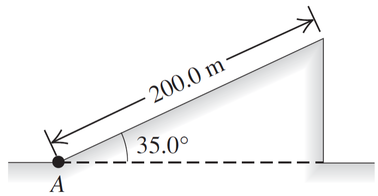
Compound Projectile Motion Problem #1

Motion in Two or Three Dimensions

Question: *A test rocket is launched by accelerating it along a 200.0-m incline at 1.25 m/s² starting from rest at point A. (See figure below.)*



The incline rises at 35.0° above the horizontal, and at the instant the rocket leaves it, its engines turn off and it is subject only to gravity (air resistance can be ignored.)

Find (a) the maximum height above the ground that the rocket reaches, and (b) the greatest horizontal range of the rocket beyond point A.

Solution: To obtain the maximum height and range of the rocket, it is necessary to know the velocity of the rocket once it leaves the ramp. We can find this quantity by separating the motion of the rocket into multiple stages.

Let stage 1 consist of the 1D movement of the rocket along the length of the ramp, and let stage 2 be the 2D projectile motion of the rocket once it no longer touches the ramp.

The final velocity at the end of the ramp in stage 1 is found from the following kinematic equation:

$$v_f^2 = v_i^2 + 2aL \tag{1}$$

Where v_f is the final velocity, v_i is the initial velocity, a is the rocket's acceleration, and L is the length of the ramp. Because the rocket is launched from rest, $v_i = 0$. The final velocity is:

$$v_f = \sqrt{2aL}$$

a.) The maximum height is found when the final velocity in the vertical direction is zero in stage 2. The 2D equivalent of equation (1) is given by:

$$v_{f_y}^2 = v_{i_y}^2 + 2a_y\Delta y \tag{2}$$

Where Δy is the vertical distance between the top of the ramp and the maximum height. Setting $v_{f_y} = 0$ and $a_y = -g$ allows Δy to be isolated:

$$0 = v_{i_y}^2 - 2g\Delta y$$

$$2g\Delta y = v_{i_y}^2$$

$$\Delta y = \frac{v_{i_y}^2}{2g}$$

The initial velocity in stage 2 is the final velocity from stage 1, but now multiplied by the sine of the ramp angle (due to the rocket moving 2-dimensionally in this stage):

$$\Delta y = \frac{(\sqrt{2aL}\sin(\theta))^2}{2g}$$

$$= \frac{\cancel{2}aL\sin^2(\theta)}{\cancel{2}g}$$

$$= \frac{aL\sin^2(\theta)}{g}$$

The maximum height above the ground is then the vertical distance of the ramp plus Δy:

$$h_{max} = h_{ramp} + \Delta y$$

$$= L\sin(\theta) + \Delta y$$

$$= (200.0\,m)\sin(35.0°) + \frac{(1.25\frac{m}{s^2})(200.0\,m)\sin^2(35.0°)}{(9.80\frac{m}{s^2})}$$

$$\boxed{\approx 123\,m}$$

To find the maximum range, we need to know the time of flight for the rocket in stage 2. Begin by starting with the kinematic equation for vertical displacement:

$$y_f = y_i + v_{i_y}t + \frac{1}{2}at^2 \tag{3}$$

When the rocket smashes into the ground, $y_f = 0$. Replace y_i with the height of the ramp, v_{i_y} with the vertical velocity at the start of stage 2, and a with the acceleration due to gravity:

$$0 = L\sin(\theta) + \sqrt{2aL}\cdot\sin(\theta)t - \frac{1}{2}gt^2$$

This result is a quadratic expression in t. The quadratic coefficients are:

$$A = -\frac{1}{2}g$$

$$B = \sqrt{2aL}\sin(\theta)$$

$$C = L\sin(\theta)$$

Plugging these into the quadratic equation and solving for t:

$$t = \frac{-(\sqrt{2aL}\sin(\theta)) \pm \sqrt{(\sqrt{2aL}\sin(\theta))^2 - 4(-\frac{1}{2}g)(L\sin(\theta))}}{\cancel{2}(-\frac{\cancel{1}}{\cancel{2}}g)}$$

$$= \frac{-(\sqrt{2aL}\sin(\theta)) \pm \sqrt{2aL\sin^2(\theta) + 2gL\sin(\theta)}}{-g}$$

$$= \frac{-(\sqrt{2aL}\sin(\theta)) \pm \sqrt{2L\sin(\theta)(a\sin(\theta) + g)}}{-g}$$

$$= \frac{\sqrt{2aL}\sin(\theta) \mp \sqrt{2L\sin(\theta)(a\sin(\theta) + g)}}{g}$$

Using the minus sign results in a negative time, which can be discarded. The positive result is:

$$= \frac{\sqrt{2(1.25\frac{m}{s^2})(200.0\,m)}\sin(35.0°) + \sqrt{2(200.0\,m\cdot\sin(35.0°)(1.25\frac{m}{s^2}\sin(35.0°) + 9.80\,\frac{m}{s^2})}}{9.80\,\frac{m}{s^2}}$$

$$\approx 6.32\,s$$

b.) With the time of flight in stage 2 known, the horizontal version of equation (3) can be used to obtain the final distance of the rocket:

$$x_f = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \tag{4}$$

Equation (4) will be applied once the rocket leaves the ramp.

The initial horizontal displacement x_i is the horizontal length of the ramp traveled in stage 1, and the rocket experiences no horizontal acceleration throughout the entire trip:

$$x_f = L\cos(\theta) + \sqrt{2aL}\cos(\theta)t$$

$$= (200.0\,m)\cos(35.0°) + \sqrt{2(1.25\,\frac{m}{s^2})(200.0\,m)}\cos(35.0°)$$

$$\boxed{\approx 280\,m}$$