2: HIGHER ORDER LINEAR ODES

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CHAPTER OVERVIEW

2: Higher order linear ODEs

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

2.1: Second order linear ODEs

Let us consider the general second order linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x),$$

where $p(x) = \frac{B(x)}{A(x)}$, $q(x) = \frac{C(x)}{A(x)}$, and $f(x) = \frac{F(x)}{A(x)}$. The word *linear* means that the equation contains no powers nor functions of y, y', and y''.

In the special case when $f(x) = 0$ we have a so-called *homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{2.1.1}$$

We have already seen some second order linear homogeneous equations:

$$\begin{array}{lll} y'' + k^2 y = 0 & \text{Two solutions are:} & y_1 = \cos(kx), \quad y_2 = \sin(kx). \\ y'' - k^2 y = 0 & \text{Two solutions are:} & y_1 = e^{kx}, \quad y_2 = e^{-kx}. \end{array}$$

If we know two solutions of a linear homogeneous equation, we know a lot more of them.

Theorem 2.1.1: Superposition

Suppose y_1 and y_2 are two solutions of the homogeneous equation (2.1.1). Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

also solves (2.1.1) for arbitrary constants C_1 and C_2.

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1y_1 + C_2y_2$ a *linear combination* of y_1 and y_2. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof

Let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2) \\ &= C_1y_1'' + C_2y_2'' + C_1py_1' + C_2py_2' + C_1qy_1 + C_2qy_2 \\ &= C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2) \\ &= C_1.0 + C_2.0 = 0 \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function, which eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being linear means that $L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2$. The proof above becomes

$$Ly = L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2 = C_1.0 + C_2.0 = 0$$

Two different solutions to the second equation $y'' - k^2y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Let us remind ourselves of the definition, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions $\sinh$ and $\cosh$ are sometimes more convenient to use than the exponential. Let us review some of their properties.

$$\begin{array}{ll} \cosh 0 = 1 & \sinh 0 = 0, \\ \frac{d}{dx}[\cosh x] = \sinh x, & \frac{d}{dx}[\sinh x] = \cosh x, \\ \cosh^2 x - \sinh^2 x = 1. & \end{array}$$

Exercise 2.1.1

Derive these properties using the definitions of sinh and cosh in terms of exponentials.

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem 2.1.2: Existence and Uniqueness

Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous functions on some interval I containing a with a, b_0 and b_1 constants. The equation

$$y'' + p(x)y' + q(x)y = f(x).$$

has exactly *one* solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k}\sin(kx)$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k}\sinh(kx)$$

Using $\cosh$ and $\sinh$ in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Exercise 2.1.2

Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (2.1.1). Can every solution be written (using superposition) in the form $y = C_1 y_1 + C_2 y_2$?

Answer

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We will say y_1 and y_2 are linearly independent if one is not a constant multiple of the other.

Theorem 2.1.3

Let $p(x)$ and $q(x)$ be continuous functions and let y_1 and y_2 be two linearly independent solutions to the homogeneous equation (2.1.1). Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. If $\sin x = A \cos x$ for some constant A, we let $x = 0$ and this would imply $A = 0$.

But then $\sin x = 0$ for all x, which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2} y = 0$. Then $y_1 = x^2$ and $y_2 = \frac{1}{x}$ are solutions. To see that they are linearly independent, suppose one is a multiple of the other: $y_1 = A y_2$, we just have to find out that A cannot be a constant. In this case we have $A = \frac{y_1}{y_2} = x^3$, this most decidedly not a constant. So $y = C_1 x^2 + C_2 \frac{1}{x}$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x) y' + q(x) y = 0$ we try a second solution of the form $y_2(x) = y_1(x) v(x)$. We just need to find v. We plug y_2 into the equation:

$$\begin{aligned} 0 = y_2'' + p(x) y_2' + q(x) y_2 &= y_1'' v + 2 y_1' v' + y_1 v'' + p(x)(y_1' v + y_1 v') + q(z) y_1 v \\ &= y_1 v'' + (2 y_1' + p(x) y_1) v' + \cancelto{0}{\left(y_1'' + p(x) y_1' + q(x) y_1\right)} v. \end{aligned} \tag{2.1.2}$$

In other words, $y_1 v'' + (2 y_1' + p(x) y_1) v' = 0$. Using $w = v'$ we have the first order linear equation $y_1 w' + (2 y_1' + p(x) y_1) w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w. We then form y_2 by computing $y_1 v$. For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1} y' - x^{-2} y = 0$. The equation for w is then $x w' + 3w = 0$. We find a solution, $w = C x^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1 v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = \frac{1}{x}$. Thus, the general solution is $y = C_1 x + C_2 \frac{1}{x}$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\, dx}}{\left(y_1(x)\right)^2}\, dx$$

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x) v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: you get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in Section 2.5. We will first focus on finding general solutions to homogeneous equations.

2.2: Constant coefficient second order linear ODEs

2.2.1: Solving Constant Coefficient Equations

Suppose we have the problem

$$y'' - 6y' + 8y = 0, y(0) = -2, y'(0) = 6$$

This is a second order linear homogeneous equation with constant coefficients. Constant coefficients means that the functions in front of y'', y', and y are constants and do not depend on x.

To guess a solution, think of a function that you know stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero.

Let us try[1] a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ \underbrace{r^2e^{rx}}_{y''} - 6\underbrace{re^{rx}}_{y'} + 8\underbrace{e^{rx}}_{y} &= 0, \\ r^2 - 6r + 8 &= 0 \qquad (\text{divide through by } e^{rx}), \\ (r-2)(r-4) &= 0. \end{aligned} \tag{2.2.1}$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Exercise 2.2.1

Check that y_1 and y_2 are solutions.

Solution

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent we could write $e^{4x} = Ce^{2x}$ for some constant C, implying that $e^{2x} = C$ for all x, which is clearly not possible. Hence, we can write the general solution as

$$y = C_1e^{2x} + C_2e^{4x}$$

We need to solve for C_1 and C_2. To apply the initial conditions we first find $y' = 2C_1e^{2x} + 4C_2e^{4x}$. We plug in $x = 0$ and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2 \\ 6 &= y'(0) = 2C_1 + 4C_2 \end{aligned} \tag{2.2.2}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{2.2.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0$$

Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, we have $e^{r_1 x}$ and $e^{r_2 x}$ as solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 2.2.1

Suppose that r_1 and r_2 are the roots of the characteristic equation.

If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then (2.2.3) has the general solution

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then (2.2.3) has the general solution

$$y = (C_1 + C_2 x) e^{r_1 x}$$

For another example of the first case, take the equation $y'' - k^2 y = 0$. Here the characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions.

Example 2.2.1

Solve

$$y'' - k^2 y = 0.$$

Solution

The characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1 e^{kx} + C_2 e_{-kx}.$$

Since $\cosh s = \frac{e^s + e^{-s}}{2}$ and $\sinh s = \frac{e^s - e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Example 2.2.2:

Find the general solution of

$$y'' - 8y' + 16y = 0$$

Solution

The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = (C_1 + C_2 x) e^{4x} = C_1 e^{4x} + C_2 x e^{4x}$$

Exercise 2.2.2: Linear Independence

Check that e^{4x} and xe^{4x} *are linearly independent.*

Answer

That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0$$

We should note that in practice, doubled root rarely happens. If coefficients are picked truly randomly we are very unlikely to get a doubled root.

Let us give a short proof for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{r}2^{x}-e^{x}1^{x}}{r_2-r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case.

2.2.2: 2.2.2 Complex numbers and Euler's formula

It may happen that a polynomial has some complex roots. For example, the equation $r^2+1=0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers, (a,b). We can think of a complex number as a point in the plane. We add complex numbers in the straightforward way, $(a,b)+(c,d)=(a+c,b+d)$. We define multiplication by

$$(a,b)\times(c,d)\stackrel{\text{def}}{=}(ac-bd,ad+bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0,1)\times(0,1)=(-1,0)$.

Generally we just write (a,b) as $(a+ib)$, and we treat i as if it were an unknown. We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2=-1$. So whenever we see i^2, we replace it by -1. The numbers i and $-i$ are the two roots of $r^2+1=0$.

Note that engineers often use the letter j instead of i for the square root of -1. We will use the mathematicians' convention and use i.

Exercise 2.2.3

Make sure you understand (that you can justify) the following identities:

a. $i^2=-1, i^3=-1, i^4=1$,
b. $\frac{1}{i}=-i$,
c. $(3-7i)(-2-9i)=\cdots=-69-13i$,
d. $(3-2i)(3+2i)=3^2-(2i)^2=3^2+2^2=13$,
e. $\frac{1}{3-2i}=\frac{1}{3-2i}\frac{3+2i}{3+2i}=\frac{3+2i}{13}=\frac{3}{13}+\frac{2}{13}i$.

We can also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the very important property: $e^{x+y}=e^xe^y$. This means that $e^{a+ib}=e^ae^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called Euler's formula.

Theorem 2.2.2

Euler's Formula

$$e^{i\theta}=\cos\theta+i\sin\theta \quad \textit{and} \quad e^{-i\theta}=\cos\theta-i\sin\theta$$

In other words, $e^{a+ib}=e^a(\cos(b)+i\sin(b))=e^a\cos(b)+ie^a\sin(b)$.

Exercise 2.2.4:

Using Euler's formula, check the identities:

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Exercise 2.2.5

Double angle identities: Start with $e^{i(2\theta)} = \left(e^{i\theta}\right)^2$. Use Euler on each side and deduce:

Answer

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

For a complex number $a+ib$ we call a the real part and b the imaginary part of the number. Often the following notation is used,

$$\mathrm{Re}(a+ib) = a \quad \text{and} \quad \mathrm{Im}(a+ib) = b$$

2.2.3: 2.2.3 Complex roots

Suppose that the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

However, the exponential is now complex valued. We would need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$y_1 = e^{(\alpha+i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x}$$

Then note that

$$\begin{aligned} y_1 &= e^{ax}\cos(\beta x) + ie^{ax}\sin(\beta x) \\ y_2 &= e^{ax}\cos(\beta x) - ie^{ax}\sin(\beta x) \end{aligned} \tag{2.2.4}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{ax}\cos(\beta x) \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{ax}\sin(\beta x) \end{aligned} \tag{2.2.5}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). Therefore, we have the following theorem.

Theorem 2.2.3

For the homegneous second order ODE

$$ay'' + by' + cy = 0$$

If the characteristic equation has the roots $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1 e^{ax}\cos(\beta x) + C_2 e^{ax}\sin(\beta x)$$

Example 2.2.3

Find the general solution of $y'' + k^2 y = 0$, for a constant $k > 0$.

Solution

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$ and by the theorem we have the general solution

$$y = C_1 \cos(kx) + C_2 \sin(kx)$$

Example 2.2.4

Find the solution of $y'' - 6y' + 13y = 0, y(0) = 0, y'(0) = 10$.

Solution

The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r-3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By the theorem we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x)$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1$$

Hence $C_1 = 0$ and $y = C_2 e^{3x} \sin(2x)$. We differentiate

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x)$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. Hence the solution we are seeking is

$$y = 5e^{3x} \sin(2x)$$

2.2.4: Footnotes

[1] Making an educated guess with some parameters to solve for is such a central technique in differential equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for “initial placement of a tool at a work piece.” Yes, the Germans have a word for that.

2.3: Higher order linear ODEs

Equations that appear in applications tend to be second order, although higher order equations do appear from time to time. Hence, it is a generally assumed that the world is "second order" from a modern physics perspective. The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. The important concept of linear independence is somewhat more complicated when more than two functions are involved.

For higher order constant coefficient ODEs, the methods are also somewhat harder to apply, but we will not dwell on these complications. We can always use the methods for systems of linear equations to solve higher order constant coefficient equations. So let us start with a general homogeneous linear equation:

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x) \tag{2.3.1}$$

Theorem 2.3.1

Superposition

Suppose $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation (Equation 2.3.1). Then

$$y(x)=C_1y_1(x)+C_2y_2(x)+\ldots+C_ny_n(x)$$

also solves Equation 2.3.1 for arbitrary constants $C_1, \ldots. C_n$.

In other words, a linear combination of solutions to Equation 2.3.1 is also a solution to Equation 2.3.1. We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 2.3.2

Existence and Uniqueness

Suppose p_o through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a)=b_0, \quad y'(a)=b_1, \quad \ldots, \quad y^{(n-1)}(a)=b_{n-1}$$

2.3.1: Linear Independence

When we had two functions y_1 and y_2 we said they were linearly independent if one was not the multiple of the other. Same idea holds for n functions. In this case it is easier to state as follows. The functions $y_1, y_2, \ldots, y_n$ are linearly independent if

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0$$

has only the trivial solution $c_1=c_2=\cdots=c_n=0$, where the equation must hold for all x. If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are linearly dependent.

Example 2.3.1

Show that e^x, e^{2x}, and e^{3x} are linearly independent functions.

Solution

Let us give several ways to show this fact. Many textbooks introduce Wronskians, but that is really not necessary to solve this example. Let us write down

$$c_1e^x+c_2e^{2x}+c_3e^{3x}=0$$

We use rules of exponentials and write $z=e^x$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0$$

The left hand side is a third degree polynomial in z. It can either be identically zero, or it can have at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

This equation has to hold for all x. What we could do is divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0$$

As the equation is true for all x, let $x \to \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0$$

Rinse, repeat!

How about yet another way. We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2 e^x + 2 c_3 e^{2x} = 0$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2 c_3 e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example 2.3.2

On the other hand, the functions e^x, e^{-x} and $\cosh x$ are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad 2\cosh x - e^x - e^{-x} = 0$$

2.3.2: Constant Coefficient Higher Order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order we need to find n linearly independent solutions. It is best seen by example.

Example 2.3.3: Third order ODE with Constant Coefficients

Find the general solution to

$$y''' - 3'' - y' + 3y = 0 \tag{2.3.2}$$

Solution

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3 e^{rx}}_{y'''} - 3\underbrace{r^2 e^{rx}}_{y''} - \underbrace{r e^{rx}}_{y'} + 3\underbrace{e^{rx}}_{y} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials, but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \dots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15; we plug in $r = 0$ and get 3. That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1, then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = -1$, 1, or 0. These are easy to compute. Our polynomial happens to have two such roots, $r_1 = -1$ and $r_2 = 1$ and. There should be three roots and the last root is reasonably easy to find. The constant term in a monic [1] polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3$$

You should check that $r_3 = 3$ really is a root. Hence we know that e^{-x}, e^x, and e^{3x} are solutions to (2.3.2). They are linearly independent as can easily be checked, and there are three of them, which happens to be exactly the number we need. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 + C_3 \\ 2 &= y'(0) = -C_1 + C_2 + 3C_3 \\ 3 &= y''(0) = C_1 + C_2 + 9C_3 \end{aligned} \tag{2.3.3}$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra, see Section 3.2 or Appendix A. For now we note that the solution is $C_1 = -\frac{1}{4}$, $C_2 = 1$, and $C_3 = \frac{1}{4}$. The specific solution to the ODE is

$$y = -\frac{1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, xe^{rx}, x^2 e^{rx}, \dots, x^{k-1} e^{rx}$$

We take a linear combination of these solutions to find the general solution.

Example 2.3.4

Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0$$

Solution

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2 + C_3 x^2)e^x}_{\text{terms coming from r = 1}} + \underbrace{C_4}_{\text{from r = 0}}$$

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1 x + \cdots + C_{k-1} x^{k-1})e^{ax} \cos(\beta x) + (D_0 + D_1 x + \cdots + D_{k-1} x^{k-1})e^{ax} \sin(\beta x)$$

where $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}$ are arbitrary constants.

Example 2.3.5

Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0$$

Solution

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0 \\ (r^2 - 2r + 2)^2 &= 0 \\ ((r-1)^2 + 1)^2 &= 0 \end{aligned} \tag{2.3.4}$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2 x)e^x \cos x + (C_3 + C_4 x)e^x \sin x$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

2.3.3: Footnotes

[1] The word monic means that the coefficient of the top degree r^d, in our case r^3, is 1.

2.3.4: Outside Links

- After reading this lecture, it may be good to try Project III from the IODE website: www.math.uiuc.edu/iode/.

2.4: Mechanical Vibrations

Let us look at some applications of linear second order constant coefficient equations.

2.4.1: Some examples

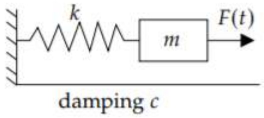

Figure 2.4.1

Our first example is a mass on a spring. Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps there is a damper connected).

Let x be the displacement of the mass ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

$$mx'' + cx' + kx = F(t)$$

This is a linear second order constant coefficient ODE. We set up some terminology about this equation. We say the motion is

i. forced, if $F \not\equiv 0$ (if F is not identically zero),
ii. unforced or free, if $F \equiv 0$ (if F is identically zero),
iii. damped, if $c > 0$, and
iv. undamped, if $c = 0$.

This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.

Here is an example for electrical engineers. Consider the pictured RLC circuit. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge in coulombs on the capacitor and $I(t)$ be the current in the circuit. The relation between the two is $Q' = I$. By elementary principles we find $LI' + RI + \frac{Q}{C} = E$. We differentiate to get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

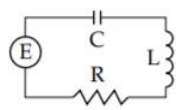

Figure 2.4.2

This is a nonhomogeneous second order constant coefficient linear equation. As L, R, and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.

Our next example behaves like a mass and spring system only approximately. Suppose a mass m hangs on a pendulum of length L. We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

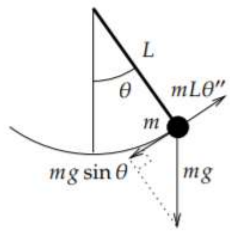

Figure 2.4.3

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m. So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg\sin\theta$ in the opposite direction. So $mL\theta'' = -mg\sin\theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin\theta \approx \theta$. This can be seen by looking at the graph. In Figure $2.4.4$ we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin\theta$ and θ are almost the same.

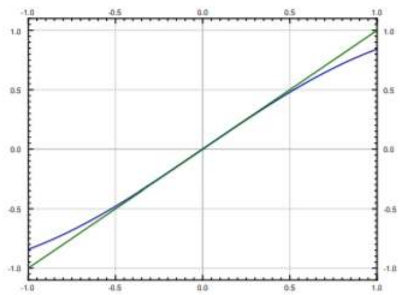

Figure $2.4.4$: The graphs of $\sin\theta$ and θ (in radians).

Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

2.4.2: Free Undamped Motion

In this section we will only consider free or unforced motion, as we cannot yet solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. We have the equation

$$mx'' + kx = 0$$

If we divide by m and let $w_0 = \sqrt{\frac{k}{m}}$, then we can write the equation as

$$x'' + w_0^2 x = 0$$

The general solution to this equation is

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

By a trigonometric identity, we have that for two different constants C and γ, we have

$$A\cos(w_0 t) + B\sin(w_0 t) = C\cos(w_0 t - \gamma)$$

It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = \dfrac{B}{A}$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C\cos(w_0 t - \gamma)$.

Exercise 2.4.1

Justify the above identity and verify the equations for C and γ. Hint: Start with $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and multiply by C. Then think what should α and β be.

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have very nice interpretation. We look at the form of the solution

$$x(t) = C\cos(w_0 t - \gamma)$$

We can see that the amplitude is C, w_0 is the (angular) frequency, and γ is the so-called phase shift. The phase shift just shifts the graph left or right. We call w_0 the natural (angular) frequency. This entire setup is usually called simple harmonic motion.

Let us pause to explain the word angular before the word frequency. The units of w_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because we know one cycle is 2π radians, the usual frequency is given by $\dfrac{w_0}{2\pi}$. It is simply a matter of where we put the constant 2π, and that is a matter of taste.

The period of the motion is one over the frequency (in cycles per unit time) and hence $\dfrac{2\pi}{w_0}$. That is the amount of time it takes to complete one full oscillation.

Example 2.4.1

Suppose that $m = 2kg$ and $k = 8\dfrac{N}{m}$. The whole mass and spring setup is sitting on a truck that was traveling at $1\dfrac{m}{s}$. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at $1\dfrac{m}{s}$, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation and what is the amplitude. The units are the mks units (meters-kilograms-seconds).

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at $1\dfrac{m}{s}$. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1$$

We can directly compute $w_0 = \sqrt{\dfrac{k}{m}} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $\dfrac{2}{2\pi} = \dfrac{1}{\pi} \approx 0.318$.

The general solution is

$$x(t) = A\cos(2t) + B\sin(2t)$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5)\sin(2t) + 2B\cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5\cos(2t) + 0.5\sin(2t)$$

A plot of $x(t)$ is shown in Figure 2.4.5.

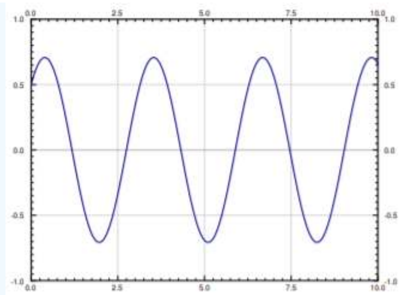

Figure 2.4.5: Simple undamped oscillation.

In general, for free undamped motion, a solution of the form

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = w_0 B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B. In the example, we have already found the amplitude C. Let us compute the phase shift. We know that $\tan\gamma = \frac{B}{A} = 1$. We take the arctangent of 1 and get approximately 0.785. We still need to check if this γ is in the correct quadrant (and add π to γ if it is not). Since both A and B are positive, then γ should be in the first quadrant, and 0.785 radians really is in the first quadrant.

Note

Many calculators and computer software do not only have the atan function for arctangent, but also what is sometimes called `atan2`. This function takes two arguments, B and A, and returns a γ in the correct quadrant for you.

2.4.3: Free Damped Motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + cx' + kx = 0$$

as

$$x'' + 2px' + w_0^2 x = 0$$

where

$$w_0 = \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}$$

The characteristic equation is

$$r^2 + 2pr + w_0^2 = 0$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - w_0^2}$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - w_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}$$

The sign of $p^2 - w_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

2.4.3.1: Overdamping

When $c^2 - 4km > 0$, we say the system is overdamped. In this case, there are two distinct real roots r_1 and r_2. Notice that both roots are negative. As $\sqrt{p^2 - w_0^2}$ is always less than P, then $-P \pm \sqrt{P^2 - w_0^2}$ is negative.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since r_1, r_2 are negative, $x(t) \to 0$ as $t \to \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions (Figure $2.4.6$).

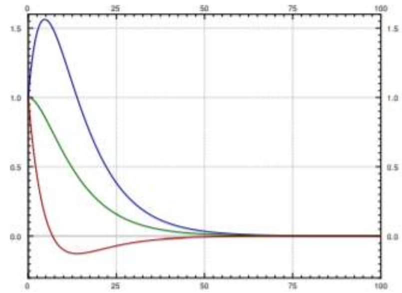

Figure $2.4.6$: Overdamped motion for several different initial conditions.

Do note that no oscillation happens. In fact, the graph will cross the x axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph will never cross the x axis, as is evident from the sample graphs.

Example $2.4.2$

Suppose the mass is released from rest. That is $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2}(r_1 e^{r_2 t} - r_2 e^{r_1 t})$$

It is not hard to see that this satisfies the initial conditions.

2.4.3.2: Critical damping

When $c^2 - 4km = 0$, we say the system is critically damped. In this case, there is one root of multiplicity 2 and this root is $-P$. Therefore, our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

2.4.3.3: Underdamping

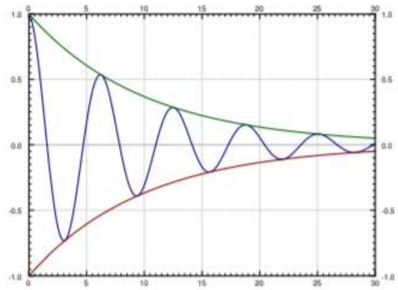

Figure 2.4.7: Underdamped motion with the envelope curves shown.

When $c^2 - 4km < 0$, we say the system is underdamped. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - w_0^2} \\ &= -p \pm \sqrt{-1}\sqrt{w_0^2 - p^2} \\ &= -p \pm i w_1 \end{aligned} \tag{2.4.1}$$

where $w_1 = \sqrt{w_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt}(A\cos(w_1 t) + B\sin(w_1 t)$$

or

$$x(t) = Ce^{-pt}\cos(w_1 t - \gamma)$$

An example plot is given in Figure $2.4.7$. Note that we still have that $x(t) \to 0$ as $t \to \infty$.

In the figure we also show the *envelope curves* Ce^{-pt} and $-Ce^{pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example if you are bungee jumping, you are really interested in computing the envelope curve so that you do not hit the concrete with your head.

The phase shift γ just shifts the graph left or right but within the envelope curves (the envelope curves do not change if γ changes).

Finally note that the angular pseudo-frequency[1] (we do not call it a frequency since the solution is not really a periodic function) w_1 becomes smaller when the damping c (and hence P) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we want w_1 to approach 0.

On the other hand when c becomes smaller, w_1 approaches w_0 (w_1 is always smaller than w_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence P) goes to 0.

2.4.4: Footnotes

[1] We do not call ω_1 a frequency since the solution is not really a periodic function.

2.5: Nonhomogeneous Equations

2.5.1: Solving Nonhomogeneous Equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? For example, the equations for forced mechanical vibrations. That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1 \tag{2.5.1}$$

We will write $Ly = 2x + 1$ when the exact form of the operator is not important. We solve (Equation 2.5.1) in the following manner. First, we find the general solution y_c to the associated homogeneous equation

$$y'' + 5y' + 6y = 0 \tag{2.5.2}$$

We call y_c the *complementary solution*. Next, we find a single particular solution y_p to (2.5.1) in some way. Then

$$y = y_c + y_p$$

is the general solution to (2.5.1). We have $Ly_c = 0$ and $Ly_p = 2x + 1$. As L is a linear operator we verify that y is a solution, $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$. Let us see why we obtain the general solution.

Let y_p and $\tilde{y}_p$ be two different particular solutions to (2.5.1). Write the difference as $w = y_p - \tilde{y}_p$. Then plug w into the left hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0$$

Using the operator notation the calculation becomes simpler. As L is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0$$

So $w = y_p - \tilde{y}_p$ is a solution to (2.5.2), that is $Lw = 0$. Any two solutions of (2.5.1) differ by a solution to the homogeneous equation (2.5.2). The solution $y = y_c + y_p$ includes all solutions to (2.5.1), since y_c is the general solution to the associated homogeneous equation.

Theorem 2.5.1

Let $Ly = f(x)$ be a linear ODE (not necessarily constant coefficient). Let y_c be the complementary solution (the general solution to the associated homogeneous equation $Ly = 0$) andlet y_p be any particular solution to $Ly = f(x)$. Then the general solution to $Ly = f(x)$ is

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we will have to choose to satisfy the initial conditions may be different, but it is the same solution.

2.5.1.1: Undetermined Coefficients

The trick is to somehow, in a smart way, guess one particular solution to (2.5.1). Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B$$

We plug in to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) \\ &= 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B) \end{aligned} \tag{2.5.3}$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore, $A = \frac{1}{3}$ and $B = -\frac{1}{9}$. That means $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x - 1}{9}$. Solving the complementary problem (exercise!) we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

Hence the general solution to (2.5.1) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}$$

Now suppose we are further given some initial conditions. For example, $y(0) = 0$ and $y'(0) = \frac{1}{3}$. First find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + \frac{1}{3}$. Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}$$

We solve to get $C_1 = \frac{1}{3}$ and $C_2 = -\frac{2}{9}$. The particular solution we want is

$$y(x) = \frac{1}{3} e^{-2x} - \frac{2}{9} e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}$$

Exercise 2.5.1

Check that y really solves the equation (2.5.1)and the given initial conditions.

Note

A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will not work. You need to first compute $y = y_c + y_p$ and only then solve for the constants using the initial conditions.

A right hand side consisting of exponentials, sines, and cosines can be handled similarly. For example,

$$y'' + 2y' + 2y = \cos(2x)$$

Let us find some y_p. We start by guessing the solution includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x)$$

We plug y_p into the equation and we get

$$\underbrace{-4A\cos(2x) - 4B\sin(2x)}_{y_p''} + 2\underbrace{\left(-2A\sin(2x) + 2B\cos(2x)\right)}_{y_p'} \\ +2\underbrace{\left(A\cos(2x) + 2B\sin(2x)\right)}_{y_p} = \cos(2x),$$

The left hand side must equal to right hand side. We group terms and we get that $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = \frac{-1}{10}$ and $B = \frac{1}{5}$. So

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}$$

Similarly, if the right hand side contains exponentials we try exponentials. For example, for

$$Ly = e^{3x}$$

we will try $y = Ae^{3x}$ as our guess and try to solve for A.

When the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right hand side. For example,

$$Ly = (1 + 3x^2)e^{-x}\cos(\pi x)$$

For this equation, we will guess

$$y_p = (A + Bx + Cx^2)e^{-x}\cos(\pi x) + (D + Ex + Fx^2)e^{-x}\sin(\pi x)$$

We will plug in and then hopefully get equations that we can solve for $A, B, C, D, E$and F. As you can see this can make for a very long and tedious calculation very quickly.

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, suppose we have

$$y'' - 9y = e^{3x}$$

We would love to guess $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}$$

There is no way we can choose A to make the left hand side be e^{3x}. The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is first we compute y_c (solution to $Ly = 0$)

$$y_c = C_1e^{-3x} + C_2e^{3x}$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y = Axe^{3x}$ and notice there is no duplication anymore. Let us try. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = \frac{1}{6}$. We can now write the general solution as

$$y = y_c + y_p = C_1e^{-3x} + C_2e^{3x} + \frac{1}{6}xe^{3x}$$

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}$$

The complementary solution is $y_c = C_1e^{3x} + C_2xe^{3x}$. Guessing $y = Axe^{3x}$ would not get us anywhere. In this case we want to guess $y_p = Ax^2e^{3x}$. Basically, we want to multiply our guess by x until all duplication is gone. But no more! Multiplying too many times will not work.

Finally, what if the right hand side has several terms, such as

$$Ly = e^{2x} + \cos x$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$.

2.5.1.2: Variation of Parameters

The method of undetermined coefficients will work for many basic problems that crop up. But it does not work all the time. It only works when the right hand side of the equation $Ly = f(x)$ has only finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x$$

Note that each new derivative of $\tan x$ looks completely different and cannot be written as a linear combination of the previous derivatives. If we start differentiating $\tan x$, we get

$$\sec^2 x, \quad 2\sec^2 x\, \tan x, \quad 4\sec^2 x\, \tan^2 x + 2\sec^4 x,$$
$$8\sec^2 x\, \tan^3 x + 16\sec^4 x\, \tan x, \quad 16\sec^2 x\, \tan^4 x + 88\sec^4 x\tan^2 x + 16\sec^6 x, \quad \ldots$$

This equation calls for a different method. We present the method of variation of parameters, which will handle any equation of the form $Ly = f(x)$, provided we can solve certain integrals. For simplicity, we restrict ourselves to second order constant coefficient equations, but the method works for higher order equations just as well (the computations become more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x$$

First we find the complementary solution (solution to $Ly_c = 0$). We get $y_c = C_1 y_1 + C_2 y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$. To find a particular solution to the nonhomogeneous equation we try

$$y_p = y = u_1 y_1 + u_2 y_2$$

where u_1 and u_2 are functions and not constants. We are trying to satisfy $Ly = \tan x$. That gives us one condition on the functions u_1 and u_2. Compute (note the product rule!)

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that $(u_1' y_1 + u_2' y_2) = 0$. This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2' \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2'') \end{aligned} \tag{2.5.4}$$

Since y_1 and y_2 are solutions to $y'' + y = 0$, we know that $y_1'' = -y_1$ and $y_2'' = -y_2$. (Note: If the equation was instead $y'' + p(x)y' + q(x)y = 0$ we would have $y_i'' = -p(x)y_i' - q(x)y_i$.) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2)$$

We have $(u_1 y_1 + u_2 y_2) = y$ and so

$$y'' = (u_1' y_1' + u_2' y_2') - y$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'$$

For y to satisfy $Ly = f(x)$ we must have $f(x) = u_1' y_1' + u_2' y_2'$.

So what we need to solve are the two equations (conditions) we imposed on u_1 and u_2

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned} \tag{2.5.5}$$

We can now solve for u_1' and u_2' in terms of $f(x), y_1$ and y_2. We will always get these formulas for any $Ly = f(x)$, where $Ly = y'' + p(x)y' + q(x)y$. There is a general formula for the solution we can just plug into, but it is better to just repeat what we do below. In our case the two equations become

$$\begin{aligned} u_1' \cos(x) + u_2' \sin(x) &= 0 \\ -u_1' \sin(x) + u_2' \cos(x) &= \tan(x) \end{aligned} \tag{2.5.6}$$

Hence

$$\begin{aligned} u_1' \cos(x)\sin(x) + u_2' \sin^2(x) &= 0 \\ -u_1' \sin(x)\cos(x) + u_2' \cos^2(x) &= \tan(x)\cos(x) = \sin(x) \end{aligned} \tag{2.5.7}$$

And thus

$$\begin{aligned} u_2'(\sin^2(x) + \cos^2(x)) &= \sin(x) \\ u_2' &= \sin(x) \\ u_1' &= \frac{-\sin^2(x)}{\cos(x)} = -\tan(x)\sin(x) \end{aligned} \tag{2.5.8}$$

Now we need to integrate u_1' and u_2' to get u_1 and u_2.

$$\begin{aligned} u_1 &= \int u_1' dx = \int -\tan(x)\sin(x) dx = \frac{1}{2}\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \sin(x) \\ u_2 &= \int u_2' dx = \int \sin(x) dx = -\cos(x) \end{aligned} \tag{2.5.9}$$

So our particular solution is

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 = \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \cos(x)\sin(x) - \cos(x)\sin(x) \\ &= \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| \end{aligned} \tag{2.5.10}$$

The general solution to $y'' + y = \tan x$ is, therefore,

$$y = C_1\cos(x) + C_2\sin(x) + \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right|$$

- - Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

2.6: Forced Oscillations and Resonance

Let us consider to the example of a mass on a spring. We now examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t)$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

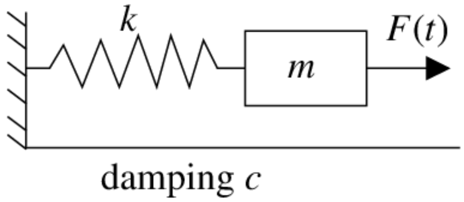

Figure 2.6.1

What we are interested in is periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in Chapter 4, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

2.6.1: Undamped Forced Motion and Resonance

First let us consider undamped $c = 0$ motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t)$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the *natural frequency* (angular), which is the frequency at which the system "wants to oscillate" without external interference.

Let us suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. Note that we need not have sine in our trial solution as on the left hand side we will only get cosines anyway. If you include a sine it is fine; you will find that its coefficient will be zero.

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

or written another way

$$x = C \cos(\omega_0 t - y) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Hence it is a superposition of two cosine waves at different frequencies.

Example 2.6.1

Take

$$0.5x'' + 8x = 10\cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0$$

Let us compute. First we read off the parameters: $\omega = \pi, \omega_0 = \sqrt{\frac{8}{0.5}} = 4, F_0 = 10, m = 0.5$. The general solution is

$$x = C_1\cos(4t) + C_2\sin(4t) + \frac{20}{16-\pi^2}\cos(\pi t)$$

Solve for C_1 and C_2 using the initial conditions. It is easy to see that $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$$

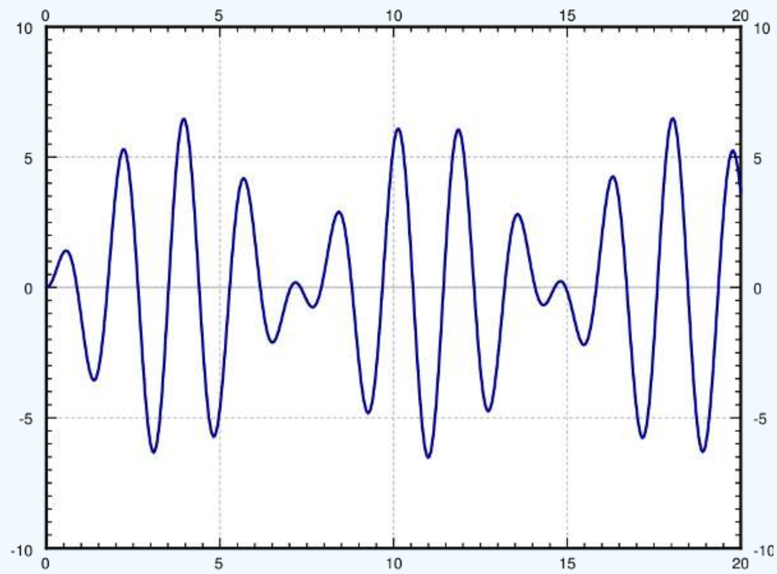

Figure 2.6.2: Graph of $\frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$.

Notice the “beating” behavior in Figure 2.6.2. First use the trigonometric identity

$$2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2}) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16-\pi^2}(2\sin(\frac{4-\pi}{2}t)\sin(\frac{4+\pi}{2}t))$$

Notice that x is a high frequency wave modulated by a low frequency wave.

Now suppose that $\omega_0 = \omega$. Obviously, we cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we need to try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we do need the sine term since the second derivative of $t\cos(\omega t)$ does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m}\cos(\omega t)$$

Plugging x_p into the left hand side we get

$$2B\omega\cos(\omega t) - 2A\omega\sin(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}t\sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t)$$

The important term is the last one (the particular solution we found). We can see that this term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure $2.6.3$ we see the graph with $C_1 = C_2 = 0, F_0 = 2, m = 1, \omega = \pi$.

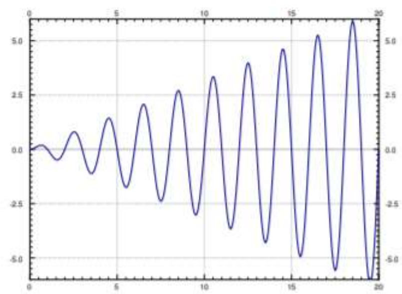

Figure 2.6.3: Graph of $\frac{1}{\pi} t \sin(\pi t)$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called resonance or perhaps pure resonance. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the “correct frequency”? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play.[1]

2.6.2: Damped Forced Motion and Practical Resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{2.6.1}$$

for some $c > 0$. We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We replace equation (2.6.1) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as we have seen before. That is,

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & \text{if } c^2 > 4km, \\ C_1 e^{pt} + C_2 t e^{-pt}, & \text{if } c^2 = 4km, \\ e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)), & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we can see that $x_c(t) \to 0$ as $t \to \infty$. Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$((\omega_0^2-\omega^2)B-2\omega pA)\sin(\omega t)+((\omega_0^2-\omega^2)A+2\omega pB)\cos(\omega t)=\frac{F_0}{m}\cos(\omega t)$$

We get that

$$A=\frac{(\omega_0^2-\omega^2)F_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}$$

$$B=\frac{2\omega pF_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}$$

We also compute $C=\sqrt{A^2+B^2}$ to be

$$C=\frac{F_0}{m\sqrt{(2\omega p)^2+\left(\omega_0^2-\omega^2\right)^2}}$$

Thus our particular solution is

$$x_P=\frac{(\omega_0^2-\omega^2)F_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}\cos(\omega t)+\frac{2\omega pF_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}\sin(\omega t)$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega\neq\omega_0$)

$$\tan\gamma=\frac{B}{A}=\frac{2\omega p}{\omega_0^2-\omega^2}$$

Hence we have

$$x_p=\frac{F_0}{m\sqrt{(2\omega p)^2+\left(\omega_0^2-\omega^2\right)^2}}\cos(\omega t-\gamma)$$

If $\omega=\omega_0$ we see that $A=0, B=C=\frac{F_0}{2m\omega p}$, and $\gamma=\frac{\pi}{2}$.

The exact formula is not as important as the idea. Do not memorize the above formula, you should instead remember the ideas involved. For different forcing function F, you will get a different formula for x_p. So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we call $x_c$the transient solution and denote it by x_{tr}. We call the x_p we found above the steady periodic solution and denote it by x_{sp}. The general solution to our problem is

$$x=x_c+x_p=x_{tr}+x_{sp}$$

We note that $x_c=x_{tr}$ goes to zero as $t\to\infty$, as all the terms involve an exponential with a negative exponent. Hence for large t, the effect of x_{tr} is negligible and we will essentially only see x_{sp}. Hence the name transient. Notice that x_{sp} involves no arbitrary constants, and the initial conditions will only affect x_{tr}. This means that the effect of the initial conditions will be negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. See Figure $2.6.4$ for a graph of different initial conditions.

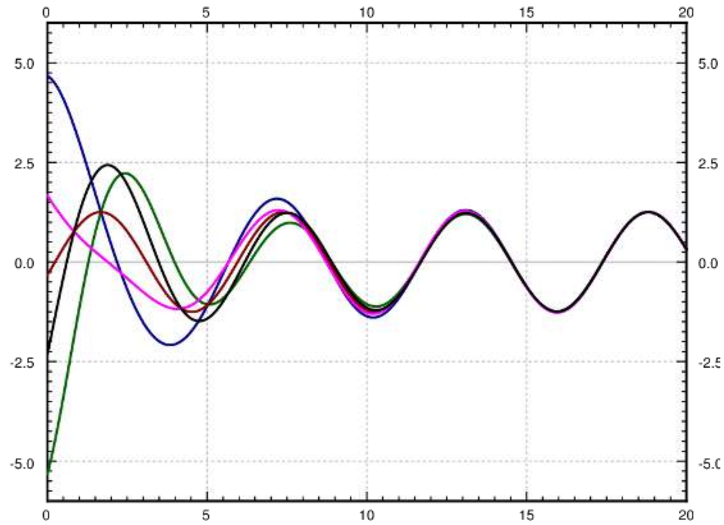

Figure 2.6.4: Solutions with different initial conditions for parameters $k=1, m=1, F_0=1, c=0.7,$ and $\omega=1.1$.

Notice that the speed at which x_{tr} goes to zero depends on P (and hence c). The bigger P is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This agrees with the observation that when $c=0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. What we will look at however is the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed) we can find its maximum. We call the ω that achieves this maximum the practical resonance frequency. We call the maximal amplitude $C(\omega)$ the practical resonance amplitude. Thus when damping is present we talk of practical resonance rather than pure resonance. A sample plot for three different values of c is given in Figure 2.6.5. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

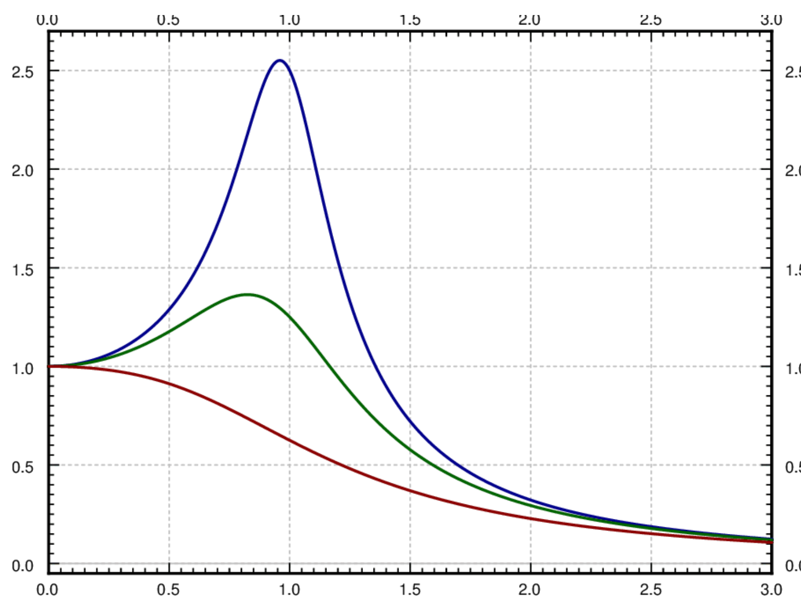

Figure 2.6.5: Graph of $C(\omega)$ showing practical resonance with parameters $k=1, m=1, F_0=1$. The top line is with $c=0.4$, the middle line with $c=0.8$, and the bottom line with $c=1.6$.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega)=\frac{-4\omega(2p^2+\omega^2-\omega_0^2)F_0}{m\left((2\omega p)^2+(\omega_0^2-\omega^2)\right)^{3/2}}$$

This is zero either when $\omega=0$ or when $2p^2+\omega^2-\omega_0^2=0$. In other words, $C'(\omega)=0$ when

$$\omega = \sqrt{\omega_0^2 - 2p^2} \text{ or } \omega = 0$$

It can be shown that if $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal, note that in this case $C'(\omega) > 0$ for small ω). If $\omega = 0$ is the maximum, then essentially there is no practical resonance since we assume that $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence P) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega \to \infty$, then $\omega \to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

2.6.3: Footnotes

[1]K. Billah and R. Scanlan, Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks, American Journal of Physics, 59(2), 1991, 118–124, http://www.ketchum.org/billah/Billah-Scanlan.pdf

2.E: Higher order linear ODEs (Exercises)

These are homework exercises to accompany Libl's "Differential Equations for Engineering" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

2.E.1: 2.1: Second order linear ODEs

Exercise $2.E.2.1.1$

Show that $y = e^x$ and $y = e^{2x}$ are linearly independent.

Exercise $2.E.2.1.2$

Take $y'' + 5y = 10x + 5$. Find (guess!) a solution.

Exercise $2.E.2.1.3$

Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1 = f(x)$ and y_2 is a solution to $Ly_2 = g(x)$ (same linear operator L). Show that $y = y_1 + y_2$ solves $Ly = f(x) + g(x)$.

Exercise $2.E.2.1.4$

For the equation $x^2 y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y = x'$.

Equations of the form $ax^2 y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy-Euler equations*. They are solved by trying $y = x^r$ and solving for r (we can assume that $x \geqslant 0$ for simplicity).

Exercise $2.E.2.1.5$

Suppose that $(b-a)^2 - 4ac > 0$.

a. Find a formula for the general solution of $ax^2 y'' + bxy' + cy = 0$. Hint: Try $y = x^r$ and find a formula for r.
b. What happens when $(b-a)^2 - 4ac = 0$ or $(b-a)^2 - 4ac < 0$?

We will revisit the case when $(b-a)^2 - 4ac < 0$ later.

Exercise $2.E.2.1.6$

Same equation as in Exercise $2.E.2.1.5$. Suppose $(b-a)^2 - 4ac = 0$. Find a formula for the general solution of $ax^2 y'' + bxy' + cy = 0$. Hint: Try $y = x^r \ln x$ for the second solution.

Exercise $2.E.2.1.7$: reduction of order

Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. Show that

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx \tag{2.E.1}$$

is also a solution.

Note: If you wish to come up with the formula for reduction of order yourself, start by trying $y_2(x) = y_1(x)v(x)$. Then plug y_2 into the equation, use the fact that y_1 is a solution, substitute $w = v'$, and you have a first order linear equation in w. Solve for w and then for v. When solving for w, make sure to include a constant of integration. Let us solve some famous equations using the method.

Exercise $2.E.2.1.8$: Chebyshev’s equation of order 1

Take $(1-x^2)y''-xy'+y=0$.

a. Show that $y=x$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise $2.E.2.1.9$: Hermite’s equation of order 2

Take $y''-2xy'+4y=0$.

a. Show that $y=1-2x^2$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise $2.E.2.1.10$

Are $\sin(x)$ and e^x linearly independent? Justify.

Answer

Yes. To justify try to find a constant A such that $\sin(x)=Ae^x$ for all x.

Exercise $2.E.2.1.11$

Are e^x and e^{x+2} linearly independent? Justify.

Answer

No. $e^{x+2}=e^2e^x$.

Exercise $2.E.2.1.12$

Guess a solution to $y''+y'+y=5$.

Answer

$y=5$

Exercise $2.E.2.1.13$

Find the general solution to $xy''+y'=0$. Hint: Notice that it is a first order ODE in y'.

Answer

$y=C_1\ln(x)+C_2$

Exercise $2.E.2.1.14$

Write down an equation (guess) for which we have the solutions e^x and e^{2x}. Hint: Try an equation of the form $y''+Ay'+By=0$ for constants A and B, plug in both e^x and e^{2x} and solve for A and B.

Answer

$y''-3y'+2y=0$

2.E.2: 2.2: Constant coefficient second order linear ODEs

Exercise $2.E.2.2.1$

Find the general solution of $2y'' + 2y' - 4y = 0$.

Exercise $2.E.2.2.2$

Find the general solution of $y'' + 9y' - 10y = 0$.

Exercise $2.E.2.2.3$

Solve $y'' - 8y' + 16y = 0$ for $y(0) = 2, y'(0) = 0$.

Exercise $2.E.2.2.4$

Solve $y'' + 9y' = 0$ for $y(0) = 1, y'(0) = 1$.

Exercise $2.E.2.2.5$

Find the general solution of $2y'' + 50y = 0$.

Exercise $2.E.2.2.6$

Find the general solution of $y'' + 6y' + 13y = 0$.

Exercise $2.E.2.2.7$

Find the general solution of $y'' = 0$ using the methods of this section.

Exercise $2.E.2.2.8$

The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation $2y' + 3y = 0$ using the methods of this section.

Exercise $2.E.2.2.9$

Let us revisit Euler's equations of Exercise $2.E.1$. Suppose now that $(b-a)^2 - 4ac < 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Note that $x^r = e^{r \ln x}$.

Exercise $2.E.2.2.10$

Find the solution to $y'' - (2\alpha)y' + \alpha^2 y = 0$, $y(0) = a$, $y'(0) = b$, where α, a, and b are real numbers.

Exercise $2.E.2.2.11$

Construct an equation such that $y = C_1 e^{-2x} \cos(3x) + C_2 e^{-2x} \sin(3x)$ is the general solution.

Exercise $2.E.2.2.12$

Find the general solution to $y'' + 4y' + 2y = 0$.

Answer

$$y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$$

Exercise $2.E.2.2.13$

Find the general solution to $y''-6y'+9y=0$.

Answer

$y=C_1e^{3x}+C_2xe^{3x}$

Exercise $2.E.2.2.14$

Find the solution to $2y''+y'+y=0, y(0)=1, y'(0)=-2$.

Answer

$y=e^{-x/4}\cos\left((\frac{\sqrt{7}}{4})x\right)-\sqrt{7}e^{-x/4}\sin\left((\frac{\sqrt{7}}{4})x\right)$

Exercise $2.E.2.2.15$

Find the solution to $2y''+y'-3y=0, y(0)=a, y'(0)=b$.

Answer

$y=\frac{2(a-b)}{5}e^{-3x/2}+\frac{3a+2b}{5}e^x$

Exercise $2.E.2.2.16$

Find the solution to $z''(t)=-2z'(t)-2z(t), z(0)=2, z'(0)=-2$.

Answer

$z(t)=2e^{-t}\cos(t)$

Exercise $2.E.2.2.17$

Find the solution to $y''-(\alpha+\beta)y'+\alpha\beta y=0$, $y(0)=a$, $y'(0)=b$, where α, (\beta\), a, and b are real numbers, and $\alpha\neq\beta$.

Answer

$y=\frac{\alpha\beta-b}{\beta-\alpha}e^{\alpha x}+\frac{b-a\alpha}{\beta-\alpha}e^{\beta x}$

Exercise $2.E.2.2.18$

Construct an equation such that $y=C_1e^{3x}+C_2e^{-2x}$ is the general solution.

Answer

$y''-y'-6y=0$

2.E.3: 2.3: Higher order linear ODEs

Exercise $2.E.2.3.1$

Find the general solution for $y''' - y'' + y' - y = 0$.

Exercise $2.E.2.3.2$

Find the general solution for $y^{(4)} - 5y''' + 6y'' = 0$.

Exercise $2.E.2.3.3$

Find the general solution for $y''' + 2y'' + 2y' = 0$.

Exercise $2.E.2.3.4$

Suppose the characteristic equation for a differential equation is $(r-1)^2(r-2)^2 = 0$.

a. Find such a differential equation.
b. Find its general solution.

Exercise $2.E.2.3.5$

Suppose that a fourth order equation has a solution $y = 2e^{4x} x \cos x$.

a. Find such an equation.
b. Find the initial conditions that the given solution satisfies.

Exercise $2.E.2.3.6$

Find the general solution for the equation of Exercise $2.E.2.3.5$.

Exercise $2.E.2.3.7$

Let $f(x) = e^x - \cos x, g(x) = e^x + \cos x$ and $h(x) = \cos x$. Are $f(x), g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $2.E.2.3.8$

Let $f(x) = 0, g(x) = \cos x$, and $h(x) = \sin x$. Are $f(x), g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $2.E.2.3.9$

Are $x, x^2, and x^4$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $2.E.2.3.10$

Are $e^x, xe^x, and x^2 e^x$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $2.E.2.3.11$

Find an equation such that $y = xe^{-2x} \sin(3x)$ is a solution.

Exercise $2.E.2.3.12$

Find the general solution of $y^{(5)} - y^{(4)} = 0$.

Answer

$y = C_1e^x + C_2x^3 + C_3x^2 + C_4x + C_5$

Exercise $2.E.2.3.13$

Suppose that the characteristic equation of a third order differential equation has roots $3 \pm 2i$.

a. What is the characteristic equation?
b. Find the corresponding differential equation.
c. Find the general solution.

Answer

a. $r^3 - 3r^2 + 4r - 12 = 0$
b. $y''' - 3y'' + 4y' - 12y = 0$
c. $y = C_1e^{3x} + C_2\sin(2x) + C_3\cos(2x)$

Exercise $2.E.2.3.14$

Solve $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0, y(0) = 0, y'(0) = 0, y''(0) = 0$.

Answer

$y = 0$

Exercise $2.E.2.3.15$

Are $e^x, e^{x+1}, e^{2x}, \sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

No. $e^1e^x - e^{x+1} = 0$.

Exercise $2.E.2.3.16$

Are $\sin(x), x, x\sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x\sin(x)$ cannot be multiples of each other.)

Exercise $2.E.2.3.17$

Find an equation such that $y = \cos(x), y = \sin(x), y = e^x$ are solutions.

Answer

$y''' - y'' + y' - y = 0$

2.E.4: 2.4: Mechanical Vibrations

Exercise $2.E.2.4.1$

Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$.

a. Set up and find the general solution of the system.
b. Is the system underdamped, overdamped or critically damped?
c. If the system is not critically damped, find a c that makes the system critically damped.

Exercise $2.E.2.4.2$

Do Exercise $2.E.2.4.1$ for $m = 3, k = 12, and c = 12$.

Exercise $2.E.2.4.3$

Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant $4\dfrac{N}{m}$. You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion.

a. You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?
b. Find a formula for the mass m given the frequency w in Hz.

Exercise $2.E.2.4.4$

Suppose we add possible friction to Exercise $2.E.2.4.3$. Further, suppose you do not know the spring constant, but you have two reference weights 1 kg and 2 kg to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kg weight you measured 1.1 Hz, for the 2 kg weight you measured 0.8 Hz.

a. Find k (spring constant) and c (damping constant).
b. Find a formula for the mass in terms of the frequency in Hz. Note that there may be more than one possible mass for a given frequency.
c. For an unknown object you measured \(0.2\text{ Hz}\), what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.

Exercise $2.E.2.4.5$

Suppose you wish to measure the friction a mass of 0.1 kg experiences as it slides along a floor (you wish to find c). You have a spring with spring constant $k = 5\dfrac{N}{m}$. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency 1 Hz. What is the friction?

Exercise $2.E.2.4.6$

A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time $t = 0$ the mass is kicked and starts traveling at 2 meters per second. How large does k have to be to so that the mass does not go further than 3 meters from the rest position?

Answer

$k = \frac{8}{9}$ (and larger)

Exercise $2.E.2.4.7$

Suppose we have an RLC circuit with a resistor of 100 miliohms (0.1 ohms), inductor of inductance of 50 millihenries (0.05 henries), and a capacitor of 5 farads, with constant voltage.

a. Set up the ODE equation for the current I.
b. Find the general solution.

c. Solve for $I(0) = 10$ and $I'(0) = 0$.

Answer

a. $0.05I'' + 0.1I' + \left(\frac{1}{5}\right) I = 0$
b. $I = Ce^{-t}\cos(\sqrt{3}t - \gamma)$
c. $I = 10e^{-t}\cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$

? Exercise $2.E.2.4.8$

A 5000 kg railcar hits a bumper (a spring) at $1\frac{m}{s}$, and the spring compresses by 0.1 m. Assume no damping.

a. Find k.
b. Find out how far does the spring compress when a 10000 kg railcar hits the spring at the same speed.
c. If the spring would break if it compresses further than 0.3 m, what is the maximum mass of a railcar that can hit it at $1\frac{m}{s}$?
d. What is the maximum mass of a railcar that can hit the spring without breaking at $2\frac{m}{s}$?

Answer

a. $k = 500000$
b. $\frac{1}{5\sqrt{2}} \approx 0.141$
c. 45000 kg
d. 11250 kg

? Exercise $2.E.2.4.9$

A mass of m kg is on a spring with $k = 3\frac{\mathrm{N}}{\mathrm{m}}$ and $c = 2\frac{\mathrm{Ns}}{\mathrm{m}}$. Find the mass m_0 for which there is critical damping. If $m < m_0$, does the system oscillate or not, that is, is it underdamped or overdamped?

Answer

$m_0 = \frac{1}{2}$. If $m < m_0$, then the system is overdamped and will not oscillate.

2.E.5: 2.5: Nonhomogeneous Equations

? Exercise $2.E.2.5.1$

Find a particular solution of $y'' - y' - 6y = e^{2x}$.

? Exercise $2.E.2.5.2$

Find a particular solution of $y'' - 4y' + 4y = e^{2x}$.

? Exercise $2.E.2.5.3$

Solve the initial value problem $y'' + 9y = \cos(3x) + \sin(3x)$ for $y(0) = 2, y'(0) = 1$.

? Exercise $2.E.2.5.4$

Setup the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x$.

Exercise $2.E.2.5.5$

Setup the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$.

Exercise $2.E.2.5.6$

a. Using variation of parameters find a particular solution of $y'' - 2y' + y = e^x$.
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on? See also Exercise $2.E.2.5.9$

Exercise $2.E.2.5.7$

Find a particular solution of $y'' - 2y' + y = \sin(x^2)$. It is OK to leave the answer as a definite integral.

Exercise $2.E.2.5.8$

For an arbitrary constant c find a particular solution to $y'' - y = e^{cx}$. Hint: Make sure to handle every possible real c.

Exercise $2.E.2.5.9$

a. Using variation of parameters find a particular solution of $y'' - y = e^x$
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on?

Exercise $2.E.2.5.10$

Find a polynomial $P(x)$, so that $y = 2x^2 + 3x + 4$ solves $y'' + 5y' + y = P(x)$.

Exercise $2.E.2.5.11$

Find a particular solution to $y'' - y' + y = 2\sin(3x)$

Answer

$$y = \frac{-16\sin(3x) + 6\cos(3x)}{73}$$

Exercise $2.E.2.5.12$

a. Find a particular solution to $y'' + 2y = e^x + x^3$.
b. Find the general solution.

Answer

a. $y = \frac{2e^x + 3x^3 - 9x}{6}$
b. $y = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \frac{2e^x + 3x^3 - 9x}{6}$

Exercise $2.E.2.5.13$

Solve $y'' + 2y' + y = x^2, y(0) = 1, y'(0) = 2$.

Answer

$$y(x) = x^2 - 4x + 6 + e^{-x}(x - 5)$$

Exercise $2.E.2.5.14$

Use variation of parameters to find a particular solution of $y'' - y = \dfrac{1}{e^x + e^{-x}}$.

Answer

$y = \frac{2xe^x - (e^x + e^{-x})\log(e^{2x}+1)}{4}$

Exercise $2.E.2.5.15$

For an arbitrary constant c find the general solution to $y'' - 2y = \sin(x+c)$.

Answer

$y = \frac{-\sin(x+c)}{3} + C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$

Exercise $2.E.2.5.16$

Undetermined coefficients can sometimes be used to guess a particular solution to other equations than constant coefficients. Find a polynomial $y(x)$ that solves $y' + xy = x^3 + 2x^2 + 5x + 2$.

Note: Not every right hand side will allow a polynomial solution, for example, $y' + xy = 1$ does not, but a technique based on undetermined coefficients does work, see Chapter 7.

Answer

$y = x^2 + 2x + 3$

2.E.6: 2.6: Forced Oscillations and Resonance

Exercise $2.E.2.6.1$

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \sin(\omega t)$. Assume $c > 0$.

Exercise $2.E.2.6.2$

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$. Assume $c > 0$.

Exercise $2.E.2.6.3$

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $m > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of c (solve in terms of m, k, and F_0) will there be no practical resonance (that is, for what values of c is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise $2.E.2.6.4$

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $c > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of m (solve in terms of c, k, and F_0) will there be no practical resonance (that is, for what values of m is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise $2.E.2.6.5$

Suppose a water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as a mass and the support acts as the spring, where the induced vibrations are horizontal. Suppose that the container with water has a mass of $m = 10,000kg$. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving).

Suppose that an earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$.

a. What is the natural frequency of the water tower?
b. If ω is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.
c. Suppose $A = 1$ and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter, the tower collapses. Will the tower collapse?

Exercise $2.E.2.6.6$

A mass of 4 kg on a spring with $k = 4$ and a damping constant $c = 1$. Suppose that $F_0 = 2$. Using forcing of $F_0 \cos(\omega t)$. Find the ω that causes practical resonance and find the amplitude.

Answer

$$\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984 \quad C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$$

Exercise $2.E.2.6.7$

Derive a formula for x_{sp} for $mx'' + cx' + kx = F_0 \cos(\omega t) + A$ where A is some constant. Assume $c > 0$.

Answer

$$x_{sp} = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t) + \frac{A}{k}$$, where $p = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

Exercise $2.E.2.6.8$

Suppose there is no damping in a mass and spring system with $m = 5, k = 20,$ and $F_0 = 5$. Suppose that ω is chosen to be precisely the resonance frequency.

a. Find ω.
b. Find the amplitude of the oscillations at time $t = 100$.

Answer

a. $\omega = 2$
b. 25