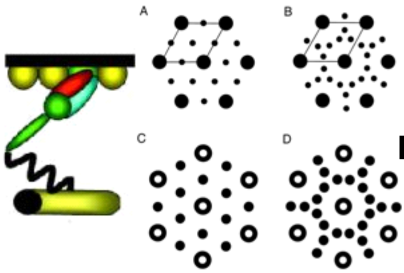
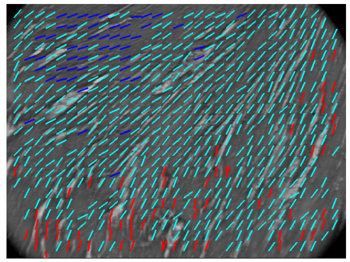
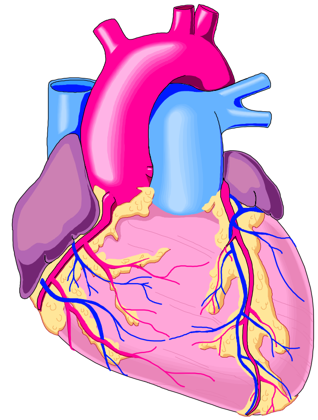
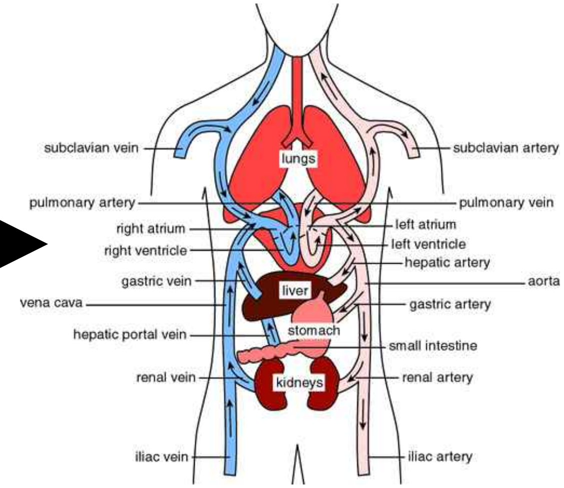
Biomechanics









cross-bridges **3-D myocardium** **ventricles** **circulation**

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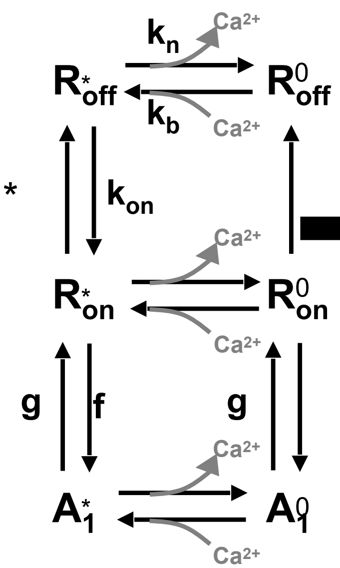
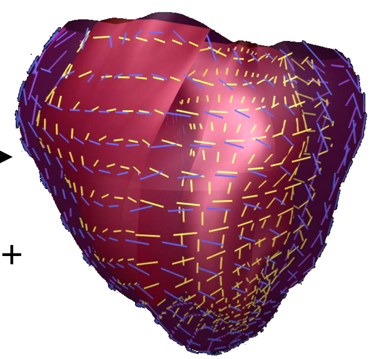
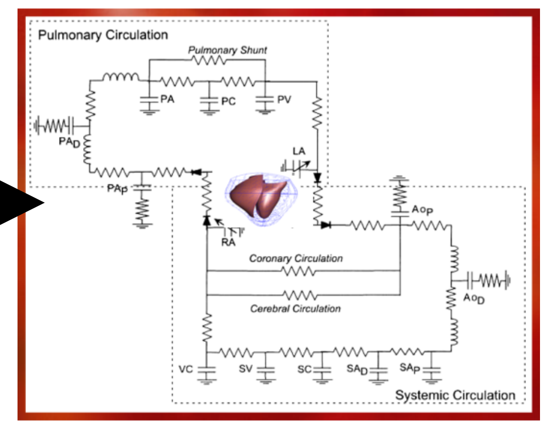


$$W = \frac{1}{2} C e^{Q}$$

where

$$Q = b_1 E_{\Theta\Theta}^2 + b_2 E_{ZZ}^2 + b_3 E_{RR}^2 + 2b_4 E_{\Theta\Theta} E_{ZZ} + 2b_5 E_{ZZ} E_{RR} + 2b_6 E_{RR} E_{\Theta\Theta} + b_7 E_{\Theta Z}^2 + b_8 E_{ZR}^2 + b_9 E_{R\Theta}^2$$





Myofilament kinetic model

Hyperelastic constitutive equation

3-D finite element model of ventricular stress

Boundary conditions: circulatory systems model

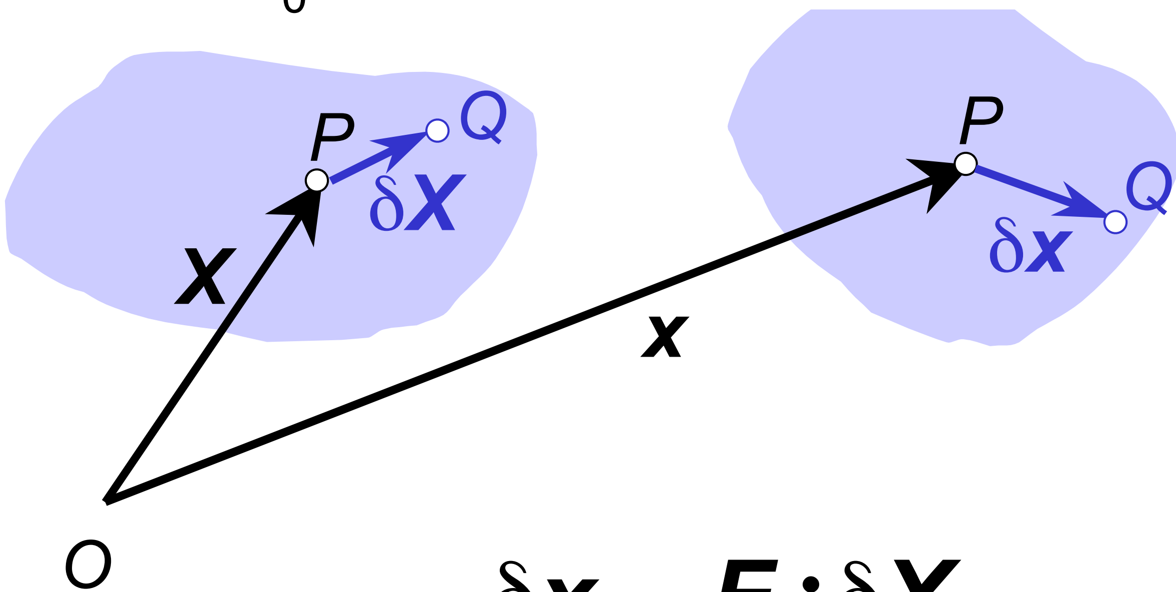
Biomechanics: Mechanics$\leftrightarrow$Physiology

Continuum Mechanics	Physiology
Geometry and structure	Anatomy and morphology
Boundary conditions	Environmental influences
Conservation laws	Biological principles
• mass	• mass transport, growth
• energy	• metabolism and energetics
• momentum	• motion, flow, equilibrium
Constitutive equations	Structure-function relations

Therefore, continuum mechanics provides a mathematical framework for integrating the structure of the cell and tissue to the mechanical function of the whole organ

Kinematics (motion)

	Reference State	**Current State**
	("undeformed")	**("deformed")**
Time:	t=0	t=t
Region:	R_0	R



$$\delta \boldsymbol{x} = \boldsymbol{F} \cdot \delta \boldsymbol{X}$$

Chain rule: $\delta x_i = \dfrac{\partial x_i}{\partial X_R} \delta X_R \quad \Rightarrow \quad \boxed{F_{iR} = \dfrac{\partial x_i}{\partial X_R}}$

Polar Decomposition Theorem: Deformation consists of rotation and stretch (shape change)

For $\boldsymbol{F}$ non-singular and square

$$\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U} = \boldsymbol{V}\cdot\boldsymbol{R}$$

where

$\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}, \boldsymbol{V} = \boldsymbol{V}^{\mathrm{T}}$ are the right and left stretch tensors

$\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{R} = \boldsymbol{I}$ is the orthogonal rotation tensor

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{F} = (\boldsymbol{R}\cdot\boldsymbol{U})^{\mathrm{T}}\cdot(\boldsymbol{R}\cdot\boldsymbol{U}) = \boldsymbol{U}^{\mathrm{T}}\cdot\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}\cdot\boldsymbol{U} = \boldsymbol{U}^{2}$$

is the right Cauchy-Green Deformation tensor (Lagrangian)

$$\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^{\mathrm{T}} = (\boldsymbol{V}\cdot\boldsymbol{R})\cdot(\boldsymbol{V}\cdot\boldsymbol{R})^{\mathrm{T}} = \boldsymbol{V}\cdot\boldsymbol{R}\cdot\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}\cdot\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}^{2}$$

is the left Cauchy-Green Deformation tensor (Eulerian)

(Finite) Strain Tensors

Strain is a measure of *change of shape* independent of rotation. Change of shape corresponds to change of *length* (i.e. stretch)

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right)$$

Lagrangian Green's Strain Tensor

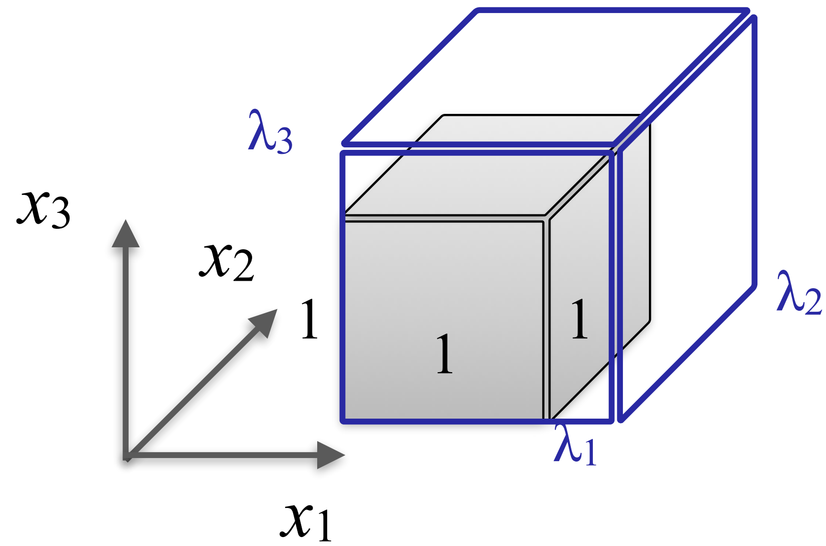
$$E_{RS} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S} - \delta_{RS}\right)$$

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right)$$

Eulerian Almansi's Strain Tensor

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_R}{\partial x_i}\frac{\partial X_R}{\partial x_j}\right)$$

Simple extension



$$x_1 = \lambda_1 X_1 \qquad x_2 = \lambda_2 X_2 \qquad x_3 = \lambda_3 X_3$$

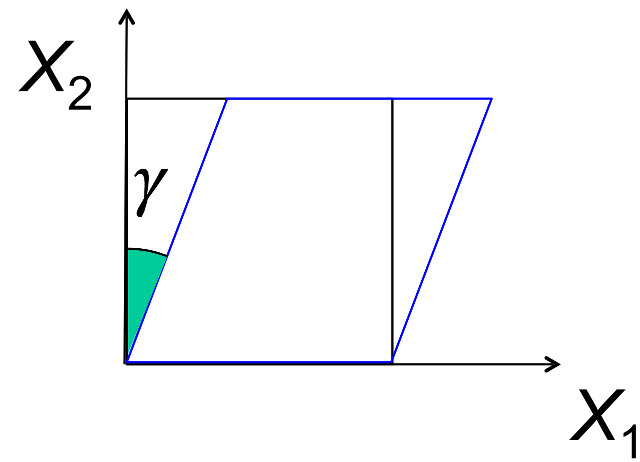
$$[F] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad [B] = [C] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$[E] = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix} \quad [e] = \frac{1}{2}\begin{bmatrix} 1 - 1/\lambda_1^2 & 0 & 0 \\ 0 & 1 - 1/\lambda_2^2 & 0 \\ 0 & 0 & 1 - 1/\lambda_3^2 \end{bmatrix}$$

Simple Shear

Simple Shear

$$x_1 = X_1 + X_2 \tan\gamma$$
$$x_2 = X_2 \quad x_3 = X_3$$



$$[F] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow [E] = \frac{1}{2}\begin{bmatrix} 0 & \tan\gamma & 0 \\ \tan\gamma & \tan^2\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

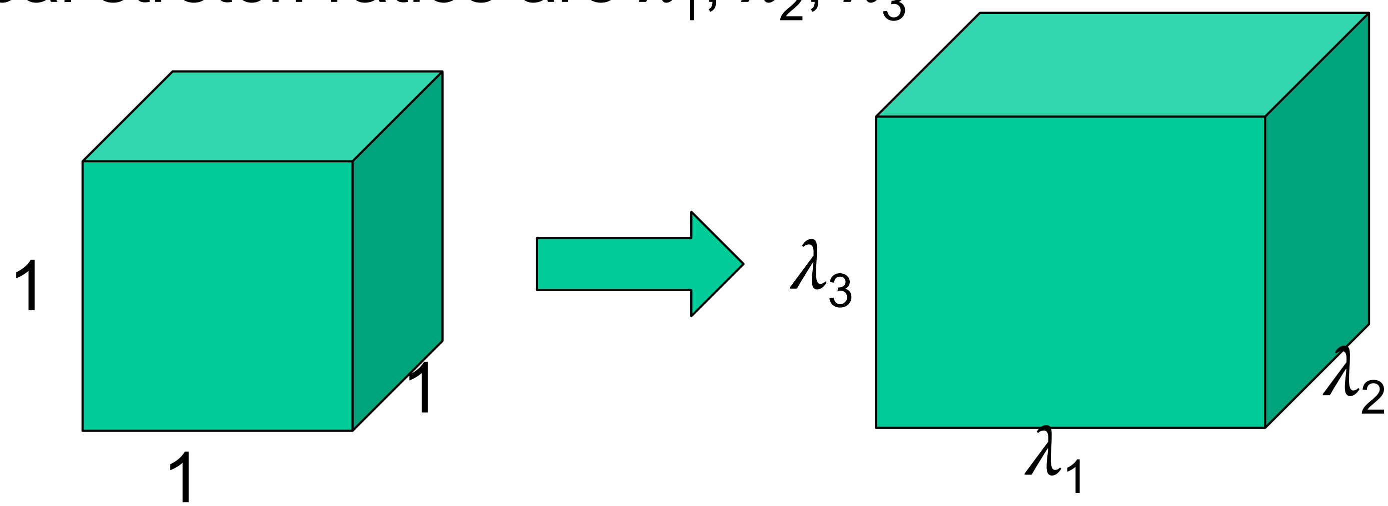
$\tan^2\gamma$ vanishingly small for small strain

Volume Change

- ***Elements of volume*** in the deformed dv and undeformed dV states are related by the determinant of $\boldsymbol{F}$:

$$dv = \det\boldsymbol{F}\, dV$$

- E.g. Consider a unit cube that is deformed so that its principal stretch ratios are λ_1, λ_2, λ_3



$dV = 1$

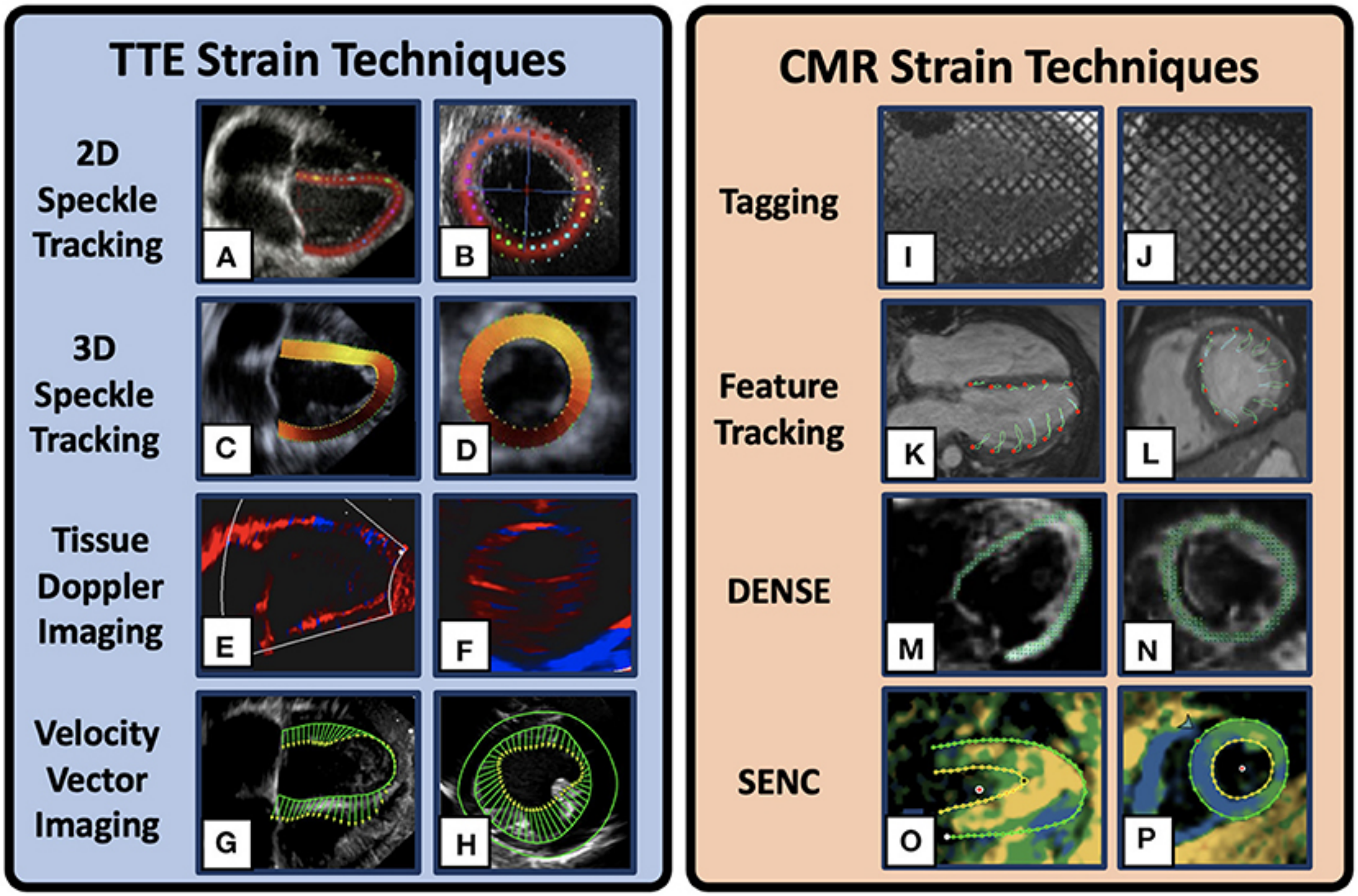
$$dv = \lambda_1\lambda_2\lambda_3 dV$$

$$= \det\boldsymbol{U}.dV = \det\boldsymbol{F}.dV$$

(since det$\boldsymbol{R}$ = 1)

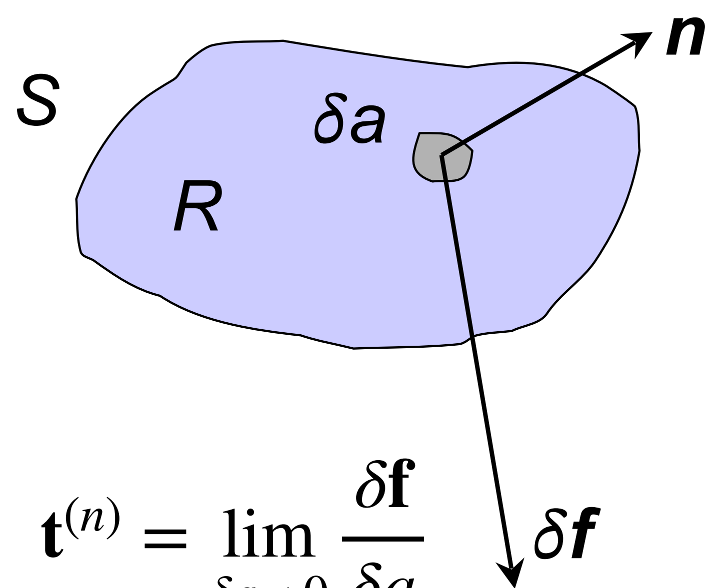
Measurement of Myocardial strain

- Radiopaque beads and biplane x-ray
- piezoelectric crystals
- video imaging of markers
- ultrasound
- MRI tagging

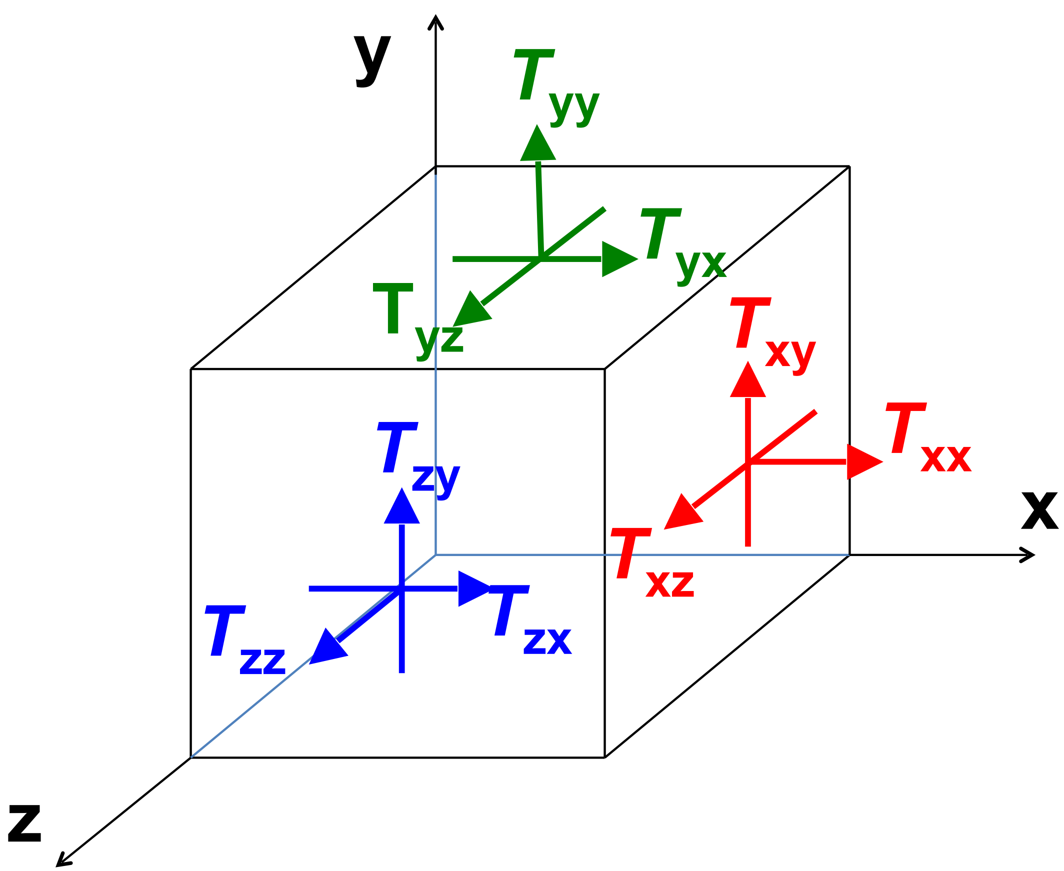


Earl CC *et al*. (2022) *Front. Cardiovasc. Med., Sec. Cardiovascular Imaging*

The Cauchy Stress Tensor, *T*



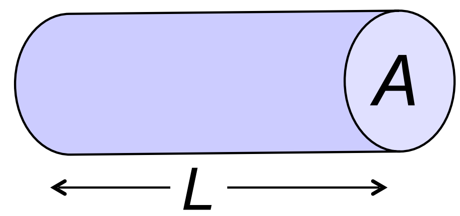
$$\mathbf{t}^{(n)} = \lim_{\delta a \to 0} \frac{\delta \mathbf{f}}{\delta a}$$



$$\boldsymbol{T} = \begin{bmatrix} T_{xx} & T_{yx} & T_{zx} \\ T_{xy} & T_{yy} & T_{zy} \\ T_{xz} & T_{yz} & T_{zz} \end{bmatrix}$$

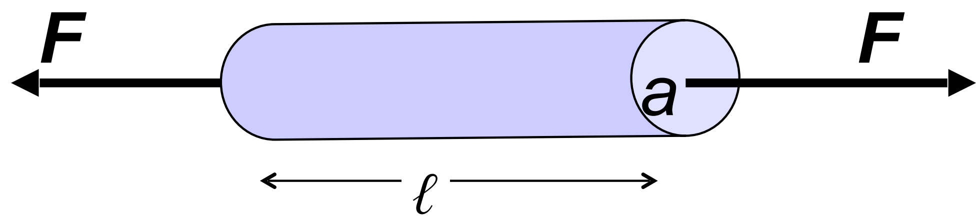
Example: Uniaxial Stress

undeformed length = L
undeformed area = A



deformed length = ℓ

deformed area = a



Cauchy Stress

$$T = \frac{F}{a}$$

Nominal Stress

$$S = \det \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{T} = \frac{\ell a}{LA} \frac{L}{\ell} \frac{F}{a} == \frac{F}{A}$$

Second Piola-Kirchhoff Stress

$$P = \boldsymbol{S} \cdot \boldsymbol{F}^{-1} = S \frac{L}{\ell} = \frac{F}{A} \frac{L}{\ell}$$

Governing Equations

- **Conservation Laws**
  - Conservation of Mass
  - Conservation of Momentum
    - *Linear*
    - *Angular*
  - Conservation of Energy
- **Constitutive Laws -** material properties

Conservation of Mass: Lagrangian

"The mass δm ($=\rho_0 \delta V$) of the material in the material volume element δV remains constant as the element deforms to volume δv with density ρ, (for δV arbitrarily small)"

$$\iiint \rho_0 \, \mathrm{d}V = \iiint \rho \, \mathrm{d}v$$

$$\iiint \rho_0 \, \mathrm{d}X_1 \, \mathrm{d}X_2 \, \mathrm{d}X_3 = \iiint \rho \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \iiint \rho \left| \frac{\partial x_i}{\partial X_R} \right| \mathrm{d}X_1 \, \mathrm{d}X_2 \, \mathrm{d}X_3$$

Hence: $\dfrac{\rho_0}{\rho} = \dfrac{dv}{dV} = det\mathbf{F} = det\mathbf{U} = \lambda_1 \,.\, \lambda_2 \,.\, \lambda_3 = \sqrt{I_3} = J$

Thus, for an *incompressible* solid: $\rho = \rho_0 \Rightarrow \det \boldsymbol{F} = 1$

Conservation of Linear Momentum

"The rate of change of linear momentum of the particles in region *R* equals the resultant of the body forces *b* per unit mass plus the resultant of the surface tractions $t^{(n)}$ acting on the surface *S*"

$$\frac{D}{Dt}\iiint_R \rho \mathbf{v}\, dV = \iiint_R \rho b\, dV + \iint_S \mathbf{t}^{(n)}\, dS$$

$$\rightarrow \quad \boxed{\begin{gathered} \rho \frac{D\boldsymbol{v}}{Dt} = \operatorname{div}\boldsymbol{T} + \rho\boldsymbol{b} \\ \text{or} \\ \rho\frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{\partial x_k} = \frac{\partial T_{ji}}{\partial x_j} + \rho b_i \end{gathered}}$$

Conservation of Angular Momentum

"The rate of change of angular momentum of the particles in a fixed region *R* equals the resultant couple about the origin of the body forces *b* per unit mass and the surface tractions $t^{(n)}$ acting on *S*".

Assuming no distributed body or surface couples, this leads simply to the symmetry of the stress tensor:

$$\boxed{\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}}$$

Conservation of Energy

"The rate of change of kinetic plus internal energy in the region *R* equals the rate at which mechanical work is done by the body forces *b* and surface tractions $t^{(n)}$ acting on the region plus the rate at which heat enters *R* across *S*".

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\iiint_R \rho\mathbf{v}\cdot\mathbf{v}\,\mathrm{dV}+\iiint_R \rho e\,\mathrm{dV}\right)=\iiint_R \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{dV}+\iint_S \mathbf{t}^{(\mathrm{n})}\cdot\mathbf{v}\,dS-\iint_S \mathbf{q}\cdot\mathbf{n}\,dS$$

With some manipulation, this leads to:

$$\boxed{\rho\frac{\mathrm{De}}{\mathrm{Dt}}=\mathrm{tr}(\boldsymbol{T}\,.\,\boldsymbol{D})-\mathrm{div}\,\boldsymbol{q}=T_{\mathrm{ji}}\frac{\partial v_{\mathrm{i}}}{\partial x_{\mathrm{j}}}-\frac{\partial q_{\mathrm{i}}}{\partial x_{\mathrm{i}}}}$$

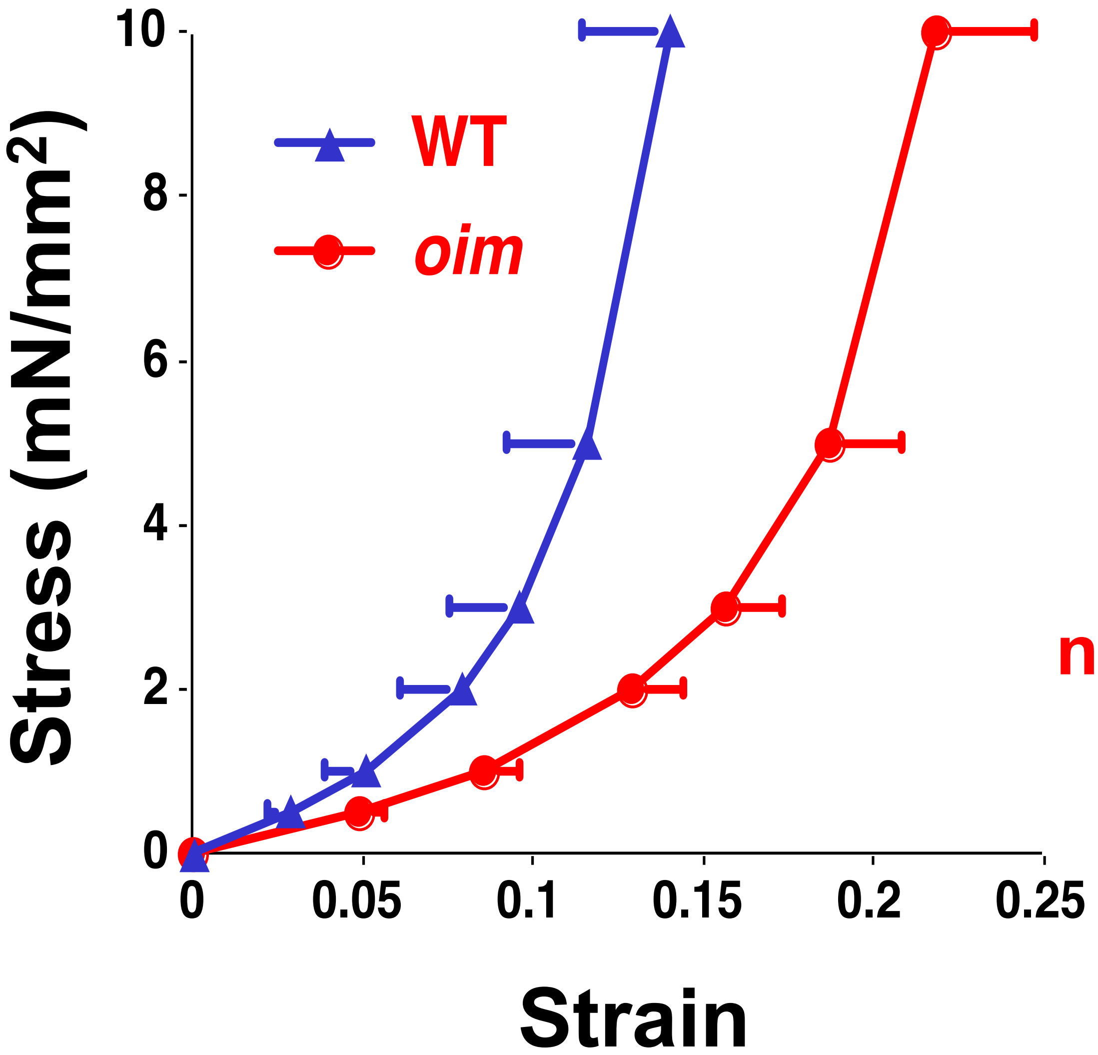
where: ***e*** is the *internal energy density*

q is the *heat flux vector*

The Constitutive Law

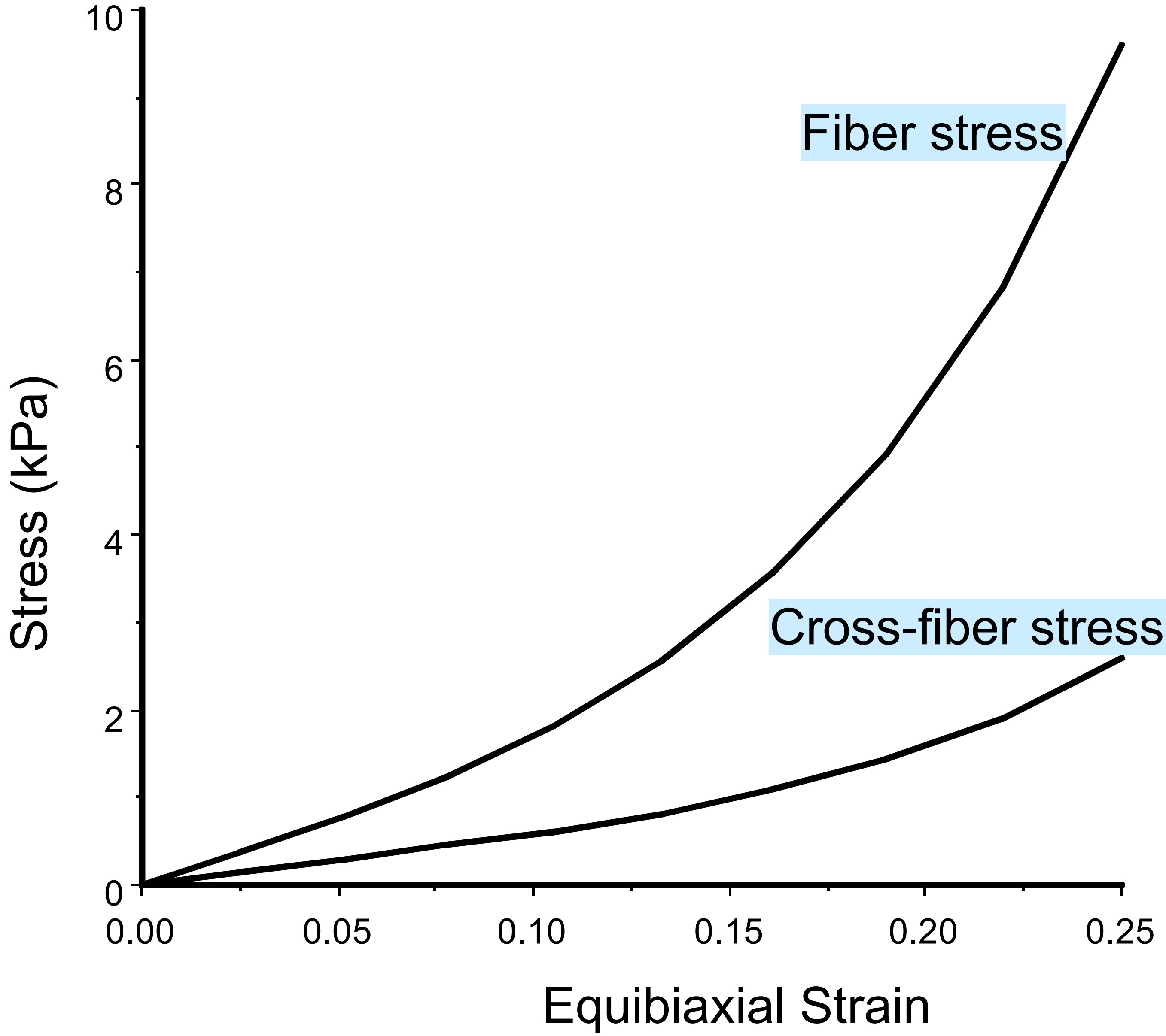
- describes the *mechanical properties* of a *material*, which depend on its *constituents*
- is a mathematical relation for *stress* as a function of *kinematic* quantities, such as *strain* or *strain-rate*
- is an *idealization* and an approximation
- the validity of the idealization depends not only on the material but on the mechanical *conditions*
- must typically be determined by *experiment*
- is *constrained* by thermodynamic and other physical conditions, e.g. conservation of mass and energy
- should be derived from considerations of material *microstructure*

Nonlinear Resting Stress-Strain Relation (Contribution of Type I Collagen)

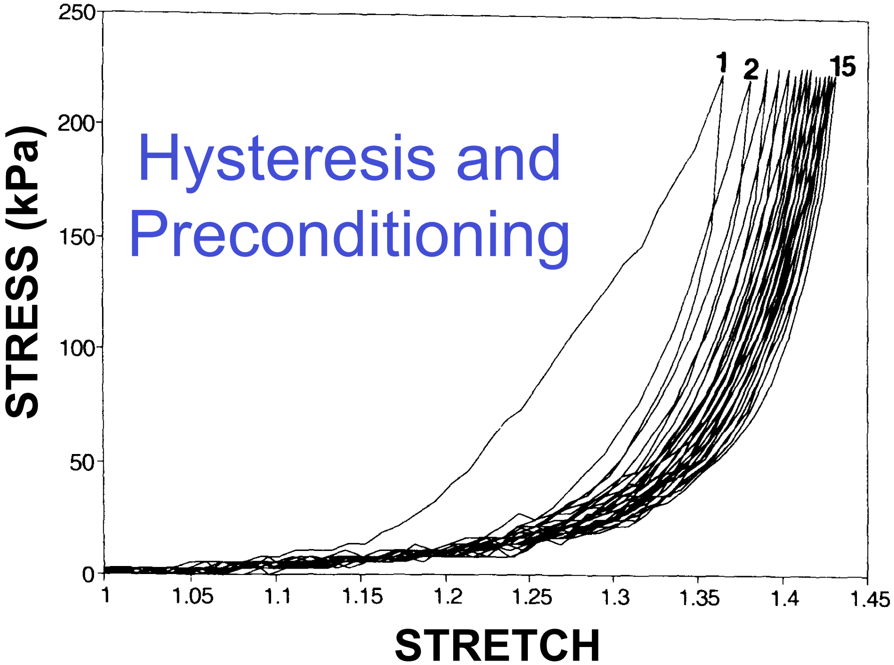


n=9 WT, n=7 *oim*
P<0.05

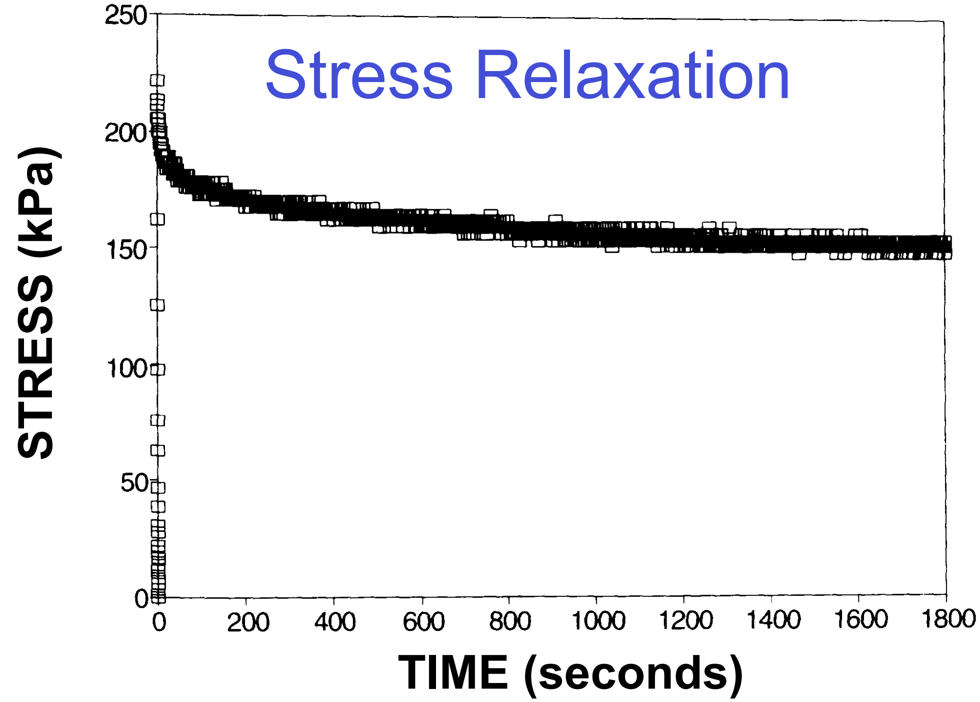
Anisotropy: Passive Biaxial Properties

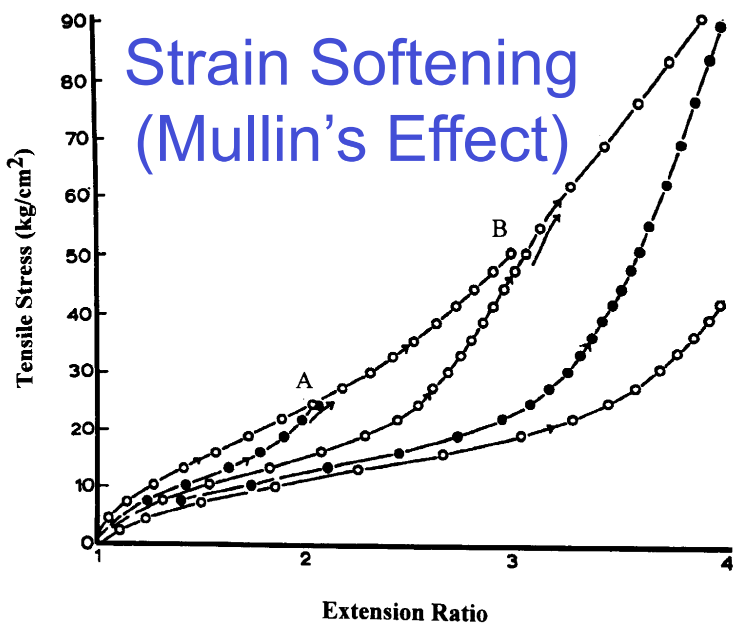


Anelastic Properties



data from Humphrey JD, Salunke N, Tippett B, 1996





Hyperelasticity

In words

*In an **elastic material** the stress depends only on the strain.*

*The work done by the stress producing strain in a **hyper-elastic material** is stored as potential energy in a thermo-dynamically reversible process.*

Mathematically

$$\boldsymbol{T} = \boldsymbol{T}(\varepsilon) \qquad \mathrm{d}W = \boldsymbol{T}\,\mathrm{d}\varepsilon \;\rightarrow\; \boldsymbol{T} = \frac{\partial W}{\partial \varepsilon}$$

$$T_{\mathrm{ij}} = T_{\mathrm{ij}}(\varepsilon_{\mathrm{kl}}) \qquad T_{\mathrm{ij}} = \frac{\partial W}{\partial \varepsilon_{\mathrm{ij}}}$$

W (Work per unit volume) is also called the *strain energy density*

Hyperelastic Constitutive Law for Finite Deformations

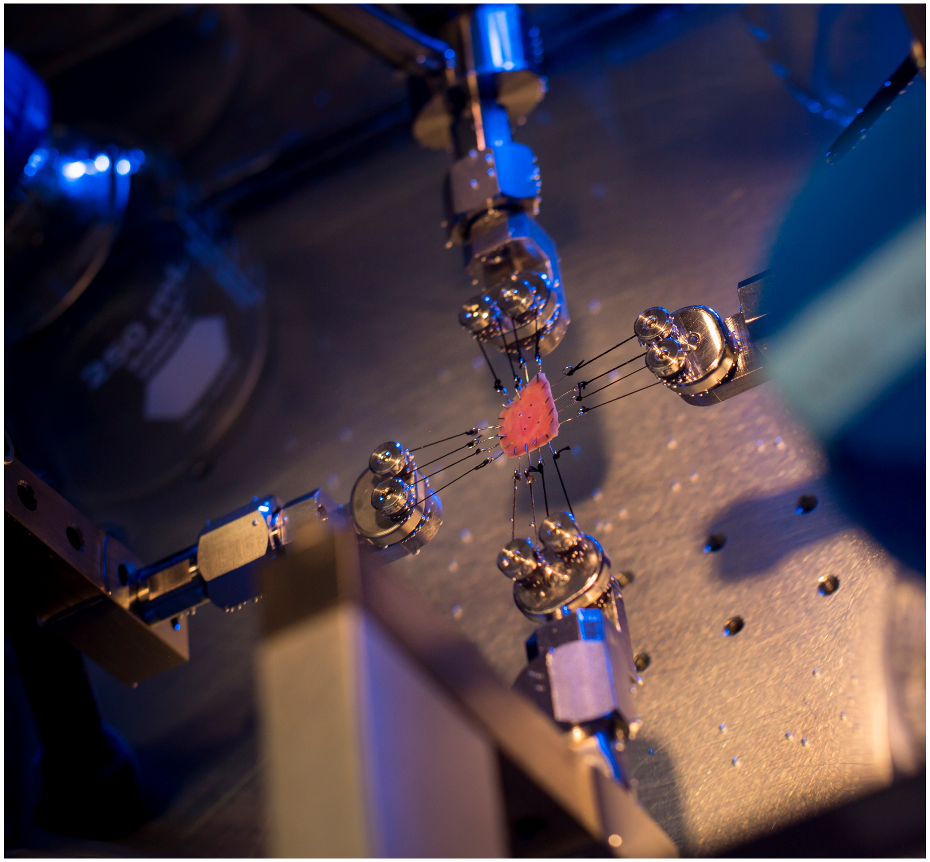
Second Piola-Kirchhoff Stress

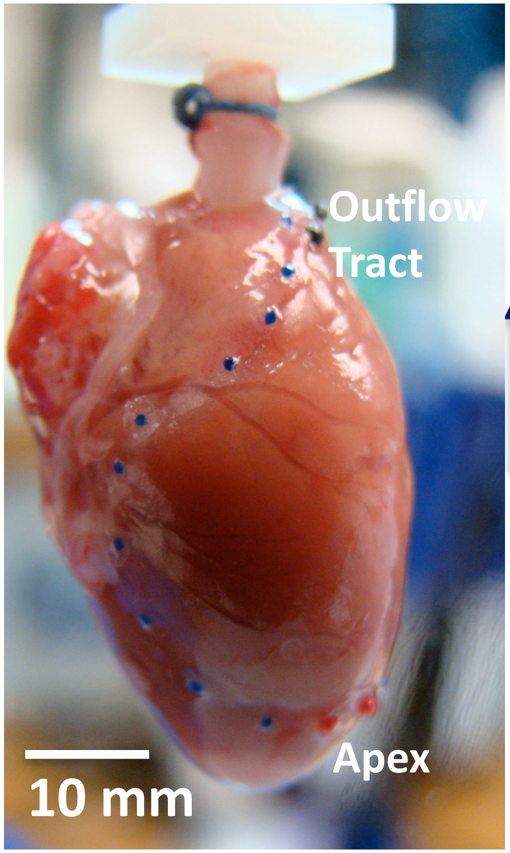
$$P_{RS} = \frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{RS}} + \frac{\partial W}{\partial E_{SR}} \right) = \frac{\partial W}{\partial E_{RS}}$$

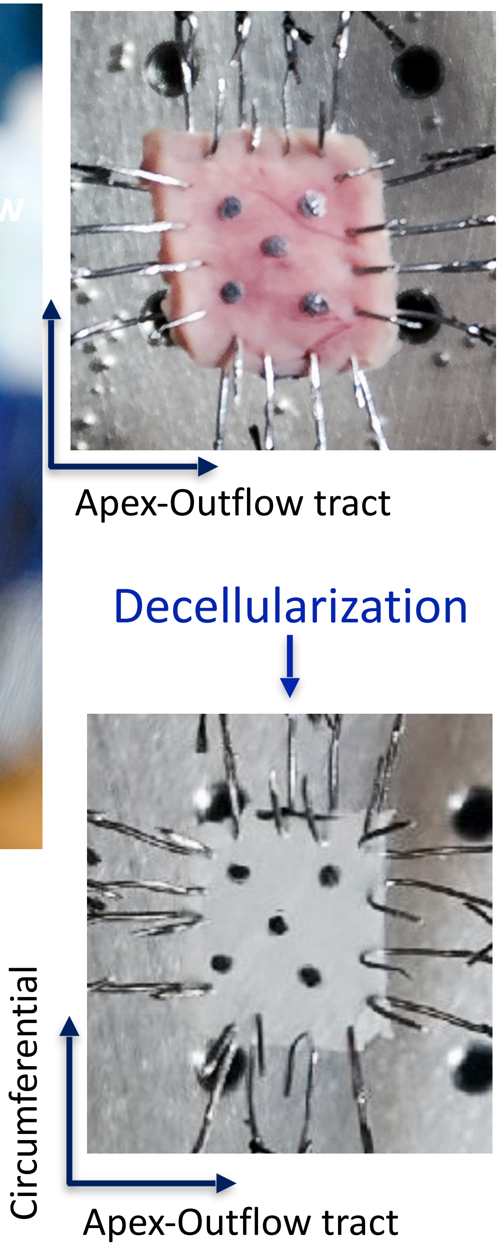
Cauchy Stress

$$T_{ij} = \frac{\rho}{\rho_o} \frac{\partial x_i}{\partial X_R} \frac{\partial x_j}{\partial X_S} \frac{\partial W}{\partial E_{RS}}$$

Biaxial Testing







Planar biaxial testing

- Equi-biaxial displacement control of intact myocardium and after decellularized ECM
- Middle region of the samples are used to compute deformation

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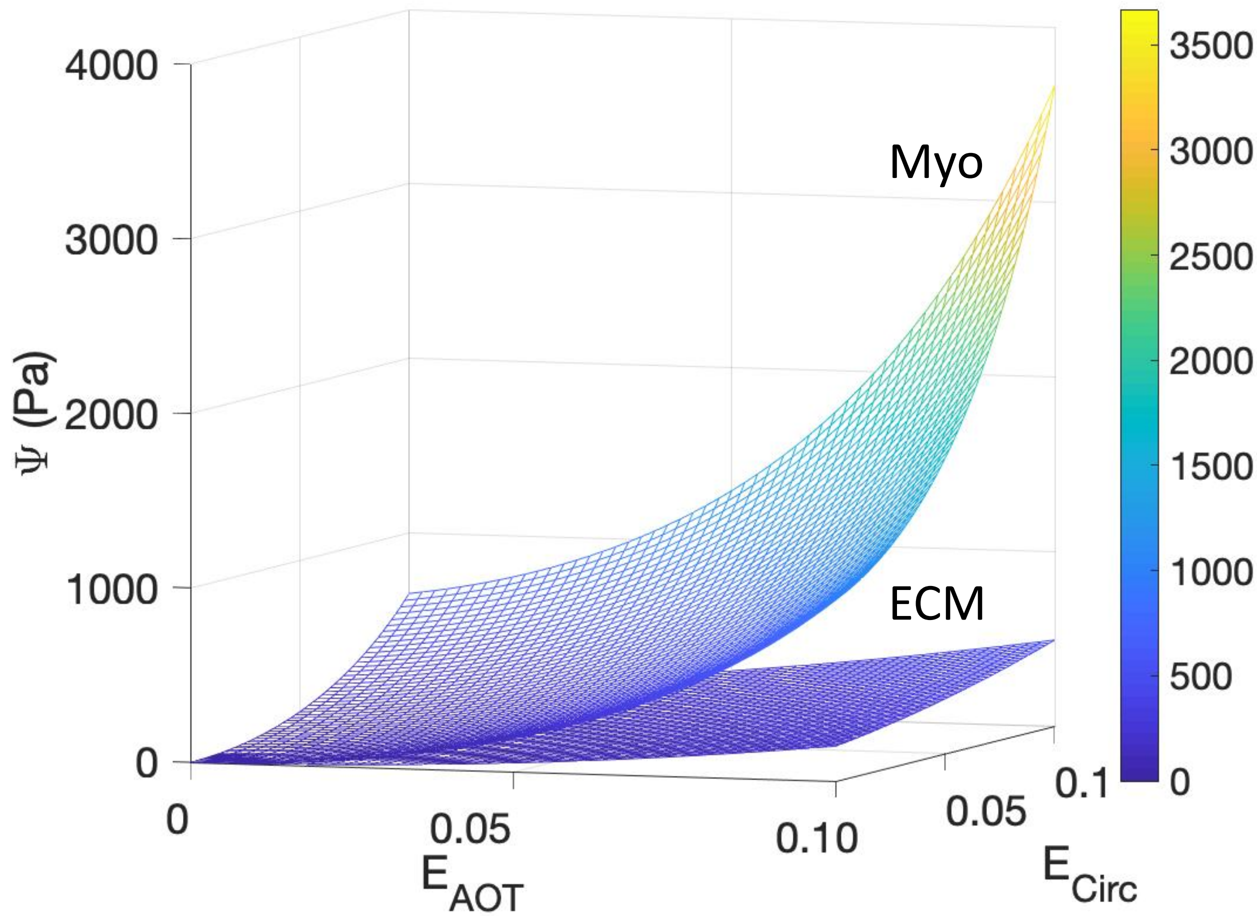
Constitutive Model

- Assuming incompressible, pseudo-hyperelastic behavior, the 2nd Piola-Kirchhoff stress is defined as

$$\mathbf{S} = \frac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}}$$

$$\boldsymbol{\Psi}(E_{11}, E_{22}) = c\left(e^{a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_3 E_{11} E_{22}} - 1\right)$$

- Stretch-control protocol designed to achieve the same strain
- RV myocardium and ECM-only stretched 10%
- Samples references to their own configurations
- Parameter estimation was based on minimizing the stress components



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Nonlinear Biomechanics: Universal Governing Equations

Kinematics

Strain-displacement relation $\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$

Deformation gradient tensor $F_{iR} = \dfrac{\partial x_i}{\partial X_R}$ $\boldsymbol{F} = \mathrm{Grad}(\boldsymbol{x})$

Conservation of Momentum

Force balance $\mathrm{Div}\boldsymbol{S}+\rho\boldsymbol{b} = \mathrm{Div}(\boldsymbol{P}\bullet\boldsymbol{F}^{\mathrm{T}})+\rho\boldsymbol{b} = \boldsymbol{0}$

Moment balance $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$

Conservation of Mass

Lagrangian form (ρ is mass density) $\rho = \rho_0 \det\boldsymbol{F} = \rho_0 J$

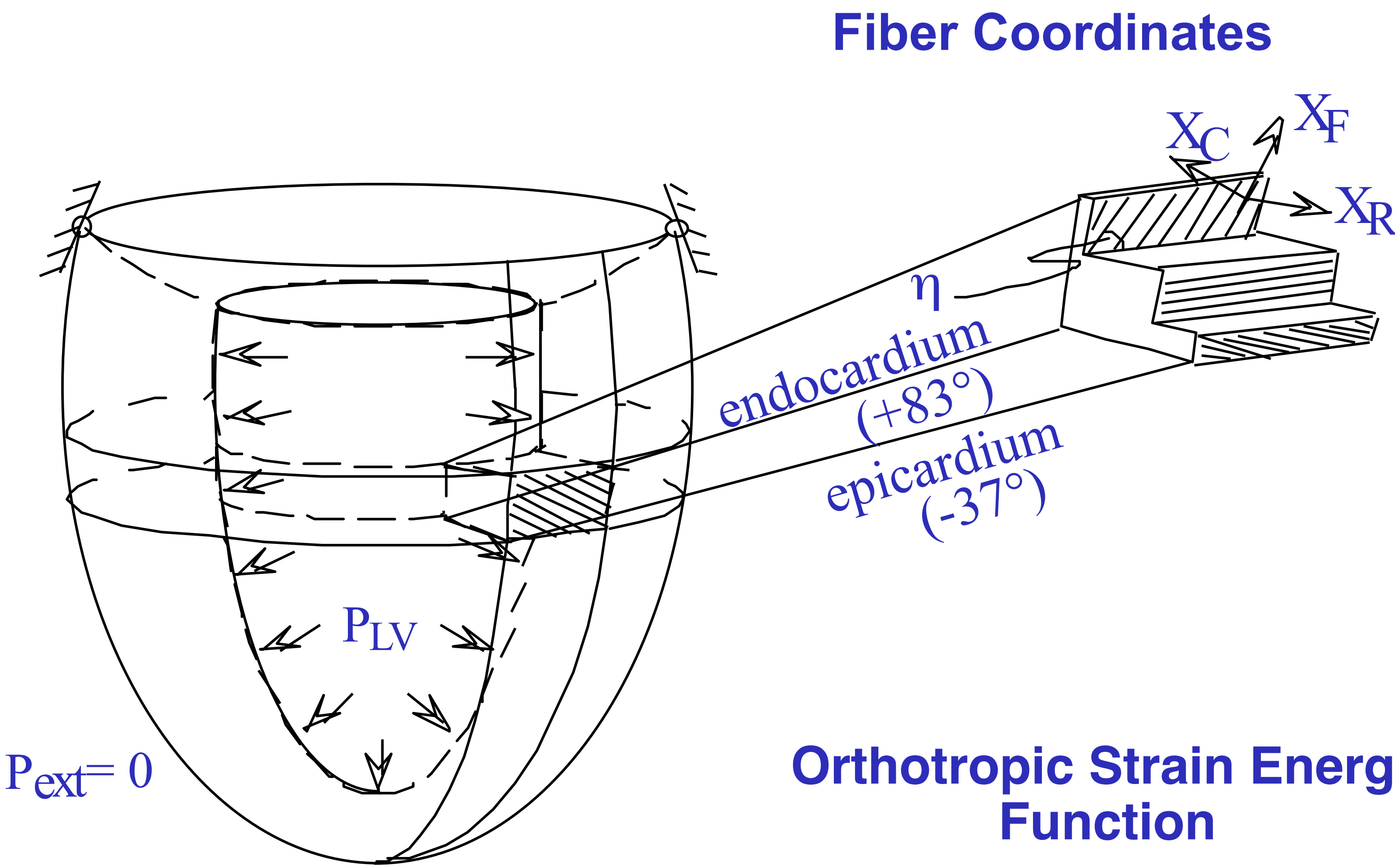
Constitutive law

Hyperelastic relation for Lagrangian 2nd Piola-Kirchoff stress (*W* is the *strain energy* function)

$$P_{RS} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{RS}} + \frac{\partial W}{\partial E_{SR}}\right)$$

Eulerian Cauchy stress

$$\boldsymbol{T} = \frac{1}{\det\boldsymbol{F}}\boldsymbol{F}\bullet\boldsymbol{P}\bullet\boldsymbol{F}^{\mathrm{T}}$$



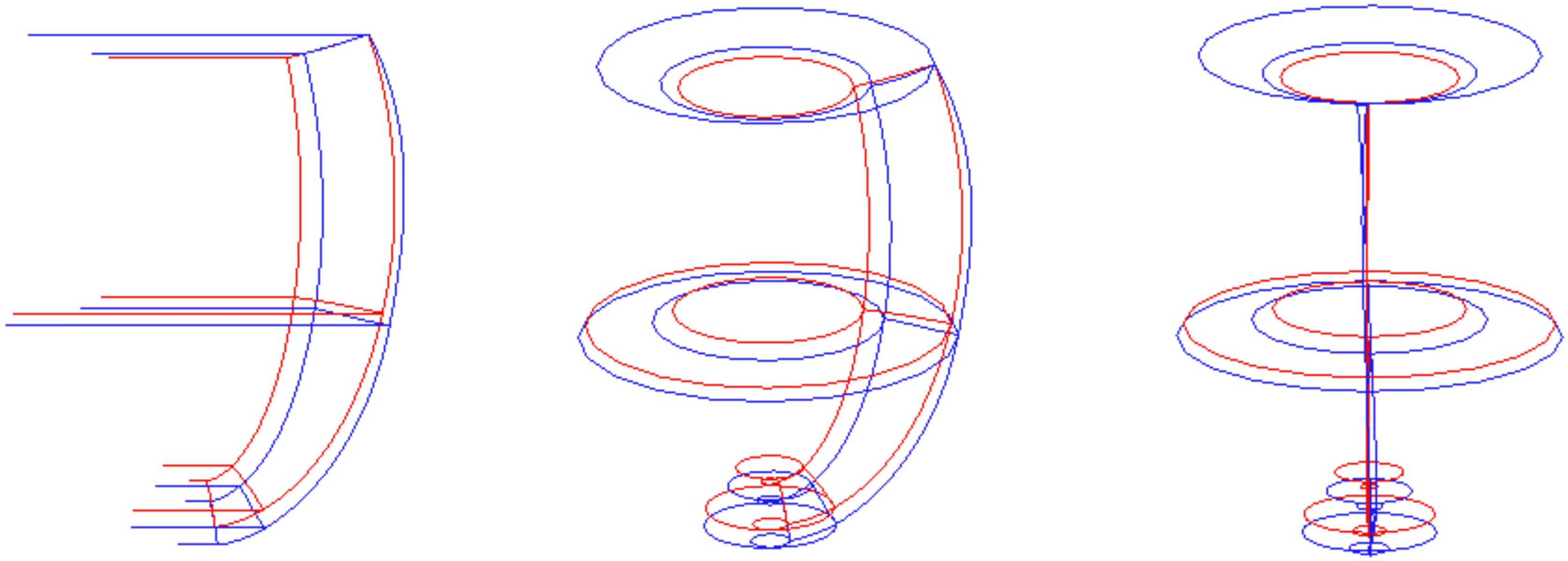
Orthotropic Strain Energy Function

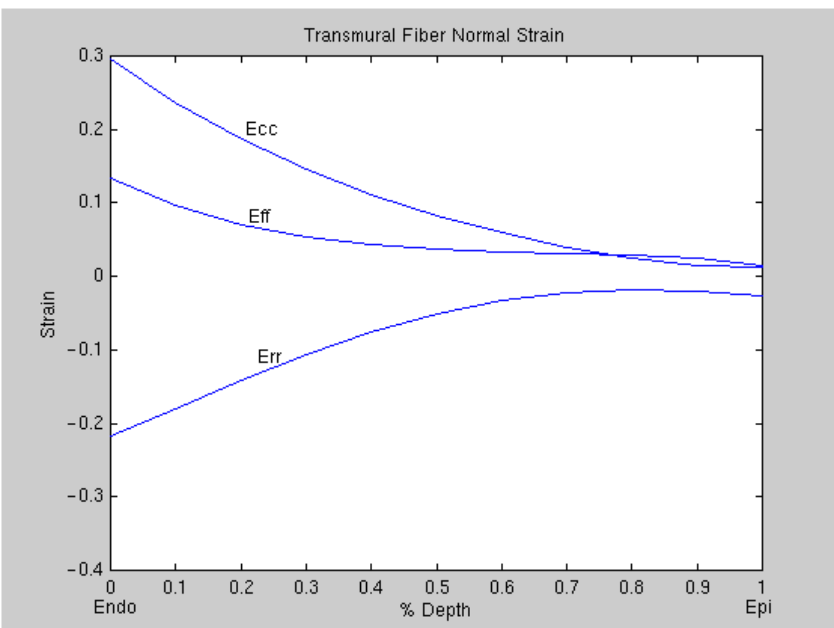
$$W = \frac{c}{2} e^{Q} + C_{compr} \left(J \ln J - J + 1 \right)$$

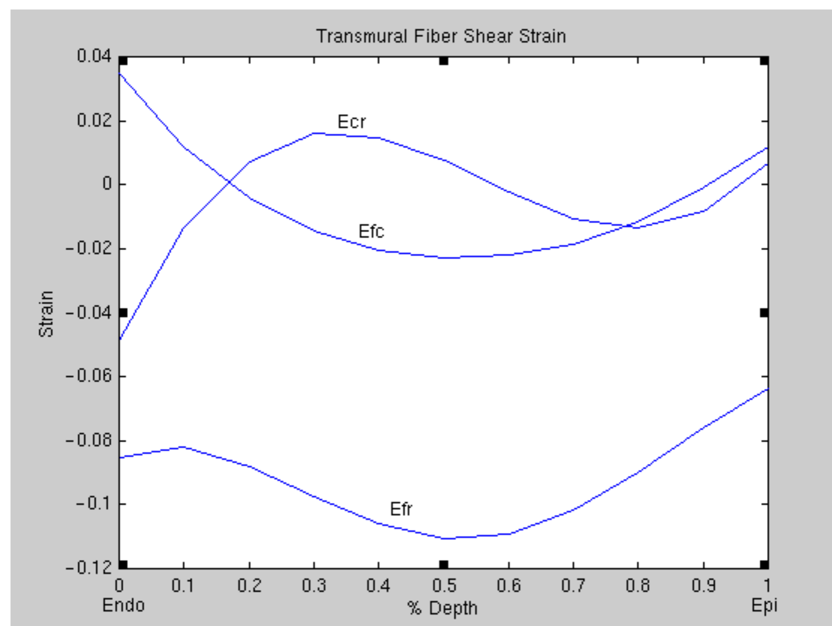
$$Q = b_1 E_{FF}^2 + b_2 E_{SS}^2 + + b_3 E_{NN}^2$$

$$+ b_4 \left(E_{FS}^2 + E_{SF}^2 \right) + b_5 \left(E_{FN}^2 + E_{NF}^2 \right) + b_6 \left(E_{SN}^2 + E_{NS}^2 \right)$$

Inflation of a High-order Passive Anisotropic Ellipsoidal Model of Canine LV

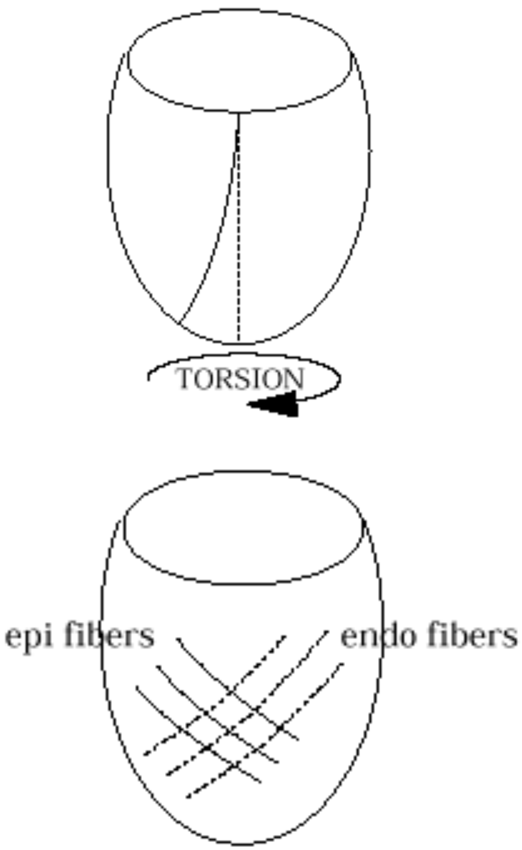


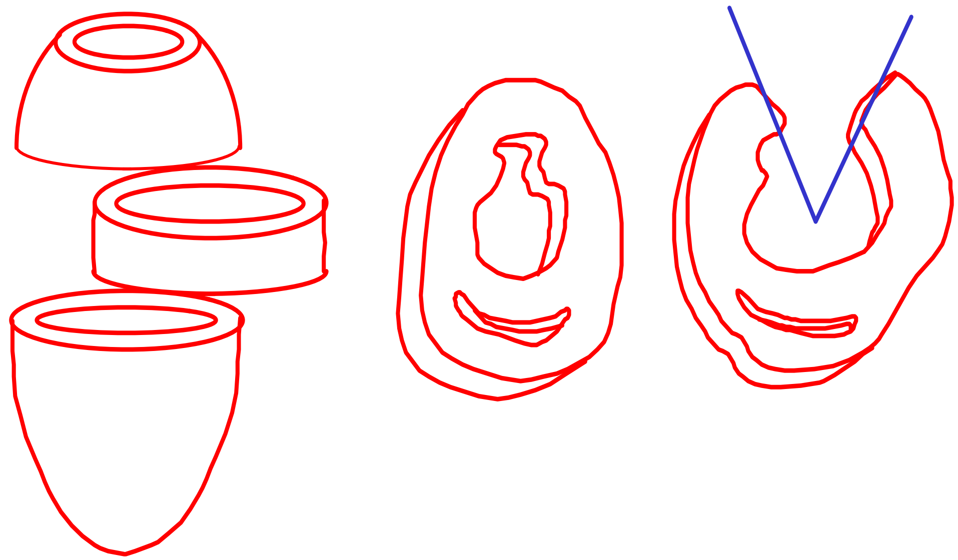




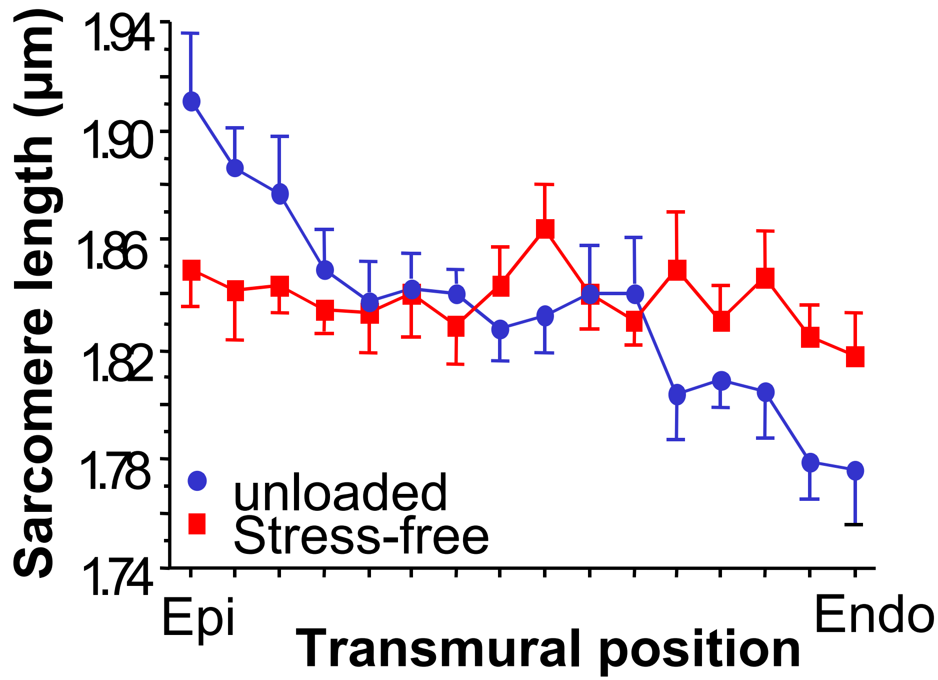
Minimizing Stress Gradients

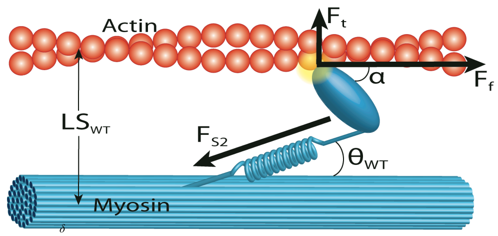
- Residual Stress
- Fiber Angles
- Torsion



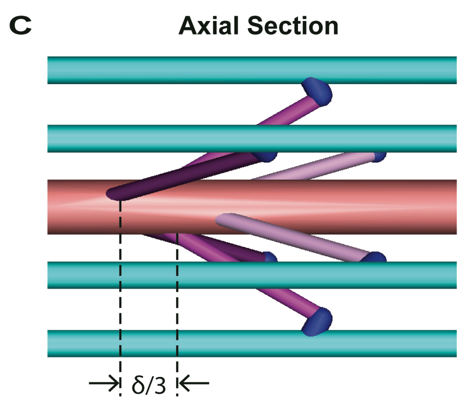


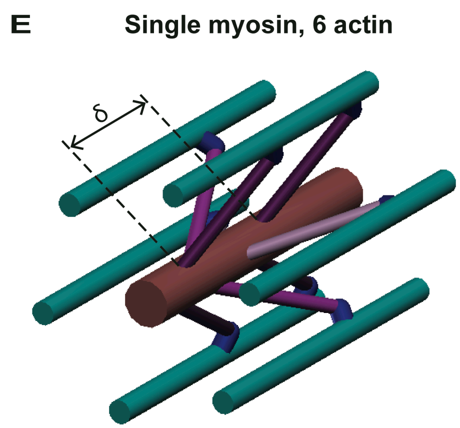
Equatorial slice
Unloaded state
Stress-free state





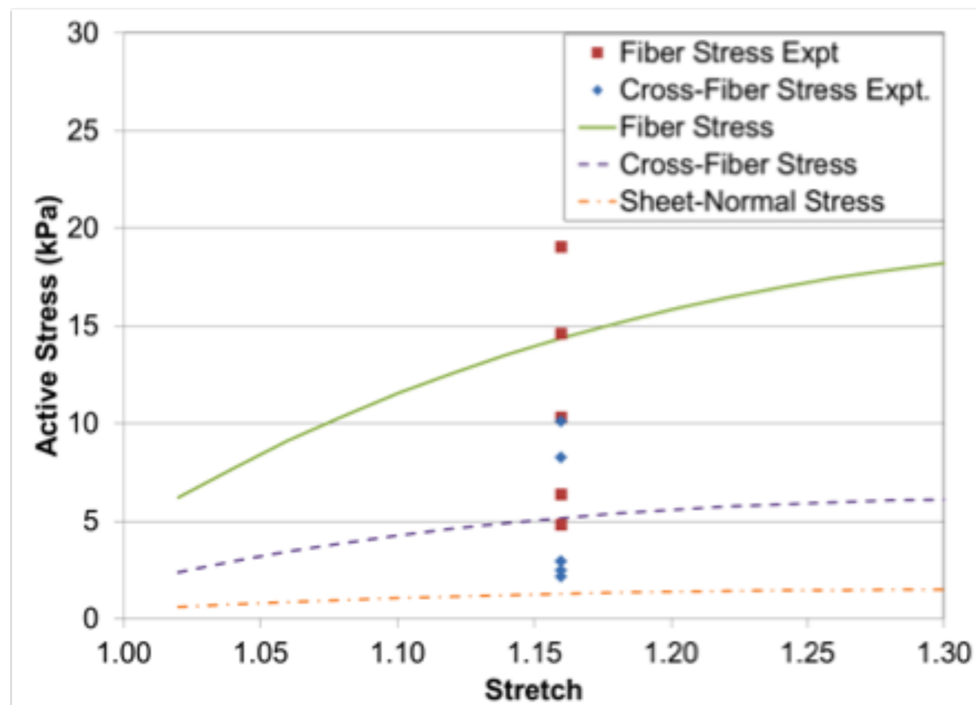
Othotropic 3-D Active Stress

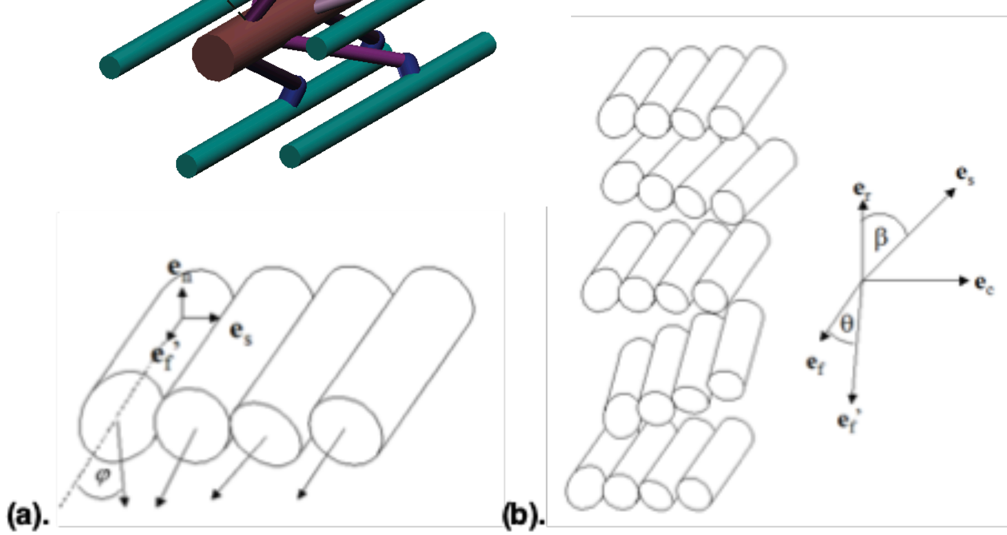




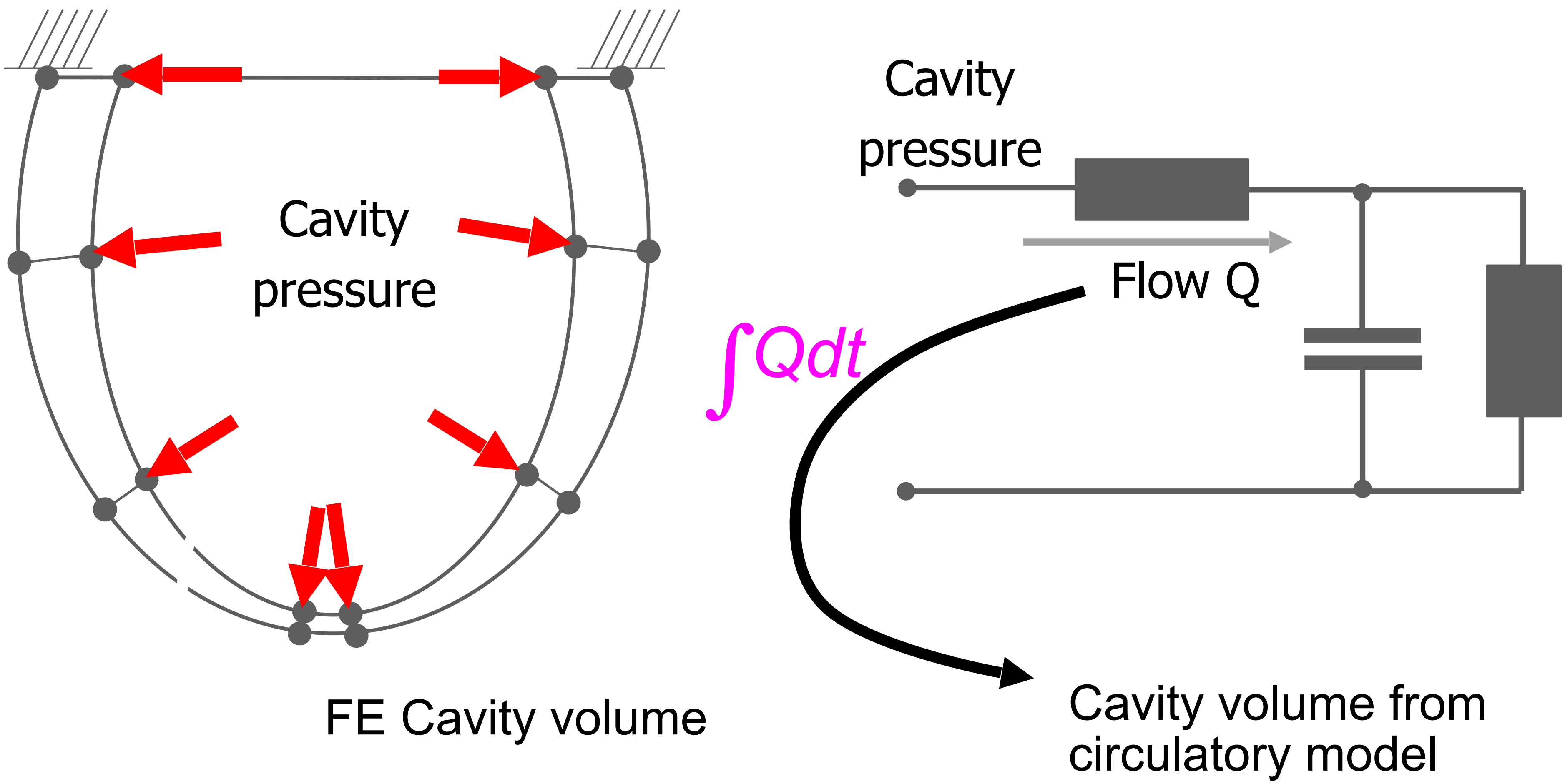
Cauchy stress tensor **T**: $\mathbf{T}_{active}$ is a function of [Ca]$_i$ and SL.

$$\mathbf{T} = \mathbf{T}^{(p)} + \mathbf{T}^{(a)}$$

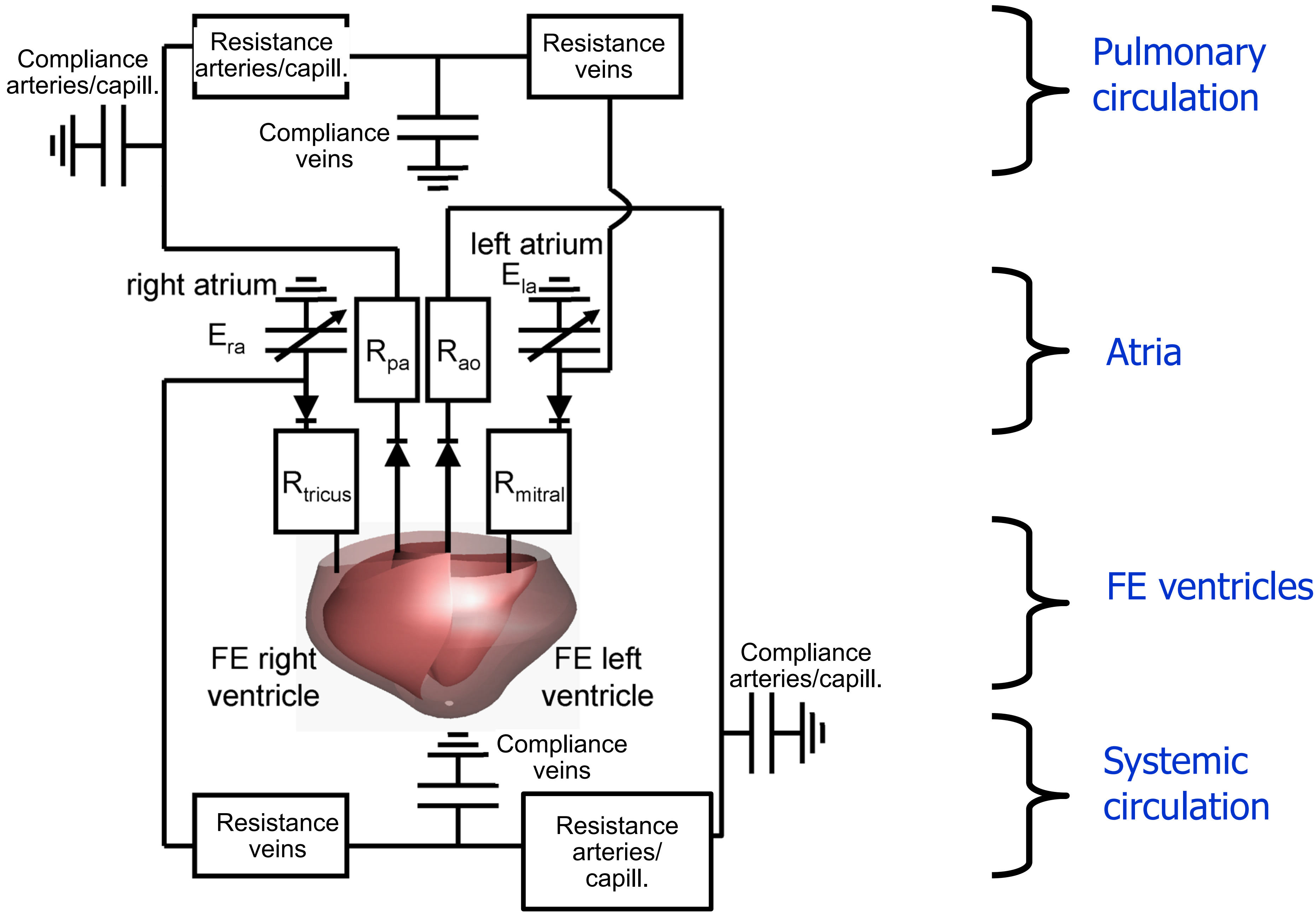




Pressure serves as hemodynamic boundary condition



Circulation



PV Relations