# Mathematical Foundations for Intelligent Engineering

## First-Order Ordinary Differential Equations

Sinan Xiao （肖思男）
Shien-Ming Wu School of Intelligent Engineering
South China University of Technology
Email: sinanxiao@scut.edu.cn

吴贤铭智能工程学院
SHIEN-MING WU SCHOOL OF INTELLIGENT ENGINEERING

# Review

- Introduction to Differential Equations
  - Differential equations are widely used in science and engineering
  - Solutions of differential equations – general & particular solutions
  - Fundamental Equations
- Classification of Differential Equations
  - ODE & PDE
  - Order of differential equations
  - Linear & nonlinear of differential equations
  - Homogenous differential equations
  - Constant coefficients differential equations
  - Autonomous differential equations

# Content

- Solution by Integral
- Separable Equations
- Slope/Direction Fields

# Solution by Integral

# Ordinary Differential Equation (ODE)

An ordinary differential equation (ODE) is an equation an equation that contains one or several derivatives of an unknown function $y(t)$ (or $y(x)$). The equation may also contain $y$ itself and constants.

For example:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos(t)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 9y = e^{-2t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^3y}{\mathrm{d}t^3} - \frac{3}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 0$$

# Ordinary Differential Equation (ODE)

An ordinary differential equation (ODE) is an equation an equation that contains one or several derivatives of an unknown function $y(t)$ (or $y(x)$). The equation may also contain $y$ itself and constants.

General form:

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}t^n}\right) = 0$$

$$F\left(t, y, y', y'', \dots, y^{(n)}\right) = 0$$

# First-Oder ODE

The order is simply the highest derivate that appears in the equation

A first-order ODE can be written as

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = 0 \qquad \text{implicit form}$$

or often in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \qquad \text{explicit form}$$

# Solution by Integral

In general, there is no simple formula or procedure one can follow to find solutions.

Here, let us assume that $f$ is a function of $t$ alone, that is, the equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

How to get the solution?

We could integrate both sides with respect to $t$.

$$\int \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = \int f(t)\mathrm{d}t + C$$

$$\boxed{y(t) = \int f(t)\mathrm{d}t + C}$$

# Example

Solve the first-order ODE $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3t^2$

The general solution should be

$$y(t) = \int 3t^2 \mathrm{d}t + C \quad = t^3 + C$$

Given an initial condition $y(0) = 1$, we can obtain a particular solution

$$1 = y(0) = 0^3 + C = C$$

$$y(t) = t^3 + 1$$

# Solution by Integral

Differential equation: $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$

General solution: $y(t) = \int f(t)\mathrm{d}t + C$

Initial condition: $y(t_0) = y_0$

Particular solution: $y(t) = \int_{t_0}^{t} f(s)\mathrm{d}s + y_0$

# Solution by Integral

First-order ODE $$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$$

Now, let us assume that $f$ is a function of $y$ alone, that is, the equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

How to get the solution?

Switch the roles of $t$ and $y$ $$\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{f(y)}$$

Integral with respect to $y$ $$t(y) = \int \frac{\mathrm{d}t}{\mathrm{d}y}\mathrm{d}y = \int \frac{1}{f(y)}\mathrm{d}y + C$$

Solve for $y(t)$ by inversion

# Example

Solve the first-order ODE $\dfrac{\mathrm{d}y}{\mathrm{d}t} = ky$

First, we note that $y = 0$ is also a solution

Then, for $y \neq 0$

Switch the roles of $t$ and $y$ $\dfrac{\mathrm{d}t}{\mathrm{d}y} = \dfrac{1}{ky}$

Integral with respect to $y$ $t = \displaystyle\int \frac{1}{ky}\mathrm{d}y + D = \frac{1}{k}\ln y + D$

Solve for $y$ $y = e^{kt-kD} = e^{-kD}e^{kt}$

Replace $e^{-kD}$ with an arbitrary constant $C$ $y = Ce^{kt}$

# Example of First-Order Nonlinear ODE

Solve the first-order nonlinear ODE $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2$

First, we note that $y = 0$ is also a solution

When $y \neq 0$

Switch the roles of $t$ and $y$ $\dfrac{\mathrm{d}t}{\mathrm{d}y} = \dfrac{1}{y^2}$

Integral $t = \displaystyle\int \frac{1}{y^2}\,\mathrm{d}y + C = -\frac{1}{y} + C$

Solve for $y$ $y = \dfrac{1}{C - t}$

# Example of First-Order Nonlinear ODE

First-order nonlinear ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$

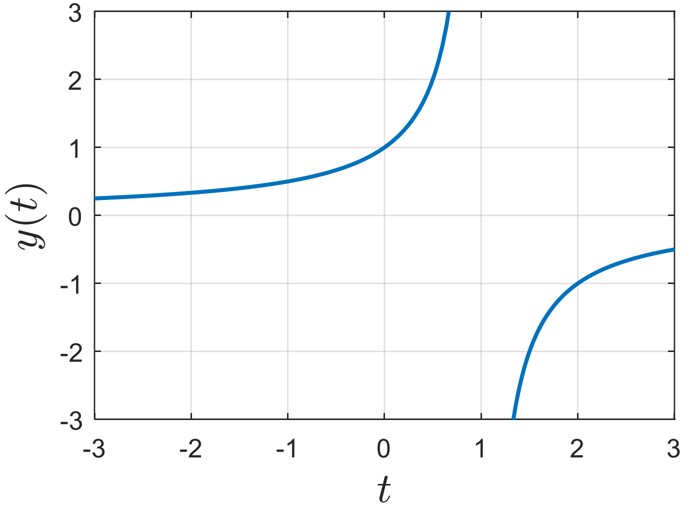
General solution $y = \frac{1}{C-t}$ or $y = 0$

There is singularity when $t = C$

The solution is only defined on some interval $(-\infty, C)$ or $(C, \infty)$

We can consider the solution on the interval $(-\infty, C)$ or $(C, \infty)$

$C = 1$



# Exercise

Suppose a car drives at a speed $e^{t/2}$ meters per second, where $t$ is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

# Exercise

Suppose a car drives at a speed $e^{t/2}$ meters per second, where $t$ is time in seconds. How far did the car ger in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Let $x$ denote the distance the car travelled. The equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^{t/2}$$

Integrate with respect to $t$

$$x(t) = 2e^{\frac{t}{2}} + C$$

Initial condition: when $t = 0$, then $x = 0$,

$$0 = x(0) = 2e^{\frac{0}{2}} + C = 2 + C \qquad C = -2$$

$$x(t) = 2e^{\frac{t}{2}} - 2$$

# Exercise

Suppose a car drives at a speed $e^{t/2}$ meters per second, where $t$ is time in seconds. How far did the car ger in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

$$x(t) = 2e^{\frac{t}{2}} - 2$$

$$x(2) = 2e^{\frac{2}{2}} - 2 \approx 3.4366$$

$$x(10) = 2e^{\frac{10}{2}} - 2 \approx 294.8263$$

# Separable Equations

# First-Oder ODE

General form of first-order ODE $$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$$

Integrating both sides yields $$y = \int f(t, y)\mathrm{d}t + C$$

Due to the dependence on $y$ in the integral, it is not easy to get an explicit solution

# Separable Equations

General form of first-order ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$

Suppose the equation on the right-hand side is separable, that is, the ODE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

$f(t)$ and $g(y)$ are functions of $t$ and $y$ separately

Rewrite the equation as

$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

# Separable Equations

$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

Integrate on both sides with respect to $t$

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = \int f(t)\mathrm{d}t + C$$

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\mathrm{d}t + C$$

Method of separating variables

# Example

Consider ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = ty$

First, note that $y = 0$ is a solution

When $y \neq 0$,

Rewrite the equation as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = t$

Take the integration $\int \frac{\mathrm{d}y}{y} = \int t\mathrm{d}t + C$

Solve for $y$ $\ln(|y|) = \frac{t^2}{2} + C$

$$|y| = e^{\frac{t^2}{2}+C} = e^{\frac{t^2}{2}}e^{C} = De^{\frac{t^2}{2}} \qquad D > 0$$

$y = De^{\frac{t^2}{2}}$ for any number $D$ (including zero or negative)

# Exercise

Solve the ODE $\quad t^2 \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - t^2 + y^2 - t^2 y^2, \quad y(1) = 0$

# Exercise

Solve the ODE $$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - t^2 + y^2 - t^2 y^2, \qquad y(1) = 0$$

Rewrite the right-hand side $$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} = (1 - t^2)(1 + y^2)$$

Separate variables $$\frac{1}{(1 + y^2)} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(1 - t^2)}{t^2}$$

Integral $$\int \frac{1}{(1 + y^2)} \mathrm{d}y = \int \left( \frac{1}{t^2} - 1 \right) \mathrm{d}t + C$$

$$\arctan(y) = -\frac{1}{t} - t + C$$

# Exercise

Solve the ODE $$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - t^2 + y^2 - t^2 y^2, \qquad y(1) = 0$$

$$\arctan(y) = -\frac{1}{t} - t + C$$

$$y = \tan\left(-\frac{1}{t} - t + C\right)$$

Initial condition $y(1) = 0$

$$0 = \tan\left(-\frac{1}{1} - 1 + C\right) \qquad C = 2 \qquad \text{(or } C = 2 + k\pi\text{)}$$

$$y = \tan\left(-\frac{1}{t} - t + 2\right)$$

# Implicit Solutions

Sometimes, we might get stuck even if we can do the integration.

Consider the equation $$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ty}{y^2+1}$$

Separate variables $$\frac{y^2+1}{y}\mathrm{d}y = t\mathrm{d}t$$

$$\left(y+\frac{1}{y}\right)\mathrm{d}y = t\mathrm{d}t$$

Take the integration $$\frac{y^2}{2} + \ln(|y|) = \frac{t^2}{2} + C$$

$$y^2 + 2\ln(|y|) = t^2 + D \qquad D = 2C$$

# Implicit Solutions

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ty}{y^2 + 1}$$

$$y^2 + 2\ln(|y|) = t^2 + D$$

It is not easy to find the solution explicitly as it is hard to solve for $y$.

We leave the solution in this form and call it an implicit solution.

Check the solution:

Differentiate with respect to $t$ $\quad \left(2y + \frac{2}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 2t$ $\quad$ Chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ty}{y^2 + 1}$$

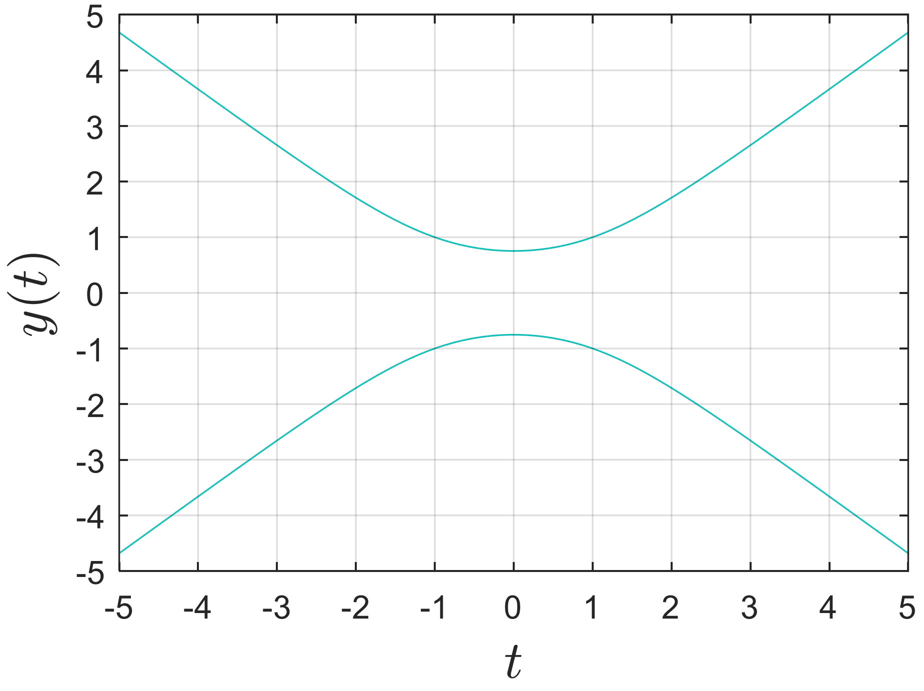
# Implicit Solutions

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ty}{y^2 + 1}$$

$$y^2 + 2\ln(|y|) = t^2 + D$$

It is not easy to get an explicit solution, but we can plot the solution using mathematical software, such as MATLAB.

The implicit solution $y^2 + 2\ln(|y|) = t^2$



# Slope/Direction Fields

# Slope/Direction Field

First-order ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behaviour of the solutions, or if we could find approximate solutions

The derivative $\frac{\mathrm{d}y(x)}{\mathrm{d}x}$ of $y(x)$ is the slope of $y(x)$. A solution curve that passes through a point $(x_0, y_0)$ must have the slope $\frac{\mathrm{d}y(x_0)}{\mathrm{d}x_0}$ equal to the $f(x_0, y_0)$

$$\frac{\mathrm{d}y(x_0)}{\mathrm{d}x_0} = f(x_0, y_0)$$

# Slope/Direction Field

First-order ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

For a given a point $(x_0, y_0)$ $$\frac{\mathrm{d}y(x_0)}{\mathrm{d}x_0} = f(x_0, y_0)$$

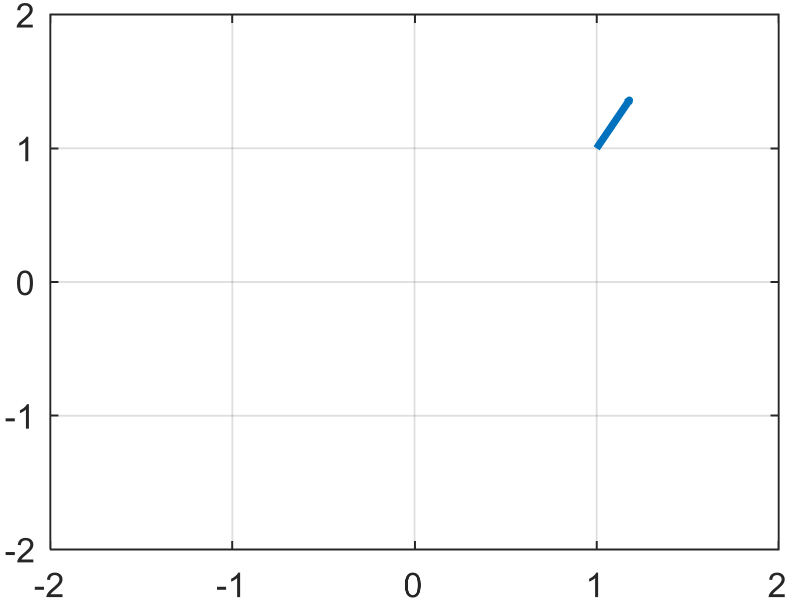
We can show directions of solution curves by drawing short straight-line segments (lineal elements) in the $xy$-plane. This gives a slope field or direction field.

# Slope/Direction Field

Consider the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x$

At point $(1,1)$, the slope of a solution curve is $1 + 1 = 2$

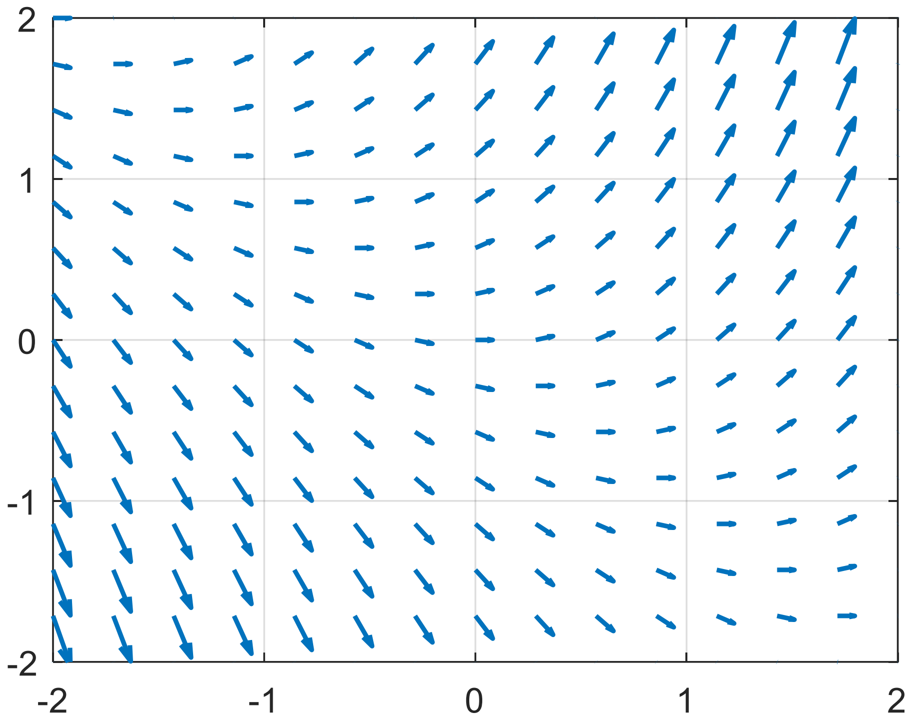
The slope $\frac{\mathrm{d}y}{\mathrm{d}x} = y + x$ at $(1,1)$



# Slope/Direction Field

Consider the ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = y + x$$

Slope field of $\frac{\mathrm{d}y}{\mathrm{d}x} = y + x$




Plot the slope at lots of points in the plane

Geometric meaning of the equation

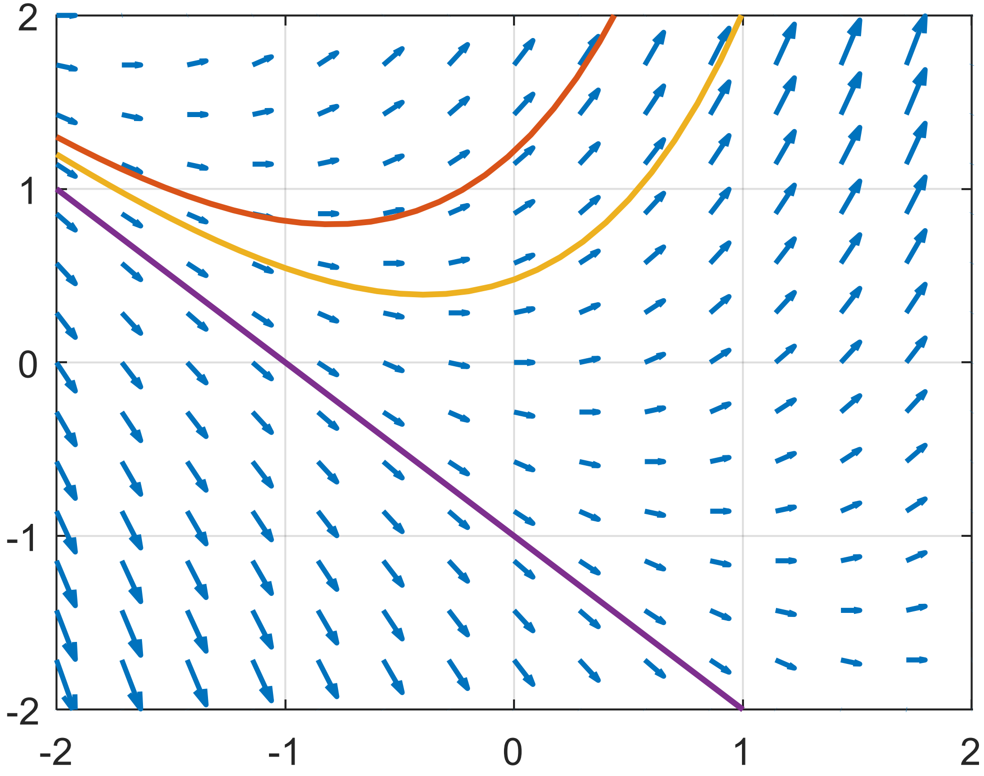
# Slope/Direction Field

Consider the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x$

The slope field provides a graphic or numeric methods for obtaining approximate solutions of ODEs

It is useful when analytic solution is not available

Slope field of $\frac{\mathrm{d}y}{\mathrm{d}x} = y + x$ and three approximate solution curves passing through (-2,1.3), (-2,1.2), (-2,1)



# Existence and uniqueness

For an ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0$$

- Does a solution exist?
- Is the solution unique (if it exists)?

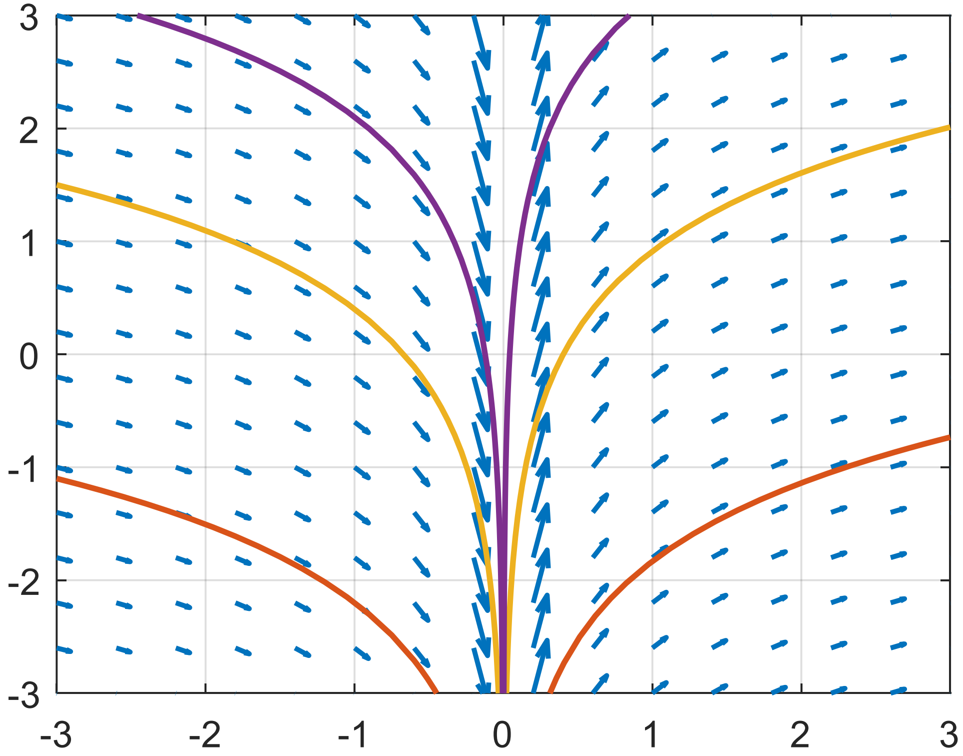
In practice, the answer is yes to both for many cases. But there are cases when the answer to either question can be no.

# Existence and uniqueness

Consider the ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}, \qquad y(0) = 0$$

General solution $$y = \ln(|x|) + C$$

The solution does not exist at $x = 0$



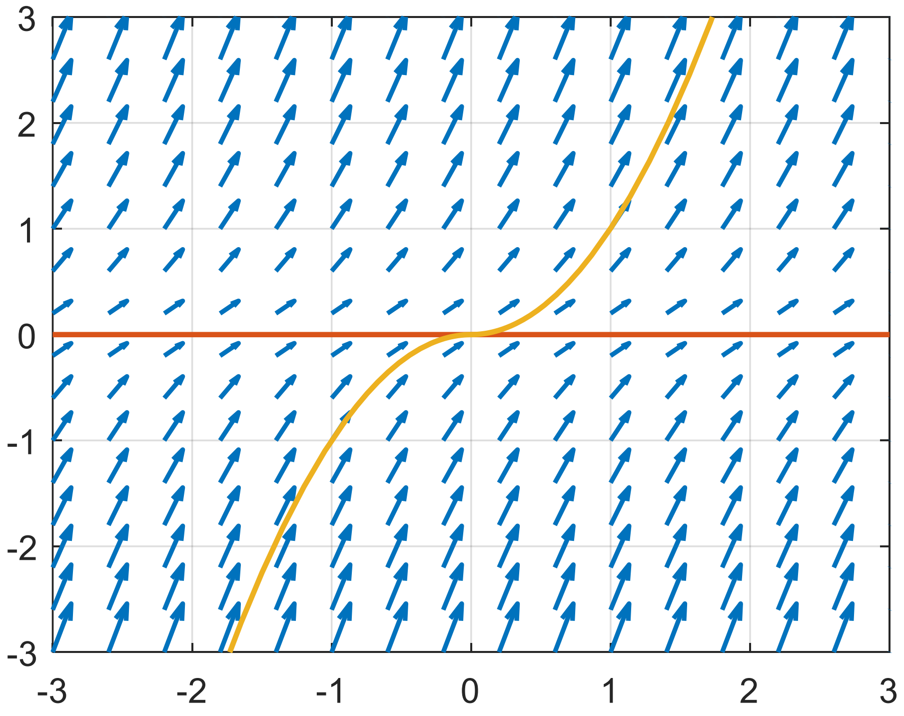
# Existence and uniqueness

Consider the ODE $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{|y|}, \qquad y(0) = 0$$

$y = 0$ is a solution

Another solution:

$$y(x) = \begin{cases} x^2, x > 0 \\ -x^2, x < 0 \end{cases}$$



# Picard's Theorem on Existence and Uniqueness

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some $(x_0, y_0)$, then a solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0$$

exists (at least for $x$ in some small interval) and is unique.

# Summary

- Solution by Integral
  - $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$
  - $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$
- Separable Equations
  - $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$
- Slope/Direction Fields
  - Geometric meaning
  - Graphic or numeric methods
  - Existence and uniqueness