Datormācības eksāmens

Sindija Vītola

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The equations become

$$\frac{dp_n(t)}{dt} = \lambda p_n(t) + \lambda p_{n-1}(t) \qquad for \quad n \geq 1,$$

$$\frac{dp_0(t)}{dt} = \lambda p_0(t).$$

If initially $p_n(0) = 1$ if $n = 0$ and is equal to 0 if $n \geq 1$, which means that the system begins empty, the

$$p_0(t) = e^{-\lambda t},$$

since the solution is given by $p_0(t) = ce^{-\lambda t}$ where c, the constant of integration, must be equal to one ($p_0(0) = 1 = ce^0$). Furthermore, from

$$\frac{dp_1(t)}{dt} = \lambda p_1(t) + \lambda e^{-\lambda t}$$

the solution is clearly

$$p_1(t) = \lambda t e^{-\lambda t}$$

since application of the product rule $(uv)' = udv + vdu$ with $u = \lambda t$ and $v = e^{-\lambda t}$ yields

$$\lambda t(-\lambda e^{-\lambda t}) + \lambda e^{-\lambda t} = -\lambda p_1(t) + \lambda e^{-\lambda t}$$

Similarly, continuing by inducion, we obtain

$$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \geq 0, \quad t \geq 0.$$

This is now seen as the formula for he Poisson distribution with mean λt. In other words, the Poisson process is a pure birth process with constant birth rate λ.

11.3.2 Steady-State Solution

In the staedy state, which we assume exists,

$$\frac{dp_n(t)}{dt} = 0,$$

and hence Equation (11.8) becomes

$$0 = \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} - (\lambda_n + \mu_n)p_n, \quad n \geq 1,$$

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \Rightarrow p_1 = \frac{\lambda_0}{\mu_1}p_0,$$

where p_n is defined as the limiting probability, $p_n = \lim_{t\to\infty} p_n(t)$, the probability that the system conains n customers once it reaches steady state where all influence of the starting state has been erased. These equations are called the global balance equations. They may also be written as

$$p_{n+1} = \frac{\lambda_n + \mu_n}{\mu_{n+1}}p_n - \frac{\lambda_{n-1}}{\mu_{n+1}}p_{n-1}, \quad n \geq 1,$$

$$p_1 = \frac{\lambda_0}{\mu_1}p_0.$$

11.3 General Birth-Death Processes 415

The equations become

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Similarly, continuing by induction, we obtain

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In the steady state, which we assume exists,

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and hence Equation (11.8) becomes

$$0 = \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} - (\lambda_n + \mu_n)p_n, \quad n \geq 1, \tag{11.9}$$

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \implies p_1 = \frac{\lambda_0}{\mu_1}p_0,$$

where p_n is defined as the limiting probability, $p_n = \lim_{t\to\infty} p_n(t)$, the probability that the system contains n customers once it reaches steady state where all influence of the starting state has been erased. These equations are called the global balance equations. They may also be written as

$$p_{n+1} = \frac{\lambda_n + \mu_n}{\mu_{n+1}}p_n - \frac{\lambda_{n-1}}{\mu_{n+1}}p_{n-1}, \quad n \geq 1,$$

$$p_1 = \frac{\lambda_0}{\mu_1}p_0.$$