Mathematical Foundations for Intelligent Engineering

Introduction

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- Course Information
- Introduction to Differential Equations
- Classification of Differential Equations

Course Information

Course Information

- Part I: Ordinary Differential Equation (ODE)
- Part II: Partial Differential Equation (PDE)
- Part III: Optimization and Learning

Course Information

3 credits; 48 hours

References:

- Ref.[1] Gilbert Strang,
  Differential Equations and Linear Algebra.
  Wellesley-Cambridge Press， 2014.
- Ref.[2] Erwin Kreyszig,
  Advanced Engineering Mathematics
  John Wiley & Sons, Inc. 2011
- Ref.[3] Jiří Lebl
  Differential Equations for Engineers (Lebl) - Mathematics LibreTexts

Course Information

- **Instructor:**
  - Sinan Xiao (肖思男)
  - Office: D1-b409
  - Email: sinanxiao@scut.edu.cn
  - Office hours: 14:00 - 16:00, Thursday
- **Teaching Assistant (TA):**
  - TBA

Math software (strongly recommended!)

- MATLAB
  https://ww2.mathworks.cn/products/matlab.html
- Mathematica
  https://www.wolfram.com/mathematica/
- Maple
  https://www.maplesoft.com/products/maple/

Very useful and powerful!

Introduction to Differential Equations

Basic Concepts: Modelling

Physical System → Mathematical Model → Mathematical Solution → Physical Interpretation

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Variables, Functions, and Equations

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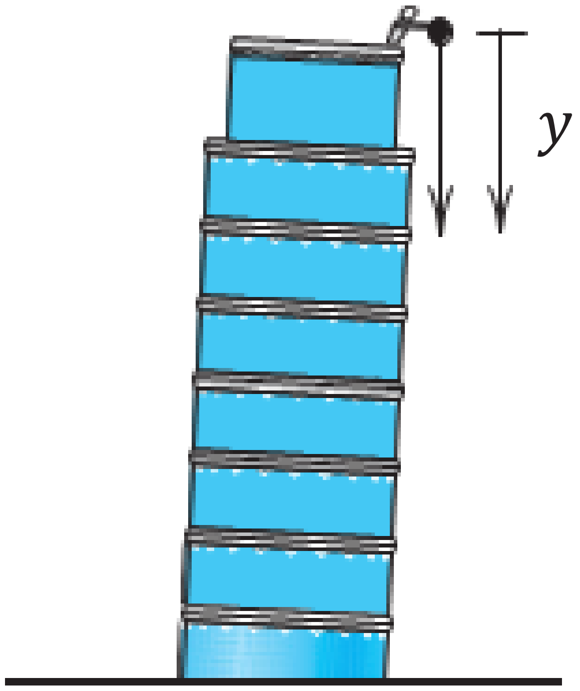
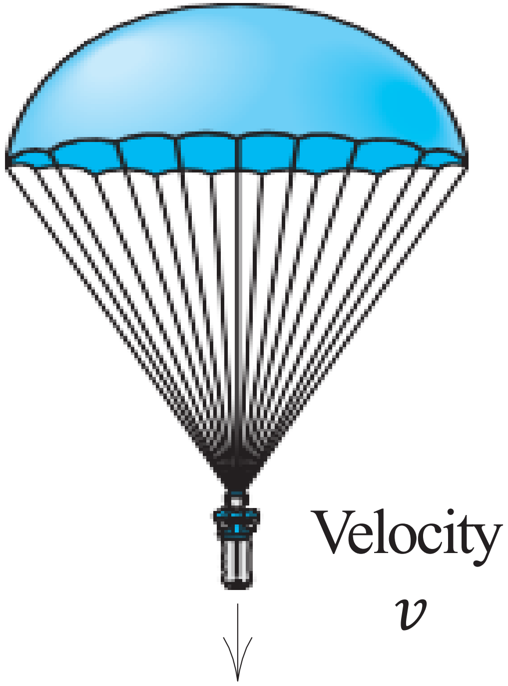
ODE
PDE
Machine Learning

Some Examples



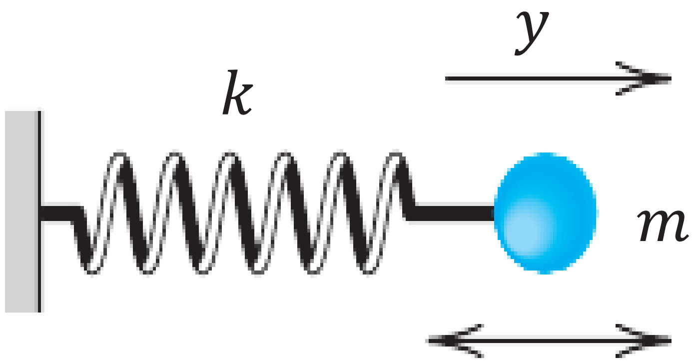
Falling stone

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g = const.$$



Parachutist

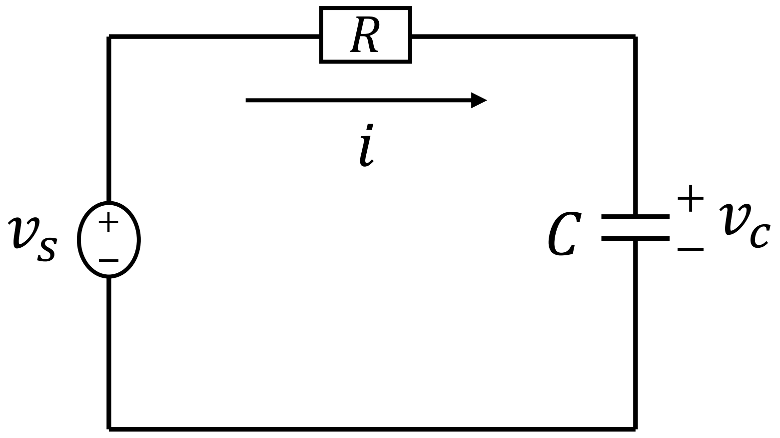
$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv^2$$



Vibrating mass on a spring

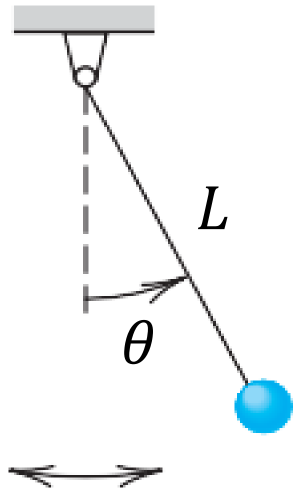
$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + ky = 0$$

Some Examples



Voltage v_c in an RC circuit

$$\frac{\mathrm{d}v_c}{\mathrm{d}t} + \frac{1}{RC} v_c = \frac{1}{RC} v_s$$



Pendulum

$$L \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g \sin\theta = 0$$

Differential Equations

The laws of physics are generally written down as differential equations.

Mathematics is the language of science, and differential equations are one of the most important parts of this language.

An example:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos(t)$$

y is the dependent variable and t is the independent variable

Solutions of Differential Equations

Solving the differential equation means finding y in terms of t, that is, the function $y(t)$.

If we plug the solution into the differential equation, it holds, that is, the left-hand side equals to the right-hand side.

We claim that

$$y = y(t) = \cos(t) + \sin(t)$$

is a solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos(t)$$

Solutions of Differential Equations

Differential equation: $\frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos(t)$

A solution: $y = y(t) = \cos(t) + \sin(t)$

Check the solution – plug the solution into the differential equation

left-hand side:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = \underbrace{(-\sin(t) + \cos(t))}_{\frac{\mathrm{d}y}{\mathrm{d}t}} + \underbrace{(\cos(t) + \sin(t))}_{y}$$

$$= 2\cos(t)$$

left-hand side equals to the right-hand side

Solutions of Differential Equations

Differential equation: $\frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos(t)$

Other solutions?

We claim $y = \cos(t) + \sin(t) + e^{-t}$ is also a solution.

left-hand side:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = (-\sin(t) + \cos(t) - e^{-t}) + (\cos(t) + \sin(t) + e^{-t})$$

$$= 2\cos(t)$$

left-hand side equals to the right-hand side

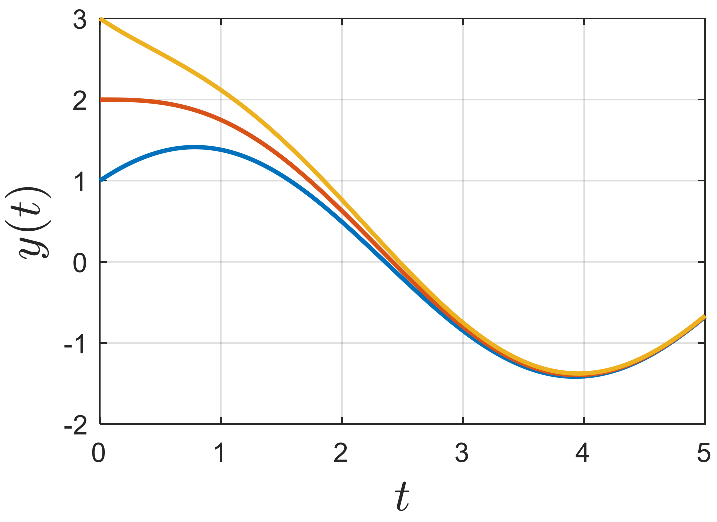
Solutions of Differential Equations

Differential equation: $\frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos(t)$

There can be many different solutions to this differential equation.

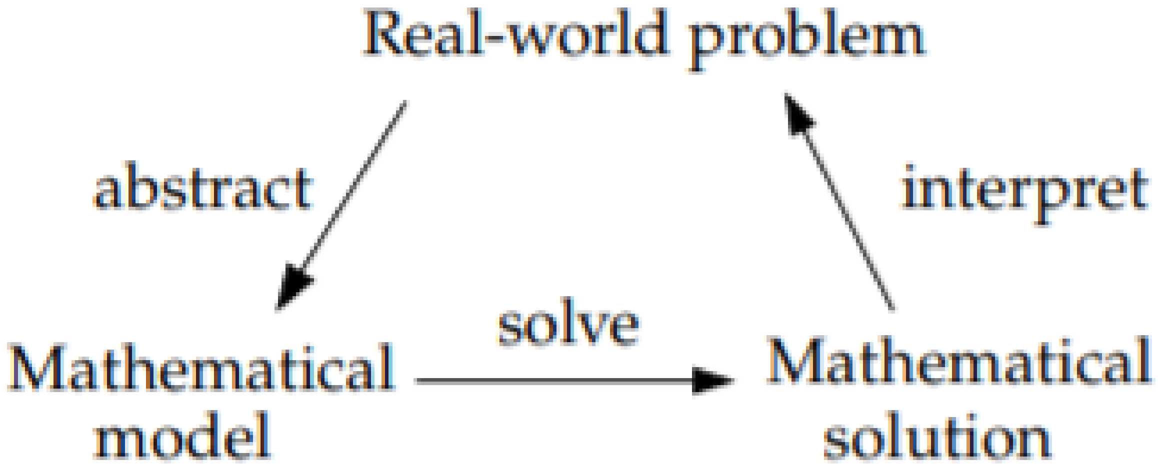
For this equation, all solutions can be written in the form

$$y = \cos(t) + \sin(t) + Ce^{-t}, \qquad C \text{ is a constant}$$



Differential Equations in Practice

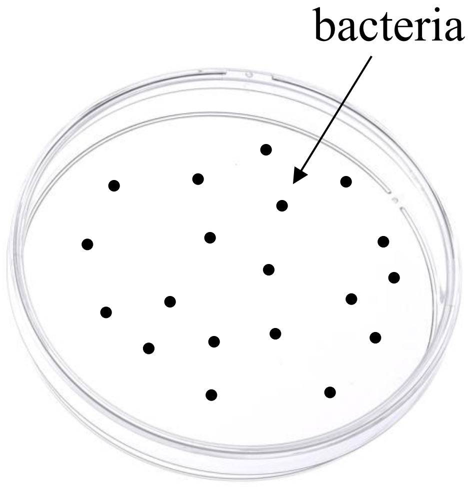
How do we use differential equations in science and engineering?



In this course, we will focus mostly on the mathematical analysis.

Example: Exponential Growth Model

Let P denote the population of some bacteria on a Petri dish.



We assume that there is enough food and enough space.

The rate of growth of bacteria is proportional to the population
- a larger population grows quicker.

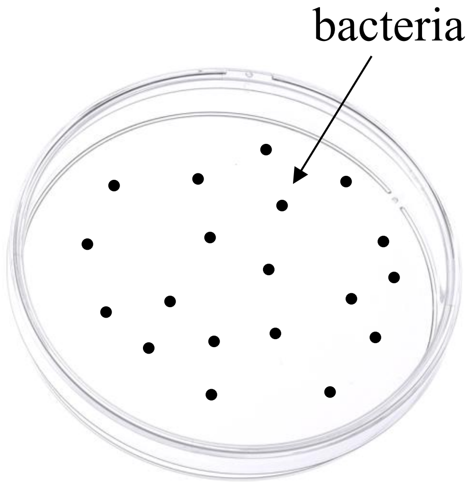
$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

k is a positive constant

Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later.
How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

Example: Exponential Growth Model

Model: $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$



We claim that a solution is given by

$$P(t) = Ce^{kt}$$

Check:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = Cke^{kt} = kP$$

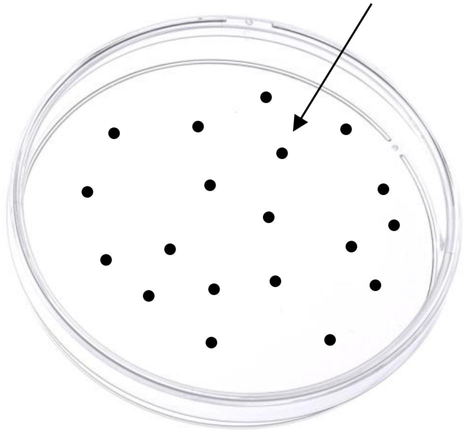
C is unknown, k is unknown, they need to be determined.

What we know: $P(0) = 100$ and $P(10) = 200$

Example: Exponential Growth Model

Model: $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Solution: $$P(t) = Ce^{kt}$$



Conditions: $$100 = P(0) = Ce^{k0} = C$$

$$200 = P(10) = 100e^{k10}$$

Then:

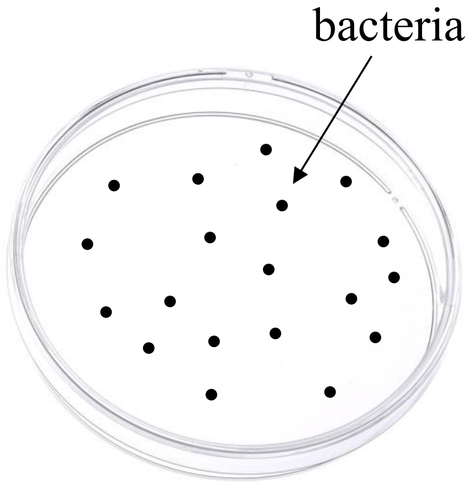
$$2 = e^{k10} \qquad \frac{\ln 2}{10} = k \qquad k \approx 0.069$$

$$P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}$$

Example: Exponential Growth Model

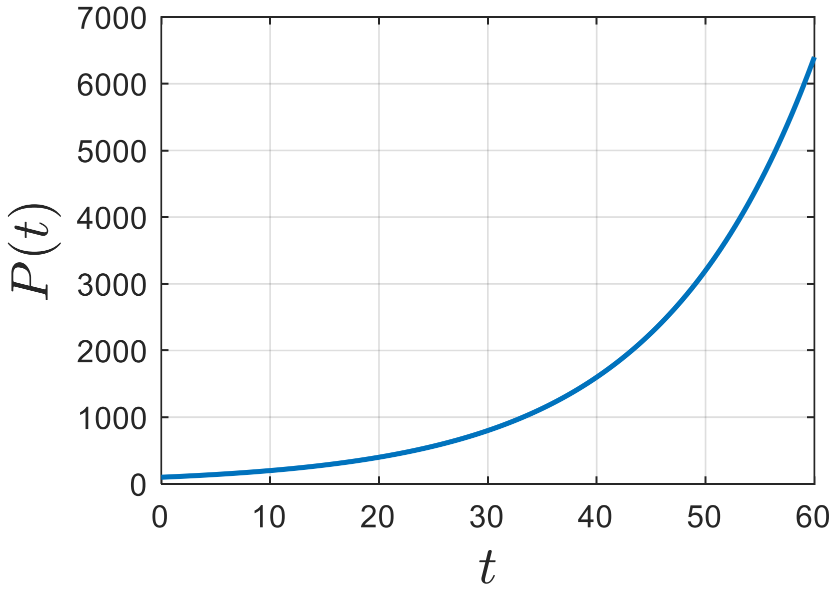
Model: $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Solution: $$P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}$$



At one minute, $t = 60$, the population is

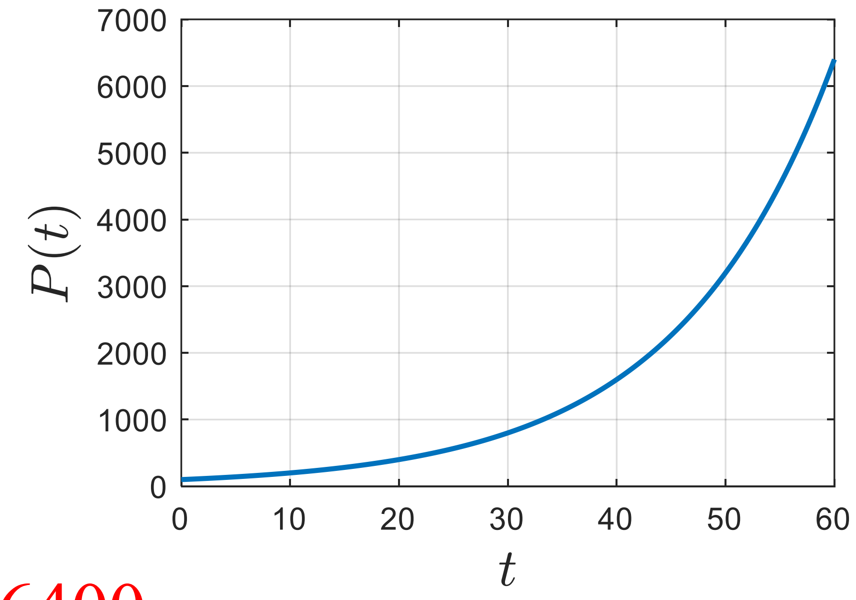
$$P(60) = 6400$$



Result Analysis

Solution: $P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}$

$P(60) = 6400$



Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s?

$P(61) \approx 6859.35$ Make sense?

Initial Condition

Model: $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

The constant k in this model is usually known, and we want to solve the equation for different initial conditions.

For example:

Take $k = 1$, and we want to solve the following equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P, \qquad P(0) = 1000 \text{ (initial condition)}$$

Solution:

$$P(t) = 1000e^{t}$$

General Solution & Particular Solution

Model: $$\frac{\mathrm{d}P}{\mathrm{d}t} = P, \qquad P(0) = 1000 \text{ (initial condition)}$$

Without initial condition

$$P(t) = Ce^t$$ *general solution*

With initial condition

$$P(t) = 1000e^t$$ *particular solution*

Fundamental Equations

There are a few equations appearing often and it is useful to just memorize what their solutions are – fundamental equations.

Differential equation	Solution
$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$	$y(t) = Ce^{kt}$
$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky$	$y(t) = Ce^{-kt}$

Fundamental Equations

Differential equation | Solution

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -k^2 y \qquad\qquad y(t) = C_1 \cos(kt) + C_2 \sin(kt)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -C_1 k \sin(kt) + C_2 k \cos(kt)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -C_1 k^2 \cos(kt) - C_2 k^2 \sin(kt) \quad = -\, k^2 y$$

Fundamental Equations

Differential equation

Solution

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = k^2 y$$

$$y(t) = C_1 e^{kt} + C_2 e^{-kt}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = C_1 k e^{kt} - C_2 k e^{-kt}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = C_1 k^2 e^{kt} + C_2 k^2 e^{-kt} \quad = k^2 y$$

Example

Consider the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0$$

The general solution

$$y = C_1 t + C_2$$

Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$

$$2 = y(0) = C_1 0 + C_2 = C_2$$
$$3 = y'(0) = C_1$$

The particular solution $y = 3t + 2$

Classification of Differential Equations

ODE & PDE

Classification based on the derivatives

- Ordinary differential equation (ODE)

  The derivatives are taken with respect to only one variable, that is, there is only one independent variable

- Partial differential equation (PDE)

  There are partial derivatives of several variables, that is, there are several independent variables.

ODE & PDE

Examples of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$ (Exponential growth)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A - y)$$ (Newton's law of cooling)

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f(t)$$ (Mechanical vibrations)

ODE & PDE

Examples of partial differential equations (PDEs)

$$\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0$$

(Transport equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$$

(Heat equation)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

(Wave equation in 2 dimensions)

System of Differential Equations

If there are several equations working together, we have a so-called system of differential equations.

System of ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = y$$

System of PDEs (Maxwell's equations for electromagnetics)

$$\nabla \cdot \vec{D} = \rho, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Order of Differential Equations

Classification based on the order of the equation (or system)

The order is simply the highest derivate that appears in the equation

First-order differential equation

The highest derivative is the first derivative

Second-order equation

The highest derivative is the second derivative

Nth-order equation

The highest derivative is the Nth derivative

Order of Differential Equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

First-order equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f(t)$$

Second-order equation

$$a^4\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0$$

Fourth-order equation

Linear & Nonlinear Differential Equations

Classification based on how the dependent variables appear in the equation (or system)

Linear equation

The dependent variable (or variables) and their derivatives appear linearly – first powers, not multiplied together, no other functions of dependent variables

Nonlinear equation

Not a linear equation

Linear & Nonlinear Differential Equations

A general linear ODE

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = b(t)$$

A special linear ODE

$$e^t\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \sin(t)\frac{\mathrm{d}y}{\mathrm{d}t} + t^2 y = \frac{1}{t}$$

Linear & Nonlinear Differential Equations

A nonlinear PDE

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial t} = \nu\frac{\partial^2 y}{\partial t^2}$$

A nonlinear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$$

Homogenous Differential Equations

A linear equation may further be called homogenous if all terms depend on the dependent variable - no term is a function of the independent variables alone.

If a linear equation is not homogeneous, it is called nonhomogeneous or inhomogeneous.

A homogeneous linear ODE

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = 0$$

Constant Coefficients Differential Equations

An equation is said to have constant coefficients if the its coefficients are constant functions.

A constant coefficient ODE

$$a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b(t)$$

Autonomous Differential Equations

An equation (or system) is called autonomous if the equation does not depend on the independent variable

Autonomous equation means an equation that does not change with time

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A - y)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$$

Summary

- Introduction to Differential Equations
  - Differential equations are widely used in science and engineering
  - Solutions of differential equations – general & particular solutions
  - Fundamental Equations
- Classification of Differential Equations
  - ODE & PDE
  - Order of differential equations
  - Linear & nonlinear of differential equations
  - Homogenous differential equations
  - Autonomous differential equations