# Computational Methods
# Sample Exam Questions

PDEs: Sample exam questions

10 02 2017

**Question 1**

(a) A partial differential equation (PDE) is to be solved inside a region $\Omega$. A boundary condition is given which specifies that the solution must be zero on the boundary $\partial\Omega$. State whether this represents a Dirichlet boundary condition or a Neumann boundary condition. Give a reason for your answer.

> **Solution:**
>
> Dirichlet conditions fix the *value* of the function on the boundary $\partial\Omega$ (of the region $\Omega$ where we need to solve the PDE.
>
> Neumann conditions fix the gradient.

(b) The partial differential equation (PDE) $u_{xx} + 5xu_y - 7x = 0$ in the function $u(x, y)$ can be classified as a second-order, linear and non-homogeneous.

Classify the PDEs below in the same way, i.e. state the order, whether they are linear or non-linear and whether they are homogeneous or non-homogeneous.

i. $g_{yy} + 5g_y g_x - 2e^x = 0$ in the function $g(x, y)$

> **Solution:**
>
> second-order, non-linear, non-homogeneous

ii. $\frac{\partial h}{\partial z} + \frac{\partial h}{\partial t} = 3h$ :

in the function $h(z, t)$.

> **Solution:**
>
> first-order, linear, homogeneous

iii. $f_{yy} - 5e^{x+y} f_{yy} - 3xf_x = 0$

in the function $f(x, y)$.

**Solution:**

second-order, linear, homogeneous

(c) Both of the following two functions

$$f(x,y) = (3x - 2y)^2 \qquad g(x,y) = e^{(3x-2y)}$$

can be used as a solution of the PDE $2w_x + 3w_y = 0$. Prove this separately for each function.

**Solution:**

Setting $w = f(x,y)$ gives

$$w_x = 6(3x - 2y)$$
$$w_y = -4((3x - 2y)$$

so that

$$2w_x + 3w_y = 12(2x - 3y) - 12(2x - 3y) = 0$$

Setting $w = g(x,y)$ gives

$$w_x = 3e^{(3x-2y)}$$
$$w_y = -2e^{(3x-2y)}$$

so that

$$2w_x + 3w_y = 6e^{(3x-2y)} - 6e^{(3x-2y)} = 0$$

(d) Explain what the *principle of superposition* means for the solutions of a linear homogeneous PDE. Illustrate this for the two solutions of the PDE in part (c).

**Solution:**

The principle of superposition states that a linear combination of two solutions of a PDE gives another solution of the PDE.

In the case of part (c) this can be illustrated by saying that

$$\alpha f + \beta g$$

is also a solution of $w_{xx} + w_{yy} = 0$.

The principle of superposition applies to *linear* and *homogeneous* PDEs.

**Question 2**

Laplace's equation is solved in a square region with a Dirichlet boundary condition applied to the edge. The lines below represent pseudocode (with a MatLab flavour) that describe how Jacobi iteration can be used to estimate the solution. The lines are, however, jumbled up. Re-arrange them so that the pseudocode makes sense. You might want to cut up a printed copy of this page and do this manually.

```
end
```

```
Inputs:
```

```
 for i in 2..N-1, j in 2..N-1
```

```
V: Array with estimate values of solution
```

```
V: NxN array with initial values set at boundary
```

```
 v = ( V(i-1,j) + V(i,j-1) + V(i+1,j) + V(i,j+1) ) / 4
```

```
while i < maxI and max(V_copy - V) > eps
```

```
eps:  A small value.  Stop iteration if difference
between
successive estimates is below this value
```

```
 i = i + 1
```

```
 end
```

```
 V_copy = V
```

```
return V
```

```
 V_copy(i,j) = v
```

```
i = 0
```

```
maxI: Maximum number of iterations.
```

```
 V = V_copy
```

```
Output:
```

**Solution:**

The lines in the correct order are:

```
Inputs:

V: NxN array with initial values set at boundary

eps: A small value. Stop iteration if difference between
     successive estimates is below this value

maxI: Maximum number of iterations.

Output:

V: Array with estimate values of solution

i = 0

while i < maxI and max(V_copy - V) > eps


  V_copy = V

  for i in 2..N-1, j in 2..N-1

    v = ( V(i-1,j) + V(i,j-1) + V(i+1,j) + V(i,j+1) ) / 4

    V_copy(i,j) = v
  end

  V = V_copy

  i = i + 1

end


return V
```

**Question 3**

The air pressure in a pipe is modelled by a function $p = p(x,t)$, which depends on spatial location $x$ and time $t$ and satisfies the following PDE

$$p_t - 5p_{xx} + p = 0$$

A numerical scheme is to be used to estimate a solution $p$ of the PDE. It uses a discrete set of locations represented by an array $P(i,j)$ to estimate $p$ where $i$ and $j$ represent indices for position and time respectively. The distance between adjacent locations in the array $P$ is $\delta x$ and the interval between successive timepoints is $\delta t$.

(a) Write a forward difference estimate for the time derivative $p_t$ in terms of the elements of the array $P$.

**Solution:**

The $p_t$ estimate can be written

$$\frac{P(i,j+1) - P(i,j)}{\delta t}$$

(b) Derive a *central* difference estimate for $p_{xx}$ in terms of the elements of the array $P$.

**Solution:**

$p_{xx}$ can be written:

$$\frac{P(i+1,j) - 2\,P(i,j) + P(i-1,j)}{\delta x^2}$$

More than one way of deriving this expression.

(c) Use the finite difference expressions for the partial derivatives in the parts (a) and (b) to write a numerical representation of the PDE.

Use this to derive an update formula that can be used to solve the PDE in a forward marching scheme.

**Solution:**

The original PDE can be written

$$
\begin{aligned}
&\frac{P(i,j+1) - P(i,j)}{\delta t} \\
&\quad - 5\left[\frac{P(i+1,j) - 2\,P(i,j) + P(i-1,j)}{\delta x^2}\right] \\
&\quad + \; P(i,j) = 0
\end{aligned}
$$

This can be rearranged to give the forward marching update formula

$$P(i,j+1) = \alpha\, P(i+1,j) \; + \; \beta\, P(i,j) \; + \; \gamma\, P(i-1,j)$$

where

$$\alpha = \frac{5\,\delta t}{\delta x^2} \qquad \beta = 1 - \frac{10\,\delta t}{\delta x^2} - \delta t \qquad \gamma = \frac{-5\,\delta t}{\delta x^2}$$

**Question 4**

Consider the boundary value problem (BVP), defined on the upper-half plane, i.e. the region $\Omega = \{(x,y) : y \geq 0\}$, with a boundary on the $x$ axis:

$$
\begin{aligned}
&u_{xx} + u_{yy} = 0 && \text{PDE} \\
&u(x,0) = 0 \quad \text{for all } x && \text{Boundary condition}
\end{aligned}
$$

(a) Verify that the function $f(x,y) = (e^y - e^{-y})\cos x$ is a solution of the PDE and that it satisfies the boundary condition.

**Solution:**

The first order partial derivatives are

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= -(e^y - e^{-y})\,\sin x \\
\frac{\partial f}{\partial y} &= (e^y + e^{-y})\,\cos x
\end{aligned}
$$

The second partials are

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -(e^y - e^{-y})\,\cos x \\
\frac{\partial^2 f}{\partial y^2} &= (e^y - e^{-y})\,\cos x
\end{aligned}
$$

Therefore the function satisfies the PDE because

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -(e^y - e^{-y})\,\cos x + (e^y - e^{-y})\,\cos x = 0$$

On the $x$ axis we have $y = 0$ and

$$f(x,0) = (e^0 - e^{-0})\cos x = (1-1)\cos x = 0$$

which shows that the function also meets the boundary condition.

(b) Verify that a second function $g(x,y) = (e^x + e^{-x})\,\sin y$ can also be a solution to the BVP.

**Solution:**

Using shorthand notation, we have

$$\begin{aligned} g_{xx} &= (e^x + e^{-x})\,\sin y \\ g_{yy} &= -(e^x + e^{-x})\,\sin y \end{aligned}$$

so that

$$g_{xx} + g_{yy} = (e^x + e^{-x})\,\sin y - (e^x + e^{-x})\,\sin y = 0$$

We also have $g(x,0) = (e^x + e^{-x}) \times \sin 0 = 0$, so that the boundary value is satisfied.

(c) Verify, for the given BVP, that the principle of superposition holds, i.e. that, if $\alpha$ and $\beta$ are any scalar values then the function formed from a linear combination (superposition) of the solutions $f$ and $g$

$$h(x,y) = \alpha f(x,y) + \beta g(x,y)$$

will also be a solution.

*Hint*: It is possible to use the formulas for the functions $f$ and $g$ given in the previous parts to do this. But we don't actually need to, we can just use the fact that they satisfy the PDE. You can use either approach.

**Solution:**

$$\begin{aligned} h_{xx} + h_{yy} &= \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \\ &= \frac{\partial^2}{\partial x^2}(\alpha f + \beta g) + \frac{\partial^2}{\partial y^2}(\alpha f + \beta g) \\ &= \alpha f_{xx} + \beta g_{xx} + \alpha f_{yy} + \beta g_{yy} \\ &= \alpha(f_{xx} + f_{yy}) + \beta(g_{xx} + g_{yy}) \\ &= \alpha \times 0 + \beta \times 0 = 0 \end{aligned}$$

where, in the last line we have used the fact that $f$ and $g$ both satisfy the PDE.

$h$ also satisfies the boundary condition because we also have $h(x,0) = \alpha f(x,0) + \beta g(x,0) = 0$ because $f$ and $g$ individually satisfy the boundary condition, i.e. $f(x,0) = g(x,0) = 0$ from the previous parts.

**Question 5**

(a) The partial differential equation $e^x w_t + 12w\,w_x = 0$ in the function $w(x,t)$ can be classified as a **first-order**, **non-linear** and **homogeneous** PDE.

Provide a similar classification for the PDEs below in the functions $f(x,y)$, $g(x,t)$ and $w(x,y,z)$ respectively:

i. $g\,g_t + g_x + \sin t = 0$

ii. $v_{xx} + y^3 v_{yy} - v_z \sin x = 0$

iii. $\frac{\partial g}{\partial x} - 12\frac{\partial g}{\partial y} + 15g = 0$

**Solution:**

i. $g\, g_t + g_x + \sin t = 0$ is first-order, non-linear and non-homogeneous

ii. $v_{xx} + y^3 v_{yy} - v_z \sin x = 0$ is second-order, linear and homogeneous

iii. $\frac{\partial g}{\partial x} - 12\frac{\partial g}{\partial y} + 15g = 0$ is first-order, linear and homogeneous

(b) For a PDE to be *well-posed*, auxiliary data need to be specified in the form of either initial conditions, boundary conditions or both. Describe the key differences between initial conditions and boundary conditions, give examples of each.

**Solution:**

Initial conditions relate to PDEs of time-dependent functions.

Boundary conditions can relate to any function, not just those that have a time dependency.

Initial conditions specify the state of the system at $t = 0$, i.e. if the function $u(x, t)$ depends on $x$ and $t$, then the initial condition says what values $u$ has for all $x$ (in the region of interest) at time $t = 0$. E.g. $u(x, 0) = x^2$ at $t = 0$.

For a time-dependent function,boundary conditions specify the values of the function at the boundary of the region or interval of interest and these conditions hold for *all* time. E.g. if the function $u(x, t)$ has one spatial and one time variable, then we can write, as an example, $u(0, t) = u(L, t) = 0$ (or some other constant value) where 0 and $L$ represent the interval we are trying to solve for.

As described, boundary conditions can apply to functions that depend only on spatial variables. E.g. $w(x, y)$, where we want to solve in a region $R$, can have boundary conditions specified on the boundary $\partial R$ of $R$. E.g. we may specify that $w(x, y) = c$ for $(x, y)$ on $\partial R$ where $c$ is some fixed value.

(c) Describe the difference between Neumann and Dirichlet boundary conditions.

**Solution:**

For a PDE to be solved in a region $\Omega$, Dirichlet boundary conditions specify the value that the solution(s) should take at each point on the boundary $\partial\Omega$. Neumann conditions, on the other hand, specify the gradient of the solution across the boundary.

(d) For each of the following second order linear PDEs, determine their status according the categories: {Parabolic, Elliptic, Hyperbolic}. Give a proof in each case.

Show for each PDE, whether the chosen category fits for *all* locations in the $(x, y)$ plane[1] or whether it fits one category in part of the plane and another category(ies) elsewhere.

[1] or the $(x, t)$ plane

1. The heat equation with coefficient $k = 1$, $u_t = u_{xx}$
2. The Laplace Equation in 2-D $u_{xx} + u_{yy} = 0$
3. The Equation $yu_{xx} + u_{yy} = 0$

**Solution:**

In order to decide, the equations need to be expressed in the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

This might need to have $y$ replaced by $t$ if we are considering a PDE in $x$ and $t$.

1. The heat equation with coefficient $k = 1$ $u_t = u_{xx}$ can be written $u_{xx} - u_t = 0$.

   So we have $A(x,t) = 1$, $B(x,t) = 0$ and $C(x,t) = 0$. I.e. all are constant functions. The discriminant is $B^2 - 4AC = 0$ everywhere in the $(x,t)$ plane.

   This means the Heat Equation is **parabolic** (everywhere).

2. The Laplace Equation in 2-D $u_{xx} + u_{yy} = 0$ has $A(x,y) = 1$, $B(x,y) = 0$, $C(x,y) = 1$ so that the discriminant $B^2 - 4AC = -4$. The discriminant is -4 for all $(x,y)$ so the Laplace Equation in 2-D is **elliptic** (everywhere).

3. The Equation $yu_{xx} + u_{yy} = 0$ has $A(x,y) = y$, $B(x,y) = 0$ and $C(x,y) = 1$ so that the discriminant $D = B^2 - 4AC = -4y$. This shows that the discriminant varies in the $(x,y)$ plane. Its sign is opposite to the sign of the $y$ coordinate, so that we have three cases:

   - For $(x,y)$ with $y > 0 \Rightarrow D < 0$ : PDE is elliptic
   - For $(x,y)$ with $y = 0 \Rightarrow D = 0$ : PDE is parabolic (i.e. on the $x$-axis.
   - For $(x,y)$ with $y < 0 \Rightarrow D > 0$ : PDE is hyperbolic

## Question 6

(a) For each of the following three partial differential equations (PDEs), determine if the PDE is separable or not. If it is, show how it can be written in separable form.

- $w\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial t} = 0$ in the function $w(x,t)$
- $f_{xx} - f_{yy} = 0$ in the function $f(x,y)$.
- $v_{xx}(x,y) + v_y(x,y) = x\,y\,v(x,y)$ in the function $v(x,y)$.

**Solution:**

$w\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial t} = 0$ is separable. We can write $w(x,t) = X(x)T(t)$ so that $\frac{\partial w}{\partial x} = X'T$ and $\frac{\partial w}{\partial t} = XT'$. Substituting in gives

$$(XT)(X'T) + (XT') = 0$$

so that

$$XX'T^2 = -XT'$$

which can be separated as

$$X' = \frac{-T'}{T^2}$$

$f_{xx} - f_{yy} = 0$ is separable. We can write $f(x, y) = X(x)Y(y)$ so that $f_{xx} = X''(x)Y(y)$ and $f_{yy} = X(x)Y''(y)$. Substituting in gives

$$X''(x)Y(y) - X(x)Y''(y) = 0$$

so that we get the separable form

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)}$$

$v_{xx}(x, y) + v_y(x, y) = x\, y\, v(x, y)$ is not separable.

The following partial differential equation in $u(x, y)$ relates to parts (b) to (e):

$$u_x = u_y + u$$

(b) By writing $u(x, y)$ as the product of two functions $f(x)$ and $g(y)$, show that the PDE above is separable. Do not solve the equation, just show that it is separable.

**Solution:**

Writing

$$u(x, y) = f(x)\, g(y)$$

we have

$$u_x = f'\, g \quad \text{and} \quad u_y = f\, g'$$

Substituting these in gives

$$f'(x)\, g(y) = f(x)\, g'(y) + f(x)\, g(y)$$

So that

$$f'(x)\, g(y) = f(x)\, (g'(y) + g(y))$$

dividing both sides by $f g$ gives

$$\frac{f'(x)}{f(x)} = \frac{g'(y) + g(y)}{g(y)}$$

In the last equation, the left hand side depends only on $x$ and the right hand side depends only on $y$. This means that the original PDE is separable.

(c) Use the result of part (b) to define two *ordinary* differential equations (ODEs). You will need to introduce a constant to do this.

**Solution:**

As each side of

$$\frac{f'(x)}{f(x)} = \frac{g'(y) + g(y)}{g(y)}$$

depends on a different variable and they are always equal, they must equal a constant.

Set this constant to $\lambda$. This gives two ODEs

$$\frac{f'(x)}{f(x)} = \lambda \quad \text{and} \quad \frac{g'(y) + g(y)}{g(y)} = \lambda$$

These can be simplified to give

$$f'(x) = \lambda\, f(x) \quad \text{and} \quad g'(y) + g(y) = \lambda\, g(y)$$

or

$$f'(x) = \lambda\, f(x) \quad \text{and} \quad g'(y) = (\lambda - 1)\, g(y)$$

(d) Find general solutions for the ODEs of part c and therefore derive a general solution for the original partial differential equation. Your solution may include a constant(s).

**Solution:**

Solving

$$f'(x) = \lambda\, f(x) \quad \text{and} \quad g'(y) = (\lambda - 1)\, g(y)$$

gives

$$f(x) = Ae^{\lambda\, x} \quad \text{and} \quad g(y) = Be^{(\lambda-1)\, y}$$

where $A$, $B$ and $\lambda$ are constants.

This means a solution of the original PDE can be given by

$$u(x, y) = f(x)\, g(y) = Ae^{\lambda\, x}\, Be^{(\lambda-1)\, y}$$

which can be written as

$$u(x, y) = Ce^{\lambda\, x + (\lambda-1)\, y}$$

for some constant $C$.

(e) Verify that the solution for $u(x, y)$ obtained in part (d) satisfies the original PDE.

**Solution:**

For

$$u(x, y) = Ce^{\lambda\, x + (\lambda-1)\, y}$$

we have

$$u_x = \lambda Ce^{\lambda\, x + (\lambda-1)\, y} = \lambda u$$

and

$$u_y = (\lambda - 1)Ce^{\lambda\, x + (\lambda-1)\, y} = (\lambda - 1)u$$

so that

$$u_y + y = (\lambda - 1)u + u = \lambda u = u_x$$

as required.

**Question 7**

(a) Explain why auxiliary data such as boundary conditions or initial conditions are useful for solving partial differential equations (PDEs).

**Solution:**

Without auxiliary conditions, a PDE can have many possible solutions. With auxiliary data, a particular solution can be found.

(b) The partial differential equation (PDE) $w_{xx} - 4xw_y + 3x = 0$ can be classified as a second-order, linear and non-homogeneous.

Classify the PDEs below in the same way, i.e. state the order, whether they are linear or non-linear and whether they are homogeneous or non-homogeneous.

i. $v_x + v_y - v\,v_z = 0$ in the function $v(x, y, z)$

**Solution:**

First order, non-linear, homogeneous.

ii. $\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial y} + xy = 0$ in the function $g(x, y)$.

**Solution:**

First order, linear, non-homogeneous.

iii. $h_t - t^2 h_{xx} + \sin x = 0$ in the function $h(x, t)$.

**Solution:**

Second order, linear, non-homogeneous.

(c) Prove that both of the two functions

$$f(x, y) = \cos x(e^y - e^{-y}) \qquad g(x, y) = \sin y(e^x + e^{-x})$$

can be used as a solution of the PDE $w_{xx} + w_{yy} = 0$.

**Solution:**

Setting $w = f(x, y)$ gives

$$\begin{aligned} w_x &= -\sin x(e^y - e^{-y}) \\ w_{xx} &= -\cos x(e^y - e^{-y}) \\ w_y &= \cos x(e^y + e^{-y}) \\ w_{yy} &= \cos x(e^y - e^{-y}) \end{aligned}$$

so that

$$w_{xx} + w_{yy} = -\cos x(e^y - e^{-y}) + \cos x(e^y - e^{-y}) = 0$$

Setting $w = g(x, y)$ gives

$$\begin{aligned} w_x &= \sin y(e^x - e^{-x}) \\ w_{xx} &= \sin y(e^x + e^{-x}) \\ w_y &= \cos y(e^x + e^{-x}) \\ w_{yy} &= -\sin y(e^x + e^{-x}) \end{aligned}$$

so that

$$w_{xx} + w_{yy} = \sin y(e^x + e^{-x}) - \sin y(e^x + e^{-x}) = 0$$

(d) Explain what is meant by the *principle of superposition* for the solutions of PDE. Illustrate this with the two solutions of part (c).

State the type of PDE that the principle of superposition applies to.

**Solution:**

The principle of superposition states that a linear combination of two solutions of a PDE gives another solution of the PDE.

In the case of part (c) this can be illustrated by saying that

$$\alpha f + \beta g$$

is also a solution of $w_{xx} + w_{yy} = 0$.

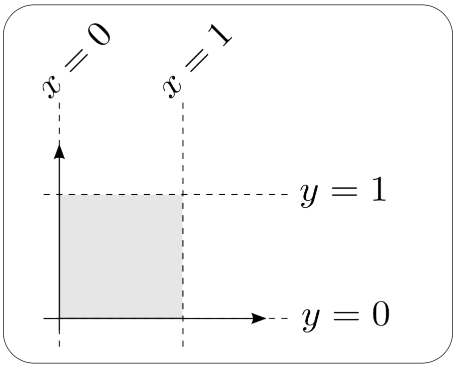
The principle of superposition applies to *linear* and *homogeneous* PDEs.

**Question 8**

In a 2-D region not containing any charges, the electric potential at a point $v(x, y)$ in the region satisfies Laplace's Equation:

$$v_{xx} + v_{yy} = 0$$

We will focus on the region defined by the unit square The separation of variables method assumes that the solution $v$ can be written as a product of two functions, each on a single variable, i.e. that we can write $v(x, y) = F(x)\,G(y)$.




(a) Show how separation of variables can convert the partial differential equation (PDE) into a pair of ordinary differential equations (ODEs) in $x$ and $y$:

$$F''(x) = kF(x) \qquad \text{and } G''(y) = -kG(y)$$

where you can assume that $k$ is a positive constant. You can use any suitable notation for the partial derivatives: $F''(x) = F_{xx} = \partial^2 F/\partial x^2$ and $G''(y) = G_{yy} = \partial^2 G/\partial y^2$.

**Solution:**

Assume that $v$ is separable, i.e. that $v(x,y) = F(x)G(y)$ for functions $F$ and $G$ that each depend on a single variable. We have $v_{xx} = F''(x)G(y)$ and $v_{yy} = F(x)G''(y)$ so that

$$F''G + FG'' = 0$$

Rearranging gives

$$\frac{F''}{F} = -\frac{G''}{G}$$

and for this equality of functions in different variables to work, we must have each side equal to a constant. So we can obtain a pair of ODEs setting this to a positive constant $k$ of our choice.

$$\frac{F''}{F} = k \qquad \text{and} \qquad \frac{G''}{G} = -k$$

This then provides the pair of ODEs in the form given by the question

$$F''(x) = kF(x) \qquad \text{and } G''(y) = -kG(y)$$

(b) Assume $k = 1$ in the previous part. Work out a general solution for each of the ODEs (which may include further constants) and hence write a general solution for $v(x,y)$.

**Solution:**

$k = 1$ in the previous part gives

$$F''(x) = F(x) \qquad \text{and} \qquad G''(y) = -G(y)$$

which gives, based on standard ODE solutions,

$$F(x) = Ae^{x} + Be^{-x} \qquad \text{and} \qquad G(y) = C\sin y + D\cos y$$

for some constants $A$, $B$, $C$, $D$. So the general solution can be written

$$v(x,y) = \left(Ae^{x} + Be^{-x}\right)(C\sin y + D\cos y)$$

*N.B.* An alternative solution for $G$ can be written as

$$G(y) = Ce^{iy} + De^{-iy}$$

for a suitably adjusted pair of constants $C$ and $D$, and this leads to an equivalent form of the general solution for $v(x,y)$ above.

$$v(x,y) = \left(Ae^{x} + Be^{-x}\right)\left(Ce^{iy} + De^{-iy}\right)$$

The use of trigonometric functions in the first form helps avoid the use of complex terms in the solution and can make the manipulation of the equation easier. Note that the constants $C$ and $D$ in the previous form would need to be different from the $C$ and $D$ in the trigonometric form.

(c) Some auxiliary information is given below for one of the square's edges, it can be used as a boundary condition for the PDE:

$$v(x, y) = 0 \text{ along the line } y = 0$$

Using the boundary condition, calculate what it implies for the constants in your general solution for $v(x, y)$ from the previous part. Show your working and update your solution to incorporate this information.

**Solution:**

The boundary condition states that $v(x, 0) = 0$. This can be written in full using the general solution from the previous part

$$0 = v(x, 0) = \left(Ae^{x} + Be^{-x}\right)\left(C\sin(0) + D\cos(0)\right)$$

which gives

$$0 = D\left(Ae^{x} + Be^{-x}\right)$$

As this equality is true for any $x$ value, we can conclude that $D = 0$.

So, based on the general solution in part (b), the condition that $D = 0$ implies that it can now be written

$$v(x, y) = C\sin y\ \left(Ae^{x} + Be^{-x}\right)$$

*N.B.* In case anyone used the complex form for $G$ earlier, the same form of solution can still be obtained Making the substitution into

$$v(x, y) = \left(Ae^{x} + Be^{-x}\right)\left(Ce^{iy} + De^{-iy}\right)$$

gives

$$0 = v(x, 0) = \left(Ae^{x} + Be^{-x}\right)(C + D)$$

so that $C + D = 0$ and we can substitute $D = -C$ to get

$$v(x, y) = C\left(Ae^{x} + Be^{-x}\right)\left(e^{iy} - e^{-iy}\right)$$

which is the same as the trigonometric form if we make the adjustment on the constant of $C \to C/2$.

(d) Here is some further possible auxiliary data. A measurement of the charge is made and it is found that $v = 0.1$ is taken at the point $(0, 0.5)$.

Show how you can find a set of constants for your general solution from the previous part that will make $v(x, y)$ satisfy this constraint. There will be many possible sets of constants, you just have to choose one set. It is better if no constant is set to zero.

**Solution:**

Many possible solutions.

The Equation $v(0, 0.5) = 0.1$ can be written out from the previous part as

$$0.1 = v(0, 0.5) = C\sin(0.5)\ \left(Ae^{0} + Be^{-0}\right)$$

i.e. that

$$0.1 = C\sin(0.5)\ (A + B)$$

We have three constants, $A$, $B$ and $C$, which need to satisfy a single constraint. So we can choose two of them freely and this will determine the third. For example, we can set $C = 1$ to give

$$0.1 = \sin(0.5)\ (A + B) \qquad \Rightarrow \qquad \frac{0.1}{\sin(0.5)} = A + B$$

We could choose anything for $A$ and determine $B$. If we choose to make $A$ and $B$ equal, then we have

$$A = B = \frac{1}{2}\frac{0.1}{\sin(0.5)} = \frac{0.05}{\sin(0.5)}$$

Many alternative solutions are possible, e.g. we could have started off by setting $C = 1/\sin(0.5)$ which would have led to $A + B = 0.1$ and in the latter equation we could have set $A = B = 0.05$ or $A = 0.01$ and $B = 0.09$ etc..

(e) *The following is not really an exam question. It is more of a practical task to use MatLab to check the above work.* Visualise your solution as we have done before for scalar fields. Use the functions meshgrid and contour or imagesc. After plotting ensure the correct orientation of your image array. It might help to use a colour bar to show the scale of values.

**Solution:**

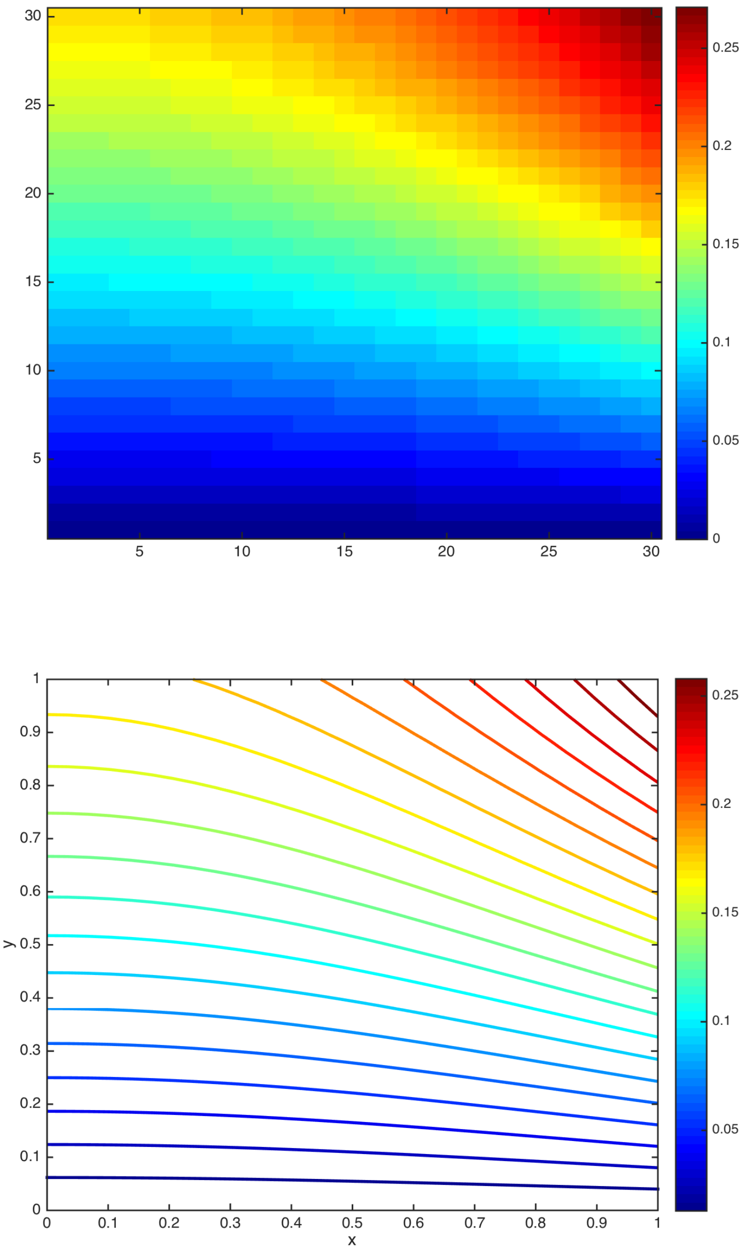
Using the choices above for $A$, $B$ and $C$, the code for calculating the solution can be
A = 0.05 / sin(0.5); B = A; C = 1; D = 0;

F = @(x) A * exp(x) + B * exp(-1 * x); G = @(y) C * sin(y) + D * cos(y);

Now we need a grid to calculate over N = 30;

x = linspace(0,1,N); y = linspace(0,1,N); [x,y] = meshgrid(x,y); Finally we can calculate the values of the function. v = F(x) .* G(y); Now some visualisation: imagesc(v) axis xy colorbar; colormap('jet') Or, we can use contour contour(x, y, v, 20, 'LineWidth', 2) colorbar xlabel('x') ylabel('y')

The results of the visualisation with the above choices are shown below




## Question 9

The partial differential equation (PDE) $4y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ is to be solved for $u(x, y)$. A curve $\vec{c}(t)$ in the $xy$ plane can be written

$$\vec{c}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

(a) Explain what is required for the curve $\vec{c}(t)$ to be a characteristic curve of the partial differential equation. Write an expression in the solution $u$ that represents this.

**Solution:**

1 mark: Reasonable description.

1 mark: Constraint on $u$.

$\vec{c}(t)$ is a characteristic curve if the solution $u$ is constant along the curve.

I.e., as $t$ varies, $u$ remains constant.

Therefore the constraint can be written as $\frac{du}{dt} = 0$.

(b) Using the chain rule or otherwise, derive an expression for the general form of the characteristic curves of the PDE $4y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$.

**Solution:**

1 mark: Application of chain rule expression to $\frac{du}{dt}$ .

1 mark: Identification terms in expression with PDE.

1 mark: Derivation of $\frac{dy}{dx}$ .

1 mark: Formula of characteristics with constant term.

The constraint $\frac{du}{dt} = 0$ can be written

$$\frac{du}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} = 0$$

Identifying with the PDE $4y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ gives

$$\frac{dx}{dt} = 4y \qquad \frac{dy}{dt} = 1$$

So

$$\frac{dy}{dx} = \frac{1}{4y}$$

which, through integration or otherwise, gives the characteristic curves

$$x = 2y^2 + A$$

for any constant $A$.

(c) Using the property of characteristic curves or otherwise, show that any function of $x - 2y^2$ satisfies the PDE. In other words, that $u(x, y) = g(x - 2y^2)$ where $g$ can be any differentiable function.

**Solution:**

4 marks: Any reasonable demonstration of the property of $u$. Two possible methods are outlined below. -1 mark for each incorrect statement.

Characteristics can be represented $x = 2y^2 + A$, so we can write them as $x - 2y^2 = A$. This means that the value of $x - 2y^2$ is constant on any characteristic. Any function of $x - 2y^2$ is therefore a function of a constant value along the curve. So, for a fixed choice of $A$, if $u = g(x - 2^y 2) = g(A)$, depends on $A$ alone, i.e. we can write it as $g(A)$ or, alternatively as $g(x - 2y^2)$.

More explicitly, we can use the expression of $u$ in the original PDE. Finding derivatives of $u = g(x - 2y^2)$,

$$\frac{\partial u}{\partial x} = g'(x - 2y^2)\frac{\partial}{\partial x}(x - 2y^2) = g'(x - 2y^2)$$

and

$$\frac{\partial u}{\partial y} = g'(x - 2y^2)\frac{\partial}{\partial y}(x - 2y^2) = -4y\, g'(x - 2y^2)$$

substituting into the PDE,

$$\begin{aligned} 4y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 4y\, g'(x - 2y^2) + \big(-4y\, g'(x - 2y^2)\big) \\ &= 0 \end{aligned}$$

i.e. $g(x - 2y^2)$ is a solution of the PDE for any $g$.

**Question 10**

(a) The partial differential equation $v_t + 3v\, v_x = 0$ can be classified as a first-order, non-linear and homogeneous PDE.

Provide a similar classification for the PDEs below in the functions $f(x, y)$, $g(x, t)$ and $w(x, y, z)$ respectively:

i. $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 5f = 0$

ii. $g_t - g\, g_x + \sin t = 0$

iii. $w_{xx} + (x - y)^2 w_{yy} - w_z \sin x = 0$

**Solution:**

i. $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 5f = 0$ : First-order, linear and homogeneous

ii. $g_t - g\, g_x + \sin t = 0$ : First-order, non-linear and non-homogeneous.

iii. $w_{xx} + (x - y)^2 w_{yy} - w_z \sin x = 0$ : Second-order, linear and homogeneous

A function $u = u(x,t)$, which depends on spatial location $x$ and time $t$, is known to satisfy the PDE

$$u_t - 2u_{xx} + 3u = 0$$

A numerical scheme is to be used to estimate a solution $u$ of the PDE. The scheme estimates $u$ at a discrete set of locations represented by an array $U(i,j)$ where $i$ indexes the spatial location and $j$ indexes the time point.

$\delta x$ denotes the distance between two adjacent array locations and $\delta t$ represents the time interval between successive estimates.

(b) Write a forward difference estimate for the first-order temporal derivative $u_t$ in terms of the elements of the array $U$.

**Solution:**

The $u_t$ estimate can be written

$$\frac{U(i,j+1) - U(i,j)}{\delta t}$$

(c) Work out a *central* difference estimate for second order spatial derivative $u_{xx}$ in terms of the elements of the array $U$.

**Solution:**

One possible way of writing the estimate of $u_{xx}$ is:

$$\frac{U(i+1,j) - 2\,U(i,j) + U(i-1,j)}{\delta x^2}$$

More than one way of deriving this expression is possible. One way is to use a Taylor expansion estimates for $u(x+\delta x, t)$ and $u(x - \delta x, t)$

$$u(x+\delta x, t) \approx u(x,t) + \delta x\, u_x(x,t) + \frac{1}{2}\delta x^2\, u_{xx}(x,t)$$

$$u(x-\delta x, t) \approx u(x,t) - \delta x\, u_x(x,t) + \frac{1}{2}\delta x^2\, u_{xx}(x,t)$$

and subsequently to add the expressions and re-arrange.

(d) Re-write the original PDE in terms of the finite difference expressions for the partial derivatives you obtained in the parts (b) and (c).

Re-arrange to obtain an update formula that can be used in a forward marching scheme to solve the PDE.

**Solution:**

The original PDE can be written

$$\begin{aligned}&\frac{U(i,j+1)-U(i,j)}{\delta t}\\&\quad-2\left[\frac{U(i+1,j)-2\,U(i,j)+U(i-1,j)}{\delta x^2}\right]\\&\quad+3\,U(i,j)=0\end{aligned}$$

A re-arrangement to a update formula for a forward marching scheme can be

$$U(i,j+1)=A\,\left[U(i+1,j)+U(i-1,j)\right]+B\,U(i,j)$$

where

$$A=\frac{2\,\delta t}{\delta x^2}\qquad\qquad B=1-\frac{4\,\delta t}{\delta x^2}+3\,\delta t$$

(e) A program is written to estimate the PDE solution for $0\le x\le L$ and $0\le t\le 1$ using the update formula of part (d). Describe one or more auxiliary conditions that could be used in order to carry this out.

**Solution:**

The solution may be estimated if the PDE is treated as an initial boundary-value problem.

One way to achieve this is to specify auxiliary conditions that consist of an initial condition specifying the value of $u(x,0)$ for $0\le x\le L$ and a boundary condition specifying the values of both $u(0,t)$ and $u(L,t)$ for $0\le t\le 1$.

An equivalent formulation can be given in terms of the discrete array referred to in the previous parts.

**Question 11**

The partial differential equation (PDE) $w_x+2xw_y=0$ is to be solved for $w(x,y)$. A curve in the $xy$ plane, parameterised by the variable $s$, can be written

$$\vec{r}(s)=\begin{pmatrix}x(s)\\y(s)\end{pmatrix}$$

(a) Explain what it means to say that the curve $\vec{r}(s)$ is a *characteristic curve* for the PDE. Write a constraint on the solution $w$ that expresses this.

**Solution:**

$\vec{r}(s)$ is a characteristic curve if the solution $w$ is constant along the curve.

I.e., as $s$ varies, $w$ remains constant.

Therefore the constraint can be written as $\frac{dw}{ds}=0$.

(b) Using the chain rule or otherwise, derive an expression for the general form of the characteristic curves of the PDE $w_x + 2xw_y = 0$.

**Solution:**

The constraint $\frac{dw}{ds} = 0$ can be written

$$\begin{aligned}\frac{dw}{ds} &= \frac{dx}{ds}\frac{\partial w}{\partial x} + \frac{dy}{ds}\frac{\partial w}{\partial y} \\ &= \frac{dx}{ds}w_x + \frac{dy}{ds}w_y = 0\end{aligned}$$

Identifying with the PDE gives

$$\frac{dx}{ds} = 1 \qquad \frac{dy}{ds} = 2x$$

So

$$\frac{dy}{dx} = 2x$$

which gives the form of the characteristic curves

$$y = x^2 + K$$

for any constant $K$.

(c) Sketch some of the characteristic curves for the PDE $w_x + 2xw_y = 0$.

You are given the auxiliary condition

$$w(x, y) = y + 1$$

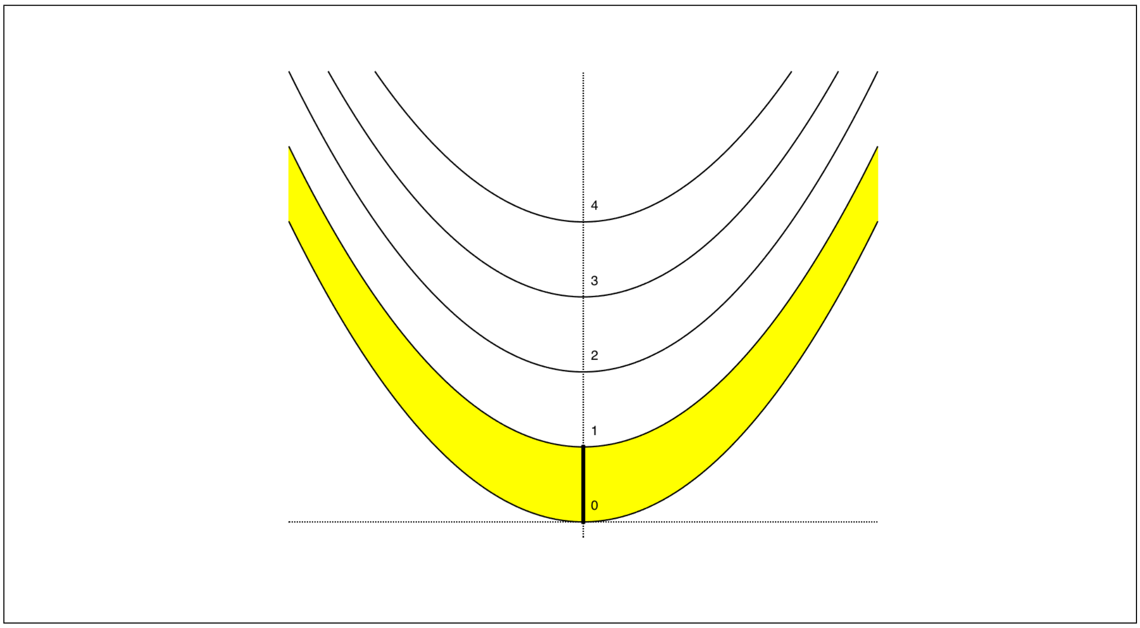
the line segment between (0,0) and (0,1). Shade the region on your plot where this condition will determine the solution of the PDE.

**Solution:**

Curves for some values of constant $K$.

The line segment where the condition applies is shown in bold.

The region where the solution is determined is shaded.




(d) Using the formula for the characteristics or otherwise, show that any function of $y - x^2$ satisfies the PDE, i.e. that we can write $w(x, y) = f(y - x^2)$ for any $f$.

**Solution:**

Any reasonable argument to demonstrate the result. Two possible approaches are given below.

As the characteristics can be represented $y = x^2 + K$, we can write them as $y - x^2 = K$ for different choices of $K$. A solution $w(x, y)$ is constant on each curve so depends on $K$ alone, i.e. we can write it as $f(K)$ or, alternatively as $f(y - x^2)$.

Alternatively, substituting $w(x, y) = f(y - x^2)$ into the PDE,

$$\begin{aligned} w_x + 2xw_y &= f'(y - x^2)\frac{\partial}{\partial x}(y - x^2) + 2xf'(y - x^2)\frac{\partial}{\partial y}(y - x^2) \\ &= f'(y - x^2)\,(-2x) \;+\; f'(y - x^2)\,2x\,(1) \\ &= f'(y - x^2)\,[\,-2x + 2x\,] = 0 \end{aligned}$$

i.e. $f(x - y^2)$ is a solution of the PDE for any $f$.

### Question 12

(a) Let $u(x, t)$ represents the density of a chemical in a long thin pipe at location $x$ at time $t$. The following equation represents a fundamental conservation law that we can use to derive partial differential equations (PDEs) in $u$

$$u_t(x, t) \;+\; \Phi_x(x, t) \;=\; f(x, t)$$

where $\Phi$ denotes the flux and $f$ represents a source term.

State the assumptions that need to be made about the flux term and the source term in order to derive the advection equation

$$u_t + cu_x = 0$$

Show in particular how the coefficient $c$ relates to these assumptions.

**Solution:**

1. We need to assume that the source term is zero, i.e.

$$f = 0$$

2. We need to assume that the flux $\Phi$ is proportional to the density $u$, i.e. that

$$\Phi = cu \Rightarrow \Phi_x = cu_x$$

I.e. the coefficient $c$ represents the constant of proportionality between the flux and density.

Substituting these into the conservation equation gives the advection equation.

Assume the advection coefficient is 3, i.e. that we have $u_t + 3u_x = 0$. We want to solve the equation on a discrete grid in the $(x, t)$ domain with distance measured in mm and time in seconds.

Let the solution be represented by the matrix $U$ where the element $U(i, j)$ represents the value at the $i^{\text{th}}$ location at the $j^{\text{th}}$ time step. Assume the locations have a spacing of 2mm and the time interval between successive values is 0.3 seconds.

(b) Write expression to approximate each of the partial derivatives, $u_t$ and $u_x$ at the location $(i, j)$.

Use a *forward* difference estimate for $u_t$ and a *backward* difference estimate for $u_x$. Write your estimates in terms of the elements of array $U$ and the grid spacing and time interval given.

**Solution:**

Using MatLab style index notation, the forward difference estimate for $u_t$ can be written as:

$$u_t \approx \frac{U(i, j+1) - U(i, j)}{0.3}$$

Backward difference estimate for $u_x$:

$$u_x \approx \frac{U(i, j) - U(i-1, j)}{2}$$

(c) Describe what is meant by a forward marching method for solving a PDE.

Derive a forward marching scheme for the advection PDE $u_t + 3u_x = 0$ using the partial derivative estimates obtained in part (b).

**Solution:**

A forward marching scheme can be used to solve a PDE with an iterative update formula. The iterative update should estimate the solution at one time point based on value(s) at a previous time point(s).

Substitution of first order partial derivative estimates into $u_t + 3u_x = 0$ gives

$$\frac{U(i,j+1) - U(i,j)}{0.3} + 3\left[\frac{U(i,j) - U(i-1,j)}{2}\right] \approx 0$$

which can be rearranged to give the update formula

$$U(i,j+1) \approx U(i,j) - 3 \times 0.3\left[\frac{U(i,j) - U(i-1,j)}{2}\right]$$

or

$$U(i,j+1) \approx U(i,j) - 0.45\left[U(i,j) - U(i-1,j)\right]$$

(d) Describe any suitable auxiliary conditions that allow a solution of the PDE $u_t + 3u_x = 0$ to be estimated with the forward marching scheme you obtained in part (c).

**Solution:**

The forward marching scheme estimates the value of $U(i,j+1)$ in terms of $U(i,j)$ and $U(i-1,j)$. So an initial condition for $j = 0$ and a boundary condition for $i = 0$ will suffice.

I.e. any conditions of the form $u(0,t) = f(t)$ and $u(x,0) = g(x)$ for some functions $f$ and $g$ will suffice (including constant valued functions).

**Question 13**

This question applies the Fourier transform to the heat equation in one spatial dimension, i.e. find $u = u(x,t)$ where, for some known function $h(x)$,

$$u_t = cu_{xx} \quad \text{the initial condition that} \quad u(x,0) = h(x)$$

(a) We can find the Fourier transform of the PDE with respect to $x$ or with respect to $t$. One of these will lead to an easier system to solve in the Fourier domain.

Decide which of these leads to an easier equation in the Fourier domain and prove this by showing both routes. You do not have to solve the system yet.

**Solution:**

Applying a Fourier transform with respect to $x$ we get

$$\int_{-\infty}^{\infty} u_t(x,t)\, e^{-i\,\omega\, x}\, dx = \int_{-\infty}^{\infty} cu_{xx}(x,t)\, e^{-i\,\omega\, x}\, dx$$

Applying the Leibniz rule for differentiation under the integral sign

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\, e^{-i\,\omega\, x}\, dx = \int_{-\infty}^{\infty} cu_{xx}(x,t)\, e^{-i\,\omega\, x}\, dx$$

Now apply the Fourier transform for the $x$ variable to both sides to get

$$\frac{\partial \tilde{u}(\omega,t)}{\partial t} = -c\omega^2 \tilde{u}(\omega,t)$$

Which gives a first-order PDE in the function $\tilde{u}$.

Now we can look at applying a Fourier transform with respect to $t$ we get

$$\int_{-\infty}^{\infty} u_t(x,t)\, e^{-i\,\omega\,t}\, dt = \int_{-\infty}^{\infty} cu_{xx}(x,t)\, e^{-i\,\omega\,t}\, dt$$

Applying the Leibniz rule for differentiation under the integral sign

$$\int_{-\infty}^{\infty} u_t(x,t)\, e^{-i\,\omega\,t}\, dt = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} cu(x,t)\, e^{-i\,\omega\,t}\, dt$$

Now apply the Fourier Transform to both sides for the $t$ variable:

$$i\theta\tilde{u}(x,\theta) = c\frac{\partial^2\tilde{u}(x,\theta)}{\partial x^2}$$

Which gives a *second-order* PDE in the function $\tilde{u}$.

In conclusion, applying the Fourier transform with respect to $x$ leads to the easier PDE to solve because it is first order.

(b) For the Fourier transform that leads to the easier system, solve the resulting equation in the Fourier domain. Recall that the solution may contain a constant type term that actually depends on a variable other than the main variable in the equation (See notes).

**Solution:**

Taking the first of the transformed PDEs in the previous part:

$$\frac{\partial\tilde{u}(\omega,t)}{\partial t} = -c\omega^2\tilde{u}(\omega,t)$$

Write $U(t) = \tilde{u}(\omega,t)$ for a fixed value of $\omega$ to give

$$\frac{\partial U(t)}{\partial t} = -c\omega^2 U(t)$$

which is a standard ODE in $U(t)$ that can be solved by

$$U(t) = A\, e^{-c\omega^2\, t}$$

This was for a fixed value of $\omega$, so we should stress that the 'constant' $A$ can actually depend on $\omega$, i.e.

$$U(t) = A(\omega)\, e^{-c\omega^2\, t}$$

This represents our solution in the Fourier domain.

(c) Show how the Fourier transform of $h(x)$ given in the initial condition relates to the constant type term in the Fourier domain equation of the previous part.

**Solution:**

As we are using a Fourier transform with respect to $x$, if we apply the same transform to our initial condition $u(x,0) = h(x)$ we obtain the direct Fourier transform of $h$ which we write as $\tilde{h}(\omega)$.

This applies when $t = 0$, so using our Fourier domain solution from the previous part and setting $t = 0$ we have

$$\tilde{h}(\omega) = U(0) = A(\omega)\, e^{-c\omega^2 \times 0} = A(\omega)$$

Therefore we have shown that the 'constant' type term $A(\omega)$ actually equals the Fourier transform of the function $h(x)$, which defines the initial condition.

**Question 14**

An advection partial differential equation in one dimension is written as an initial value problem

$$\begin{aligned} &\text{find } u(x,t) \text{ satisfying } u_t + 0.5u_x = 0 \\ &\qquad\qquad \text{where } u(x,0) = \sin(\pi x), \qquad \text{for } 0 \le x \le 2 \end{aligned} \tag{1}$$

$\vec{r}(s) = (x(s), t(s))^T$ is a curve in the $xt$ plane that is parametrised by $s$, i.e. as $s$ varies, we can move along the curve.

(a) Explain what it means for the curve $\vec{r}(s)$ to be described as a *characteristic* for the PDE.

**Solution:**

$\vec{r}(x)$ is a characteristic curve for the PDE is if the solution $u(x,y)$ is constant as we move along it.

(b) A solution for the PDE given by Equation 1 is constrained along a characterstic. Write a mathematical expression for this constraint and then use it to find a general expression for the family of *all* characteristic curves for the given advection PDE.

**Solution:**

Along a characteristic curve, $\vec{r}(s)$, we have $u$ is constant so that

$$\frac{du}{ds} = 0$$

so that

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}$$

The PDE is written as $\frac{\partial u}{\partial t} + 0.5\frac{\partial u}{\partial x} = 0$ so we can make the identification

$$\frac{dt}{ds} = 1 \qquad \text{and} \qquad \frac{dx}{ds} = 0.5$$

which gives $\frac{dx}{dt} = 0.5$ on a characteristic. So that $x = 0.5t + K$ for some constant $K$.

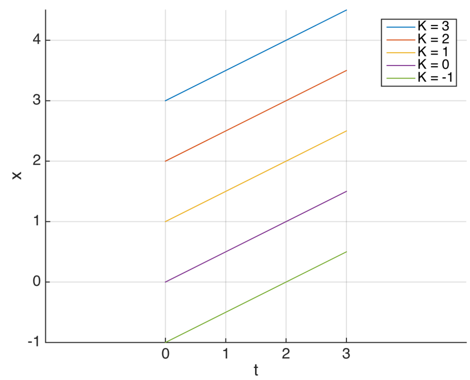
I.e. along any 'curve' of the form $x = 0.5t + K$, the solution will be constant. The family of characteristic curves is determined by the constant $K$.

Each curve can be written as $x - 0.5t = K$.

(c) Sketch a diagram of the $xt$ plane showing some of the characteristics of the advection equation above.

**Solution:**

Sketch showing some of the 'curves' in the $K$ parameter family $x - 0.5t = K$.

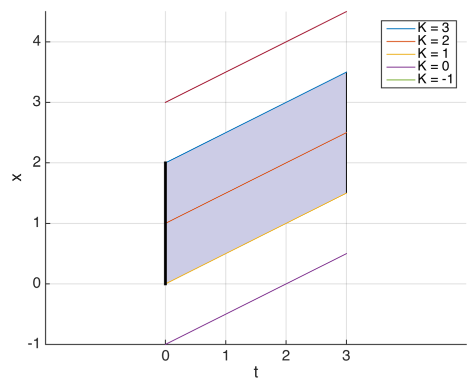



(d) You are given that the advection equation has an initial condition on the function $u$. This initial condition is that $u(x, 0) = \sin(\pi x)$, for $0 \leq x \leq 2$

Sketch a diagram to illustrate the region of the $xt$ plane where the solution is determined by this initial condition.

**Solution:**

The region is defined as the intersection of the characteristic curves with the line segment where the boundary condition applies (i.e. $0 \leq x \leq 2$). The line segment is shown in bold in the $t - x$ plot below and the regions is shaded up to $t = 3$.




(e) Write down the particular solution for the advection equation that applies in this region.

**Solution:**

We have the function $\sin(\pi x)$ satisfying the PDE when $t = 0$ and characteristic curves of the form $x - 0.5t = K$. So we can make the substitution $x \leftarrow x - 0.5t$ to obtain a particular solution in the region where the solution is determined by the initial condition. I.e. we have

$$u(x, t) = \sin(\pi(x - 0.5t))$$

which matches the initial condition on the line segment $0 \leq x \leq 2$ when $t = 0$.