1819-108-C1-lappuse

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February 2019

[You will find it convinient to use

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \frac{2n!\sqrt{\pi}}{2^{2n} n!}$$

for integer $n \geq 0$.]

18.9 By initially writing y(x) as $x^{1/2} f(x)$ and then making subsequent changes of variable, reduse Stokes' equation,

$$\frac{d^2y}{dx^2} + \lambda xy = 0,$$

to Bessel's equation. Hence show that a solution that is finite at x = 0 is a multiple of $x^{1/2} J_{1/3}(\frac{2}{3}\sqrt{\lambda x^3})$

18.10 By choosing a suitable form for h in their generating function,

$$G(z,h) = exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) h^n$$

show that integral repesentations of the Bessel functions of the first kind are given, for integral m, by

$$J_2 m(z) = \frac{(-1)^m}{2\pi}\int_0^{2\pi} cos(z\ cos\ \theta)\ cos\ 2m\theta\ d\theta \qquad m \geq 1,$$

$$J_2 m+1(z) = \frac{(-1)^m}{2\pi}\int_0^{2\pi} sin(z\ cos\ \theta)\ cos(2m+1)\theta d\ \theta \qquad m \geq 0.$$

18.11 Identify the series for the following hypergeometric functions, writing them in terms of better known functions:

(a) F$(a,b,b;z)$,
(b) F(1,1,2;-x),
(c) $F(\frac{1}{2},1,\frac{3}{2};-x^2)$,
(d) $F(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2)$,
(e) $F(-a,a,\frac{1}{2};sin^2 x)$; this is a much more difficult exercise.

18.12 By making the substitution $z = (1-x)/2$ and suitable choices for a, b and c, convert the hypergeometric equation,

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0,$$

into the Legendre equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \ell(\ell+1)y = 0$$

Hence, using the hypergeometric series, generate the Legendre polynomials $P_\ell(x)$ for the integer values $\ell = 0, 1, 2, 3$. Comment on their normalisations.

18.13 Find a change of variable that will allow the integral

$$I = \int_1^{\infty} \frac{\sqrt{u-1}}{(u+1)^2} du$$

to be expressed in terms of the beta funtion, and so evaluate it.

18.14 Prove that, if m and n are both greater than -1, then

$$I = \int_0^{\infty} \frac{u^m}{(au^2+b)^{(m+n+2)/2}} du = \frac{\Gamma[\frac{1}{2}(m+1)]\Gamma[\frac{1}{2}(n+1)]}{2a^{(m+1)/2} b^{(n+1)/2} \Gamma[\frac{1}{2}(m+n+2)]}.$$