Module-IV

Some Special Types of Matrices

Symmetric matrix: A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g.$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Skew-Symmetric matrix: A square matrix is called symmetric matrix if $A = -A^T$

i.e. $a_{ij} = -a_{ji}$. The diagonal elements of a skew-symmetric matrix are zero because $a_{ii} = -a_{ii}$ if and only if $a_{ii} = 0$

e.g. $\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

Conjugate Matrix: A matrix $\bar{A}$ obtained by replacing all the elements of matrix A by their conjugate numbers is called conjugate matrix of A.

e.g. $A = \begin{bmatrix} 1+3i & 2 & 2-5i \\ 2 & 1 & -2+4i \\ 2+6i & -2 & 1+i \end{bmatrix}$

Then

$$\bar{A} = \begin{bmatrix} 1-3i & 2 & 2+5i \\ 2 & 1 & -2-4i \\ 2-6i & -2 & 1-i \end{bmatrix}$$

Tranjugate Matrix: Transpose of conjugate matrix is called tranjugate matrix.

$$A^{\theta} \quad or \quad A^{*} = \left(\overline{A}\right)' = \begin{bmatrix} 1-3i & 2 & 2-6i \\ 2 & 1 & -2 \\ 2+5i & -2-4i & 1-i \end{bmatrix}$$

Orthogonal matrix: A square matrix A is said to be orthogonal if

$$AA^{T} = A^{T}A = I.$$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Unitary matrix: A square matrix A is said to be Unitary if

$$A^{\theta}A = AA^{\theta} = I$$

where $A^{\theta} = \left(\overline{A}\right)^{T}$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Note: Every orthogonal matrix is unitary.

Hermitian matrix: A square matrix A is said to be Hermitian matrix if

$A^{\theta} = A$ i.e. $a_{ij} = \overline{a_{ji}}$

Diagonal elements of a Hermitian matrix are real numbers.

e.g. $A = \begin{bmatrix} 1 & 2+3i & 5-6i \\ 2-3i & 2 & 9-6i \\ 5+6i & 9+6i & -11 \end{bmatrix}$

Skew Hermitian matrix: A square matrix A is said to be skew Hermitian matrix if

$A^{\theta} = -A$ i.e. $a_{ij} = -\overline{a_{ji}}$

Diagonal elements of a skew Hermitian matrix

are either zero or purely imaginary numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & -5-6i \\ -2+3i & 2 & -9+6i \\ 5-6i & 9+6i & -11 \end{bmatrix}$$

Similar matrices: A square matrix A is said to be similar to a square matrix B if there exists an invertible matrix P such that $A = P^{-1}BP$. P is called similarity matrix.

This relation of similarity is a symmetric relation.

1. Prove $\left(A^{\theta}\right)^{\theta} = A$

Proof: $\left(A^{\theta}\right)^{\theta} = \overline{(\bar{A}')'} = \overline{\left(\overline{(A')'}\right)} = \overline{(\bar{A})} = A$

2. Prove that $A + A^{\theta}$ is a Hermitian matrix.

Proof: Now

$$\left(A+A^{\theta}\right)^{\theta}=A^{\theta}+\left(A^{\theta}\right)^{\theta}=A^{\theta}+A=A+A^{\theta}$$

Hence $A+A^{\theta}$ is a Hermitian matrix.

3. Prove that $A-A^{\theta}$ is a Skew-Hermitian matrix.

Proof: Now

$$\left(A-A^{\theta}\right)^{\theta}=A^{\theta}-\left(A^{\theta}\right)^{\theta}=A^{\theta}-A=-\left(A-A^{\theta}\right)$$

Hence $A-A^{\theta}$ is a Skew-Hermitian matrix.

4. Prove that every square matrix A can be expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

Proof: A can be written as

$$A=\frac{1}{2}\left(A+A^{\theta}\right)+\frac{1}{2}\left(A-A^{\theta}\right) \text{ ----------- (I)}$$

Claim 1: $\frac{1}{2}\left(A+A^{\theta}\right)$ is a Hermitian matrix

Pf: Now

$$\frac{1}{2}\left(A+A^{\theta}\right)^{\theta}=\frac{1}{2}\left(A^{\theta}+\left(A^{\theta}\right)^{\theta}\right)=\frac{1}{2}\left(A^{\theta}+A\right)=$$

$$\frac{1}{2}\left(A+A^{\theta}\right)$$

Hence $\frac{1}{2}\left(A+A^{\theta}\right)$ is a Hermitian matrix.

Claim 2: $\frac{1}{2}\left(A-A^{\theta}\right)$ is a Skew-Hermitian matrix

Pf: Now

$$\frac{1}{2}\left(A-A^{\theta}\right)^{\theta}=\frac{1}{2}\left(A^{\theta}-\left(A^{\theta}\right)^{\theta}\right)=\frac{1}{2}\left(A^{\theta}-A\right)=$$

$$-\frac{1}{2}\left(A-A^{\theta}\right)$$

Hence $\frac{1}{2}\left(A+A^{\theta}\right)$ is a Skew-Hermitian matrix.

From (I), Claim 1 and Claim 2 we get that A has been expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

5. Prove eigen value of a Hermitian matrix is real.

Sol. Let A be a Hermitian matrix.

Therefore $A^{\theta} = A - - - - - - -(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{yields} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{yields} X^{\theta} A^{\theta} = \bar{\alpha} X^{\theta}$$

$\xrightarrow{yields} X^{\theta} A = \bar{\alpha} X^{\theta}$ (using (1))

Post multiplying both sides by X, we get

$$X^{\theta}(AX) = \bar{\alpha}\left(X^{\theta} X\right) \xrightarrow{yields} X^{\theta}\ \alpha X = \bar{\alpha}\left(X^{\theta} X\right)$$

$$\xrightarrow{yields} \alpha\left(X^{\theta} X\right) = \bar{\alpha}\left(X^{\theta} X\right) \xrightarrow{yields} \alpha = \bar{\alpha}$$

Hence α is a real number. Therefore Eigen value of a Hermitian matrix is real.

6. Prove eigen value of a Skew-Hermitian matrix is purely imaginary.

Sol. Let A be a Skew-Hermitian matrix.

Therefore $A^{\theta} = -A - - - - - - -(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{yields} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{yields} X^{\theta} A^{\theta} = \bar{\alpha} X^{\theta}$$

$\xrightarrow{yields} X^{\theta}(-A) = \bar{\alpha} X^{\theta}$ (using (1))

Post multiplying both sides by X, we get

$$X^{\theta}(-AX) = \bar{\alpha}\left(X^{\theta}X\right) \xrightarrow{yields} X^{\theta}\,(-\alpha X) = \bar{\alpha}\left(X^{\theta}X\right)$$

$$\xrightarrow{yields} -\alpha\left(X^{\theta}X\right) = \bar{\alpha}\left(X^{\theta}X\right) \xrightarrow{yields} -\alpha = \bar{\alpha}$$

Hence α is purely imaginary or zero.

(Because if $\alpha = x + iy \xrightarrow{yields} \bar{\alpha} = x - iy$

Now $-\alpha = \bar{\alpha} \xrightarrow{yields} -x - iy = x - iy$

Comparing real and imaginary parts, we get

$$-x = x \xrightarrow{yields} x = 0$$

And $y = y$

Hence α is either zero or purely imaginary)

Therefore Eigen value of a Hermitian matrix is purely imaginary or zero.

7. Prove that product of two orthogonal matrices is orthogonal matrix

Sol. Let A and B be two orthogonal matrices. Therefore

$AA^T = A^T A = I$ and $BB^T = B^T B = I$

Now $(AB)(AB)^T = ABB^T A^T = AIA^T = AA^T = I$ and

$(AB)^T (AB) = B^T A^T \, AB = BIB^T = BB^T = I$

Hence AB is an orthogonal matrix. Therefore product of two orthogonal matrices is orthogonal

matrix.

8. Prove that transpose of an orthogonal matrix is orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^T A = I$

Now $A^T (A^T)^T = A^T A = I$ and

$(A^T)^T A^T$=A$A^T = I$

Hence A^T is an orthogonal matrix

Therefore transpose of an orthogonal matrix is orthogonal matrix.

9. Prove that inverse of an orthogonal matrix is an orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^T A = I$

Now $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^T A)^{-1} = I^{-1} = I$

and

$(A^{-1})^T A^{-1}$=$(A^T)^{-1}A^{-1} = (AA^T)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

10. Prove that determinant of an orthogonal matrix is ±1.

Sol. Let A be orthogonal matrix. Therefore

$$AA^T = A^T A = I$$

Taking determinant on both sides

$$|AA^T| = |I| \xrightarrow{yields} |A||A^T| = 1 \xrightarrow{yields} |A||A| = 1 \xrightarrow{yields} |A|^2 = 1 \xrightarrow{yields} |A| = \pm 1$$

(Because $|CD| = |C||D|, \quad |I| = 1, \quad |A| = |A^T|$)

11. Prove that inverse of a unitary matrix is a unitary matrix.

Sol. Let A be unitary matrix. Therefore

$A^\theta A = AA^\theta = I$ where $A^\theta = \left(\overline{A}\right)^T$

Now $A^{-1}(A^{-1})^\theta = A^{-1}\left(A^\theta\right)^{-1} = \left(A^\theta A\right)^{-1} = I^{-1} = I$

and

$$(A^{-1})^{\theta}A^{-1}=\left(A^{\theta}\right)^{-1}A^{-1}=\left(AA^{\theta}\right)^{-1}=I^{-1}=I$$

Hence A^{-1} is a unitary matrix

Therefore inverse of a unitary matrix is a unitary matrix.

Note 1: The product of Eigen values of a matrix A is equal to determinant of A

Note 2: The sum of Eigen values of a matrix A is equal to trace of A i.e. equal to sum of diagonal elements of A.