MATH 242 Metric_Space_and_Calculus

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To Du, To Do

Chapter 1

1.1 The definition of a metric space

Definition 1.1.1 distance

Let X be a set. A **distance** on X is a function
$d : X \times X \to \mathbb{R}_{\geq 0}$
to the non-negative real numbers(because the distance should be a positive number) that satisfies the following axioms:

M1: For every $x, y \in X, \quad d(x, y) = 0 \ \ iff \ x = y$
In natural language: if there are 2 points in set X, and the distance between these 2 points is 0, then these 2 point is equal.

M2 Symmetry: For every $x, y \in X, \quad d(x, y) = d(y, x)$
In natural language: if there are 2 points in set X, the distance from x to y is equal to the distance from y to x.

M3 Triangle inequality: For every $x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$
In natural language: if there are 3 points, the distance between x and z must less or equal to the sum of distance between x with y and y with z.

In Tao's Analysis 2, exist M4
M4 Positivity: For any $distinct\ x, y \in X$, we have $d(x, y) > 0$.
In natural language: the distance must be non-negative.

Proof M4
If we set $x = z$ in **M3**, and take account of **M1** and **M2**, we can find that:
$d(x, y) = |x - y| = |y - x|, \ x, y \in \mathbb{R}$

For some more trivial metric,
thus the distance must be non-negtive. And in $\mathbb{R}$, $d(x, y) = |x - y| = |y - x|$
**M4 is not important in this module.*

Definition 1.1.2 Metric Space

A $metric\ space$ is a pair (X,d), where X is a set and d is a distance on X.

Example 1.1.3(The example of $Cross\ product(Cartesian\ product)$)

If $X = \{Dog, Cat, Horse\}, \ Y = \{1, 2\}$
Then the $cartesian\ product\ of\ X\ and\ Y\ is$
$X \times Y = \{(Dog, 1), (Cat, 1), (Horse, 1), (Dog, 2), (Cat, 2), (Horse, 2)\}$

Definition 1.1.4

The number of elements in a finite set X is denoted by $\#X$.
propoties: $\#(X \times Y) = \#X \times \#Y$

Now let $\mathbb{R}$ be the set of real numbers, we have:

1. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ the $real\ plane(2 - dimensional)$.Element $x \in \mathbb{R}^2$are vectors of real numbers that we write as $x = (x_1, x_2) \in R^2$

2. $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ the $real\ space(3-dimensional)$.Element $x \in \mathbb{R}^3$are vectors of real numbers that we write as $x = (x_1, x_2, x_3) \in R^3$
3. ......

We can consider there are n vectors span the space(or plane) independently, then the space(or plane) is R^n, and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
In linear algebra, these vectors are called the base vector of the space.

Definition 1.1.5 d_1 **distance(**$\equiv absolute\ value \equiv manhattan\ distance$**).**

Let $X = \mathbb{R}$ and define the function
$d_1(x, y) := |x - y|$
M1 and **M2** is obviously.
Proof M3:
we need introduce a $Lemma$
#####Lemma 1.1.6
$For\ \forall a, b, c, d \in \mathbb{R},\ we\ have$
$max(a + b, c + d) \leq max(a, c) + max(b, d)$
Proof:
$Use\ scaling\ method\ could\ proof\ a + b \leq max(a, c) + max(b, d)$
$Similarly, c + d \leq max(a, c) + max(b, d)$

$$Thus, Lemma\ is\ true.$$

back to M3 Proof,
$Assume\ A = x - y,\ B = y - z,\ C = y - x,\ D = z - y$
$d(x, z) = |x - z| = max(x - z, z - x)\ \ ---\ by\ M4$
$\leq max(x - y, y - x) + max(y - z, z - y)\ \ ---\ by\ Lemma\ 1.1.6$
$= |x - y| + |y - z|\ \ ---\ by\ definition\ 1.1.5$
$= d(x, y) + d(y, z)\ \ ---\ by\ definition\ 1.1.5$

$$M3\ Proof\ end.$$

For d_1, we can assume the block of manhatten, we can only go by the road, and unable to cross the building or over the block (we cannot fly).
More precise and mathematical, we can consider the distance is a type of observation, it reflect the position relations between 2 points in a set. The "1" of d_1 is the dimension of the observation. That means we can only consider the difference of position on the base vector direction. Thus d_1 is the sum of difference of all base vector directions.

Proposition 1.1.8. ---The function d_1 is as distance on $\mathbb{R}^2$
Proof:
Before we verify the 3 properties,we have
$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$
M1, M2 is obviously.
Proof of M3
$Similar\ with\ Lemma\ 1.1.6$

$$d_1(x, z) = |x_1 - z_1| + |x_2 - z_2|$$
$$\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$$
$$= d_1(x, y) + d_1(y, z)$$

In more ordinary cases:
$If\ \forall\ x, y \in \mathbb{R}^n, we\ can\ define\ d_1\ as:$
$d_1 = \sum_{i=1}^{n} |x_i - y_i|$

Definition 1.1.9. d_2 **distance(**$\equiv Euclidean\ Distance$**)**

$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Proposition 1.1.10. --- $The\ function\ d_2\ is\ a\ distance\ on\ \mathbb{R}^2$

Proof:
$As\ usual,\ we\ begin\ by\ observing\ that\ d_2(x, y) \in \mathbb{R}_{\geq 0}.$
M1,M2 is obviously.
M3:
$define: a_1 = x_1 - y_1,\ a_2 = x_2 - y_2,\ b_1 = y_1 - z_1,\ b_2 = y_2 - z_2$

$$d_2(x, z)^2 = (a_1 + b_1)^2 + (a_2 + b_2)^2$$
$$= a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2(a_1 b_1 + a_2 b_2)$$
$$\leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \quad --- Cauchy - Schwarz\ inequality$$
$$= (\sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2})^2$$
$$= (d_2(x, y) + d_2(y, z))^2$$

Definition 1.1.10 $d_\infty\ distance\ in\ \mathbb{R}^2$

$d_\infty(x, y) = max(|x_1 - y_1|, |x_2 - y_2|)$

Proposition 1.1.11 $d_\infty\ is\ a\ distance\ on\ \mathbb{R}^2$

M1,M2 as before
M3 same as the $d_1\ 's$ **M3** proof.
For more organize case, d_∞ definded by:
$d_\infty = max_{1 \leq i \leq n}|x_i - y_i|\ is\ a\ distance\ on\ \mathbb{R}^n$

Definition 1.1.12 $d_p\ distance$

$d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}$

Proposition $d_p\ definds\ a\ distance\ on\ \mathbb{R}^n$

Proof
$The\ Triangle\ inequality\ defines\ on\ L_p\ is\ Minkovski\ inequality.\ We\ will\ discuss\ it\ later.$