Compound Projectile Motion Problem #5

Motion in Two or Three Dimensions

A projectile is thrown from a point P. It moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown. You can ignore air resistance.

Solution

Let P be the origin of a 2D coordinate system. As the projectile travels, its distance from P is given by the magnitude of its time-dependent position vector:

$$|\vec{r}(t)| = \sqrt{x(t)^2 + y(t)^2} \tag{1}$$

Where $x(t)$ and $y(t)$ represent the time-dependent kinematic equations for displacement:

$$x(t) = x_\mathrm{i} + v_\mathrm{i}\cos(\theta)t + \frac{1}{2}a_x t^2$$
$$y(t) = y_\mathrm{i} + v_\mathrm{i}\sin(\theta)t + \frac{1}{2}a_y t^2$$

Where $x_\mathrm{i} = 0$ and $y_\mathrm{i} = 0$ due to the motion starting from the origin at P.
Additionally, $a_x = 0$ and $a_y = -g$. Therefore:

$$x(t) = v_\mathrm{i}\cos(\theta)t$$
$$y(t) = v_\mathrm{i}\sin(\theta)t - \frac{1}{2}gt^2$$

Substituting the equations above into (1):

$$|\vec{r}(t)| = \sqrt{\Big(v_\mathrm{i}\cos(\theta)t\Big)^2 + \Big(v_\mathrm{i}\sin(\theta)t - \frac{1}{2}gt^2\Big)^2}$$

If the projectile's distance from P is always increasing, then this means the derivative of $|\vec{r}(t)|$ with respect to time is always positive:

$$\frac{\mathrm{d}|\vec{r}(t)|}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\Big(v_i\cos(\theta)t\Big)^2 + \Big(v_i\sin(\theta)t - \frac{1}{2}gt^2\Big)^2}\right) \tag{2}$$

This restricts the location of the projectile on the xy plane to the first quadrant.

Expansion and simplification of (2) yields:

$$= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{{v_{\mathrm{i}}}^2\cos^2(\theta)t^2 + {v_{\mathrm{i}}}^2\sin^2(\theta)t^2 - \cancel{2}\cdot\frac{g}{\cancel{2}}v_{\mathrm{i}}\sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{{v_{\mathrm{i}}}^2t^2\cancel{\Big(\cos^2(\theta)+\sin^2(\theta)\Big)} - gv_{\mathrm{i}}\sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{{v_{\mathrm{i}}}^2t^2 - gv_{\mathrm{i}}\sin(\theta)t^3 + \frac{g^2}{4}t^4}$$

To execute the derivative, apply the u-substitution technique:

Let $u = {v_{\mathrm{i}}}^2t^2 - gv_{\mathrm{i}}\sin(\theta)t^3 + \frac{g^2}{4}t^4$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = 2{v_{\mathrm{i}}}^2t - 3gv_{\mathrm{i}}\sin(\theta)t^2 + g^2t^3$ such that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{{v_{\mathrm{i}}}^2t^2 - gv_{\mathrm{i}}\sin(\theta)t^3 + \frac{g^2}{4}t^4} &= \frac{\mathrm{d}}{\mathrm{d}u}\sqrt{u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} \\ &= \frac{1}{2\sqrt{u}}\cdot\Big(2{v_{\mathrm{i}}}^2t - 3gv_{\mathrm{i}}\sin(\theta)t^2 + g^2t^3\Big) \\ &= \frac{2{v_{\mathrm{i}}}^2t - 3gv_{\mathrm{i}}\sin(\theta)t^2 + g^2t^3}{2\sqrt{u}}\end{aligned}$$

Given that u is just a modified form of $x(t)^2 + y(t)^2$, and that $\sqrt{x(t)^2 + y(t)^2} = |\vec{r}(t)|$, then:

$$\frac{\mathrm{d}|\vec{r}\,|}{\mathrm{d}t} = \frac{2{v_i}^2t - 3gv_i\sin(\theta)t^2 + g^2t^3}{2|\vec{r}(t)|} \tag{3}$$

Since (3) is always increasing, this means that:

$$\frac{2{v_{\mathrm{i}}}^2t - 3gv_{\mathrm{i}}\sin(\theta)t^2 + g^2t^3}{2|\vec{r}(t)|} > 0$$

Multiplying both sides by $2|\vec{r}(t)|$:

$$2{v_{\mathrm{i}}}^2t - 3gv_{\mathrm{i}}\sin(\theta)t^2 + g^2t^3 > 0$$

Dividing both sides by t:

$$2{v_{\mathrm{i}}}^2 - 3gv_{\mathrm{i}}\sin(\theta)t + g^2t^2 > 0$$

The result is a quadratic expression in t. The quadratic coefficients are:

$$A = g^2$$

$$B = -3gv_{\mathrm{i}}\sin(\theta)$$

$$C = 2{v_{\mathrm{i}}}^2$$

Rather than solve the quadratic equation, take advantage of the fact that $x(t) > 0$ and $y(t) > 0$ only if $t > 0$. This implies two things:

1. The quadratic coefficient A must be positive, and
2. The quadratic discriminant ($\Delta = B^2 - 4AC$) must be negative or zero.

The sign of Δ can be found by setting up (but not solving) the quadratic equation and simplifying enough to arrive at a form that can be plotted:

$$t = \frac{-\Big(-3gv_{\mathrm{i}}\sin(\theta)\Big) \pm \sqrt{\Big(-3gv_{\mathrm{i}}\sin(\theta)\Big)^2 - 4(g^2)(2v_{\mathrm{i}})^2}}{2g^2}$$

$$= \frac{3gv_{\mathrm{i}}\sin(\theta) \pm \sqrt{\Big(-3gv_{\mathrm{i}}\sin(\theta)\Big)^2 - 8(gv_{\mathrm{i}})^2}}{2g^2}$$

$$= \frac{3gv_{\mathrm{i}}\sin(\theta) \pm \sqrt{(gv_{\mathrm{i}})^2\left[\Big(-3\sin(\theta)\Big)^2 - 8\right]}}{2g^2}$$

$$== \frac{3\cancel{g}v_{\mathrm{i}}\sin(\theta) \pm \cancel{g}v_{\mathrm{i}}\sqrt{\Big(-3\sin(\theta)\Big)^2 - 8}}{2g^{\cancel{2}}}$$

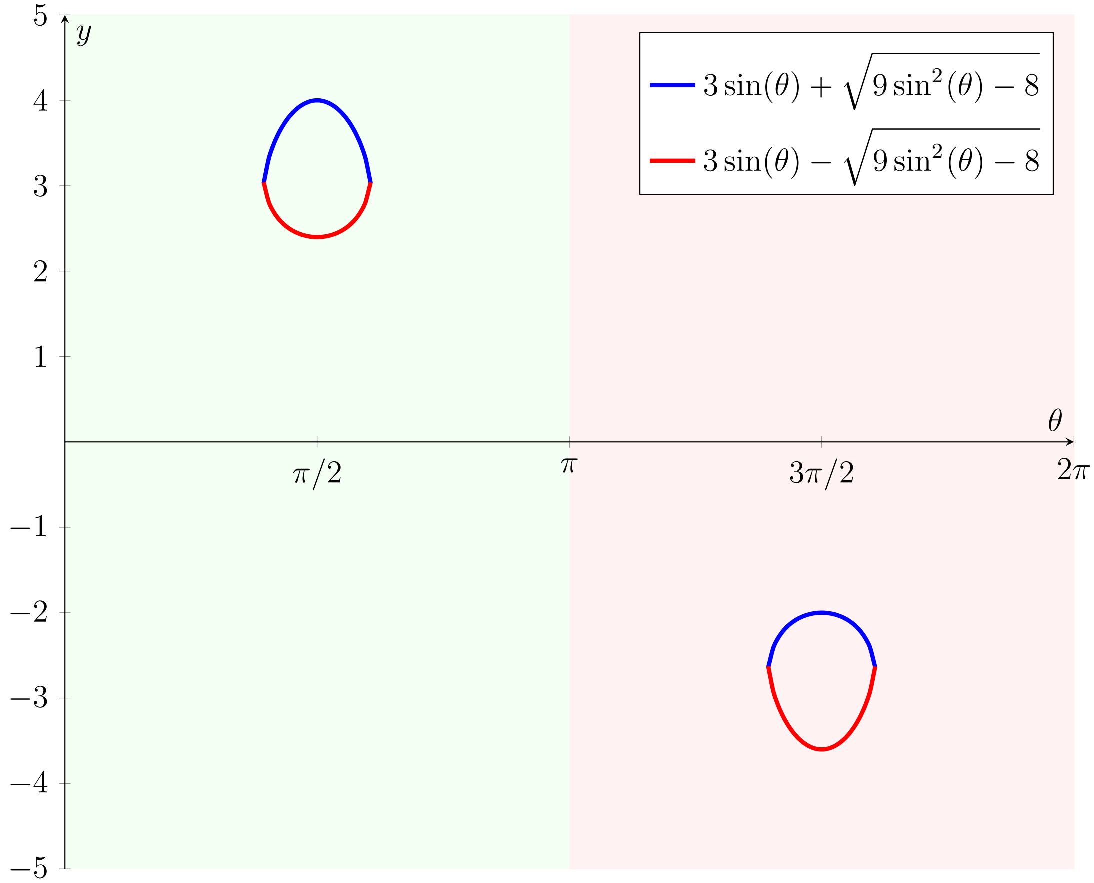
$$= \frac{v_{\mathrm{i}}}{2g}\Big(3\sin(\theta) \pm \sqrt{9\sin^2(\theta) - 8}\Big)$$

Given that $x(t) > 0$ and $y(t) > 0$ only if $t > 0$, the quadratic equation can be expressed as:

$$\frac{v_{\mathrm{i}}}{2g}\left(3\sin(\theta) \pm \sqrt{9\sin^2(\theta) - 8}\right) > 0$$

$$3\sin(\theta) \pm \sqrt{9\sin^2(\theta) - 8} > 0$$

No further simplification is possible and the inequalities are a function of angle only. Plotting these expressions demonstrates the following behavior:



The green region contains solutions that satisfy $t > 0$, whereas the red region does not. Regardless, no interaction with the x-axis occurs. Therefore $\Delta < 0$.

Let θ_{max} be the maximum angle of the projectile such that:

$$B^2 - 4AC < 0$$

$$\Big(- 3gv_{\text{i}} \sin(\theta_{\text{max}})\Big)^2 - 4(g^2)(2v_{\text{i}}^2) < 0$$

$$9g^2v_{\text{i}}^2 \sin^2(\theta_{\text{max}}) - 8g^2v_{\text{i}}^2 < 0$$

$$9\cancel{g^2}\cancel{v_{\text{i}}^2} \sin^2(\theta_{\text{max}}) < 8\cancel{g^2}\cancel{v_{\text{i}}^2}$$

$$\sin^2(\theta_{\text{max}}) < \frac{8}{9}$$

$$\sin(\theta_{\text{max}}) < \frac{2\sqrt{2}}{3}$$

$$\theta_{\text{max}} < \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\approx \boxed{70.5^\circ}$$