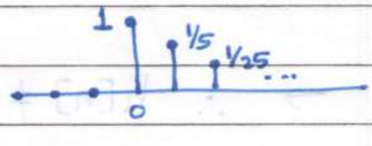
EE 210 : SIGNALS & SYSTEMS

2.28 The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answer.

(a) $h[n] = \left(\frac{1}{5}\right)^n u[n]$



• For system to be causal $h[n] = 0$ for $n < 0$

• For system to be stable $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\rightarrow$ $h[n] = 0$ for $n < 0$ $\therefore$ CAUSAL

$\rightarrow$ $\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1 - 1/5} = \frac{5}{4} < \infty$ $\therefore$ STABLE

(d) $h[n] = 5^n u[3-n]$

$u[3-n]$ is $u[n]$ flipped at 0 and shifted to right by 3.

$\therefore$ $h[n] = 0$ $n > 3$

$\rightarrow$ But $h[n] \neq 0$ $n < 0$ $\therefore$ NON-CAUSAL

$\rightarrow$ $\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=-\infty}^{3} (5)^n u[3-n] = \sum_{n=-\infty}^{3} 5^n = \frac{5^3}{1 - 1/5} < \infty$

$\therefore$ STABLE

(f) $h[n] = \left(\frac{-1}{2}\right)^n u[n] + (1.01)^n u[1-n]$

for $n<0$ $\left(\frac{-1}{2}\right)^n u[n] = 0$

But $u[1-n] \cdot (1.01)^n \neq 0$

$\rightarrow$ $\therefore$ $h[n] \neq 0$ for $n<0$ $\therefore$ NON-CAUSAL

$\rightarrow$ $\sum_{n=-\infty}^{\infty} h[n] = \sum_{n=0}^{\infty} \left|\left(-\frac{1}{2}\right)^n\right| u[n] + \sum_{n=-\infty}^{1} (1.01)^n u[1-n]$

$= \dfrac{1}{1 - 1/2} + \dfrac{1.01}{1 - \frac{1}{1.01}} < \infty$ $\therefore$ STABLE

2.29 The following are impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify.

(a) $h(t) = e^{-4t} u(t-2)$

- For system to be causal $h(t) = 0$ for $t<0$
- For system to be stable $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$\rightarrow$ $u(t-2)$ is $u(t)$ shifted to right by 2.

$\therefore$ $h(t) = 0$ for $t<2$ $\therefore$ CAUSAL

$$\rightarrow \int_{-\infty}^{\infty} |h(t)|\,dt = \int_{2}^{\infty} e^{-4t}\,dt = \frac{-1}{4}\left[e^{-4t}\right]_2^{\infty}$$

$$= \frac{1}{4} e^{-8} < \infty \quad \therefore \text{STABLE}$$

(b) $h(t) = e^{-6t} u(3-t)$

$\rightarrow \quad h(t) \neq 0 \quad \text{for } t < 3 \quad \therefore$ NON-CAUSAL

$$\rightarrow \int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{3} e^{-6t}\,dt = \infty \quad \therefore \text{NOT STABLE}$$

(e) $h(t) = e^{-6|t|} \quad = e^{-6t} u[t] + e^{-6t} u[-t] \quad t \neq 0$

$\qquad\qquad\qquad\quad = 1 \qquad\qquad\qquad\qquad t = 0$

$\rightarrow h(t) \neq 0 \quad \text{for } t < 0 \quad \therefore$ NON-CAUSAL

$$\rightarrow \int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} e^{-6t}\,dt + \int_{-\infty}^{0} e^{6t}\,dt = \frac{1}{3} < \infty$$

$\therefore$ STABLE

2.30 Consider the first order difference equation

$$y[n] + 2y[n-1] = x[n]$$

Assuming the condition of initial rest (i.e. $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response of a system whose input and output are related by this difference equation. You may solve the problem by rearranging the difference equations so as to express $y[n]$ in terms of $y[n-1]$ and $x[n]$ and generating values of $y[0]$, $y[+1]$, $y[+2]$...

Sol^n:

We need to find the output $y[n]$ when the input $x[n] = \delta[n]$.

Assuming initial rest condition

$$y[n] = 0 \quad \text{for } n < 0$$

$$y[n] = x[n] - 2y[n-1]$$

$$\therefore y[0] = x[0] - 2y[-1] = 1 - 0 = 1$$

$$y[1] = x[1] - 2y[0] = 0 - 2$$

$$y[2] = -2y[1] = 4$$

$$y[3] = -2y[2] = -8$$

$$\vdots$$

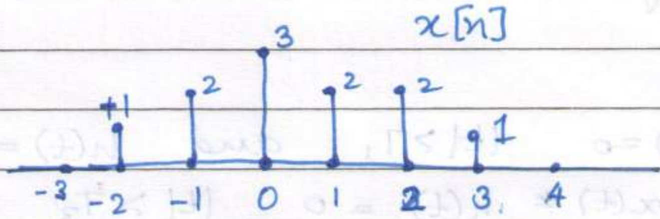
$$y[n] = (-2)^n u[n]$$

This is the impulse response of the system

2.31 Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2x[n-2]$$

Find the response of the system to the input depicted below by solving the difference equation recursively



Sol^n: Initial ~~inv~~ rest condition implies

$$y[n] = 0 \text{ for } n < -2$$

Now

$$y[n] = x[n] + 2x[n-2] - 2y[n-1]$$

$\therefore y[-2] = 1$

$y[-1] = 0$

$y[0] = 5$

$y[1] = -4$, $y[2] = 16$, ~~$y[3] = -26$~~ $y[3] = -27$

$y[4] = 58$, $y[5] = -114$

for $n \geq 5$ $\quad y[n] = -2y[n-1]$

$$\therefore y[n] = -114(-2)^{n-5} \quad n > 5$$

2.44 $\therefore\ y[n] = $

1	$n=-2$
0	-1
5	0
-4	1
16	2
-27	3
58	4
-114	5

$$y[n] = -114\,(-2)^{n-5} \qquad n > 5$$

2.44 (a) If $x(t)=0 \quad |t|>T_1$ and $h(t)=0 \quad |t|>T_2$ then $x(t) * h(t) = 0 \quad |t| > T_3$ for some T_3. Express T_3 in terms of T_1 and T_2

$$\rightarrow \quad x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-T_1}^{T_1} x(\tau)\, h(t-\tau)\, d\tau$$

Note that $h(-t)=0$ for $|t|>T_2$

$\therefore\ h(t-\tau) = 0$ for $\tau > t + T_2$ and $\tau < -T_2 + t$

$\therefore$ Integral evaluates to zero either if
$T_1 < -T_2 + t$ or $T_2 - t < -T_1$

$\therefore$ Integral $=0$ for $|t| > T_1 + T_2$

2.44 (b) A discrete-time LTI system has input $x[n]$, impulse response $h[n]$ and output $y[n]$.
If $h[n]$ is known to be zero everywhere outside $N_0 \le n \le N_1$ and $x[n] \ne 0$ only for $N_2 \le n \le N_3$ then output $y[n] = 0$ everywhere except $N_4 \le n \le N_5$

i) Determine N_4 and N_5 in terms of N_0, N_1, N_2, N_3

ii) If $N_0 \le n \le N_1$ has length M_h and
$N_2 \le n \le N_3$ has length M_x
& $N_4 \le n \le N_5$ has length M_y
Find M_y in terms of M_x and M_h.

→ i) $y[n] = x[n] * h[n] = \sum_{k=+N_0}^{N_1} x[k]\, h[n-k]$

$$= \sum_{k=N_0}^{N_1} h[k]\, x[n-k]$$

$x[-k] \ne 0$ for $-N_3 \le n \le -N_2$

$\therefore x[n-k] \ne 0$ for $-N_3 + n \le k \le -N_2 + n$

$\therefore$ Sum non zero if $-N_3 + n \le N_1$ and
$-N_2 + n \ge N_0$

$\therefore y[n]$ non-zero for $n \le N_1 + N_3$ and
$n \ge N_0 + N_2$

ii) As we can see $M_y = N_1 + N_3 - (N_0 + N_2) + 1$

$$= (N_1 - N_0) + 1 + (N_3 - N_2) + 1 - 1$$

$$\boxed{M_y = M_x + M_h - 1}$$

2.44 (c) Consider a discrete time LTI system with the property that if input $x[n] = 0$ for all $n \geq 10$ then output $y[n] = 0$ for all $n \geq 15$.
What condition must $h[n]$ satisfy for this?

$\rightarrow$

$h[n]$ is convolved with $x[n]$ to get $y[n]$

$\therefore\ y[n] = 0 \quad \forall\ n \geq 15$ given
$x[n] = 0 \quad \forall\ n \geq 10$

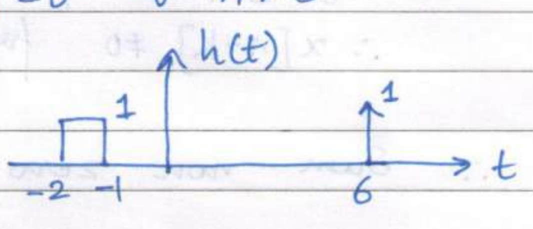
~~$\Rightarrow h[n] = 0$~~

~~$y[16]\ h$~~ $y[15] = x[9]\,h[6] + x[8]\,h[7] \ldots$

(Expanding convolution)

Clearly $h[n] = 0 \quad \forall\ n > 5$

2.44 (d) Given $h(t)$



Over what range should we know $x(t)$ to know $y(0)$ ~~completely~~ ?

$\rightarrow y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-2}^{-1} x(t-\tau) + x(t-6)$

$\therefore\ y(0) = \int_{-2}^{-1} x(-\tau)\,d\tau + x(-6)$

$\therefore\ \boxed{t = -6,\ (1,2)}$

2.49 Consider an LTI system with impulse response $h[n]$ that is not absolutely summable i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \infty$$

We will show that absolutely summability is a necessary condition for stability. Sufficiency of A.S. for stability is known and used in solving problem 2.28.

(a) Suppose $x[n] = \begin{cases} 0 & \text{if } h[-n] = 0 \\ \dfrac{h[-n]}{|h[-n]|}, & \text{o.w.} \end{cases}$

Is $x[n]$ Bounded. Find the upper bound on $|x[n]|$

$$|x[n]| = \begin{cases} 0 & \text{if } h[-n] = 0 \\ 1 & \text{if } h[-n] \neq 0 \end{cases}$$

$$\therefore |x[n]| \leq 1$$

(b) Calculate the output at $n=0$ for this particular input. Does the result prove anything

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]\, h[k] = \sum_{k=-\infty}^{\infty} \frac{(h[k])^2}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]|$$

$$= \infty$$

$\therefore$ Output is not Bounded. $\therefore$ System is not stable.

P.T.O.

∴ Failure of A.S. causes instability.

→ Clearly this proves that A.S. is a necessary condition for stability.

2.49 (C) For a continuous-time LTI system show that absolute integrability (A.I.) is necessary for stability of system.

→ Consider a LTI system with $\int_{-\infty}^{\infty} |h(t)| = \infty$

Consider a particular input

$$x(t) = \begin{cases} 0 & \text{if } h(-t) = 0 \\ \dfrac{h(-t)}{|h(-t)|} & \text{o.w.} \end{cases}$$

Clearly $|x(t)| \leq 1$ ∴ Input to system is Bounded.

Consider $y(0) = \int_{-\infty}^{\infty} x(-\tau)\, h(\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} \frac{h(\tau)^2}{|h(\tau)|} = \int_{-\infty}^{\infty} |h(t)|\, dt = \infty$$

∴ Output is unbounded

∴ System is unstable.

∴ A.I. is a necessary condition for stability