Compound Projectile Motion Problem #5

Motion in Two or Three Dimensions

Question: *A projectile is thrown from a point P. It moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown. You can ignore air resistance.*

Solution: Let P be the origin of a 2D coordinate system.

The projectile's distance from P is given by the position vector $\vec{r}$:

$$|\vec{r}\,| = \sqrt{x^2 + y^2}$$

Where x and y are the kinematic equations for displacement (with x_i & y_i equal to zero):

$$x = v_i cos(\theta)t$$
$$y = v_i sin(\theta)t - \frac{1}{2}gt^2$$

Substituting these into the position vector:

$$|\vec{r}\,| = \sqrt{\Big(v_i cos(\theta)t\Big)^2 + \Big(v_i sin(\theta)t - \frac{1}{2}gt^2\Big)^2} \tag{1}$$

Given that the projectile's distance from P is always increasing, this would imply that its rate of change of $\vec{r}$ with respect to time is always positive:

$$\frac{d|\vec{r}\,|}{dt} = \frac{d}{dt}\left(\sqrt{\Big(v_i cos(\theta)t\Big)^2 + \Big(v_i sin(\theta)t - \frac{1}{2}gt^2\Big)^2}\right)$$

Expanding the part under the square root:

$$= \frac{d}{dt}\sqrt{{v_i}^2 cos^2(\theta)t^2 + {v_i}^2 sin^2(\theta)t^2 - \cancel{2} \cdot \frac{g}{\cancel{2}} v_i sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

$$= \frac{d}{dt}\sqrt{{v_i}^2 t^2 \cancel{\Big(cos^2(\theta) + sin^2(\theta)\Big)} - g v_i sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

$$= \frac{d}{dt}\sqrt{{v_i}^2 t^2 - g v_i sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

At this point, let $u = {v_i}^2t^2 - gv_isin(\theta)t^3 + \frac{g^2}{4}t^4$ and $\frac{du}{dt} = 2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3$ such that:

$$\begin{aligned}
\frac{d}{dt}\sqrt{{v_i}^2t^2 - gv_isin(\theta)t^3 + \frac{g^2}{4}t^4} &= \frac{d}{du}\sqrt{u} \cdot \frac{du}{dt} \\
&= \frac{1}{2\sqrt{u}} \cdot 2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3 \\
&= \frac{2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3}{2\sqrt{{v_i}^2t^2 - gv_isin(\theta)t^3 + \frac{g^2}{4}t^4}}
\end{aligned}$$

We can shorten this by recognizing that u is simply $x^2 + y^2$ and $\sqrt{x^2 + y^2} = r$. Thus:

$$\frac{d|\vec{r}\,|}{dt} = \frac{2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3}{2r} \tag{2}$$

Since (2) is always increasing, this means we can apply the following condition:

$$\frac{2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3}{2r} > 0$$

$$2{v_i}^2t - 3gv_isin(\theta)t^2 + g^2t^3 > 0$$

$$2{v_i}^2 - 3gv_isin(\theta)t + g^2t^2 > 0$$

This is a quadratic expression in t. The quadratic coefficients are:

$$\begin{aligned}
&A = g^2 \\
&B = -3gv_isin(\theta) \\
&C = 2{v_i}^2
\end{aligned}$$

Plugging these into the quadratic equation:

$$t = \frac{-\Big(-3gv_i sin(\theta)\Big) \pm \sqrt{\Big(-3gv_i sin(\theta)\Big)^2 - 4(g^2)(2v_i{}^2)}}{2(g^2)}$$

$$= \frac{3gv_i sin(\theta) \pm \sqrt{9\Big(gv_i sin(\theta)\Big)^2 - 8(gv_i)^2}}{2g^2}$$

Rather than solve the quadratic completely, we can take a shortcut and inspect the individual parts of the expression for t. The $3gv_i sin(\theta)$ term in the numerator and the $2g^2$ term in the denominator are both positive, and the overall result for t <u>must</u> be positive as well (since projectiles don't move backwards in time).

If the square root term is *also* positive, then it's possible for $t \leq 0$ if the determinant is a large enough number. This would violate our requirement of $t > 0$.

Therefore, it must be that:

$$\sqrt{9\Big(gv_i sin(\theta)\Big)^2 - 8(gv_i)^2} \leq 0$$

Using this to solve for the maximum angle θ:

$$9\Big(gv_i sin(\theta)\Big)^2 - 8(gv_i)^2 \leq 0$$

$$9\cancel{g^2 v_i^2} sin^2(\theta) \leq 8\cancel{g^2 v_i^2}$$

$$9 sin^2(\theta) \leq 8$$

$$sin^2(\theta) \leq \frac{8}{9}$$

$$sin(\theta) \leq \frac{2\sqrt{2}}{3}$$

$$\theta \leq sin^{-1}\Big(\frac{2\sqrt{2}}{3}\Big)$$

Thus, the maximum angle of θ that causes t to be strictly positive is $\boxed{70.5°.}$