Practice Problem #6
Newton's Laws of Motion

Question: *An object of mass m is at rest in equilibrium at the origin. At t = 0, a new force $\vec{F}$ is applied that has components:*

$$F_x(t) = k_1 + k_2 y \qquad F_y(t) = k_3 t$$

where k_1, k_2, and k_3 are constants.

Calculate the position $\vec{r}(t)$ and velocity $\vec{v}(t)$ vectors as functions of time.

Solution: Using the vector form of Newton's 2nd Law, we can obtain the expression for $\vec{a}$:

$$\vec{F} = m\vec{a}$$

$$\frac{\vec{F}}{m} = \vec{a}$$

$$\frac{(k_1 + k_2 y)\hat{\boldsymbol{\imath}} + (k_3 t)\hat{\boldsymbol{\jmath}}}{m} = \vec{a}$$

$$\left(\frac{k_1 + k_2 y}{m}\right)\hat{\boldsymbol{\imath}} + \left(\frac{k_3 t}{m}\right)\hat{\boldsymbol{\jmath}} = \vec{a}$$

The idea is to integrate this expression with respect to time for $\vec{v}(t)$, and then once more for $\vec{r}(t)$. There is an issue, however, due to the $\hat{\imath}$-component of $\vec{F}$ being expressed in terms of y. This means that we cannot integrate both components simultaneously.

What we can do is integrate the $\hat{\jmath}$-component of the acceleration to obtain the expression for y. With that known, we can then plug that into the $\hat{\imath}$-component and integrate as intended.

Starting with the integration of the $\hat{j}$-component of $\vec{a}$:

$$\begin{aligned}
v_y(t) &= \int a_y(t)\,dt + v_{iy} \\
&= \int \left(\frac{k_3 t}{m}\hat{\boldsymbol{j}}\right) dt + 0 \\
&= \frac{k_3}{m}\hat{\boldsymbol{j}} \int t\,dt \\
&= \frac{k_3}{m}\hat{\boldsymbol{j}} \left[\frac{t^2}{2}\right] \\
&= \frac{k_3 t^2}{2m}\hat{\boldsymbol{j}}
\end{aligned}$$

Integrating once more:

$$\begin{aligned}
y(t) &= \int v_y(t) dt + y_i \\
&= \int \left(\frac{k_3 t^2}{2m}\hat{\boldsymbol{j}}\right) dt + 0 \\
&= \frac{k_3}{2m}\hat{\boldsymbol{j}} \int t^2 dt \\
&= \frac{k_3}{2m}\hat{\boldsymbol{j}} \left[\frac{t^3}{3}\right] \\
&= \frac{k_3 t^3}{6m}\hat{\boldsymbol{j}}
\end{aligned}$$

Plugging this value of y into the $\hat{\imath}$-component of $\vec{a}$:

$$a_x(t) = \left(\frac{k_1 + k_2\left(\frac{k_3 t^3}{6m}\right)}{m}\right)\hat{\boldsymbol{\imath}}$$

$$= \left(\frac{k_1}{m} + \frac{k_2 k_3 t^3}{6m^2}\right)\hat{\boldsymbol{\imath}}$$

Integrating this component for the velocity:

$$v_x(t) = \int a_x(t)\, dt + v_{ix}$$

$$= \int \left[\left(\frac{k_1}{m} + \frac{k_2 k_3 t^3}{6m^2}\right)\hat{\boldsymbol{\imath}}\right] dt + 0$$

$$= \left[\frac{k_1 t}{m} + \frac{k_2 k_3 t^4}{24m^2}\right]\hat{\boldsymbol{\imath}}$$

Integrating once more:

$$x(t) = \int v_x(t)\, dt + x_i$$

$$= \int \left[\left(\frac{k_1 t}{m} + \frac{k_2 k_3 t^4}{24m^2}\right)\hat{\boldsymbol{\imath}}\right] + 0$$

$$= \left[\frac{k_1 t^2}{2m} + \frac{k_2 k_3 t^5}{120m^2}\right]\hat{\boldsymbol{\imath}}$$

Putting it all together for both vectors:

$$\vec{v}(t) = \boxed{\left[\frac{k_1 t}{m} + \frac{k_2 k_3 t^4}{24m^2}\right]\hat{\boldsymbol{\imath}} + \frac{k_3 t^2}{2m}\hat{\boldsymbol{\jmath}}}$$

$$r\vec{(t)} = \boxed{\left[\frac{k_1 t^2}{2m} + \frac{k_2 k_3 t^5}{120m^2}\right]\hat{\boldsymbol{\imath}} + \frac{k_3 t^3}{6m}\hat{\boldsymbol{\jmath}}}$$