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The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work So Well

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State-of-the-art stochastic volatility models generate a "volatility smirk" that explains why out-of-the-money index puts have high prices relative to the Black-Scholes benchmark. These models also adequately explain how the volatility smirk moves up and down in response to changes in risk. However, the data indicate that the slope and the level of the smirk fluctuate largely independently. Although single-factor stochastic volatility models can capture the slope of the smirk, they cannot explain such largely independent fluctuations in its level and slope over time. We propose to model these movements using a two-factor stochastic volatility model. Because the factors have distinct correlations with market returns, and because the weights of the factors vary over time, the model generates stochastic correlation between volatility and stock returns. Besides providing more flexible modeling of the time variation in the smirk, the model also provides more flexible modeling of the volatility term structure. Our empirical results indicate that the model improves on the benchmark Heston stochastic volatility model by 24% in-sample and 23% out-of-sample. The better fit results from improvements in the modeling of the term structure dimension as well as the moneyness dimension.

Key words: stochastic correlation; stochastic volatility; equity index options; multifactor model; persistence; affine; out-of-sample
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1. Introduction

An extensive empirical literature has documented the empirical biases of the Black and Scholes (1973) option valuation model for the purpose of the valuation of equity index options. Most prominently among these biases, observed market prices for out-of-the-money put prices (and in-the-money call prices) are higher than Black-Scholes prices. This stylized fact is known as the volatility "smirk." Implied volatilities for at-the-money options also contain a term structure effect that cannot be explained by the Black-Scholes model.

Perhaps the most popular approach to modeling the smirk is the use of stochastic volatility models that allow for negative correlation between the level of the stock return and its variance.[1] This negative correlation captures the stylized fact that decreases in the stock price are associated with larger increases in variance than similar stock price increases (Black 1976, Christie 1982). This stylized fact is known as the leverage effect. The leverage effect is important for equity index option valuation because it increases the probability of a large loss and consequently the value of out-of-the-money put options. The leverage effect induces negative skewness in stock returns, which in turn yields a volatility smirk.

The stochastic volatility models of Hull and White (1988), Melino and Turnbull (1990), and Heston (1993) allow for nonzero correlation between the level of the stock return and its variance. Several papers have documented that stochastic volatility models are helpful in modeling the smirk and that the modeling of the leverage effect is critical in this regard (e.g., see Bakshi et al. 1997, Bates 2000, Chernov and

[1] For early stochastic volatility models see, for example, Hull and White (1987), Scott (1987), and Wiggins (1987).

Ghysels 2000, Jones 2003, Nandi 1998, Pan 2002). Stochastic volatility models can also address term structure effects by modeling mean reversion in the variance dynamic. As a consequence, many papers use a single-factor stochastic volatility model as the starting point for more complex models.[2]

Single-factor stochastic volatility models can generate smiles and smirks. However, these models are overly restrictive in their modeling of the relationship between the volatility level and the slope of the smirk. The data suggest that the shape of the smile is largely independent of the volatility level. There are low volatility days with a steep volatility slope as well as a flat volatility slope and also high volatility days with steep and flat volatility slopes. A single-factor stochastic volatility model can generate steep smirks or flat smirks at a given volatility level but cannot generate both for a given parameterization. In a purely cross-sectional analysis, this is not a problem because we can estimate different parameter values for the one-factor model to calibrate the time-varying nature of the cross-section. However, when estimating model parameters using multiple cross-sections of option contracts, a one-factor model has a structural problem. If its parameters are geared toward explaining a slope of the smirk that is on average high over the sample, it will result in large model error on those days that the slope of the smirk is relatively flat, and vice versa. Another way to understand this restrictiveness is that in single-factor stochastic volatility models, the correlation between variance and stock returns is constant over time, and this limits the model's ability to capture the time-varying nature of the smirk. To date, we do not have a good understanding of how incorporating stochastic correlation can improve the performance of benchmark stochastic volatility models such as Heston (1993).

This paper uses a straightforward way to incorporate a stochastic correlation by using multiple stochastic volatility factors. We demonstrate that two-factor models have much more flexibility in controlling the level and slope of the smirk. An additional advantage is that two-factor models also provide more flexibility to model the volatility term structure. In our empirical estimates, one of the factors has high mean reversion and determines the correlation between short-term returns and variance. The other factor has lower mean reversion and determines the correlation between long-term returns and variance. We implement and test a two-factor stochastic volatility model that builds on the valuation results in Heston (1993) to maintain a closed-form solution for option prices and remain computationally tractable. We test this model in- and out-of-sample, and we pay particular attention to model parsimony in order to improve out-of-sample performance.

We implement and test the multifactor volatility model using 1990–2004 data on European call options on the S&P 500. We split up our data set into 15 samples that each contain one year of option data. We therefore perform 15 in-sample exercises and then evaluate the first 14 sets of parameter estimates one-year out of sample. We find that in-sample, the implied volatility root mean squared error of the multifactor model is 24% lower than that of the one-factor Heston (1993) model. Out-of-sample, the two-factor model improves on the one-factor model by 23%. This remarkable consistency between the in-sample and out-of-sample results suggests that the more richly parameterized two-factor model's improvement over the one-factor model does not merely arise from overfitting.

To provide more insight into the differences in pricing performance, we extensively investigate along which dimensions the estimated two-factor model differs from the one-factor model. We find that the two-factor model substantially improves on the one-factor model in the term structure dimension as well as in the moneyness dimension. We also demonstrate that the modeling of conditional skewness and kurtosis in the standard one-factor model is extremely restrictive, and that estimated conditional higher moments are highly correlated with the estimated conditional variance. In contrast, the two-factor model allows for more flexibility in modeling conditional skewness and kurtosis for given levels of conditional variance, which is consistent with the finding that the slope of the smirk evolves largely independently from the level of the volatility.

In the option literature, Taylor and Xu (1994) use a two-factor model to uncover short-run and long-run variance expectations in foreign exchange markets. Bates (2000) conducts an extensive empirical analysis of a large class of option pricing models using 1988–1993 S&P 500 futures option data. Among other things, he documents the negative skew in post-1987 index option data and compares the performance of stochastic volatility models with the performance of stochastic volatility models augmented with Poisson-normal jumps. His investigation is related to this paper: He evaluates the empirical performance of a two-factor stochastic volatility model and investigates whether jumps and additional stochastic volatility factors are substitutes. Bates (2000) concludes that stochastic volatility models cannot reconcile return

[2] See, for instance, the extensive literature on jump models. Bakshi et al. (1997), Bates (1996, 2000), Broadie et al. (2007), Eraker (2004), and Pan (2002) compare the empirical fit of the Heston (1993) model with more complex models that contain different types of jumps in returns and volatility.

data and option data, even with two stochastic volatility factors, and that jumps are needed.

Our focus is on explaining why a two-factor model works better than a one-factor model for the purpose of option pricing by emphasizing its features for modeling the moneyness and term structure dimensions. We directly focus on the modeling of the risk-neutral distribution and do not discuss the link between physical and risk-neutral distribution, therefore sidestepping one of the main motivations for including jumps in the model, as discussed by Bates (2000). Although we do compare the statistical fit of the models, we put more emphasis on the underlying stylized facts that result in an improved statistical fit. In line with this approach, we conduct an extensive out-of-sample exercise to corroborate that the improved in-sample fit is due to the improved modeling of these stylized facts rather than to a simple increase in the number of model parameters.[3]

It is perhaps somewhat surprising that multifactor models have not yet become more popular in the option valuation literature. In the yield curve literature, which uses models with a mathematical structure similar to those in the option valuation literature, the use of multifactor models of the short rate is widespread. In fact, it is widely accepted in the literature that one-factor is not sufficient to capture the time variation and cross-sectional variation in the term structure. The consensus seems to be that a minimum of three factors is needed.[4] Option valuation and term structure modeling have a lot in common: In both cases one faces the demanding task of providing a good empirical fit to the time-series as well as to the cross-sectional dimension using tractable, parsimonious models. We therefore believe that the use of multifactor models is as critical for the equity option valuation literature as it is for the term structure literature and that in the future multifactor models may become as widespread in the option valuation literature as they now are in the term structure literature.[5]

In the equity index option valuation literature, the deficiencies of the one-factor stochastic volatility model have traditionally been addressed by adding a jump process to the return dynamic.[6] This paper does not question the usefulness of this approach. Instead, we surmise that adding additional factors to the volatility process is an alternative way of addressing model deficiencies.[7] Our paper does not investigate whether multifactor models or jump processes are more appropriate for modeling option data or if indeed both features are useful. That particular empirical question has to be decided by an out-of-sample comparison between jump models and multifactor stochastic volatility models, and such a comparison is beyond the scope of this paper.

The paper proceeds as follows. In §2, we conduct a principal component analysis as well as some regression analysis to motivate the use of multiple volatility factors and discuss some stylized facts in the data. In §3, we present a two-factor stochastic volatility process that has the potential to match the empirical regularities found in §2. We then use the model to illustrate some critical differences between one- and two-factor models. In §4, we present the estimation strategy and assess the empirical fit of the model using S&P 500 index options in- and out-of-sample. Section 5 further explores the empirical results, and §6 concludes. We include additional conditional moment diagnostics in the online appendix, which is available in the e-companion.[8]

2. Option Data Exploration

In this section, we first describe the option data and then perform a principal components analysis on the implied volatility surface. Finally, we study time series patterns in the level and slope of the option implied volatility smirk.

[3] See also Alizadeh et al. (2002), Chernov et al. (2003), and Christoffersen et al. (2008b) for related work. Eraker (2004) and Duffie et al. (2000) suggest the potential usefulness of our approach. Carr and Wu (2007) model stochastic skewness in currency options using a different approach.

[4] See Litterman and Scheinkman (1991) for a characterization of the number of factors needed to model the term structure. See Pearson and Sun (1994) for an early example of multifactor term structure models and Duffee (1999) and Dai and Singleton (2000, 2002, 2003) for further applications. Duffie and Kan (1996) and Dai and Singleton (2000) provide widely used classifications of multifactor term structure models.

[5] If anything, modeling equity options is probably more challenging than modeling the term structure because modeling the cross-sectional dimension for equity options requires the modeling of moneyness effects as well as maturity effects. On the other hand, the multifactor stochastic volatility models considered in this paper differ from multifactor term structure models in the sense that the one-factor stochastic volatility model can itself be considered as a two-factor model, with the first factor being provided by the stock return.

[6] See, for example, Andersen et al. (2002), Bakshi et al. (1997), Bates (1996, 2000), Broadie et al. (2007), Carr and Wu (2004), Chernov et al. (2003), Eraker et al. (2003), Eraker (2004), Pan (2002), Santa-Clara and Yan (2009), and Huang and Wu (2004). Huang and Wu (2004) consider a two-factor volatility model driven by Levy processes.

[7] Interestingly, recent research has investigated the importance of jump processes for modeling the term structure of interest rates. See, for example, Johannes (2004). This paper complements this line of research by using an approach that is more typical of the existing empirical research on term structure models and applying it to the valuation of equity options.

[8] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

Table 1 S&P 500 Index Call Option Data, 1990–2004

	$DTM < 30$	$30 < DTM < 90$	$90 < DTM < 180$	$DTM > 180$	All
		Panel A. Number of call option contracts			
$S/X < 0.975$	625	3,859	3,188	3,770	11,442
$0.975 < S/X < 1$	1,247	3,189	1,174	987	6,597
$1 < S/X < 1.025$	1,319	2,685	1,023	811	5,838
$1.025 < S/X < 1.05$	1,027	1,937	821	562	4,347
$1.05 < S/X < 1.075$	778	1,475	688	500	3,441
$S/X > 1.075$	1,338	2,324	1,625	1,306	6,593
All	6,334	15,469	8,519	7,936	38,258
		Panel B. Average call price			
$S/X < 0.975$	3.41	10.01	16.78	30.40	18.25
$0.975 < S/X < 1$	9.63	21.96	33.45	56.22	26.80
$1 < S/X < 1.025$	19.89	31.57	40.45	56.29	33.92
$1.025 < S/X < 1.05$	31.72	41.38	47.41	62.11	42.92
$1.05 < S/X < 1.075$	42.71	52.18	54.40	66.88	52.62
$S/X > 1.075$	55.76	61.99	63.42	72.13	63.09
All	28.54	31.97	36.81	47.67	35.74
		Panel C. Average implied volatility from call options			
$S/X < 0.975$	16.22	16.06	16.25	16.89	16.39
$0.975 < S/X < 1$	15.95	17.08	17.30	18.14	17.06
$1 < S/X < 1.025$	17.48	17.93	17.77	17.93	17.80
$1.025 < S/X < 1.05$	19.44	18.83	18.45	18.09	18.81
$1.05 < S/X < 1.075$	22.80	19.95	18.98	18.55	20.20
$S/X > 1.075$	32.39	22.48	19.75	18.87	23.10
All	21.17	18.28	17.68	17.67	18.50

Notes. Our sample consists of European call options written on the S&P 500 index. We use closing quotes on every Wednesday during the period January 1, 1990–December 31, 2004. The moneyness and maturity filters used by Bakshi et al. (1997) are applied here as well. The implied volatilities are extracted using the Black-Scholes formula.

2.1. Option Data

For our empirical investigation, we use data on European S&P 500 call options from 1990 through 2004. The data from 1990 through 1995 are from the Berkeley Option Database and the data from 1996 through 2004 are from OptionMetrics. We use closing option quotes from the Chicago Board Options Exchange each Wednesday. From the bid and ask quotes, mid-quotes are calculated as simple averages. Each option quote is matched with the underlying index level, which is adjusted for dividends by subtracting the present value of the future realized flow of dividends between the quote date and the maturity date of the option.[9] T-bill rates are used for this purpose. The risk-free rate for each option maturity is calculated via interpolation of available T-bill rates. Options with fewer than seven days to maturity are omitted from the sample, as are options with extreme moneyness and options that violate various no-arbitrage conditions. These filtering rules follow Bakshi et al. (1997).

Table 1 summarizes our data set of 38,258 contracts. Panels A–C in Table 1 are split up into four (calendar) days-to-maturity (DTM) categories and six moneyness (S/X) categories. Panel A reports the number of contracts in each category, panel B reports the average call price in each category, and panel C gives the average Black-Scholes implied volatility in each category. The systematic and well-known volatility "smirk" across moneyness is evident from each column in panel C. Although the smirk is most dramatic for the short-maturity options, it is present in each maturity category.

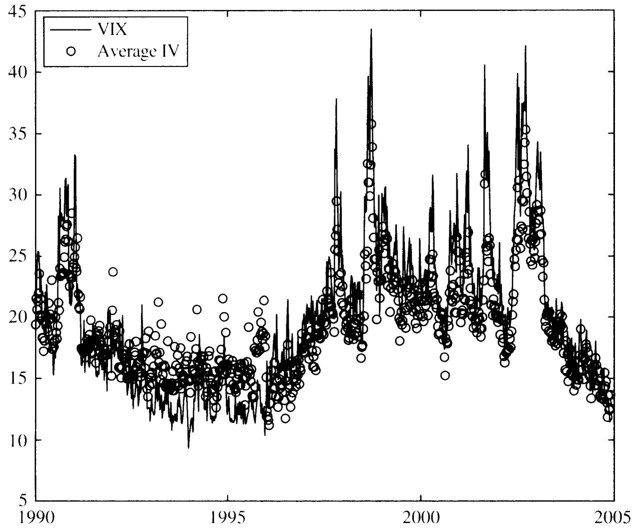
Figure 1 presents our option data from a different perspective. For each of the 776 available Wednesdays in the 1990–2004 sample, the circles represent the average implied Black-Scholes volatility. The average is taken across maturities and strike prices. For comparison, the solid line represents the one-month, at-the-money volatility index, VIX, from cboe.com. It is clear that our sample adequately captures the time variation in the overall market, showing high volatility around the first Gulf War as well as during the 1998 LTCM debacle and the 2000–2002 stock market downturn. The VIX and average implied volatilities (IVs) are slightly different because the IV from our data is a simple average across maturities and moneyness each day.

2.2. Principal Components Analysis

Our objective here is to investigate whether the data support multiple variance factors, without relying

[9] This procedure follows Harvey and Whaley (1992a, b).

Figure 1 Average Implied Volatility in S&P 500 Option Data and the CBOE VIX



Notes. The circles plot the average implied Black-Scholes volatility each Wednesday during 1990–2004. The average is taken across maturities and strike prices using the call options in our data set. For comparison, the solid line shows the one-month, at-the-money VIX volatility index retrieved from http://www.cboe.com.

on any particular option valuation model. This is not straightforward, because by definition the stock return variance is a latent factor. We circumvent the unobservability of the return variance by using an observable proxy. In particular, we investigate the number of factors in the implied Black-Scholes variance.[10] Although this approach clearly has some limitations, it is meant to provide an exploration of the need for multiple factors and is not meant to substitute for a more detailed statistical analysis.

Table 2 reports the results of a principal component analysis of Black-Scholes implied variances. To facilitate the interpretation of the principal component analysis, we performed it not on the raw data but on a standardized variance surface. This variance surface is constructed as follows. In a first stage, we fit a quadratic polynomial in maturity and moneyness for each day in the data set. In the second stage, we use these estimates to generate a standardized variance surface using five different levels of moneyness (0.9, 0.95, 1, 1.05, and 1.1) and five different maturities (30 days, 60 days, 90 days, 180 days, and 270 days).

Table 2 reports the loadings on the first four principal components and the fraction of the variance

Table 2 Principal Component Analysis of Implied Variance

		Principal components			
S/X	Maturity (days)	First	Second	Third	Fourth
0.90	30	0.139	0.072	0.307	0.286
0.90	60	0.141	0.027	0.275	0.179
0.90	90	0.143	−0.014	0.244	0.075
0.90	180	0.147	−0.105	0.151	−0.190
0.90	270	0.139	−0.107	0.049	−0.312
0.95	30	0.167	0.111	0.285	0.205
0.95	60	0.168	0.048	0.249	0.082
0.95	90	0.168	−0.009	0.214	−0.030
0.95	180	0.166	−0.137	0.121	−0.256
0.95	270	0.151	−0.162	0.039	−0.260
1.00	30	0.199	0.176	0.206	0.118
1.00	60	0.199	0.085	0.169	−0.012
1.00	90	0.198	0.004	0.135	−0.118
1.00	180	0.189	−0.175	0.055	−0.259
1.00	270	0.166	−0.227	0.005	−0.119
1.05	30	0.236	0.285	0.024	0.053
1.05	60	0.235	0.149	−0.002	−0.069
1.05	90	0.233	0.030	−0.026	−0.152
1.05	180	0.216	−0.222	−0.067	−0.157
1.05	270	0.185	−0.309	−0.063	0.147
1.10	30	0.277	0.463	−0.341	0.069
1.10	60	0.277	0.249	−0.329	−0.023
1.10	90	0.274	0.071	−0.320	−0.060
1.10	180	0.251	−0.289	−0.275	0.117
1.10	270	0.212	−0.422	−0.188	0.596
Explained by PC:		90.790	4.759	2.885	0.887

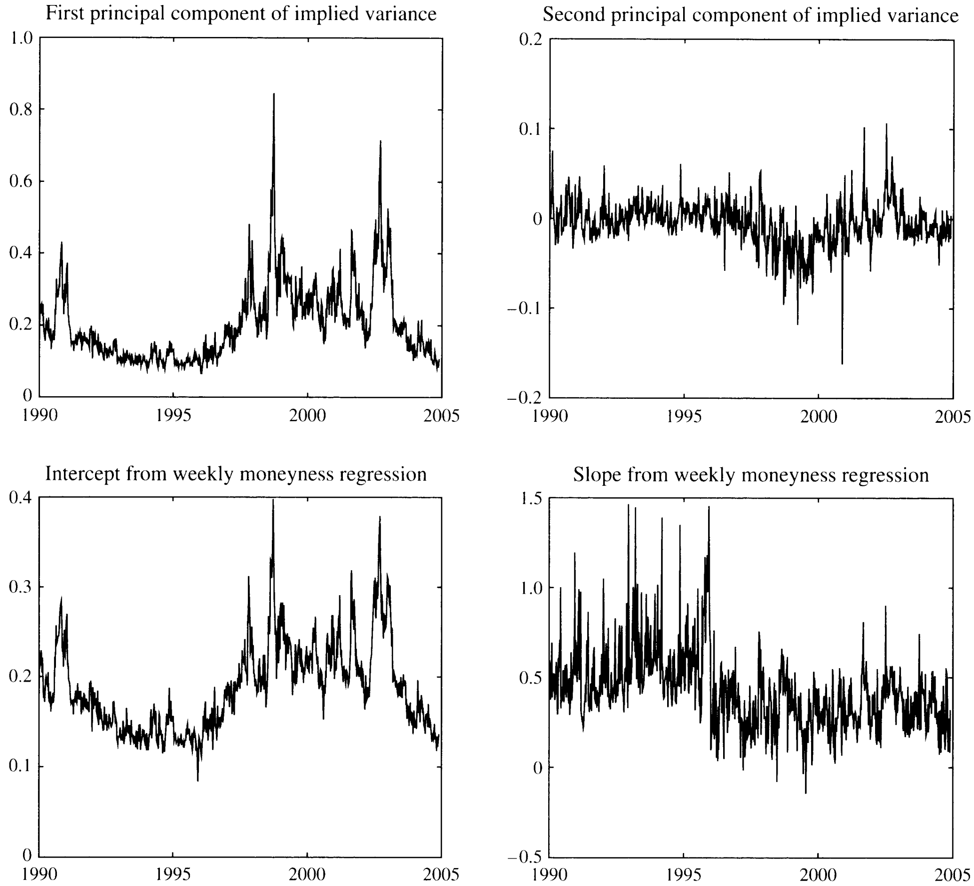
Notes. We report factor loadings and percentage of variance explained by the first four principal components. The principal component analysis is performed using implied variances for each Wednesday during 1990–2004. In a first stage, a quadratic polynomial in maturity and moneyness is fit for each Wednesday, using data for all available moneyness and maturity. In a second stage, these estimates are used to generate a variance surface with standardized moneyness and maturity. The principal component analysis is performed on this standardized variance surface.

explained by each of these four components. The most important conclusion is that the first component explains more than 90% of the variation in the data and the first two components together explain more than 95% of the variation in the data. The results therefore seem to suggest that a two-factor model may be a good representation of the data.

The first principal component has relatively similar weights for all 25 data series. The top left panel of Figure 2 represents this component over time, and it is closely related to the average implied volatility (see Figure 1). Recall that the level of the implied variance is simply the average of the implied variances across moneyness and maturity on a given day. The top right panel of Figure 2 shows the second component over time. The loadings of the second principal component on the 25 data series are not as uniform as is the case for the first principal component. Table 2 shows that it has large positive loadings on in-the-money calls with short maturities and negative loadings on most other options. It is therefore to be expected that the

[10] Although the Black-Scholes implied standard deviation is more extensively used as a measure of stock return variability than the Black-Scholes implied variance, we report a principal component analysis of variance because the latent factors in the subsequent model are variances and not standard deviations. An analysis of the Black-Scholes implied volatilities yielded very similar results. See Skiadopoulos et al. (1999) for further discussion.

Figure 2 Principal Components for Implied Variance and Weekly Regressions on Moneyness



Notes. The top two panels plot the first and second principal component for the implied S&P 500 index variance during 1990–2004. The principal component analysis is performed using implied variances for each Wednesday during 1990–2004. In a first stage, a quadratic polynomial in maturity and moneyness is fit for each Wednesday, using data for all available moneyness and maturity. In a second stage, these estimates are used to generate a variance surface with standardized moneyness and maturity. The principal component analysis is performed on this standardized variance surface. The bottom two panels are obtained by regressing implied volatilities for all available contracts on a given day on an intercept and log moneyness. The resulting coefficients can be interpreted as the volatility level and the slope of the smirk on that day.

second principal component combines maturity and moneyness effects.

In summary, subject to the caveat that this approach can only be interpreted as a preliminary, model-free gauge of the option price dynamics, we find that a principal component analysis of implied Black-Scholes variances reveals that a stochastic volatility model with two factors is likely to capture a lot of the variation in the data. Empirically, the question is whether a richer model results in a better fit than a one-factor model. It is especially important to consider out-of-sample exercises. After all, the two-factor model nests the one-factor model, and therefore it should provide a better in-sample fit. Whether this more richly parameterized model provides reliable enough improvements to improve the out-of-sample fit constitutes a much more stringent test.

2.3. Level of Volatility and Slope of the Smirk

In order to illustrate the advantages offered by a two-factor model, we start by characterizing a simple stylized fact of the volatility smirk: The slope of the smirk is largely independent of the level of volatility. This was first noticed by Derman (1999), who documents that the slope of the smirk changes little when volatility changes. However, to the best of our knowledge, this stylized fact is not extensively documented elsewhere, and we therefore perform a simple empirical exercise to provide additional evidence.

On each Wednesday in the 1990–2004 sample, we regress the implied volatilities of all option contracts on that Wednesday on a constant and the natural logarithm of the contract's moneyness. The estimate of the intercept can be interpreted as the volatility level for that day, and the estimated coefficient on the moneyness can be interpreted as the steepness of the slope. The bottom two panels of Figure 2 present the results of this analysis. First, comparing the time series of the intercepts in the bottom left panel with Figure 1 reveals that the estimated volatility levels are very reliable. A visual inspection of the bottom two panels suggests that the slope evolves quite independently from the volatility level. The estimated slope coefficients vary significantly over time, but these changes are not necessarily related to sharp increases or decreases in the volatility level. Moreover, when the index falls (for example, in 1998 and 2001–2002), the volatility increases, but there does not seem to be a change in the estimated slope coefficients. We computed the correlation between the intercept and the slope on an expanding window of data starting with the first 30 weekly observations. The absolute value of the correlation is never very large, and it is at times positive and at other times negative. The maximum over the 15 years turns out to be 21%, and the minimum is −26%.

Comparing the top and bottom row of panels in Figure 2, we see that to a large extent the level and slope factors correspond to the principal components. The time-series correlation between the first principal component (top left) and the volatility level factor (bottom left) is 98%. The correlation between the second principal component (top right) and the volatility slope factor (bottom right) is 65%. This level and slope interpretation gives a more intuitive understanding of the fluctuations in option prices. The challenge is to build a model that explains these features in economic terms.

3. The Two-Factor Model

We now present the two-factor model and provide some intuition for the model. Consider first the one-factor Heston (1993) model, which is one of the most popular models in the option valuation literature. This model is given by[11]

$$dS = rSdt + \sqrt{V}Sdz_1, \tag{1}$$

$$dV = (a - bV)dt + \sigma\sqrt{V}dz_2, \tag{2}$$

where the correlation between z_1 and z_2 is ρ.[12]

Suppose now instead that the variance of the risk-neutral, ex-dividend stock price process is determined by two factors:

$$dS = rSdt + \sqrt{V_1}Sdz_1 + \sqrt{V_2}Sdz_2, \tag{3}$$

$$dV_1 = (a_1 - b_1V_1)dt + \sigma_1\sqrt{V_1}dz_3, \tag{4}$$

$$dV_2 = (a_2 - b_2V_2)dt + \sigma_2\sqrt{V_2}dz_4. \tag{5}$$

[11] Because of our choice of estimation method, we only require the risk-neutral process. Risk-neutralization in this model can be motivated in the usual way by specifying a representative agent with logarithmic utility. See, for instance, Lewis (2000).

[12] The square root process can access zero with positive probability if $\sigma^2 > 2a$. In this case, we assume a standard reflecting barrier at the origin.

We assume z_1 and z_2 are uncorrelated. Note that the variance of the stock return is the sum of the two variance factors:

$$\mathrm{Var}_t[dS/S] = (V_1 + V_2)dt \equiv Vdt. \tag{6}$$

In addition, we assume the following stochastic structure: z_1 has correlation ρ_1 with z_3, and z_2 has correlation ρ_2 with z_4, but z_1 is uncorrelated with z_4, z_2 is uncorrelated with z_3, and z_3 is uncorrelated with z_4. In other words, the variance is the sum of two uncorrelated factors that may be individually correlated with stock returns.

For option valuation, we need to determine the characteristic function of the log-spot price, $x = \ln(S)$. Generalizing the results in Heston (1993), the characteristic function satisfies

$$E_t[\exp(i\phi x(t+\tau))] = S(t)^{i\phi} f(V_1, V_2, \tau, \phi), \tag{7}$$

where

$$\begin{aligned}
f(V_1,V_2,\tau,\phi) &= \exp(A(\tau,\phi)+B_1(\tau,\phi)V_1+B_2(\tau,\phi)V_2),\\
A(\tau,\phi) &= r\phi i\tau + \frac{a_1}{\sigma_1^2}\left[(b_1-\rho_1\sigma_1\phi i+d_1)\tau - 2\ln\left[\frac{1-g_1\exp(d_1\tau)}{1-g_1}\right]\right]\\
&\quad + \frac{a_2}{\sigma_2^2}\left[(b_2-\rho_2\sigma_2\phi i+d_2)\tau - 2\ln\left[\frac{1-g_2\exp(d_2\tau)}{1-g_2}\right]\right],\\
B_j(\tau,\phi) &= \frac{b_j-\rho_j\sigma_j\phi i+d_j}{\sigma_j^2}\left[\frac{1-\exp(d_j\tau)}{1-g_j\exp(d_j\tau)}\right],\\
g_j &= \frac{b_j-\rho_j\sigma_j\phi i+d_j}{b_j-\rho_j\sigma_j\phi i-d_j},\\
d_j &= \sqrt{(\rho_j\sigma_j\phi i-b_j)^2+\sigma_j^2(\phi i+\phi^2)}.
\end{aligned}$$

Note that the $B_j(\tau, \phi)$ terms are identical to their one-dimensional counterpart in Heston (1993), and $A(\tau, \phi) = r\phi i\tau$ plus the sum of two terms that are identical to their one-dimensional counterpart.

Using these results, European call options can be valued via Fourier inversion by inserting the probabilities

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi\ln(S(t)/X)} f(V_1, V_2, \tau, \phi+1)}{i\phi S(t)e^{r\tau}}\right] d\phi, \tag{8}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi\ln(S(t)/X)} f(V_1, V_2, \tau, \phi)}{i\phi}\right] d\phi \tag{9}$$

into the option valuation formula

$$C(t) = S(t)P_1 - Xe^{-r\tau}P_2. \tag{10}$$

The two integrals can be combined into one, which may yield a more computationally efficient call option valuation formula. We get

$$C(t) = \frac{1}{2}(S(t) - Xe^{-r\tau}) + \frac{e^{-r\tau}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi\ln(S(t)/X)}f(\phi+1)}{i\phi} - \frac{Xe^{i\phi\ln(S(t)/X)}f(\phi)}{i\phi}\right]d\phi. \quad (11)$$

See, for example, Heston and Nandi (2000). Lewis (2000) suggests a further simplification of the integrand, which requires integration in the complex plane.

3.1. Stochastic Correlation

For each factor, the covariance with the stock return is given by

$$\mathrm{Cov}_t[dS/S, dV_j] = \sigma_j \rho_j V_j dt. \quad (12)$$

The covariance of stock returns with overall variance is given by

$$\mathrm{Cov}_t[dS/S, dV] = (\sigma_1\rho_1 V_1 + \sigma_2\rho_2 V_2)dt, \quad (13)$$

and the variance of the variance is

$$\mathrm{Var}_t[dV_t] = (\sigma_1^2 V_1 + \sigma_2^2 V_2)dt. \quad (14)$$

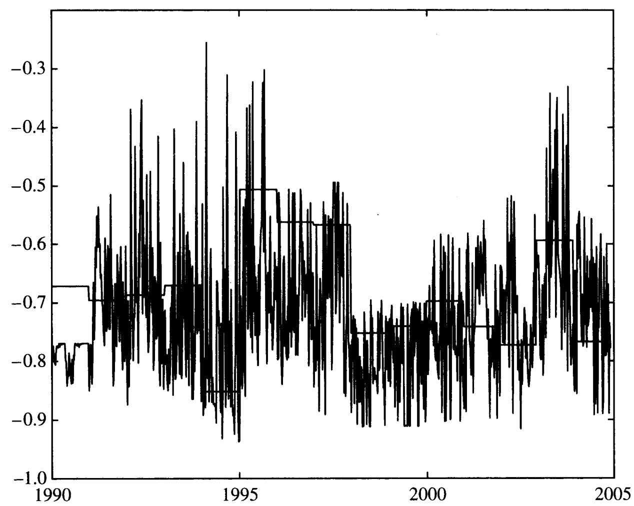
Note that although ρ_1 and ρ_2 on the one hand, and σ_1 and σ_2 on the other hand, may not differ much for certain parameter sets, these relatively small differences yield large fluctuations in the paths of $\mathrm{Cov}_t(dS/S, dV)$ and $\mathrm{Var}_t(dV)$ and also in the higher moments of returns. The online appendix contains plots of these conditional moments using the parameter values estimated below.

The correlation between the stock return and variance is determined by ρ_1 and ρ_2 and depends on the current levels of the factors. Note that this implies that the leverage correlation in the two-factor model is given by

$$\mathrm{Corr}_t[dS/S, dV] = \frac{\sigma_1\rho_1 V_1 + \sigma_2\rho_2 V_2}{\sqrt{\sigma_1^2 V_1 + \sigma_2^2 V_2}\sqrt{V_1 + V_2}}dt. \quad (15)$$

Although this model is conceptually fairly straightforward, it holds promise to resolve existing biases in option valuation because of its flexibility. For the purpose of modeling moneyness effects, note that the correlation of stock returns with overall variance depends on the current levels of the factors. Hence, this model displays not only stochastic variance but also *stochastic correlation* between stock return and variance, and this feature potentially enables the model to capture fluctuations in option skewness. For the purpose of modeling term structure effects, one of the factors can have relatively "fast" mean reversion (high b) to determine short-run variance, whereas the other factor can have relatively "slow" mean reversion (low b) to determine long-run variance. The different implications of these factors can also be influenced by interactions between mean reversion parameters and the parameters that determine the third and fourth moment of returns (ρ and σ).

Figure 3 Conditional Correlation Between Stock Returns and Volatility



Notes. We plot the correlation between stock returns and variance. We compute the correlation on a year-by-year basis using the parameter estimates from Table 3. We report the time-varying correlation for the two-factor model as well as the constant (within-year) correlation for the one-factor model.

Figure 3 provides additional intuition for the two-factor model's properties by documenting how the correlation between stock returns and variance changes over time, given the estimates obtained in Table 3, which will be discussed in detail in §4.2. Each year uses a different set of parameter estimates corresponding to the 15 sets of results reported in Table 3. We also graph the constant within-year correlation for the one-factor model.

Figure 3 clearly demonstrates that time-varying correlation is an important model feature. Note from (15) that the correlation between stock returns and variance is not restricted to lie between the correlations between the stock return and the respective variance factors. Note that defining the variance ratio $V_R = V_1/V$ and recalling $V = V_1 + V_2$, we can write

$$\mathrm{Corr}_t[dS/S, dV] = \frac{V[V_R(\sigma_1\rho_1 - \sigma_2\rho_2) + \sigma_2\rho_2]}{V\sqrt{V_R(\sigma_1^2 - \sigma_2^2) + \sigma_2^2}} = \frac{V_R(\sigma_1\rho_1 - \sigma_2\rho_2) + \sigma_2\rho_2}{\sqrt{V_R(\sigma_1^2 - \sigma_2^2) + \sigma_2^2}}, \quad (16)$$

which is independent of the level of V. If $\sigma_1\rho_1 < \sigma_2\rho_2$, then the correlation has its minimum at ρ_1 when

Table 3 Parameter Estimates and Option Fit

Panel A. One-factor stochastic volatility model

	Parameter estimates				IVRMSE		Ratio: 2F/1F		
Year	b	a/b	σ	ρ	In-sample	Out-of-sample	In-sample	Out-of-sample	Number of obs.
1990	1.9561	0.0593	0.8516	−0.6717	2.1149	NA	0.8757	NA	2,857
1991	2.4240	0.0442	0.5834	−0.6957	2.1080	2.3052	0.7488	0.8681	2,974
1992	2.5476	0.0375	0.5519	−0.6865	2.0033	2.0432	0.7127	0.8861	3,345
1993	2.6846	0.0254	0.5105	−0.6703	2.8983	3.0412	0.8016	0.7843	3,578
1994	4.4324	0.0233	0.4560	−0.8519	2.8183	2.8504	0.7950	0.8004	4,297
1995	2.5070	0.0190	0.5597	−0.5061	2.3416	2.5352	0.6360	0.6873	4,701
1996	3.1798	0.0298	0.5823	−0.5619	1.0477	1.3212	0.6109	0.6231	1,558
1997	2.1672	0.0528	0.6018	−0.5666	1.3907	1.7610	0.7520	0.6234	2,214
1998	1.8315	0.1029	0.8079	−0.7521	1.5289	2.1847	0.7945	0.7943	2,062
1999	2.1310	0.1091	0.7552	−0.7404	1.1587	1.3933	0.8605	0.7681	1,883
2000	2.5751	0.0678	0.6561	−0.6975	0.9527	1.5845	0.5892	0.7406	1,817
2001	3.8191	0.0564	0.6489	−0.7410	0.9880	1.0848	0.7101	0.7499	1,627
2002	3.3760	0.0532	0.5973	−0.7725	1.1585	1.2022	0.6364	0.6365	1,609
2003	1.7201	0.0691	0.6837	−0.5939	0.8788	1.0039	0.6868	0.6581	1,845
2004	1.6048	0.0464	0.3796	−0.7670	1.1839	1.5452	0.7205	0.6362	1,891
Total					**1.9945**	**2.1670**	**0.7587**	**0.7721**	38,258

Panel B. Two-factor stochastic volatility model

	First factor estimates				Second factor estimates				IVRMSE	
Year	b_1	a_1/b_1	σ_1	ρ_1	b_2	a_2/b_2	ρ_2	σ_2	In-sample	Out-of-sample
1990	0.2370	0.0227	1.0531	−0.7695	8.4983	0.0273	0.6827	−0.8417	1.8520	NA
1991	0.2966	0.0197	1.8157	−0.8575	4.4513	0.0319	0.3360	−0.6057	1.5784	2.0012
1992	0.2022	0.0051	6.2755	−0.9670	0.7424	0.0684	0.2740	−0.8040	1.4277	1.8104
1993	0.2000	0.0052	5.2500	−0.9666	0.6131	0.0569	0.2123	−0.8216	2.3233	2.3852
1994	0.1668	0.0050	9.4346	−0.9877	0.2098	0.1633	0.1706	−0.9364	2.2406	2.2816
1995	0.2061	0.0050	6.8941	−0.9206	1.4677	0.0242	0.2413	−0.7512	1.4893	1.7424
1996	0.2101	0.0052	2.0149	−0.9684	0.5561	0.0575	0.1868	−0.7978	0.6400	0.8233
1997	0.1397	0.0053	1.5423	−0.9914	0.1878	0.1648	0.1239	−0.8928	1.0457	1.0979
1998	0.1374	0.0051	2.1196	−0.9917	0.6247	0.1733	0.3965	−0.9117	1.2147	1.7354
1999	0.1388	0.0051	1.9895	−0.9917	0.7322	0.1736	0.3828	−0.9108	0.9970	1.0702
2000	0.1404	0.0052	1.9382	−0.9915	0.3542	0.1690	0.2292	−0.9024	0.5614	1.1735
2001	0.1433	0.0054	1.9115	−0.9911	0.2347	0.1655	0.2047	−0.8983	0.7016	0.8135
2002	0.1491	0.0058	1.9754	−0.9902	0.1855	0.1607	0.1715	−0.8896	0.7373	0.7652
2003	0.1638	0.0032	8.8078	−0.9838	0.4625	0.1198	0.3976	−0.6569	0.6036	0.6607
2004	0.1500	0.0059	1.9829	−0.9902	0.2335	0.1621	0.1971	−0.8918	0.8529	0.9830
Total									**1.5133**	**1.6733**

Notes. Each model is estimated year by year using the Wednesday closing option quotes from Table 1. The structural parameters reported above and the weekly spot volatilities are estimated using the iterative two-step method outlined in §4. The in-sample root mean squared errors (RMSE) are calculated using the Black-Scholes Vega approximation to IVRMSE. The ratio IVRMSE is calculated by normalizing each two-factor IVRMSE with the IVRMSE from the one-factor model.

$V_R = 1$. The maximum can be found by taking the first derivative of the correlation w.r.t. V_R and setting it to zero. We get

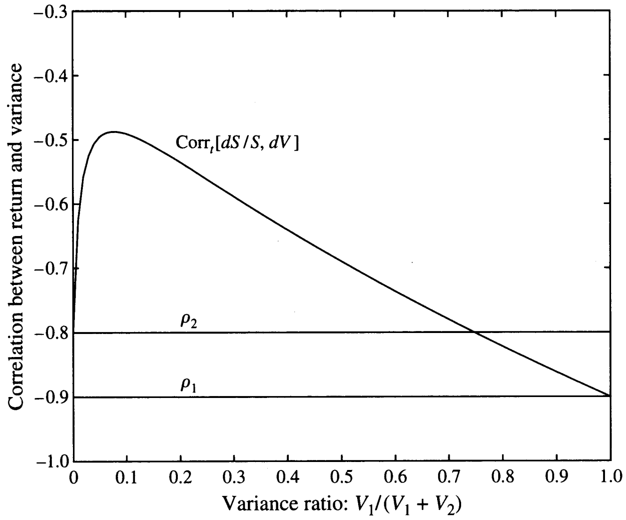
$$V_R^* = \frac{0.5(\sigma_1^2-\sigma_2^2)\sigma_2\rho_2-(\sigma_1\rho_1-\sigma_2\rho_2)\sigma_2^2}{0.5(\sigma_1\rho_1-\sigma_2\rho_2)(\sigma_1^2-\sigma_2^2)}. \qquad (17)$$

Figure 4 shows the correlation as a function of V_R using the following parameters, $\sigma_1 = 2.0$, $\rho_1 = -0.9$, $\sigma_2 = 0.2$, and $\rho_2 = -0.8$. Notice in particular that although the parameter estimates for ρ_1 and ρ_2 may both be close to -1, the overall conditional correlation $\mathrm{Corr}_t[dS/S, dV]$ is not restricted to be between these parameter estimates.

3.2. Expected Future Spot Variances

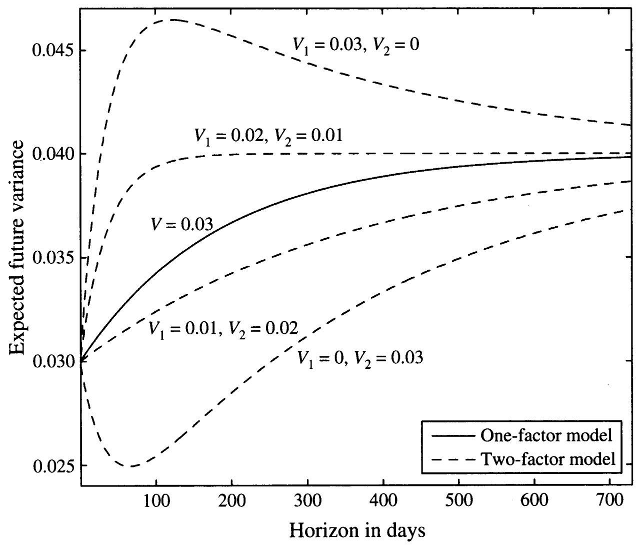
The two-factor model has the potential to improve on the fit of the one-factor model by allowing for richer modeling of maturity and moneyness effects. Although the improvements in the moneyness dimension are a bit more subtle, the improvements in the maturity dimension are relatively easy to understand. Because the one-factor model has only one parameter to capture mean reversion to the unconditional

Figure 4 Conditional Correlation as a Function of the Variance Ratio



Notes. The conditional correlation between return and variance, $\mathrm{Corr}_t[dS/S, dV]$, is plotted against the variance ratio, V_1/V, where $V = V_1 + V_2$. The parameter values are $\sigma_1 = 2.0$, $\rho_1 = -0.9$, $\sigma_2 = 0.2$, and $\rho_2 = -0.8$.

Figure 5 Expected Future Variance in the One-Factor (Solid) and Two-Factor (Dashed) Models



Notes. We plot the expected future spot variance over a two-year horizon using parameterizations for the one-factor and two-factor models. For the one-factor model, the long-run variance a/b is 0.04, the mean reversion coefficient b is 2, and the spot variance V is 0.03. For the two-factor model, the long-run variance is also 0.04, the mean reversion coefficient for the first variance factor b_1 is 2, and for the second variance component b_2 is 10. The long-run mean is 0.02 for both factors. The spot variances for the first factor V_1 are 0, 0.01, 0.02, and 0.03, respectively, and the second spot variance V_2 is $0.03 - V_1$ in all cases.

variance, the patterns for the term structure of conditional variance are rather limited. The two-factor model has two parameters capturing the mean reversion of each of the factors. Depending on the size of each of the two factors, this can lead to very rich patterns in the term structure of the conditional variance.

The formula for expected future spot variance in the two-factor model is

$$E_t[V(t+\tau)] = \frac{a_1}{b_1} + \left(V_1 - \frac{a_1}{b_1}\right)\exp(-b_1\tau) + \frac{a_2}{b_2} + \left(V_2 - \frac{a_2}{b_2}\right)\exp(-b_2\tau).$$

Figure 5 presents a parametric example where we plot the expected future variance over a two-year horizon using a number of different combinations of initial spot variances V_1 and V_2 in the two-factor model.

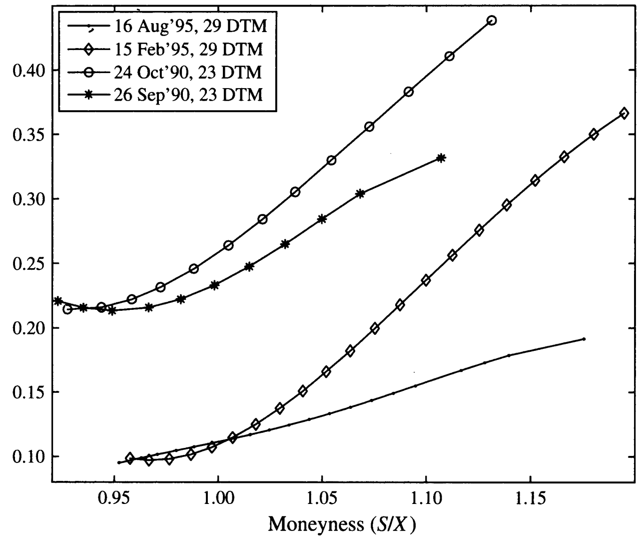
For the one-factor model (solid line), we use a long-run variance a/b of 0.04 and a mean reversion coefficient b of 2. For this particular example, the spot variance is taken to be 0.03. For the one-factor model, the expected future variance converges monotonically to the long-run variance. To make the comparison with the two-factor model as straightforward as possible, we let the mean reversion coefficients for the variance factors be $b_1 = 2$ and $b_2 = 10$, but we take the long-run variance to be 0.02 for each factor so that the total long-run variance is the same as in the one-factor model.

We generate different term structures of expected future variances for the two-factor model (dashed lines) by simply varying the levels of the two spot variances. The spot variance for the first factor V_1 is 0, 0.01, 0.02, and 0.03, respectively, and the second spot variance $V_2 = 0.03 - V_1$ in all cases so as to fix the overall spot variance at 0.03 as in the one-factor model. It can be seen that the two-factor model can lead to many different patterns for expected future variances, including monotonically increasing expected variances but also expected variances that first increase and subsequently decrease and vice versa.

3.3. Can Stochastic Volatility Models Capture Independent Movements in the Level and Slope of the Volatility Smirk?

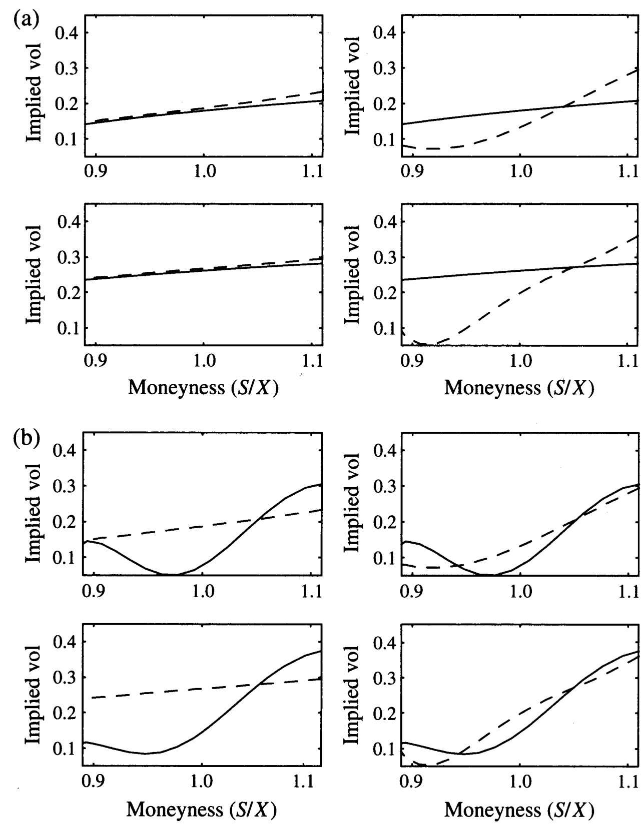
Movements in the level and slope of the volatility smirk are largely uncorrelated. We perform a few simple simulation exercises to demonstrate that one-factor stochastic volatility models have difficulty modeling this simple stylized fact. The model's problems can be illustrated in several ways, but a particularly simple example is the following. Consider the volatility smirks on four different days in the sample in Figure 6. The smirk is subject to strong term structure effects, but the smirks in Figure 6 are for options with either 23 or 29 days to maturity and are therefore comparable. The figure includes two relatively low volatility days: August 16, 1995, and February 15, 1995. October 24, 1990, and September 26, 1990, are

Figure 6 Volatility Smirks for Selected Days and Maturities



Notes. We plot the volatility smirk for call options with either 23 or 29 days to maturity on four different days: August 16, 1995, a low volatility day with a flat smirk; February 15, 1995, a low volatility day with a steep smirk; September 26, 1990, a high volatility day with a flat smirk; and October 24, 1990, a high volatility day with a steep smirk.

Figure 7 Volatility Smirks for the One-Factor (Solid) and Two-Factor (Dashed) Models



Notes. The dashed lines represent the volatility smirk for options with 30 days to maturity using the following parameterization of the two-factor model: $b_1 = 2.597$, $a_1/b_1 = 0.053$, $\sigma_1 = 0.280$, $\rho_1 = -0.834$, $b_2 = 2.597$, $a_2/b_2 = 0.053$, $\sigma_2 = 3.667$, $\rho_2 = -0.957$. In the top pictures of each panel, the initial variance is 0.03; in the bottom pictures of each panel, the initial variance is 0.07. In the pictures on the left the spot variance factors are $V_1 = V$ and $V_2 = 0$; in the pictures on the right we have $V_2 = V$ and $V_1 = 0$. In (a), the one-factor model is calibrated on the two-factor data represented in the pictures on the left. In (b), the one-factor model is calibrated on the two-factor data represented in the pictures on the right.

relatively high volatility days. Figure 6 therefore illustrates that to fit these cross-sections of option contracts simultaneously, a model has to be able to accommodate both high and low smirk slopes on low volatility days as well as high and low smirk slopes on high volatility days.

Figure 7 demonstrates that the two-factor volatility model has the potential to provide this flexibility, whereas the one-factor model does not. The dashed lines in Figure 7 represent implied volatilities for the two-factor model. In each case, the parameterization for the model is the following: $b_1 = 2.597$, $a_1/b_1 = 0.053$, $\sigma_1 = 0.280$, $\rho_1 = -0.834$, $b_2 = 2.597$, $a_2/b_2 = 0.053$, $\sigma_2 = 3.667$, and $\rho_2 = -0.957$, which are motivated by the empirical results below. The only difference for the two-factor model is that for the two pictures on the left, the factor spot variances are $V_1 = V$ and $V_2 = 0$, whereas in the two pictures on the right, they are $V_2 = V$ and $V_1 = 0$. We conduct two experiments for each parameterization: In the two top pictures of each panel, the initial variance V is 0.03, and in the two bottom pictures of each panel, the initial variance V is 0.07.

The one-factor model cannot capture these moneyness patterns. In Figure 7(a), the one-factor model is calibrated on the two-factor data represented in the pictures on the left, in which the first component drives the results. In Figure 7(b), the one-factor model is calibrated on the two-factor data represented in the pictures on the right, in which the second component drives the results. When the one-factor model is calibrated to capture a steep smirk, it cannot capture a flat smirk, and vice versa. In other words, the one-factor model can generate steep smirks or flat smirks at a given volatility level but cannot generate both for a given parameterization. In a purely cross-sectional analysis, this is not a problem, because we can estimate different parameter values for the one-factor model to calibrate the time-varying nature of the cross-section. However, in most recent empirical exercises in the academic finance literature, the emphasis is (justifiably) on demonstrating the ability of the model to capture a variety of different cross-sections with fixed model parameters. Parameters are estimated using multiple cross-sections of option contracts while iterating on the underlying return data to link the cross-sections. A one-factor model has a structural problem in this type of exercise. If its parameters are geared towards explaining a slope of the smirk that is on average high over the sample, it will result in large model error on those days that the slope of the smirk is relatively flat, and vice versa.

4. Empirical Methodology and Results

In this section, we implement the two-factor stochastic volatility (SV) model using our data set on S&P 500 options, and we compare the models' empirical performance with that of the standard one-factor Heston (1993) model.

4.1. Estimation Methodology

When implementing the SV models, one is confronted with the challenge of jointly estimating the model's structural parameters, $\Theta = \{a_i, b_i, \sigma_i, \rho_i\}_{i=1,2}$, as well as the spot volatilities $\{V_i(t)\}_{i,t}$. Various approaches are available in the literature. One popular approach treats the spot volatilities as just another parameter that is reestimated daily. This approach is followed, for example, by Bakshi et al. (1997). Other approaches filter volatility using the time series of underlying returns, which ensures consistency between the physical and risk-neutral measures. This is done in a Bayesian setting in Jones (2003) and Eraker (2004). Andersen et al. (2002) and Chernov and Ghysels (2000) use an Efficient Method of Moments approach, Pan (2002) uses the Generalized Method of Moments, Carr and Wu (2007) use a Kalman filter approach, and Christoffersen et al. (2008a) and Johannes et al. (2009) use particle filtering.

In this paper, we use a modification of the approach taken by Bates (2000), who estimates the structural parameters and spot volatilities using option data only in an iterative two-step procedure. This approach is also used by Huang and Wu (2004). Consider a sample of T Wednesdays of option data. In our implementation we choose $T = 52$, which corresponds to a calendar year. Given a set of starting values for Θ and $\{V_i(t)\}$, the iterative procedure proceeds as follows.

Step 1. For a given set of structural parameters, Θ, solve T sum of squared pricing error optimization problems of the form

$$\{\widehat{V}_1(t), \widehat{V}_2(t)\} = \arg\min \sum_{j=1}^{N_t} (C_{j,t} - C_j(\Theta, V_1(t), V_2(t)))^2/\text{Vega}_{j,t}^2, \quad t = 1, 2, \ldots, T, \quad (18)$$

where $C_{j,t}$ is the quoted price of contract j on day t, and $C_j(\Theta, V_1(t), V_2(t))$ is the corresponding model price. N_t is the number of contracts available on day t. $\text{Vega}_{j,t}$ is the Black-Scholes sensitivity of the option computed using the implied volatility from the market price of the option, $C_{j,t}$.

Step 2. For a given set of spot variances $\{V_1(t), V_2(t)\}$ obtained in Step 1, solve one aggregate sum of squared pricing error optimization problem of the form

$$\widehat{\Theta} = \arg\min \sum_{j,t}^{N} (C_{j,t} - C_j(\Theta, V_1(t), V_2(t)))^2/\text{Vega}_{j,t}^2, \quad (19)$$

where $N = \sum_{t=1}^{T} N_t$.

The procedure iterates between Step 1 and Step 2 until no further significant decreases in the overall objective in Step 2 are obtained. Note that although each function evaluation requires recomputing the model option price for every option involved, the closed-form characteristic functions above guarantee that these calculations can be done in an expedient fashion. Furthermore, the two-step procedure is remarkably well behaved. Convergence is achieved in just a few iterations within each step, and overall convergence also requires only few iterations between Step 1 and Step 2.[13]

The modification vis-à-vis Bates (2000) arises from the rescaling of option pricing errors by $1/\text{Vega}_{j,t}$. The resulting objective function can be viewed as an approximation to implied volatility errors. We can think of the model price of the option as an approximation to the market price using a first-order approximation around implied Black-Scholes volatility:

$$C_j(\Theta, V_1(t), V_2(t)) \approx C_{j,t} + \text{Vega}_{j,t}(\sigma_{j,t} - \sigma_{j,t}(\Theta, V_1(t), V_2(t))),$$

where $\sigma_{j,t}$ and $\sigma_{j,t}(\Theta, V_1(t), V_2(t))$ are implied Black-Scholes volatilities from the market price and the model price respectively, and where $\text{Vega}_{j,t}$ is the Black-Scholes sensitivity of the option price with respect to volatility, computed using $\sigma_{j,t}$. Using this approximation, we can assess model fit using the implied volatility root mean squared error (IVRMSE) loss function as follows:

$$\text{IVRMSE} \equiv \sqrt{\frac{1}{N}\sum_{j,t}^{N}(\sigma_{j,t} - \sigma_{j,t}(\Theta, V_1(t), V_2(t)))^2} \approx \sqrt{\frac{1}{N}\sum_{j,t}^{N}(C_{j,t} - C_j(\Theta, V_1(t), V_2(t)))^2/\text{Vega}_{j,t}^2}.$$

The approximation to the implied volatility errors is extremely useful in large scale empirical estimation exercises such as ours, where the Black-Scholes inversion of model prices becomes very costly numerically. This approximation has also been used in Carr and Wu (2007) and Trolle and Schwartz (2009), among others.

[13] One potential disadvantage of this approach vis-à-vis some of the other procedures discussed above is that no consistency is imposed between the properties of the variance factors and the estimated time series of spot variances.

4.2. Parameter Estimates and In-Sample Results

We present parameter estimates and in-sample results for the one-factor and the two-factor model. The iterative two-step estimation routine is applied for each of the 15 calendar years in our sample. The resulting 30 sets of parameter estimates are reported in Table 3. Note that for expositional reasons, Table 3 reports on the ratio of coefficients a/b rather than a. The ratio a/b is equal to the unconditional annual variance for the one-factor model and to the mean of the variance factor in the two-factor case.

Consider first the one-factor model. The one-factor model does an excellent job of capturing the overall level of volatility. The weekly time-series correlation between the volatility level factor (from §2.2) and estimates of the one-factor V is 95%. But the one-factor model does not capture the slope. The correlation between the volatility slope factor and the one-factor V is only 1%.

The parameters for the one-factor model are intuitively plausible and quite stable across estimation years. The mean-reversion of variance parameter b is typically between 1.5 and 3.5, which means that the half-life of variance shocks is between three and eight months. The unconditional variance a/b is between 0.02 and 0.05, the volatility of variance σ is between 0.4 and 0.8, and the correlation ρ between returns and return variance is between -0.8 and -0.5. The overall IVRMSE as a percentage is between 0.879% (in 2003) and 2.898% (in 1993). The overall in-sample IVRMSE is 1.995% across the six years, which is quite impressive in a data set with an average observed implied volatility of 18.5% (see Table 1).

Consider next the two-factor model. The purpose of the second factor is to capture independent movements in option prices. The correlation between the first principal component (from §2.2) and the sum of the factor estimates $V_1 + V_2$ is 88%, and the correlation between the volatility level factor (from §2.3) and the sum of the factors is 87%. The two factors are able to explain more than just the volatility level. The correlation between the second principal component and the difference of the two factors $V_1 - V_2$ is 62%, and the correlation between the slope factor and the difference of the two factors is 51%. Because this correlation is only 1% in the one-factor model, the second factor is important to capture these independent movements in option prices. Once again, we want to emphasize that the objective of the principal components analysis is merely to provide a crude and model-free way to capture option prices' dynamics. Nevertheless, we think the relationships between the principal components and the stochastic volatility factors are suggestive of the flexibility provided by the two-factor model.

A consistent finding across estimation years is that one of the factors is slowly mean reverting, with b between 0.14 and 0.30, and that the second factor mean reverts more quickly—sometimes dramatically so—in all of the sample years. Notice also that the slowly mean-reverting first factor has a higher volatility of variance than the second factor in each of the estimation years. The in-sample fit of the two-factor model is impressive. The penultimate column in panel B of Table 3 reports the IVRMSE and the third-last column in panel A normalizes the two-factor IVRMSE by the IVRMSE of the one-factor model. The IVRMSE improvement is between 41% in 2000 and 12% in 1990. The overall IVRMSE of the two-factor model is 1.51%, or 24.1% below the overall IVRMSE of the one-factor model.

Bates (2000) estimates a two-factor model using 1988–1993 S&P 500 futures option prices. He obtains one set of estimates using the entire sample, and the estimates are as follows: $a_1 = 0.028$, $b_1 = 0.00$, $\sigma_1 = 1.039$, $\rho_1 = -0.775$, $a_2 = 0.130$, $b_2 = 5.58$, $\sigma_2 = 0.667$, and $\rho_2 = -0.382$. In comparison, the averages of our estimates in Table 3 are $a_1/b_1 = 0.007$, $b_1 = 0.179$, $\sigma_1 = 3.667$, $\rho_1 = -0.957$, $a_2/b_2 = 0.114$, $b_2 = 1.303$, $\sigma_2 = 0.280$, $\rho_2 = -0.834$. The estimates of Bates (2000) imply $a_2/b_2 = 0.233$, but we cannot compute his a_1/b_1. There are of course many potential reasons why his estimates differ from ours, not in the least the very different data sets. However, there are some striking similarities. In both cases, the factor with the highest mean reversion has a smaller σ and a smaller (in absolute value) ρ, indicating this factor is a less important driver of skewness and kurtosis. However, in our case the differences between σ_1 and σ_2 are larger than in Bates (2000) and the differences between ρ_1 and ρ_2 are smaller. It is also noteworthy that our (absolute values of) estimates of ρ_1 and ρ_2 are much larger than those of Bates (2000), perhaps indicating more negative skewness in the data.

Table 4 reports on in-sample IVRMSE by moneyness and maturity. We report the IVRMSE for the one-factor model (panel A), the IVRMSE for the two-factor model (panel B), and the ratio of the two-factor to one-factor IVRMSE (panel C). Notice that the ratio IVRMSE is below one for every maturity category (last row of panel C) and also below one for every moneyness category (last column of panel C). Looking across all moneyness and maturity bins, it is clear that the improvements tend to be the largest for short-maturity options that are not close to at-the-money. The one-factor model performs best only for in-the-money calls with 90–180 days to maturity. The two-factor model dominates in all the other moneyness/maturity combinations.

Figure 8 further documents the differences between the in-sample IVRMSEs of the models by graphing

Table 4 IVRMSE and Ratios by Moneyness and Maturity, 1990–2004, In-Sample

	$DTM < 30$	$30 < DTM < 90$	$90 < DTM < 180$	$DTM > 180$	All
		Panel A. IVRMSE from one-factor model			
$S/X < 0.975$	1.9400	1.2418	0.8903	0.9328	1.1105
$0.975 < S/X < 1$	1.6819	1.0346	0.8160	0.9331	1.1406
$1 < S/X < 1.025$	1.4368	0.9272	0.8328	0.9080	1.0478
$1.025 < S/X < 1.05$	2.3232	0.9579	0.8722	0.9930	1.3983
$1.05 < S/X < 1.075$	3.9814	1.4389	0.9198	1.1553	2.1988
$S/X > 1.075$	7.1687	3.0466	1.4555	1.7891	3.8545
All	3.8776	1.5689	1.0087	1.1343	1.9945
		Panel B. IVRMSE from two-factor model			
$S/X < 0.975$	1.3562	0.8718	0.7250	0.8457	0.8596
$0.975 < S/X < 1$	1.3198	0.7753	0.6754	0.7530	0.8865
$1 < S/X < 1.025$	1.2765	0.7554	0.7425	0.7854	0.9016
$1.025 < S/X < 1.05$	1.8250	0.8851	0.8930	0.8582	1.1755
$1.05 < S/X < 1.075$	2.6039	1.2217	1.0500	0.9725	1.5908
$S/X > 1.075$	5.1843	2.2435	1.4724	1.5963	2.8754
All	2.8132	1.1871	0.9514	1.0022	1.5133
		Panel C. IVRMSE ratio: Two-factor model over one-factor model			
$S/X < 0.975$	0.6991	0.7021	0.8143	0.9066	0.7741
$0.975 < S/X < 1$	0.7847	0.7494	0.8277	0.8069	0.7772
$1 < S/X < 1.025$	0.8884	0.8147	0.8915	0.8650	0.8605
$1.025 < S/X < 1.05$	0.7855	0.9240	1.0238	0.8643	0.8407
$1.05 < S/X < 1.075$	0.6540	0.8490	1.1416	0.8417	0.7235
$S/X > 1.075$	0.7232	0.7364	1.0116	0.8923	0.7460
All	0.7255	0.7567	0.9432	0.8836	0.7587

Notes. We use the parameter estimates in Table 3 to compute the IVRMSE for various moneyness and maturity bins. Panel A reports the IVRMSE for the one-factor model. Panel B reports IVRMSE from the two-factor model. Panel C reports the ratio of the IVRMSE from the two-factor model to the IVRMSE from the one-factor model.

the average weekly IVRMSE over the 1990–2004 sample for the one-factor model and the ratio of the two-factor IVRMSE to the one-factor IVRMSE. The analysis is in-sample: For each year, the corresponding parameters from Table 3 are used. Notice that the ratio IVRMSEs in the lower panel are almost always less than one, so the improvement in the two-factor model is not confined to a particular time period in our sample. Moreover, episodes of 50% improvement in IVRMSE over the one-factor model occur throughout the sample.

4.3. Out-of-Sample Results

Our out-of-sample results are obtained as follows. For each but the final year of the sample, we use the in-sample structural parameters Θ in Table 3 and Step 1 described in §4.1 to compute the out-of-sample spot volatilities for the following year. The overall sum of squared pricing errors is then simply calculated as the sum of the 52 sums of squares from Step 1. This out-of-sample implementation follows Huang and Wu (2004).

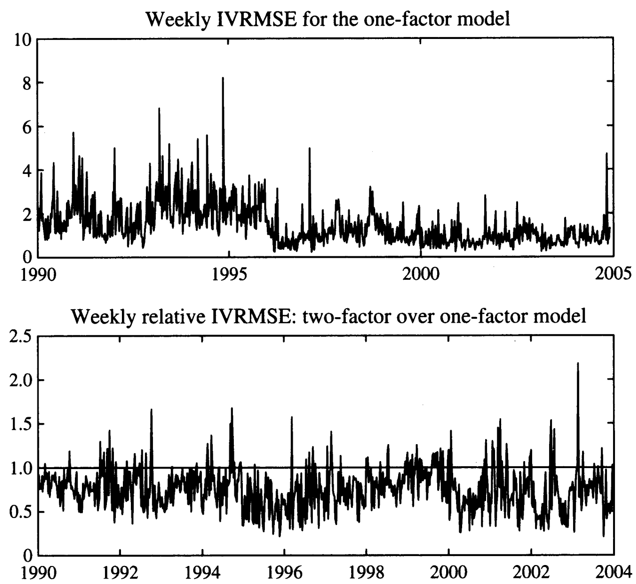
Table 3 shows the out-of-sample IVRMSEs year by year for the two models. Consider first the benchmark one-factor model. Note first that the in-sample IVRMSEs are always lower than any of the corresponding out-of-sample IVRMSEs in the same row. This is reassuring, because it demonstrates that the estimation routine is satisfactory. More interestingly, note also that the out-of-sample IVRMSEs are often quite close to the in-sample value.

Comparing the one-factor model with the two-factor model in the penultimate column of panel A in Table 3, we see that the two-factor model performs better than the one-factor model out-of-sample in each of the 14 years in our out-of-sample period. The average improvement in IVRMSE offered by the two-factor model is 23.8%. The smallest improvement occurs in 1991, at 13.2%, and the largest improvement occurs in 1996, at 37.7%.

4.4. Ad Hoc Benchmark Models

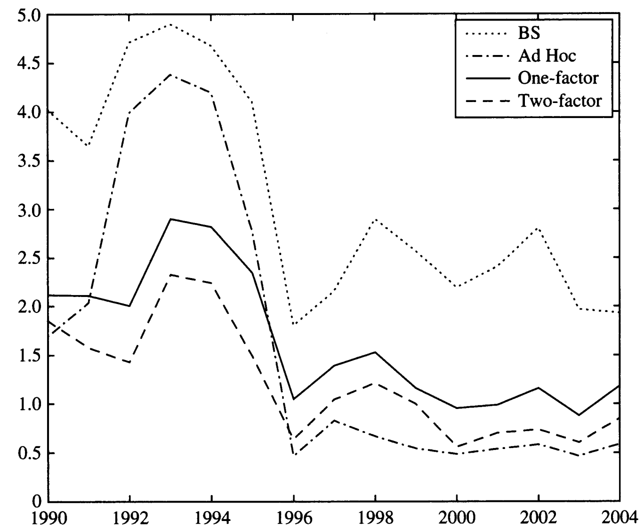
Although the out-of-sample performance of the two-factor model is impressive relative to the one-factor model, the question arises whether the one-factor model is a good choice of benchmark. There are two ways to address this question. First, there are not many structural option pricing models available in the literature that robustly improve upon the out-of-sample performance of the Heston (1993) model. Models that contain Poisson jumps in returns and/or volatility may significantly improve on the in-sample

Figure 8 Weekly In-Sample Root Mean Squared Error (IVRMSE) 1990–2004



Notes. The top panel shows the weekly IVRMSE for the one-factor model. The bottom panel shows the ratio of the IVRMSEs for the two-factor and one-factor models.

Figure 9 IVRMSE from Stochastic Volatility Models vs. Black-Scholes and Ad Hoc Models



Notes. We plot the in-sample IVRMSE year by year for four models. The Black-Scholes benchmark is calculated using a different volatility each week but keeping that volatility constant across the contracts observed in a given week. The Ad Hoc benchmark is calculated as in Dumas et al. (1998) by regressing implied volatility on a second-order polynomial in the strike price and maturity.

performance of the Heston (1993) model, but the out-of-sample improvements are modest or nonexistent (see, for example, Bakshi et al. 1997, Eraker 2004).[14] Option valuation models that are based on Levy processes for the underlying asset seem to be more successful out-of-sample (see Huang and Wu 2004, Carr and Wu 2007). Although it is always difficult to compare the performance of models estimated using different techniques, as well as using different data samples, our simple two-factor model seems to improve on the performance of the one-factor Heston (1993) model by at least the same amount as the most sophisticated jump models.

Figure 9 addresses the appropriateness of the benchmark in a different way, by comparing the annual in-sample IVRMSE from Table 3 with two often used ad hoc benchmark models. The first ad hoc benchmark, labeled "Black-Scholes (BS)," is implemented by setting the spot volatility each Wednesday equal to the average implied BS volatility for that day. Thus, while retaining the structure of the BS pricing formula, it allows for time-varying volatility via weekly reestimation. Clearly, both stochastic volatility models we consider perform well in comparison with this benchmark.

The second benchmark, labeled "Ad Hoc," is the model used in Dumas et al. (1998), which regresses implied BS volatilities on a second-order polynomial in strike price and maturity. Dumas et al. (1998) find that this method outperforms the deterministic volatility models they consider in their paper. Notice that the two stochastic volatility models we consider outperform the ad hoc BS benchmark in the early part of the sample but the Ad Hoc model slightly outperforms the SV models in the latter part of the sample.

Christoffersen and Jacobs (2004) show that when the ad hoc model is implemented via minimization of the loss function subsequently used in model evaluation, the fit of the ad hoc model improves drastically. Note that we are using an (approximate) implied volatility loss function when estimating the SV models, and therefore the estimation and evaluation criteria are closely aligned when using the standard ad hoc model from Dumas et al. (1998).

5. Model Properties

We have argued that two-factor models are more flexible than one-factor models for the purpose of modeling moneyness effects as well as volatility term structure. In §4, we compared the pricing performance of the one-factor and two-factor models, and our empirical results confirm that the two-factor models provide a better fit. Although these types of comparisons are critical, they do not highlight the model features that enable the model to fit the data better.

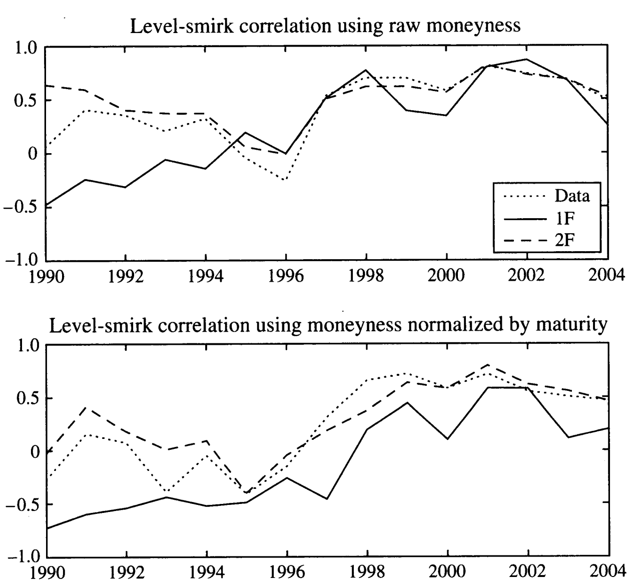
Figure 3 documents the time-varying correlation property of the two-factor model. In the online appendix, we show that although the spot variance

[14] Broadie et al. (2007) show that when restricting certain parameters to be equal to estimates from historical returns, adding Poisson jumps to a stochastic volatility model improves option fit.

paths are similar across the one-factor and two-factor models, the spot covariance between returns and variance differs across models as does the spot variance of variance. We also show in the online appendix that the one-factor and two-factor models differ substantially in terms of the flexibility offered to model the conditional skewness and kurtosis at various horizons, and the one-factor model seems constrained in this respect. Conditional dynamics therefore suggest that the improved fit of the two-factor model is partly due to the improved modeling of higher conditional moments.

However, we have not yet directly related the improvement in fit to the modeling of the smirk and the volatility term structure. In this section, we explore the empirical results in more detail and try to provide more intuition for the differences in empirical performance by discussing the models' ability to capture these stylized facts. The dotted line in the top panel of Figure 10 shows the correlation between the volatility level and the slope of the volatility smirk by regressing implied volatilities on a given day on an intercept and log moneyness year by year. The dashed and solid lines show the correlation based on the same regression analysis, except that the implied volatilities used in the regressions are not those implied by the data but are based on the option prices predicted by the one-factor and two-factor models, respectively. This analysis therefore investigates whether the two-factor model better matches the correlations in the data. Figure 10 shows that the two-factor model matches the empirical correlation much better than the one-factor model, particularly in the early parts of the sample when the data-based correlation is positive and the one-factor model correlation is negative.[15]

Figure 10 **Correlation Between the Volatility Level and the Slope of the Volatility Smirk**



Notes. We compute the correlation between the volatility level and the slope of the smirk on a year-by-year basis. The volatility level and the slope are obtained by regressing implied volatility on a measure of moneyness. In the top panel, implied volatilities are regressed on simple log moneyness. In the bottom panel, implied volatilities are regressed on log moneyness normalized by maturity. The dots denote correlations from data IVs, the solid line denotes IVs from the one-factor model, and the dashes denote IVs from the two-factor model.

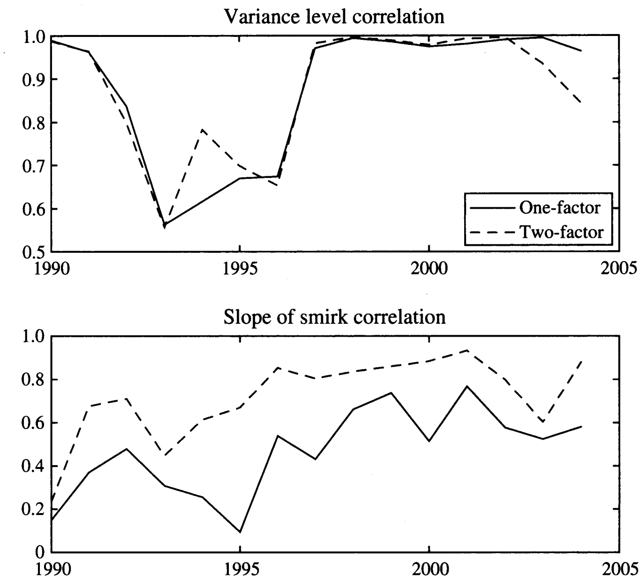
It could be argued that regressing implied volatilities on log moneyness may lead to noisy results because significant maturity effects are not filtered out. The bottom panel of Figure 10 therefore repeats the analysis of the top panel, but implied data and model volatilities are regressed on $\ln[S/(X\exp(-r\tau))]/\sqrt{\tau}$, to remove maturity effects. Although the correlations are somewhat different, the two-factor model again captures the patterns in the data much better than the one-factor model. The one-factor model consistently generates correlations between volatility level and slope that are below the correlations in the data. The two-factor model tracks the data correlation very well throughout the 15-year sample.

In Figure 11, we compute the absolute correlation between the variance factors and the time series for the level and slope of the smirk by regressing the implied volatilities on a given Wednesday on moneyness corrected for maturity $\ln[S/(X\exp(-r\tau))]/\sqrt{\tau}$. We report the results on a year-by-year basis because the parameter estimates are on a year-by-year basis.

The top panel of Figure 11 reports results for the variance level. The solid line indicates that the variance factor in the one-factor model is highly correlated with the time series for the level. The dashed line indicates that in the two-factor model, the multiple correlation between the variance level and the two factors is high as well. The bottom panel reports the correlation between the slope of the smirk and the variance factors. The solid line again denotes the one-factor model and the dashed line the multiple correlation for the two-factor model. Note that the slope of the smirk is captured quite well when using the two-factor model but considerably less convincingly by the one-factor model. Altogether, Figure 11 indicates that the two-factor model is better at capturing the slope of the smirk, even if the multiple correlation coefficient indicates that some of the variation in the smirk remains unexplained, especially in the early part of the sample.

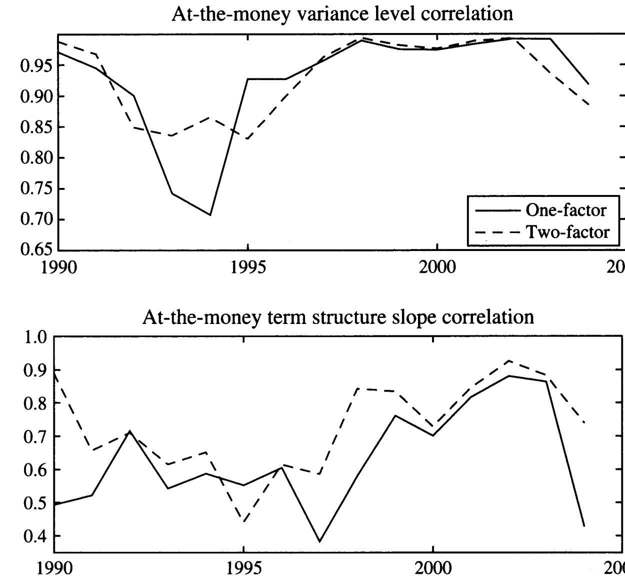
[15] We also compared the performance of the one-factor and two-factor models by conducting a principal component analysis on the fitted data. The variation in the fitted data explained by the first, second, and third factor is 95.95%, 3.35%, and 0.36% in the case of the one-factor model, and 94.62%, 3.42%, and 1.35% in the case of the two-factor model. The patterns generated by the two-factor model are therefore closer to those in the data (see Table 2), but the differences between the models are not large along this dimension.

Figure 11 **Absolute Correlation of Variance Factors with Volatility Level and Slope of the Smirk**



Notes. On a year-by-year basis, we compute the absolute correlation between the time-series of the variance factors and the level and slope of the smirk obtained by regressing implied volatility on moneyness. Moneyness is normalized by maturity. For the two-factor model, we compute the multiple correlation coefficient obtained by regressing either level or slope of the smirk on both factors.

Figure 12 **Absolute Correlation of Variance Factors with Volatility Level and At-the-Money Term Structure Slope**



Notes. On a year-by-year basis, we compute the absolute value of the correlation between the time-series of the variance factors and the level and slope of the at-the-money term structure obtained by regressing implied volatility on maturity. Only contracts with moneyness between 0.96 and 1.04 are used in the regressions. For the two-factor model, we compute the multiple correlation coefficient obtained by regressing either level or slope of the term structure on both factors.

Figure 12 reports on a similar analysis. Instead of regressing implied volatilities on moneyness, we regress on a constant and maturity using only at-the-money options. Some important conclusions obtain. Again the two models are fairly close in capturing the level of at-the-money variance (top panel). However, the two-factor model (dashed line) is better than the one-factor model (solid line) at capturing the slope of the term structure (bottom panel), particularly in the early and middle parts of the sample.

When combining the findings from Figures 11 and 12, we arrive at the following conclusions. First, the two-factor model offers more flexibility than the one-factor model for modeling the maturity dimension as well as the moneyness dimension. Second, although the two-factor model substantially outperforms the one-factor model, the two-factor model seems to encounter some difficulty in capturing moneyness effects in the earlier part of the sample. This suggests that an even richer model may be needed.

6. Summary and Conclusion

This paper investigates a tractable model for equity index option valuation that allows for rich modeling of term structure and moneyness effects. It is important to have simple yet robust models that are relevant from a theoretical as well as a practical perspective, and we believe that the two-factor SV model satisfies this criterion. We find that adding volatility factors to an existing framework and exploiting the pricing results of Heston (1993) greatly improves the model's flexibility to capture the volatility term structure. Moreover, we demonstrate that the two-factor model is more flexible in capturing largely independent fluctuations in the level and the slope of the volatility smirk, which are inextricably linked in the one-factor Heston model.

We are not the first to suggest the use of multiple volatility factors, but our discussion of the role of multiple factors in capturing term structure and moneyness effects is novel. One-factor models are unable to account for some important stylized facts in index option data. The in-sample and out-of-sample performance of the two-factor model, which is at par with the most sophisticated models currently available in the literature, forcefully illustrates the power of the multifactor approach.

The paucity of multifactor volatility models in the option valuation literature is remarkable when one considers the related empirical literature on yield curve modeling. The theoretical models and empirical techniques used in the option valuation literature are closely related to those used in the yield curve literature. Interestingly, almost every paper in the yield curve literature uses a multifactor model, and in fact three-factor models have become the standard. We speculate that in the future, multifactor models may become as important for the equity option literature as they are for the term structure literature.

A number of extensions to the analysis in this paper may prove worthwhile. First, it may prove interesting to compare the relative value of adding jump components and additional variance factors to a stochastic volatility model. The resulting models may have different implications for the modeling of term structure and moneyness effects, and their performance may differ in-sample as well as out-of-sample. Second, an integrated analysis of multifactor models using option data as well as underlying returns ought to be done. Following the observations of Bates (1996) and Broadie et al. (2007), it will be of particular interest to investigate the model's pricing performance when imposing consistency between physical and risk-neutral estimates. Third, the focus of our empirical analysis is to convince the reader that a second factor allows for more realistic modeling of conditional higher moments, which improves the modeling of term structure and moneyness effects. We leave open the question whether additional factors are needed and how they would improve pricing performance. The out-of-sample performance of such models is of particular interest because often richly parameterized models perform poorly out-of-sample. Fourth, our analysis also does not address the interesting question of how the dynamics of the different factors ought to be specified. We intentionally choose a simple specification to obtain a closed-form solution. In the term structure literature, recent empirical studies have demonstrated that multifactor models with some square root factors and some Gaussian factors outperform multifactor models with multiple square root factors. Also, although the variance factors in the model are assumed to be uncorrelated in order to obtain a closed form solution, correlated factors may improve model fit.

In summary, our paper argues that we need multifactor models to capture some of the most salient stylized facts in index option prices. We hope that our results will lead to a more extensive search for an even better multifactor model.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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