Partial Differential Equations (PDE's)

Biomedical Engineering 5CCYB070

COMPUTATIONAL METHODS

Lecture 14 Vector Calculus and PDEs II

01 Tools to solve PDEs

02 Methods to solve PDEs

Learning objectives

Tools to solve PDEs

Basic integral calculus

Fundamental conservation law

Methods to solve PDEs

Review the ABC in ODEs

Separation of variables

Integral transforms: Fourier and Laplace

Learning objectives

- Toolkit to solve PDEs
  - Identify the **basic integral calculus** necessary for PDEs; e.g. surface integrals or the divergence theorem
  - Become familiar with a **fundamental conservation law** and use it to derive PDEs in specific physical contexts (e.g. diffusion)
- Methods to solve PDEs
  - Review the ABC in ODEs
  - Separation of variables
  - Integral transforms: Fourier and Laplace

Fundamental Theorem of Calculus

- The FTC relates a function and its derivative

$$f(x) = \frac{dF(x)}{dx} \quad \Leftrightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a)$$

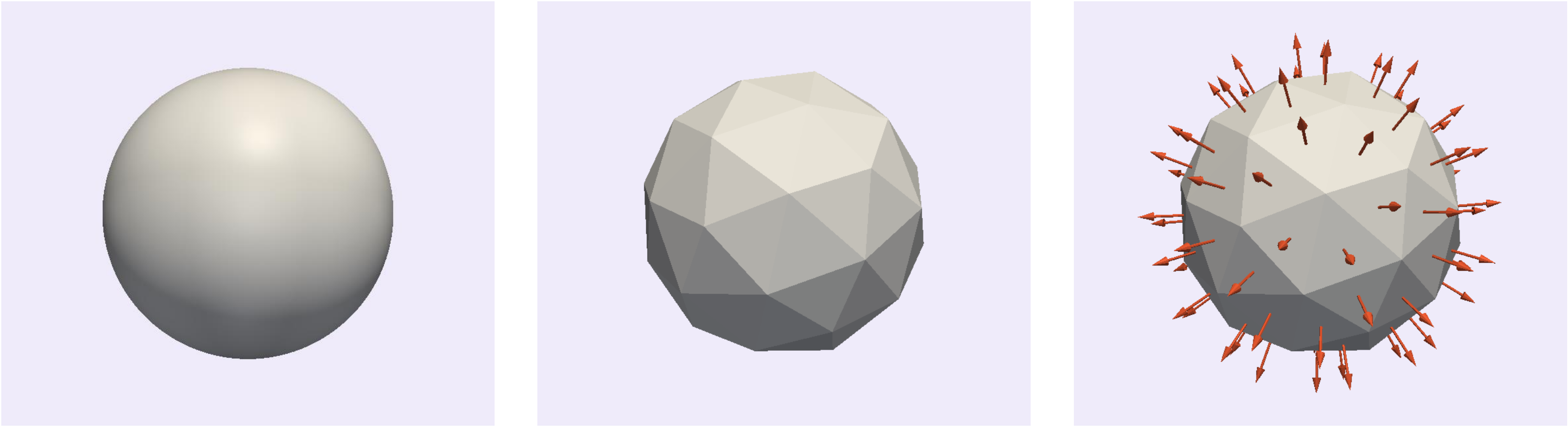
- Still works if functions have more than one variable, i.e. if we use partial derivatives, for example:

$$g(x,y,z) = \frac{\partial G(x,y,z)}{\partial y} \quad \Leftrightarrow \int_{y=\alpha}^{y=\beta} g(x,y,z)\,dy = G(x,\beta,z) - G(x,\alpha,z)$$

Note the 'free' variable(s)

Surface integrals

- To compute this, we need to model a surface
- Need an ‘ideal’ surface but this is not practical
- Approximate an ideal surface using small *area elements*
- Each element has unit normal vector



Surface integral of scalar field 

- A surface S : set of surface elements: $\delta S_1, \delta S_2, \ldots, \delta S_N$
- Scalar field $\varphi(x,y,z)$, value at each element: $\varphi_1, \varphi_2, \ldots, \varphi_N$
- Can approximate the integral of φ over S

$$\int_S \phi \, dS \approx \sum_{i=1}^{N} \phi_i \ \mathrm{area}(\delta S_i)$$

$$\int_S \phi \, dS \approx \sum_{i=1}^{N} \phi_i \, \delta S_i$$

- Simplify notation: assume δS_i represents the **area** of the element

Gives a scalar result

Surface integral of scalar field

- The parts

$$\int_S \phi \, dS$$

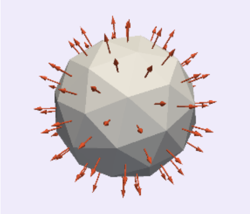
Specific surface region over which to integrate, e.g. the surface of a sphere

Scalar field to integrate, depends on *x,y,z*

'Infinitesimal' surface area element

- Not actually a single variable integration (even if it looks like one)
- The surface can be parametrised by **two** variables
- or modelled by a discrete number of polygons such as triangles

Surface integral with normals



- Can take surface normals into account, E.g.

$$\sum_{i=1}^{N} \phi_i \, \hat{n}_i \, \delta S_i \qquad \text{instead of} \qquad \sum_{i=1}^{N} \phi_i \, \delta S_i$$

Vector result — Scalar result

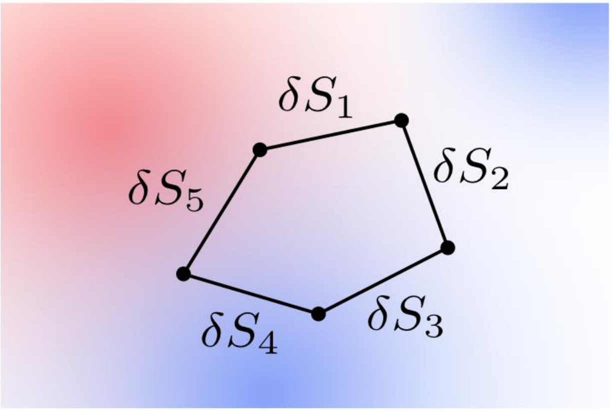
- Textbooks sometimes write $\hat{n}_i \delta S_i$ as $\delta \vec{S}_i$

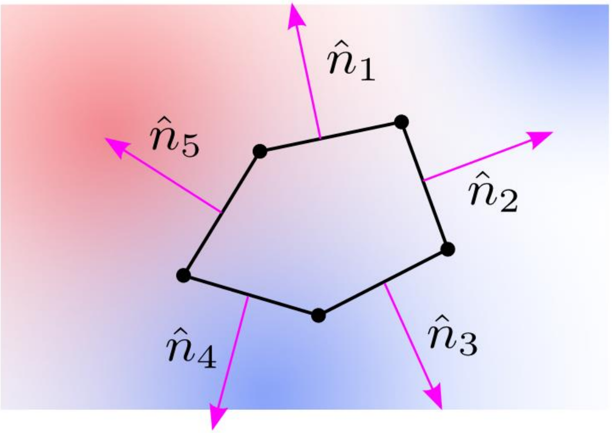
- so can re-write $$\sum_{i=1}^{N} \phi_i \, \hat{n}_i \, \delta S_i = \sum_{i=1}^{N} \phi_i \, \delta \vec{S}_i$$

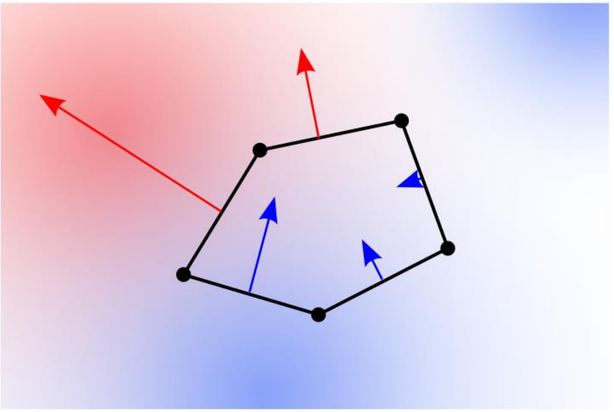
- Approximation for $$\int_S \phi \, \hat{n} \, dS \quad \text{or} \quad \int_S \phi \, d\vec{S}$$

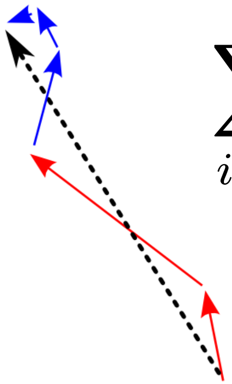
Surface integral of scalar field with normals

- 2-D schematic illustration, 'surface' represented by polygon.









$$\sum_{i=1}^{N} \phi_i \, \hat{n}_i \, \delta S_i = \sum_{i=1}^{N} \phi_i \, \delta \vec{S}_i$$

Gives a vector result

Scalar field represented by colour cloud (red=positive), 'surface elements' are all same size

Surface integral of a vector field

- Can integrate vector field (VF) over a surface in different ways
- **One way:** use the dot product of VF with the surface normals

- A surface S : set of surface elements $\delta S_1, \delta S_2, \ldots, \delta S_N$
- Normals $\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_N$
- A vector field $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$

$$\sum_{i=1}^{N} \vec{v}_i \cdot \hat{n}_i \, \delta S_i = \sum_{i=1}^{N} \vec{v}_i \cdot \delta \vec{S}_i$$

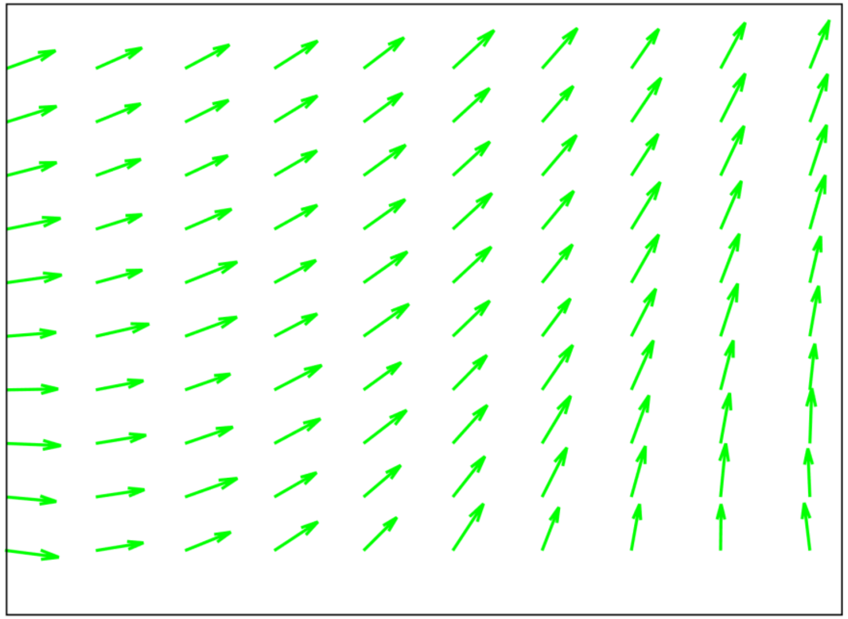
Gives a SCALAR result

Approximate the dot product integral of vector field over S

- Either of the following are equivalent

Surface integral of a vector field

- 2-D schematic illustration



Equivalent forms for the continuous integral

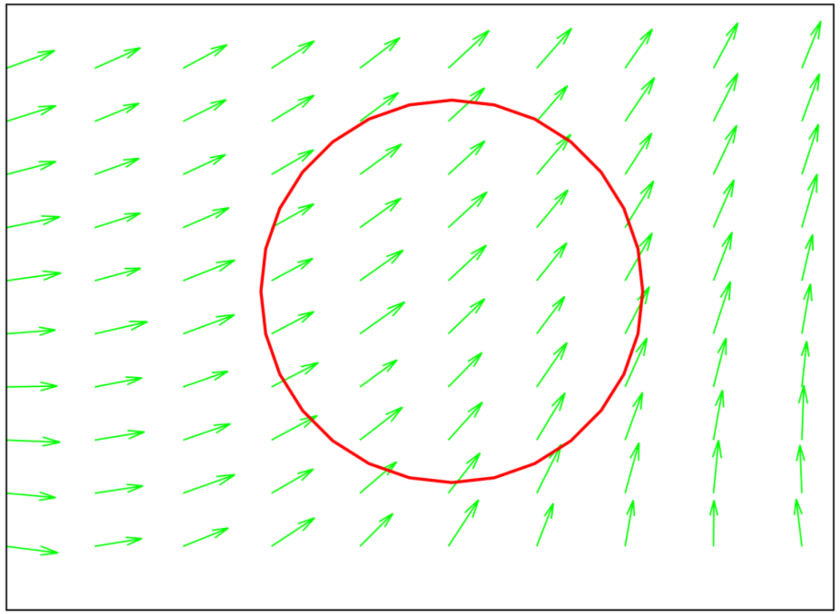
$$\int_S \vec{v} \cdot d\vec{S}$$

$$\int_S \vec{v} \cdot \hat{n}\, dS$$

$$\sum_{i=1}^{N} \vec{v}_i \cdot \hat{n}_i\, \delta S_i = \sum_{i=1}^{N} \vec{v}_i \cdot \delta \vec{S}_i$$

Surface integral of a vector field

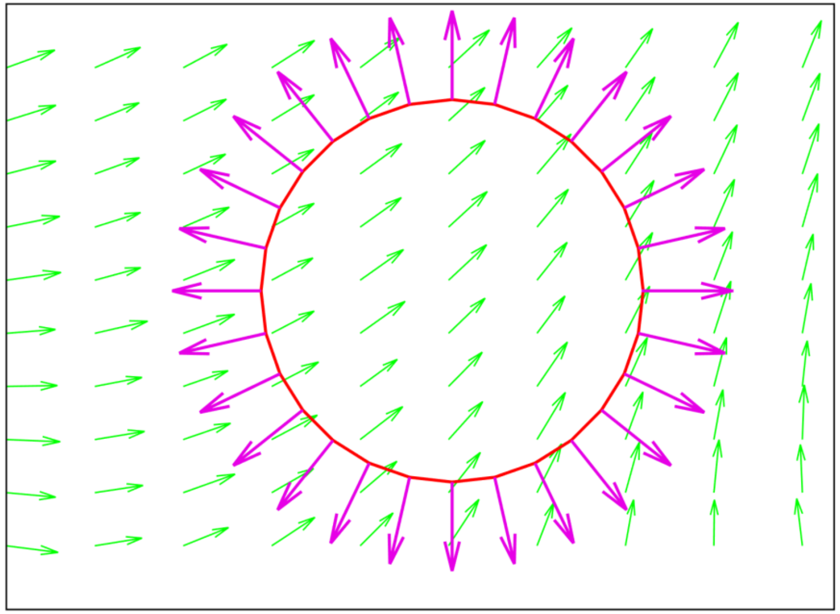
- 2-D schematic illustration



$$\sum_{i=1}^{N} \vec{v}_i \cdot \hat{n}_i \, \delta S_i = \sum_{i=1}^{N} \vec{v}_i \cdot \delta \vec{S}_i$$

Surface integral of a vector field

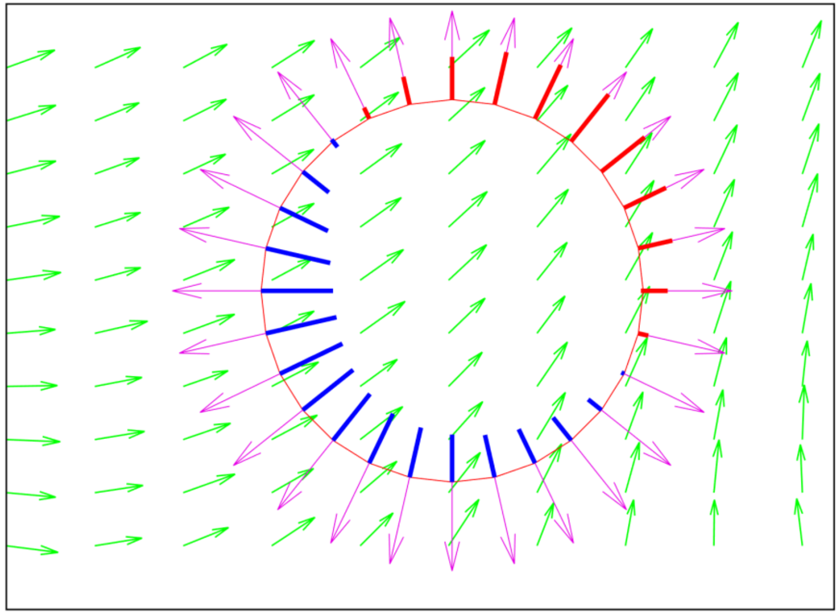
- 2-D schematic illustration



$$\sum_{i=1}^{N} \vec{v}_i \cdot \hat{n}_i \, \delta S_i = \sum_{i=1}^{N} \vec{v}_i \cdot \delta \vec{S}_i$$

Surface integral of a vector field

- 2-D schematic illustration



Equivalent forms for the continuous integral

$$\int_S \vec{v} \cdot d\vec{S}$$

$$\int_S \vec{v} \cdot \hat{n}\, dS$$

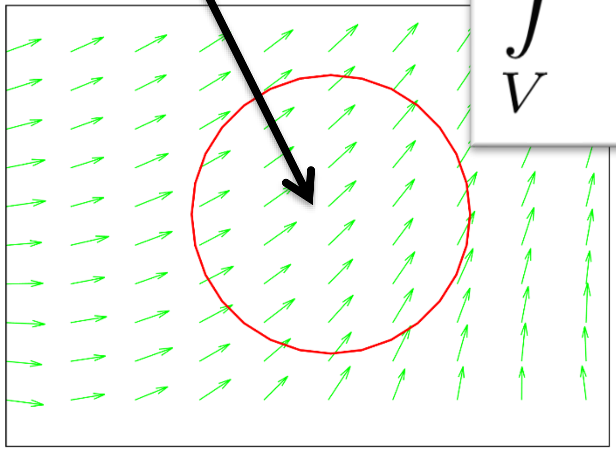
$$\sum_{i=1}^{N} \vec{v}_i \cdot \hat{n}_i\, \delta S_i = \sum_{i=1}^{N} \vec{v}_i \cdot \delta \vec{S}_i$$

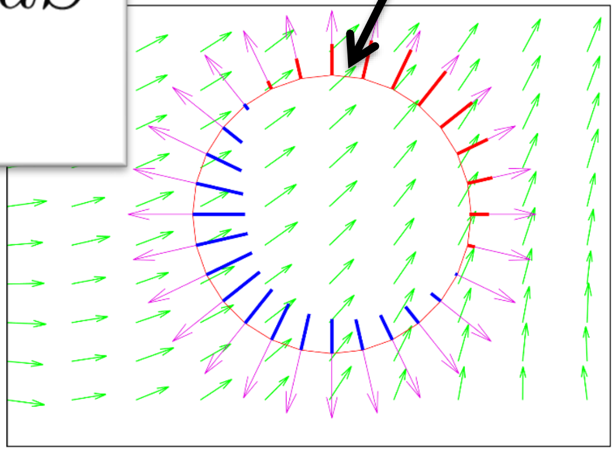
Divergence theorem

- Total Divergence of a vector field v **inside a region**
- Integral of v **over boundary** (with the dot product)

Integral of divergence inside the region **=** *Integral of the scalar components over the boundary surface*

$$\int_V \nabla \cdot \vec{a}\, dV = \oint_{\partial V} \vec{a} \cdot d\vec{S}$$





Divergence theorem

- The parts

Integral over a **closed** surface

Volume integral = Surface Integral

Infinitesimal surface element area multiplied by its normal

$$\int_V \nabla \cdot \vec{a}\, dV = \oint_{\partial V} \vec{a} \cdot d\vec{S}$$

Region for the volume integral

Divergence of vector field

Dot product of vector field with surface element normal

Infinitesimal volume element

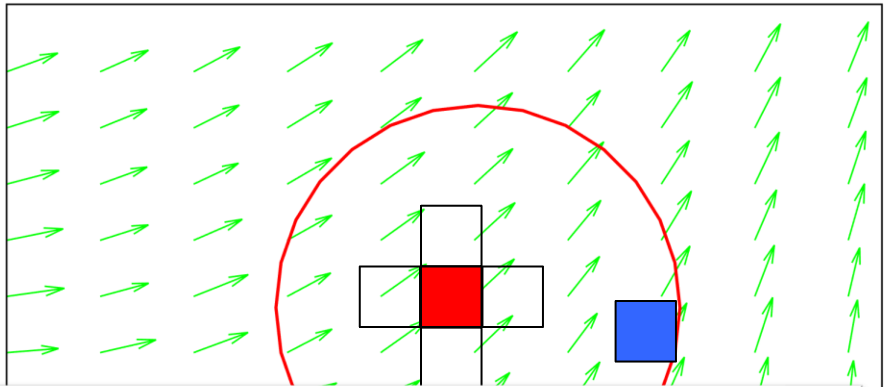
Bounding surface of the volume region

Divergence theorem

Left hand integral as a sum over small volume elements

For an internal volume element, net flow out equals flow *in* for neighbours, with **opposite** signs, so cancel in the summation

$$\int_V \nabla \cdot \vec{a}\, dV \approx \sum_i \nabla \cdot \vec{a}_i\, \delta V_i$$



Element at the boundary: Flow across the boundary does not get cancelled.
Represented by

$$\vec{a}_i \cdot \hat{n}_i \delta S_i = \vec{a}_i \cdot \delta \vec{S}_i$$

So we get

$$\sum_i \nabla \cdot \vec{a}_i\, \delta V_i \approx \sum_j \vec{a}_j \cdot \delta \vec{S}_j \Rightarrow \int_V \nabla \cdot \vec{a}\, dV = \oint_{\partial V} \vec{a} \cdot d\bar{S}$$

i loops over volume elements
j loops over surface elements

Leibniz's Rule: '*differentiation under the integral sign*'

- When is it possible to 'carry' differentiation from outside an integral to inside it?
- If we have a function of ***two*** variables and integrated w.r.t. ***one*** of them, e.g.

$$\int_a^b f(x,t)\,dt$$

- And we take the derivative w.r.t the ***other*** variable

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt$$

- … the derivative can be 'carried' in ***if the limits are constant***

Leibniz's Rule: '*differentiation under the integral sign*'

- This works for either variable, e.g. if the integral is over *x*

$$\int_c^d f(x,t)\,dx$$

- if *c* and *d* are constant, then

$$\frac{d}{dt}\int_c^d f(x,t)\,dx = \int_c^d \frac{\partial f(x,t)}{\partial t}\,dx$$

- Important thing is that the derivative and integral are over different variables

Leibniz's Rule

- More variables, surface integral example, differentiated w.r.t. *t*

- Rule allows us to carry in the *t*-derivative if surface *S* is fixed.

$$\frac{d}{dt}\int_S g(x,y,z,t)\,dS = \int_S \frac{\partial g(x,y,z,t)}{\partial t}\,dS$$

Learning objectives

- Toolkit to solve PDEs
  - Identify the **basic integral calculus** necessary for PDEs; e.g. surface integrals or the divergence theorem
  - Become familiar with a **fundamental conservation law** and use it to derive PDEs in specific physical contexts (e.g. diffusion)
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Conservation law

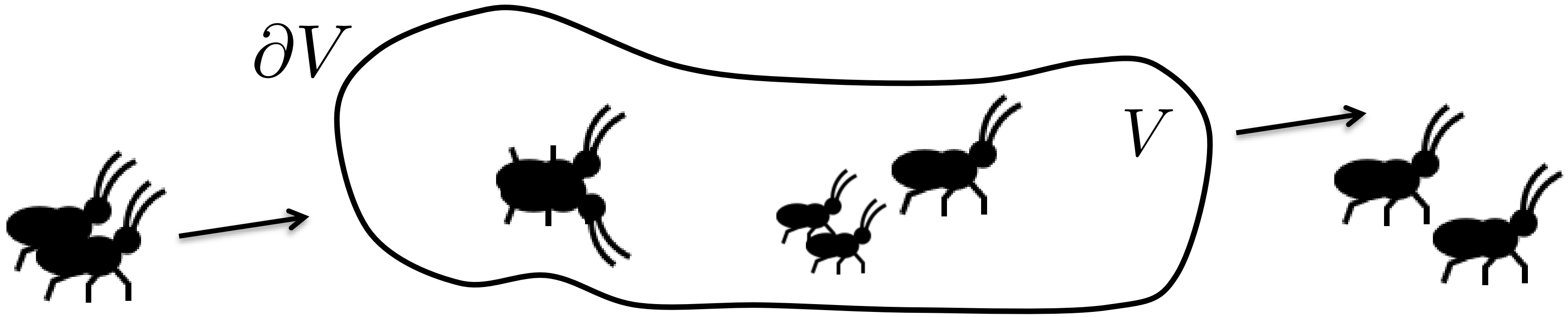
How (energy, chamical density…) is *conserved* defines a PDE

- E.g. rate of change of population in a region V depends on:
  - Flow rate into the region
  - Flow rate out of the region
  - Birth rate
  - Death rate

$\Big\}$ *Net flow rate*: **can be +ve or -ve** (flow rate into and out of the region)

$\Big\}$ *Net creation rate*: **can be +ve or -ve** (birth rate and death rate)

- Provides a PDE that the measured quantity should satisfy.

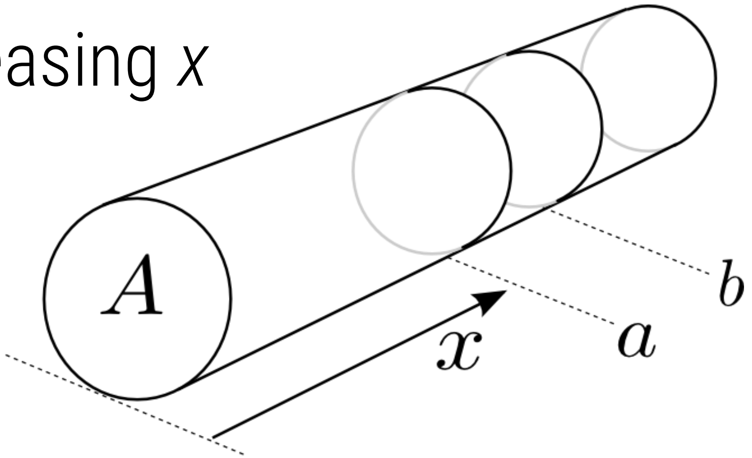


Conservation Law: A 1-D case

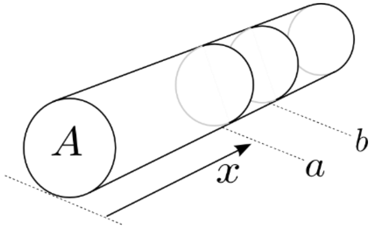
- Measure 'stuff' in a long thin bar
- $u(x,t)$ indicates **density** at location x and time t
- $f(x,t)$ indicates **net creation rate** at location x and time t
  - f is known as the *source* term
- Focus on interval between $x=a$ and $x=b$
- '*Flux*': **net flow rate** across a specific point (signed)

  +ve = to the 'right' , in the direction of increasing x

  Denoted by $\Phi(x,t)$
- A is the cross-sectional area



Conservation law: 1-D



Total amount of stuff in the interval $\int_a^b A\,u\,dx$

- Focus on rates of change:

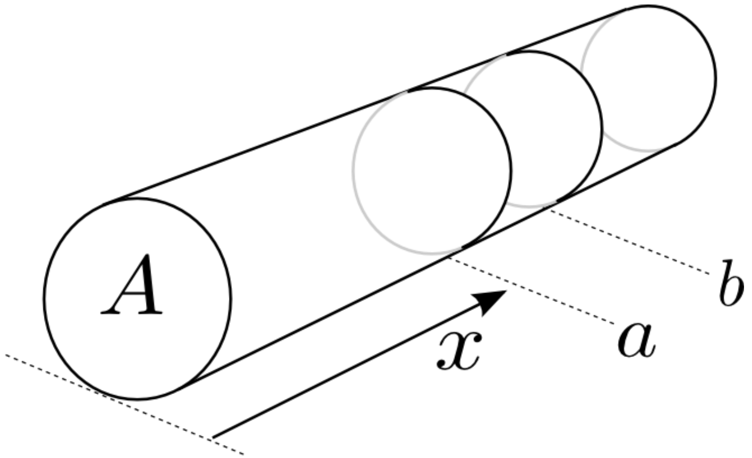
Rate of change of total amount	$\frac{\partial}{\partial t}\int_a^b A\,u\,dx$
	=
Rate of change due to net flow	$A\Phi(a,t) - A\Phi(b,t)$
	+
Rate of change due to creation	$\int_a^b A\,f(x,t)\,dx$

Conservation law: 1-D

u(*x*,*t*) density   *f* (*x*,*t*) creation rate   Φ (*x*,*t*) flux   *A* cross-sectional area

$$\frac{\partial}{\partial t}\int_a^b \cancel{A}u(x,t)\,dx = \cancel{A}\Phi(a,t) - \cancel{A}\Phi(b,t) + \int_a^b \cancel{A}f(x,t)\,dx$$

$$\int_a^b u_t(x,t)\,dx = \Phi(a,t) - \Phi(b,t) + \int_a^b f(x,t)\,dx$$

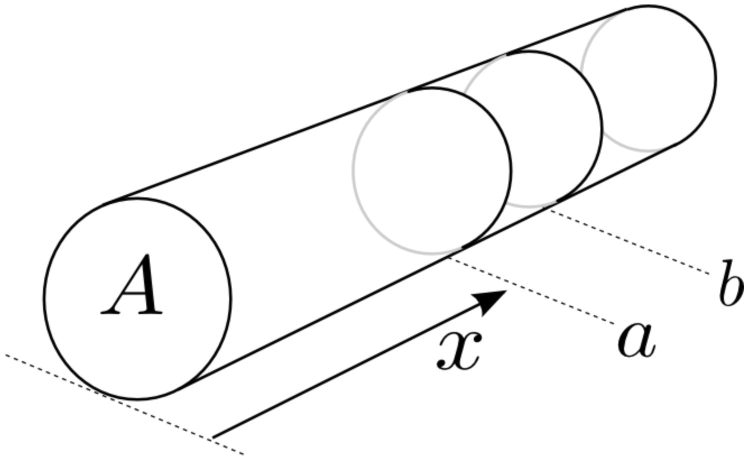


Conservation law: 1-D

u(*x*,*t*) density *f* (*x*,*t*) creation rate Φ (*x*,*t*) flux *A* cross-sectional area

$$\int_a^b u_t(x,t)\,dx = -\int_a^b \Phi_x(x,t)\,dx + \int_a^b f(x,t)\,dx$$

Get three integrals with same limits



Conservation law: 1-D

$$\int_a^b u_t(x,t)\,dx = -\int_a^b \Phi_x(x,t)\,dx + \int_a^b f(x,t)\,dx$$

- Because this is true for ***any*** interval [*a*,*b*], we get

$$u_t(x,t) = -\Phi_x(x,t) + f(x,t)$$

$$\boxed{u_t(x,t) + \Phi_x(x,t) = f(x,t)}$$

Fundamental Conservation Law

- We can decide which *f* to use and how Φ relates to *u*
  - Depends on the system we want to model.
- Allows us to write Φ in terms of *u* ⇒ Obtain a PDE in *u*

Advection model

- Start with the FCL $$\boxed{u_t(x,t) + \Phi_x(x,t) = f(x,t)}$$
- **Make Assumption**: Flux is proportional to density

$$\Phi = cu \qquad \Phi_x(x,t) = \frac{\partial \Phi}{\partial x} = c\,\frac{\partial u}{\partial x}$$

- **Make Assumption:** No material generated, source term *f* is zero
- Substitute into FCL: Provides the Advection Equation

$$\boxed{u_t + c\,u_x = 0}$$

- Example: *u* = density of concentration of chemical at a point in a flowing river

Diffusion model

- Start with the FCL

$$\boxed{u_t(x,t) + \Phi_x(x,t) = f(x,t)}$$

- **Make Assumption**: Flux is proportional to *gradient* of density (In *opposite* direction)

$$\Phi = -\alpha \frac{\partial u}{\partial x} \qquad \Phi_x(x,t) = \frac{\partial \Phi}{\partial x} = -\alpha \frac{\partial^2 u}{\partial x^2}$$

- **Make Assumption:** No material generated, source term *f* is zero

- Substitute into FCL: Provides the diffusion equation

$$\boxed{u_t - \alpha\, u_{xx} = 0}$$

- Example: *u* = Temperature of a point along a metal bar

Advection-diffusion model

- Start with the FCL $$u_t(x,t) + \Phi_x(x,t) = f(x,t)$$
- **Make Assumption**: Flux depends on *both* density and its gradient

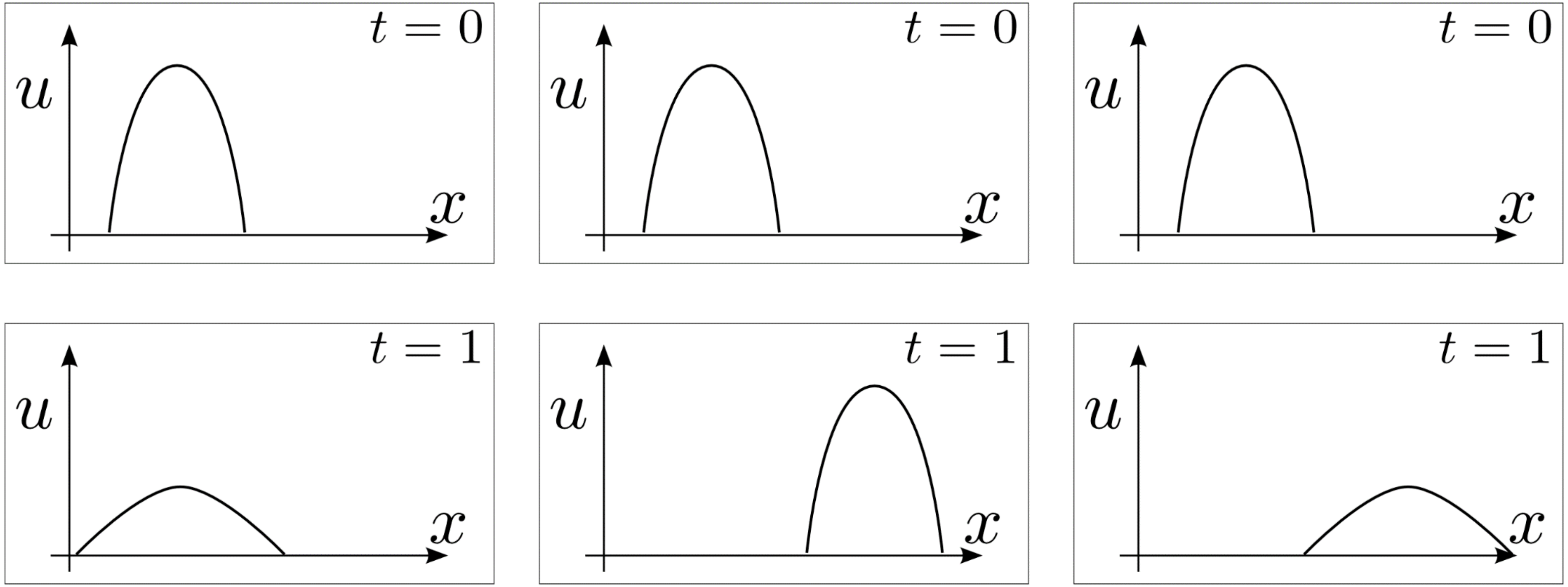
$$\Phi = cu - \alpha \frac{\partial u}{\partial x} \qquad \Phi_x(x,t) = \frac{\partial \Phi}{\partial x} = cu_x - \alpha u_{xx}$$

- **Make Assumption:** No material generated, source term *f* is zero
- Substitute into FCL: Provides the Advection-Diffusion Equation

$$\boxed{u_t + cu_x - \alpha u_{xx} = 0}$$

Diffusion, Advection and Advection-Diffusion

Diffusion | Advection | Advection-Diffusion



$$u_t - \alpha u_{xx} = 0 \qquad u_t + c u_x = 0 \qquad u_t + cu_x - \alpha u_{xx} = 0$$

How the different systems vary with time.

Learning objectives

- Toolkit to solve PDEs
  - Identify the **basic integral calculus** necessary for PDEs; e.g. surface integrals or the divergence theorem
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- Methods to solve PDEs
  - Review the ABC in ODEs
  - Separation of variables
  - Integral transforms: Fourier and Laplace

Useful results from ODEs

- Following can be useful when solving PDEs
- Especially if we can convert a PDE into one or more ODEs

$$\frac{du}{dx} = \lambda\, u$$

$$u(x) = e^{\lambda\, x}$$

$$\frac{d^2u}{dx^2} = \lambda^2\, u$$

$$u(x) = Ae^{\lambda\, x} + Be^{-\lambda\, x}$$

$$\frac{d^2u}{dx^2} = -\lambda^2\, u$$

$$u(x) = C\sin\lambda x + D\cos\lambda x$$

$$u(x) = Ae^{\lambda\, i\, x} + Be^{-\lambda\, i\, x}$$

$$\sin a = \frac{1}{2\,i}\left(e^{ia} - e^{-ia}\right)$$

$$\cos a = \frac{1}{2}\left(e^{ia} + e^{-ia}\right)$$

Analysis of solutions: Advection

- Example: Advection Eqn in 1-D $u_t + cu_x = 0$

- Consider the simple function $p(x, t) = x - ct$

$$\frac{\partial p}{\partial t} = -c \qquad \frac{\partial p}{\partial x} = 1 \qquad p_t + cp_x = -c + c \times 1 = 0$$

- So $p(x,t) = x - ct$ is a possible solution

- Now consider a **function of** p (indirectly a function of x and y)

$$F(x, y) = F(p) = F(x - ct)$$

Analysis of solutions: Advection

- Example: Advection Eqn in 1-D $u_t + cu_x = 0$

  $p(x,t) = x - ct$ $F(x, y) = F(p) = F(x - ct)$

- Partial derivatives

$$\frac{\partial F}{\partial t} = \frac{dF}{dp}\frac{\partial p}{\partial t} = -c\,F'(p) = -c\,F'(x - ct)$$

$$\frac{\partial F}{\partial x} = \frac{dF}{dp}\frac{\partial p}{\partial x} = F'(p) = F'(x - ct)$$

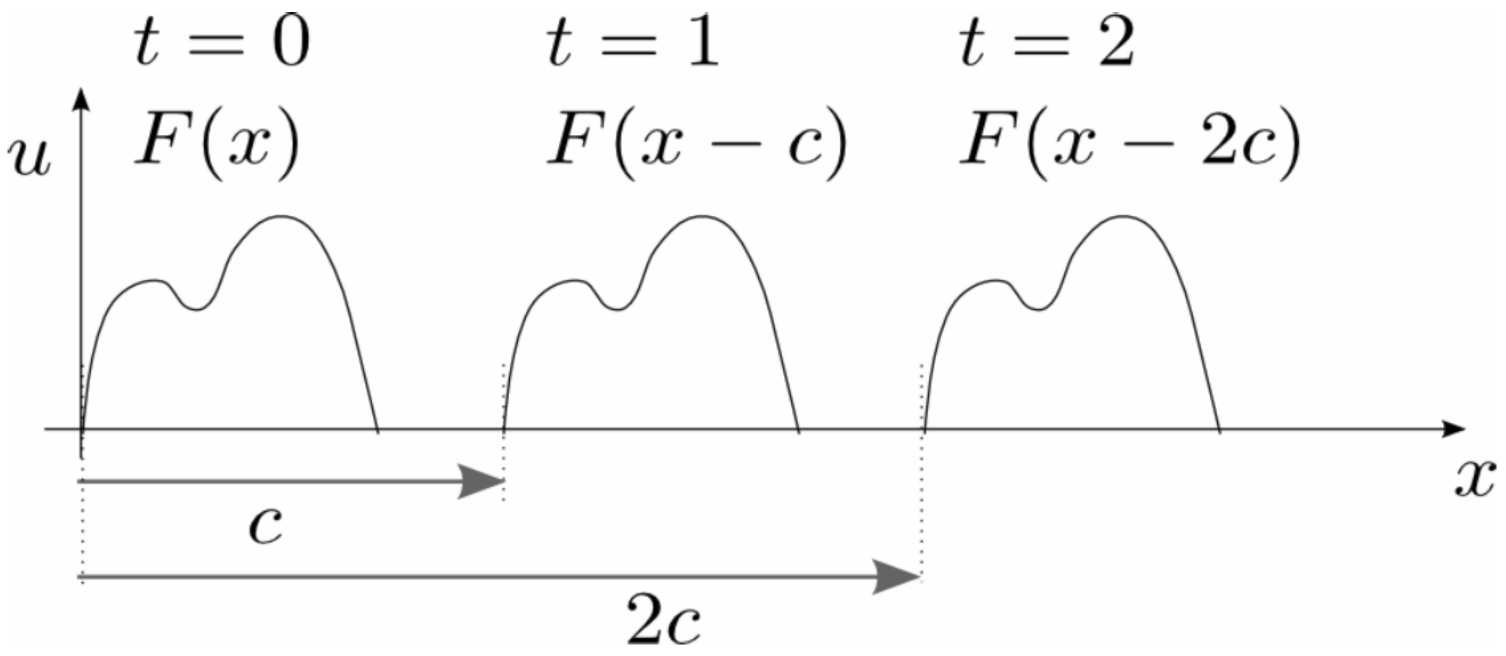
- We get

$$F_t + cF_x = -cF'(x-ct) + cF'(x - ct) = 0$$

- So that *F* is also a solution of the advection equation

Analysis of solutions: Advection

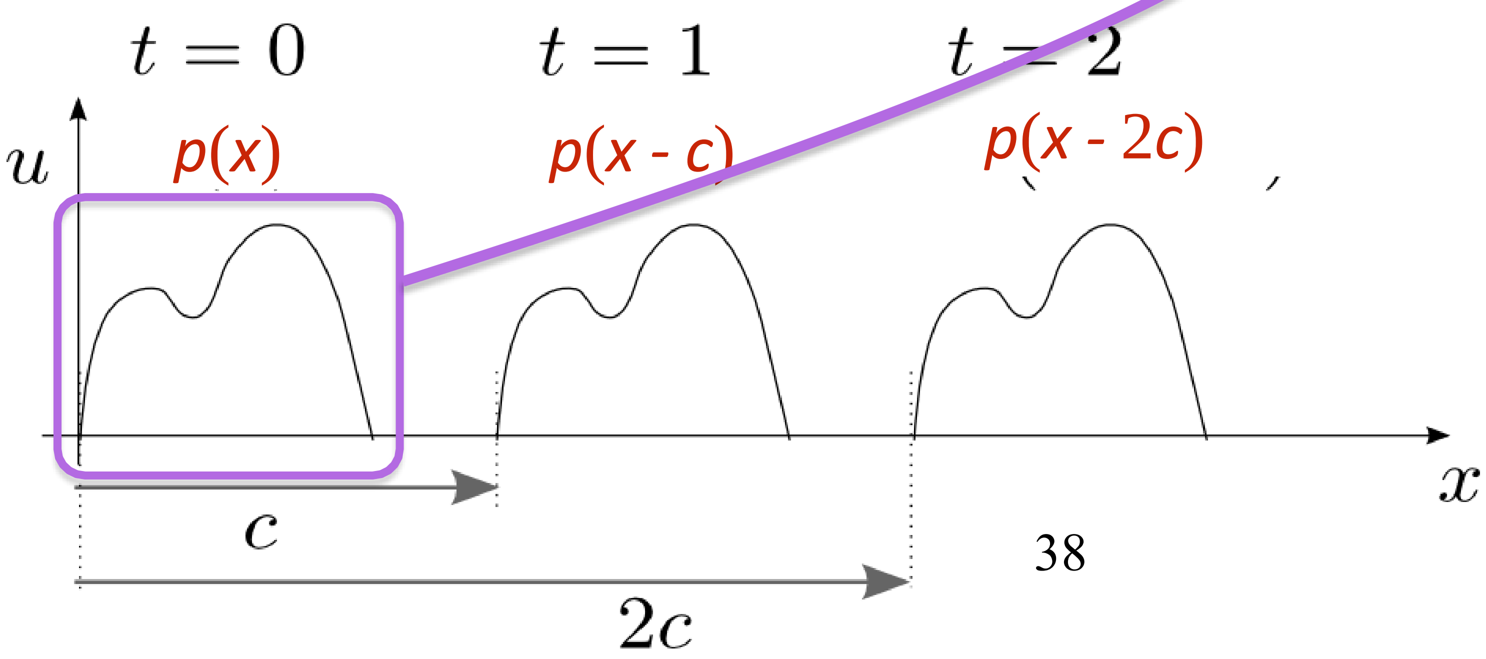
- Example: Advection Eqn in 1-D $u_t + cu_x = 0$
- F can be ***any*** arbitrary function of $p = x\text{-}ct$
- $F(x\text{-}K)$: a copy of $F(x)$ shifted by K
- $F\,(x\text{-}ct)$: shifted by a time-varying amount, i.e. travelling wave



Analysis of solutions: Advection

- Example: Advection Eqn in 1-D $u_t + cu_x = 0$
- Shape of $F(x\text{-}ct)$ depends on auxiliary conditions

Find $u(x,t)$ where $u_t + cu_x = 0$ and $\boxed{u(x,0) = p(x)}$



Method: separation of variables

- Analytical method for solving a PDE
- Assume solution to a PDE is separable, e.g.

$$f(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

- Examples

$e^x y(z+1)^2$	$xy(z+t)$	$x+y+z$
$x^2(z+3)\sin x$	$yz\cos(xy)$	$t(x+t)$
Separable	*Part separable*	*Not separable*

Separation of variables: Wave equat.

- Example: Wave equation in 1-D

$$\boxed{\frac{1}{c^2}u_{tt} = u_{xx}}$$ *PDE*

- Assume the solution is separable $u(x,t) = X(x)\,T(t)$

- Derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial X(x)}{\partial x}T(t) = X'(x)\,T(t) \qquad \frac{\partial u}{\partial t} = X(x)\frac{\partial T(t)}{\partial t} = X(x)\,T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X(x)}{\partial x^2}T(t) = X''(x)\,T(t) \qquad \frac{\partial^2 u}{\partial t^2} = X(x)\frac{\partial^2 T(t)}{\partial t^2} = X(x)\,T''(t)$$

Separation of variables: Wave equat.

- Example: Wave equation in 1-D

$$\boxed{\frac{1}{c^2} u_{tt} = u_{xx}}$$ *PDE*

$$u_{xx} = X''(x)\,T(t)$$
$$u_{tt} = X(x)\,T''(t)$$

$$\longrightarrow \quad \frac{1}{c^2} X(x)\,T''(t) = X''(x)\,T(t)$$

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K$$

- Setting $K = -1$ leads to

$$\boxed{T''(t) = -c^2 T(t) \qquad\qquad X''(x) = -X(x)}$$

Pair of ODEs

Separation of variables: Wave equat.

- Example: Wave equation in 1-D

$$\boxed{\frac{1}{c^2}u_{tt} = u_{xx}}$$ *PDE*

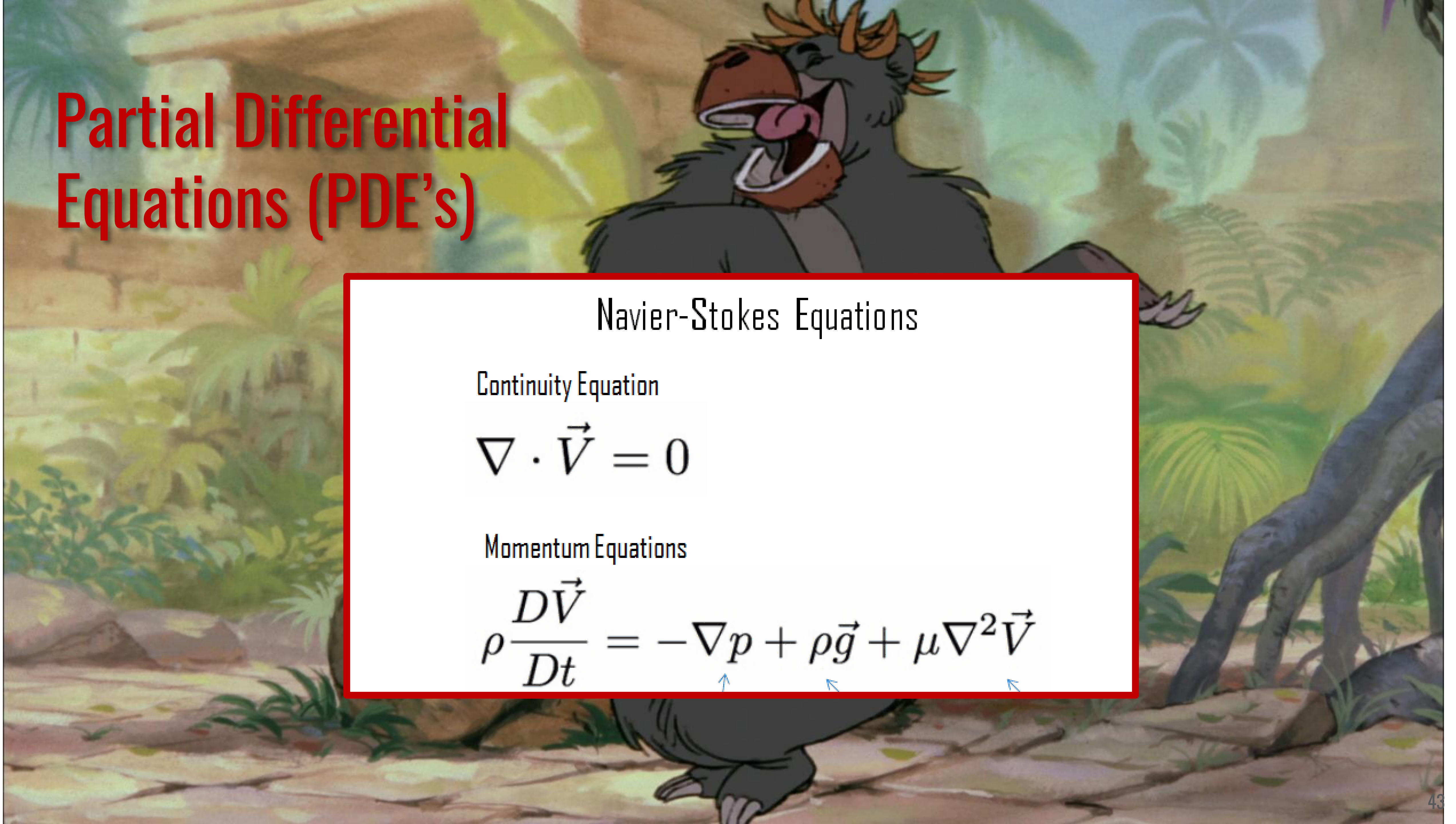
- Now we have *ODEs*

$$\begin{aligned} X''(x) &= -X(x) \\ T''(t) &= -c^2\,T(t) \end{aligned} \longrightarrow \begin{aligned} X(x) &= \alpha \sin x + \beta \cos x \\ T(t) &= \gamma \sin ct + \delta \cos ct \end{aligned}$$

$$u(x,t) = (\alpha \sin x + \beta \cos x)(\gamma \sin ct + \delta \cos ct)$$

$$u(x,t) = A \sin(x - ct) + B \sin(x + ct)$$

- Sum of **two** travelling waves, rightward and leftward
- Compare with Advection Equation's solution



Partial Differential Equations (PDE's)

Navier-Stokes Equations

Continuity Equation

$$\nabla \cdot \vec{V} = 0$$

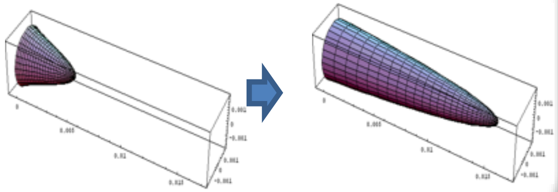
Momentum Equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

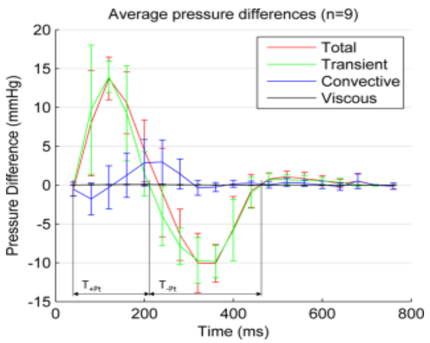
$$\nabla p = \varrho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right) - \mu \Delta \mathbf{u}$$

Transient

Acceleration in time
Heart pump & compliance



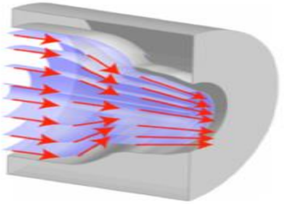
t=0.1s t=0.2s

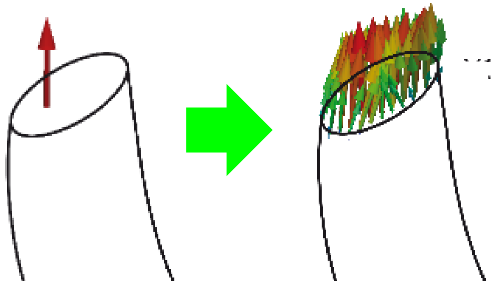


[Lamata MRM 2014]

Convective

Acceleration in space
Vessel geometry

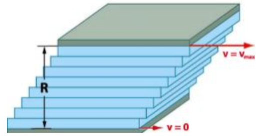


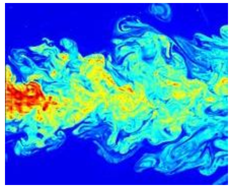


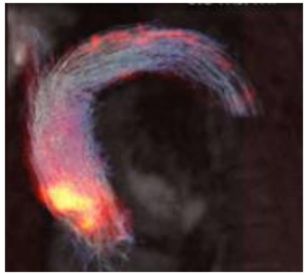
[Donati Circ.Im. 2017]

Viscous

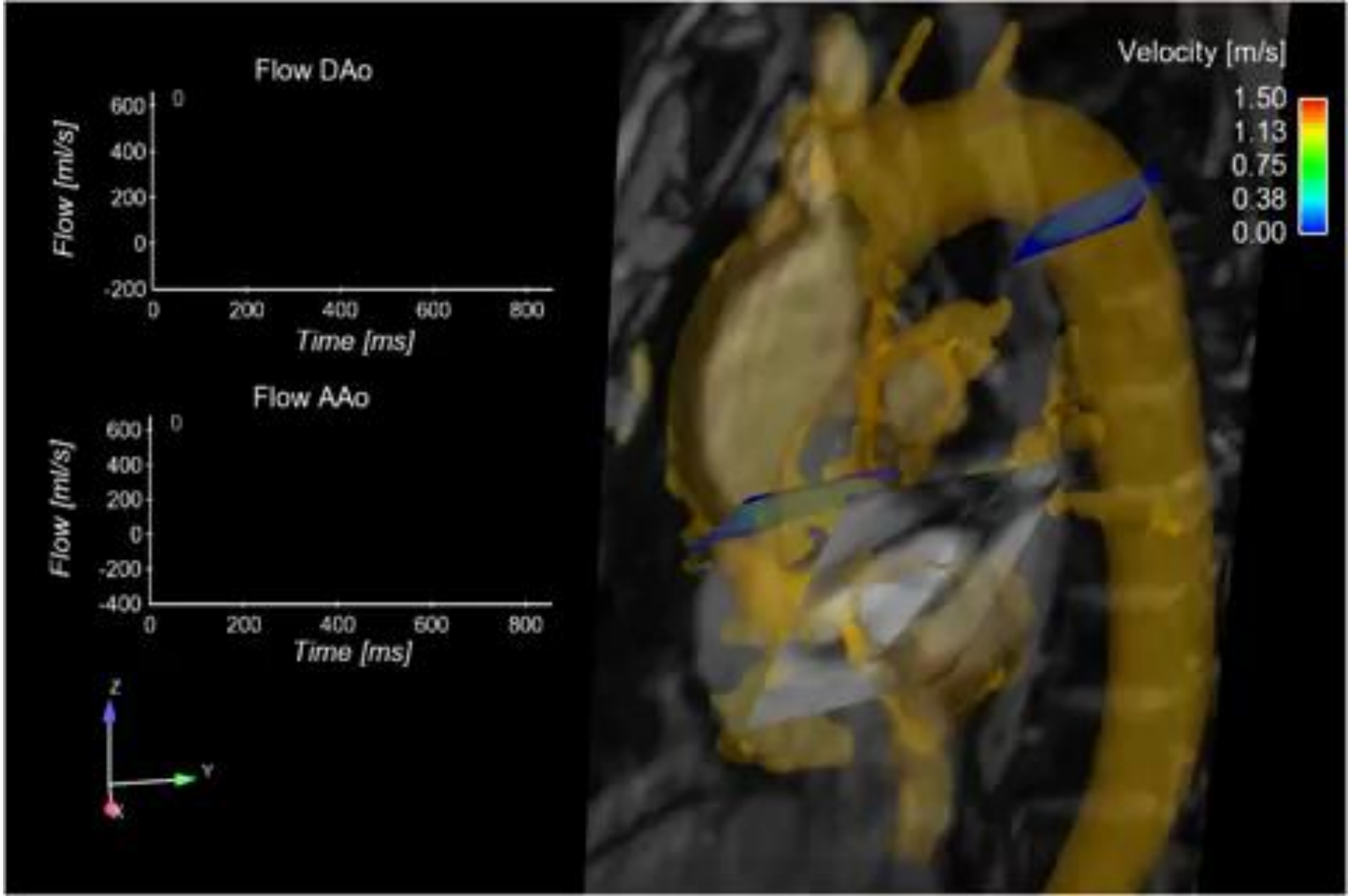
Dissipation, losses
Friction by velocity differences







[Lamata MRM 2014]



Flow DAo
Flow [ml/s]
600
400
200
0
-200
0
200
400
600
800
Time [ms]
Flow AAo
Flow [ml/s]
600
400
200
0
-200
-400
0
200
400
600
800
Time [ms]
Z
Y
X
Velocity [m/s]
1.50
1.13
0.75
0.38
0.00

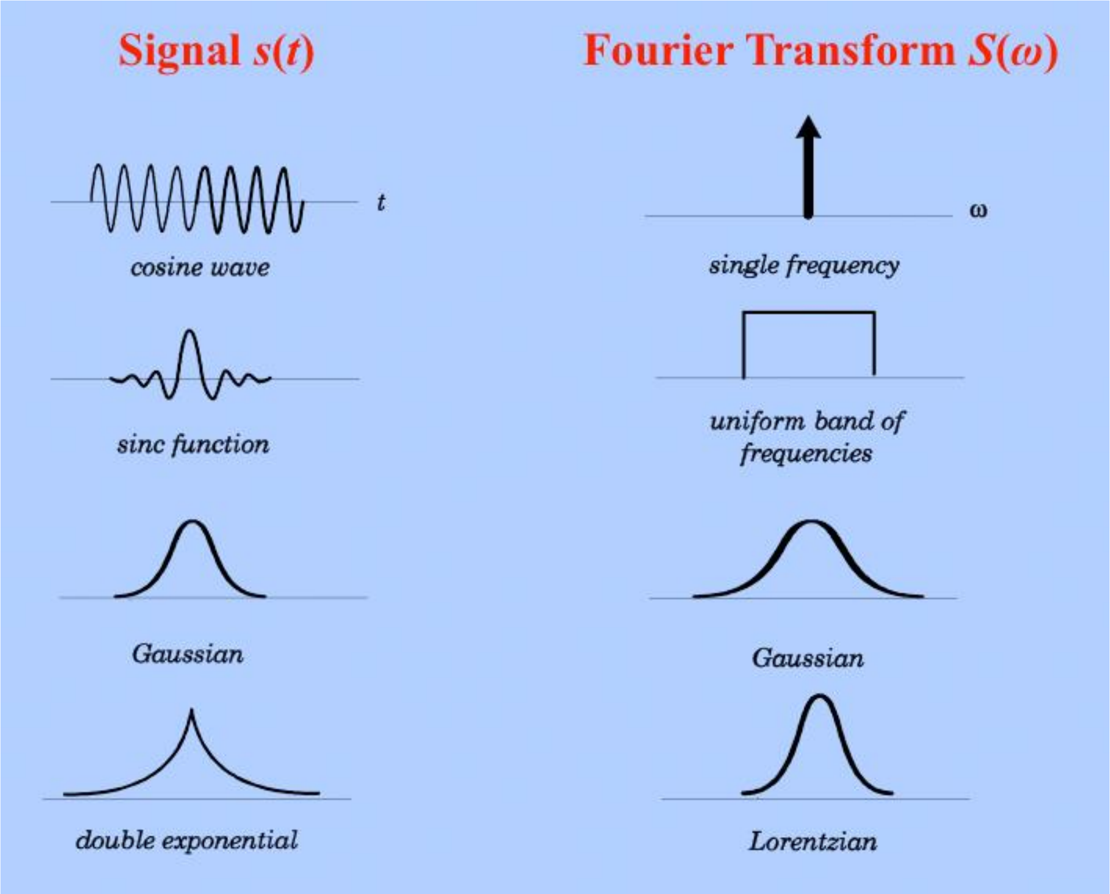
Learning objectives

- Toolkit to solve PDEs
  - Identify the **basic integral calculus** necessary for PDEs; e.g. surface integrals or the divergence theorem
  - Become familiar with a **fundamental conservation law** and use it to derive PDEs in specific physical contexts (e.g. diffusion)
- Methods to solve PDEs
  - Review the ABC in ODEs
  - Separation of variables
  - Integral transforms: Fourier and Laplace

Transforms for PDEs

- Possible to use different transforms, e.g.
  - Fourier Transform
  - Laplace Transform

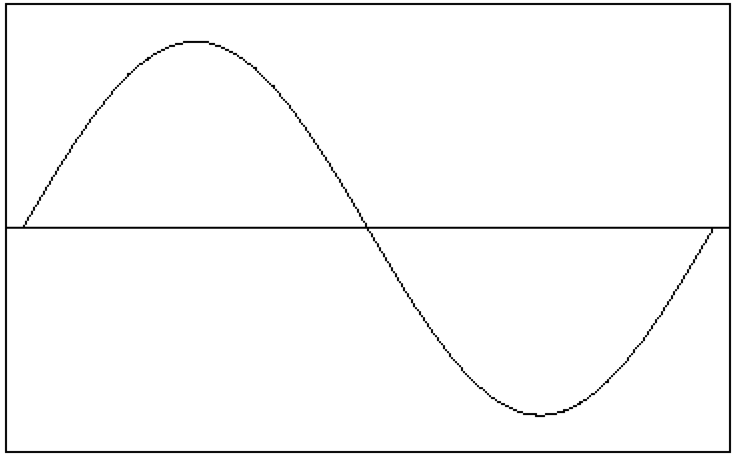
- Basic idea:
  - Convert a PDE to a new domain
  - Easier to find solution in new domain
  - Transform back to obtain solution



Fourier Transform

- Operates on functions

$$\boxed{\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt}$$



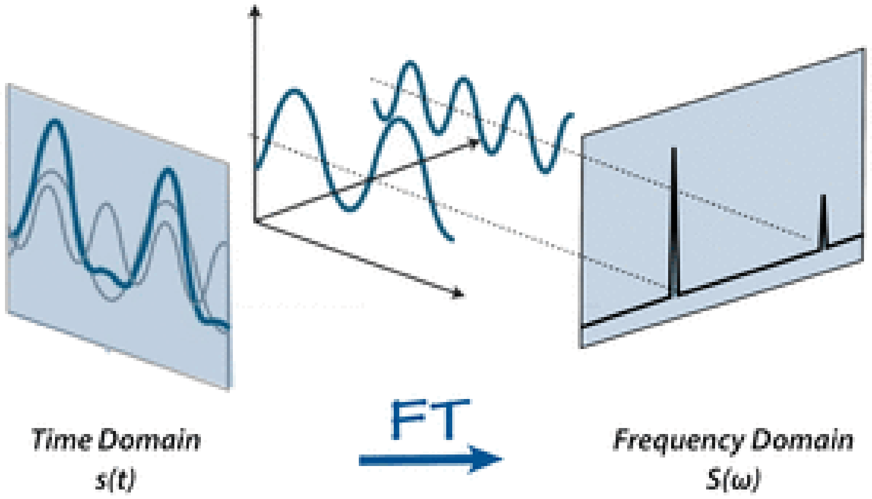
Fourier Transform

- Operates on functions

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\, dt$$

- Can go `back' from Fourier to or

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\,\omega\,t}\, d\omega$$



Fourier Transform

- Standard properties relating to differentiation

Function $\xrightarrow{\mathcal{F}}$	Fourier Transform
$f(t)$	$\tilde{f}(\omega)$
$\frac{\partial f}{\partial t}$	$i\omega \tilde{f}(\omega)$
$\frac{\partial^2 f}{\partial t^2}$	$i\omega(i\omega \tilde{f}(\omega)) = -\omega^2 \tilde{f}(\omega)$
$\vdots$	$\vdots$

Fourier Transform – multivariate function

- E.g two variables, give a choice of integrations

$$
\begin{tikzcd}
& f = f(x,y) \arrow[dl, bend right, "\mathcal{F}_x"'] \arrow[dr, bend left, "\mathcal{F}_y"] & \\
\begin{array}{c} \displaystyle\int_{-\infty}^{\infty} f(x,y)\, e^{-i\,\omega\,x}\, dx \\[2em] \tilde{f}(\omega,y) \end{array} \arrow[dr, bend right, "\mathcal{F}_y"'] & & \begin{array}{c} \displaystyle\int_{-\infty}^{\infty} f(x,y)\, e^{-i\,\omega\,y}\, dy \\[2em] \tilde{f}(x,\theta) \end{array} \arrow[dl, bend left, "\mathcal{F}_x"] \\
& \tilde{f}(\omega,\theta) &
\end{tikzcd}
$$

Fourier Transform – Applied to PDE

- General idea recap
  - Start with a PDE
  - Apply the Fourier transform
  - Get an easier equation to solve in the frequency domain
  - Solve it ‘over there’
  - Apply the inverse Fourier transform to ‘get back’
  - We have a solution to our original problem

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

$$\mathcal{F}_x \swarrow \qquad\qquad \searrow \mathcal{F}_x$$

$$\int_{-\infty}^{\infty} u_{tt}(x,t)\, e^{-i\,\omega\,x}\, dx = \int_{-\infty}^{\infty} c^2 u_{xx}(x,t)\, e^{-i\,\omega\,x}\, dx$$

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension

$$\int_{-\infty}^{\infty} u_{tt}(x,t)\, e^{-i\,\omega\,x}\, dx = \int_{-\infty}^{\infty} c^2 u_{xx}(x,t)\, e^{-i\,\omega\,x}\, dx$$

$$\frac{\partial^2}{\partial t^2}\underbrace{\boxed{\int_{-\infty}^{\infty} u(x,t)\, e^{-i\,\omega\,x}\, dx}}_{\tilde{u}(\omega,t)} = c^2 \underbrace{\boxed{\int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2}\, e^{-i\,\omega\,x}\, dx}}_{-\omega^2\,\tilde{u}(\omega,t)}$$

$$\frac{\partial^2 \tilde{u}(\omega,t)}{\partial t^2} = -c^2\omega^2\tilde{u}(\omega,t)$$

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension

$$\frac{\partial^2 \tilde{u}(\omega, t)}{\partial t^2} = -c^2\omega^2\tilde{u}(\omega, t)$$

- Treat ω as a constant parameter and write $U(t) = \tilde{u}(\omega, t)$

- Obtain an ODE $$\frac{\partial^2 U(t)}{\partial t^2} = -c^2\omega^2 U(t)$$

- Solve ODE $$U(t) = \tilde{u}(\omega, t) = F\, e^{-i\omega ct} + G\, e^{i\omega ct}$$

Formula $\frac{d^2u}{dx^2} = -\lambda^2 u \quad u(x) = Ae^{\lambda i x} + Be^{-\lambda i x}$

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension
- Solution in the Fourier Domain

$$U(t) = \tilde{u}(\omega, t) = F\, e^{-i\omega c t} + G\, e^{i\omega c t}$$

- F, G constants w.r.t variable t but they **can depend on** ω! And make explicit that *all* functions live in the Fourier domain (use tilde notation)

$$\boxed{\tilde{u}(\omega, t) = \tilde{F}(\omega)\, e^{-i\omega c t} + \tilde{G}(\omega)\, e^{i\omega c t}}$$

$$\mathcal{F}_\omega^{-1}$$

$$\boxed{u(x, t) = \ ?}$$

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension
- Inverse transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\tilde{F}(\omega)\,e^{-i\omega ct}+\tilde{G}(\omega)\,e^{i\omega ct}\right)\,e^{i\,\omega\,x}\,d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega)\,e^{-i\omega ct}\,e^{i\,\omega\,x}\,d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(\omega)\,e^{i\omega ct}\,e^{i\,\omega\,x}\,d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega)\,e^{i\omega\,(x-ct)}\,d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(\omega)\,e^{i\omega\,(x+ct)}\,d\omega$$

Fourier Transform – Applied to PDE

- Example
  - Wave equation in one spatial and one time dimension
- Inverse transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega)\,e^{i\omega\,\alpha}\,d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(\omega)\,e^{i\omega\,\beta}\,d\omega$$

$$\alpha = x - ct \qquad\qquad \beta = x + ct$$

Now we have a pair of 'standard' Inverse FTs, so

$$u(x,t) = F(\alpha) + G(\beta)$$

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$

General Solution to the wave equation

Fourier Transform – Applied to PDE

Re-cap

$u_{tt}(x,t) = c^2 u_{xx}(x,t)$	Wave equation
$\mathcal{F}_x$	FT both sides
$\dfrac{\partial^2 U(t)}{\partial t^2} = -c^2\omega^2 U(t) \qquad U(t) = \tilde{u}(\omega,t)$	ODE in Fourier domain
$\tilde{u}(\omega,t) = \tilde{F}(\omega)\, e^{-i\omega ct} + \tilde{G}(\omega)\, e^{i\omega ct}$	Solution in Fourier domain
$u(x,t) = F(x-ct) + G(x+ct)$	Inverse FT to obtain General Solution

Wave Equation General Solution

$$u(x,t) = F(x - ct) + G(x + ct)$$

- Without auxiliary conditions, F and G can be *any* functions of $x - ct$ and $x + ct$, E.g., with $c = 2$

$$F(x,t) = \sin(x - 2t) \qquad G(x,t) = 7(x + 2t)^3$$

- Solution is sum of leftward and rightward travelling waves

Wave Equation

Well-posed PDE

- If auxiliary conditions are *initial* conditions on u and u_t

Find $u_{tt}(x, t) = c^2 u_{xx}(x, t)$ where $u(x, 0) = p(x)$

and $u_t(x, 0) = 0$

- Then we can use

$$F(x - ct) = \frac{1}{2}p(x - ct) \qquad G(x + ct) = \frac{1}{2}p(x + ct)$$

- Which gives the particular solution

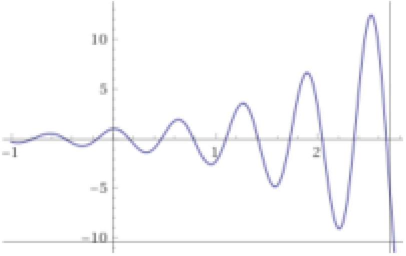
$$u(x, t) = \frac{1}{2}p(x - ct) + \frac{1}{2}p(x + ct)$$

- **Exercise**: Prove that, for any p, that this solution satisfies
  - the PDE
  - the initial conditions

Laplace transform

- Fourier for **steady state signal**
- Laplace for **transient signal**
  - Response to pulses, step functions, delta functions,
- Basis: a set of exponentials

$$\bar{f}(s) = \int_0^{\infty} f(t)\, e^{-s\,t}\, dt$$



Original domain

$$\begin{tikzcd} f(t) \arrow[r, bend left, "\boxed{\mathcal{L}}"] & \bar{f}(s) \arrow[l, bend left, "\boxed{\mathcal{L}^{-1}}"] \end{tikzcd}$$

Laplace domain

https://johnflux.com/2019/02/12/laplace-transform-visualized/

Laplace transform

- Can be looked up for common functions, e.g.

Original (time/space) domain

$f(t)$	$\bar{f}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\sin t$	$\frac{s^2}{s^2+1}$
e^{at}	$\frac{a}{s-a}$
$\vdots$	$\vdots$

Laplace (frequency) domain

Laplace transform

- Important properties relating to differential equations
- Laplace transform of first derivative

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)\, e^{-s\,t}\, dt$$

- Can use integration by parts to show that

$$\mathcal{L}[f'(t)] = -f(0) + s\mathcal{L}[f(t)]$$

- And higher order: replace *f* for *f'*

$$\mathcal{L}[f''(t)] = -f'(0) + \underbrace{s\mathcal{L}[f'(t)]}$$

$$\mathcal{L}[f''(t)] = -f'(0) \overbrace{- s\, f(0) + s^2\mathcal{L}[f(t)]}$$

Laplace transform

General idea

- Start with a PDE
- Apply Laplace transform
- Get an easier equation in the Laplace domain
- Solve it ‘over there’
- Apply the inverse Laplace transform to ‘get back’
- We have a solution to our original problem

Issues:

Very similar to use of Fourier Transform but …

Harder to analytically invert than for the FT

In the discrete case, the counterpart is the z-transform (Signal and Image processing module?)

Laplace transform: example

- ODE: First order linear, constant coefficients

$$A\frac{\partial f}{\partial t} + B f + C = 0$$

- Apply LT:

$$A\,\mathcal{L}\left[\frac{\partial f}{\partial t}\right] + B\,\mathcal{L}[\,f\,] + C\,\mathcal{L}[\,1\,] = 0$$

- Rules for LT:

$$A\left(-f(0) + s\bar{f}(s)\right) + B\bar{f}(s) + C\,\frac{1}{s} = 0$$

- Obtain a simple equation in $\bar{f}(s)$

Laplace transform: example

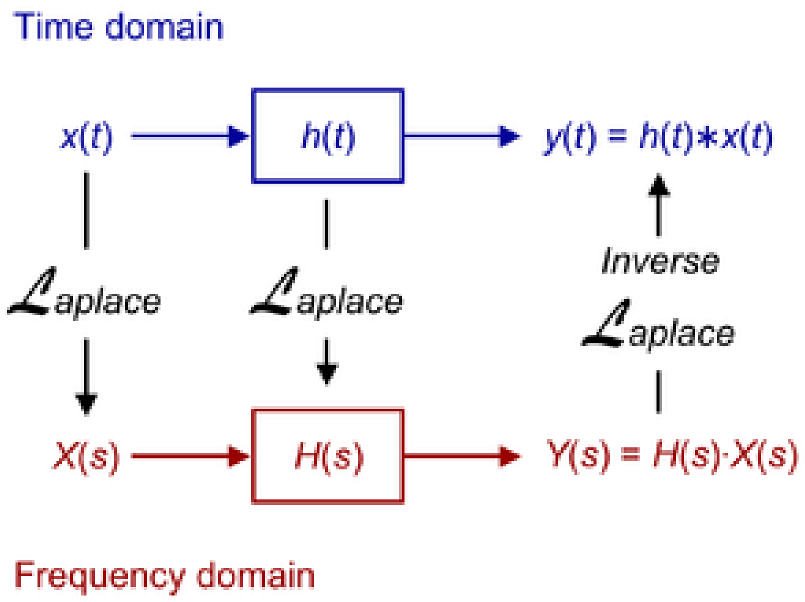
- ODE: First order linear, constant coefficients

$$A\frac{\partial f}{\partial t} + Bf + C = 0$$

- Rearrange $$A\left(-f(0) + s\bar{f}(s)\right) + B\bar{f}(s) + C\,\frac{1}{s} = 0$$

$$\bar{f}(s) = \frac{As\,f(0) - C}{s(As + B)}$$

- Now get the inverse.
- Popular in linear time invariant systems



Laplace transform: multivariate

- E.g two variables, give a choice of integrations

$$
\begin{tikzcd}
& g = g(x,y) \arrow[dl, bend right, "\mathcal{L}_x"'] \arrow[dr, bend left, "\mathcal{L}_y"] & \\
\int_0^\infty g(x,y)\, e^{-s\,x}\, dx & & \int_0^\infty g(x,y)\, e^{-s\,y}\, dy \\
\bar{g}(s,y) & & \bar{g}(x,s)
\end{tikzcd}
$$

Laplace transform: multivariate

- When applying to a PDE, choose the one that gives the simplest resulting equation in the Laplace domain.
  - Example: diffusion equation

$$
\begin{tikzcd}[column sep=small]
& u_t = \alpha\, u_{xx} \arrow[dl, "\mathcal{L}_x"'] \arrow[dr, "\mathcal{L}_t"] & \\
\displaystyle\int_0^\infty u_t\, e^{-sx}\, dx = \underbrace{\int_0^\infty \alpha\, u_{xx}\, e^{-sx}\, dx}_{\text{Hard to integrate}} & & \displaystyle\underbrace{\int_0^\infty u_t\, e^{-st}\, dt}_{\text{Have a rule for this}} = \int_0^\infty \alpha\, \underbrace{u_{xx}}_{\text{This term can be factored out}}\, e^{-st}\, dt
\end{tikzcd}
$$

Easier route

Learning objectives

- Toolkit to solve PDEs
  - Identify the **basic integral calculus** necessary for PDEs; e.g. surface integrals or the divergence theorem
  - Become familiar with a **fundamental conservation law** and use it to derive PDEs in specific physical contexts (e.g. diffusion)
- Methods to solve PDEs
  - Review the ABC in ODEs
  - Separation of variables
  - Integral transforms: Fourier and Laplace

Partial Differential Equations (PDE's)

Biomedical Engineering 5CCYB070

COMPUTATIONAL METHODS

Lecture 14 Vector Calculus and PDEs II

01 Math tools and definitions

02 Finite diferences