Week 9: Visualising Networks

Visual Data Analytics

University of Sydney

Outline

- Graph theory
  - Graphs
  - All about edges
  - All about nodes
- Bipartite graphs

Motivation

- Networks are ubiquitous in business applications
  - Supply chains
  - Social networks
  - International trade
- Networks can be understood and visualised using graph theory.

Network v Graph

- A *network* is a real system consisting of some entities and the relationships between them.
- A *graph* is an abstract mathematical construct that can describe networks.
- Note that this week we use *graph* with a specific mathematical meaning, and not in the everyday sense of graphs as different visualisations (e.g histograms and scatterplots).

Graph theory

Definition

- A graph is made up of two sets
  - A set of vertices $\mathcal{V}$
  - A set of edges $\mathcal{E}$
  - At most one edge connects exactly two vertices
- In some settings, the third condition is relaxed so that a vertex can have an edge with itself and multiple edges can connect the same two vertices

Examples

- In a social network:
  - Vertices are individuals / users
  - Edges indicate whether two individuals are *friends* on the social network
- In a food delivery app
  - Edges may be restaurants and suburbs
  - Vertices indicate whether a restaurant delivers to a suburb.

Simple example

- Four vertices: A, B, C and D
  - Edges from A to B, C and D
  - Edges from B to A and D
  - Edges from C to A and D
  - Edges from D to A, C and D
- This is adapted from the bridges of Königsberg (the first application of graph theory).

In Python

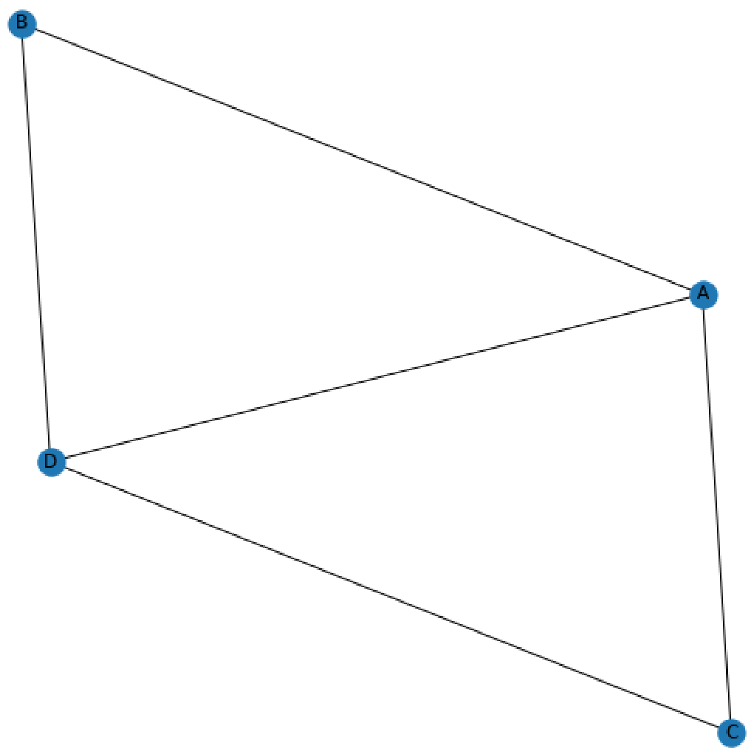
- We will use the `networkx` package to deal with graphs and networks.
- We can build a network from a pandas data frame

```
import networkx as nx
df = pd.DataFrame({ 'from':['A', 'A', 'A','B','C'], 'to':['B', 'C', 'D'
df
```

```
##   from to
## 0    A  B
## 1    A  C
## 2    A  D
## 3    B  D
## 4    C  D
```

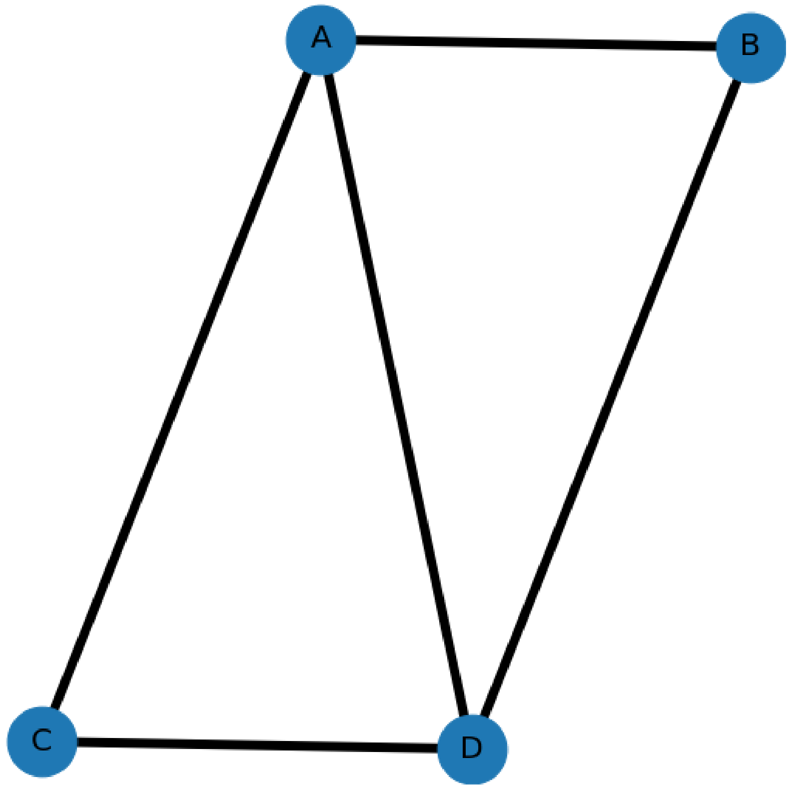
Drawing graph

```
G=nx.from_pandas_edgelist(df, 'from', 'to')
nx.draw(G, with_labels=True)
plt.show()
```



To look nicer

```
G=nx.from_pandas_edgelist(df, 'from', 'to')
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = 6)
plt.show()
```



Why are they different?

- Layout of graphs is determined by algorithms.
- The default algorithm is known as Fruchterman-Reingold.
- To understand must consider two "forces":
  - Think of edges as "springs" with an attractive force to display shorter edges.
  - Think of non-adjacent nodes as repelled by electric forces.

Fruchterman-Reingold

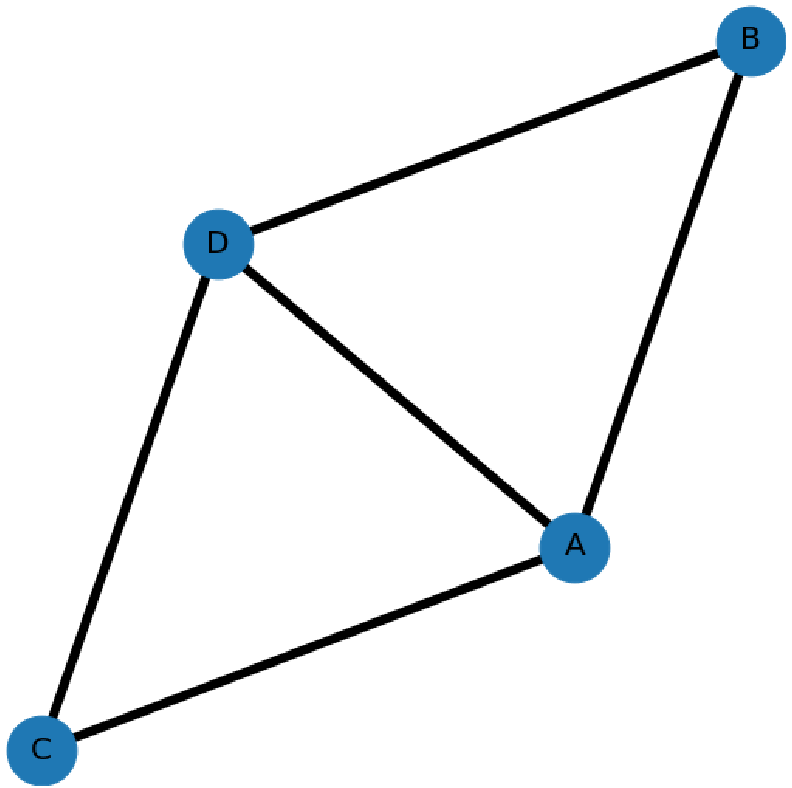
- Initialise nodes in random positions
- Compute both attracting and repelling forces for each node.
- Move all nodes by an amount proportional to net forces.
- Repeat this, each time making the adjustment smaller.

Fruchterman-Reingold

- Due to initial random step, can plot the same plot and get different results.
- To obtain a consistent layout we can
  - Fix the position of some nodes
  - Set the same random seed whenever we draw the graph.
- Usually experimenting with a few different random seeds is satisfactory.

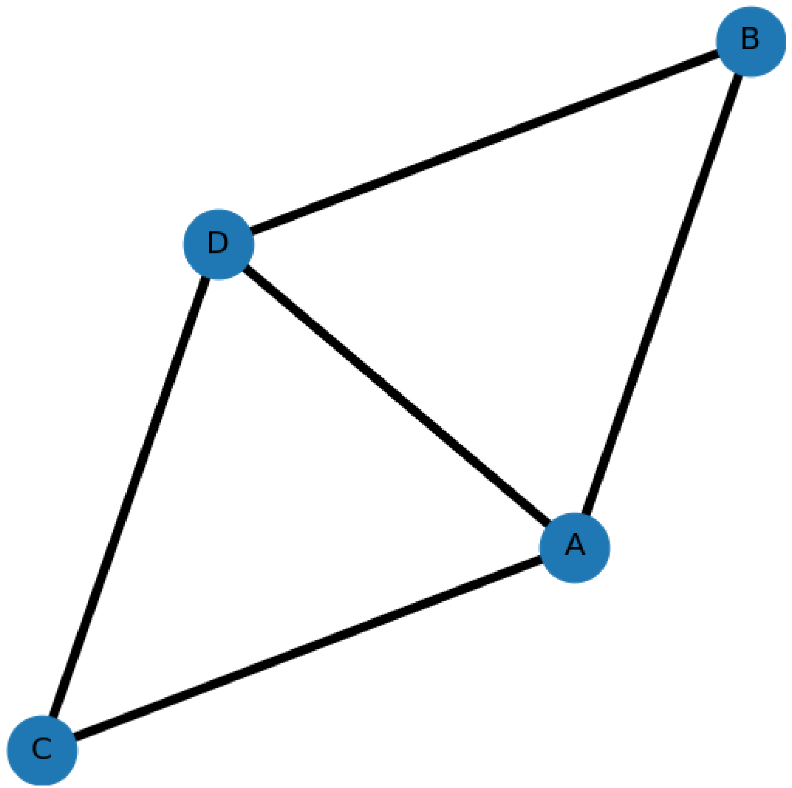
Set seed

```
G=nx.from_pandas_edgelist(df, 'from', 'to')
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = 6, p
plt.show()
```



Same layout

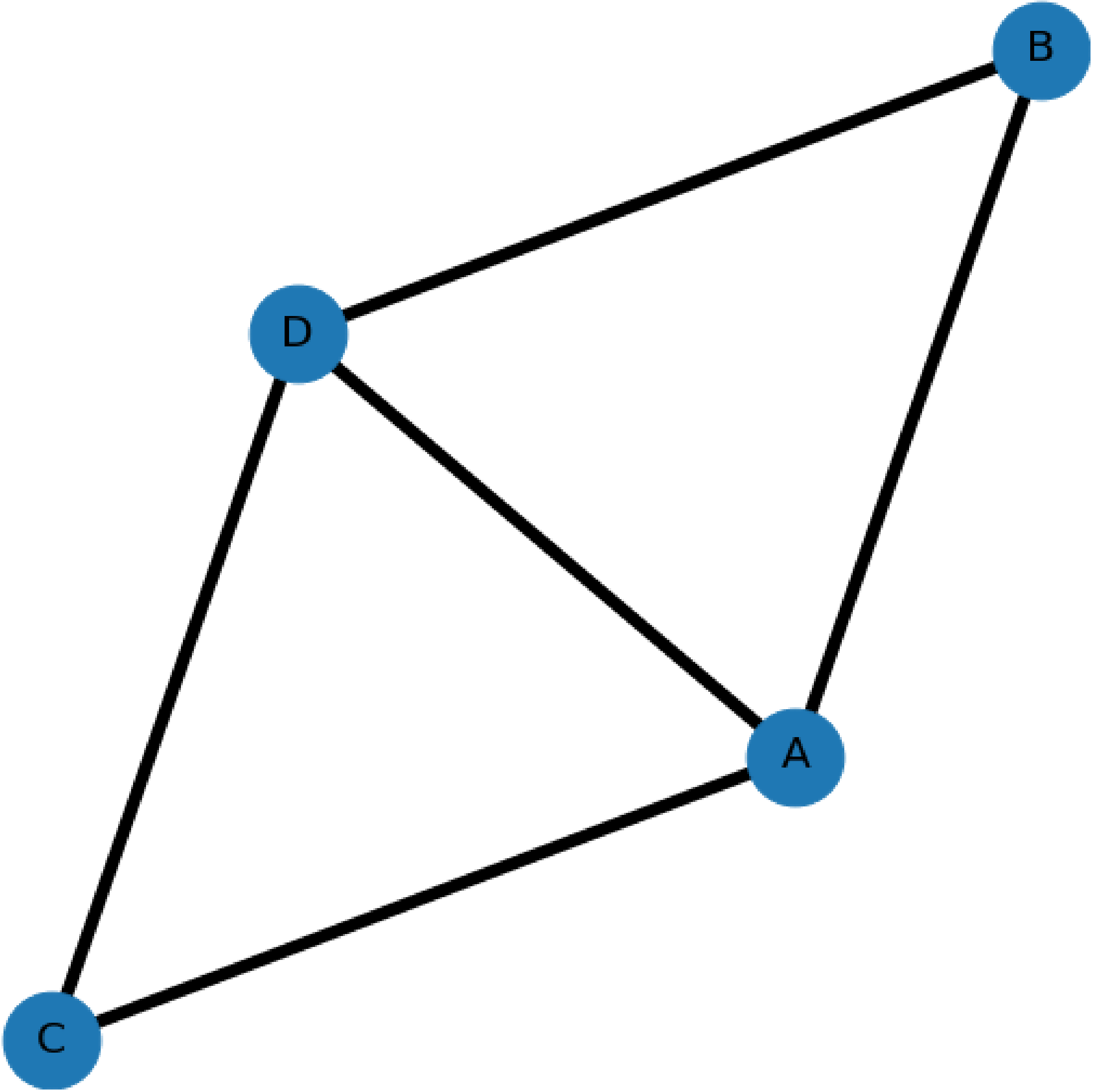
```
G=nx.from_pandas_edgelist(df, 'from', 'to')
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = 6, p
plt.show()
```



Adjacency matrix

- The adjacency matrix is defined so that
  - The element $a_{ij} = 1$ if two vertices are connected by an edge (they are *adjacent*)
  - The element $a_{ij} = 0$ otherwise
- Let's see an example

Graph again

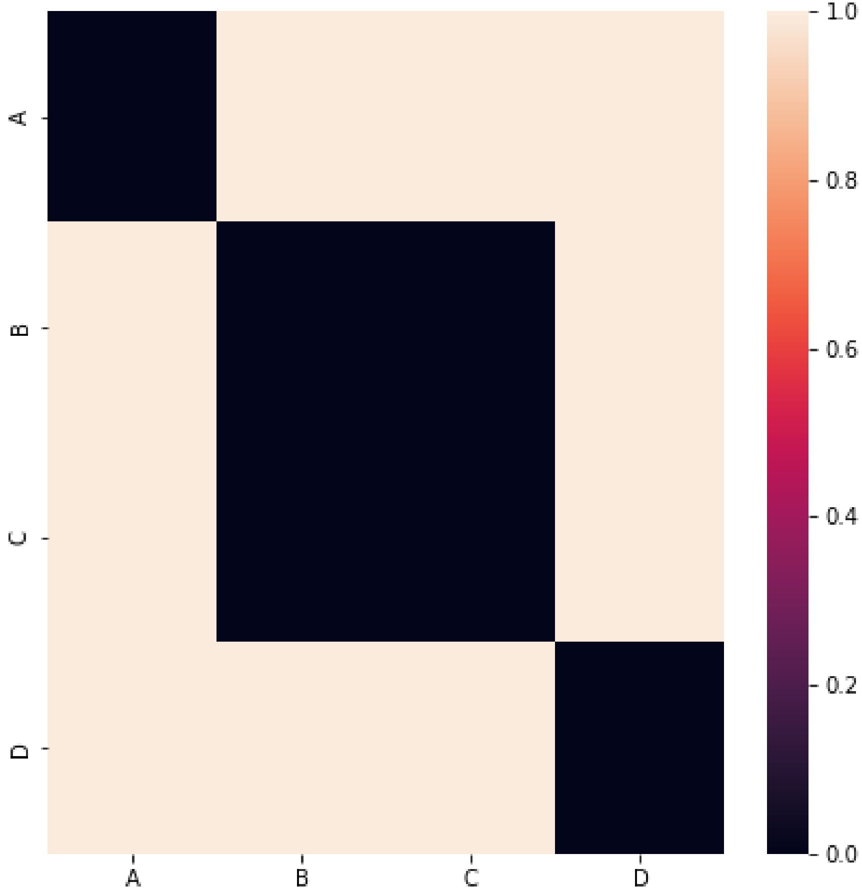


Adjacency matrix

Adjacency matrix given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

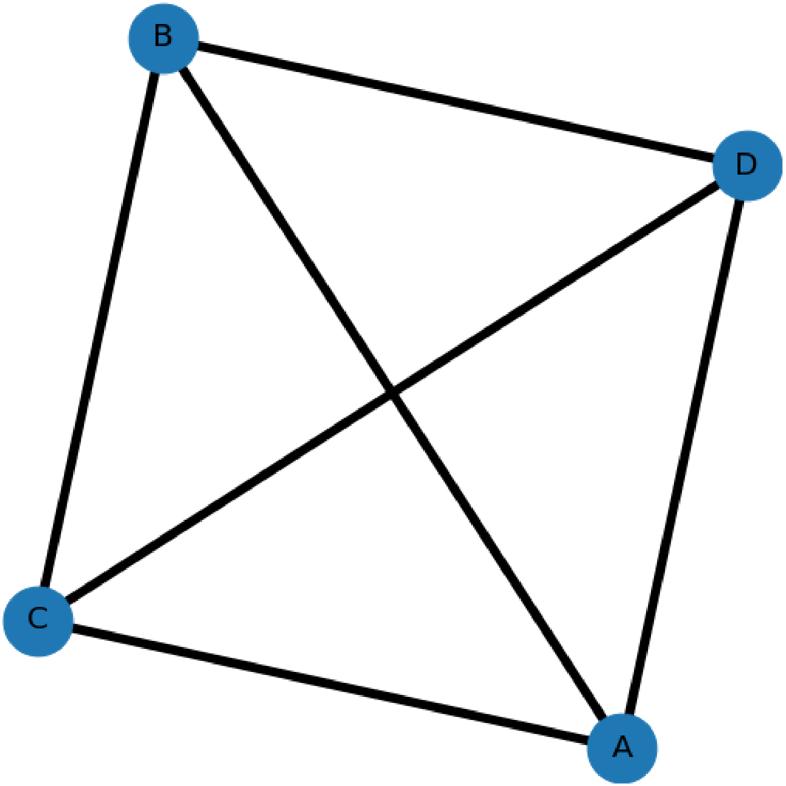
As heatmap



Complete graphs

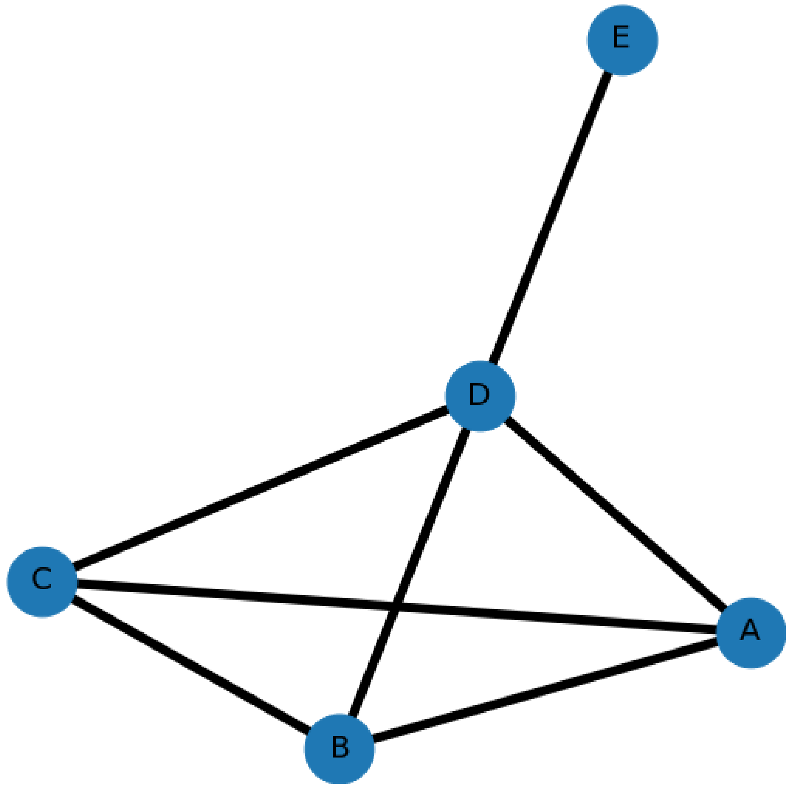
- If there are edges between every pair of nodes the graph is *complete* (then $\boldsymbol{A}$ only has ones off diagonal).
- If a subset of vertices has this property this is a called a *clique*.
- The number of edges divided by the number of edges of a complete graph is a measure of graph *density*.

Complete graph



Clique

A,B,C and D form a clique



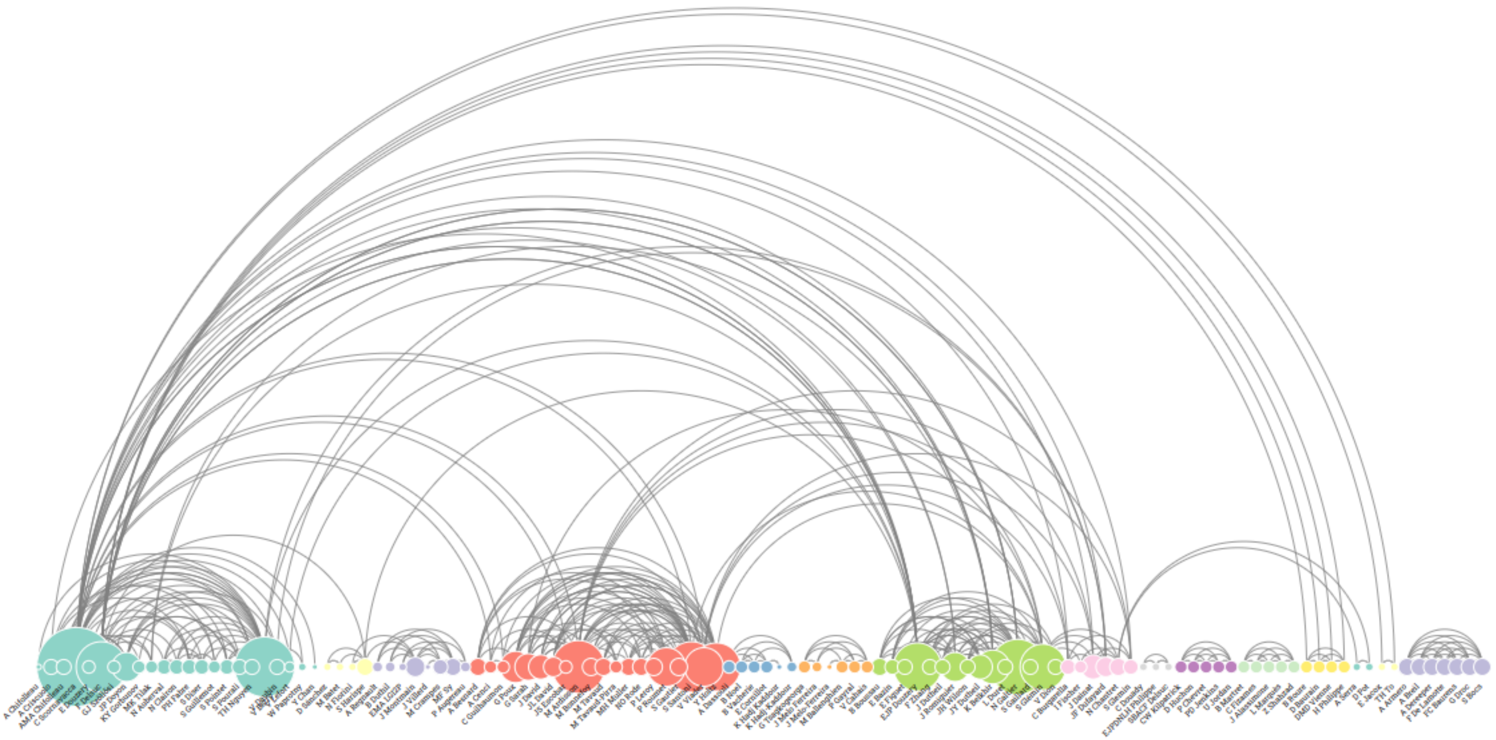
What is the density of this graph?

Larger networks

- The Fruchterman Reingold method works well for graphs with hundreds of nodes.
- For larger graphs different algorithms have been developed.
- This exploits the fact that larger graphs have low density.
- Break up networks into smaller smaller subgraphs/cliques.
- Larger graphs benefit from interactivity which is possible (but challenging) using plotly.

Arc plot

An alternative is the arc plot.



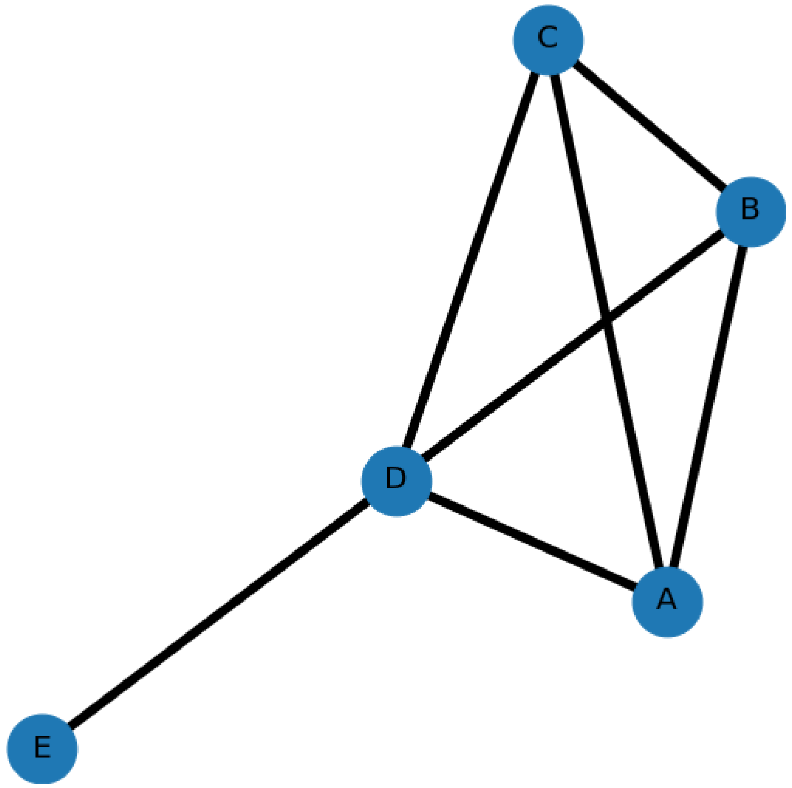
Paths and edges

Paths

- If connect one vertex to another without visting the same vertex twice, this is known as a *path*.
- Finding the *shortest path* has many applications.
- This gives a measure of *distance* between nodes
- We can visualise paths as well

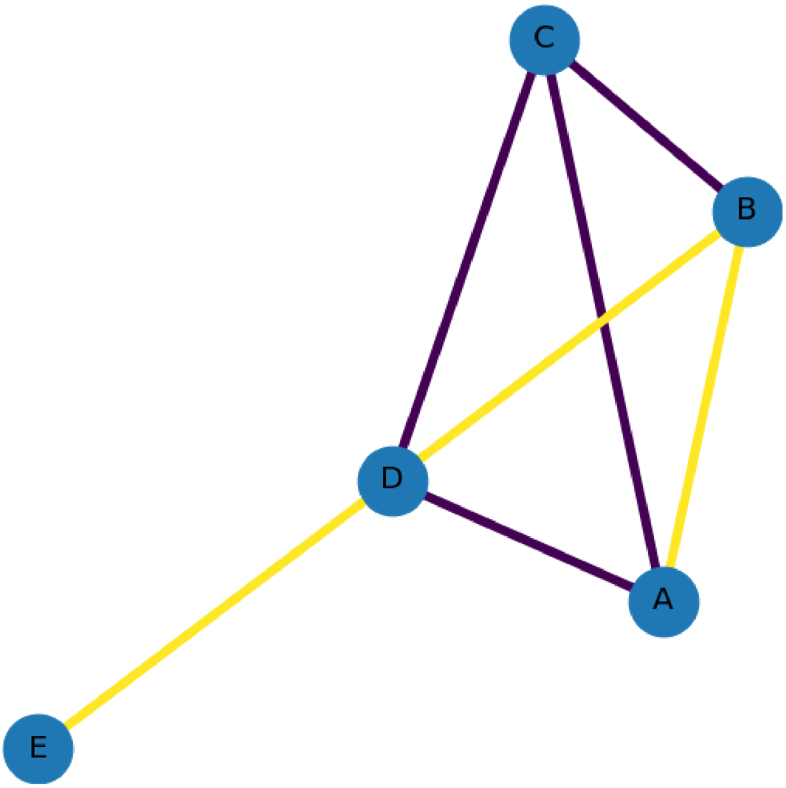
Paths

Path from A to E



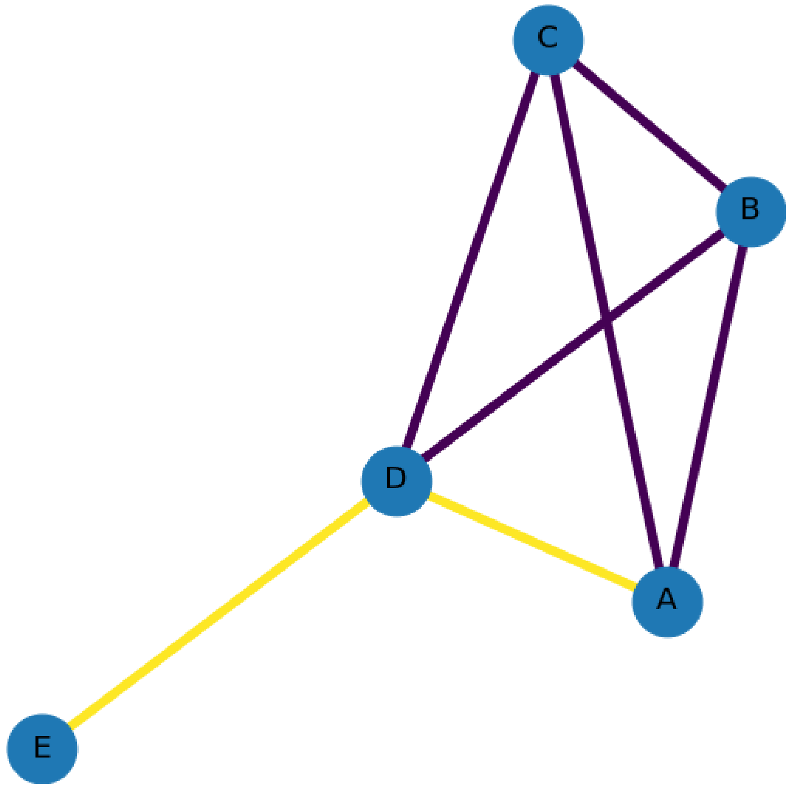
Paths

Path from A to E



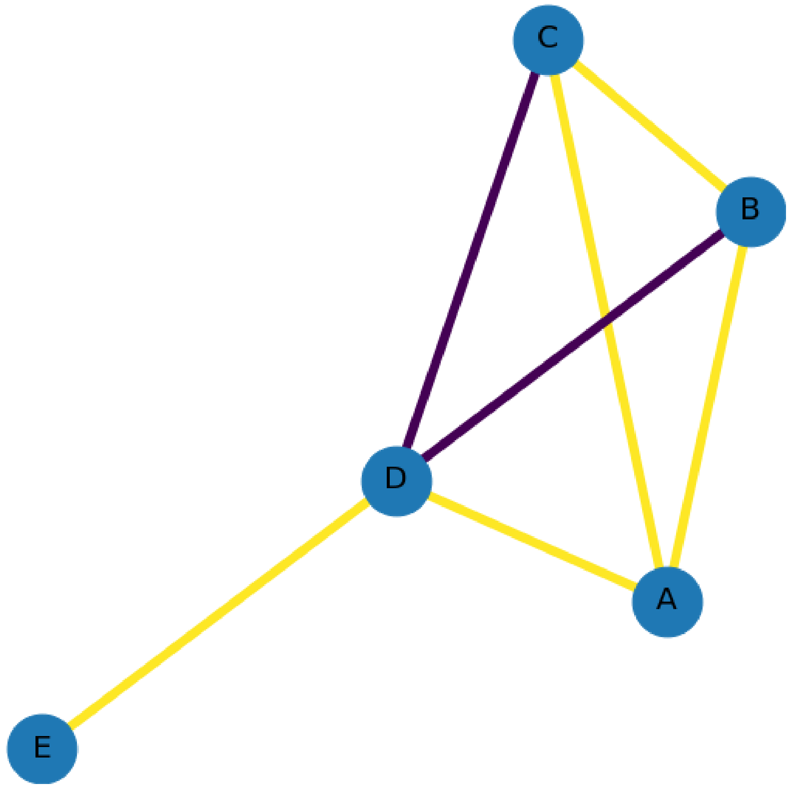
Paths

Another path from A to E



Paths

Not a path

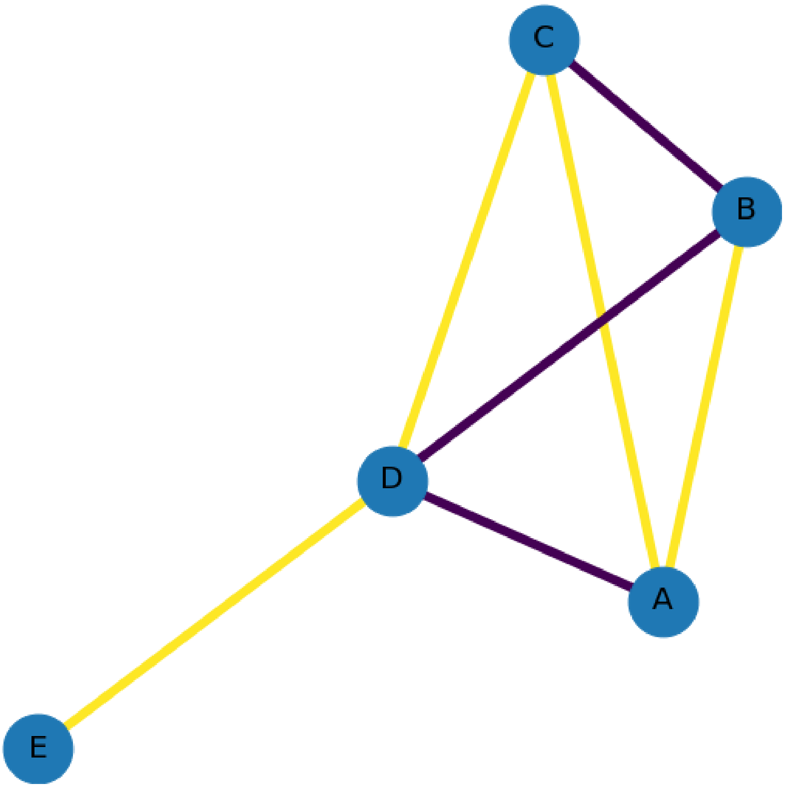


Hamiltonian Path

- Hamiltonian paths pass through every vertex exactly once.
- Finding the shortest Hamiltonian path has many applications in logisitics.
- This is similar to the Travelling Salesman problem.
- In the TSP however, you must return to the first node

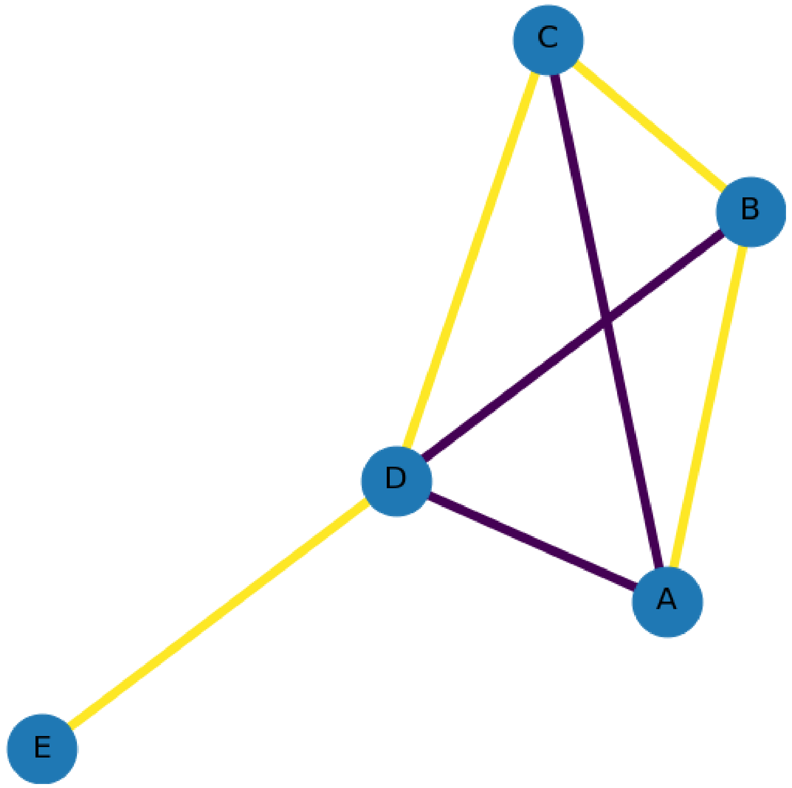
Paths

Hamiltonian Path



Paths

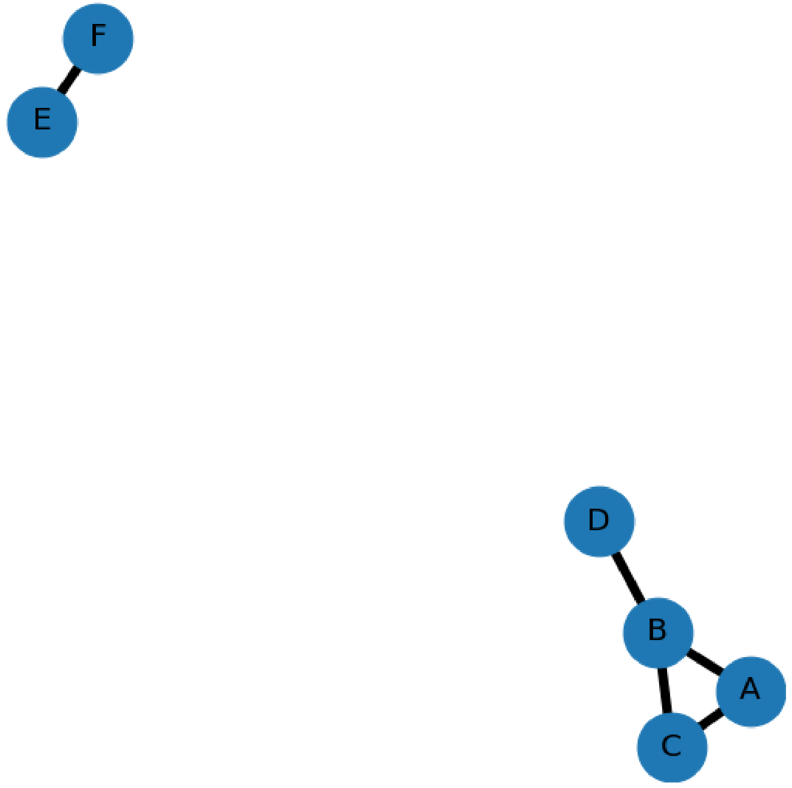
Another Hamiltonian Path



Connected graphs

- If there is a path between all pairs of vertices then a graph is said to be *connected*.
- Otherwise it is *disconnected*
- If a graph is disconnected we usually study its constiuent parts in isolation.

Disconnected graph

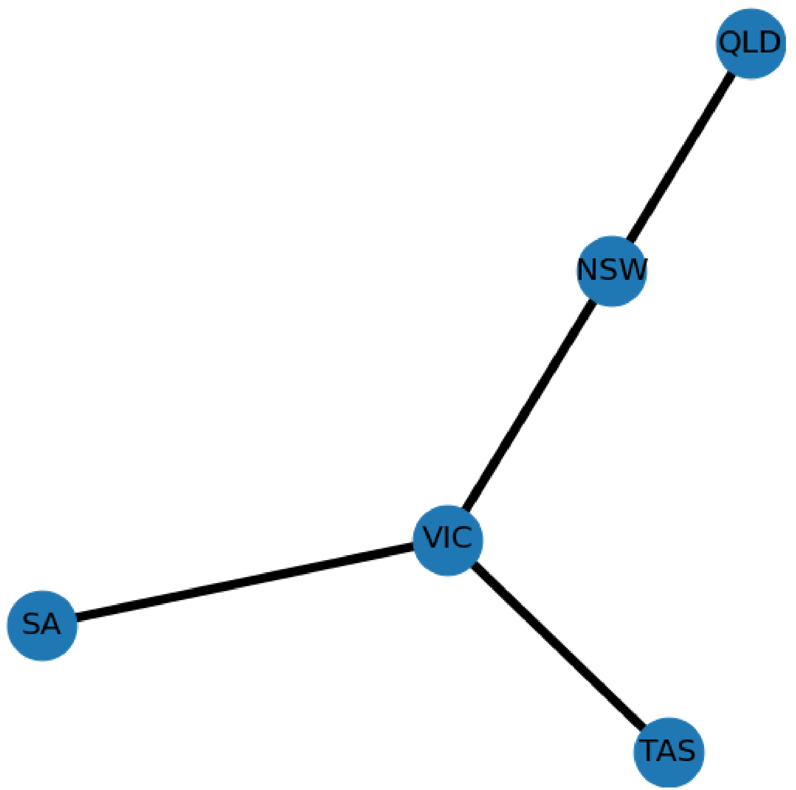


A real example

- The Australian National Electricy Market (NEM) was formed by connecting 5 state markets
- The are a limited number of interconnections between states.

As a graph

```
nem = pd.DataFrame({ 'from':['QLD', 'NSW', 'VIC','VIC'], 'to':['NSW', 'V
G=nx.from_pandas_edgelist(nem, 'from', 'to')
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = 6, p
plt.show()
```



Weighted Graph

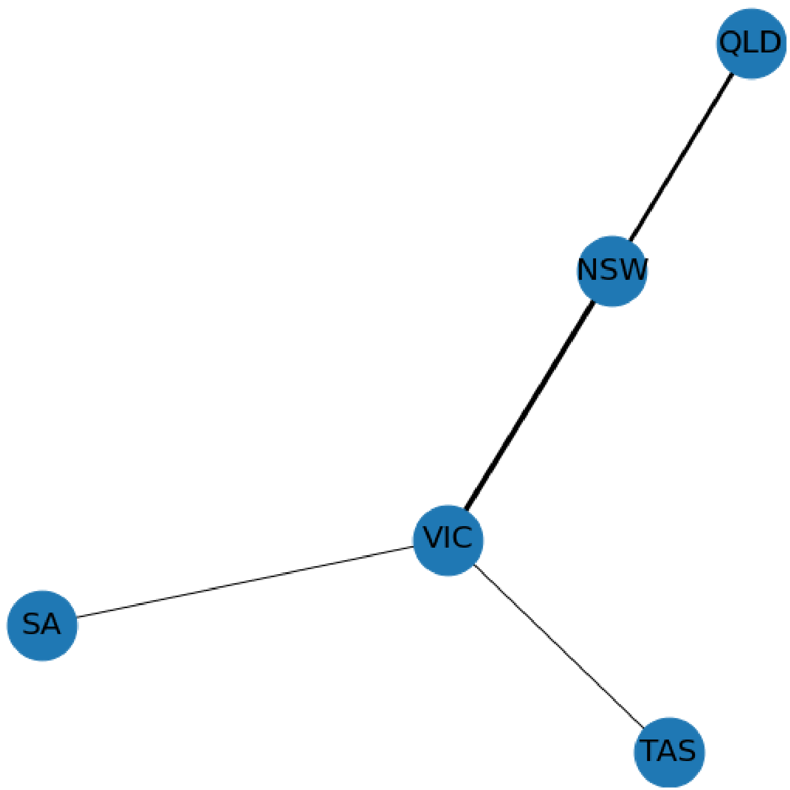
- We may assign a number to each edge known as a *weight*.
- This is called a *weighted graph*
- In a network the weight often measures some distance between locations.
- Mathematically the ones in the adjacency matrix become any positive number instead.
- Therefore weights can represent capacities or flows along a network instead.

Visualising weighted graphs

- When drawing graphs the edge weight can be shown using the thickness of the line representing each edge.
- In this way, we can also represent any data about our edges.
- In the following, we will consider the NEM example and represent the capacity of each interconnector.
- Note that we potentially lose the interpretation of distance when we assign arbitrary data to the weights.

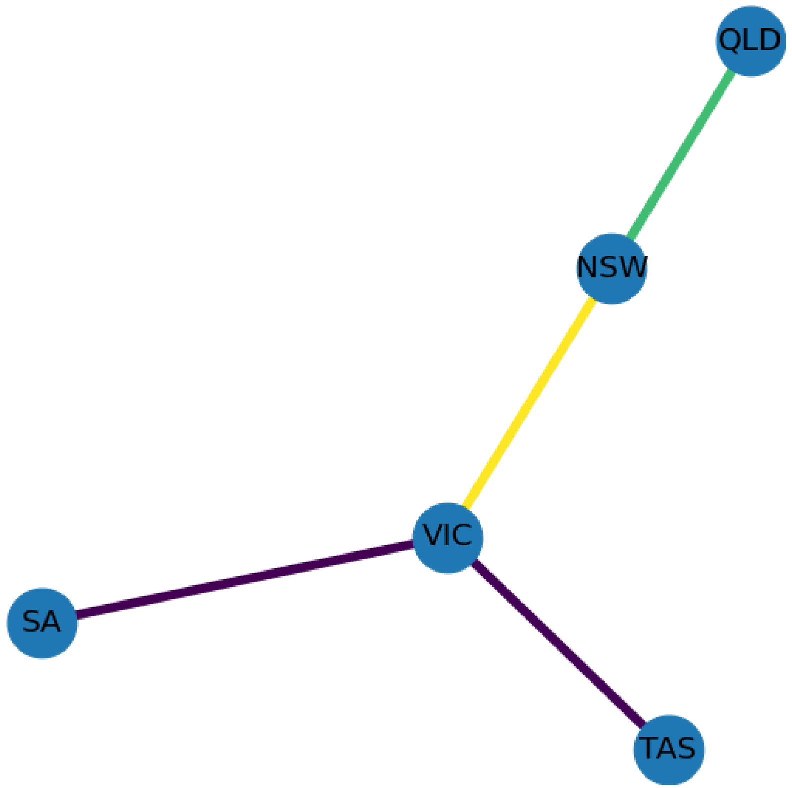
Weighted NEM graph

```
nem = pd.DataFrame({ 'from':['QLD', 'NSW', 'VIC','VIC'], 'to':['NSW', '
'capacity':[1380,1770,495, 500]})
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = nem[
plt.show()
```



Weighted NEM graph

```
nx.draw(G, with_labels=True, node_size=2000, font_size=20, width = 6, e
plt.show()
```



Directed graph

- Another generalisation of graphs allows for edges to have a direction.
- An edge from A to B is different from an edge from B to A
- This is usually visualised using an arrow on the edge

NEM data

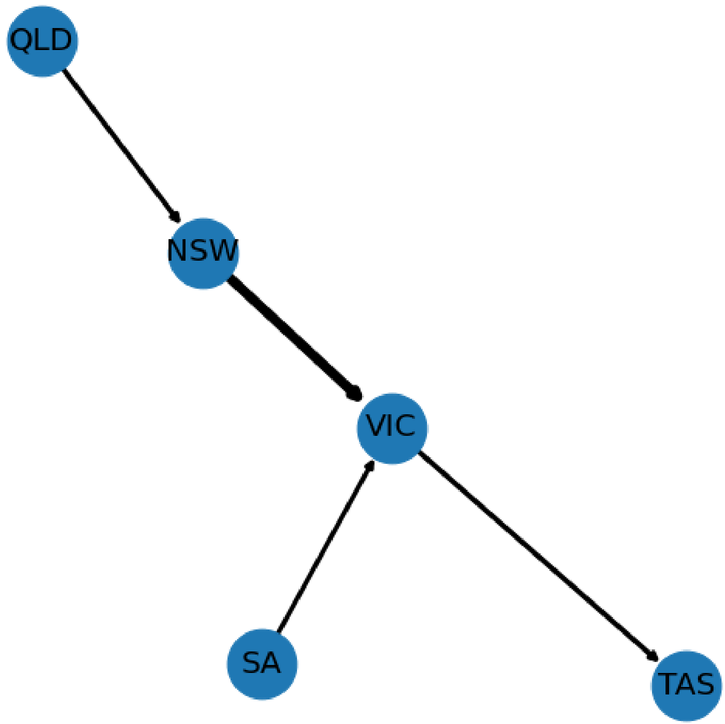
- Consider an application for the NEM data.
- Instead of capacity of the interconnectors consider the flow of electricity across different regions
- These data are based on a single snapshot on time and sourced from the NEM.

Construct graph

```
nem = pd.DataFrame({
  'from':['QLD', 'NSW', 'SA','VIC'],
  'to':['NSW', 'VIC', 'VIC','TAS'],
  'flow':[332, 644, 339, 308]})
Gd=nx.from_pandas_edgelist(
  nem,
  'from',
  'to',
  create_using=nx.DiGraph()
  )
```

Directed graph

```
nx.draw(Gd, arrows=True, with_labels=True, node_size=2000, font_size=20
plt.show()
```



Other examples

- Directed graphs can be used in social networks where one user *follows* another.
- Road systems with one way roads.
- Directed graphs also used to describe causal relationships

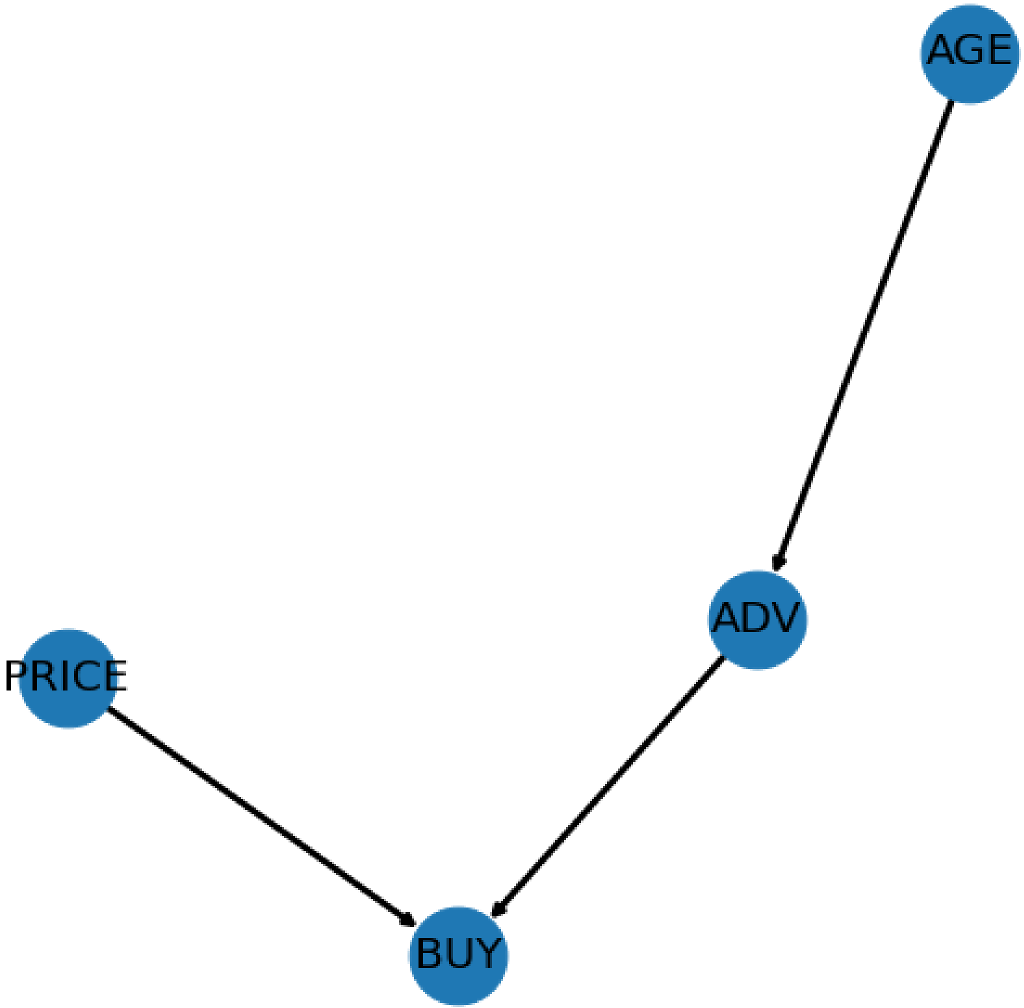
Purchase decision

- Consider the following variables
  - Age of customer
  - Advertising exposure
  - Price
  - Buy a product

Purchase decision

- Suppose age does not directly influence the decision to purchase.
- However, age may influence the exposure to the advertising.
  - Advertising may be on legacy media more likely to be seen by older individuals.
- Price directly influence decision to buy.

As a graph



Directed Graphs and models

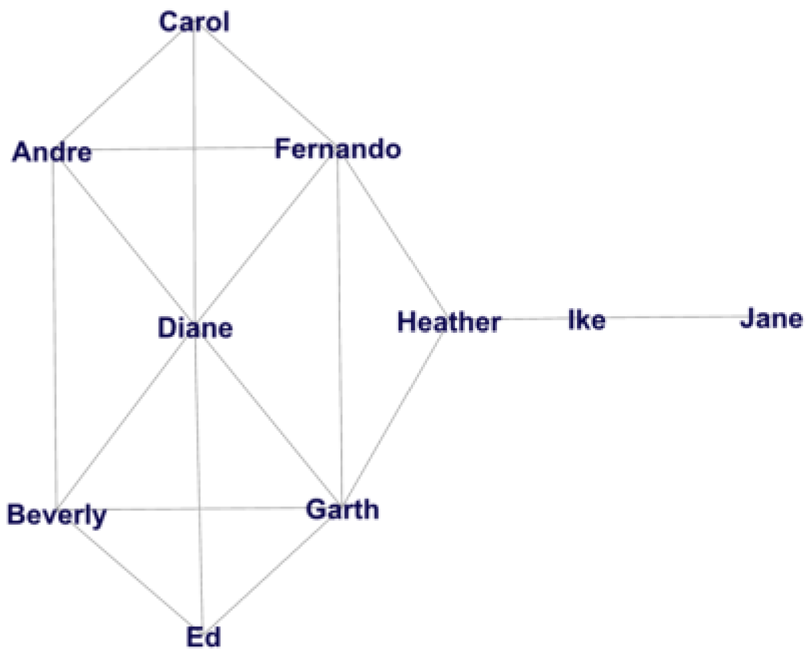
- Directed graphs can be used to build statistical models.
- These are known as *Bayesian belief nets*
- One condition is that the graphs cannot contain cycles.
- The statistical model estimates conditional probabilities of some nodes conditional on other nodes.
- For example we may be able to estimate the probability that a 25 year old customer purchases a product, given the price is reduced.

All about vertices

Characterising vertices

- The *degree* of a vertex is the number of nodes it shares an edge with.
- The *eccentricity* of a vertex is the greatest distance to any other node.
- The *diameter* is the largest *eccentricity* over the whole graph

Example



Carol
Andre
Fernando
Diane
Heather
Ike
Jane
Beverly
Garth
Ed

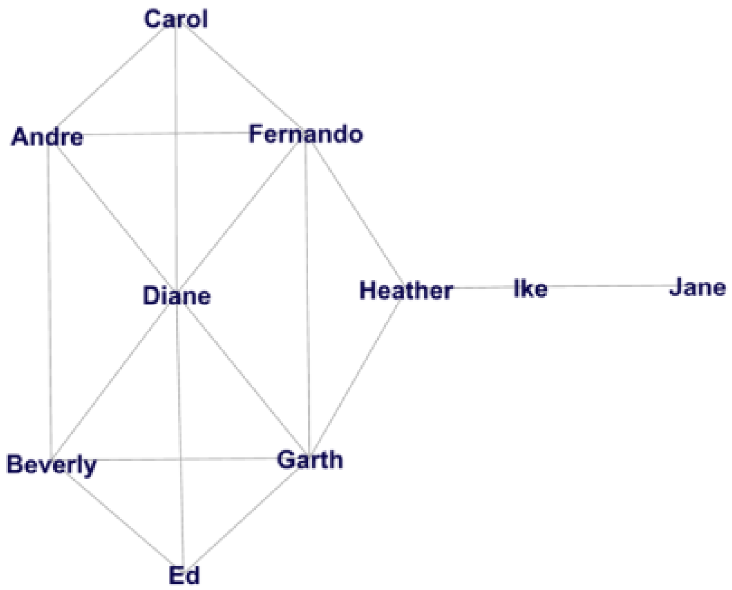
Questions

- This is known as Krackhardt's kite. Answer the following
  - What is the degree of Andre?
  - What is the degree of Ike?
  - What is the eccentricity of Fernando?
  - What is the diameter of the graph?

Graph centrality

- Graph centrality is aimed at determining the most 'important' vertex in the graph.
- In a social network this can be used to identify 'influencers'.
- There are multiple ways to understand centrality and Krackhardt's kite demonstrates the strengths and weaknesses of each method.

Degree Centrality



- One measure is to find the vertex with the highest degree.
- This is Diane with a degree of 6.

Closeness Centrality

Closeness centrality is the inverse of all shortest paths

$$c_j = \frac{n-1}{\sum_{i \neq j} d_{i,j}}$$

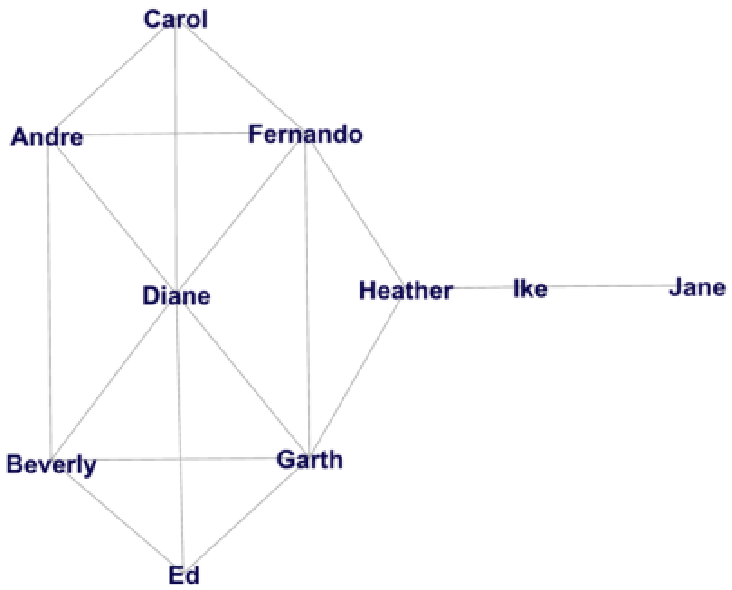
where $d_{i,j}$ is the length of the shortest path from vertex i to vertex j and n is the number of vertices.

Closeness Centrality of Ike

- Ike is a distance of
  - 1 from Ike
  - 2 from Heather
  - 3 from Fernando and Garth
  - 4 from everyone else
- Closeness centrality is

$$10/(1+2+2\times 3+5\times 4)\approx 0.349$$

Closeness centrality



- Fernando and Garth have the highest closeness centrality
- They are in the best position to monitor information flow on the network.

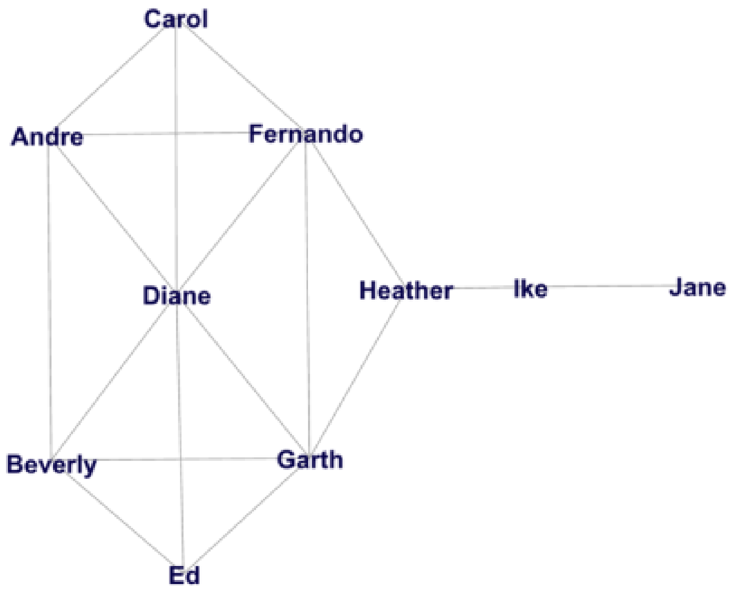
Betweeness centrality

- Betweeness centrality gives a measure of how critical a node is in connecting different parts of the graph.
- For a vertex j
  - Take a pair of nodes $i, l \neq j$
  - Compute all *shortest paths* between these nodes
  - Find the proportion of these shortest paths that include node j
  - Add up over all possible pairs i, l

An example

- Compute betweeness centrality for Fernando.
  - Shortest paths between Andre and Ed. None of these include Fernando (add 0/2)
  - Shortest paths between Carol and Heather. Only one that goes through Fernando (add 1/1)
  - Shortest paths from Diane to Ike. There are two and one goes through Fernando (add 1/2)
- You would need to do these for all pairs and add, i.e. $(0/2) + (1/1) + (1/2)$

Betweeness centrality



- Heather has the highest betweeness centrality
- She is in a position to control information along the network.

Eigenvalue Centrality

- We will conclude with the notion of *Eigenvalue Centrality*. The following must hold for all j.

$$c_j = \frac{1}{\lambda} \sum_{i \neq j} a_{ij} c_i$$

I am important if my friends are important and my friends are important if I am important

Solution

Must find some value $\boldsymbol{c}$ such that

$$\boldsymbol{A}\boldsymbol{c} = \lambda \boldsymbol{c}$$

This is a famous problem in math known as the eigenvalue problem.

There will in general be multiple solutions. However only one solution satisfies the condition that \lamba and all centrality measures are positive.

Eigenvalue centrality

- Perhaps the most famous application of this is the PageRank algorithm.
- This was the original algorithm used on google search.
- The idea was to use hyperlinks between websites to construct a network.
- PageRank was used to order the importance of websites.
- If you had learnt about this 25 years ago you could be billionaires (sorry!)

Bipartite Graph

Bipartite Graph

- A *bipartite graph* is a special graph where the vertices can be separated into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$.
  - An example may be that one set is products and the other customers.
- All edges join one vertex from $\mathcal{V}_1$ to one vertex in $\mathcal{V}_2$.

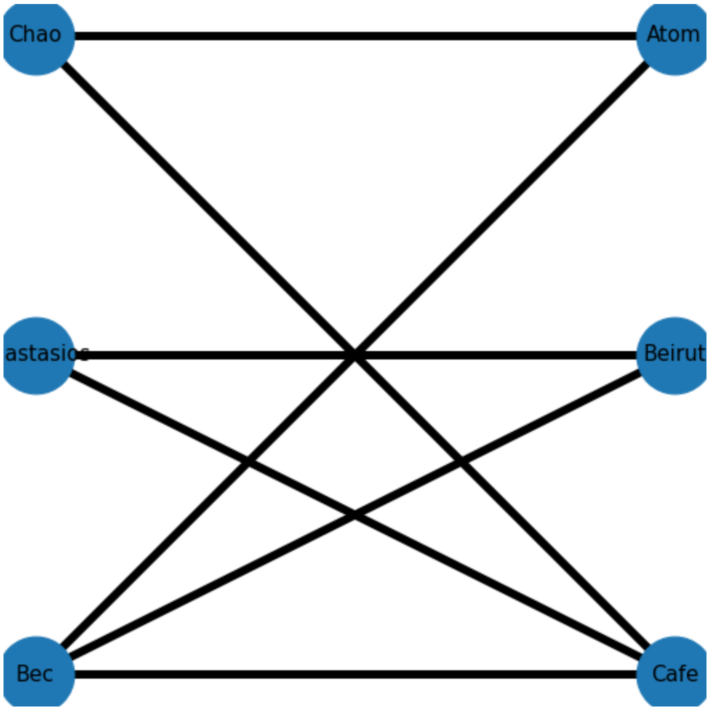
Example

- Consider three local restaurants
  - *Atom* Thai
  - *Beirut* Felafel
  - *Cafe* Abercrombie
- Three customers
  - Anastasios
  - Bec
  - Chao

```
df = pd.DataFrame({  'customer':['Bec', 'Chao', 'Anastasios', 'Bec','An
G=nx.from_pandas_edgelist(df, 'customer', 'restaurant')
```

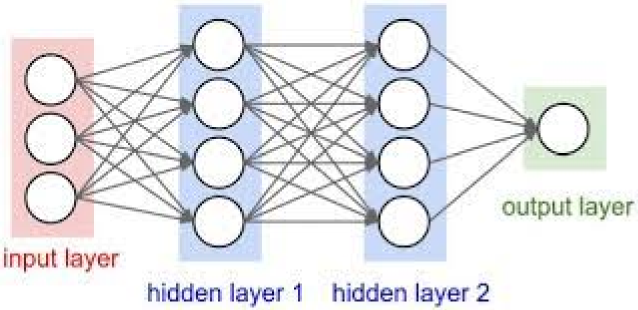
Bipartite graph

```
nx.draw(G, with_labels=True, node_size=3000, font_size=15, width = 6, p
plt.show()
```



Multipartite graphs

- The idea can be extended to multipartite graphs
- These are used in feedforward deep neural networks



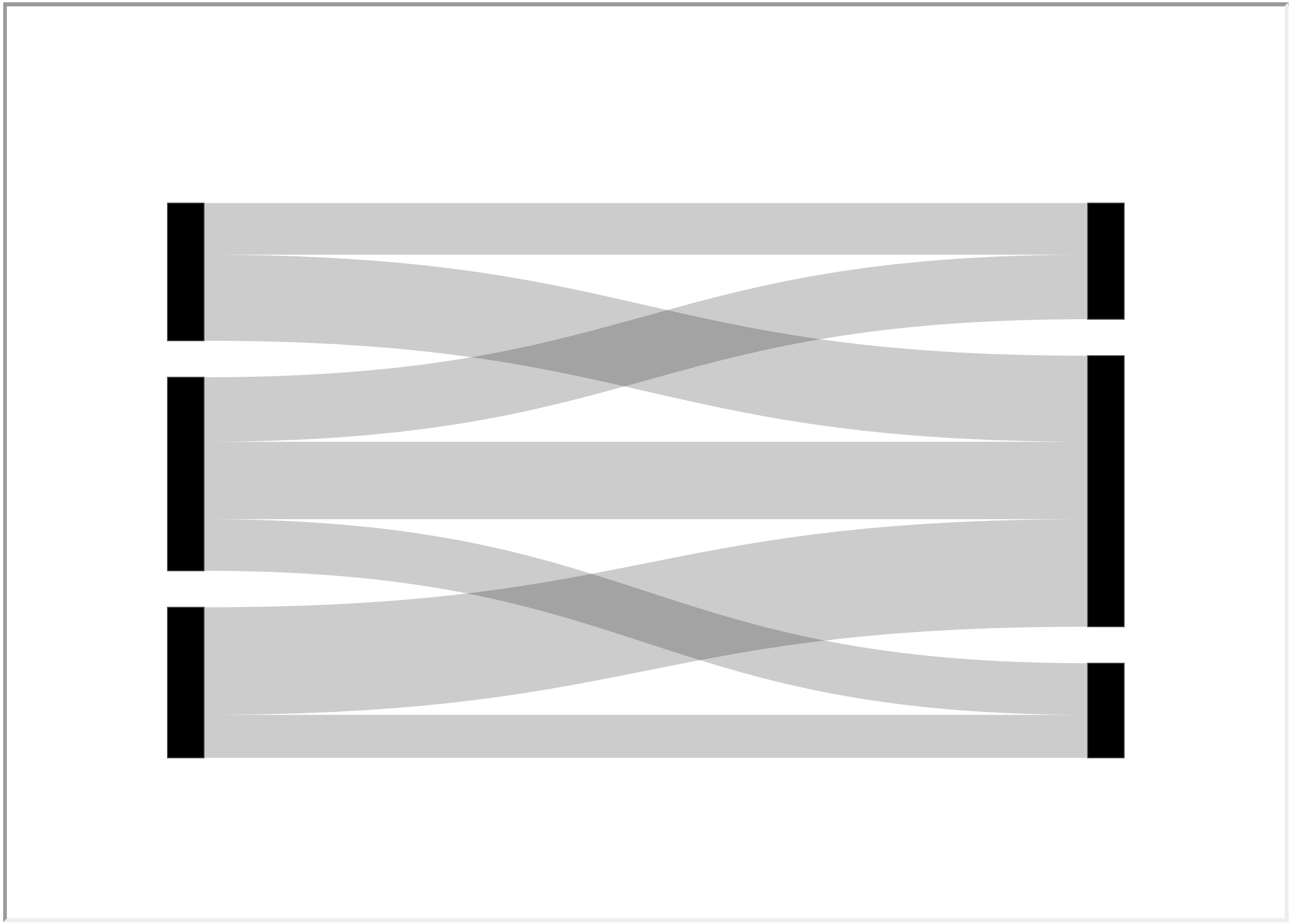
Sankey diagram

- Multipartite graphs can have weighted edges.
- Where these represent flows, a useful visualisation is the Sankey diagram.
- We will constuct a basic example using `plotly`
- A more detailed example can be found here

Setup

```python
import plotly.graph_objects as go
source = [0,0,1,1,1,2,2]
target = [4,5,3,4,5,3,5]
value = [10,25,15,12,18,12,20]
link = dict(source = source, target = target, value = value)
data = go.Sankey(link = link)
fig = go.Figure(data)
fig.write_html('sankey.html')
```

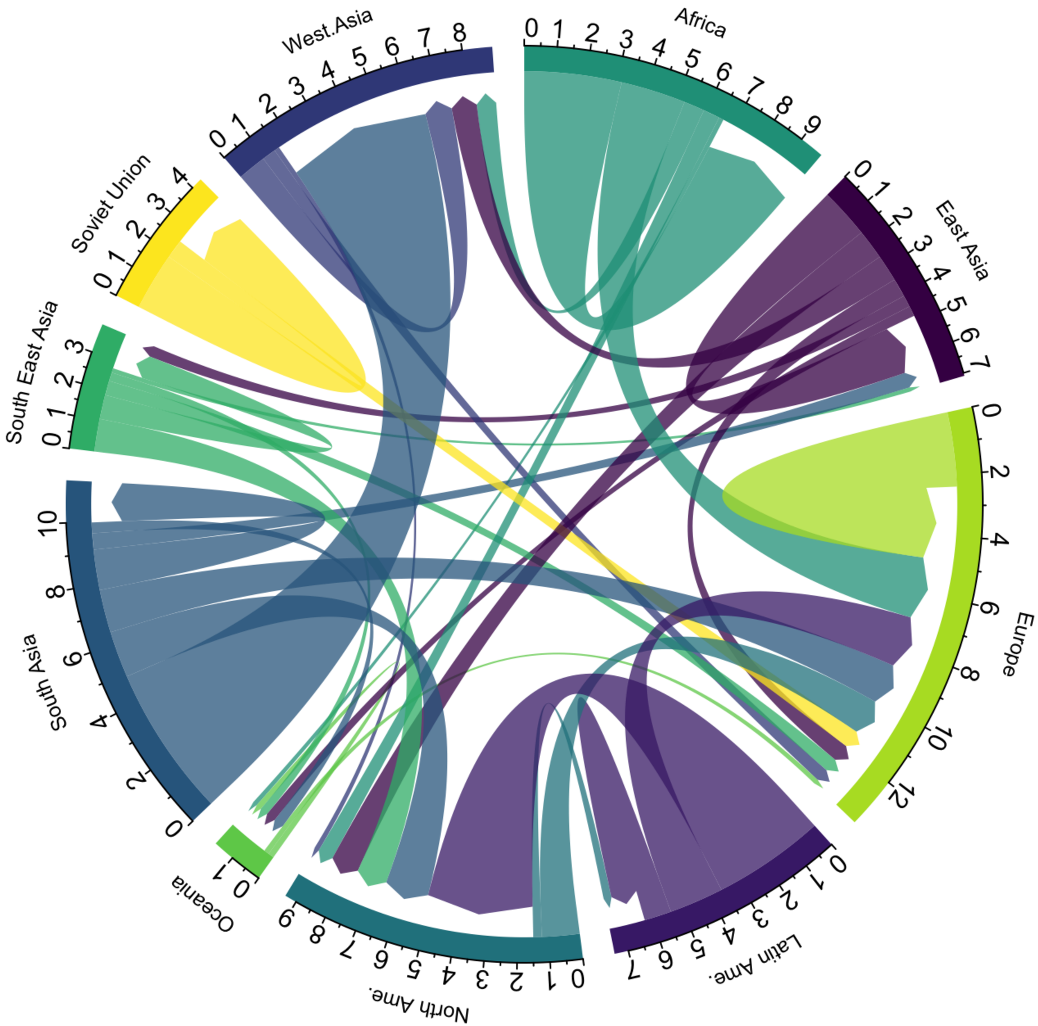
Sankey diagram



Chord diagram

- A similar idea to a Sankey diagram but for a complete graph is a chord chart.
- The nodes are arranged in a circle.
- Different colours often used for each node.
- Works for directed and undirected graphs.
- Example on next slide uses a migration dataset.

Chord diagram



Wrap-up

Conclusions

- In graph theory
  - We can characterise paths between vertices
  - Add more information to edges by using weighted and directed graphs.
  - Find most critical vertices
  - Undertand different types of graphs
- Throughout we saw examples of visualisation.

Questions