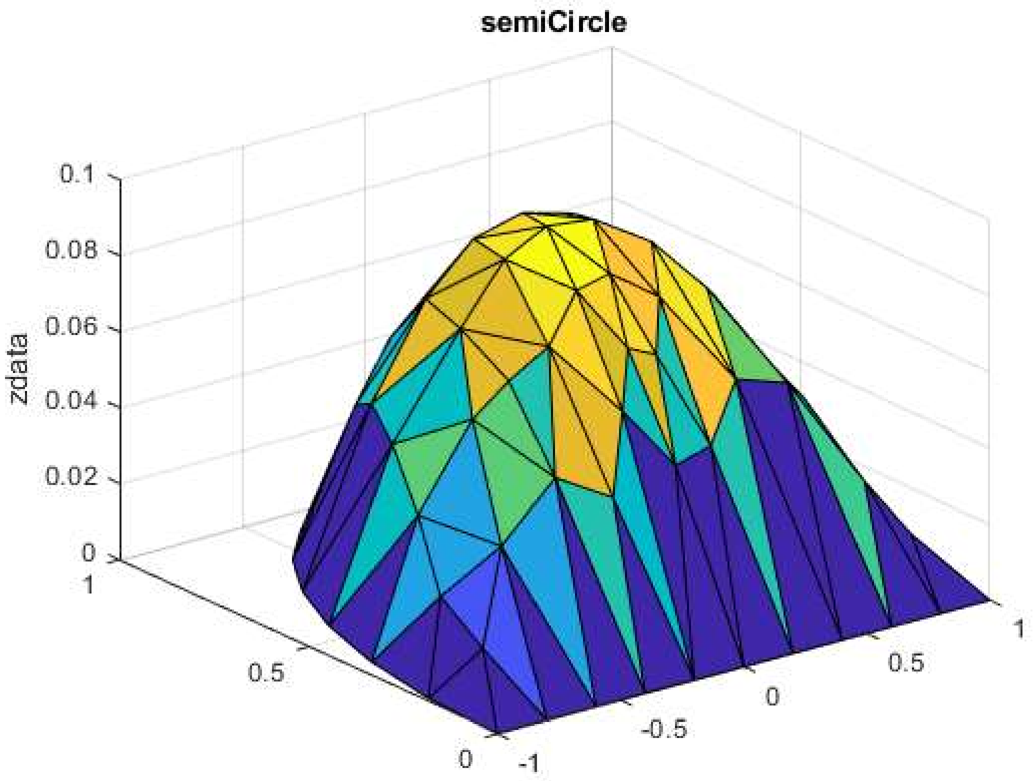
```
% end

u = poisson(pSq,eSq,tSq);
mypdeplot(pSq,eSq,tSq,"square1",u)
```




```
u = poisson(pSec, eSec, tSec);
mypdeplot(pSec, eSec, tSec,"sector",u)
```



```
u = poisson(pSemiCirc, eSemiCirc, tSemiCirc);
mypdeplot(pSemiCirc, eSemiCirc, tSemiCirc,"semiCircle",u)
```



Adaptive poisson solver

We want to refine the mesh where is is the most necessary. There are many different rules which can be followed, for example in the lecture we discussed the red-green nodes stategie with a local error estimate.

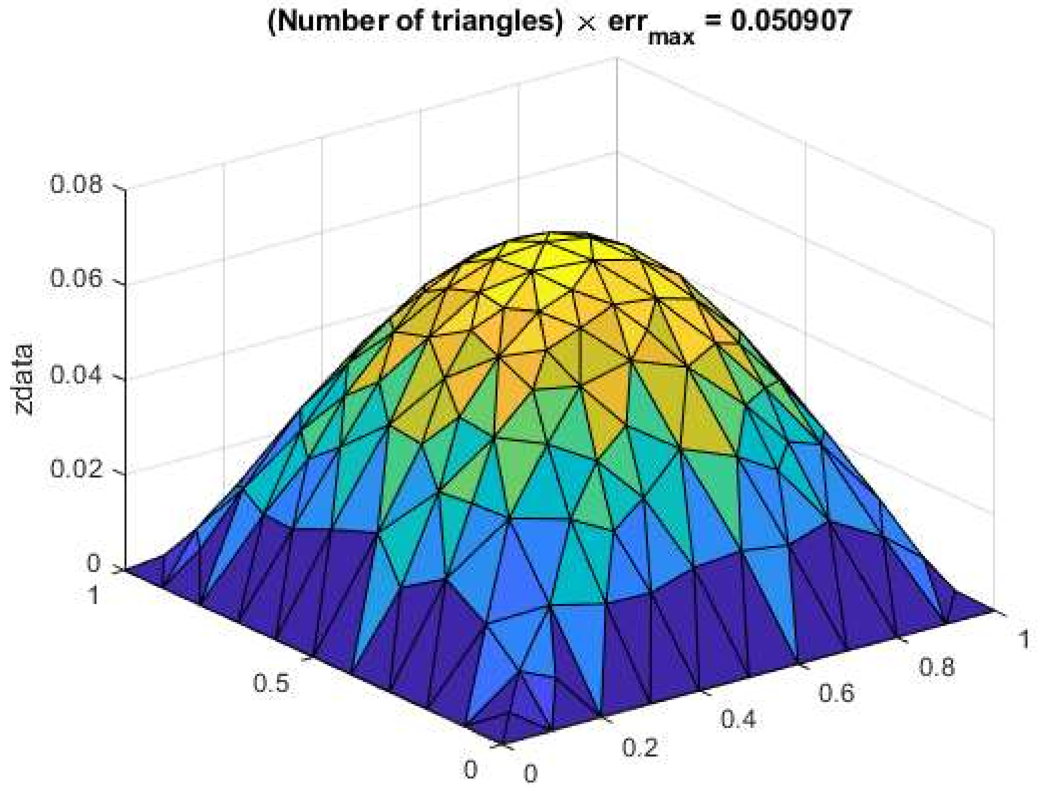
In this exercise sheet we also have a local error for a traingle t_k given by

$$\sum_{t_l \text{ is neighbour of } t_k} |S_{\text{kl}}|^2 \,|\nabla u_{h,k}(s_{\text{kl}}) - \nabla\, u_{h,l}(s_{\text{kl}})|^2$$

This error is calculated for the whole geometry and stored in a vector with the function `poisson_error.m (See end of the PDF)`

```
tol = 0.1;

[err_sq, pSq, eSq, tSq] = poisson_adapt('squareg',tol);
```

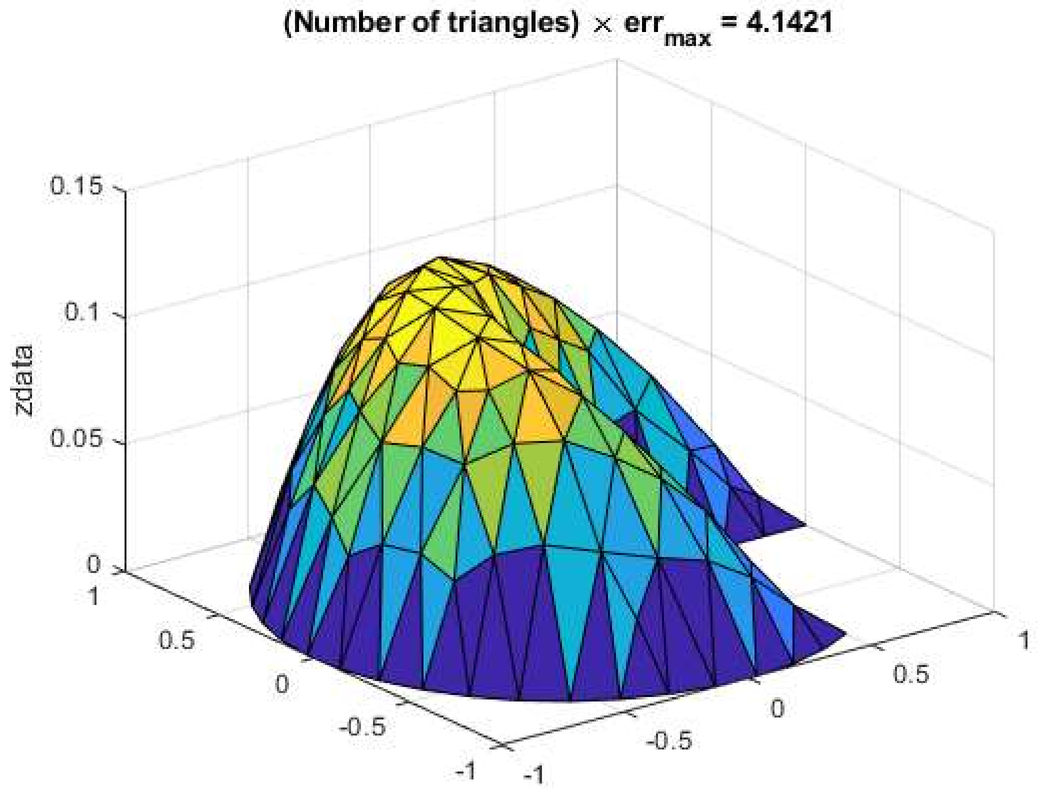
```
nt = 328
```




```
refine_indices =

  1×0 empty double row vector
```

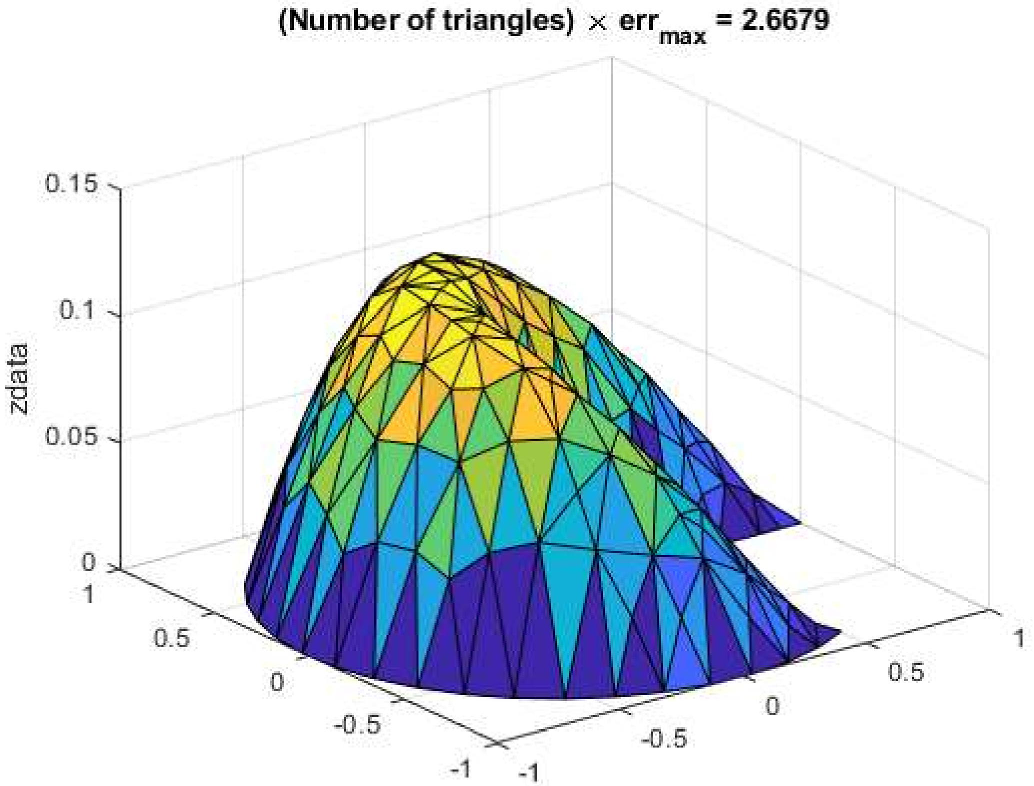
```
[err_Sec, pSec, eSec, tSec] = poisson_adapt('sectorg',tol);
```

```
nt = 243
```

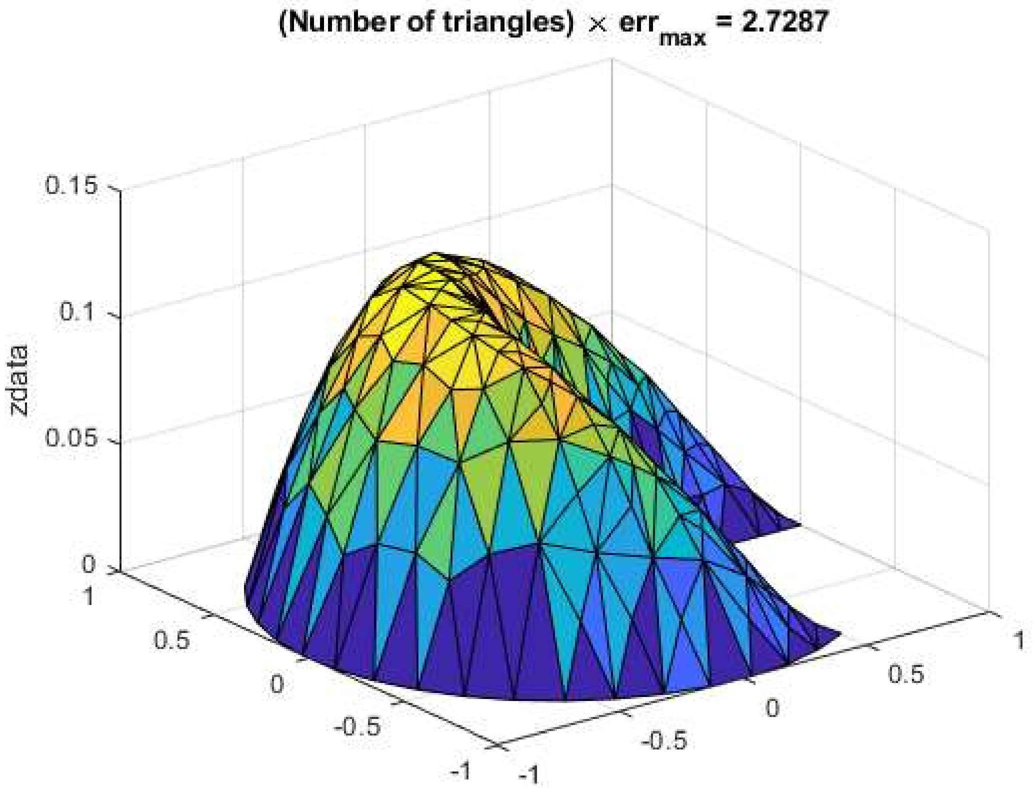



```
refine_indices = 1×64
     2     3     4     5     7     8    11    12    31    32    33    40    50    52    63    6
```

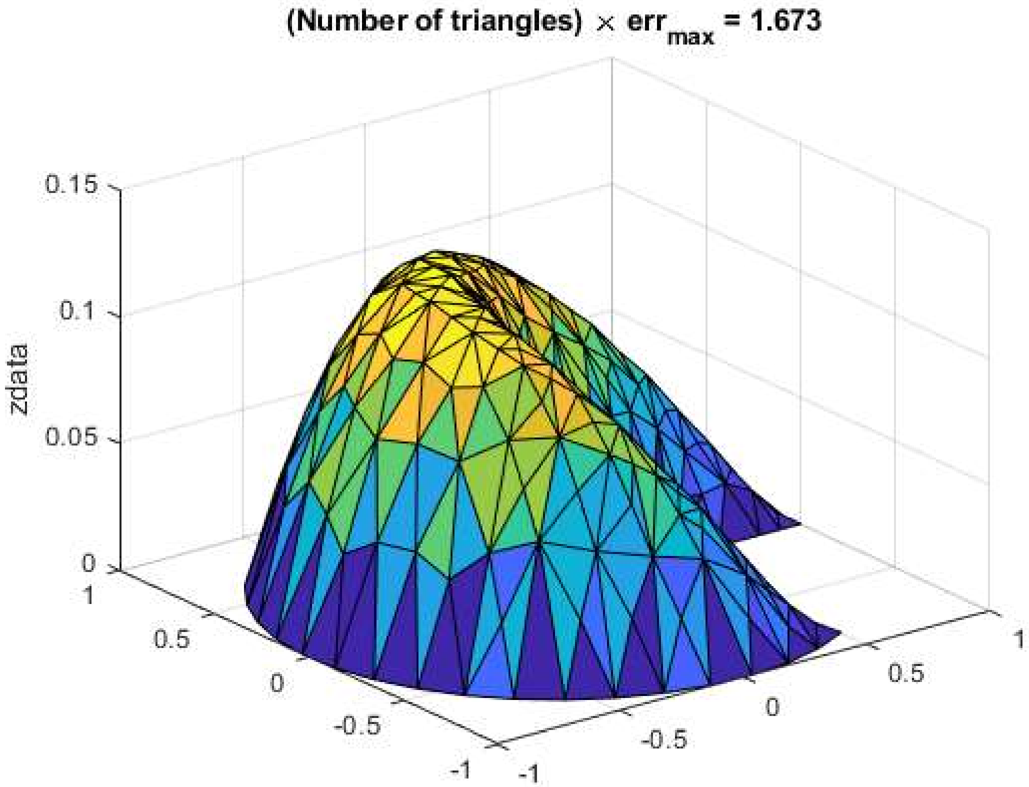
```
nt = 331
```



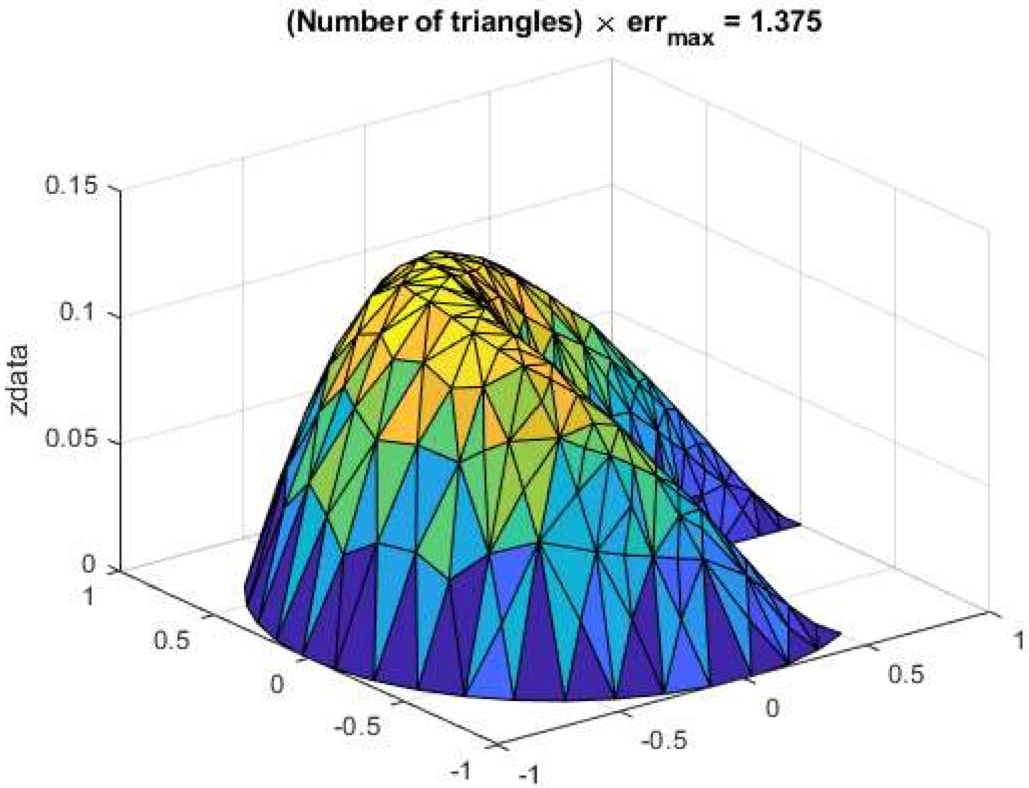

```
refine_indices = 1×56
     1     3     4     5    11    13    16    39    40    62    68    80    82    84    94    9
nt = 417
```



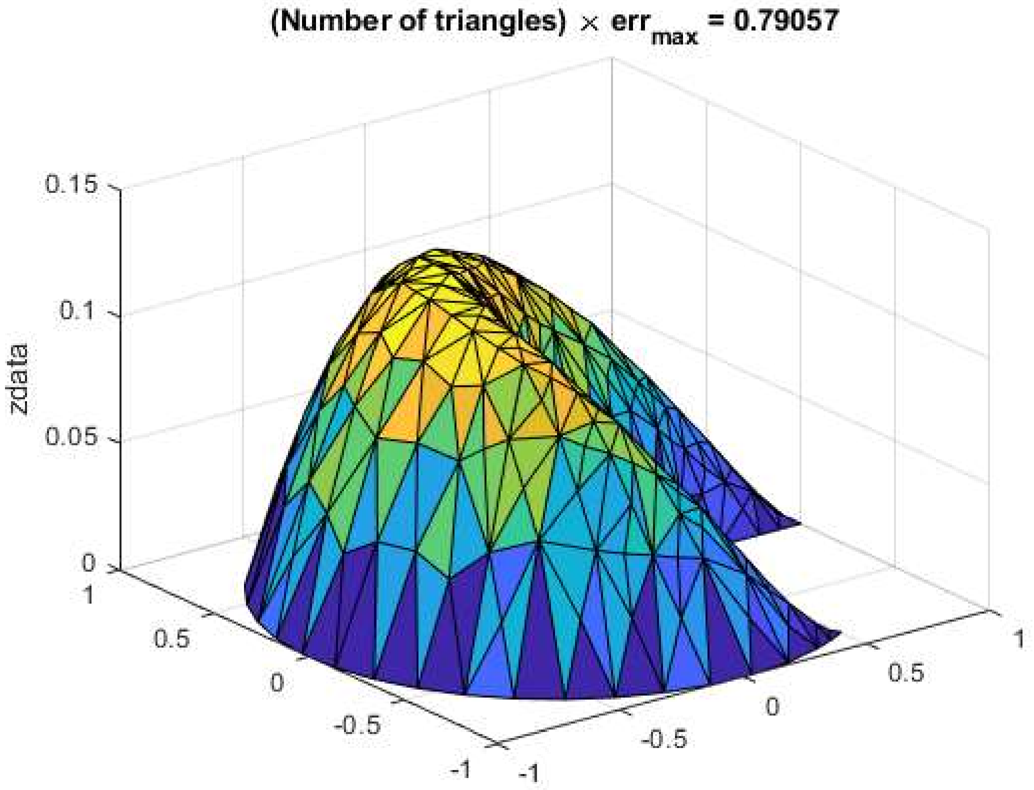

```
refine_indices = 1×50
     3     5    21    26    32    63    64    69    71    79    93   109   118   121   124   13
nt = 490
```



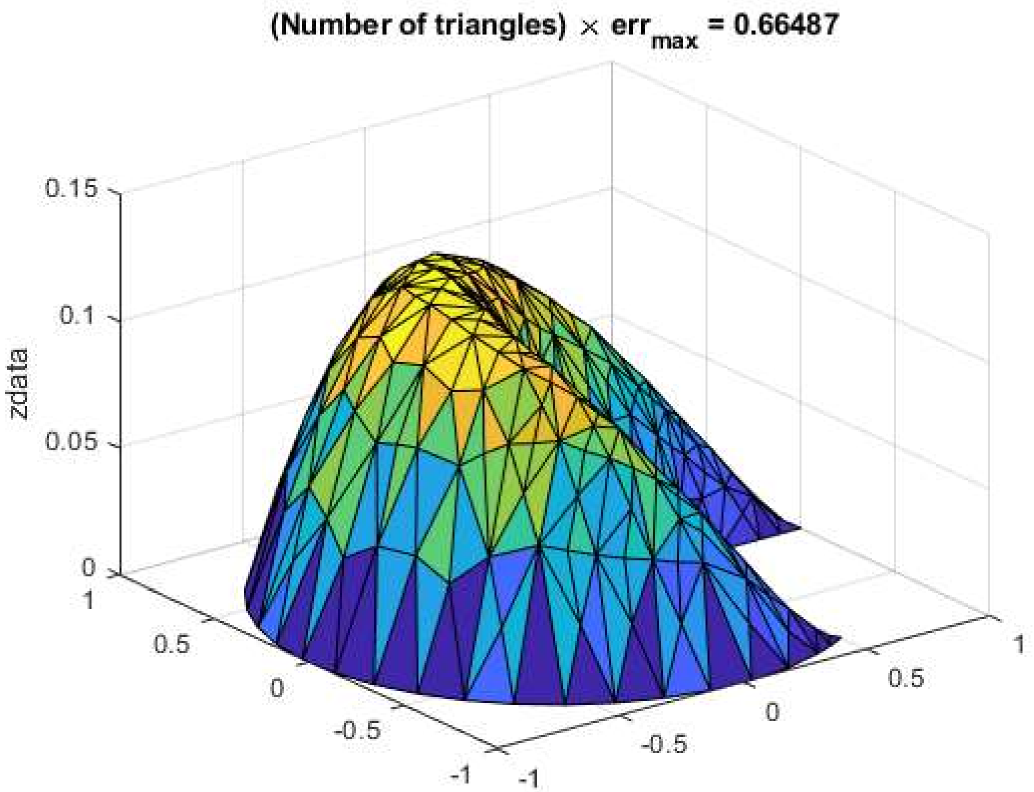

```
refine_indices = 1×37
     1     5     6     7     8    88    89    94    95    98   110   112   113   160   175   21
nt = 546
```



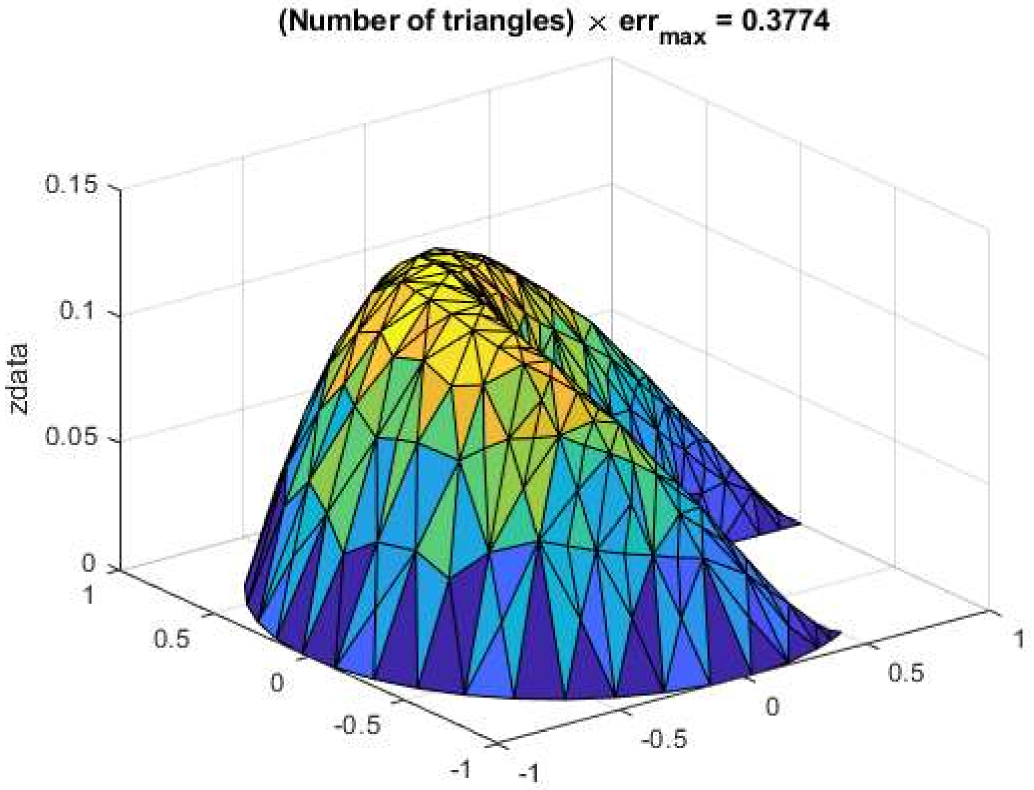

```
refine_indices = 1×28
     5    12    19    29    31    40   118   164   200   215   249   259   333   339   359   39
nt = 589
```



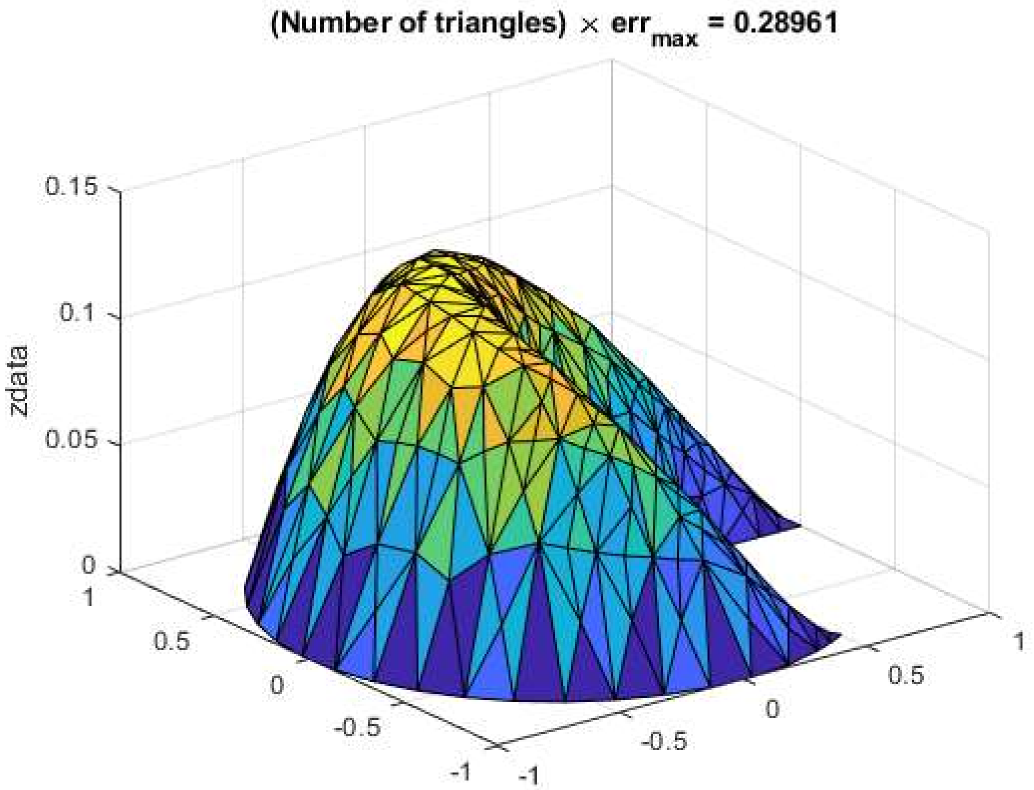

```
refine_indices = 1×27
     1     4    25   140   158   198   199   332   339   342   346   397   419   420   480   49
nt = 632
```



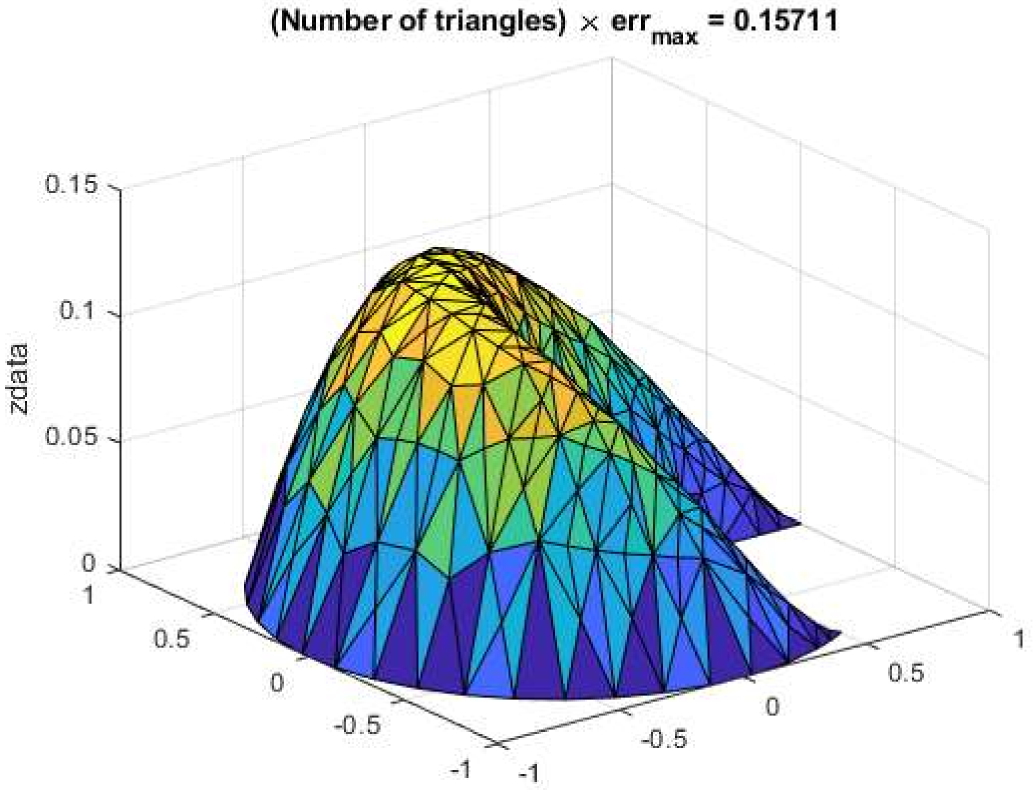

```
refine_indices = 1×20
    35   192   241   268   274   284   314   377   471   520   536   544   580   584   588   61
nt = 663
```



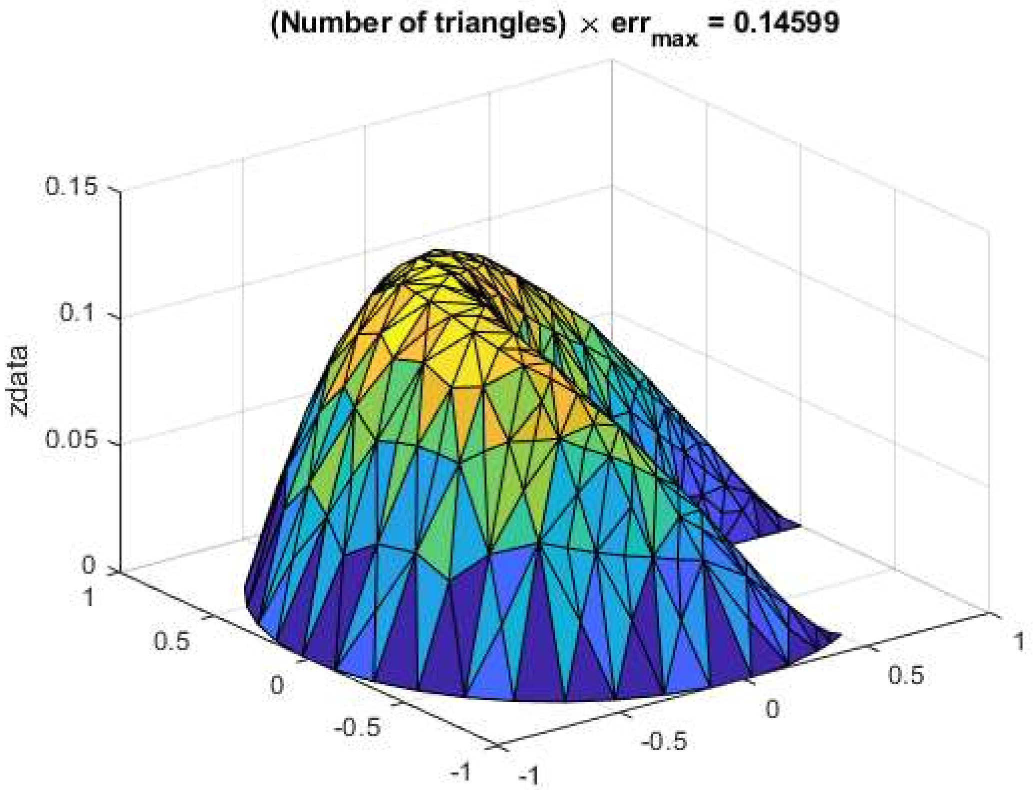

```
refine_indices = 1×16
    16   129   172   301   317   366   370   377   573   584   619   621   650   654   658   66
nt = 687
```



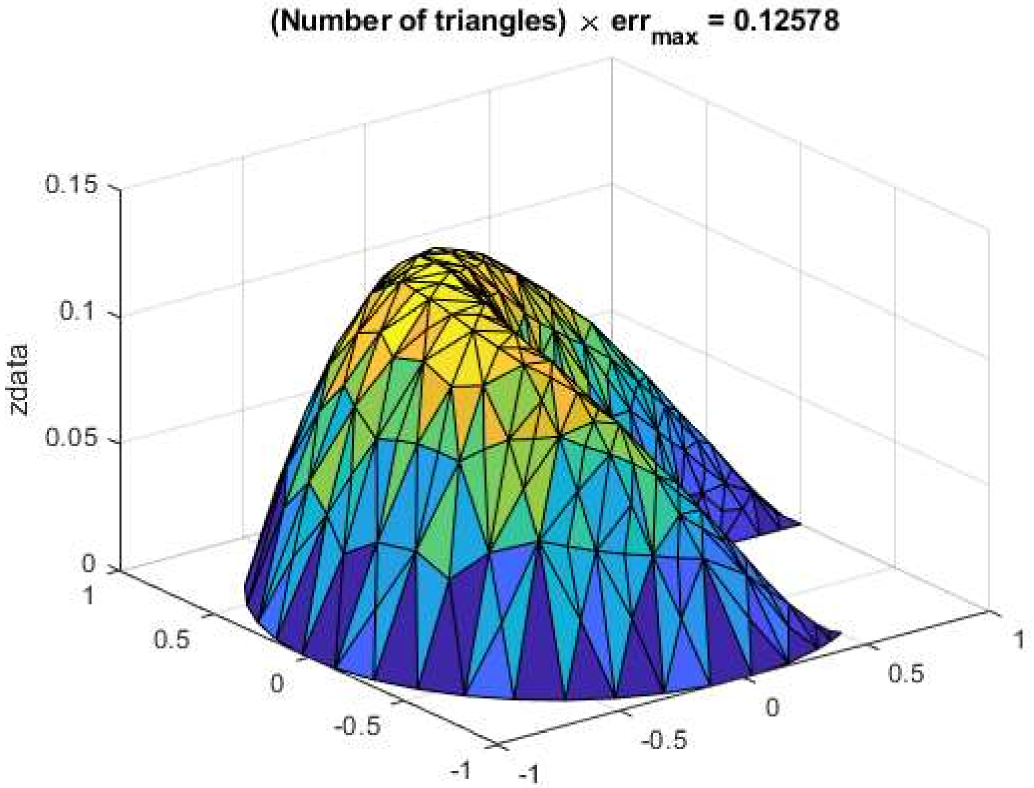

```
refine_indices = 1×13
    55    79    96   143   231   422   444   450   625   658   678   682   686
nt = 707
```



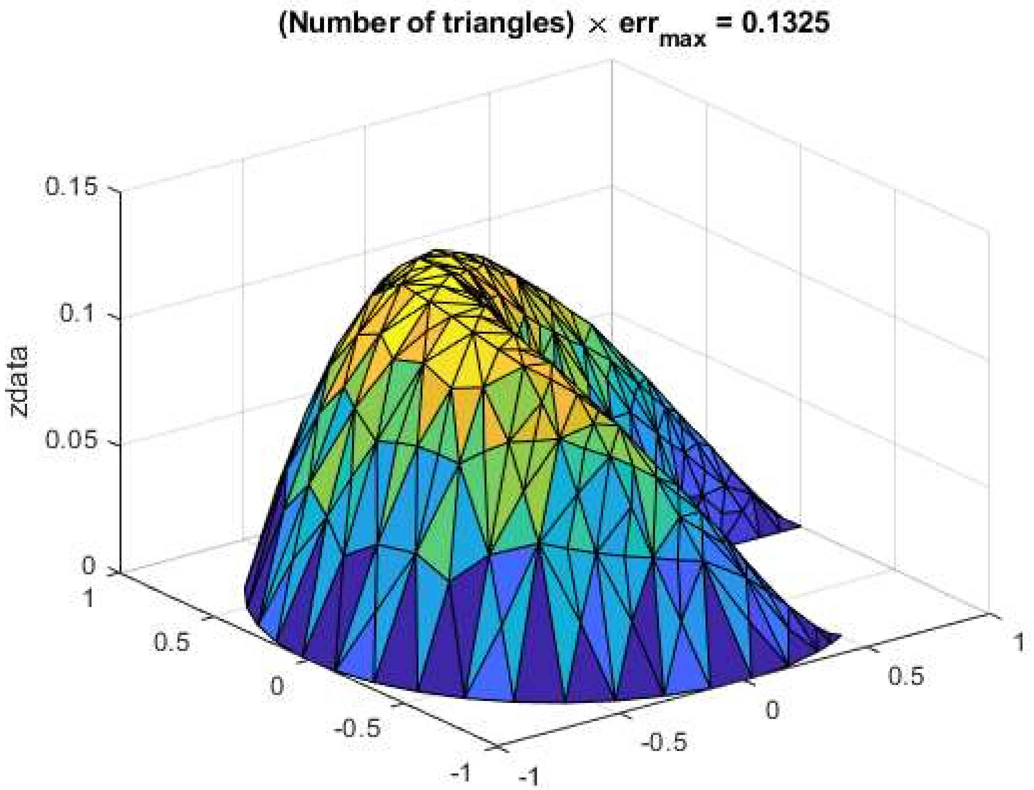

```
refine_indices = 1×6
   253    643    656    662    686    704
nt = 716
```



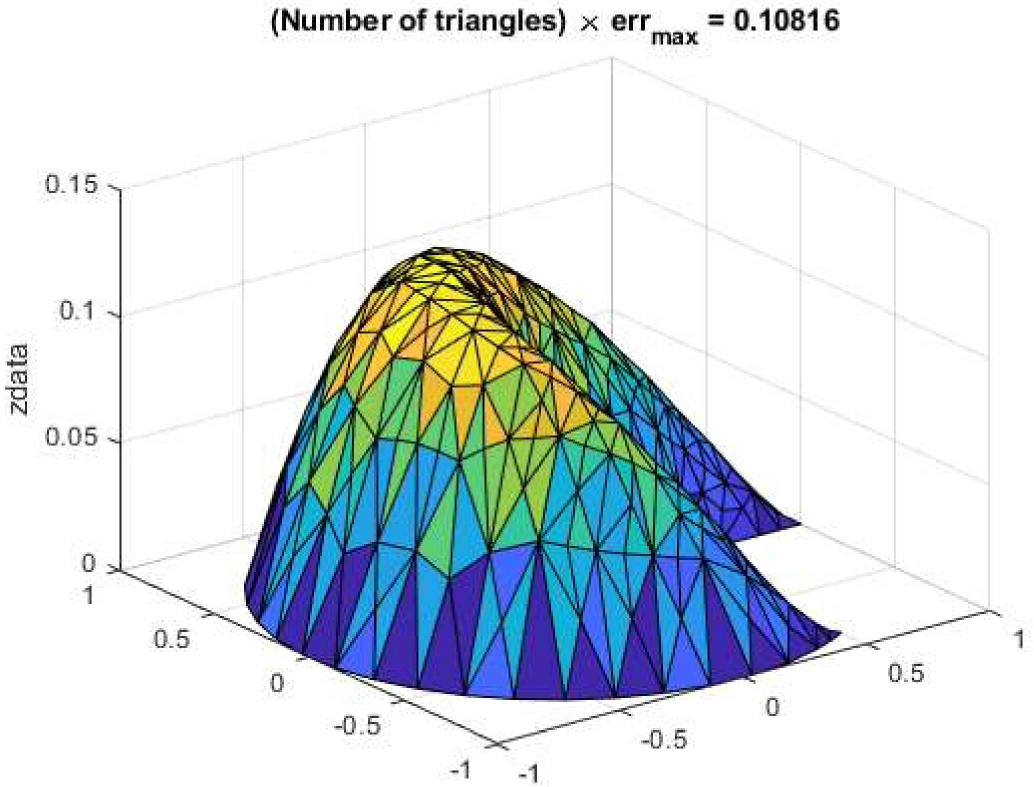

```
refine_indices = 1×5
   295    298    616    678    713
nt = 723
```



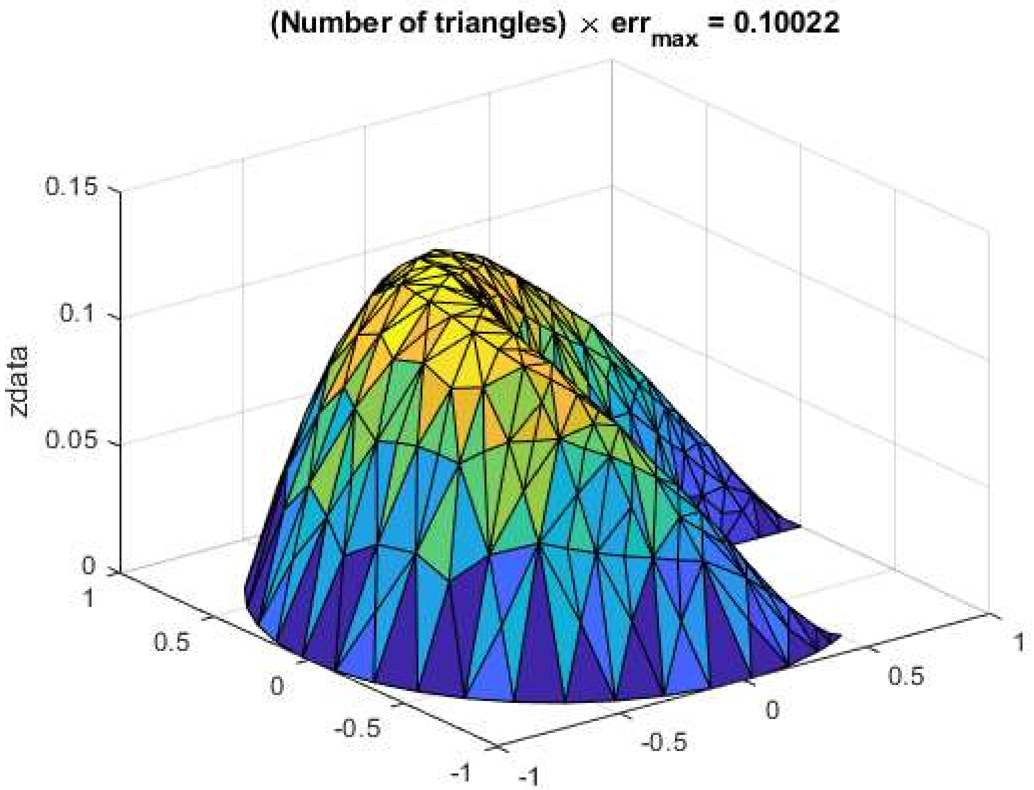

```
refine_indices = 1×3
    81   683   702
nt = 729
```



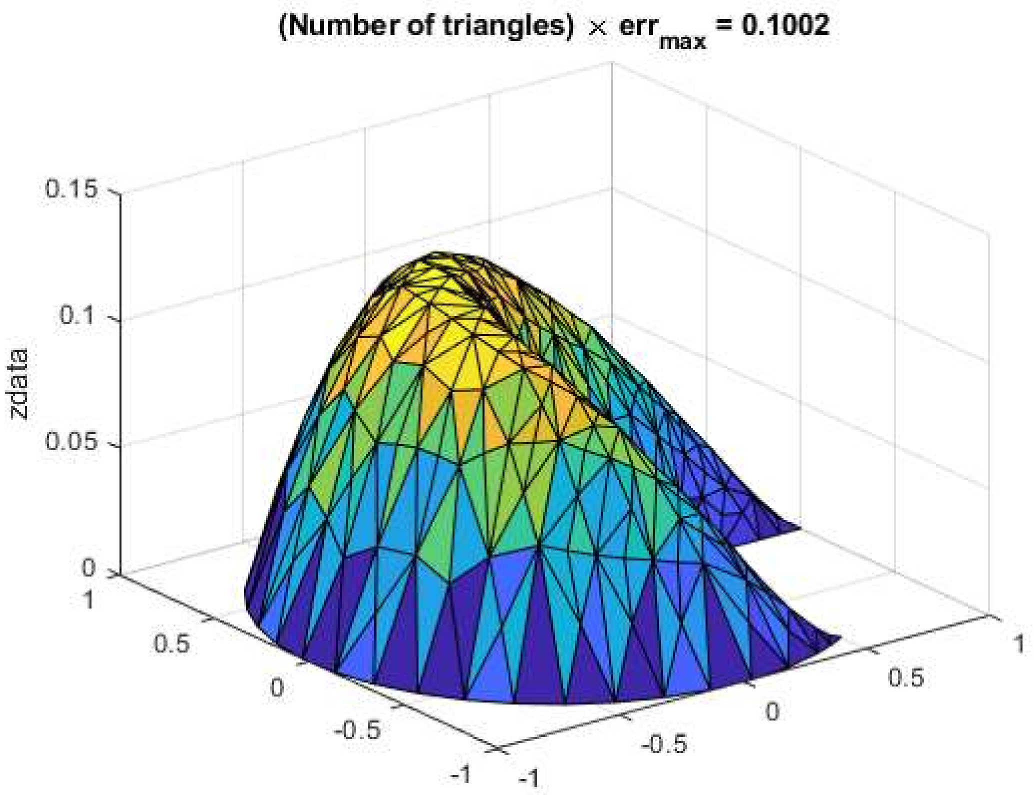

```
refine_indices = 1×2
   704   728
nt = 731
```



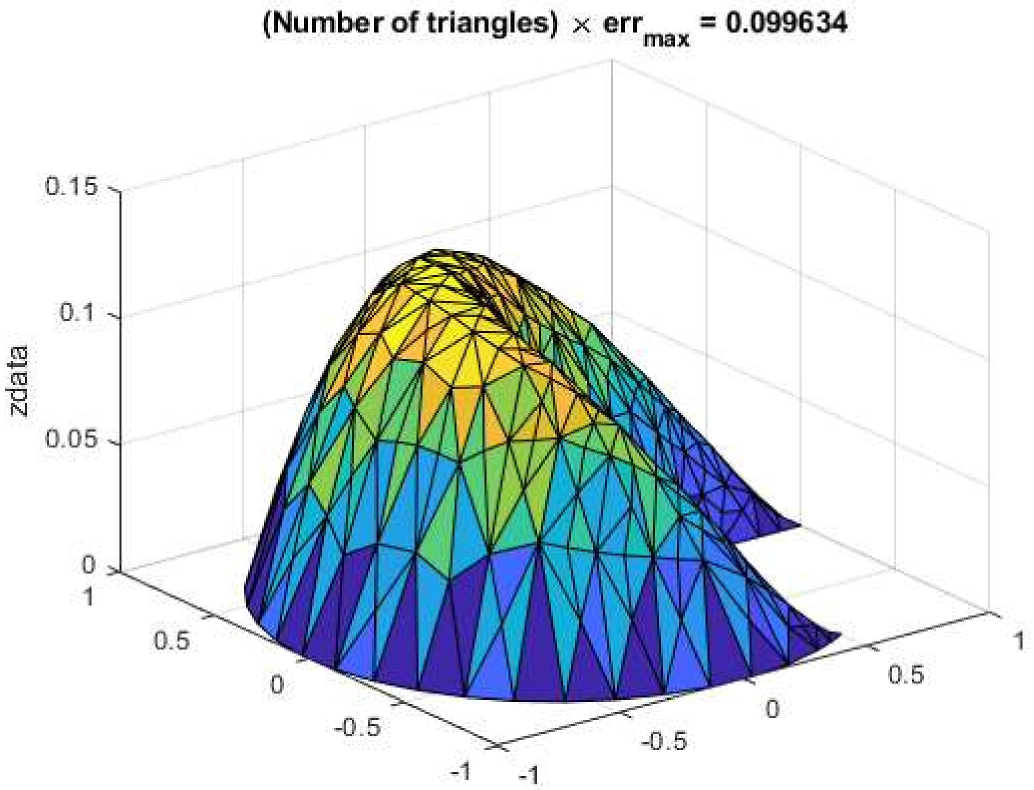

```
refine_indices = 1×2
   706   715
nt = 735
```




```
refine_indices = 207
nt = 737
```



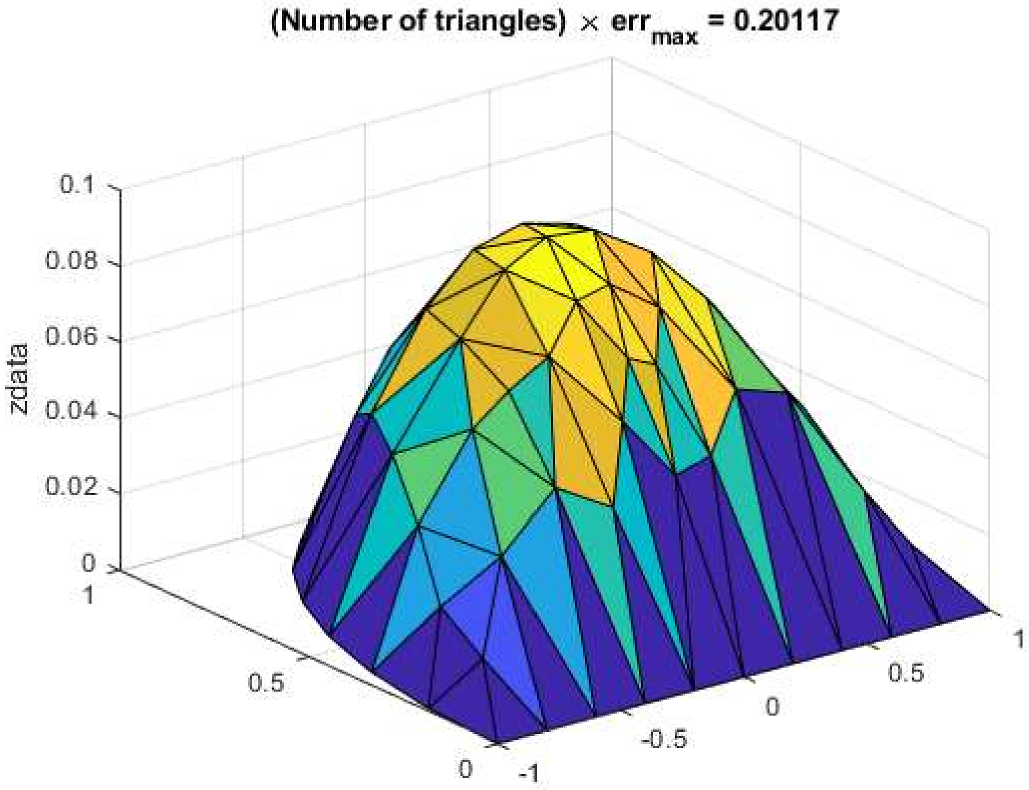

```
refine_indices = 258
nt = 739
```



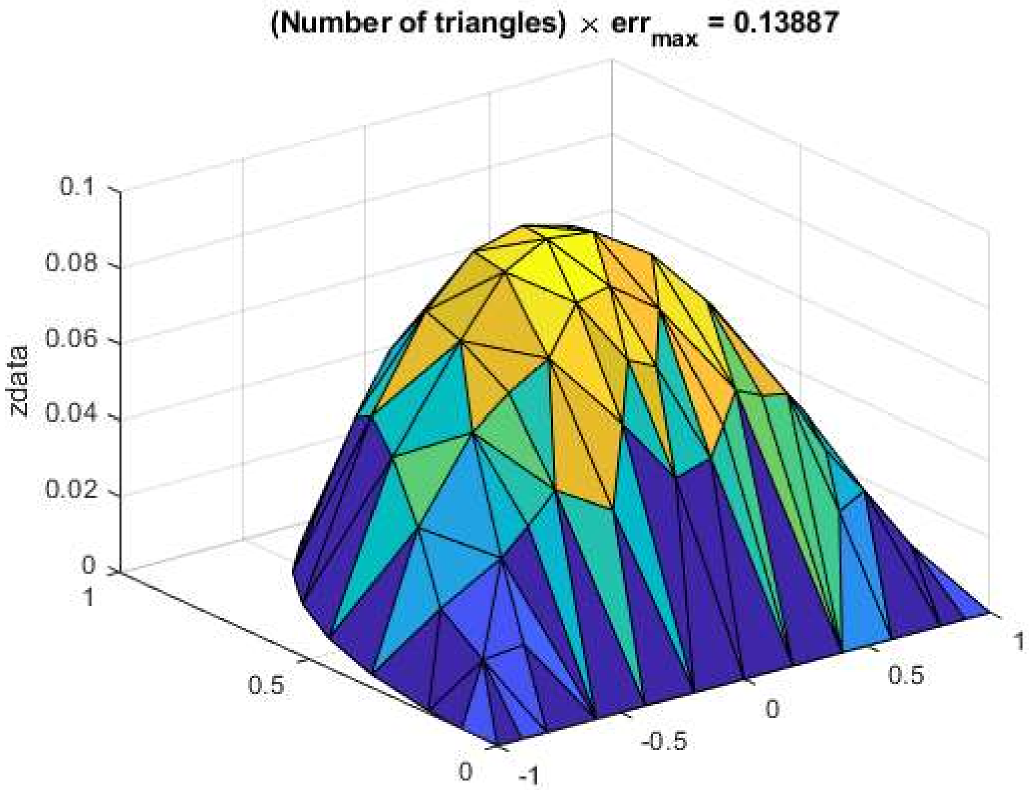

```
refine_indices =

  1×0 empty double row vector
```

```
[err_SemiCirc, pSemiCirc, eSemiCirc, tSemiCirc] = poisson_adapt('semicircleg',tol);
```

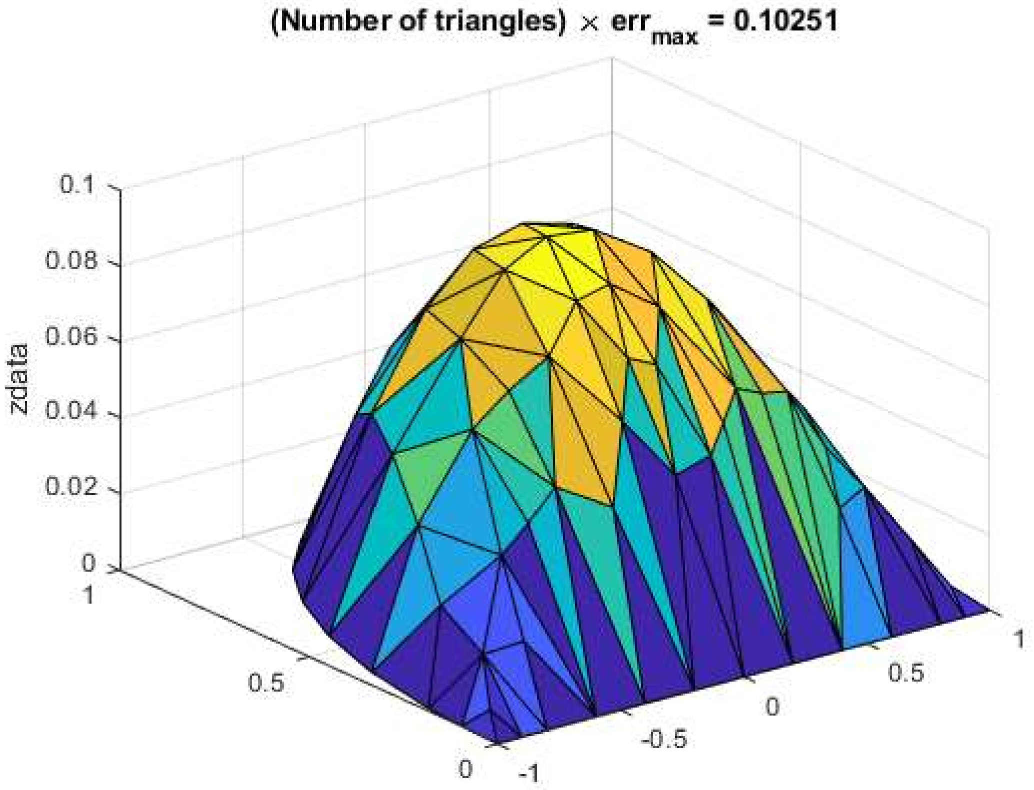
```
nt = 128
```



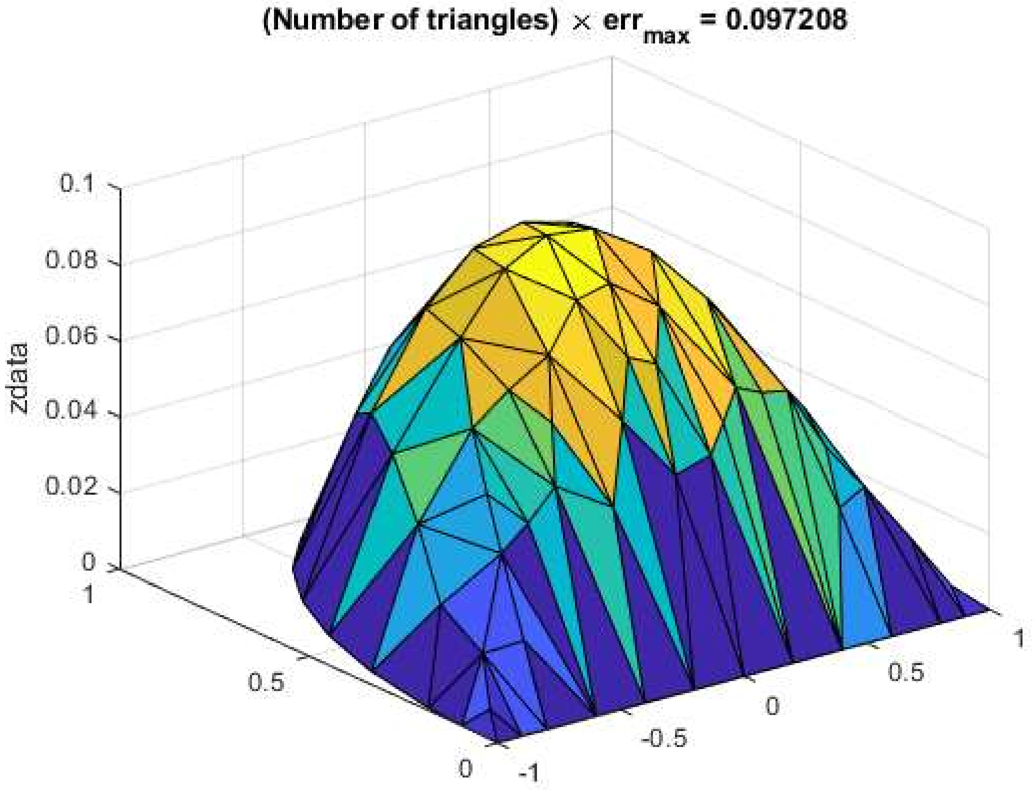

```
refine_indices = 1×12
     2     4     8     9    10    19    23    56    64    76    83    86
nt = 144
```




```
refine_indices = 1×4
     9    23   129   132
nt = 148
```




```
refine_indices = 1×2
    42    74
nt = 152
```




```
refine_indices =

  1×0 empty double row vector
```

Functions

Poisson init

```
function [A,b] = poisson_init(p,e,t)
%Computes the stiffness matrix A and the right hand side b
%The matrix A is sparse
n = size(p,2);
num_elem = size(t,2);
A = sparse(n,n);
b = zeros(n,1);
reference = @(x,y) [1-x-y, x, y];
grad_reference = [-1, 1, 0;...
                  -1, 0, 1];

%in edge e boundaries from p1 to p2 are stored
bound_points = union(e(1,:),e(2,:));


%We iterate over all triangle elements and add the components
%to the assembled matrix
%If it is a boundary we will set it to g(.) = 0
%Calculation according to NUMPDE skript p.104-106
    for t_elem = 1:num_elem
        B = [p(:,t(2,t_elem)) - p(:,t(1,t_elem)), p(:,t(3,t_elem))- p(:,t(1,t_elem))];
        det_B = abs(det(B));
        inv_B = inv(B);
        %Area of the reference element T = 1/2 and Trafo B since const.
        %functions
        A_help = 1/2 * det_B * grad_reference'*inv_B*inv_B'*grad_reference;



        %For each triangular element there are three corners(c1,c2,c3), so the
        %element has an influence at A_(c1,c1) ... A_(c1,c2) ... A(c1,c3)
        %                             A_(c2,c1) ... A_(c2,c2) ... A(c2,c3)
        %                             A_(c3,c1) ... A_(c3,c2) ... A(c3,c3)
        %The reference mapping g from local to global coordinates is
        %defined in t(.)

        for i = 1:3
            ih = t(i,t_elem);
            b(ih) = b(ih) + 1/3 * det_B * 1/2;
            %Assemble A
            for j = 1:3
                jh = t(j,t_elem);
                A(ih,jh) = A(ih,jh) + A_help(i,j);

            end
        end

%         for i = 1:3
%             ih = t(i,t_elem);
%             if ~any(ih == bound_points)
%                 %Calculate b
%                 b(ih) = b(ih) + 1/3 * det_B * 1/2;
%                 %Assemble A
%                 for j = 1:3
%                     jh = t(i,t_elem);
%                     if ~any(jh == bound_points)
%                         A(ih,jh) = A(ih,jh) + A_help(i,j);
```

```
%                     end
%                 end
%             end
%         end



    end

    A(:,bound_points) = [];
    A(bound_points,:) = [];
    b(bound_points) = [];


end
```

Poisson_error

```
function err = poisson_error(p,e,t,u)
nt = size(t,2);
err = zeros(1,nt);
for k = 1:(nt-1)
       B_k = [p(:,t(2,k)) - p(:,t(1,k)), p(:,t(3,k)) - p(:,t(1,k))];
       grad_k = inv(B_k)' * [u(t(2,k)) - u(t(1,k)); u(t(3,k))- u(t(1,k))];

       for i = 1:3
           %Both properties should be fulfilled
           nb = find(sum((t(1:3,(k+1):nt)==t(i,k)) +...
                         (t(1:3,(k+1):nt)==t(mod(i,3)+1,k)))==2);
           l = k + nb;
           if ~isempty(l)
               B_l = [p(:,t(2,l)) - p(:,t(1,l)), p(:,t(3,l)) - p(:,t(1,l))];
               grad_l = inv(B_l') * [u(t(2,l)) - u(t(1,l)); u(t(3,l))- u(t(1,l))];


               s_kl = norm(p(1:2,t(i,k)) - p(1:2,t(mod(i,3) + 1,k)));
               err_kl = (s_kl *norm(grad_l - grad_k))^2;

               %Update Error
               err(k) = err(k) + err_kl;
               err(l) = err(l) + err_kl; % employ symmetry of error estimator
           end
       end
end
```

Poisson_adapt

```
function [err,p,e,t] = poisson_adapt(g,tol)
[p,e,t] = initmesh(g);

while true
    nt = size(t,2)

    u = poisson(p,e,t);
    err = poisson_error(p,e,t,u);


    mypdeplot(p,e,t,'zdata',u)
```

```
    title("(Number of triangles) \times err_{max} = " + nt*max(err))


    refine_indices = find(err>(tol/nt))
    if isempty(refine_indices)
        break;
    end
    [p,e,t] = bisect(g,p,e,t,refine_indices);
end
end
```