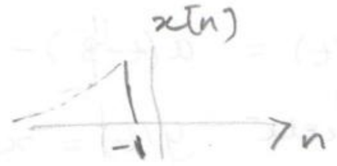
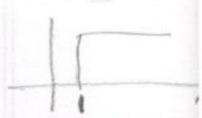
HW 4

$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1]$



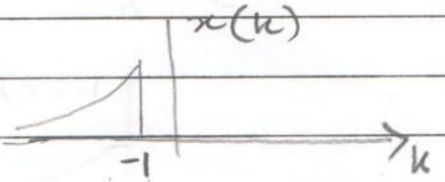
$h[n] = u[n-1]$ =



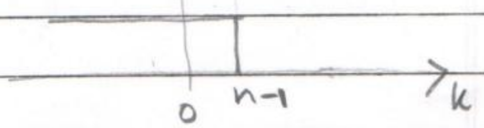
2.6. Compute & plot the convol^n $y[n] = x[n] * h[n]$

where $x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1]$ & $h[n] = u[n-1]$

Sol^n: From the sketches above, we see that there are 2 distinct regions for nature of overlap of :



$h[n-k] = u[n-1-k]$



1) $y[n \geq 0] = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} = \frac{1}{3} \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^{p} = \frac{1}{2}$

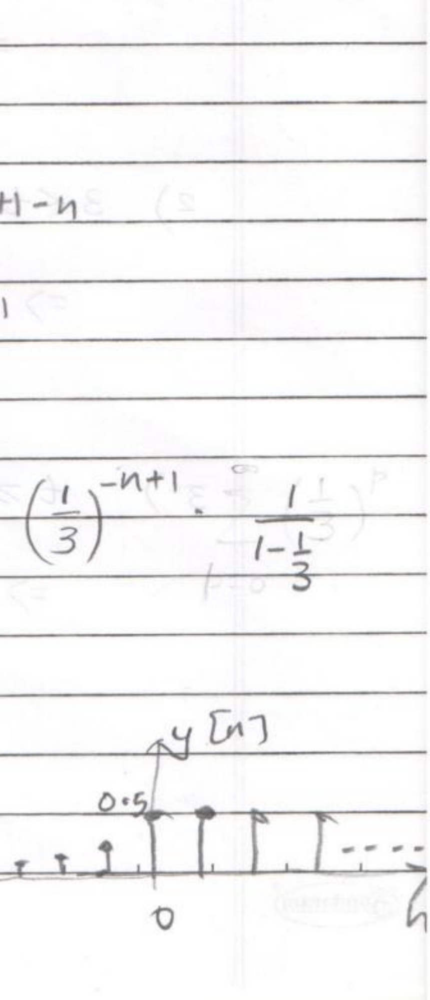
2) $y[n < 0] = \sum_{k=-\infty}^{-1+n} \left(\frac{1}{3}\right)^{-k}$

Let $p = k+1-n \Rightarrow -k = -p+1-n$

$\therefore y[n<0] = \sum_{p=-\infty}^{0} \left(\frac{1}{3}\right)^{-p} \left(\frac{1}{3}\right)^{-n+1}$

$= \left(\frac{1}{3}\right)^{-n+1} \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^{p} = \left(\frac{1}{3}\right)^{-n+1} \cdot \frac{1}{1-\frac{1}{3}}$

$= \frac{3^n}{2}$



$y[n] = \frac{3^n}{2}$ for $n < 0$

$= \frac{1}{2}$ for $n \geq 0$

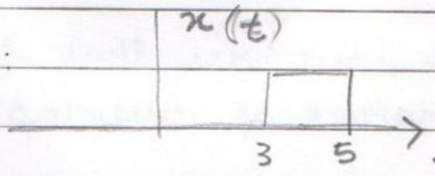
2.11. Let $x(t) = u(t-3) - u(t-5)$ & $h(t) = e^{-3t}u(t)$

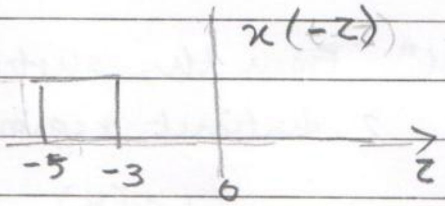
a) Compute $y(t) = x(t) * h(t)$

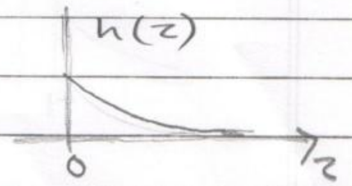
b) Compute $g(t) = \frac{d\,x(t)}{dt} * h(t)$

c) Relate $g(t)$ to $y(t)$

Soln.







a) We can see that we have 3 distinct regions for type of overlap:

1) $t < 3$ => no overlap between $x(t-\tau)$ & $h(\tau)$

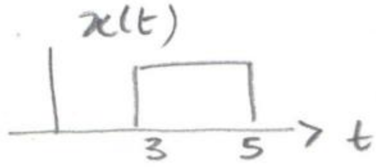
$$\Rightarrow y(t) = 0, \quad t < 3$$

2) $3 \leq t < 5$ => partial overlap

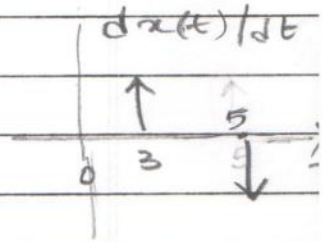
$$\Rightarrow y(t) = \int_0^{t-3} e^{-3\tau}\, d\tau = \frac{1 - e^{-3(t-3)}}{3}$$

3) $t \geq 5$ => full overlap

$$\Rightarrow y(t) = \int_{t-5}^{t-3} e^{-3\tau}\, d\tau = \frac{(1-e^{-6})\,e^{-3(t-5)}}{3}$$



b) $\frac{d\,x(t)}{dt} = \delta(t-3) - \delta(t-5)$



we need $g(t) = \frac{d\,x(t)}{dt} * h(t)$

Solⁿ: $g(t) = h(t) * \left[\delta(t-3) - \delta(t-5)\right]$

$= h(t-3) - h(t-5)$

we have 3 regions

1) $t<3$, $g(t) = 0$

2) $3 \le t < 5$

$g(t) = h(t-3) = e^{-3(t-3)}$

3) $t > 5$

$g(t) = e^{-3(t-3)} - e^{-3(t-5)}$

$= (e^{-6} - 1)\, e^{-3(t-5)}$

c) From a), we have $\frac{dy}{dt} = 0$, $t \le 3$

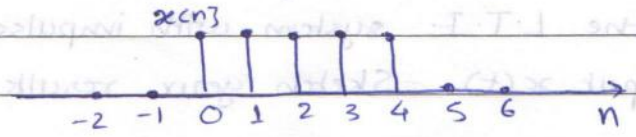
$= e^{-3(t-3)}$, $3 \le t < 5$

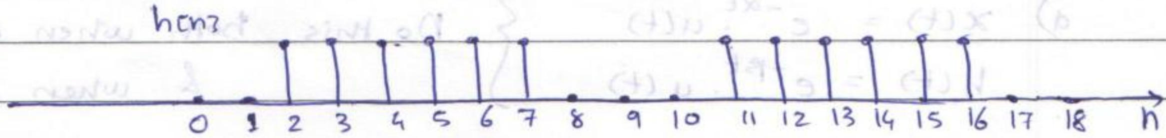
$= -(1-e^{-6})\, e^{-3(t-5)}$, $t > 5$

We note that

$\frac{dx(t)}{dt} * h(t) = g(t) = \frac{dy(t)}{dt} = \frac{d}{dt}\left(x(t) * h(t)\right)$

2.21) d) $x[n]$ and $h[n]$ are as shown in the figures:





Compute the convolution

$$y[n] = x[n] * h[n]$$

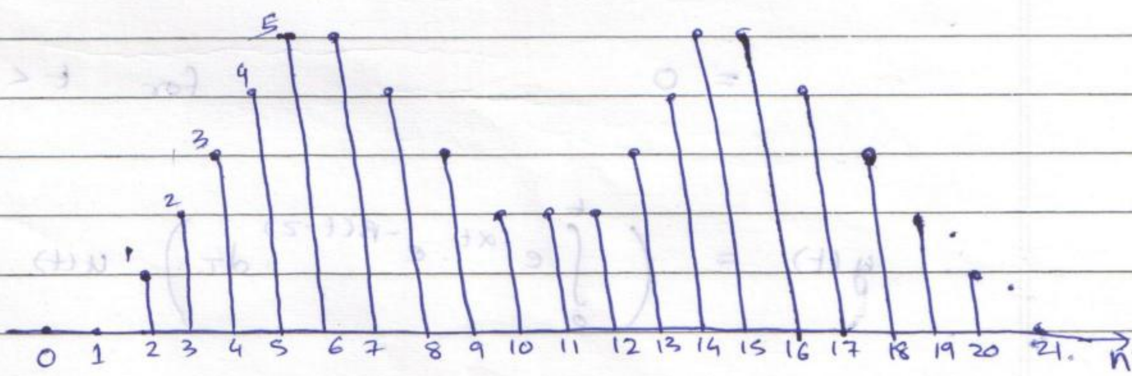
$$\rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= x[0] \cdot h[n] + x[1] \cdot h[n-1] + x[n] \cdot h[n-2] + x[3] \cdot h[n-3] + x[4] \cdot h[n-4]$$

$$= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$$

$\therefore$ $y[n]$ is the sum of the current and previous 4 samples of $h[n]$

$y[n]$:

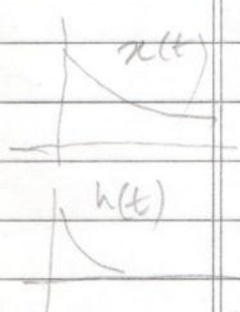


2.22) For each of the following pairs of waveforms, find the response $y(t)$ of the L.T.I. system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

d) $x(t) = e^{-\alpha t} \cdot u(t)$
$h(t) = e^{-\beta t} \cdot u(t)$
Do this both when $\alpha \neq \beta$ & when $\alpha = \beta$

$$\rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau}\, e^{-\beta(t-\tau)} \cdot u(\tau) \cdot u(t-\tau) \cdot d\tau$$



Now, if $t \geq 0$

$u(\tau) \cdot u(t-\tau) = 1$ for $0 \leq \tau \leq t$

$= 0$ otherwise.

if $t < 0$

$u(\tau) \cdot u(t-\tau) = 0$ for all t

$$\therefore \quad y(t) = \int_{0}^{t} e^{-\alpha\tau} \cdot e^{-\beta(t-\tau)} \cdot d\tau \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

$$\therefore \quad y(t) = \left(\int_{0}^{t} e^{-\alpha t} \cdot e^{-\beta(t-\tau)} \cdot d\tau \right) \cdot u(t)$$

Case 1: $\alpha \neq \beta$

$$y(t) = \left(\int_0^t e^{-\alpha\tau} \cdot e^{-\beta(t-\tau)} \cdot d\tau \right) \cdot u(t)$$

$$= \left(e^{-\beta t} \cdot \int_0^t e^{-(\alpha-\beta)\tau} \cdot d\tau \right) \cdot u(t)$$

$$= e^{-\beta t} \cdot \left[\frac{e^{-(\alpha-\beta)\tau}}{-(\alpha-\beta)} \right]_0^t \cdot u(t)$$

$$= \frac{e^{-\beta t} \cdot [e^{-(\alpha-\beta)t} - 1] \cdot u(t)}{\beta - \alpha}$$

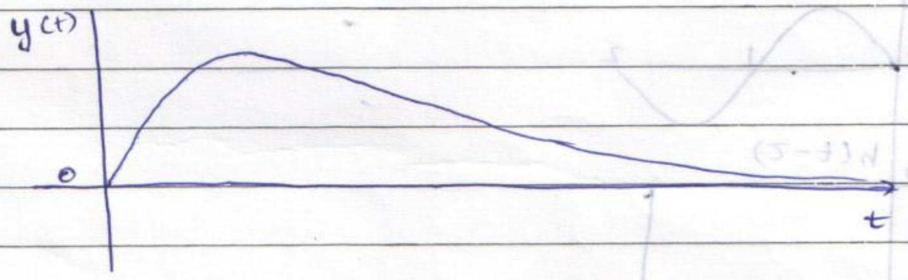
Case 2: $\alpha = \beta$

$$y(t) = \left(\int_0^t e^{-\beta\tau} \cdot e^{-\beta(t-\tau)} \cdot d\tau \right) \cdot u(t)$$

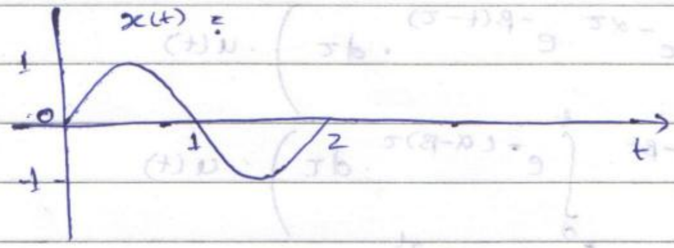
$$= \left(\int_0^t e^{-\beta t} \cdot d\tau \right) \cdot u(t)$$

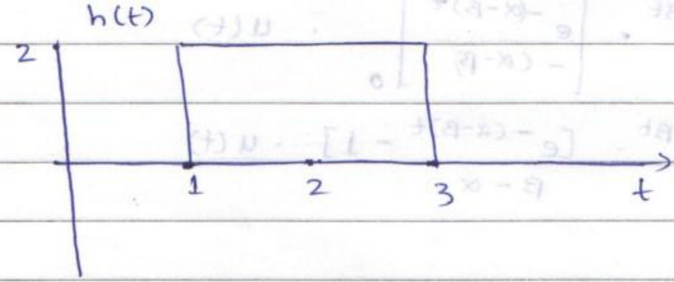
$$= \left(e^{-\beta t} \int_0^t 1 \cdot d\tau \right) u(t)$$

$$= t \cdot e^{-\beta t} \cdot u(t)$$



c) $x(t)$ and $h(t)$ are as given by the figures:





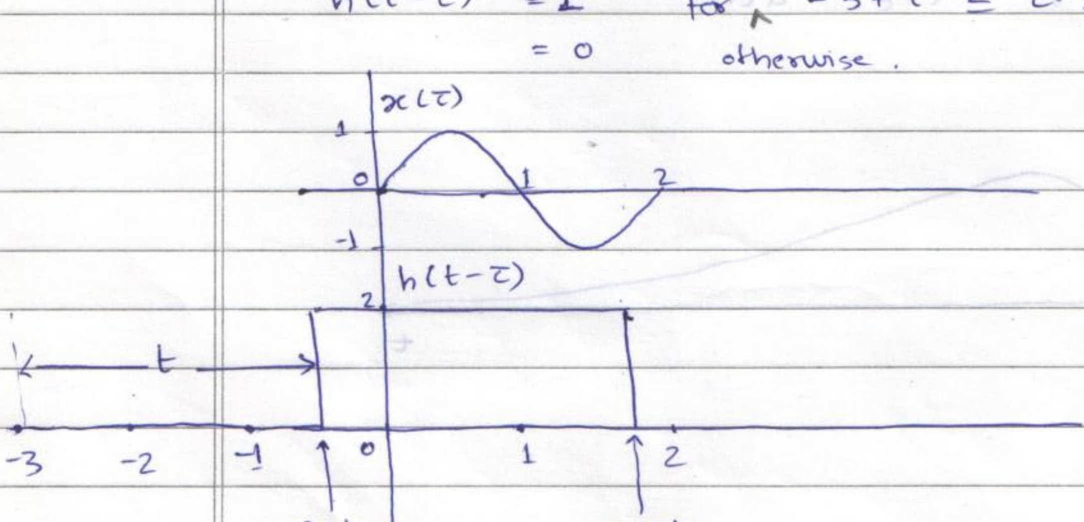
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

$$= \int_{0}^{2} x(\tau)\cdot h(t-\tau)\cdot d\tau \qquad (\because x(\tau)=0 \text{ for } \tau<0 \ \& \ \tau>2)$$

$$= \int_{0}^{2} \sin(\pi\tau)\cdot h(t-\tau)\cdot d\tau$$

$$(1<t-\tau<3 \Rightarrow 1-t <= \tau < 3-t)$$

$h(t-\tau) = 2$ for $-3+t \le \tau \le -1+t$
$= 0$ otherwise.



for $t < 1$:

$y(t) = 0$ (as there is no overlap between $x(\tau)$ & $h(t-\tau)$)

for $1 \leq t < 3$:

$$y(t) = \int_{0}^{t-1} 2 \cdot \sin(\pi t) \cdot dt$$

$$= \frac{2}{\pi} \cdot [1 - \cos(\pi(t-1))]$$

for $3 \leq t \leq 5$:

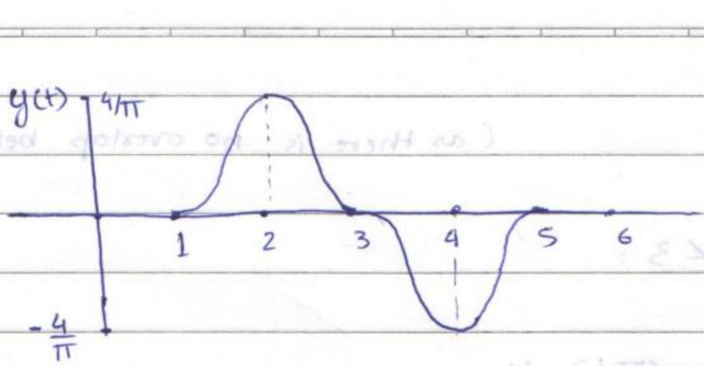
$$y(t) = \int_{-3+t}^{2} 2 \cdot \sin(\pi t) \cdot dt$$

$$= \frac{2}{\pi} \cdot [\cos(\pi(t-3)) - 1]$$

for $t > 5$:

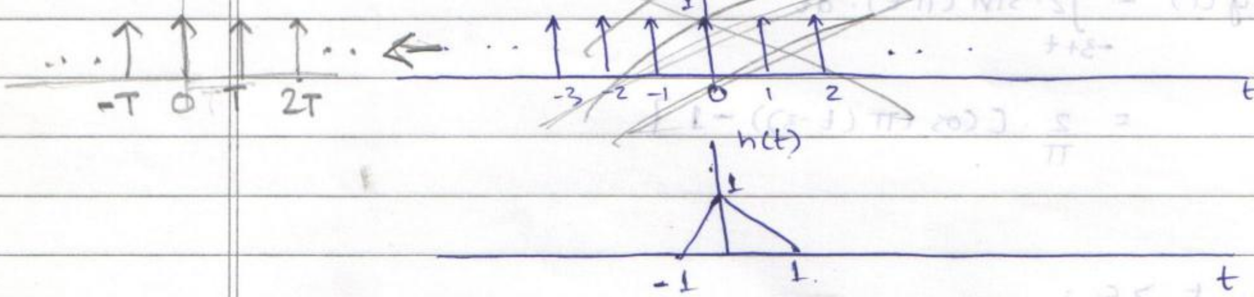
$y(t) = 0$ (as again, there is no overlap between $x(\tau)$ & $h(t-\tau)$)

$$\therefore \quad y(t) = \begin{cases} 0 & \text{for } t < 1 \\ 2/\pi \, [1 - \cos(\pi(t-1))] & \text{for } 1 \leq t < 3 \\ 2/\pi \, [\cos(\pi(t-3)) - 1] & \text{for } 3 \leq t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$



2.23) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$ is an impulse train

& h(t) is a triangular pulse as shown in the figures below.



Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T

a) $T = 4$ b) $T = 2$ c) $T = 3/2$ d) $T = 1$

$\longrightarrow$

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

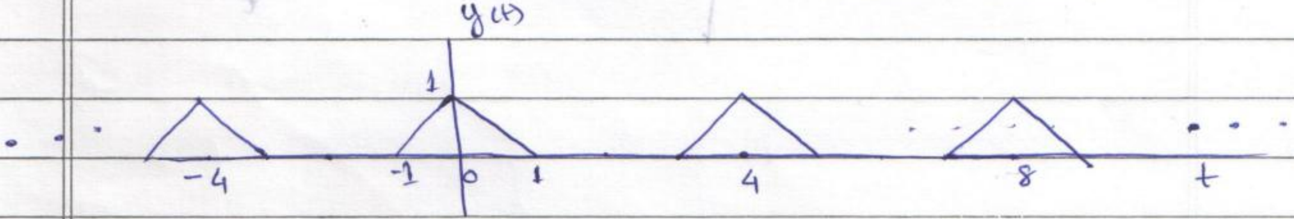
$y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \delta(\tau - kT) \right) \cdot h(t-\tau) \cdot d\tau$$

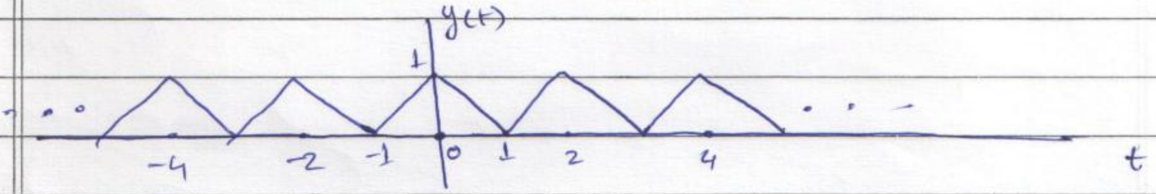
$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\tau - kT) \cdot h(t-\tau) \cdot d\tau$$

$$= \sum_{k=-\infty}^{\infty} h(t-kT) \quad \text{— (By sifting property of Dirac delta function)}$$

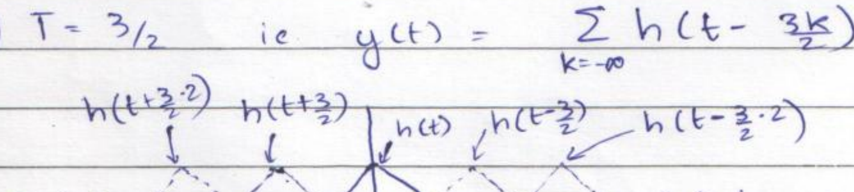
a) $T = 4$ ie. $y(t) = \sum_{k=-\infty}^{\infty} h(t-4k)$

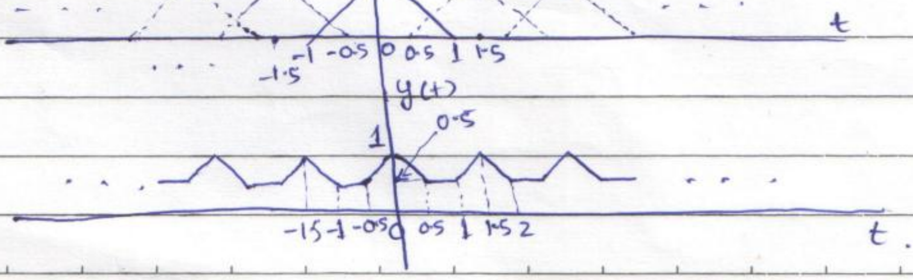


b) $T = 2$ ie. $y(t) = \sum_{k=-\infty}^{\infty} h(t-2k)$



c) $T = 3/2$ ie $y(t) = \sum_{k=-\infty}^{\infty} h(t - \frac{3k}{2})$





d) $T = 1$ ie. $y(t) = \sum_{k=-\infty}^{\infty} h(t-k)$