1: FIRST ORDER ODES

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CHAPTER OVERVIEW

1: First order ODEs

1.1: Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y)$$

or just

$$y' = f(x, y)$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x) \tag{1.1.1}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)dx = \int f(x)dx + C$$

that is

$$y(x) = \int f(x)dx + C$$

This $y(x)$ is actually the general solution. So to solve Equation 1.1.1, we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called **indefinite integral**. The indefinite integral is really the **antiderivative** (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)dx + C$ as

$$\int_{x_0}^{x} f(t)dt + C$$

Hence the terminology to integrate when we may really mean to antidifferentiate. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should always think of the definite integral.

Example 1.1.1

Find the general solution of $y' = 3x^2$.

Solution

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check: $y' = 3x^2$. We have gotten precisely our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbersx_0 and y_0 x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)ds + y_0 \tag{1.1.2}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, Equation 1.1.2 is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 1.1.2

Solve

$$y' = e^{-x^2}, \quad y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2}\, ds + 1.$$

Solution

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y)$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx} = f(y)$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dy}{dx} = \frac{1}{f(y)}$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)} dy + C$$

Finally, we try to solve for y.

Example 1.1.3

Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln|y| + D$$

where D is an arbitrary constant. Now we solve for$y$ (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD}e^{kx}$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y=0$. We get the same general solution as we guessed before, $y=Ce^{kx}$.

Example 1.1.4

Find the general solution of $y'=y^2$.

Solution

First we note that $y=0$ is a solution. We can now assume that $y\neq 0$. Write

$$\frac{dx}{dy}=\frac{1}{y^2}$$

We integrate to get

$$x=\frac{-1}{y}+C$$

We solve for $y=\dfrac{1}{C-x}$. So the general solution is

$$y=\frac{1}{C-x}\ \textit{or}\ y=0$$

Note the singularities of the solution. If for example $C=1$, then the solution as we approach $x=1$. See Figure 1.1.1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y'=y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty,C)$ or (C,∞). Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0)=1$, then the solution is $y=\frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty,1)$. In the figure, it is the left side of the graph.

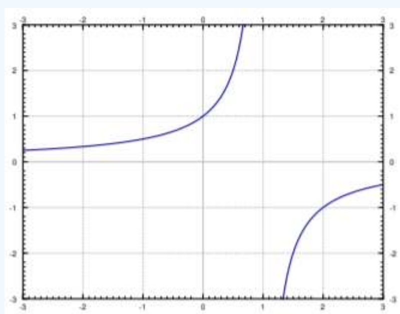

Figure 1.1.1: Plot of $y=\frac{1}{1-x}$.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

Example 1.1.5

Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t=0$)? How far in 10 seconds?

Solution

Let x denote the distance the car traveled. The equation is

$$x'=e^{t/2}$$

We can just integrate this equation to get that

$$x(t)=2e^{t/2}+C$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44\text{~meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294\text{~meters}$$

Example 1.1.6

Suppose that the car accelerates at a rate of $t^2 \frac{m}{s^2}$. At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 $^{\mathrm{m}}/_{\mathrm{s}}$. Where is the car at time $t = 10$.

Solution

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \quad v(0) = 10$$

Once we solve for v, we can integrate and find x.

1.2: Slope fields

The general first order equation we are studying looks like

$$y' = f(x, y)$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or if we could find approximate solutions. At this point it may be good to first try the Lab I and/or Project I from the IODE website.

1.2.1: Slope fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. And this is the slope a solution $y(x)$ would have at x if its value was y. In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y). At a point (x, y), we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See Figure $1.2.1$.

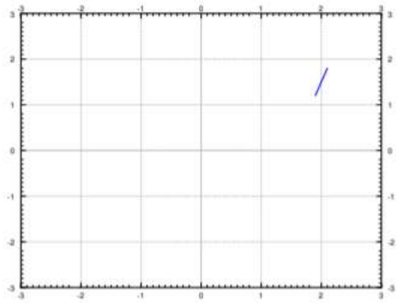

Figure $1.2.1$: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the of the equation. See Figure $1.2.2$ for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer do the drawing.

Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure $1.2.3$. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

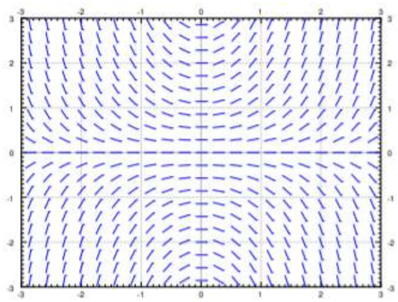

Figure $1.2.3$: Slope field of $y' = xy$.

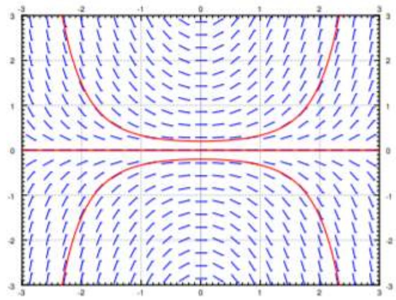

Figure 1.2.3: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure $1.2.3$ we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see Figure $1.2.4$. If we think of moving from left to right (perhaps x is time and time is usually increasing), then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

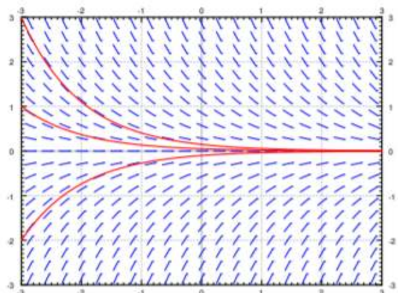

Figure 1.2.4: Slope field of $y' = -y$ with a graph of a few solutions.

1.2.2: Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

i. Does a solution exist?
ii. Is the solution unique (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

✓ Example 1.2.1

Attempt to solve

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Solution

Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure $1.2.5$ on the following page. The equation may have been written as the seemingly harmless $xy' = 1$.

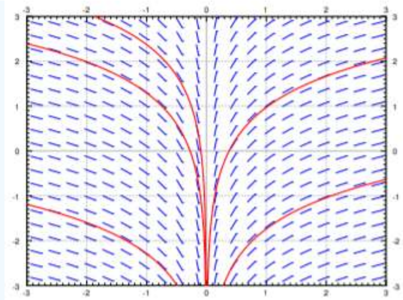

Figure 1.2.5: Slope field of $y' = \frac{1}{x}$.

✓ Example 1.2.2

Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

Solution

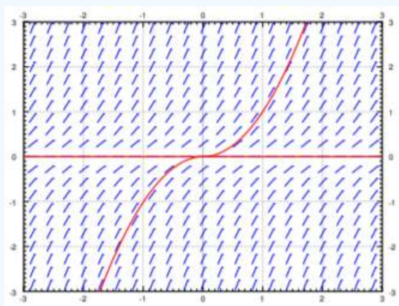

Figure 1.2.6: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as **Picard's theorem**.[1]

Theorem 1.2.1

Picard's theorem on existence and uniqueness

If $f(x, y)$ is *continuous* (*as a function of two variables*) *and* $\frac{\partial f}{\partial y}$ *exists and is continuous near some* (x_0, y_0), then a solution to

$$y' = f(x, y), \quad y(x_0) = y_0$$

exists (*at least for* x *in some small interval*) *and is unique*.

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example 1.2.3

For some constant A, solve:

$$y' = y^2 \quad y(0) = A$$

.

Solution

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = -\frac{1}{y} + C$, so $y = \frac{1}{C - x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

1.2.3: Footnotes

[1] Named after the French mathematician Charles Émile Picard (1856 – 1941)

1.3: Separable Equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)dx + C$$

Notice the dependence on y in the integral.

1.3.0.1: Separable equations

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y)$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 1.3.1

Take the equation

$$y' = xy$$

First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C$$

Or

$$|y| = e^{\frac{x^2}{2} + C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}}$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}}$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy$$

Yay!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y)$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

We integrate both sides with respect to x.

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx + C$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)}dy = \int f(x)dx + C$$

And we are done.

1.3.1: Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2+1}$$

We separate variables,

$$\frac{y^2+1}{y}dy = \left(y + \frac{1}{y}\right)dy = x\,dx$$

We integrate to get

$$\frac{y^2}{2} + ln\,|y| = \frac{x^2}{2} + C$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2ln\,|y| = x^2 + D$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y + \frac{2}{y}\right) = 2x$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y) that are solutions to $y^2 + 2\ln|y| = x^2$, you find the two curves in Figure $1.3.1$. This is not quite a graph of a function. For each x there are two choices of y. To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1)=1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

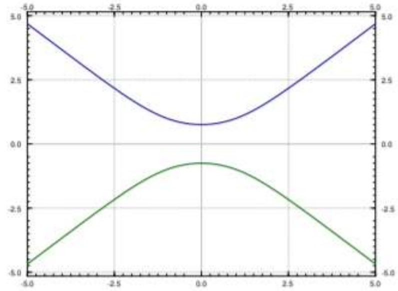

Figure 1.3.1: The implicit solution $y^2+2\ln|y|=x^2$ to $y'=\frac{xy}{y^2+1}$.

The equation above also has the solution $y=0$. So the general solution is

$$y^2+2\ln|y|=x^2+C, \qquad \text{and} \qquad y=0.$$

These outlying solutions such as $y=0$ are sometimes called *singular solutions*.

Example 1.3.2

Solve $x^2y'=1-x^2+y^2-x^2y^2$, $y(1)=0$.

Solution

First factor the right hand side to obtain

$$x^2y'=\left(1-x^2\right)\left(1+y^2\right)$$

Separate variables, integrate, and solve for y

$$\begin{aligned} \frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2}-1, \\ \arctan(y) &= -\frac{1}{x^2}-x+C, \\ y &= \tan\left(-\frac{1}{x}-x+C\right) \end{aligned} \tag{1.3.1}$$

Solve for the initial condition, $0=\tan(-2+C)$ to get $C=2$ (or $C=2+\pi$, or $C=2+2\pi$, etc.). The particular solution we seek is, therefore,

$$y=\tan\left(\frac{-1}{x}-x+2\right).$$

Example 1.3.3

Bob made a cup of coffee, and Bob likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t=0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Solution

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned} \frac{1}{T-A}\frac{dT}{dt} &= -k, \\ \ln(T-A) &= -kt + C, \quad (\text{note that} T - A > 0) \\ T - A &= De^{-kt}, \\ T &= A + De^{-kt} \end{aligned} \tag{1.3.2}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 1.3.2.

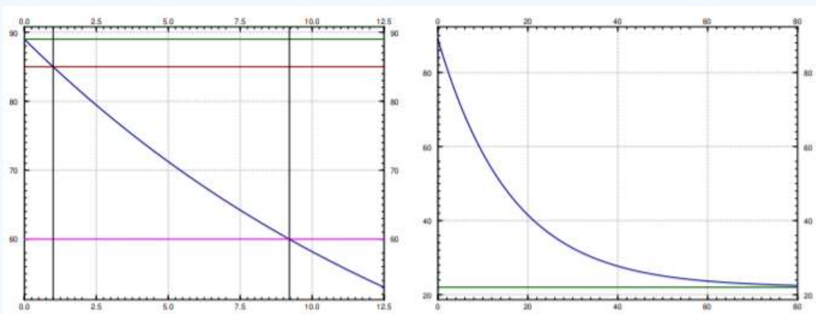

Figure 1.3.2: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22.

Example 1.3.4

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\begin{aligned} -\frac{3}{y^2}y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{\frac{x^2}{2} + C} = \frac{6}{x^2 + 2C}. \end{aligned} \tag{1.3.3}$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \quad \text{and} \quad y = 0.$$

1.4: Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called linear equations. In fact, the majority of the course is about linear equations. In this lecture we focus on the first order linear equation. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{1.4.1}$$

Here the word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations. The trick is to rewrite the left hand side of (1.4.1) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y]$$

This is the left hand side of (1.4.1) multiplied by $r(x)$. So if we multiply (1.4.1) by $r(x)$, we obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the integrating factor and the method is called the integrating factor method.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx}y' + e^{\int p(x)dx}p(x)y &= e^{\int p(x)dx}f(x), \\ \frac{d}{dx}\left[e^{\int p(x)dx}y\right] &= e^{\int p(x)dx}f(x), \\ e^{\int p(x)dx}y &= \int e^{\int p(x)dx}f(x)dx + C, \\ y &= e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}f(x)dx + C\right). \end{aligned} \tag{1.4.2}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 1.4.1

Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1$$

Solution

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2}y' + 2xe^{x^2}y &= e^{x-x^2}e^{x^2}, \\ \frac{d}{dx}\left[e^{x^2}y\right] &= e^{x}. \end{aligned} \tag{1.4.3}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned} \tag{1.4.4}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

? Exercise 1.4.1

Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above. An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

. Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{\int -_{x_0}^{x} p(s)ds} \left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)ds} f(t)dt + y_0 \right) \tag{1.4.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

? Exercise 1.4.2

Check that $y(x_0) = y_0$ in formula (1.4.5).

? Exercise 1.4.3

Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \quad y(0) = 10$$

Note

Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (1.4.5) for the solution. Second, it follows from the formula (1.4.5) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 1.2.1.

✓ Example 1.4.2

Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

Figure 1.4.1

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Solution

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in x concentration in})\Delta t - (\text{rate out x concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in x concentration in}) - (\text{rate out x concentration out})$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned} \tag{1.4.6}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \text{ x } 0.1) - \left(3 \frac{x}{60+2t}\right)$$

Or in the form (1.4.1)

$$\frac{dx}{dt} + \frac{3}{60+2t} x = 0.5$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t} dt\right) = \exp\left(\frac{3}{2} \ln(60+2t)\right) = (60+2t)^{3/2}$$

We multiply both sides of the equation to get

$$\begin{aligned} (60+2t)^{3/2} \frac{dx}{dt} + (60+2t)^{3/2} \frac{3}{60+2t} x &= 0.5(60+2t)^{3/2}, \\ \frac{d}{dt}\left[(60+2t)^{3/2} x\right] &= 0.5(60+2t)^{3/2}, \\ (60+2t)^{3/2} x &= \int 0.5(60+2t)^{3/2} dt + C, \\ x &= (60+2t)^{-3/2} \int \frac{(60+2t)^{3/2}}{2} dt + C(60+2t)^{-3/2}, \\ x &= (60+2t)^{-3/2} \frac{1}{10} (60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ x &= \frac{(60+2t)}{10} + C(60+2t)^{-3/2}. \end{aligned} \tag{1.4.7}$$

We need to find C. We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2}$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60 + 2t = 100$, or when $t = 20$. So

$$\begin{aligned} x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2} \\ &\approx 10 + 1859.03(100)^{-3/2} \approx 11.86. \end{aligned} \tag{1.4.8}$$

See Figure $1.4.2$ for the graph of x over t.

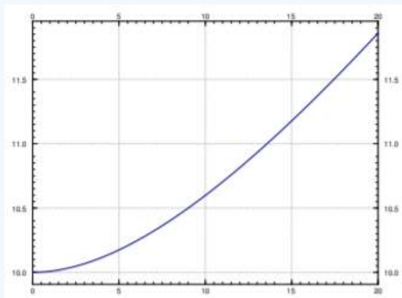

Figure $1.4.2$: Graph of the solution x kilograms of salt in the tank at time t.

The concentration at the end is approximately $0.1186\,{}^{\text{kg}}/_{\text{liter}}$ and we started with $\frac{1}{6}$ or $0.167\,{}^{\text{kg}}/_{\text{liter}}$.

1.5: Substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

1.5.1: Substitution

The equation

$$y' = (x - y + 1)^2 \tag{1.5.1}$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try

$$v = x - y + 1. \tag{1.5.2}$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2 \tag{1.5.3}$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2} dv = dx \tag{1.5.4}$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \quad \text{or} \quad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \quad \text{or} \quad \frac{v+1}{v-1} = De^{2x},$$

for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to “unsubstitute” to obtain

$$\frac{x - y + 2}{x - y} = De^{2x} \tag{1.5.5}$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.

$$\begin{aligned} x - y + 2 &= (x - y)\, De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned} \tag{1.5.6}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general things to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Usually you try to substitute in the “most complicated” part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

1.5.2: Bernoulli Equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called Bernoulli equation.[1]

$$y' + p(x)\, y = q(x)\, y^n \tag{1.5.7}$$

This equation looks a lot like a linear equation except for the y^n. If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 1.5.1: Bernoulli Equation

Solve

$$xy' + y(x+1) + xy^5 = 0, \;\; y(1) = 1$$

Solution

First, the equation is Bernoulli $p(x) = \dfrac{x+1}{x}$ (and $q(x) = -1$). We substitute

$$v = y^{1-5} = y^{-4}, \quad v' = -4y^{-5}y'$$

In other words, $\left(\dfrac{-1}{4}\right) y^5 v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4} v' + y(x+1) + xy^5 &= 0, \\ \frac{-x}{4} v' + y^{-4}(x+1) + x &= 0, \\ \frac{-x}{4} v' + v(x+1) + x &= 0, \end{aligned} \tag{1.5.8}$$

and finally

$$v' - \frac{4(x+1)}{x} v = 4$$

Now the equation is linear. We can use the integrating factor method. In particular, we use formula (1.4.17). Let us assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is $x = 1$. Let us compute the integrating factor. Here $p(s)$ from formula (1.4.17) is $\dfrac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s)ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s} ds\right) = e^{-4x-4ln(x)+4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s)ds} &= e^{4x+4ln(x)-4} = e^{4x-4}x^4 \end{aligned} \tag{1.5.9}$$

We now plug in to (1.4.17)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s)ds}\left(\int_1^x e^{\int_1^t p(s)ds} 4dt + 1\right), \\ &= e^{4x-4}x^4\left(\int_1^x 4\frac{e^{-4t+4}}{t^4} dt + 1\right) \end{aligned} \tag{1.5.10}$$

Note that the integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$\begin{aligned} y^{-4} &= e^{4x-4}x^4\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt+1\right), \\ y &= \frac{e^{-x+1}}{x\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt+1\right)^{1/4}} \end{aligned} \tag{1.5.11}$$

1.5.3: Homogeneous Equations

Another type of equations we can solve by substitution are the so-called homogeneous equations. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right) \tag{1.5.12}$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv' \tag{1.5.13}$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v)-v} = \frac{1}{x} \tag{1.5.14}$$

Hence an implicit solution is

$$\int \frac{1}{F(v)-v}dv = \ln|x| + C \tag{1.5.15}$$

Example 1.5.2

Solve

$$x^2y' = y^2 + xy, \quad y(1) = 1$$

Solution

We put the equation into the form $y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$. We substitute $v = \dfrac{y}{x}$ to get the separable equation

$$xv' = v^2 + v - v = v^2$$

$$v^{-2}dv = \frac{1}{x}dx$$

which has a solution

$$\begin{aligned} \int \frac{1}{v^2}dv &= ln\,|x| + C, \\ \frac{-1}{v} &= ln\,|x| + C, \\ v &= \frac{-1}{ln\,|x| + C}. \end{aligned} \tag{1.5.16}$$

We unsubstitute

$$\begin{aligned} \frac{y}{x} &= \frac{-1}{ln\,|x| + C}, \\ y &= \frac{-x}{ln\,|x| + C} \end{aligned} \tag{1.5.17}$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{ln\,|1| + C} = \frac{-1}{C}$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{ln\,|x| - 1}$$

1.5.4: Footnotes

[1] There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

- - Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

1.6: Autonomous equations

Let us consider general differential equation problems of the form

$$\frac{dx}{dt} = f(x)$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called **autonomous** equations. If we think of t as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (see Example 1.3.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x - A)$$

where x is the temperature, t is time, k is some constant, and A is the ambient temperature. See Figure $1.6.1$ for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the equilibrium solutions. The points on the x axis where $f(X) = 0$ are called critical points. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \to \infty$ we get $x \to A$. We call such critical points stable. In this simple example it turns out that all solutions in fact go to A as $t \to \infty$. If a critical point is not stable we would say it is unstable.

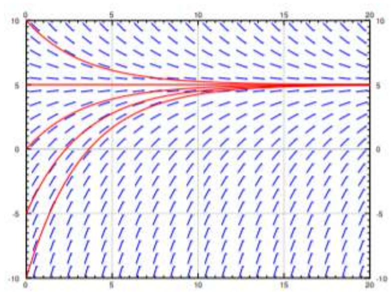

Figure $1.6.1$: The slope field and some solutions of $x' = 0.3(5 - x)$.

Let us consider the logistic equation

$$\frac{dx}{dt} = kx(M - x)$$

for some positive k and M. This equation is commonly used to model population if we know the limiting population M, that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math (see Figure $1.6.2$ for an example).

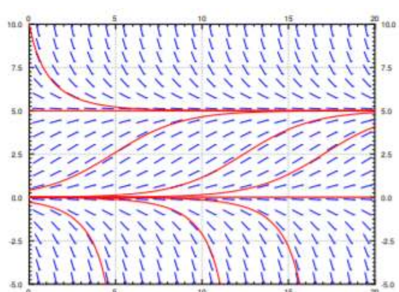

Figure $1.6.2$: The slope field and some solutions of $x' = 0.1x(5 - x)$.

Note two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$. is stable. On the other hand the critical point at $x = 0$. is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the solutions. For example, from the above slope field plot, we can easily see that

$$\lim_{t\to\infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Where DNE means "does not exist." From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞. Think of the equation $x' = x^2$, we have seen that it only exists for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the or , which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x. We draw the x-axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.

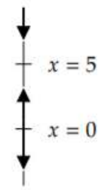

Figure 1.6.3

Armed with the phase diagram, it is easy to sketch the solutions approximately: As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

Exercise 1.6.1

Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.

Once we draw the phase diagram, we can easily classify critical points as stable or unstable.[1]

Figure 1.6.4

Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose$x$ is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done and $k > 0$ is some constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h$$

We expand the right hand side and solve for critical points

$$\frac{dx}{dt} = -kx^2 + kMx - h$$

Solving for the critical points A and B from the quadratic equations:

$$A=\frac{kM+\sqrt{(kM)^2-4hk}}{2k}, \quad B=\frac{kM-\sqrt{(kM)^2-4hk}}{2k}$$

? Exercise 1.6.2

Sketch a phase diagram for different possibilities. Note that these possibilities are $A>B$, or $A=B$, or A and B both complex (i.e. no real solutions). Hint: Fix some simple k and M and then vary h.

For example, let $M=8$ and $k=0.1$. When $h=1$, then A and B are distinct and positive. The slope field we get is in Figure $1.6.5$. As long as the population starts above B, which is approximately 1.55 million, then the population will not die out. It will in fact tend towards $A\approx 6.45$ million. If ever some catastrophe happens and the population drops below B, humans will die out, and the fast food restaurant serving them will go out of business.

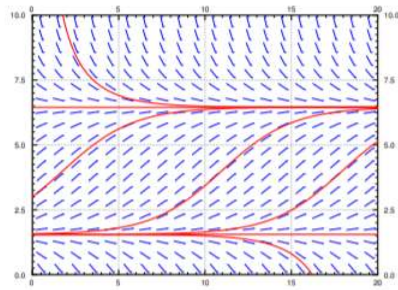

Figure $1.6.5$: Slope field and some solutions of $x'=0.1x(8-x)-1$.

When $h=1.6$, then $A=B=4$ and there is only **one** critical point and it is **unstable**. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business; there is no room for error (see Figure $1.6.6$).

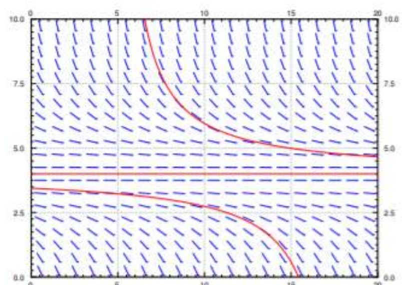

Figure $1.6.6$: The slope field and some solutions of $x'=0.1x(8-x)-1.6$.

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts (see Figure $1.6.7$).

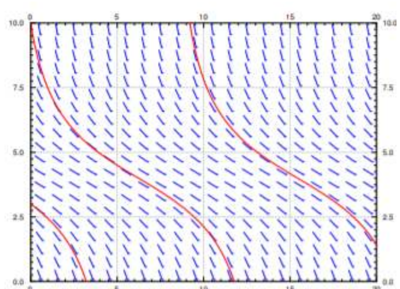

Figure $1.6.7$: Slope field and some solutions of $x'=0.1x(8-x)-2$.

1.6.1: Footnotes

[1] Unstable points with one of the arrows pointing towards the critical point are sometimes called semistable.

1.6.2: References

1. Paul W. Berg and James L. McGregor, Elementary Partial Differential Equations, Holden-Day, San Francisco, CA, 1966.
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1.7: Numerical methods: Euler's method

At this point it may be good to first try the Lab II and/or Project II from the IODE website: www.math.uiuc.edu/iode/. As we said before, unless $f(x,y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is Euler's Method.[1] It works as follows: Take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. Follow the line for an interval of length h on the x-axis. Hence if $y = y_0$ at x_0, then we say that y_1 (the approximate value of y at $x_1 = x_0 + h$) is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2, etc. Consider the equation $y' = \frac{y^2}{3}$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1, & y_1 &= y_0 + h\, f(x_0, y_0) = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1.333, \\ x_2 &= x_1 + h = 1 + 1 = 2, & y_2 &= y_1 + h\, f(x_1, y_1) = \frac{4}{3} + 1 \cdot \frac{\left(\frac{4}{3}\right)^2}{3} = \frac{52}{27} \approx 1.926. \end{aligned} \tag{1.7.1}$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0), (x_1, y_1), (x_2, y_2),…. For the first two steps of the method see Figure $1.7.1$.

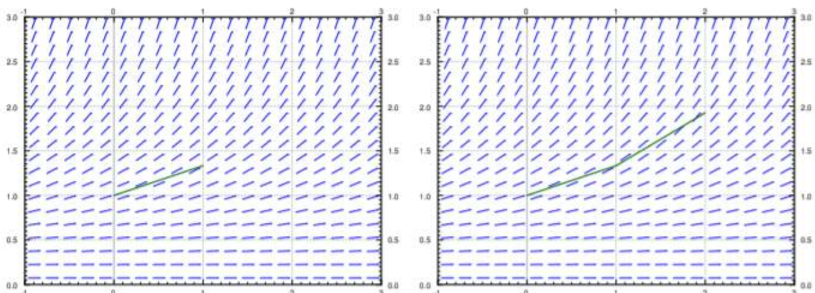

Figure $1.7.1$: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

More abstractly, for any $i = 0, 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\, f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure $1.7.2$ for the plot of the real solution and the approximation.

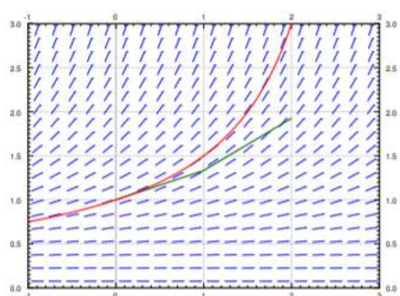

Figure $1.7.2$: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

We continue with the equation $y' = \frac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures $1.7.1$ and $1.7.2$ we have graphically approximated $y(2)$ with step size 1. With step size 1, we have $y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table $1.7.1$ gives the values computed for various parameters.

Exercise 1.7.1

Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly …what is the point of doing the approximation?

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table $1.7.1$: Euler's method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a first order method. In the IODE Project II you are asked to implement a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $\frac{1}{4} = \frac{1}{2}\text{x}\frac{1}{2}$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

Exercise 1.7.2

In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table $1.7.2$ for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table $1.7.2$: Attempts to use Euler's to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

Another case when things can go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \mathrm{x} \frac{1}{2} \mathrm{x} \frac{1}{2} \mathrm{x} \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.
- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse.
- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called stiff. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

1.7.1: Footnotes

[1] Named after the Swiss mathematician Leonhard Paul Euler (1707–1783). The correct pronunciation of the name sounds more like "oiler."

1.8: Exact Equations

Another type of equation that comes up quite often in physics and engineering is an *exact equation*. Suppose $F(x, y)$ is a function of two variables, which we call the *potential function*. The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x, y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x, y) = C$, for some constant C. In our example, the curves $x^2 + y^2 = C$ are circles. See Figure 1.8.1.

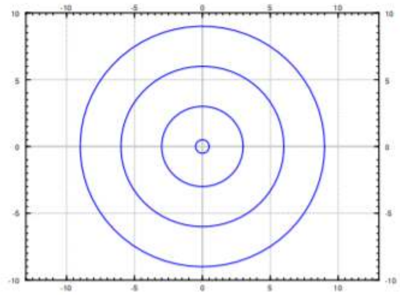

Figure 1.8.1: Solutions to $F(x, y) = x^2 + y^2 = C$ for various C.

We take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x\, dx + 2y\, dy.$$

We apply the total derivative to $F(x, y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M\, dx + N\, dy = 0, \qquad \text{or} \qquad M + N \frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F. In our simple example, we obtain the equation

$$2x\, dx + 2y\, dy = 0, \qquad \text{or} \qquad 2x + 2y \frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x. In our example,

$$y = \pm\sqrt{C^2 - x^2}.$$

An interpretation of the setup is that at each point $\vec{v} = (M, N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x, y), we have a *vector field*. The particular vector field $\vec{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x, y)$, such that

$$\vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2). If we think of $\vec{v}$ as force, then the work required to move along γ is

$$\int_\gamma \vec{v}(\vec{r}) \cdot d\vec{r} = \int_\gamma M\, dx + N\, dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\vec{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x,y)=C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

1.8.1: Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N, but we do not know F. That is, we may have just started with $2x+2y\frac{dy}{dx}=0$, or perhaps even

$$x+y\frac{dy}{dx}=0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F, the equation $F\big(x,y(x)\big)=C$ gives an implicit solution of the ODE.

✓ Example 1.8.1

Let us find the general solution to $2x+2y\frac{dy}{dx}=0$. Forget we knew what F was.

Solution

If we know that this is an exact equation, we start looking for a potential function F. We have $M=2x$ and $N=2y$. If F exists, it must be such that $F_x(x,y)=2x$. Integrate in the x variable to find

$$F(x,y)=x^2+A(y), \tag{1.8.1}$$

for some function $A(y)$. The function A is the , though it is only constant as far as x is concerned, and may still depend on y. Now differentiate (1.8.1) in y and set it equal to N, which is what F_y is supposed to be:

$$2y=F_y(x,y)=A'(y).$$

Integrating, we find $A(y)=y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x,y)=x^2+y^2$. Next for a constant C, we solve

$$F\big(x,y(x)\big)=C.$$

for y in terms of x. In this case, we obtain $y=\pm\sqrt{C^2-x^2}$ as we did before.

? Exercise 1.8.1

Why did we not need to add a constant of integration when integrating $A'(y)=2y$? Add a constant of integration, say 3, and see what F you get. What is the difference from what we got above, and why does it not matter?

The procedure, once we know that the equation is exact, is:

i. Integrate $F_x=M$ in x resulting in $F(x,y)=\text{something}+A(y)$.
ii. Differentiate this F in y, and set that equal to N, so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x. Pretty easy huh? Let's try this again.

✓ Example 1.8.2

Consider now $2x+y+xy\frac{dy}{dx}=0$.

OK, so $M=2x+y$ and $N=xy$. We try to proceed as before. Suppose F exists. Then $F_x(x,y)=2x+y$. We integrate:

$$F(x,y)=x^2+xy+A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N:

$$N = xy = F_y(x,y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists.

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists

Is there an easier way to check for the existence of F, other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma.[1]

Theorem 1.8.1

Pointcaré

If M and N are continuously differentiable functions of (x,y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x,y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to Example 1.8.2 where $M = 2x + y$ and $N = xy$. Notice $M_y = 1$ and $N_x = y$, which are clearly not equal. The equation is not exact.

Example 1.8.3

Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \qquad y(0) = 1.$$

Solution

We write the equation as

$$(2x + y) + (x - 1)\frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x, we find

$$F(x,y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N, we find

$$x - 1 = x + A'(y).$$

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x,y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C. As $y(0) = 1$ then $F(0,1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

Example 1.8.4

Solve

$$-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy=0, \qquad y(1)=2.$$

Solution

We leave to the reader to check that $M_y=N_x$.

This vector field (M,N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1,0)$ and ending at $(1,0)$ going counterclockwise, then if F existed we would expect

$$0=F(1,0)-F(1,0)=\int_\gamma F_x\,dx+F_y\,dy=\int_\gamma \frac{-y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy=2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0,0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1)=2$ we start at the point $(1,2)$. Considering $x>0$ and integrating M in x or N in y, we find

$$F(x,y)=\arctan\left(\frac{y}{x}\right).$$

The implicit solution is $\arctan\left(\frac{y}{x}\right)=C$. Solving, $y=\tan(C)x$. That is, the solution is a straight line. Solving $y(1)=2$ gives us that $\tan(C)=2$, and so $y=2x$ is the desired solution. See Figure 1.8.1, and note that the solution only exists for $x>0$.

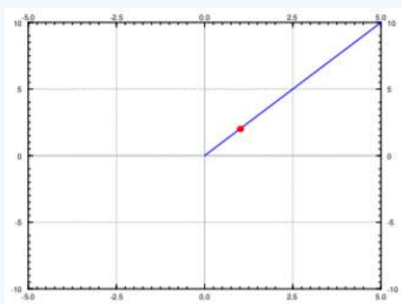

Figure 1.8.1: Solution to $-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy=0$, $y(1)=2$, with initial point marked.

Example 1.8.5

Solve

$$x^2+y^2+2y(x+1)\frac{dy}{dx}=0.$$

Solution

The reader should check that this equation is exact. Let $M=x^2+y^2$ and $N=2y(x+1)$. We follow the procedure for exact equations

$$F(x,y)=\frac{1}{3}x^3+xy^2+A(y),$$

and

$$2y(x+1)=2xy+A'(y).$$

Therefore $A'(y)=2y$ or $A(y)=y^2$ and $F(x,y)=\frac{1}{3}x^3+xy^2+y^2$. We try to solve $F(x,y)=C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (\frac{1}{3})x^3}{x+1}, \qquad \text{so} \qquad y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x+1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

1.8.2: Integrating factors

Sometimes an equation $M\,dx + N\,dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x,y)$. That is, perhaps for some nonzero function $u(x,y)$,

$$u(x,y)M(x,y)\,dx + u(x,y)N(x,y)\,dy = 0$$

is exact. Any solution to this new equation is also a solution to $M\,dx + N\,dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \qquad \text{or} \qquad \big(p(x)y - f(x)\big)\,dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x)\,dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N\frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u? Well, given an equation

$$M\,dx + N\,dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y}\big[uM\big] = u_yM + uM_y = \frac{\partial}{\partial x}\big[uN\big] = u_xN + uN_x.$$

Therefore,

$$(M_y - N_x)u = u_xN - u_yM.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x, and u_y is just zero. Then

$$\frac{M_y - N_x}{N}u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N}u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x)\,dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x)\,dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M}u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y)\,dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y)\,dy}$ is the integrating factor.

Example 1.8.6

Solve

$$\frac{x^2 + y^2}{x + 1} + 2y \frac{dy}{dx} = 0.$$

Solution

Let $M = \frac{x^2 + y^2}{x + 1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x + 1} - 0 = \frac{2y}{x + 1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x + 1} \frac{1}{2y} = \frac{1}{x + 1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x)\,dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x + 1)$ to obtain

$$x^2 + y^2 + 2y(x + 1)\frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example $1.8.5$. The solution was

$$y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x + 1}}.$$

Example 1.8.7

Solve

$$y^2 + (xy + 1)\frac{dy}{dx} = 0.$$

Solution

First compute

$$M_y - N_x = 2y - y = y.$$

As this is not zero, the equation is not exact. We observe

$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$

is a function of y alone. We compute the integrating factor

$$e^{-\int Q(y)\,dy} = e^{-\ln y} = \frac{1}{y}.$$

Therefore we look at the exact equation

$$y + \frac{xy+1}{y} \frac{dy}{dx} = 0.$$

The reader should double check that this equation is exact. We follow the procedure for exact equations

$$F(x,y) = xy + A(y),$$

and

$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y).$$

Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x,y) = xy + \ln y$. It is not possible to solve $F(x,y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:

$$xy + \ln y = C.$$

We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

1.8.3: Footnotes

[1] Named for the French polymath Jules Henri Poincaré (1854–1912).

1.9: First Order Linear PDE

We only considered ODE so far, so let us solve a linear first order PDE. Consider the equation

$$a(x,t)\,u_x + b(x,t)\,u_t + c(x,t)\,u = g(x,t), \qquad u(x,0) = f(x), \qquad -\infty < x < \infty, \quad t > 0,$$

where $u(x,t)$ is a function of x and t. The *initial condition* $u(x,0) = f(x)$ is now a function of x rather than just a number. In these problems, it is useful to think of x as position and t as time. The equation describes the evolution of a function of x as time goes on. Below, the coefficients a, b, c, and the function g are mostly going to be constant or zero. The method we describe works with nonconstant coefficients, although the computations may get difficult quickly.

This method we use is the . The idea is that we find lines along which the equation is an ODE that we solve. We will see this technique again for second order PDE when we encounter the wave equation in Section 4.8.

Example 1.9.1

Consider the equation

$$u_t + \alpha u_x = 0, \qquad u(x,0) = f(x).$$

This particular equation, $u_t + \alpha u_x = 0$, is called the *transport equation*.

The data will propagate along curves called characteristics. The idea is to change to the so-called *characteristic coordinates*. If we change to these coordinates, the equation simplifies. The change of variables for this equation is

$$\xi = x - \alpha t, \qquad s = t.$$

Let's see what the equation becomes. Remember the chain rule in several variables.

$$\begin{aligned} u_t &= u_\xi \xi_t + u_s s_t = -\alpha u_\xi + u_s, \\ u_x &= u_\xi \xi_x + u_s s_x = u_\xi. \end{aligned} \tag{1.9.1}$$

The equation in the coordinates ξ and s becomes

$$\underbrace{(-\alpha u_\xi + u_s)}_{u_t} + \alpha \underbrace{(u_\xi)}_{u_x} = 0,$$

or in other words

$$u_s = 0.$$

That is trivial to solve. Treating ξ as simply a parameter, we have obtained the ODE $\frac{du}{ds} = 0$.

The solution is a function that does not depend on s (but it does depend on ξ). That is, there is some function A such that

$$u = A(\xi) = A(x - \alpha t).$$

The initial condition says that:

$$f(x) = u(x,0) = A(x - \alpha 0) = A(x),$$

so $A = f$. In other words,

$$u(x,t) = f(x - \alpha t).$$

Everything is simply moving right at speed α as t increases. The curve given by the equation

$$\xi = \text{constant}$$

is called the characteristic. See Figure $1.9.1$. In this case, the solution does not change along the characteristic.

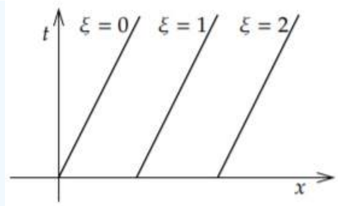

Figure 1.9.1: Characteristic curves.

In the (x,t) coordinates, the characteristic curves satisfy $t = \frac{1}{\alpha}(x-\xi)$, and are in fact lines. The slope of characteristic lines is $\frac{1}{\alpha}$, and for each different ξ we get a different characteristic line.

We see why $u_t + \alpha u_x = 0$ is called the transport equation: everything travels at some constant speed. Sometimes this is called . An example application is material being moved by a river where the material does not diffuse and is simply carried along. In this setup, x is the position along the river, t is the time, and $u(x,t)$ the concentration the material at position x and time t. See Figure $1.9.2$ for an example.

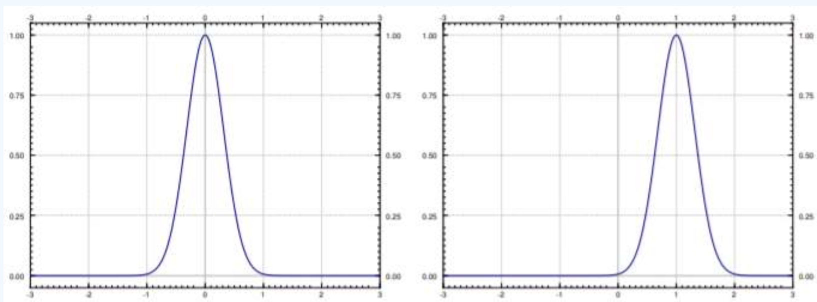

Figure 1.9.2: Example of "transport" in $u_t - u_x = 0$ (that is, $\alpha = 1$) where the initial condition $f(x)$ is a peak at the origin. On the left is a graph of the initial condition $u(x,0)$. On the right is a graph of the function $u(x,1)$, that is at time $t = 1$. Notice it is the same graph shifted one unit to the right.

We use similar idea in the more general case:

$$au_x + bu_t + cu = g, \qquad u(x,0) = f(x).$$

We change coordinates to the characteristic coordinates. Let us call these coordinates (ξ, s). These are coordinates where $au_x + bu_t$ becomes differentiation in the s variable.

Along the characteristic curves (where ξ is constant), we get a new ODE in the s variable. In the transport equation, we got the simple $\frac{du}{ds} = 0$. In general, we get the linear equation

$$\frac{du}{ds} + cu = g. \tag{1.9.2}$$

We think of everything as a function of ξ and s, although we are thinking of ξ as a parameter rather than an independent variable. So the equation is an ODE. It is a linear ODE that we can solve using the integrating factor.

To find the characteristics, think of a curve given parametrically $\big(x(s), t(s)\big)$. We try to have the curve satisfy

$$\frac{dx}{ds} = a, \qquad \frac{dt}{ds} = b.$$

Why? Because when we think of x and t as functions of s we find, using the chain rule,

$$\frac{du}{ds} + cu = \underbrace{\left(u_x \frac{dx}{ds} + u_t \frac{dt}{ds}\right)}_{\frac{du}{ds}} + cu = au_x + bu_t + cu = g.$$

So we get the ODE (1.9.2), which then describes the value of the solution u of the PDE along this characteristic curve. It is also convenient to make sure that $s = 0$ corresponds to $t = 0$, that is $t(0) = 0$. It will be convenient also for $x(0) = \xi$. See Figure 1.9.3.

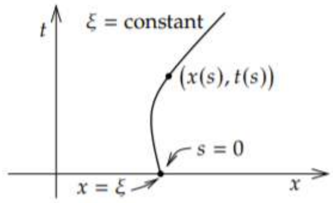

Figure 1.9.3: General characteristic curve.

Example 1.9.2

Consider

$$u_x + u_t + u = x, \qquad u(x,0) = e^{-x^2}.$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = 1, \qquad \frac{dt}{ds} = 1.$$

So

$$x = s + c_1, \qquad t = s + c_2,$$

for some c_1 and c_2. At $s = 0$ we want $t = 0$, and x should be ξ. So we let $c_1 = \xi$ and $c_2 = 0$:

$$x = s + \xi, \qquad t = s.$$

The ODE is $\frac{du}{ds} + u = x$, and $x = s + \xi$. So, the ODE to solve along the characteristic is

$$\frac{du}{ds} + u = s + \xi.$$

The general solution of this equation, treating ξ as a parameter, is $u = Ce^{-s} + s + \xi - 1$, for some constant C. At $s = 0$, our initial condition is that u is $e^{-\xi^2}$, since at $s = 0$ we have $x = \xi$. Given this initial condition, we find $C = e^{-\xi^2} - \xi + 1$. So,

$$\begin{aligned} u &= \left(e^{-\xi^2} - \xi + 1\right)e^{-s} + s + \xi - 1 \\ &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1. \end{aligned} \tag{1.9.3}$$

Substitute $\xi = x - t$ and $s = t$ to find u in terms of x and t:

$$\begin{aligned} u &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1 \\ &= e^{-(x-t)^2 - t} + (1 - x + t)e^{-t} + x - 1. \end{aligned} \tag{1.9.4}$$

See Figure 1.9.4 for a plot of $u(x,t)$ as a function of two variables.

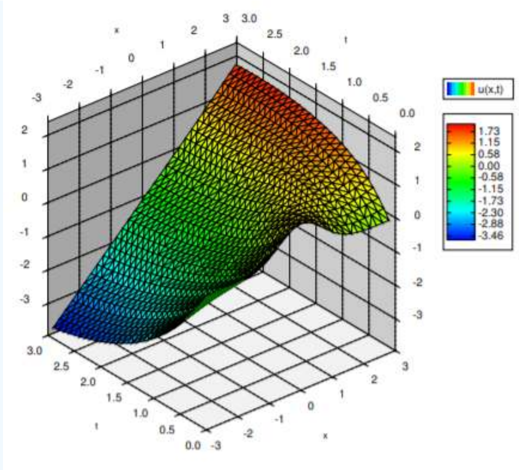

Figure 1.9.4: Plot of the solution $u(x,t)$ to $u_x + u_t + u = x$, $u(x,0) = e^{-x^2}$.

When the coefficients are not constants, the characteristic curves are not going to be straight lines anymore.

Example 1.9.3

Consider the following variable coefficient equation:

$$xu_x + u_t + 2u = 0, \qquad u(x,0) = \cos(x). \tag{1.9.5}$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = x, \qquad \frac{dt}{ds} = 1.$$

So

$$x = c_1 e^s, \qquad t = s + c_2.$$

At $s = 0$, we wish to get the line $t = 0$, and x should be ξ. So

$$x = \xi e^s, \qquad t = s.$$

OK, the ODE we need to solve is

$$\frac{du}{ds} + 2u = 0.$$

This is for a fixed ξ. At $s = 0$, we should get that u is $\cos(\xi)$, so that is our initial condition. Consequently,

$$u = e^{-2s}\cos(\xi) = e^{-2t}\cos(xe^{-t}).$$

We make a few closing remarks. One thing to keep in mind is that we would get into trouble if the coefficient in front of u_t, that is the b, is ever zero. Let us consider a quick example of what can go wrong:

$$u_x + u = 0, \qquad u(x,0) = \sin(x).$$

This problem has no solution. If we had a solution, it would imply that $u_x(x,0) = \cos(x)$, but $u_x(x,0) + u(x,0) = \cos(x) + \sin(x) \neq 0$. The problem is that the characteristic curve is now the line $t = 0$, and the solution is already provided on that line!

As long as b is nonzero, it is convenient to ensure that b is positive by multiplying by -1 if necessary, so that positive s means positive t.

Another remark is that if a or b in the equation are variable, the computations can quickly get out of hand, as the expressions for the characteristic coordinates become messy and then solving the ODE becomes even messier. In the examples above, b was always 1, meaning we got $s = t$ in the characteristic coordinates. If b is not constant, your expression for s will be more complicated.

Finding the characteristic coordinates is really a system of ODE in general if a depends on t or if b depends on x. In that case, we would need techniques of systems of ODE to solve, see Chapter 3 or Chapter 8. In general, if a and b are not linear functions or constants, finding closed form expressions for the characteristic coordinates may be impossible.

Finally, the method of characteristics applies to nonlinear first order PDE as well. In the nonlinear case, the characteristics depend not only on the differential equation, but also on the initial data. This leads to not only more difficult computations, but also the formation of singularities where the solution breaks down at a certain point in time. An example application where first order nonlinear PDE come up is traffic flow theory, and you have probably experienced the formation of singularities: traffic jams. But we digress.

1.E: First order ODEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

1.E.1: 1.1: Integrals as solutions

Exercise $1.E.1.1.1$

Solve for v, and then solve for x. Find $x(10)$ to answer the question.

Exercise $1.E.1.1.2$

Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

Exercise $1.E.1.1.3$

Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

Exercise $1.E.1.1.4$

Solve $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ for $y(0) = 0$.

Exercise $1.E.1.1.5$

Solve $y' = y^3$ for $y(0) = 1$.

Exercise $1.E.1.1.6$: (little harder)

Solve $y' = (y-1)(y+1)$ for $y(0) = 3$.

Exercise $1.E.1.1.7$

Solve $\frac{dy}{dx} = \frac{1}{y+1}$ for $y(0) = 0$.

Exercise $1.E.1.1.8$: (harder)

Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

Exercise $1.E.1.1.9$

A spaceship is traveling at the speed $2t^2 + 1$ $^{\text{km}}/_{\text{s}}$ (t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?

Exercise $1.E.1.1.10$

Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

Exercise $1.E.1.1.11$

A dropped ball accelerates downwards at a constant rate 9.8 meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground.

Exercise $1.E.1.1.12$

Find the general solution of $y' = e^x$, and then $y' = e^y$.

Exercise $1.E.1.1.13$

Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

Answer

$y = e^x + \frac{x^2}{2} + 9$

Exercise $1.E.1.1.14$

Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

Answer

$x = (3t - 2)^{1/3}$

Exercise $1.E.1.1.15$

Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

Answer

A$x = \sin^{-1}(t+1)$

Exercise $1.E.1.1.16$

Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$ the Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

Answer

170

Exercise $1.E.1.1.17$

Solve $y' = y''$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

Answer

If $n \neq 1$, then $y = ((1-n)x + 1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

Exercise $1.E.1.1.18$

The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $\frac{4}{3})\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t = 0$ minutes, the radius is 10 centimeters. After 5 minutes, the radius is 8 centimeters. At what time t will the snowball be completely melted?

Answer

The equation is $r' = -C$ for some constant C. The snowball will be completely melted in 25 minutes from time $t = 0$.

Exercise $1.E.1.1.19$

Find the general solution to $y'''' = 0$. How many distinct constants do you need?

Answer

$y = Ax^3 + Bx^2 + Cx + D$, so 4 constants.

1.E.2: 1.2: Slope fields

Exercise $1.E.1.2.1$

Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

Exercise $1.E.1.2.2$

Sketch slope field for $y' = x^2$.

Exercise $1.E.1.2.3$

Sketch slope field for $y' = y^2$.

Exercise $1.E.1.2.4$

Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.

Exercise $1.E.1.2.5$

Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

Exercise $1.E.1.2.6$

Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.

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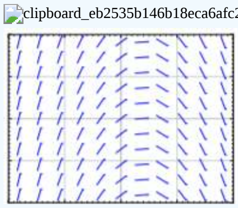

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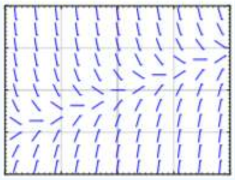
c. 

Exercise $1.E.1.2.7$: (challenging)

Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

Exercise $1.E.1.2.8$: (challenging)

Take $(y - x)y' = 0$, $x(0) = 0$.

a. Find two distinct solutions.
b. Explain why this does not violate Picard's theorem.

Exercise $1.E.1.2.9$

Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$:

a. f does not depend on y.
b. f does not depend on x.
c. $f, (t, t) = 0$ for any number t.
d. $f(x, 0) = 0$ and $f(x, 1) = 1$ for all x.

Exercise $1.E.1.2.10$

Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?

Exercise $1.E.1.2.11$

Take an equation $y' = (y - 2x)g(x, y) + 2$ for some function $g(x, y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?

Exercise $1.E.1.2.12$: (challenging)

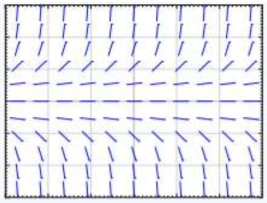
Suppose $y' = f(x, y)$ is such that $f(x, 1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.

a. Guess a solution given the initial condition $y(0) = 1$.
b. Can graphs of two solutions of the equation for different initial conditions ever intersect?
c. Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?

Exercise $1.E.1.2.13$

Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

Answer



$y = 0$ is a solution such that $y(0) = 0$

Exercise $1.E.1.2.14$

Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

Answer

Yes a solution exists. The equation is $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact, $y = 0$ is the solution.)

Exercise $1.E.1.2.15$

Is it possible to solve $y' = \frac{x}{x^2 - 1}$ for $y(1) = 0$?

Answer

No, the equation is not defined at $(x, y) = (1, 0)$.

Exercise $1.E.1.2.16$

Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify.

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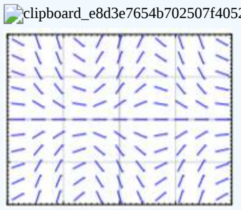
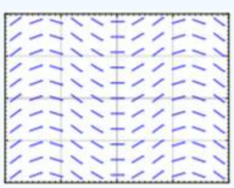
b. 

c. 

Answer

a. $y' = \cos y$
b. $y' = y\cos(x)$
c. $y' = \sin x$

Justification left to reader.

Exercise $1.E.1.2.17$: (tricky)

Suppose

$$f(y) = \begin{cases} 0 & if\ y > 0, \\ 1 & if\ y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?

Answer

Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x. But then for those x we would have $y'(x) = 0$, so clearly the derivative cannot be continuous.

Exercise $1.E.1.2.18$

Consider an equation of the form $y' = f(x)$ for some continuous function f, and an initial condition $y(x_0) = y_0$. Does a solution exist for all x? Why or why not?

Answer

The solution is $y(x) = \int_{x_0}^{x} f(s)ds + y_0$, and this does indeed exist for every x.

1.E.3: 1.3: Separable Equations

Exercise $1.E.1.3.1$

Solve $y' = \frac{x}{y}$.

Exercise $1.E.1.3.2$

Solve $y' = x^2 y$.

Exercise $1.E.1.3.3$

Solve $\frac{dx}{dt} = \left(x^2 - 1\right)$, for $x(0) = 0$.

Exercise $1.E.1.3.4$

Solve $\frac{dx}{dt} = x \sin(t)$, for $x(0) = 1$.

Exercise $1.E.1.3.5$

Solve $\frac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right hand side.

Exercise $1.E.1.3.6$

Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

Exercise $1.E.1.3.7$

Solve $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, for $y(0) = 1$.

Exercise $1.E.1.3.8$

Find an implicit solution for $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, for $y(0) = 1$.

Exercise $1.E.1.3.9$

Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

Exercise $1.E.1.3.10$

Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

Exercise $1.E.1.3.11$

Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer.

Exercise $1.E.1.3.12$

Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

Exercise $1.E.1.3.13$

Solve $y' = 2xy$.

Answer

$$y = Ce^{x^2}$$

Exercise $1.E.1.3.14$

Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Answer

$$y = e^{t^3} + 1$$

Exercise $1.E.1.3.15$

Find an implicit solution for $x' = \frac{1}{3x^2+1}$ $x(0) = 1$.

Answer

$$x^3 + x = t + 2$$

Exercise $1.E.1.3.16$

Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

Answer

$$y = \frac{1}{1 - \ln x}$$

Exercise $1.E.1.3.17$

Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

Answer

$\sin(y) = -\cos(x) + C$

Exercise $1.E.1.3.18$

Take Example 1.3.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

Answer

The range is approximately 7.45 to 12.15 minutes.

Exercise $1.E.1.3.19$

A population x of rabbits on an island is modeled by $x' = x - (\frac{1}{1000})x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

a. Find the solution to the equation with the initial condition.
b. How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer)

Answer

a. $x = \frac{1000e^t}{e^t + 24}$.
b. 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

1.E.4: 1.4: Linear equations and the integrating factor

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

Exercise $1.E.1.4.1$

Solve $y' + xy = x$.

Exercise $1.E.1.4.2$

Solve $y' + 6y = e^x$.

Exercise $1.E.1.4.3$

Solve $y' + 3x^2 y = \sin(x)\, e^{-x^3}$ with $y(0) = 1$.

Exercise $1.E.1.4.4$

Solve $y' + \cos(x)\, y = \cos(x)$.

Exercise $1.E.1.4.5$

Solve $\frac{1}{x^2+1}y' + xy = 3$ with $y(0) = 0$.

Exercise $1.E.1.4.6$

Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.

a. Find the concentration of toxic substance as a function of time in both lakes.
b. When will the concentration in the first lake be below 0.001 kg per liter?
c. When will the concentration in the second lake be maximal?

Exercise $1.E.1.4.7$

Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures).

a. Find the general solution.
b. In the long term, will the initial conditions make much of a difference? Why or why not?

Exercise $1.E.1.4.8$

Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

Exercise $1.E.1.4.9$

Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

Exercise $1.E.1.4.10$

Solve $y' + 3x^2y + x^2$.

Answer

$$y = Ce^{-x^3} + \frac{1}{3}$$

Exercise $1.E.1.4.11$

Solve $y' + 2\sin(2x)y = 2\sin(2x)$ with $y(\pi/2) = 3$.

Answer

$$y = 2e^{\cos(2x)+1} + 1$$

Exercise $1.E.1.4.12$

Suppose a water tank is being pumped out at $3\ \frac{\mathrm{L}}{\mathrm{min}}$. The water tank starts at $10\ \mathrm{L}$ of clean water. Water with toxic substance is flowing into the tank at $2\ \frac{\mathrm{L}}{\mathrm{min}}$, with concentration $20t\ \frac{\mathrm{g}}{\mathrm{L}}$ at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

Answer

250 grams

Exercise $1.E.1.4.13$

Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1\,t)P$. If $P(0) = 1000$, find the population at $t = 5$.

Answer

$P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

Exercise $1.E.1.4.14$

A cylindrical water tank has water flowing in at I cubic meters per second. Let A be the area of the cross section of the tank in meters. Suppose water is flowing from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h, the height of the water, introducing and naming constants that you need. You should also give the units for your constants.

Answer

$Ah' = I - kh$, where k is a constant with units $\frac{\mathrm{m}^2}{\mathrm{s}}$.

1.E.5: 1.5: Substitution

Hint: Answers need not always be in closed form.

Exercise $1.E.1.5.1$

Solve $y' + y(x^2 - 1) + xy^6 = 0$, with $y(1) = 1$.

Exercise $1.E.1.5.2$

Solve $2yy' + 1 = y^2 + x$, with $y(0) = 1$.

Exercise $1.E.1.5.3$

Solve $y' + xy = y^4$, with $y(0) = 1$.

Exercise $1.E.1.5.4$

Solve $yy' + x = \sqrt{x^2 + y^2}$.

Exercise $1.E.1.5.5$

Solve $y' = (x + y - 1)^2$.

Exercise $1.E.1.5.6$

Solve $y' = \dfrac{x^2 - y^2}{xy}$, with $y(1) = 2$.

Exercise $1.E.1.5.7$

Solve $xy' + y + y^2 = 0$, $y(1) = 2$.

Answer

$y = \frac{2}{3x-2}$

Exercise $1.E.1.5.8$

Solve $xy' + y + x = 0$, $y(1) = 1$.

Answer

$y = \frac{3-x^2}{2x}$

Exercise $1.E.1.5.9$

Solve $y^2 y' = y^3 - 3x$, $y(0) = 2$.

Answer

$y = \left(7e^{3x} + 3x + 1\right)^{1/3}$

Exercise $1.E.1.5.10$

Solve $2yy' = e^{y^2 - x^2} + 2x$.

Answer

$y = \sqrt{x^2 - \ln(C - x)}$

1.E.6: 1.6: Autonomous equations

Exercise $1.E.1.6.1$

Consider $x' = x^2$.

a. Draw the phase diagram, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = -1$.

Exercise $1.E.1.6.2$

Let $x' = \sin x$.

a. Draw the phase diagram for $-4\pi \le x \le 4\pi$. On this interval mark the critical points stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 1$.

Exercise $1.E.1.6.3$

Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x.

a. Draw the phase diagram for $x' = f(x)$, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t \to \infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.

Exercise $1.E.1.6.4$

Start with the logistic equation $\frac{dx}{dt} = kx(M - x)$. Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx).

a. Construct the differential equation.
b. Show that if $kM > h$, then the equation is still logistic.
c. What happens when $kM < h$?

Exercise $1.E.1.6.5$

A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x, and the number of susceptible but uninfected people, $S - x$.

a. Write down the differential equation.
b. Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t \to \infty} x(t)$.
c. Does the solution to part b) agree with your intuition? Why or why not?

Exercise $1.E.1.6.6$

Let $x' = (x - 1)(x - 2)x^2$.

a. Sketch the phase diagram and find critical points.
b. Classify the critical points.
c. If $x(0) = 0.5$ then find $\lim_{t \to \infty} x(t)$.

Answer

a. $0,\ 1,\ 2$ are critical points.
b. $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable.
c. 1

Exercise $1.E.1.6.7$

Let $x' = e^{-x}$.

a. Find and classify all critical points.
b. Find $\lim_{t \to \infty} x(t)$ given any initial condition.

Answer

a. There are no critical points.
b. ∞

Exercise $1.E.1.6.8$

Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M - x)$. Now suppose that fish are continually added at A fish per unit of time.

a. Find the differential equation for x.
b. What is the new limiting population?

Answer

a. $\frac{dx}{dt} = kx(M - x) + A$
b. $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

Exercise $1.E.1.6.9$

Suppose $\frac{dx}{dt} = (x - \alpha)(x - \beta)$ for two numbers $\alpha < \beta$.

a. Find the critical points, and classify them.

For b), c), d), find $\lim_{t \to \infty} x(t)$ based on the phase diagram.

b. $x(0) < \alpha$,
c. $\alpha < x(0) < \beta$,
d. $\beta < x(0)$.

Answer

1. α is a stable critical point, β is an unstable one.
2. α
3. α
4. ∞ or DNE.

1.E.7: 1.7: Numerical methods: Euler's method

Exercise $1.E.1.7.1$

Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

Exercise $1.E.1.7.2$

Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$.

a. Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$.
b. Solve the equation exactly.
c. Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

Exercise $1.E.1.7.3$

Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

Exercise 1.E.1.7.4

Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the Runge-Kutta method. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f(x_i + \frac{h}{2}, y_i + k_1\frac{h}{2}) \qquad\qquad x_{i+1} = x_i + h, \\ k_3 &= f(x_i + \frac{h}{2}, y_i + k_2\frac{h}{2}) \qquad y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \tag{1.E.1}$$

Exercise 1.E.1.7.5

Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.

a. Use Runge-Kutta (see above) with step sizes $h = 1$ and $h = \frac{1}{2}$ to approximate $y(1)$.
b. Use Euler's method with $h = 1$ and $h = \frac{1}{2}$.
c. Solve exactly, find the exact value of $y(1)$, and compare.

Exercise 1.E.1.7.6

Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

Answer

Approximately: $1.0000,\ 1.2397,\ 1.382$

Exercise 1.E.1.7.7

Let $x' = 2t$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Solve exactly, and compute the errors.
c. Compute the factor by which the errors changed.

Answer

a. $0,\ 8,\ 12$
b. $x(4) = 16$, so errors are: $16,\ 8,\ 4$
c. Factors are $0.5,\ 0.5,\ 0.5$

Exercise 1.E.1.7.8

Let $x' = xe^{xt+1}$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Guess an exact solution based on part a) and compute the errors.

Answer

a. $0,\ 0,\ 0$

b. $x = 0$ is a solution so errors are: $0,\ 0,\ 0$.

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i), we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2, and then take the average of k_1 and k_2, hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\ k_2 &= f(x_i + h, y_i + k_1 h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2} h. \end{aligned} \tag{1.E.2}$$

Exercise $1.E.1.7.9$

Consider $\dfrac{dy}{dx} = x + y$, $y(0) = 1$.

a. Use the improved Euler's method (see above) with step sizes $h = \frac{1}{4}$ and $h = \frac{1}{8}$ to approximate $y(1)$.
b. Use Euler's method with $h = \frac{1}{4}$ and $h = \frac{1}{8}$.
c. Solve exactly, find the exact value of $y(1)$.
d. Compute the errors, and the factors by which the errors changed.

Answer

a. Improved Euler: $y(1) \approx 3.3897$ for $h = 1/4$, $y(1) \approx 3.4237$ for $h = 1/8$,
b. Standard Euler: $y(1) \approx 2.8828$ for $h = 1/4$, $y(1) \approx 3.1316$ for $h = 1/8$,
c. $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366
d. Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

1.E.8: 1.8 Exact Equations

Exercise $1.E.1.8.1$

Solve the following exact equations, implicit general solutions will suffice:

a. $(2xy + x^2)\,dx + (x^2 + y^2 + 1)\,dy = 0$
b. $x^5 + y^5 \frac{dy}{dx} = 0$
c. $e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0$
d. $(x + y)\cos(x) + \sin(x) + \sin(x)y' = 0$

Exercise $1.E.1.8.2$

Find the integrating factor for the following equations making them into exact equations:

a. $e^{xy}\,dx + \frac{y}{x} e^{xy}\,dy = 0$
b. $\frac{e^x + y^3}{y^2}\,dx + 3x\,dy = 0$
c. $4(y^2 + x)\,dx + \frac{2x + 2y^2}{y}\,dy = 0$
d. $2\sin(y)\,dx + x\cos(y)\,dy = 0$

Exercise $1.E.1.8.3$

Suppose you have an equation of the form: $f(x)+g(y)\frac{dy}{dx}=0$.

a. Show it is exact.
b. Find the form of the potential function in terms of f and g.

Exercise $1.E.1.8.4$

Suppose that we have the equation $f(x)\,dx-dy=0$.

a. Is this equation exact?
b. Find the general solution using a definite integral.

Exercise $1.E.1.8.5$

Find the potential function $F(x,y)$ of the exact equation $\frac{1+xy}{x}\,dx+\left(\frac{1}{y}+x\right)\,dy=0$ in two different ways.

a. Integrate M in terms of x and then differentiate in y and set to N.
b. Integrate N in terms of y and then differentiate in x and set to M.

Exercise $1.E.1.8.6$

A function $u(x,y)$ is said to be a if $u_{xx}+u_{yy}=0$.

a. Show if u is harmonic, $-u_y\,dx+u_x\,dy=0$ is an exact equation. So there exists (at least locally) the so-called function $v(x,y)$ such that $v_x=-u_y$ and $v_y=u_x$.

Verify that the following u are harmonic and find the corresponding harmonic conjugates v:

a. $u=2xy$
b. $u=e^x\cos y$
c. $u=x^3-3xy^2$

Exercise $1.E.1.8.7$

Solve the following exact equations, implicit general solutions will suffice:

a. $\cos(x)+ye^{xy}+xe^{xy}y'=0$
b. $(2x+y)\,dx+(x-4y)\,dy=0$
c. $e^x+e^y\frac{dy}{dx}=0$
d. $(3x^2+3y)\,dx+(3y^2+3x)\,dy=0$

Answer

a. $e^{xy}+\sin(x)=C$
b. $x^2+xy-2y^2=C$
c. $e^x+e^y=C$
d. $x^3+3xy+y^3=C$

Exercise $1.E.1.8.8$

Find the integrating factor for the following equations making them into exact equations:

a. $\frac{1}{y}\,dx+3y\,dy=0$
b. $dx-e^{-x-y}\,dy=0$
c. $\left(\frac{\cos(x)}{y^2}+\frac{1}{y}\right)\,dx+\frac{x}{y^2}\,dy=0$

d. $\left(2y+\frac{y^2}{x}\right) dx+(2y+x)\, dy=0$

Answer

a. Integrating factor is y, equation becomes $dx+3y^2dy=0$.
b. Integrating factor is e^x, equation becomes $e^x dx-e^{-y}dy=0$.
c. Integrating factor is y^2, equation becomes $(\cos(x)+y)dx+xdy=0$.
d. Integrating factor is x, equation becomes $(2xy+y^2)dx+(x^2+2xy)dy=0$.

? Exercise $1.E.1.8.9$

a. Show that every separable equation $y'=f(x)g(y)$ can be written as an exact equation, and verify that it is indeed exact.
b. Using this rewrite $y'=xy$ as an exact equation, solve it and verify that the solution is the same as it was in Example 1.3.1.

Answer

a. The equation is $-f(x)dx+\frac{1}{g(y)}dy$, and this is exact because $M=-f(x)$, $N=\frac{1}{g(y)}$, so $M_y=0=N_x$.
b. $-xdx+\frac{1}{y}dy=0$, leads to potential function $F(x,y)=-\frac{x^2}{2}+\ln|y|$, solving $F(x,y)=C$ leads to the same solution as the example.

1.E.9: 1.9: First Order Linear PDE

? Exercise $1.E.1.9.1$

Solve

a. $u_t+9u_x=0$, $u(x,0)=\sin(x)$,
b. $u_t-8u_x=0$, $u(x,0)=\sin(x)$,
c. $u_t+\pi u_x=0$, $u(x,0)=\sin(x)$,
d. $u_t+\pi u_x+u=0$, $u(x,0)=\sin(x)$.

? Exercise $1.E.1.9.2$

Solve $u_t+3u_x=1$, $u(x,0)=x^2$.

? Exercise $1.E.1.9.3$

Solve $u_t+3u_x=x$, $u(x,0)=e^x$.

? Exercise $1.E.1.9.4$

Solve $u_x+u_t+xu=0$, $u(x,0)=\cos(x)$.

? Exercise $1.E.1.9.5$

a. Find the characteristic coordinates for the following equations:
$u_x+u_t+u=1$, $u(x,0)=\cos(x)$,) $2u_x+2u_t+2u=2$, $u(x,0)=\cos(x)$.
b. Solve the two equations using the coordinates.
c. Explain why you got the same solution, although the characteristic coordinates you found were different.

? Exercise $1.E.1.9.6$

Solve $(1+x^2)u_t+x^2u_x+e^xu=0$, $u(x,0)=0$. Hint: Think a little out of the box.

Exercise $1.E.1.9.7$

Solve

a. $u_t - 5u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$,
b. $u_t + 2u_x = 0$, $u(x,0) = \cos(x)$.

Answer

a. $u = \frac{1}{1+(x+5t)^2}$
b. $u = \cos(x - 2t)$

Exercise $1.E.1.9.8$

Solve $u_x + u_t + tu = 0$, $u(x,0) = \cos(x)$.

Answer

$u = \cos(x-t)e^{-t^2/2}$

Exercise $1.E.1.9.9$

Solve $u_x + u_t = 5$, $u(x,0) = x$.

Answer

$u = x + 4t$
