Com S 228 Spring 2019

Project 3: Prime Factor List (250 pts)

Due at **11:59pm**

Saturday, March 30

1. Prime Factorization

A ***prime number*** is an integer greater than one and is divisible by one and itself only. The sequence of prime numbers starts with 2, 3, 5, 7, 11, 13, 17, 19, … You may check out the first 1,000 primes at the following site:

`https://en.wikipedia.org/wiki/List_of_prime_numbers#The_first_500_prime_numbers`

There are infinitely many primes. As of January 2019, the largest known prime is $2^{82,589,933} - 1$, which has 24,862,048 digits.

An integer greater than one and divisible by a third natural number besides 1 and itself is called a ***composite number***. For example, 4 is a composite number because it is also divisible by 2 in addition to 1 and itself. The sequence of composite numbers starts with 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, …

By definition, 1 is neither a prime number nor a composite number.

The Fundamental Theorem of Arithmetic states that every integer greater than one is either a prime or a product of primes. For example, the number 25480 is a product of 2, 2, 2, 5, 7, 7, 13, all prime numbers. In other words, it is factored as

$$25480 = 2^3 \cdot 5 \cdot 7^2 \cdot 13$$

In the above factorization, the prime 2 appears three times as a factor. It is said to have ***multiplicity*** 3. The prime factors 5, 7, and 13 have multiplicities 1, 2, 1, respectively.

Generally speaking, any integer $n \geq 2$ can be written as a product of powers of its prime factors. More specifically, it has the form

$$n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$$

where $p_1, p_2, \dots, p_k$ with $p_1 < p_2 < \cdots < p_k$ are the prime factors, and $\alpha_1, \alpha_2, \dots, \alpha_k > 0$ are their respective multiplicities. The process of writing n into the above product form is called ***prime factorization***.

1.1 Factorization and Encryption

Prime factorization is a very difficult problem. In 2009, several researchers successfully concluded two years of effort to factor the following 232-digit number (named RSA-768) via the use of hundreds of machines:

1230186684530117755130494958384962720772853569595334792197322452151726400507263657518745202199786469389956474942774063845925192557326303453731548268507917026122142913461670429214311602221240479274737794080665351419597459856902143413

$=$

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

$\times$

36746043666799590428244633799627952632279158164343087642676032283815739666511279233373417143396810270092798736308917

The CPU time spent on finding the two factors amounted to the equivalent of 75 years of work for a single 2.2 GHz Opteron-based computer.

No efficient prime factorization algorithm for large numbers is known. More specifically, no algorithm is known to factor all integers in polynomial time, i.e., to factor b-bit numbers in time $O(bk)$ for some constant k.

The presumed difficulty of prime factorization lies at the foundation of cryptography algorithms such as RSA. A message receiver may publish a key --- a large integer, referred to as the ***public key***, while keeping another paired key, referred to as the ***private key***, to his/her own knowledge. Messages sent to the receiver are encrypted using the public key, but are impossible to decrypt without the knowledge of the private key. The decryption process comes down to prime factorization of very large numbers --- even larger than RSA-768, a process without knowing the private key would be intractable given today's computational power.

1.2 Direct Search Factorization

In this project we will use a brute force strategy named the ***direct search factorization***. It is based on the observation that a number n, if composite, must have a prime factor less than or equal to $\sqrt{n}$. Clearly, the number cannot have more than one prime factor greater than $\sqrt{n}$. This method initializes $m \leftarrow n$ and systematically tests divisors d from 2 up to $\lfloor\sqrt{n}\rfloor$, where the floor operator $\lfloor \quad \rfloor$ gives the largest integer less than or equal to the operand $\sqrt{n}$. In your implementation, test the condition $d \cdot d \leq n$ instead of $d \leq \sqrt{n}$ for efficiency. If d divides m, then set $m \leftarrow m/d$, and repeat so with the divisor d until it no longer divides m. Then set $d \leftarrow d + 1$ to continue the testing until $d \cdot d > m$.

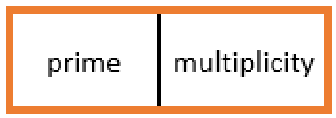
For example, to factorize 25480. We try to divide the number by 2. Since 2 is a divisor, we record it and divide 25480 by 2 to obtain 12740. We find that the number 12740 can be divided by 2 two more times. The original number thus contains the prime factor 2 with multiplicity 3. Now we work on 25480 / 8 = 3185. The next number to test is 3, which does not divide 3185.

Neither does 4. The number 5 divides 3185 only one time, yielding the quotient 637. Moving on, 6 does not divide 637. We find that 7 divides 637 twice to yield 13. The algorithm terminates since $8 \cdot 8 > 13$.

You can cut down the number of test candidates by half easily, by ignoring all the even numbers greater than 2 (because they are apparently not prime).

2. List Structure

The class `PrimeFactorization` houses a doubly-linked list structure to store the prime factors of a number. Every node on the list stores a ***distinct*** prime factor and its multiplicity, which are together packaged into an object of the `PrimeFactor` class as illustrated below.

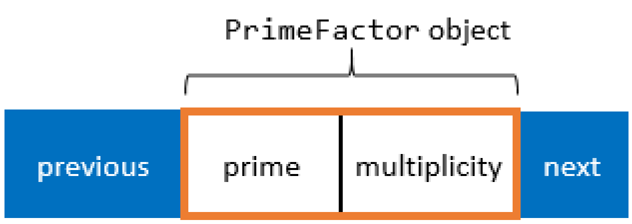


A node is implemented by the private class `Node` inside `PrimeFactorization`. The class `Node` also has two links `previous` and `next` to reference the preceding and succeeding nodes.

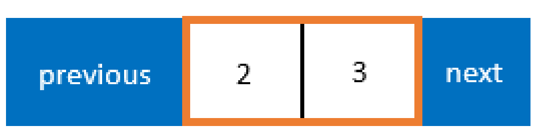
```
private class Node
{
    public PrimeFactor pFactor;
    public Node next;
    public Node previous;

    // constructors
    …
}
```

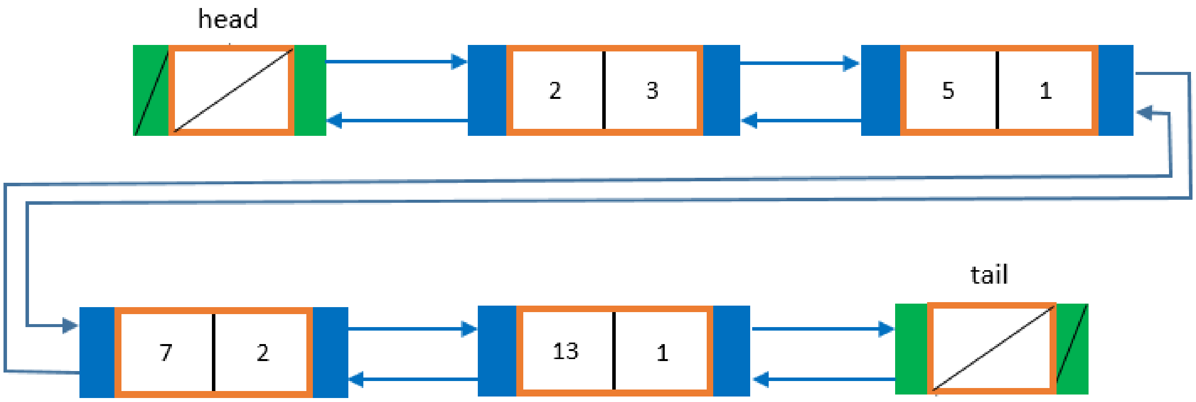
The node is illustrated below:



For example, 25480 has the prime factor 2 with multiplicity 3. This leads to the creation of the following node:



In the linked list inside the class `PrimeFactorization`, the nodes are connected by the `next` link in the ***increasing order of prime factor***. The list also has two dummy nodes: `head` and `tail`. The factorization of 25480 is represented by the list below with six nodes, four of which represent its prime factors 2, 5, 7, 13 with multiplicities 3, 1, 2, 1, respectively.

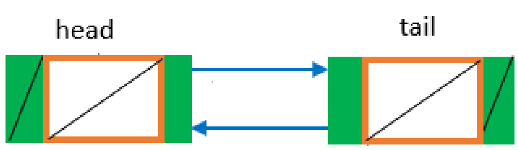


The class `PrimeFactorization` also stores the factored number in the instance variable `value` of the `long` integer type (64-bit). If this number exceeds the maximum $2^{63} - 1$ representable by the type `long`, `value` is set to be -1 (defined to be a constant `OVERFLOW` in the template code). For this reason, we refer to the ***represented integer*** by `this` object to make a distinction from `value`. The two integers are equal when `value != -1`.

This class has four constructors:

```
public PrimeFactorization()
public PrimeFactorization(long n) throws IllegalArgumentException
public PrimeFactorization(PrimeFactorization pf)
public PrimeFactorization(PrimeFactor[] pfList)
```

a) The first constructor is a default one that constructs an empty list to represent the number 1.



b) The second constructor performs the direct search factorization described in Section 1.2 on the integer parameter `n`. An exception will be thrown if `n` is less than 1.

c) The third one is a copy constructor which assumes that the argument object `pf` stores a correct prime factorization.

d) The fourth one constructs over an array of `PrimeFactor` objects. This is especially useful for the case of an overflowing number.

3. Arithmetics

The list structure of this `PrimeFactorization` object needs to be updated when its represented integer is multiplied with or divided by another integer, either of type `long` or represented by another `PrimeFactorization` object. After every arithmetic operation, the nodes in the linked list must remain in the ***increasing order*** of the prime factors they store.

3.1 Multiplication

There are three methods to carry out multiplication, accepting a long integer, a `PrimeFactorization` object, and two such objects, respectively. The linked list of this `PrimeFactorization` object, will be updated.

```
public void multiply(long n) throws IllegalArgumentException
public void multiply(PrimeFactorization pf)
public static PrimeFactorization multiply
              (PrimeFactorization pf1, PrimeFactorization pf2)
```

The first method can be implemented via a traversal of the linked list referenced by `head` using an iterator (which is an object of the private class `PrimeFactorizationIterator`) as follows. Factor `n` using the direct search described in Section 1.2. For each prime factor found (with a multiplicity), move the iterator forward from its current position to the first node storing an equal or a larger prime. In the first case, increment the `multiplicity` field of that node. In the second case, create a new node for this factor and add it to the linked list before the larger prime. If the cursor reaches the end of the list, simply add the new node as the last node.

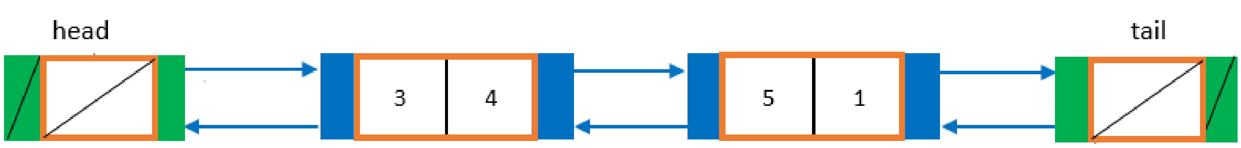
The second method updates the linked list by traversing it and the linked list of the argument object `pf` simultaneously. Essentially, it merges a copy of the second list into the first list by doing the following on the first list:

a) Copy over the nodes from the second list for prime factors that do not appear in the first list.

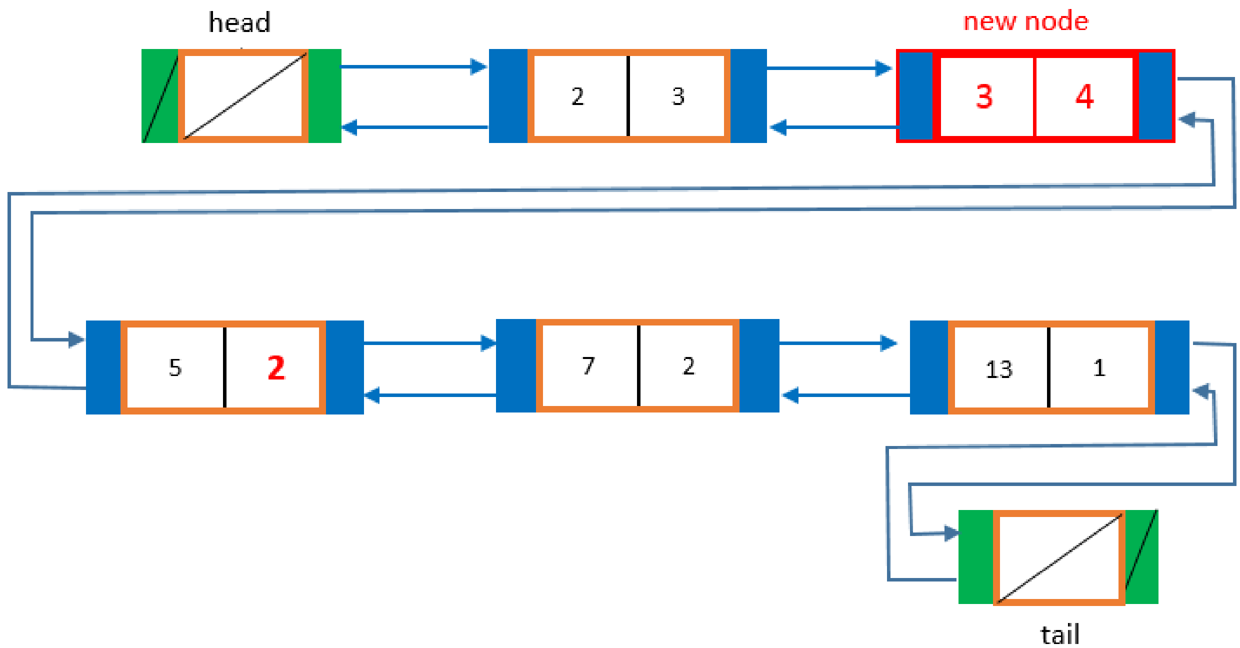
b) For each prime factor shared with the second list, increment the `multiplicity` field of the `PrimeFactor` object stored at the corresponding node in the first list.

The implementation requires the use of two iterators, one for each list, and works in a way similar to merging two sorted arrays into one.

For example, suppose a `PrimeFactorization` object for 25480 (with the linked list shown in Section 2) calls `multiply()` on another object storing the factorization of 405 shown below:



After the call, the first linked list will become



A new node (drawn in red) has been added, and the `PrimeFactor` object stored at an existing node has its `multiplicity` field updated from 1 to 2 (shown in red).

The first two `multiply()` methods also call `Math.multiplyExact()` to carry out multiplication of `this.value`, if not -1, with `n` or `pf.value`. An `ArithmeticException` will be thrown if an overflow happens. In this case, set the `value` field to -1. Otherwise, store the product in `value`. Note that `this.value == - 1` only means that the number represented by the linked list cannot be represented in Java as a `long` integer. We can still perform arithmetic operations on the large number by this `PrimeFactorization` object.

The third `multiply()` method is a static one that takes two `PrimeFactorization` objects, and returns a new object storing the product of the two numbers represented by the first two objects.

3.2 Division

The `PrimeFactorization` class provides three methods to perform division –- by an integer, by the integer encapsulated in another `PrimeFactorization` object, and of the first such object by a second object. It is updated to store the result.

```
public boolean dividedBy(long n)
public boolean dividedBy(PrimeFactorization pf)
public static PrimeFactorization dividedBy
          (PrimeFactorization pf1, PrimeFactorization pf2)
```

The first method immediately returns `false` if `this.value != -1` and `this.value < n`. The second method immediately returns `false` in one of the following two situations:

a) `this.value != -1` and `pf.value != -1 and this.value < pf.value`

b) `this.value != -1` but `pf.value == -1`

If the dividend and the divisor have equal values, then remove all the nodes storing prime factors by calling the private method `clearList()`. In this case, set `value` to 1 and return `true`.

The first `dividedBy()` method constructs a `PrimeFactorization` object for `n`, and then calls the second `dividedBy()` method with the constructed object as the argument.
When the second `divideBy()` method is called to carry out the division, make a copy of the linked list of `this` object. Traverse the list copy and the list of the object `pf` using two separate iterators, say, `iterCopy` and `iterPf`, respectively. Advance `iterCopy.cursor` until `iterCopy.cursor.pFactor.prime >= iterPf.cursor.pFactor.prime` or (`!iterCopy.cursor.hasNext() && iterPf.hasNext()`). In the latter case, returns `false` immediately. In the former case, there are three possible situations:

a) `iterCopy.cursor.pFactor.prime > iterPf.cursor.pFactor.prime`. The number is not divisible by the number represented by pf. Return `false` immediately.

b) `iterCopy.cursor.pFactor.prime == iterPf.cursor.pFactor.prime` but `iterCopy.cursor.pFactor.multiplicity < iterPf.cursor.pFactor.multiplicity`. The number is not divisible by the number represented by pf. Return `false` immediately.

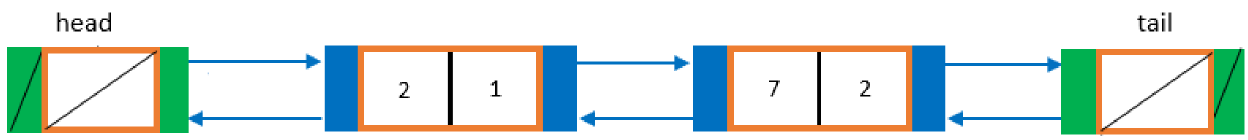
c) `Otherwise, the following two conditions must hold:`

```
iterCopy.cursor.pFactor.prime = iterPf.cursor.pFactor.prime
iterCopy.cursor.pFactor.multiplicity >=
                iterPf.cursor.pFactor.multiplicity
```

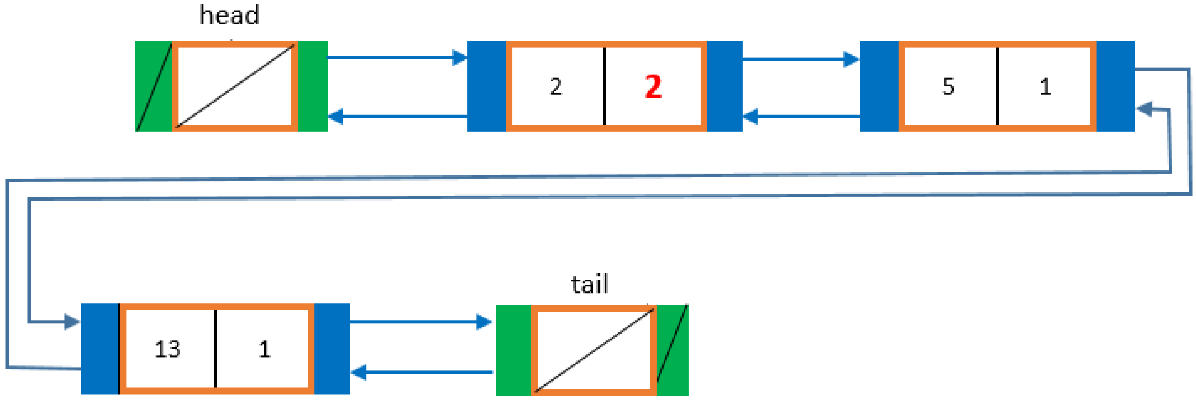
`Reset iterCopy.cursor.pFacor.multiplicity` to the difference between itself and `iterPf.cursor.pFactor.multiplicity.` If the difference is zero, the node needs to be removed. Advance both iterators.

Repeat until either `false` is returned or `iterPf.cursor == iterPf.tail` eventually holds. If the latter happens, the integer of `this` object is divisible by that of `pf`. Set the copy list to be the linked list of `this` object by updating its `head` and `tail`. Also, update the object's `size` and `value` fields accordingly. Return `true` after the updates.

As an example, suppose the earlier `PrimesFactors` object representing 25480 calls `divideBy(98)`. A `PrimesFactors` object for 98 is first constructed with a linked list illustrated below:



Then, this temporary object is provided as the argument for a call to the second `divideBy()` method. The new linked list for the original object is shown below with the multiplicity of 2 decreased to 2 (colored red) and the node for the factor 7 deleted :



In the case that `this` object represents a number that overflowed before the division, which is indicated by `this.value == -1`, the quotient to replace this number in the object may not overflow (i.e., less than or equal to $2^{63} - 1$). When that happens, the `value` field should store the quotient instead of -1. Checking can be done at the end of `divideBy()` using a method called `updateValue()`. Initialize a variable to have value 1. Keep multiplying it with the power of every prime factor (determined by its multiplicity) stored in the new linked list representing the quotient. Use `Math.multiplyExact()` to carry out the multiplications, which should stop as soon as the method throws an `ArithmeticException`.

The third `dividedBy()` method is a static one that takes two `PrimeFactorization` objects, and returns a new object storing the quotient of the first corresponding number divided by the second number, or null, if the first number is not divisible by the second one.

4. Greatest Common Divisor and Least Common Multiple

The class `PrimeFactorization` provides methods to compute the greatest common divisor and least common multiple of two positive numbers, and return the results as `PrimeFactorization` objects. The linked list in a returned `PrimeFactorization` object must have its nodes store prime factors in the ***increasing*** order**.**

4.1 The Euclidean Algorithm

A ***common divisor*** d of two positive integers a and b is a positive integer that divides both a and b. The ***greatest common divisor*** (GCD) of a and b is the biggest of all their common divisors. For example, 12 and 42 have four common divisors 1, 2, 3, 6. Their GCD is 6.

The GCD of two numbers can be efficiently computed using the Euclidean algorithm, invented by the Greek mathematician Euclid in 300 B.C. Suppose we are to find the GCD of 184 and 69. Let us divide the bigger number by the smaller one, obtaining

$$184 = 2 \cdot 69 + 46.$$

The remainder 46 is less than the divisor 69. Any common divisor of 184 and 69 must also divide $46 = 184 - 2 \cdot 69$. Therefore, the GCD of 184 and 69 is also the GCD of 69 and 46. Next, we divide 69 by 46, yielding

$$69 = 1 \cdot 46 + 23$$

The problem further boils down to finding the GCD of 46 and 23. One more round of division results in

$$46 = 2 \cdot 23 + 0$$

Since the remainder becomes 0, 23 divides 46. We have thus obtained the GCD:

$$GCD(184, 69) = GCD(69, 46) = GCD(46, 23) = 23.$$

Generally, suppose we want to find the GCD of two numbers a and b. Let $r_0 = a$ and $r_1 = b$. The Euclidean algorithm carries out the following sequence of divisions:

$$\begin{aligned} r_0 &= q_1 \cdot r_1 + r_2 & \quad 0 < r_2 < r_1 \\ r_1 &= q_2 \cdot r_2 + r_3 & \quad 0 < r_3 < r_2 \\ r_2 &= q_3 \cdot r_3 + r_4 & \quad 0 < r_4 < r_3 \\ &\vdots & \\ r_{k-1} &= q_k \cdot r_k + 0 & \end{aligned}$$

The dividend and divisor in a division were respectively the divisor and remainder in the previous division. Since the remainder is always less than the divisor, it follows that

$$r_1 > r_2 > \cdots > r_k > r_{k+1} = 0$$

The algorithm will terminate because the remainder decreases by at least one after each division step. It holds that

$$GCD(r_0, r_1) = GCD(r_1, r_2) = \cdots = GCD(r_{k-1}, r_k) = r_k.$$

The following static method implements the Euclidean algorithm:

```
public static long Euclidean(long m, long n) throws IllegalArgumentException
```

The `Euclidean()` method **must be used** by the `gcd()` method below to compute the GCD of `this.value` and another number `n` when `this.value != -1`.

```
public PrimeFactorization gcd(long n) throws IllegalArgumentException
```

4.2 Computing GCD from Prime Factorizations

If the second number is encapsulated in a `PrimeFactorization` object, the GCD computation is carried out as follows.

a) Traverse the linked lists of the two objects to find the common prime factors of the two numbers.

b) For every common prime factor p, create a new node to store p and $\min(\alpha_1, \alpha_2)$, the minimum of the multiplicities α_1 and α_2 of p stored in the two lists.

c) Link all the new nodes together to construct a `PrimeFactorization` object to represent the GCD.

The work can be done during one round of simultaneous traversals of the linked lists of both `PrimeFactorization` objects. The above procedure is implemented by the method below:

```
public PrimeFactorization gcd(PrimeFactorization pf)
```

Even when one or both of the numbers cause overflows, their GCD may not. Thus, if `this.value == -1` or `pf.value == -1`, call `updateValue()` to check if the GCD overflows, and set the `value` field of the generated `PrimeFactorization` object properly.

When `this.value ! = -1` and the second number for a GCD computation is provided as an integer, an alternative to calling the first `gcd()` method (described in Section 4.1) would be to construct a `PrimeFactorization` object `pf` over the second number, and call the second `gcd()` method above. However, it is more efficient to directly use the first `gcd()` method, even though it returns a `PrimeFactorization` object constructed through factoring the GCD. This factorization is less expensive than factoring `n`, which has to be done to construct the object `pf` to call the second `gcd()` method with. The reason is that `n` could be significantly larger than the GCD.

The third `gcd` method is static and takes two `PrimeFactorization` objects.

```
public static PrimeFactorization gcd(PrimeFactorization pf1,
                                     PrimeFactorization pf2)
```

4.3 Least Common Multiple

A ***common multiple*** m of two positive integers a and b is a positive integer that is divisible by both a and b. The two numbers have infinitely many common multiples. The ***least common multiple*** (LCM) of a and b is the smallest of all their common multiples. For example, 12 and 42 have 84 as their least common multiple. Let us examine the prime factorizations of these two numbers:

$$12 = 2^2 \cdot 3$$
$$42 = 2 \cdot 3 \cdot 7$$

The LCM can be represented as

$$84 = 2^{\max(2,1)} \cdot 3^{\max(1,1)} \cdot 7$$

We can generalize the above into a procedure for computing the LCM, to be implemented by the following method:

```
public PrimeFactorization lcm(PrimeFactorization pf)
```

The method computes the LCM and constructs a `PrimeFactorization` object representing it. If either `this.value == -1` or `pf.value == -1`, that is, either of the two numbers overflows, the constructed object must have its `value` field set to -1. Otherwise, the `lcm()` method needs to call `updateValue()` to check the computed LCM for overflow and set the `value` field either to the LCM or -1.

Traverse the two linked lists of this `PrimeFactorization` object and the object `pf` simultaneously with two cursors, comparing the `prime` factors housed by the two nodes at the cursor positions to decide which cursor to advance. (This is similar to merging two sorted arrays into one.)

a) If the two prime factors are not equal, add a copy of the node storing the smaller prime to the list being constructed for the new `PrimeFactorization` object. Advance from this node in the corresponding original list.

b) If the two prime factors are equal to, say, p, which has multiplicities α_1 and α_2 in the two nodes being compared, respectively, from the two lists. Generate a new node to store the information below

and add it to the list under construction.

The algorithm terminates when one list is fully traversed. Then, simply copy the remaining nodes from the unfinished list onto the list under construction.

You also need to implement a second version of `lcm()`, by simply calling the first version over a `PrimeFactorization` object constructed over the parameter `n`.

```
public PrimeFactorization lcm(long n) throws IllegalArgumentException
```

A third (and static) version of `lcm()` takes two `PrimeFactorization` objects as parameters and return a third object to store the result.

```
public static PrimeFactorization lcm(PrimeFactorization pf1,
                                     PrimeFactorization pf2)
```

4. Iterators

The iterator class `PrimeFactorizationIterator` implements `ListIterator<PrimeFactor>`. Refer to the Javadocs of these methods, especially of `add()` and `remove()`.

5. Submission

Write your classes in the `edu.iastate.cs228.hw3` package. **Turn in the zip file not your class files.** Please follow the guideline posted under Documents & Links on Canvas.

You are not required to submit any JUnit test cases. Nevertheless, you are encouraged to write JUnit tests for your code. Since these tests will not be submitted, feel free to share them with other students.

Include the Javadoc tag `@author` in every class source file you have made changes to. Your zip file should be named `Firstname_Lastname_HW3.zip`.