HW#5 Problem 3

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$$\frac{du}{dt} = \frac{a}{1+v^n} - u = f(u,v) \qquad (1)$$

$$\frac{dv}{dt} = \frac{a}{1+u^n} - v = g(u,v) \qquad (2)$$

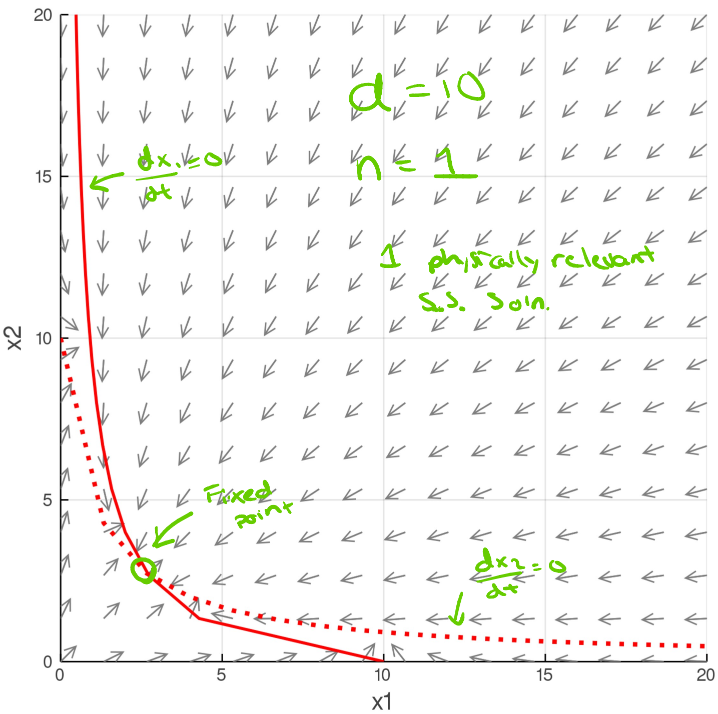
a) i) u, v

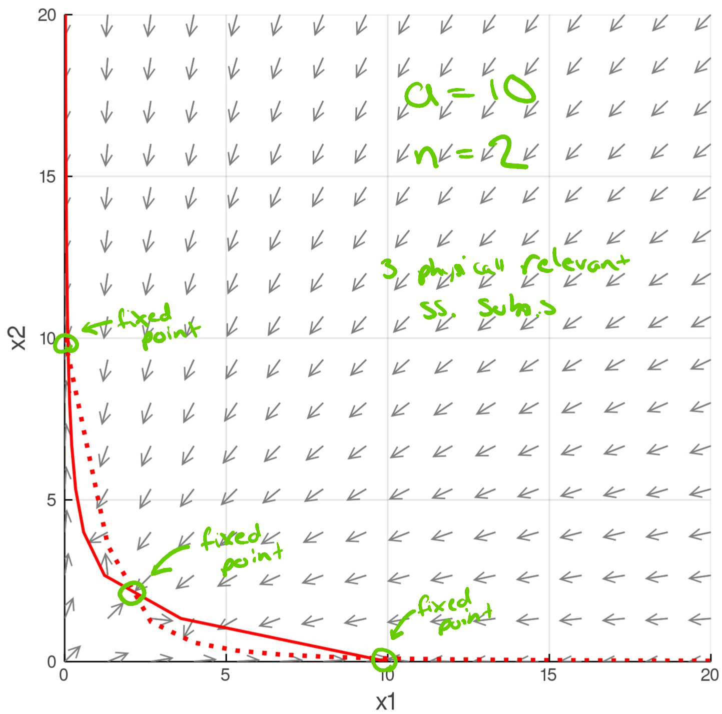
ii) a

iii) n

iv) -1

b) As the degree of cooperativity increased the S.S. soln. became unstable





c) See Problem 3 part b)

The steady state is stable (only considering physically relevant concentration ranges ie [ ] ≥ 0

d) s.s. $f(u,v) = g(u,v)$ $\quad a = 10, n = 1$

$$\frac{10}{1+v_s^2} - u_s = 0 \qquad u_s = \frac{10}{1+v_s}$$

$$\frac{10}{1+u_s^1} - v_s = 0 \Rightarrow \frac{10}{1+\frac{10}{1+v_s}} = v_s \qquad 10 = v_s + \frac{10 v_s}{1+v_s}$$

$$10 - v_s = \frac{10 v_s}{1+v_s} \qquad (1+v_s)(10-v_s) = 10 v_s \qquad 10 - v_s + \cancel{10} v_s - v_s^2 = \cancel{10} v_s$$

$v_s^2 + v_s - 10 = 0 \Rightarrow v_s = \{-3.7, 2.7\}$ $\quad v_s \Rightarrow 2.7$ only physically relevant root

$$u_s = \frac{10}{1+2.7} = 2.7$$

$$U = u - u_s \qquad V = v - v_s$$

$$\frac{dU}{dt} = f(u_s, v_s) + \left.\frac{\partial f}{\partial u}\right|_{u_s, v_s} U + \left.\frac{\partial f}{\partial v}\right|_{u_s, v_s} V = -1(U) - \frac{10}{(1+v)^2} V = -U - \frac{10V}{(1+v)^2} = -U \frac{-10V}{(1+2.7)^2}$$

$$\frac{dV}{dt} = g(u_s, v_s) + \left.\frac{\partial g}{\partial u}\right|_{u_s, v_s} U + \left.\frac{\partial f}{\partial v}\right|_{u_s, v_s} V = \frac{-10U}{(1+u)^2} - V = \frac{-10U}{(1+2.7)^2} - V$$

$$\Rightarrow \frac{dU}{dt} = \frac{-10}{13.69} V - U \qquad \frac{dV}{dt} = \frac{-10}{13.69} U - V$$

$$\dot{\bar{X}} = J \bar{X} \quad \text{where} \quad \bar{X} = \begin{pmatrix} U \\ V \end{pmatrix}, \; J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} -1 & \frac{-10}{(1+v_s)^2} \\ \frac{-10}{(1+u_s)^2} & -1 \end{pmatrix}$$

Stability criterion is that real part of eigenvalue (λ_i) is < 0

Degree of cooperativity has a large effect on stability while the rate of synthesis does not. As degree of cooperativity increases the real part of λ_i becomes increasing positive.

e) S.S. sol'n $a=10,\ n=2 \xRightarrow{\text{wolfram}} \{(0.101, 9.89), (2,2), (9.89, 0.101)\}$

Assuming (2,2) is central SS soln

For $n=1$: $J = \begin{pmatrix} -1 & -10/(1+2.7)^2 \\ -10/(1+2.7)^2 & -1 \end{pmatrix}$ $\quad \lambda_1 = \frac{-2369}{1369} \quad \lambda_2 = \frac{-369}{1369}$

$\lambda_1 = -1.73 \quad \lambda_2 = -0.27$ stable as <0

For $n=2$: $J = \begin{pmatrix} -1 & \frac{-2\cdot 10\cdot 2}{(1+2^2)^2} \\ \frac{-2\cdot 10\cdot 2}{(1+2^2)^2} & -1 \end{pmatrix}$ $\quad \lambda_1 = -2.6 \quad \lambda_2 = 0.6$

One eigenvalue greater than zero so unstable

f) Part 1

$$\frac{dR_i^*}{dt} = k_f L R_i - k_r R_i^* \qquad (3)$$

$$\frac{dN_i^*}{dt} = k_f^{ND} N_i D_j - k_r^{ND} N_i^* \qquad (4)$$

$$\frac{dD_i}{dt} = k_D R_i^* - \gamma_D D_i \qquad (5)$$

$$\frac{dR_i}{dt} = \frac{\beta^n}{K^n + N_i^{*n}} - \gamma_R R_i \qquad (6) \rightarrow \frac{dR_1}{dt} = \frac{\beta^n}{K^n + N_1^{*n}} - \gamma_R R_1$$

$$\frac{dR_2}{dt} = \frac{\beta^n}{K^n + N_2^{*n}} - \gamma_R R_2$$

Assume fast equilibrium

$$\Rightarrow 0 = k_f L R_i - k_r R_i^* \rightarrow 0 = k_f L R_1 - k_r R_1^* \qquad 0 = k_f L R_2 - k_r R_2^*$$

$$0 = k_f^{ND} N_i D_j - k_r^{ND} N_i^* \rightarrow 0 = k_f^{ND} N_1 D_2 - k_r^{ND} N_1^* \qquad 0 = k_f^{ND} N_2 D_1 - k_r^{ND} N_2^*$$

$$0 = k_D R_i^* - \gamma_D D_i \rightarrow 0 = k_D R_1^* - \gamma_D D_1 \qquad 0 = k_D R_2^* - \gamma_D D_2$$

$$N_1^* = \frac{k_f^{ND}}{k_r^{ND}} N_1 D_2 = \left(\frac{k_f}{k_r}\right)^{ND} N_1 D_2 \qquad N_2^* = \left(\frac{k_f}{k_r}\right)^{ND} N_2 D_1$$

$$D_1 = \frac{k_D R_1^*}{\gamma_D} \qquad D_2 = \frac{k_D R_2^*}{\gamma_D}$$

$$R_1^* = \frac{k_f L R_1}{k_r} \qquad R_2^* = \frac{k_f L R_2}{k_r} \rightarrow D_1 = \frac{k_D k_f L R_1}{\gamma_D k_r} \qquad D_2 = \frac{k_D k_f L R_2}{\gamma_D k_r}$$

$$\Rightarrow N_1^* = \left(\frac{k_f}{k_r}\right)^{ND} N_1 \frac{k_D k_f L R_2}{\gamma_D k_r} = \left(\frac{k_f}{k_r}\right)^{ND+1} \frac{N_1 k_D L R_2}{\gamma_D}$$

$$N_2^* = \left(\frac{k_f}{k_r}\right)^{ND+1} \frac{N_2 k_D L R_1}{\gamma_D}$$

$$\Rightarrow \frac{dR_1}{dt} = \frac{\beta^n}{K^n + \left(\left(\frac{k_f}{k_r}\right)^{ND+1} \frac{N_1 k_D L R_2}{\gamma_D}\right)^n} - \gamma_R R_1 \qquad \frac{dR_2}{dt} = \frac{\beta^n}{K^n + \left(\left(\frac{k_f}{k_r}\right)^{ND+1} \frac{N_2 k_D L R_1}{\gamma_D}\right)^n} - \gamma_R R_2$$

Part 2) Non-dimensionalize $u = R_1/K$, $v = R_2/K$, $\tau = \gamma_R t$

$$\rightarrow R_1 = uK \qquad \frac{dR_1}{dt} = K\frac{du}{dt} \qquad \longrightarrow K\gamma_R \frac{du}{d\tau} = \frac{\beta^n}{K^n + \left(\left(\frac{k_f}{k_r}\right)^{ND+1} \frac{N_1 k_D L v K}{\gamma_D}\right)^n} - \gamma_R u K$$

$$\rightarrow R_2 = vK \qquad \frac{dR_2}{dt} = K\frac{dv}{dt}$$

$$\rightarrow t = \tau/\gamma_R \qquad \frac{d}{dt} = \frac{d}{d\tau/\gamma_R} = \gamma_R \frac{d}{d\tau}$$

$$\Rightarrow \frac{dR_1}{dt} = K\gamma_R \frac{du}{d\tau} \qquad \frac{dR_2}{dt} = K\gamma_R \frac{dv}{d\tau}$$

$$\Rightarrow \frac{du}{d\tau} = -u + \frac{\beta^n}{\gamma_R K^{n+1} + \gamma_R K^{n+1}\left(\left(\frac{k_f}{k_r}\right)^{ND+1} N_1 \frac{k_D L v}{\gamma_D}\right)^n} = \frac{\beta^n}{\gamma_R K^{n+1} + \gamma_R K^{n+1}\left(\left(\frac{k_f}{k_r}\right)^{ND+1} N_1 \frac{k_D L}{\gamma_D}\right)^n v^n} - u$$

$$= \frac{\left(\beta^n / \gamma_R K^{n+1}\right)}{1 + \left(\left(\frac{k_f}{k_r}\right)^{ND+1} N_1 \frac{k_D L}{\gamma_D}\right)^n v^n} - u \longrightarrow \frac{\alpha}{1 + \ell^n v^n} - u$$

$$\frac{dv}{dt} = \frac{\alpha}{1 + \ell^n u^n} - v$$

As [L] increases the soln becomes more unstable. We can also drive the system towards instability by increasing n.