1 Peter's Interpolation

Peter's interplation function is this:

$$F(q/T) = N\left(\Xi_b I_+(q/T) + \Xi_f I_-(q/T)\right) \tag{1}$$

where N is a suitable normalization constant to make $F(0) = 1$.

Since we have SU(3) and 3 flavors,

$$\Xi_b = 2t_A = 6 \tag{2}$$
$$X_f = 2N_f = 6 \tag{3}$$

which makes

$$F(q/T) = \frac{I_+(q/T) + I_-(q/T)}{I_+(0) + I_-(0)} \tag{4}$$

The $I_\pm(Q)$ are given by

$$\begin{aligned} I_\pm(x) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(\pm 1)^{m+n-1} I_{mn}(x) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\pm 1)^{m+n-1} I_{mn}(x) + \zeta_\pm(3) \end{aligned} \tag{5}$$

where

$$I_{mn}(x) = \frac{mn}{2(m+n)^3} x^2 K_2(x\sqrt{mn}) \tag{6}$$

and

$$\begin{aligned} \zeta_+(3) &= \zeta(3) = \sum_{m=1}^{\infty} 1/m^3 \\ &= 1.2020569031595942 \end{aligned} \tag{7}$$

$$\begin{aligned} \zeta_-(3) &= \sum_{m=1}^{\infty}(-1)^{m-1}/m^3 = (3/4)\zeta(3) \\ &= 0.9015426773696957 \end{aligned} \tag{8}$$

We also know

$$I_\pm(0) = \zeta_\pm(2) = \begin{cases} \frac{\pi^2}{6} & (+) \\ \frac{\pi^2}{12} & (-) \end{cases} \tag{9}$$

Hence the normalization constant is

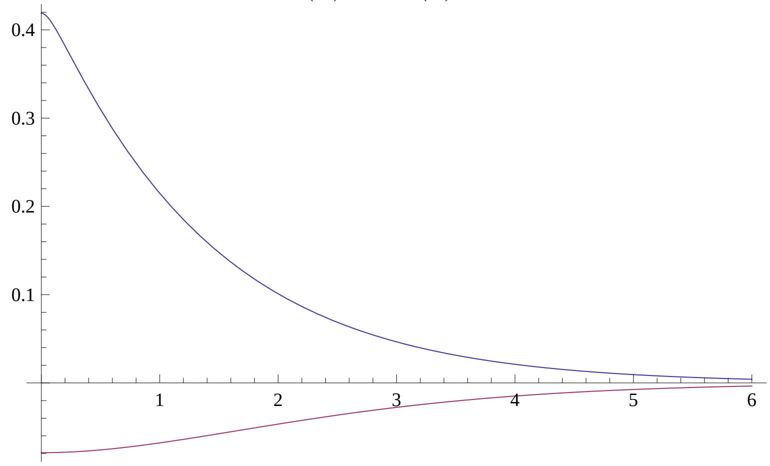
$$I_+(0) + I_-(0) = \frac{\pi^2}{4} \tag{10}$$

Asymptotically,

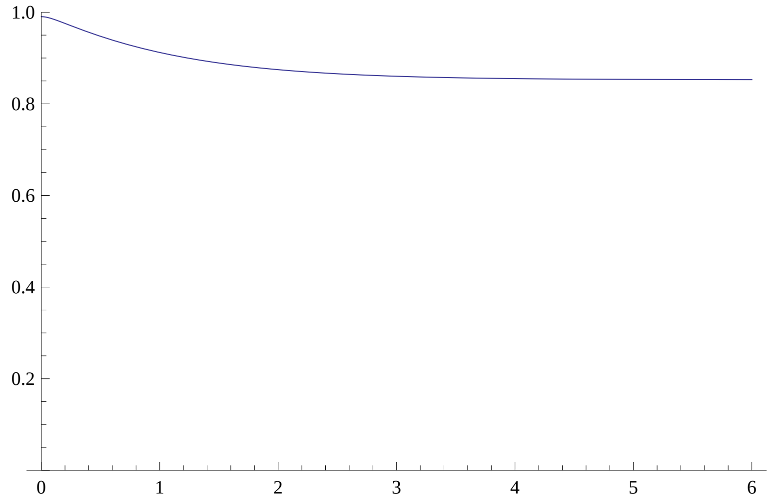
$$K_2(x) \quad = \quad \sqrt{\frac{\pi}{2x}} e^{-x} \tag{11}$$

Hence $I_\pm(x)$ approaches a constant exponentially.

Here is the plot of $I_\pm(x) - \zeta_\pm(3)$:



Here is the normalized interpolating function $F(x)$:



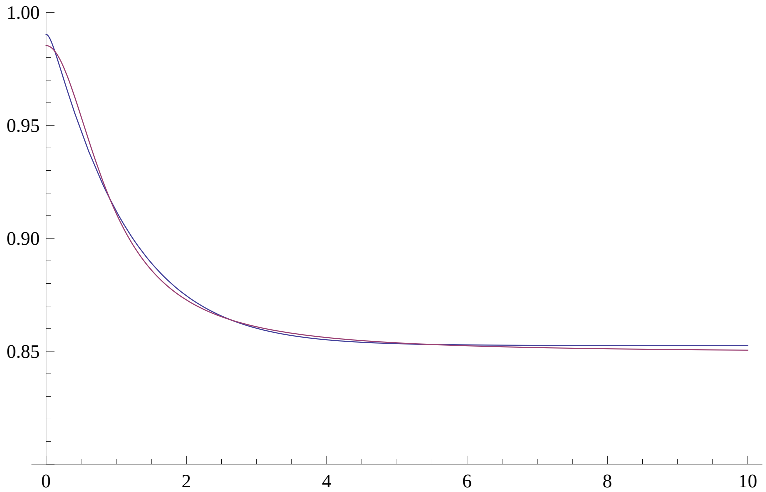
This is a monotonically decreasing function. Best fit is probably in terms of exponential. But that's not conveninent. Instead, we will fit it with

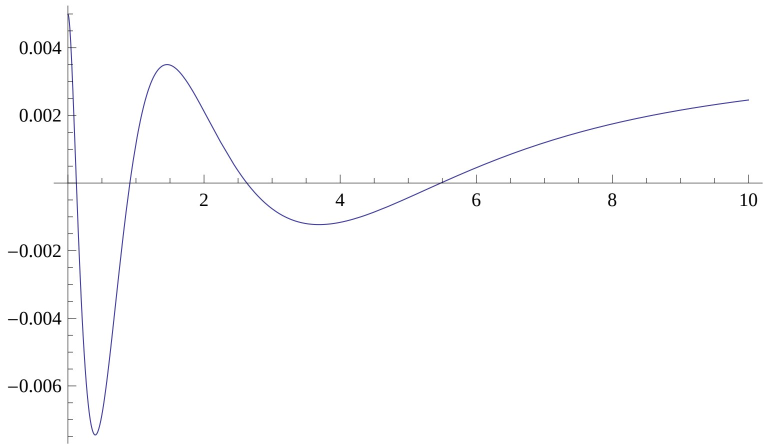
$$f_{\text{fit}}(x) = c_a + \frac{c_b}{1 + c_c x^2} \tag{12}$$

Fitting it within $0.001 < x < 6.001$, we get

$$f_{\text{fit}}(x) = 0.849346 + \frac{0.13597}{1 + 1.20524x^2} \tag{13}$$

Here is the comparison plot of the two functions and their relative error:





So the fit is good within a fraction of percent. We can live with that. Since the integral is done with $\tilde{q} = q/m_D$,

$$f(q/T) \quad = \quad f(\tilde{q}/(T/m_D)) = 0.849346 + \frac{0.13597}{1 + 1.20524(\tilde{q}/(T/m_D))^2}$$

$$
\begin{aligned}
&= 0.849346 + \frac{0.13597(T/m_D)^2}{(T/m_D)^2 + 1.20524\tilde{q}^2} \\
&= 0.849346 + \frac{0.1112816(T/m_D)^2}{0.82971(T/m_D)^2 + \tilde{q}^2} \\
&= 0.849346\left(1 + \frac{0.132827(T/m_D)^2}{0.82971(T/m_D)^2 + \tilde{q}^2}\right) \qquad (14)
\end{aligned}
$$

The integral to do naw takes the form (already scaled with m_D):

$$
\mathcal{I} = \int d\!\!\bar{}^2 q \left(\frac{1}{q^2} - \frac{1}{q^2+1}\right)\left(1 + \frac{a^2}{q^2+s^2}\right) N_{\rm fit}(1 - e^{i\mathbf{q}\cdot\mathbf{b}}) \qquad (15)
$$

where

$$
\begin{aligned}
N_{\rm fit} &= 0.849346 \qquad (16) \\
a^2 &= 0.132827(T/m_D)^2 \qquad (17) \\
s^2 &= 0.82971(T/m_D)^2 \qquad (18)
\end{aligned}
$$

Using

$$
\int_0^{2\pi} e^{iqb\cos\phi} = 2\pi J_0(qb) \qquad (19)
$$

we get

$$
\mathcal{I} = \frac{N_{\rm fit}}{2\pi}\int_0^\infty qdq\,(1 - J_0(qb))\left(\frac{1}{q^2} - \frac{1}{q^2+1} + \frac{a^2}{s^2}\left(\frac{1}{q^2} - \frac{1}{q^2+s^2}\right) - \frac{a^2}{s^2-1}\left(\frac{1}{q^2+1} - \frac{1}{q^2+s^2}\right)\right) \qquad (20)
$$

$$
\int_0^\infty qdq\left(\frac{s^2-\epsilon^2}{(q^2+\epsilon)(q^2+s^2)}\right) = \ln(s) - \ln(\epsilon) \qquad (21)
$$

$$
\begin{aligned}
\int_0^\infty qdqJ_0(qb)\left(\frac{s^2-\epsilon^2}{(q^2+\epsilon)(q^2+s^2)}\right) &= K_0(\epsilon b) - K_0(sb) \\
&\approx -\gamma_E - \ln(\epsilon) - \ln(b/2) - K_0(sb) \qquad (22)
\end{aligned}
$$

using

$$
\int_0^\infty qdq\,\frac{J_0(bq)}{q^2+m^2} = K_0(bm) \qquad (23)
$$

We get

$$
\begin{aligned}
\mathcal{I} &= \frac{N_{\mathrm{fit}}}{2\pi}\int_0^\infty qdq\,(1-J_0(qb))\left(\frac{1}{q^2}-\frac{1}{q^2+1}+\frac{a^2}{s^2}\left(\frac{1}{q^2}-\frac{1}{q^2+s^2}\right)-\frac{a^2}{s^2-1}\left(\frac{1}{q^2+1}-\frac{1}{q^2+s^2}\right)\right)\\
&= \frac{N_{\mathrm{fit}}}{2\pi}\Bigg[\ln 1-\ln\epsilon+K_0(\epsilon b)-K_0(b)\\
&\quad+\frac{a^2}{s^2}\left(\ln s-\ln\epsilon+K_0(\epsilon b)-K_0(sb)\right)\Bigg]\\
&\quad-\frac{a^2}{s^2-1}\left(\ln(s)-\ln 1+K_0(b)-K_0(sb)\right)\\
&= \frac{N_{\mathrm{fit}}}{2\pi}\Bigg[-\ln\epsilon+\gamma_E+\ln\epsilon+\ln(b/2)+K_0(b)\\
&\quad+\frac{a^2}{s^2}\left(\ln s-\ln\epsilon+\gamma_E+\ln\epsilon+\ln(b/2)+K_0(sb)\right)\Bigg]\\
&\quad-\frac{a^2}{s^2-1}\left(\ln s-K_0(b)+K_0(sb)\right)\\
&= \frac{N_{\mathrm{fit}}}{2\pi}\Bigg[\gamma_E+\ln(b/2)+K_0(b)\\
&\quad+\frac{a^2}{s^2}\left(\gamma_E+\ln(sb/2)+K_0(sb)\right)\Bigg]\\
&\quad-\frac{a^2}{s^2-1}\left(\ln s-K_0(b)+K_0(sb)\right)\\
&= \frac{N_{\mathrm{fit}}}{2\pi}\Bigg[\gamma_E+\ln(b/2)+K_0(b)\\
&\quad+\frac{a^2}{s^2}\left(\gamma_E+\ln(sb/2)+K_0(sb)\right)\\
&\quad-\frac{a^2}{s^2-1}\left(\ln s-(K_0(b)+\gamma_E+\ln(b/2))+(K_0(sb)+\gamma_E+\ln(sb/2))+\ln(b/2)-\ln(sb/2)\right)\Bigg]\\
&= \frac{N_{\mathrm{fit}}}{2\pi}\Bigg[\gamma_E+\ln(b/2)+K_0(b)\\
&\quad+\frac{a^2}{s^2}\left(\gamma_E+\ln(sb/2)+K_0(sb)\right)\\
&\quad-\frac{a^2}{s^2-1}\left(-(K_0(b)+\gamma_E+\ln(b/2))+(K_0(sb)+\gamma_E+\ln(sb/2))\right)\Bigg]
\end{aligned}
$$

$$= \frac{N_{\rm fit}}{2\pi}\left[\left(1+\frac{a^2}{s^2-1}\right)K_G(b)-\left(\frac{a^2}{s^2(s^2-1)}\right)K_G(sb)\right] \qquad (24)$$

where we defined

$$K_G(x) = K_0(x) + \gamma_E + \ln(x/2) \qquad (25)$$

which is regular at $x = 0$.

So the bottom line: We replace $K_G(b)$ in Guy's code with

$$K_G(b) \to N_{\rm fit}\left[\left(1+\frac{a^2}{s^2-1}\right)K_G(b)-\left(\frac{a^2}{s^2(s^2-1)}\right)K_G(sb)\right] \qquad (26)$$

where

$$\begin{aligned} N_{\rm fit} &= 0.849346 && (27)\\ a^2 &= 0.132827(T/m_D)^2 && (28)\\ s^2 &= 0.82971(T/m_D)^2 && (29) \end{aligned}$$