Exponential Distribution

- Definition: Exponential distribution with parameter λ:
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
- The cdf:
$$F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
- Mean $E(X) = 1/\lambda$.
- Moment generating function:
$$\phi(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$
- $E(X^2) = \frac{d^2}{dt^2}\phi(t)|_{t=0} = 2/\lambda^2$.
- $Var(X) = E(X^2) - (E(X))^2 = 1/\lambda^2$.

- Properties
  1. Memoryless: $P(X > s + t | X > t) = P(X > s)$.

$$\begin{aligned} & P(X > s + t | X > t) \\ = \ & \frac{P(X > s + t, X > t)}{P(X > t)} \\ = \ & \frac{P(X > s + t)}{P(X > t)} \\ = \ & \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ = \ & e^{-\lambda s} \\ = \ & P(X > s) \end{aligned}$$

     – Example: Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes, $\lambda = 1/10$. What is the probability that a customer will spend more than 15 minutes in the bank? What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?
     Solution:

$$P(X > 15) = e^{-15\lambda} = e^{-3/2} = 0.22$$
$$P(X > 15 | X > 10) = P(X > 5) = e^{-1/2} = 0.604$$

– *Failure rate* (hazard rate) function $r(t)$

$$r(t) = \frac{f(t)}{1 - F(t)}$$

– $P(X \in (t, t + dt) | X > t) = r(t)dt$.

– For exponential distribution: $r(t) = \lambda$, $t > 0$.

– Failure rate function uniquely determines $F(t)$:

$$F(t) = 1 - e^{-\int_0^t r(t)dt} .$$

2. If X_i, $i = 1, 2, ..., n$, are iid exponential RVs with mean $1/\lambda$, the pdf of $\sum_{i=1}^{n} X_i$ is:

$$f_{X_1+X_2+\cdots+X_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!},$$

gamma distribution with parameters n and λ.

3. If X_1 and X_2 are independent exponential RVs with mean $1/\lambda_1$, $1/\lambda_2$,

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

4. If X_i, $i = 1, 2, ..., n$, are independent exponential RVs with rate μ_i. Let $Z = \min(X_1, ..., X_n)$ and $Y = max(X_1, ..., X_n)$. Find distribution of Z and Y.

   – Z is an exponential RV with rate $\sum_{i=1}^{n} \mu_i$.

$$\begin{aligned} P(Z > x) &= P(\min(X_1, ..., X_n) > x) \\ &= P(X_1 > x, X_2 > x, ..., X_n > x) \\ &= P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) \\ &= \prod_{i=1}^{n} e^{-\mu_i x} = e^{-(\sum_{i=1}^{n} \mu_i)x} \end{aligned}$$

   – $F_Y(x) = P(Y < x) = \prod_{i=1}^{n}(1 - e^{-\mu_i x})$.

Poisson Process

- *Counting process*: Stochastic process $\{N(t), t \geq 0\}$ is a counting process if $N(t)$ represents the total number of "events" that have occurred up to time t.
  - $N(t) \geq 0$ and are of integer values.
  - $N(t)$ is nondecreasing in t.
- *Independent increments*: the numbers of events occurred in *disjoint* time intervals are independent.
- *Stationary increments*: the distribution of the number of events occurred in a time interval only depends on the length of the interval and does not depend on the position.

- A counting process $\{N(t), t \geq 0\}$ is a *Poisson process* with rate λ, $\lambda > 0$ if

  1. $N(0) = 0$.
  2. The process has independent increments.
  3. The process has staionary increments and $N(t+s)-N(s)$ follows a Poisson distribution with parameter λt:

$$P(N(t+s)-N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, ...$$

- Note: $E[N(t+s) - N(s)] = \lambda t$.
  $E[N(t)] = E[N(t+0) - N(0)] = \lambda t$.

Interarrival and Waiting Time

- Define T_n as the elapsed time between $(n-1)$st and the nth event.

$$\{T_n, n = 1, 2, ...\}$$

is a sequence of *interarrival times*.

- **Proposition 5.1**: T_n, $n = 1, 2, ...$ are independent identically distributed exponential random variables with mean $1/\lambda$.

- Define S_n as the *waiting time* for the nth event, i.e., the arrival time of the nth event.

$$S_n = \sum_{i=1}^{n} T_i \ .$$

- Distribution of S_n:

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \, ,$$

gamma distribution with parameters n and λ.

- $E(S_n) = \sum_{i=1}^{n} E(T_i) = n/\lambda$.

- Example: Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day. (a) What is the expected time until the tenth immigrant arrives? (b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds 2 days?

  Solution:

  Time until the 10th immigrant arrives is S_{10}.

$$E(S_{10}) = 10/\lambda = 10\,.$$

$$P(T_{11} > 2) = e^{-2\lambda} = 0.133\ .$$

Further Properties

- Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ. Each event belongs to two types, I and II. The type of an event is independent of everything else. The probability of being in type I is p.
- Examples: female vs. male customers, good emails vs. spams.
- Let $N_1(t)$ be the number of type I events up to time t.
- Let $N_2(t)$ be the number of type II events up to time t.
- $N(t) = N_1(t) + N_2(t)$.

- **Proposition 5.2**: $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$. Furthermore, the two processes are independent.

- Example: If immigrants to area A arrive at a Poisson rate of 10 per week, and if each immigrant is of English descent with probability $1/12$, then what is the probability that no people of English descent will immigrate to area A during the month of February?
  Solution:
  The number of English descent immigrants arrived up to time t is $N_1(t)$, which is a Poisson process with mean $\lambda/12 = 10/12$.
  $$P(N_1(4) = 0) = e^{-(\lambda/12)\cdot 4} = e^{-10/3} .$$

- **Conversely**: Suppose $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes having respective rates λ_1 and λ_2. Then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. For any event occurred with unknown type, independent of everything else, the probability of being type I is $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and type II is $1 - p$.

- Example: On a road, cars pass according to a Poisson process with rate 5 per minute. Trucks pass according to a Poisson process with rate 1 per minute. The two processes are indepdendent. If in 3 minutes, 10 veicles passed by. What is the probability that 2 of them are trucks?
  Solution:
  Each veicle is independently a car with probability $\frac{5}{5+1} = \frac{5}{6}$ and a truck with probability $\frac{1}{6}$. The probability that 2 out of 10 veicles are trucks is given by the binomial distribution:
  $$\binom{10}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8$$

Conditional Distribution of Arrival Times

- Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ. Up to t, there is exactly one event occurred. What is the conditional distribution of T_1?
- Under the condition, T_1 uniformly distributes on $[0, t]$.
- Proof

$$
\begin{aligned}
& P(T_1 < s | N(t) = 1) \\
= \ & \frac{P(T_1 < s, N(t) = 1)}{P(N(t) = 1)} \\
= \ & \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)} \\
= \ & \frac{P(N(s) = 1) P(N(t) - N(s) = 0)}{P(N(t) = 1)} \\
= \ & \frac{(\lambda s e^{-\lambda s}) \cdot e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
= \ & \frac{s}{t} \qquad \text{Note: cdf of a uniform}
\end{aligned}
$$

- If $N(t) = n$, what is the joint conditional distribution of the arrival times S_1, S_2, ..., S_n?
- S_1, S_2, ..., S_n is the *ordered statistics* of n independent random variables uniformly distributed on $[0, t]$.
- Let Y_1, Y_2, ..., Y_n be n RVs. $Y_{(1)}$, $Y_{(2)}$,..., $Y_{(n)}$ is the ordered statistics of Y_1, Y_2, ..., Y_n if $Y_{(k)}$ is the kth smallest value among them.
- If Y_i, $i = 1, ..., n$ are iid continuous RVs with pdf f, then the joint density of the ordered statistics $Y_{(1)}$, $Y_{(2)}$,..., $Y_{(n)}$ is

$$f_{Y_{(1)},Y_{(2)},\ldots,Y_{(n)}}(y_1, y_2, \ldots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f(y_i) & y_1 < y_2 < \cdots < y_n \\ 0 & \text{otherwise} \end{cases}$$

- We can show that

$$f(s_1, s_2, ..., s_n \mid N(t) = n) = \frac{n!}{t^n}$$
$$0 < s_1 < s_2 \cdots < s_n < t$$

Proof

$$\begin{aligned}
& f(s_1, s_2, ..., s_n \mid N(t) = n) \\
= \ & \frac{f(s_1, s_2, ..., s_n, n)}{P(N(t) = n))} \\
= \ & \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \cdots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\
= \ & \frac{n!}{t^n}, \quad 0 < s_1 < \cdots < s_n < t
\end{aligned}$$

- For n independent uniformly distributed RVs on $[0, t]$, Y_1, ..., Y_n:

$$f(y_1, y_2, ..., y_n) = \frac{1}{t^n} .$$

- **Proposition 5.4**: Given $S_n = t$, the arrival times S_1, S_2, ..., S_{n-1} has the distribution of the ordered statistics of a set $n-1$ independent uniform $(0, t)$ random variables.

Generalization of Poisson Process

- *Nonhomogeneous Poisson process*: The counting process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$ if

  1. $N(0) = 0$.
  2. The process has independent increments.
  3. The distribution of $N(t+s)-N(t)$ is Poisson with mean given by $m(t+s) - m(t)$, where

  $$m(t) = \int_0^t \lambda(\tau) d\tau \ .$$

- We call $m(t)$ *mean value function*.
- Poisson process is a special case where $\lambda(t) = \lambda$, a constant.

- *Compound Poisson process*: A stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i \ , \ t \geq 0$$

where $\{N(t), t \geq 0\}$ is a Poisson process and $\{Y_i, i \geq 0\}$ is a family of independent and identically distributed random variables which are also independent of $\{N(t), t \geq 0\}$.

- The random variable $X(t)$ is said to be a *compound Poisson random variable*.

- Example: Suppose customers leave a supermarket in accordance with a Poisson process. If Y_i, the amount spent by the ith customer, $i = 1, 2, ...$, are independent and identically distributed, then $X(t) = \sum_{i=1}^{N(t)} Y_i$, the total amount of money spent by customers by time t is a compound Poisson process.

- Find $E[X(t)]$ and $Var[X(t)]$.
- $E[X(t)] = \lambda t E(Y_1)$.
- $Var[X(t)] = \lambda t (Var(Y_1) + E^2(Y_1))$
- Proof

$$\begin{aligned}
E(X(t)|N(t) = n) &= E(\sum_{i=1}^{N(t)} Y_i | N(t) = n) \\
&= E(\sum_{i=1}^{n} Y_i | N(t) = n) \\
&= E(\sum_{i=1}^{n} Y_i) = nE(Y_1)
\end{aligned}$$

$$\begin{aligned}
E(X(t)) &= E_{N(t)} E(X(t)|N(t)) \\
&= \sum_{i=1}^{\infty} P(N(t) = n) E(X(t)|N(t) = n) \\
&= \sum_{i=1}^{\infty} P(N(t) = n) n E(Y_1) \\
&= E(Y_1) \sum_{i=1}^{\infty} n P(N(t) = n) \\
&= E(Y_1) E(N(t)) \\
&= \lambda t E(Y_1)
\end{aligned}$$

$$
\begin{aligned}
Var(X(t)|N(t)=n) &= Var(\sum_{i=1}^{N(t)} Y_i|N(t)=n) \\
&= Var(\sum_{i=1}^{n} Y_i|N(t)=n) \\
&= Var(\sum_{i=1}^{n} Y_i) \\
&= nVar(Y_1)
\end{aligned}
$$

$$
\begin{aligned}
& Var(X(t)|N(t)=n) \\
=\ & E(X^2(t)|N(t)=n) - (E(X(t)|N(t)=n))^2
\end{aligned}
$$

$$
\begin{aligned}
& E(X^2(t)|N(t)=n) \\
=\ & Var(X(t)|N(t)=n) + (E(X(t)|N(t)=n))^2 \\
=\ & nVar(Y_1) + n^2E^2(Y_1)
\end{aligned}
$$

$$
\begin{aligned}
& Var(X(t)) \\
= \ & E(X^2(t)) - (E(X(t)))^2 \\
= \ & \sum_{i=1}^{\infty} P(N(t) = n) E(X^2(t) | N(t) = n) - (E(X(t)))^2 \\
= \ & \sum_{i=1}^{\infty} P(N(t) = n)(nVar(Y_1) + n^2 E^2(Y_1)) - (\lambda t E(Y_1))^2 \\
= \ & Var(Y_1)E(N(t)) + E^2(Y_1)E(N^2(t)) - (\lambda t E(Y_1))^2 \\
= \ & \lambda t Var(Y_1) + \lambda t E^2(Y_1) \\
= \ & \lambda t (Var(Y_1) + E^2(Y_1)) \\
= \ & \lambda t E(Y_1^2)
\end{aligned}
$$