A Geometric Characterization of Leader-Follower Formation Control

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***Abstract*— The paper focuses on leader-follower formations of nonholonomic mobile robots. A formation control alternative to those existing in the literature is introduced. We show that the geometry of the formation imposes a bound on the maximum admissible curvature of leader trajectory. An peculiar feature of the proposed strategy is that the followers position is not rigidly fixed with respect to the leader reference frame but varies in suitable cones centered in the leader reference frame. Our approach also applies to hierarchical multirobot formations described by rooted tree graphs. Simulation experiments confirm the effectiveness of the proposed control schemes.**

I. INTRODUCTION

In the last few years formation control became one of the leading research areas in mobile robotics. By formation control we simply mean the problem of controlling the relative position and orientation of the robots in a group while allowing the group to move as a whole [2]. The use of robot formations ranges from military to civilian applications such as terrain and utilities inspection, disaster monitoring, environmental surveillance, search and rescue and planetary exploration. Different robot formation typologies have been studied in the literature: ground vehicles [5], [7], [12], [13], unmanned aerial vehicles (UAVs) [3], [10], aircraft [8], [9], and surface and underwater autonomous vehicles [6], [14]. Existing approaches to robot formation control generally fall into three categories: behavior based, virtual structure and leader following.

In the behavior based approach [1], [11] several desired behaviors (e.g. collision avoidance, formation keeping, target seeking) are prescribed to each robot. Robot final action is derived by weighting the relative importance of each behavior. The theoretical formalization and mathematical analysis of this approach is difficult and consequently it is not easy to guarantee the convergence of the formation to a desired configuration.

The virtual structure approach [15] considers the robot formation as a single virtual rigid structure so that the behavior of the robotic system is assimilable to that of a physical object. Desired trajectories are not assigned to each single robot but to the entire formation as a whole. In this case the behavior of the robot formation is predictable and consequently the control of the robot formation is straightforward. Nevertheless a large inter-robot communication bandwidth is required.

In the leader-follower approach a robot of the formation, designed as the leader, moves along a predefined trajectory while the other robots, the followers, are to maintain a desired posture (distance and orientation) to the leader [5], [16]. The main criticism to the leader-follower approach is that the formation does not tolerate leader faults and exhibits poor disturbance rejection features. In spite of these deficiencies the leader-follower approach is particularly appreciated because of its simplicity and scalability.

The leader-follower formation control of nonholonomic mobile robots is the subject of this work. We propose a leader-follower setup that is alternative to those existing in the literature [5], [13]. The main difference is that the desired angle between the leader and the follower is measured in the follower frame instead of the leader frame. In this framework, we design a formation control strategy that generates smoother trajectories while guaranteeing lower control effort (especially for large distances between the robots) with respect to other controllers proposed in the literature as shown in [4].

The main contribution of this work is that of showing how the geometric properties of the formation affect the set of the admissible curvatures of leader trajectory and the velocity bounds of the followers. Differently from [5], in our setting the followers are not rigidly disposed with respect to the leader reference frame, but their relative positions vary in time in suitable cones centered in the leader frame thus making the formation more flexible.

Basic results are extended to multirobot hierarchical formations described by rooted tree graphs. To the best of our knowledge this coordination scheme, a generalization of leader following (of which inherits pros and cons), has not been investigated yet in the literature.

The rest of the paper is organized as follows. Sect. II is devoted to the problem formulation. In Sect. III and IV the exact formation control problem and the stabilization problem are studied. In Sect. V basic results are extended to multirobot hierarchical formations. In Sect. VI simulation experiments confirm the effectiveness of the proposed control schemes. In Sect. VII the major contributions of the paper are summarized and future research lines are highlighted.

Notation: $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$; $\forall\, t \geq 0$, $\mathrm{sign}(t) = 1$; $\forall\, t < 0$, $\mathrm{sign}(t) = -1$; $\forall\, a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$; $\forall\, x, y \in \mathbb{R}^n$ $(n \geq 1)$, $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$, $\|x\| = \sqrt{\langle x, x\rangle}$; $\forall\, x \in \mathbb{R}^2 \backslash \{0\}$, $\arg(x) = \theta$, where $\theta \in [0, 2\pi)$ and $x = \|x\|(\cos\theta, \sin\theta)^T$; $\forall\, \theta \in \mathbb{R}$, $\tau(\theta) = (\cos\theta,\, \sin\theta)^T$, $\nu(\theta) = (-\sin\theta,\, \cos\theta)^T$.

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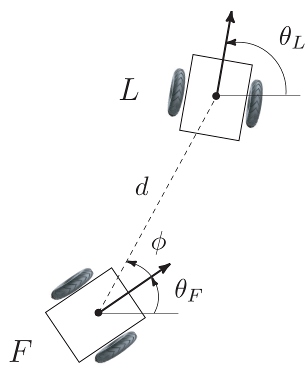


Fig. 1. Basic leader-follower setup.

II. PROBLEM FORMULATION

The basic leader-follower setup considered in this paper is presented in Fig. 1. It consists of a leader robot R_L and a follower R_F whose kinematics is described by the unicycle model

$$\begin{aligned}\dot{x}_L &= v_L \cos\theta_L \\ \dot{y}_L &= v_L \sin\theta_L \\ \dot{\theta}_L &= \omega_L\end{aligned} \tag{1}$$

and

$$\begin{aligned}\dot{x}_F &= v_F \cos\theta_F \\ \dot{y}_F &= v_F \sin\theta_F \\ \dot{\theta}_F &= \omega_F\end{aligned} \tag{2}$$

where the vectors $L = (x_L, y_L)$, $F = (x_F, y_F)$ represent the position of the leader and respectively the follower. Analogously θ_L, θ_F are the orientation of the leader and follower with respect to the reference system (x, y). Finally, v_L, v_F and ω_L, ω_F are the linear and angular velocities of the robots. With reference to Fig. 1, consider the following.

Definition 1: Set $d > 0$ and $\phi : |\phi| < \frac{\pi}{2}$; the robots R_L and R_F make a (d, ϕ)-formation, if, $\forall\, t \geq 0$:

$$\|L(t) - F(t)\| \;=\; d \tag{3}$$

$$\arg(L(t) \,-\, F(t)) - \theta_F(t) \;=\; \phi\,. \tag{4}$$

It is required that the velocities of the leader and the follower verify the following constraints:

$$\begin{array}{ccccc} 0 & < & v_L & \leq & V_L \\ -K & \leq & \omega_L/v_L & \leq & K \end{array} \tag{5}$$

$$\begin{array}{ccccc} 0 & \leq & v_F & \leq & V_F \\ -\Omega_F & \leq & \omega_F & \leq & \Omega_F \end{array} \tag{6}$$

$$\liminf_{t\to\infty}\; v_F(t) \;>\; 0$$

where V_L, V_F, $\Omega_F \in \mathbb{R}^+$ are the leader and follower maximum linear velocity and the follower maximum angular velocity, and K represents the leader trajectory maximum curvature (note that for a unicycle robot the instantaneous trajectory curvature is given by ω_L/v_L).

The first problem we discuss can be stated as follows.

*Problem 1 (**Exact formation control**):* Find the conditions on leader motion that guarantee the existence of the control functions v_F, ω_F, and an initial state such that R_L and R_F make a (d, ϕ)-formation, that is equations (3) and (4) hold for all t.

The second problem we deal with, is that of stabilizing these trajectories.

*Problem 2 (**Stabilization**):* Find the conditions on leader motion and the control functions v_F, ω_F, such that, starting from an arbitrary initial state, equations (3)-(4) are asymptotically satisfied.

In Sect. V we extend Definition 1 to multirobot hierarchical formations (or (D, Φ)-formations) and a generalization of Problem 1 is discussed.

III. EXACT FORMATION CONTROL

The following Theorem gives a solution to Problem 1.

Theorem 1: In the previous hypotheses and notation, let $d > 0$ and $\phi : |\phi| < \frac{\pi}{2}$ be given. For any robot R_L verifying conditions (5) there exist initial conditions $x_F(0), y_F(0), \theta_F(0)$ and controls $v_F(t), \omega_F(t)$ such that R_L and R_F are in (d, ϕ)-formation and bounds (6) are verified if and only if

$$Kd \leq 1 \tag{7}$$

$$V_L\, K \leq \Omega_F \tag{8}$$

$$V_L\,\cos\big(0 \vee |\phi| - \arcsin(Kd\cos\phi)\big) \leq V_F\,\cos\phi\,. \tag{9}$$

Furthermore v_F and ω_F are given by

$$\begin{aligned} v_F &= v_L \frac{\cos(\beta - \phi)}{\cos\phi} \\ \omega_F &= v_L\, \frac{\sin\beta}{d\cos\phi} \end{aligned} \tag{10}$$

where $\beta = \theta_L - \theta_F$ and, $\forall\, t \geq 0$

$$|\beta(t)| \leq \arcsin(Kd\cos\phi)\,. \tag{11}$$

Proof: see [4], Theorem 1. ■

Remark 1: From inequality (11) it follows that β is bounded. This implies the following geometric property

$$L(t) - F(t) \in \mathcal{C}\big(\theta_L(t) + \phi,\, \arcsin(Kd\cos\phi)\big) \tag{12}$$

where $\forall\, \theta,\, \gamma \in [0, 2\pi]$,

$$\mathcal{C}(\theta, \gamma) = \big\{x \in \mathbb{R}^2 \big| \langle\, x\;,\, \tau(\theta)\rangle \geq \|x\|\cos\gamma\big\}$$

is a cone of aperture 2γ centered in the origin, whose symmetry axis is given by $\tau(\theta) = (\cos\theta, \sin\theta)^T$ (see Fig. 2). Differently from [5], the followers are not rigidly disposed with respect to the leader reference frame, but their relative positions vary in time in suitable cones and only these cones remain stable with respect to the leader reference frame. To prove (12) remark that, from (11)

$$\begin{aligned}\langle L - F\;,\, \tau(\theta_L + \phi)\rangle &= \langle d\,\tau(\theta_F + \phi),\, \tau(\theta_L + \phi)\rangle \\ &= d\cos\beta \geq d\cos(\arcsin Kd\cos\phi).\end{aligned}$$

With respect to the cone, β is the angle between the line connecting the leader to the follower and the symmetry axis of the cone.

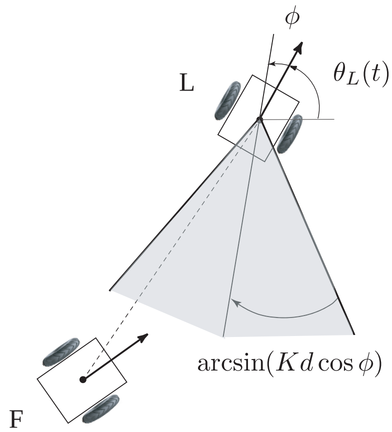


Fig. 2. The cone $\mathcal{C}(\theta_L(t)+\phi,\, \arcsin(K d\cos\phi))$.

IV. STABILIZATION

We now suppose that the leader and the follower start from arbitrary initial conditions. It is assumed that the leader velocity is bounded from below by a positive constant V_{0L}, namely, $0 < V_{0L} \le v_L \le V_L$. The control strategy consists of two steps.

In the first step the follower rotates with zero translational velocity until its direction is sufficiently close to that of the leader in order to satisfy the condition $|\beta| \le \alpha_\chi$, where α_χ is a suitable positive constant. In the second step the follower performs the control defined in the previous section with an added stabilizing term in order to reduce the error asymptotically to zero. The stabilizing term is chosen accurately in order to satisfy the input bounds (6).

Theorem 2: Suppose that

$$K d < 1 \tag{13}$$

$$V_L\, K < \Omega_F \tag{14}$$

$$V_L\, \cos\big(0 \vee |\phi| - \arcsin(K d\cos\phi)\big) < V_F\, \cos\phi \tag{15}$$

and let $\chi_1,\, \chi_2$ be any constants such that

$$K d < \chi_1 < \chi_2 < 1. \tag{16}$$

Set the following control functions

$$v_F = \begin{cases} 0 & \text{if } |\beta| > \alpha_\chi \\ \dfrac{\eta\,\langle E,\, \tau(\theta_F+\phi)\rangle + v_L \cos(\beta-\phi)}{\cos\phi} & \text{if } |\beta| \le \alpha_\chi \end{cases}$$

$$\omega_F = \begin{cases} \mathrm{sign}(\beta)\, \Omega_F & \text{if } |\beta| > \alpha_\chi \\ \dfrac{\eta\,\langle E,\, \nu(\theta_F)\rangle + v_L \sin\beta}{d\cos\phi} & \text{if } |\beta| \le \alpha_\chi \end{cases} \tag{17}$$

where η is given by

$$\eta = \frac{(\Omega_F - K v_L)\, d\cos\phi}{|\langle E,\, \nu(\theta_F)\rangle|} \wedge \frac{(v_L\, \chi_1 - |\omega_L|)\, d\cos\phi}{|\langle E,\, \nu(\theta_F)\rangle|} \wedge \frac{V_F\, \cos\phi - v_L \cos(\beta-\phi)}{|\langle E,\, \tau(\theta_F+\phi)\rangle|} \wedge \frac{v_L\, \cos(\beta-\phi)}{|\langle E,\, \tau(\theta_F+\phi)\rangle|} \wedge M \tag{18}$$

M is a positive constant and $E(t) = L(t) - F(t) - d\,\tau(\theta_F(t)+\phi)$ is the error vector. Then for any initial state $x_L(0),\, y_L(0),\, \theta_L(0),\, x_F(0),\, y_F(0),\, \theta_F(0)$, the solution of systems (1), (2), with $v_F,\, \omega_F$ given by (17), is such that $\lim_{t\to+\infty} E(t) = 0$, (6) hold and there exists $t_0 \ge 0$, such that $|\beta(t)| \le \alpha_\chi = \arcsin(\chi_2\, d\cos\phi),\ \forall\, t \ge t_0$.

Proof: Let $x_L,\, y_L,\, \theta_L,\ x_F,\, y_F,\, \theta_F$ be the solution of systems (1), (2) with $v_F,\, \omega_F$ given by (17) and initial conditions $x_L(0),\, y_L(0),\, \theta_L(0),\ x_F(0),\, y_F(0),\, \theta_F(0)$.
Let $\mathcal{A}^+ = \{t \,|\, \beta(t) \ge \alpha_\chi\}$ and $\mathcal{A}^- = \{t \,|\, \beta(t) \le -\alpha_\chi\}$. First it will be shown that

$$\begin{cases} \dot\beta(t) < -c, & \forall\, t \in \mathcal{A}^+ \\ \dot\beta(t) > c, & \forall\, t \in \mathcal{A}^- \end{cases} \tag{19}$$

where $c = (\Omega_F - K V_L) \wedge (\chi_2 - \chi_1)\, V_{0L}$ which is positive by (14) and (16). In fact, suppose that $t_0 \in \mathcal{A}^+$ and $\beta(t_0) > \alpha_\chi$, then, by (17), $\dot\beta(t_0) = \omega_L - \Omega_F \le -(\Omega_F - K\, V_L) \le -c$ if $\beta(t_0) = \alpha_\chi$, then, by (17) and (18)

$$\begin{aligned} \dot\beta(t_0) &= \omega_L(t_0) - \omega_F(t_0) \\ &= \omega_L(t_0) - \frac{\eta(t_0)\langle E(t_0),\, \nu(\theta_F(t_0))\rangle + v_L(t_0)\sin\alpha_\chi}{d\cos\phi} \\ &\le -\left(\frac{\Omega_F}{V_L}\, v_L(t_0) - \omega_L(t_0)\right) + \frac{\eta(t_0)\langle E(t_0),\, \nu(\theta_F(t_0))\rangle}{d\cos\phi} \\ &\le -(\chi_2\, v_L(t_0) - \omega_L(t_0)) + \\ &\quad + \mathrm{sign}\langle E(t_0),\, \nu(\theta_F(t_0))\rangle \big(\chi_1\, v_L(t_0) - |\omega_L(t_0)|\big) \\ &\le -(\chi_2 - \chi_1)\, V_{0L} \le -c\,. \end{aligned}$$

Therefore, by (19) there exists a unique $t_0 \ge 0$ such that

$$|\beta(t)| > \alpha_\chi,\ \ \forall\, t \in [0,\ t_0)\,,$$
$$|\beta(t)| \le \alpha_\chi,\ \ \forall\, t \ge t_0.$$

Then by definition of $v_F,\, \omega_F$ and η, the follower bounds are satisfied, being $\forall\, t \ge t_0$ (by (17), (18)):

$$\begin{aligned} |\omega_F| &\le \frac{\eta\,\langle E,\, \nu(\theta_F)\rangle + v_L \sin\alpha_\chi}{d\cos\phi} \le \Omega_F - K v_L + K v_L \le \Omega_F, \\ 0 &\le \frac{-\eta\,|\langle E,\, \tau(\theta_F+\phi)\rangle| + v_L \cos(\beta-\phi)}{\cos\phi} \le V_F \le v_F \\ &\le \frac{\eta\,|\langle E,\, \tau(\theta_F+\phi)\rangle| + v_L \cos(\beta-\phi)}{\cos\phi} \le V_F\ . \end{aligned}$$

Finally, from (18) it follows that

$$\eta \ge \frac{\Omega_F - K V_L}{\|E\|} \wedge \frac{(K-\chi_1)\, V_{0L}}{\|E\|} \wedge \frac{(V_F - \cos(0 \vee \arcsin(K\, d\,\cos\phi))\, (\cos\phi)^{-1})\cos\phi}{\|E\|} \wedge \frac{V_{0L}\cos(\phi+\alpha_\chi)}{\|E\|} \wedge M = \frac{c}{\|E\|} \wedge M\,.$$

Therefore $\forall\, t \ge t_0$,

$$\dot{\overline{\|E(t)\|^2}} \le -\big(c\, \|E(t)\|^{-1} \wedge M\big)\, \|E(t)\|^2$$

which implies that $\lim\limits_{t\to\infty} \|E(t)\| = 0$. ■

V. HIERARCHICAL MULTIROBOT FORMATIONS

This section extends the definition of (d, ϕ)-formation given in Sect. II for two robots R_L and R_F, to hierarchical multirobot formations and generalizes the results in Sect. III.

Definition 2: Let $D = \{d_i : d_i > 0, i = 1, \dots n\}$, $\Phi = \{\phi_i : |\phi_i| < \frac{\pi}{2}, i = 1, \dots, n\}$ be two sets of given parameters. Let $n_i,\ i = 1, \dots, n$ be such that $n_i \in \{0, \dots, i-1\}$. The set of $n+1$ robots $R_0,\ R_1,\ \dots, R_n$ make a (D, Φ)-formation (with leaders $R_0,\ R_{n_1},\ \dots, R_{n_n}$) if $\forall\, t \geq 0$, for $i = 1, \dots, n$

$$\begin{pmatrix} x_{n_i} \\ y_{n_i} \end{pmatrix}(t) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}(t) +\ d_i\, \tau(\theta_i + \phi_i)\ . \tag{20}$$

Note that the structure just introduced with the notion of (D, Φ)-formation represents a rooted tree in the context of group theory (see Fig. 3). In particular, if $n = 1$ and $n_1 = 0$ we find again the (d, ϕ)-formation defined in Sect. II, while if $n_i = i - 1$ for $i = 1, \dots, n$ we obtain a convoy-like formation. The following result is a generalization of Theorem 1.

Theorem 3: Let R_0 be a robot such that there exist three constants $V_0,\ K_0^-,\ K_0^+$ with the property:

$$0 < v_0(t) \leq V_0, \quad K_0^- \leq \frac{\omega_0(t)}{v_0(t)} \leq K_0^+,\ \forall\, t \geq 0\ . \tag{21}$$

Let $n_i,\ d_i,\ \phi_i$ be as in Definition 2. Define recursively the following $2n$ real extended constants,

$$K_i^{\pm} = \begin{cases} \left(\mathrm{sign}(K_{n_i}^{\pm}) \sqrt{\frac{1}{(K_{n_i}^{\pm})^2} - d_i^2 \cos^2 \phi_i} + d_i \sin \phi_i\right)^{-1} \\ \qquad\qquad\qquad\qquad \text{if} \quad \frac{1}{(K_{n_i}^{\pm})^2} - d_i^2 \cos^2 \phi_i > 0 \\ \mathrm{sign}(K_{n_i}^{\pm}) \cdot \infty \qquad \text{if} \quad \frac{1}{(K_{n_i}^{\pm})^2} - d_i^2 \cos^2 \phi_i \leq 0 \end{cases} \tag{22}$$

$i = 1, \dots, n$, with the convention that $(+1) \cdot \infty = +\infty$, $(-1) \cdot \infty = -\infty$. Let $x_i,\ y_i,\ \theta_i,\ i = 1, \dots, n$ be the solution of the following system

$$\begin{cases} \dot{x}_i = v_i \cos \theta_i \\ \dot{y}_i = v_i \sin \theta_i \\ \dot{\theta}_i = \omega_i \\ v_i = v_{n_i} \dfrac{\cos(\beta_i - \phi_i)}{\cos \phi_i} \\ \omega_i = v_{n_i} \dfrac{\sin \beta_i}{d_i \cos \phi_i} \\ x_i(0) = x_i^0,\ y_i(0) = y_i^0,\ \theta_i(0) = \theta_i^0 \end{cases}$$

where $\beta_i(t) = \theta_{n_i}(t) - \theta_i(t)$ and $x_i^0,\ y_i^0,\ \theta_i^0$ are assigned constants. Suppose that

$$|K_{n_i}^-| \vee |K_{n_i}^+| < \frac{1}{d_i}\,, \quad \forall\, i = 1, \dots, n, \tag{23}$$

then, $\forall\, i = 1, \dots, n$

$$-\infty < K_i^- < K_i^+ < +\infty\ . \tag{24}$$

Furthermore, if the robots are in (D, Φ)-formation at time $t = 0$, namely, for $i = 1, \dots, n$,

$$\begin{pmatrix} x_{n_i}^0 \\ y_{n_i}^0 \end{pmatrix} = \begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix} +\ d_i \begin{pmatrix} \cos(\theta_i^0 + \phi_i) \\ \sin(\theta_i^0 + \phi_i) \end{pmatrix}$$

where $x_{n_i}^0,\ y_{n_i}^0$ are assigned values, and

$$\arcsin(K_{n_i}^- d_i \cos \phi_i) \leq \beta_i(0) \leq \arcsin(K_{n_i}^+ d_i \cos \phi_i)$$

then the formation is held $\forall\, t \geq 0$, that is equation (20) is satisfied $\forall\, t \geq 0$. Moreover $\forall\, t \geq 0$ the following bounds are satisfied,

$$\arcsin(K_{n_i}^- d_i \cos \phi_i) \leq \beta_i(t) \leq \arcsin(K_{n_i}^+ d_i \cos \phi_i) \tag{25}$$

$$0 < v_i(t) \leq V_i \tag{26}$$

$$K_i^- \leq \frac{\omega_i(t)}{v_i(t)} \leq K_i^+ \tag{27}$$

where

$$V_i = \frac{V_{n_i}}{\cos \phi_i} \cos\big(0 \wedge \arcsin(K_{n_i}^+ d_i \cos \phi_i) - \phi_i \wedge \phi_i - \arcsin(K_{n_i}^- d_i \cos \phi_i)\big)\ .$$

Remark 2: Since the internal dynamics β_i is bounded, the following generalization of (12) holds,

$$\begin{pmatrix} x_{n_i} \\ y_{n_i} \end{pmatrix}(t) - \begin{pmatrix} x_i \\ y_i \end{pmatrix}(t) \in \mathcal{C}\big(\theta_{n_i}(t) + \phi_i,\ \arcsin(K_{n_i}^{\pm} d_i \cos \phi_i)\big)$$

where $\mathcal{C}\big(\theta_{n_i}(t) + \phi_i,\ \arcsin(K_{n_i}^{\pm} d_i \cos \phi_i)\big)$ represents a cone having right, left semiaperture $\arcsin(K_{n_i}^+ d_i \cos \phi_i)$, $\arcsin(K_{n_i}^- d_i \cos \phi_i)$ and axis $\tau(\theta_{n_i}(t) + \phi_i)$.

Proof of Theorem 3: The following equation holds,

$$\dot{\beta}_i = \dot{\theta}_{n_i} - \dot{\theta}_i = \frac{v_{n_i}}{d_i \cos \phi_i} \left[\frac{\omega_{n_i}}{v_{n_i}} d_i \cos \phi_i - \sin \beta_i\right].$$

Since, by hypothesis, $\arcsin(K_{n_i}^- d_i \cos \phi_i) \leq \beta_i(0) \leq \arcsin(K_{n_i}^+ d_i \cos \phi_i)$, by (23),

$$\arcsin(K_{n_i}^- d_i \cos \phi_i) \leq \beta_i(t) \leq \arcsin(K_{n_i}^+ d_i \cos \phi_i) \tag{28}$$

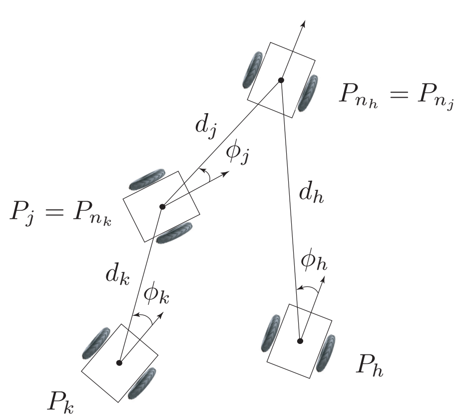
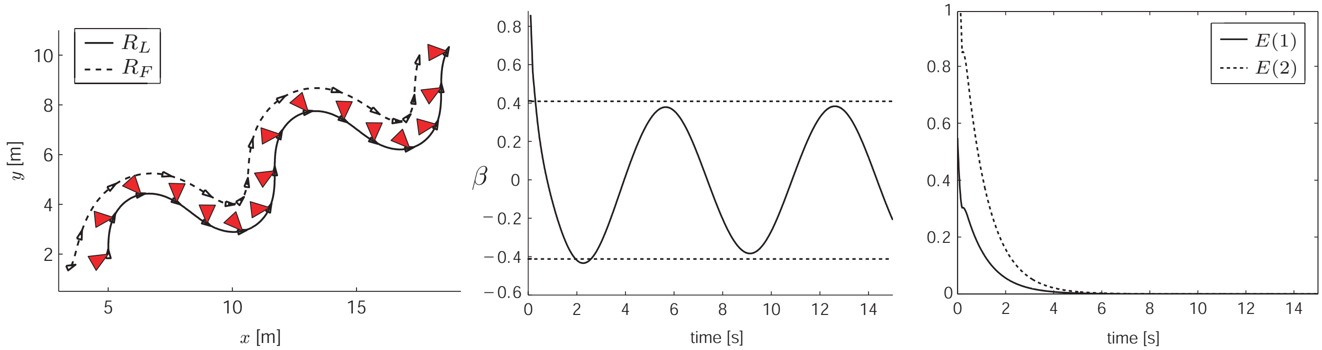


Fig. 3. Notation used in a sample (D, Φ)-formation: $P_h = (x_h,\, y_h)$.



(a) (b) (c)

Fig. 4. (a) Trajectory of the robots and cones, (robots are drawn each second); (b) Angle β and bounds $\pm\alpha$; (c) Error vector E.

for any $t \geq 0$ and therefore (25) holds. To verify (26), remark that it is true for $i = 0$, by hypothesis (21); suppose that it is true $\forall j : 0 \leq j \leq i-1$, then it is true for $j = i$. In fact, owing to (28) and (23)

$$
\begin{aligned}
v_i(t) = v_{n_i}(t)\,\frac{\cos(\beta_i(t)-\phi_i)}{\cos\phi_i} \geq \frac{v_{n_i}(t)}{\cos\phi_i}\big(\,|\cos\beta_i(t)|\cos\phi_i - \\
|\sin\beta_i(t)\cos\phi_i|\big) \geq v_{n_i}(t)\Big[\sqrt{1-\big(d_i(|K^-_{n_i}|\vee|K^+_{n_i}|)\cos\phi_i\big)^2} \\
- d_i\left(|K^-_{n_i}|\vee|K^+_{n_i}|\right)|\sin\phi_i|\Big] > 0
\end{aligned}
$$

and

$$
\begin{aligned}
v_i(t) = {} & v_{n_i}(t)\,\frac{\cos(\beta_i(t)-\phi_i)}{\cos\phi_i} \\
\leq {} & \frac{V_{n_i}}{\cos\phi_i}\cos\big(0\wedge\arcsin(K^+_{n_i}d_i\cos\phi_i)-\phi_i \\
& \wedge\,\phi_i-\arcsin(K^-_{n_i}d_i\cos\phi_i)\big)\,.
\end{aligned}
$$

To verify property (27), remark that it is true for $i = 0$, by hypothesis (21); suppose that it is true $\forall j : 0 \leq j \leq i-1$, then it is true for $j = i$. In fact the curvature of the path of the ith vehicle is given by

$$
\frac{\omega_i(t)}{v_i(t)} = \frac{\sin\beta_i}{d_i\cos(\beta_i-\phi_i)} = \frac{1}{d_i(\cot\beta_i\cos\phi_i+\sin\phi_i)}\,.
$$

Being the function

$$
f(\beta_i) = \begin{cases} \dfrac{1}{d_i\left[\cot\beta_i\cos\phi_i+\sin\phi_i\right]} & \text{if } \beta_i \neq 0 \\ 0 & \text{if } \beta_i = 0 \end{cases}
$$

monotone increasing, it follows by (28) that

$$
\begin{aligned}
\frac{1}{d_i[\cot(\arcsin(K^-_{n_i}d_i\cos\phi_i))\cos\phi_i+\sin\phi_i]} \leq \frac{\omega_i(t)}{v_i(t)} \leq \\
\frac{1}{d_i[\cot(\arcsin(K^+_{n_i}d_i\cos\phi_i))\cos\phi_i+\sin\phi_i]}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sqrt{\frac{1}{(K^-_{n_i})^2}-d_i^2\cos^2\phi_i}+d_i\sin\phi_i} \leq \frac{\omega_i(t)}{v_i(t)} \leq \\
\frac{1}{\sqrt{\frac{1}{(K^+_{n_i})^2}-d_i^2\cos^2\phi_i}+d_i\sin\phi_i}
\end{aligned}
$$

(remark that the denominator is not zero since (23) holds) and the proof of the theorem is over. ■

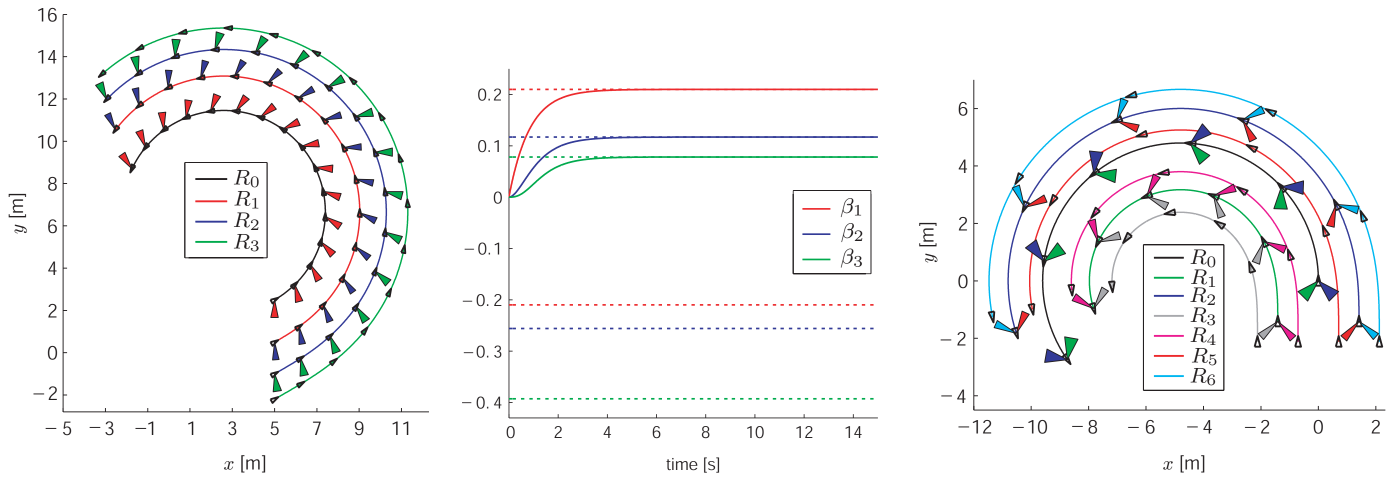
VI. SIMULATION RESULTS

Figs. 4 and 5 show the results of the simulation experiments we carried out to evaluate to the effectiveness of the proposed control strategies.

In Fig. 4 we used the stabilizing controller presented in Theorem 2. The initial conditions are $(x_L(0),\, y_L(0),\, \theta_L(0))^T = (5,\,2,\,\pi/2)^T$, $(x_F(0),\, y_F(0),\, \theta_F(0))^T = (3,\,1.5,\,\pi/8)^T$ and the desired values $d = 1.2$ m, $\phi = -\pi/3$ rad. We set $v_L(t) = 1.5$ m/s, $\omega_L(t) = -\sin(0.9\,t)$ rad/s, $K = 1/1.5 = 0.6667$ rad/m, $V_L = 1.5$ m/s, $V_F = 4$ m/s, $\Omega_F = \pi$ rad/s. With these values, conditions (13)-(15) are satisfied. The controller parameters are $M = 1$, $\chi_1 = 0.85$, $\chi_2 = 0.9$ (hence condition (16) is verified). Fig. 4(a) shows the trajectory of R_L, R_F and a limited portion of the infinite cones defined in Remark 1. The formation converges to the desired configuration and R_F keeps inside the cones at steady state. In Fig. 4(b) the time history of angle β (solid) and bounds $\pm\,\alpha$ (dash) is shown. The error vector E is zero after about 5 seconds (Fig. 4(c)).

In Fig. 5 we used the control scheme presented in Theorem 3. Two (D,Φ)-formations are considered: a *convoy-like formation* and a *tree formation*.

The convoy-like formation consists of four robots, $R_0,\ldots,R_3$, with $n_1 = 0$, $n_2 = 1$, $n_3 = 2$. At time $t = 0$ the robots are in formation. We set $v_0(t) = 1.2$ m/s, $\omega_0(t) = 0.25$ rad/s and we chose $V_0 = 1.2$ m/s, $K_0^{\pm} = \pm\,0.25/1.2 = \pm\,0.2083$ rad/m. The desired values are $d_1 = 2$ m, $d_2 = 1.5$ m, $d_3 = 1.2$ m, $\phi_i = \pi/3$ rad, $i = 1,2,3$. We propagated the curvature bounds according to equation (22). With these values condition (23) is satisfied



(a) (b) (c)

Fig. 5. *Convoy-like formation*: (a) Trajectory of the robots and cones; (b) Angles β_i and bounds $\arcsin(K^{\pm}_{n_i} d_i \cos\phi_i)$, $i = 1, 2, 3$. *Tree formation*: (c) Trajectory of the robots and cones.

and inequality (24) holds. Fig. 5(a) shows the trajectory of the four robots and the cones defined in Remark 2. In Fig. 5(b) the time history of the angles β_i (solid) and bounds $\arcsin(K^{\pm}_{n_i} d_i \cos\phi_i)$ (dash), $i = 1, 2, 3$ is given. Since β_i keep inside the respective bounds, condition (25) is satisfied. Analogously, v_i and ω_i/v_i satisfy conditions (26), (27).

The tree formation consists of seven robots, $R_0, \ldots, R_6$, with $n_1 = n_2 = 0$, $n_3 = n_4 = 1$ and $n_5 = n_6 = 2$. We set $v_0(t)$, $\omega_0(t)$, V_0 and $K^{\pm}_0$ as in the previous case. The desired values are $d_1 = d_2 = 2\,\text{m}$, $d_3 = d_4 = d_5 = d_6 = 1\,\text{m}$, $\phi_1 = \phi_3 = \phi_5 = -\pi/4$ rad, $\phi_2 = \phi_4 = \phi_6 = \pi/4$ rad. Fig. 5(c) shows the trajectory of the robots and the cones. The time history of the angles β_i, $i = 1, \ldots, 6$, is neglected being similar to that of the convoy-like formation.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper we study leader-follower formations of nonholonomic mobile robots and we propose a setup and a control strategy that are alternative to those existing in the literature. The formation control problem has been formalized in a geometric framework able to explain how the geometric properties of the formation affect the set of the admissible curvatures of the leader trajectory. We prove that during the control, the vector connecting the follower to the leader remains in a given cone centered in the leader reference frame. Basic results are extended to hierarchical multirobot formations described by rooted tree graphs. Simulation experiments show the effectiveness of our designs.

Future research lines include the experimental validation of our control strategies and the extension of Theorem 2 to hierarchical multirobot formations.

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