Computational Methods
Sample Exam Questions

PDEs: Sample maths practice questions

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The questions below are intended to help you practice the maths required for the PDE parts of the module. There is also some practice in the use of MatLab functions that are useful for tasks such as visualising scalar and vector fields.

Question 1

A scalar function ϕ and a vector field $\vec{v}$ are defined as

$$\phi(x, y, z) = x \sin z \qquad \vec{v}(x, y, z) = \begin{pmatrix} x + y \\ z^2 \\ 2y \end{pmatrix}$$

(a) Write an expression for a new vector field obtained by multiplying ϕ and $\vec{v}$

> **Solution:**
>
> $$\phi\vec{v} = \begin{pmatrix} (x + y)\, x \sin z \\ z^2\, x \sin z \\ 2y\, x \sin z \end{pmatrix}$$

(b) Find the divergence of the new vector field, i.e. find $\nabla \cdot (\phi\vec{v})$

> **Solution:**
>
> $$\begin{aligned} \nabla \cdot (\phi\vec{v}) &= \nabla \cdot \begin{pmatrix} (x + y)\, x \sin z \\ z^2\, x \sin z \\ 2y\, x \sin z \end{pmatrix} \\ &= \frac{\partial}{\partial x}\left((x + y)\, x \sin z\right) + \frac{\partial}{\partial y}\left(z^2\, x \sin z \right) + \frac{\partial}{\partial z}\left(2y\, x \sin z \right) \\ &= (2x + y)\, \sin z + 0 + 2xy \cos z \\ &= (2x + y)\, \sin z + 2xy \cos z \end{aligned}$$
>
> Answer

(c) Calculate expressions for the gradient of ϕ and for the divergence of $\vec{v}$, i.e. for $\nabla\phi$ and $\nabla \cdot \vec{v}$

Solution:

$$
\begin{aligned}
\nabla\phi &= \begin{pmatrix} \frac{\partial}{\partial x}\,(x\,\sin z) \\ \frac{\partial}{\partial y}\,(x\,\sin z) \\ \frac{\partial}{\partial z}\,(x\,\sin z) \end{pmatrix} \\
&= \hat{\imath}(\sin z) \,+\, \hat{\jmath}(0) \,+\, \hat{k}(x\cos z) \\
&= \hat{\imath}\sin z \,+\, \hat{k}x\cos z
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\vec{v} &= \nabla\cdot\begin{pmatrix} x+y \\ z^2 \\ 2y \end{pmatrix} \\
&= \frac{\partial}{\partial x}(x+y) \,+\, \frac{\partial}{\partial y}(z^2) \,+\, \frac{\partial}{\partial z}(2y) \\
&= 1+0+0 = 1
\end{aligned}
$$

Answer

(d) Use your answers from part (c) to verify that the divergence of the product obtained in part (b) satisfies the following formula

$$\nabla\cdot(\phi\,\vec{v}) \;=\; \nabla\phi\cdot\vec{v} + \phi\;\nabla\cdot\vec{v}$$

Solution:

From part (b), we have the divergence of the product, which is the left hand side of the formula:

$$\nabla\cdot(\phi\vec{v}) = (2x+y)\,\sin z + 2xy\cos z$$

Now we use the answers from part (c) to calculate the right hand side:

$$
\begin{aligned}
\nabla\phi\cdot\vec{v} + \phi\;\nabla\cdot\vec{v} &= \Big(\hat{\imath}\sin z \,+\, \hat{k}x\cos z\Big)\cdot\vec{v} + (\phi)\;(1) \\
&= \Big(\hat{\imath}\sin z \,+\, \hat{k}x\cos z\Big)\cdot\Big((x+y)\hat{\imath} + z^2\hat{\jmath} + 2y\hat{k}\Big) + (x\sin z)\;(1) \\
&= (x+y)\sin z + 2xy\cos z + x\sin z \\
&= (2x+y)\sin z + 2xy\cos z
\end{aligned}
$$

The last expression is equal to the expression from the left hand side, which verifies the formula.

Question 2

(a) A vector field defined in 2-D by the formula

$$\vec{f}(x,y) = \frac{1}{5}\begin{pmatrix} -x \\ y \end{pmatrix}$$

A grid of coordinates is defined at integer locations (x_i, y_j) where both x_i and y_j values range from -2 to 3.

Write a MatLab script to visualise this vector field.

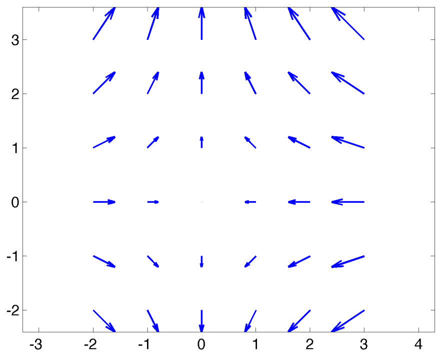
In your script, define a suitable grid of x and y coordinates using the `meshgrid` function (See `doc meshgrid` and the Manual for the Computer Programming course chapter 8).

Define two arrays `u` and `v` to contain the x and y components of the vector field function $\vec{f}$.

Read `doc quiver` and work out how to visualise the vector field for the grid of points in the range given.

Solution:

Illustration using a step size of 1 in the x and y directions for the meshgrid call. Different step sizes will give variations but overall shape should be the same.

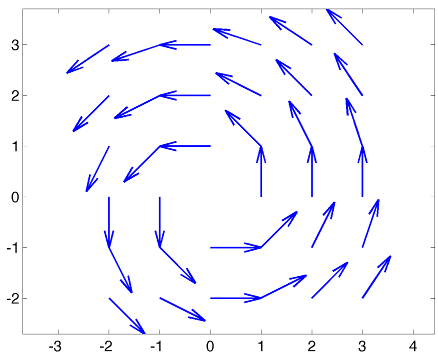


(b) Repeat the last part with the vector field

$$\vec{g}(x, y) = \frac{-y\,\hat{i} + x\,\hat{j}}{\sqrt{x^2 + y^2}}$$

Solution:

Again, an illustration with a step size of 1.



Question 3

A scalar field ϕ is given by

$$\phi(x, y, z) = (x + y^2)\,\sin z$$

(a) Find the gradient of ϕ.

Solution:

$$\begin{aligned}\nabla\phi &= \frac{\partial}{\partial x}((x+y^2)\ \sin z)\,\hat{\imath} + \frac{\partial}{\partial y}((x+y^2)\ \sin z)\,\hat{\jmath} + \frac{\partial}{\partial z}((x+y^2)\ \sin z)\,\hat{k} \\ &= \sin z\,\hat{\imath} + 2y\ \sin z\,\hat{\jmath} + (x+y^2)\ \cos z\,\hat{k}\end{aligned}$$

(b) The result of the part (a) provides a vector field. Work out the divergence of this vector field. What is the common name for the divergence of a gradient?

Solution:

$$\begin{aligned}\nabla\cdot\nabla\phi &= \nabla\cdot\left[\sin z\,\hat{\imath} + 2y\ \sin z\,\hat{\jmath} + (x+y^2)\ \cos z\,\hat{k}\right] \\ &= \left[\frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}\right]\cdot\left[\hat{\imath}\sin z + 2y\ \sin z\,\hat{\jmath} + (x+y^2)\ \cos z\,\hat{k}\right] \\ &= \frac{\partial}{\partial x}(\sin z) + \frac{\partial}{\partial y}(2y\ \sin z) + \frac{\partial}{\partial z}((x+y^2)\ \cos z) \\ &= 0 + 2\sin z - (x+y^2)\ \sin z \\ &= (2-x-y^2)\ \sin z\end{aligned}$$

The Laplacian is the common term for $\nabla\cdot\nabla\phi$, which can also be written $\nabla^2\phi$ or $\Delta\phi$

(c) Work out the curl of the vector field result from part (a).

You should notice something about the result. Explain why it happens.

Solution:

$$\begin{aligned}\nabla\times\nabla\phi &= \nabla\times\left[\hat{\imath} + 2y\ \sin z\,\hat{\jmath} + y^2\ \cos z\,\hat{k}\right] \\ &= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & 2y\ \sin z & (x+y^2)\ \cos z\end{vmatrix}\end{aligned}$$

so

$$\begin{aligned}\nabla\times\nabla\phi = \hat{\imath}&\left[\frac{\partial}{\partial y}((x+y^2)\ \cos z) - \frac{\partial}{\partial z}(2y\ \sin z)\right] \\ -\,\hat{\jmath}&\left[\frac{\partial}{\partial x}((x+y^2)\ \cos z) - \frac{\partial}{\partial z}(\sin z)\right] \\ +\,\hat{k}&\left[\frac{\partial}{\partial x}(2y\ \sin z) - \frac{\partial}{\partial y}(\sin z)\right] \\ = \hat{\imath}&\ [(2y\ \cos z) - 2y\ \cos z] - \hat{\jmath}\ [\cos z - \cos z] + \hat{k}\ [0-0] \\ = 0&\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} \\ = \vec{0}&\end{aligned}$$

The result of evaluating the curl is the zero vector $\vec{0}$. This is expected for the curl of any gradient field. I.e. all gradient fields are irrotational.

Question 4

(a) An unknown scalar field in 2-D, $h(x, y)$ has the following gradient

$$\nabla h = \begin{pmatrix} 6x + y \\ x + \cos y \end{pmatrix}$$

Find an expression for $h(x, y)$ that has this gradient. The expression may contain an arbitrary constant.

Hint: see the gradient examples given in the PDE notes on KEATS.

Solution:

We have $h_x = 6x + y$. Integrating this with respect to x gives an expression for h

$$\frac{\partial h}{\partial x} = 6x + y \Rightarrow h = \int 6x + y \, dx = 3x^2 + xy + g(y)$$

where $g(y)$ is an unknown function that depends on y only.

Differentiating this expression with respect to y gives

$$h_y = \frac{\partial}{\partial y}\left(3x^2 + xy + g(y)\right) = x + g'(y)$$

We are given that $h_y = x + \cos y$, so we can equate

$$x + g'(y) = x + \cos y \Rightarrow g(y) = \sin y \, + \, C$$

For some constant C.

Putting this together gives h as

$$h(x, y) = 3x^2 + xy + g(y) = 3x^2 + xy + \sin y \, + \, C$$

(b) A function $g(x, y, z)$ has a gradient given by

$$\nabla g = y \, \hat{\imath} + (x + e^y) \, \hat{\jmath} + 2z \, \hat{k}$$

Find an expression for g in terms of x, y and z that satisfies this equation (up to a constant).

Solution:

We have $g_x = y$ so that $g = xy + u(y, z)$ where u is an unknown function that does not depend on x and only depends on y and z.

Differentiating this expression with respect to y gives

$$g_y = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(xy + u(y, z)\right) = x + \frac{\partial u}{\partial y}$$

The given gradient tells us that $g_y = x + e^y$ so we can equate

$$x + e^y = x + \frac{\partial u}{\partial y} \Rightarrow e^y = \frac{\partial u}{\partial y}$$

Integrating the partial derivative of the function $u(y, z)$ with respect to y gives

$$\int \frac{\partial u}{\partial y} dy = \int e^y dy \Rightarrow u(y, z) = e^y + v(z)$$

For some yet-to-be-determined function v that depends only on z and does not depend on y.

Substituting for $u(y, z)$ in what we have so far for $g(x, y, z)$ gives

$$g(x, y, z) = xy + e^y + v(z)$$

Now we can differentiate with respect to z and equate with what the gradient tells we should get for g_z (which is $g_z = 2z$)

$$\frac{\partial g}{\partial z} = \frac{\partial}{\partial z}\left(xy + e^y + v(z)\right) = 0 + 0 + v'(z) = v'(z)$$

So we know that $v'(z) = 2z$ from which we obtain $v(z) = z^2 + C$ for some constant C.

Substituting this in gives

$$g(x, y, z) = xy + e^y + v(z) = xy + e^y + z^2 + C$$

which determines g up to an arbitrary constant.

Question 5

This question requires you to make up your own scalar function and visualise it using MatLab on a suitable grid of points.

(a) Make up scalar function ϕ that depends on two variables, if we call them x and y we have $\phi = \phi(x, y)$. Write your answer as a mathematical formula and as a MatLab command(s).

Solution:

Many possible answers. Ensure mathematical expression and MatLab formulation match. Ensure correct use of element-wise operations in the MatLab formulation.

(b) Choose a grid of points over which to visualise the function. This grid should be suitable in that the function should vary sufficiently over the range of x and y values represented by the grid.

Visualise the function on the grid using the `imagesc` function. You can save the visualisation using a command like the following

```
print('-r200', '-dpng', 'myFileName.png')
```

This allows you to make an image file. This can be useful when saving an image for an electronic submission or to simply to print out a plot you have made.

Solution:

The result will depend on the choice of function which should demonstrate a reasonable amount of variation over the window in which it is visualised and also not vary to significantly (i.e. without very large or very small values which make for difficult visualisation).

See notes and slides from lecture for examples.

The resolution of the grid should also be appropriate, it should not really lead to a blocky appearance.

(c) Now write code to visualise the components of the *gradient* of your function. You can make *two* images, one for each partial derivative. See the example in the slides for Lecture 14 as a hint, the example used the built in `grad` function which you should check in `doc`.

Solution:

Gradient should match function given in previous part. MatLab calls to gradient function should be checked for correctness and should incorporate the correct spacings in the x and y directions.

See notes and slides from lecture for examples.

(d) Visualise the gradient in one step using the `quiver` function to show it as a set of arrows.

Try to overlay this on to the original scalar field. Again, have a look at the examples in the slides to get an idea of what to aim for.

Question 6

Consider the following functions

(i) $f(x, y) = x^2\, y$
(ii) $g(x, y) = x^2 + y^2 + 4$
(iii) $h(x, y) = \sin\,(x\,/\,y)$
(iv) $m(x, y) = \arctan\,(y\,/\,x)$
(v) $p(u, v, w) = 5uv + (1 - v)^2 e^{3w}$

(a) Find all the first partial derivatives for the functions above.

Solution:

Marks as shown in parts, -1 for each error.

(i) $f(x, y) = x^2\, y$

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2$$

(ii) $g(x, y) = x^2 + y^2 + 4$

$$\frac{\partial g}{\partial x} = 2x \qquad \frac{\partial g}{\partial y} = 2y$$

(iii) $h(x,y) = \sin(x\,/\,y)$

$$\frac{\partial h}{\partial x} = \frac{1}{y}\cos(x\,/\,y) \qquad \frac{\partial h}{\partial y} = \frac{-x}{y^2}\cos(x\,/\,y)$$

(iv) $m(x,y) = \arctan(y\,/\,x)$

$$\begin{aligned}\frac{\partial m}{\partial x} = \frac{\partial}{\partial x}\arctan(y\,/\,x) &= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\\ &= \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)\\ &= \frac{-y}{x^2+y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial m}{\partial y} = \frac{\partial}{\partial y}\arctan(y\,/\,x) &= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\\ &= \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)\\ &= \frac{x}{x^2+y^2}\end{aligned}$$

(v) $p(u,v,w) = 5uv + (1-v)^2 e^{3w}$

$$\frac{\partial p}{\partial u} = 5v$$

$$\begin{aligned}\frac{\partial p}{\partial v} &= 5u + 2(1-v)(-1)e^{3w}\\ &= 5u + 2(v-1)e^{3w}\end{aligned}$$

$$\frac{\partial p}{\partial w} = 3(1-v)^2 e^{3w}$$

(b) For (i) and (iii), find

$$\frac{\partial^2(\cdot)}{\partial x^2} \qquad \frac{\partial^2(\cdot)}{\partial x \partial y} \qquad \frac{\partial^2(\cdot)}{\partial y^2}$$

where $(\cdot)$ can be replaced by f or h.

Solution:

Marks as shown in parts, -1 for each error.

(i)

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2xy) = 2y \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x^2\right) = 0$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2\right) = 2x$$

(iii)

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{y}\cos\left(x\,/\,y\right)\right) = \frac{-1}{y^2}\sin\left(x\,/\,y\right)$$

Using the product rule, we can find the second partial derivative w.r.t. y:

$$\begin{aligned}\frac{\partial^2 h}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{-x}{y^2}\cos\left(x\,/\,y\right)\right)\\ &= \frac{2x}{y^3}\cos(x/y) + \left(\frac{x}{y^2}\right)\left(\frac{-x}{y^2}\right)\sin(x/y)\\ &= \frac{2x}{y^3}\cos(x/y) - \frac{x^2}{y^4}\sin(x/y)\\ &= \frac{x}{y^4}\left(\,2y\cos(x/y) - x\sin(x/y)\,\right)\end{aligned}$$

Using a similar approach, we can obtain:

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{-x}{y^2}\cos\left(x\,/\,y\right)\right) = \frac{1}{y^3}\left[x\sin(x/y) - y\cos(x/y)\right]$$

(c) For (iv), verify that the second-order mixed partials are equal.

$$\frac{\partial^2 m}{\partial x \partial y} = \frac{\partial^2 m}{\partial y \partial x}$$

Solution:

$$\frac{\partial m}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial m}{\partial y} = \frac{x}{x^2+y^2}$$

So

$$\begin{aligned}\frac{\partial^2 m}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial m}{\partial y}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\\ &= \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 m}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial m}{\partial x}\right)\\ &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\\ &= \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

Which shows that the mixed partials are equal and that the order of differentiation does not matter.

Question 7

(a) For each of the functions below, state what it takes as input, what it gives as output and which type of function it is, a vector field, a scalar field, or a curve.

(i) $\vec{m}(x, y) = (\, x \,, -y \,)^T$ for $(x, y) \in \mathbb{R}^2$

(ii) $\vec{g}(s) = \hat{\imath}\cos s \, + \hat{\jmath}\sin s \, + \hat{k}s$

(iii) $\vec{a}(\mu) = (2\mu, \, \cos\mu, \, -1)$ for $\mu \in [0, 10]$.

(iv) $\vec{b}(x, y, z) = (x + y)\hat{\imath} + (y - z)^2\hat{k}$

(v) $\vec{d}(\vec{r}) = r\vec{r}$ where $\vec{r} = (x, y)^T \in \mathbb{R}^2$ and $r = \|\vec{r}\|$.

(vi) $h(\vec{x}) = \left[2\hat{\imath} - \hat{\jmath} - \hat{k}\right] \cdot \vec{x}$ where $\vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \in \mathbb{R}^3$.

Solution:

(i) $\vec{m}(x, y) = (\, x \,, -y \,)^T$ for $(x, y) \in \mathbb{R}^2$

- input: Position vector (x, y)
- output: Vector
- type: **Vector field**

(ii) $\vec{g}(s) = \hat{\imath}\cos s \, + \hat{\jmath}\sin s \, + \hat{k}s$

- input: Scalar, s
- output: Vector
- type: **Curve**

(iii) $\vec{a}(\mu) = (2\mu, \, \cos\mu, \, -1)$ for $\mu \in [0, 10]$.

- input: Scalar
- output: Vector
- type: **Curve**

(iv) $\vec{b}(x, y, z) = (x + y)\hat{\imath} + (y - z)^2\hat{k}$

- input: Vector
- output: Vector
- type: **Vector field**

(v) $\vec{d}(\vec{r}) = r\vec{r}$ where $\vec{r} = (x, y)^T \in \mathbb{R}^2$ and $r = \|\vec{r}\|$.

- input: Vector
- output: Vector
- type: **Vector field**

(vi) $h(\vec{x}) = \left[2\hat{\imath} - \hat{\jmath} - \hat{k}\right] \cdot \vec{x}$ where $\vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \in \mathbb{R}^3$.

- input: Position Vector
- output: Scalar
- type: **Scalar field**

Show all your working for the following parts and express the results in terms of the 'Del' operator ∇ and how it is applied to each function.

(b) Calculate the gradient of the scalar field $f(x, y, z) = x + z \sin y$.

Solution:

$f(x, y, z) = x + z \sin y$:

So

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= (1)\hat{i} + (z \cos y)\hat{j} + (\sin y)\hat{k} \\ &= \hat{i} + \hat{j} z \cos y + \hat{k} \sin y\end{aligned}$$

which can be written

$$\nabla f = \begin{pmatrix} 1 \\ z \cos y \\ \sin y \end{pmatrix}$$

$$\nabla f = (1\,,\, z \cos y\,,\, \sin y)^T$$

(c) Find the Laplacian of the scalar field f of part b.

Solution:

The Laplacian can be obtained from the divergence of the gradient.

$$\begin{aligned}\Delta f = \nabla \cdot \nabla f = \nabla \cdot &\begin{pmatrix} 1 \\ z \cos y \\ \sin y \end{pmatrix} \\ &= \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(z \cos y) + \frac{\partial}{\partial z}(\sin y) + \\ &= 0 + (-z \sin y) + 0 \\ &= -z \sin y\end{aligned}$$

Question 8

This question is similar to the exercise on visualising a scalar field but this time you will need to visualise a vector field.

Make up your own vector field $\vec{v}$ that depends on two spatial variables, x and y. In other words, we have $\vec{v} = (v_1(x, y), v_2(x, y))^T$ and you need to make up *two* scalar fields, one for each component.

For this question, you will need to use the `meshgrid` and `quiver` functions to visualise the vector field. See `doc quiver` to learn how to use this function.

Hint: choose a suitable vector field function and a good grid to visualise it on, i.e. one where the vector field shows a reasonable amount of variation. Credit will be given for a vector field that not a simple uniform field, i.e. one that genuinely does vary over the region.

See slides and notes for examples.

Solution:

Many possible answers. Correct use of quiver function is needed and appropriate use of scaling if necessary.

See slides and notes for examples.