EE 210 : SIGNALS & SYSTEMS

HW 3 SOLN

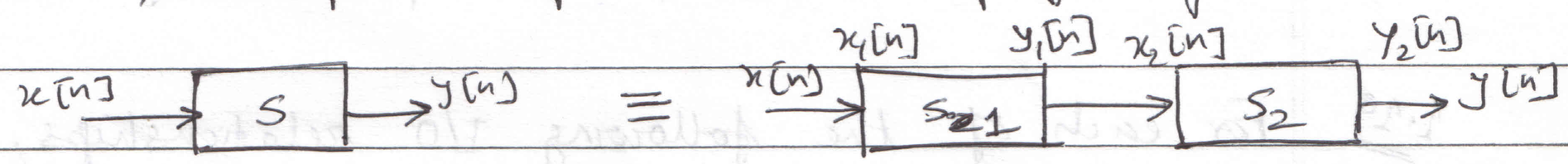
1.15 Consider a system S with input $x[n]$ & output $y[n]$. This system is obtained through a series interconnection of a system S_1 followed by a system S_2. The input-output relationships for S_1 & S_2 are:

$$S_1 : \; y_1[n] = 2x_1[n] + 4x_1[n-1],$$

$$S_2 : \; y_2[n] = x_2[n-2] + \tfrac{1}{2}x_2[n-3],$$

where $x_1[n]$ & $x_2[n]$ denote input signals.

(a) Determine the input-output relationship for system S



The signal $x_2[n]$, input of system S_2 is same as $y_1(n)$, the output of system S_1

$$\therefore \; y_2[n] = x_2[n-2] + \tfrac{1}{2}x_2[n-3]$$

$$\Rightarrow y[n] = y_2[n] = y_1[n-2] + \tfrac{1}{2}y_1[n-3]$$

$$\Rightarrow y[n] = \left(2x_1[n-2] + 4x_1[n-3]\right) + \frac{1}{\cancel{2}}\left(\cancel{2}\,x_1[n-3] + {}^{2}\cancel{4}\,x_1[n-4]\right)$$

$$\Rightarrow y[n] = 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4]$$

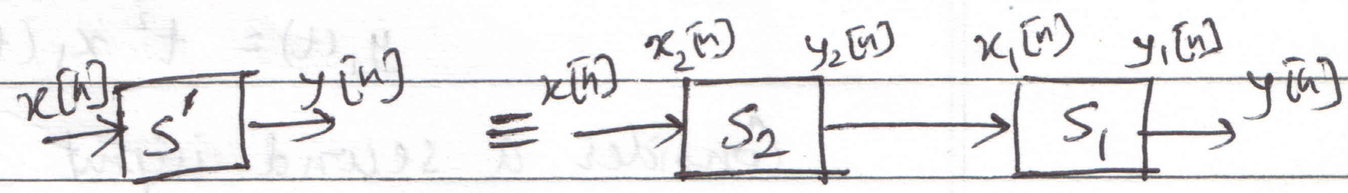
But as $x_1[n] \equiv x[n]$

I/O relationship for system S is

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

(b) Does the input-output relationship of System S change if the order in which S_1 & S_2 are connected is reversed?

Consider system S' :



Let us determine its I/O relationship

Signal $x_1[n]$ is same as $y_2[n]$

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$\therefore \quad y_1[n] = 2x_1[n] + 4x_1[n-1]$

$\Rightarrow y[n] = y_1[n] = 2\, y_2[n] + 4\, y_2[n-1]$

$\Rightarrow \quad y[n] = 2\left(x_2[n-2] + \frac{1}{2} x_2[n-3]\right) + 4\left(x_2[n-3] + \frac{1}{2} x_2[n-4]\right)$

$\Rightarrow \quad y[n] = 2x_2[n-2] + 5x_2[n-3] + 2x_2[n-4]$

$\Rightarrow \quad y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$

which is same as for system S

$\therefore$ I/O relationship is independent of the ordering of S_1, S_2

1.19 For each of the following I/O relationships, determine whether the corresponding system is linear, time-invariant or both.

(a) $y(t) = t^2 x(t-1)$

Consider two arbitrary inputs $x_1(t)$ & $x_2(t)$

$x_1(t) \longrightarrow t^2 x_1(t-1) = y_1(t)$

$x_2(t) \longrightarrow t^2 x_2(t-1) = y_2(t)$

Let $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ where α, β are arbitrary scalars

Let $x_3(t) \longrightarrow y_3(t)$

$y_3(t) = t^2 x_3(t-1)$

$= t^2[\alpha x_1(t-1) + \beta x_2(t-1)]$

$= \alpha t^2 x_1(t-1) + \beta t^2 x_2(t-1) = \alpha y_1(t) + \beta y_2(t)$

Hence it is linear

Now, consider arbitrary input $x_1(t)$

$y_1(t) = t^2 x_1(t-1)$

Consider a second input $x_2(t) = x_1(t-t_0)$

The output, $y_2(t) = t^2 x_2(t-1)$

$= t^2 x_1(t-t_0-1)$ — (1)

Also, $y_1(t-t_0) = (t-t_0)^2 x_1(t-t_0-1)$ — (2)

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Since (1) & (2) are not equivalent, the system is not time-invariant.

(d) $y(t) = Od\{x(t)\} = \left[\frac{x(t) - x(-t)}{2}\right]$

Consider arbitrary inputs $x_1(t)$ & $x_2(t)$

$x_1(t) \longrightarrow y_1(t) = Od\{x_1(t)\} = \frac{1}{2}(x_1(t) - x_1(-t))$

$x_2(t) \longrightarrow y_2(t) = Od\{x_2(t)\} = \frac{1}{2}(x_2(t) - x_2(-t))$

Consider $x_3(t) = \alpha x_1(t) + \beta x_2(t)$; α, β are arbitrary scalars.

Let $x_3(t) \longrightarrow y_3(t) = Od\{x_3(t)\}$

$$= \frac{1}{2}(x_3(t) - x_3(-t))$$

$$= \frac{1}{2}\left(\alpha x_1(t) + \beta x_2(t) - (\alpha x_1(-t) + \beta x_2(-t))\right)$$

$$= \frac{1}{2}\left[\alpha(x_1(t) - x_1(-t)) + \beta(x_2(t) - x_2(-t))\right]$$

$$= \alpha\, Od\{x_1(t)\} + \beta\, Od\{x_2(t)\}$$

$\therefore$ it is linear.

Consider arbitrary input $x_1(t)$.

$$y_1(t) = Od\{x_1(t)\} = \frac{x_1(t) - x_1(-t)}{2} \quad \text{— (1)}$$

Consider an input $x_2(t) = x_1(t - t_0)$

$$y_2(t) = Od\{x_2(t)\} = \frac{x_2(t) - x_2(-t)}{2} = \frac{x_1(t-t_0) - x_1(-t-t_0)}{2} \quad \text{— (2)}$$

$$= \frac{x_1(t-t_0) - x_1(-t-t_0)}{2}$$

Also consider if we delay (1) by t_0

$$y_1(t - t_0) = \frac{x_1(t-t_0) - x_1(-t+t_0)}{2} \quad \text{— (3)}$$

(2) & (3) are not equivalent. Hence system is not time invariant.

1.27 In this chapter we introduced a number of general properties of systems. In particular, a system may / may not be:
(1) Memoryless (2) Time invariant (3) Linear (4) causal (5) Stable.
Determine which of these properties hold and which do not for each of the following systems.

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1.27

(a) $y(t) = x(t-2) + x(2-t)$

i) Memoryless? <u>No</u>. $y(t)$ Depends on instants other than "t" e.g. on $(t-2)$

ii) Time-invariant? <u>No</u>

Proof:

Consider a delayed input $x_1(t) = x(t-t_0)$

$\Rightarrow \quad y_1(t) = x_1(t-2) + x_1(2-t)$

$= x(t-2-t_0) + x(2-t-t_0)$

But $y(t-t_0) = x((t-t_0)-2) + x(2-(t-t_0))$

$= x(t-t_0-2) + x(2-t+t_0)$

we see that $y_1(t)$ (resp. to $x(t-t_0)$) $\neq y(t-t_0)$

iii) Linear? <u>Yes</u>

Proof: Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$ ... (1)

we need to show that $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$

Now: $y_3(t) = x_3(t-2) + x_3(2-t)$

$= a_1 x_1(t-2) + a_2 x_2(t-2) + a_1 x_1(2-t) + a_2 x_2(2-t)$

$= a_1 x_1(t-2) + a_1 x_1(2-t) + a_2 x_2(t-2) + a_2 x_2(2-t)$

$= a_1 y_1(t) + a_2 y_2(t)$ ✓

iv) Causal? <u>No</u>. Ex. $y(-1) = x(-3) + \underline{x(3)}$ ← future ip.

iv) Stable? <u>Yes</u>. For $|x(t)| \leq B$, we have $|y(t)| \leq 2B$.

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(b) $y(t) = [\cos(3t)]\, x(t)$

i) Memoryless ? Yes . $y(t)$ does not depend on x at other instants

ii) Time-invariant ? No

Proof :

Consider $x_1(t) = x(t - t_0)$

$\therefore \quad y_1(t) = [\cos(3t)]\, x_1(t) = [\cos(3t)]\, x(t - t_0)$

But $y(t - t_0) = [\cos(3(t - t_0))] \cdot x(t - t_0)$

$y_1(t) \neq y(t - t_0)$

iii) Linear ? Yes

Proof : Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$\Rightarrow y_3(t) = [\cos(3t)]\, x_3(t)$

$= [\cos(3t)](a_1 x_1(t) + a_2 x_2(t))$

$= a_1 [\cos(3t)]\, x_1(t) + a_2 [\cos(3t)]\, x_2(t)$

$= a_1 y_1(t) + a_2 y_2(t)$ ✓

iv) Causal ? Yes $y(t)$ depends only on current instant "t"

v) Stable ? Yes

Proof : $|x(t)| \leq B \;\; \forall t \Rightarrow |y(t)| \leq |\cos(3t)| \cdot B$

$\leq B \quad \forall t$

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(c) $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau$

(i) Memoryless? No. $y(t)$ depends on ip values upto $2t$ (future)

(ii) Time-invariant? No

Proof: Consider $x_1(t) = x(t-t_0)$

$$\Rightarrow \quad y_1(t) = \int_{-\infty}^{2t} x_1(\tau)\, d\tau = \int_{-\infty}^{2t} x(\tau - t_0)\, d\tau$$

Let $\tau_1 = \tau - t_0$

$\Rightarrow d\tau_1 = d\tau$; $\tau = -\infty \Rightarrow \tau_1 = -\infty$; $\tau = 2t \Rightarrow \tau_1 = 2t-$

$\therefore$ we can write $y_1(t) = \int_{-\infty}^{2t-t_0} x(\tau_1)\, d\tau_1$

$$= \int_{-\infty}^{2t-t_0} x(\tau)\, d\tau$$

But $y(t-t_0) = \int_{-\infty}^{2(t-t_0)} x(\tau)\, d\tau \neq y_1(t)$

iii) Linear? Yes

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$$\Rightarrow y_3(t) = \int_{-\infty}^{2t} x_3(\tau)\, d\tau = \int_{-\infty}^{2t} a_1 x_1(\tau) + a_2 x_2(\tau)\, d\tau$$

$$= a_1 \int_{-\infty}^{2t} x_1(\tau)\, d\tau + a_2 \int_{-\infty}^{2t} x_2(\tau)\, d\tau$$

$$= a_1 y_1(t) + a_2 y_2(t) \quad \checkmark$$

iv) Causal? No. $y(t)$ depends on future instants of ip e.g. 2

v) Stable? No. Ex: $x(t) = 1 \ \forall t \Rightarrow y(t) \to \infty$ as $t \to \infty$.

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(d) $y(t) = 0 \quad , \quad t < 0$

$\quad = x(t) + x(t-2) \quad , \quad t \geq 0$

i) Memoryless ? <u>No</u> $y(t)$, at some t, depends on $x(t-2)$

ii) Time-invariance ? <u>No</u>

Proof Let $x_1(t) = x(t-t_0)$

$\Rightarrow y_1(t) = 0 \quad , \quad t < 0$

$\quad = x_1(t) + x_1(t-2) \quad , \quad t \geq 0$

$\quad = x(t-t_0) + x(t-2-t_0) \quad , \quad t \geq 0$

But $y(t-t_0) = 0 \quad , \quad t - t_0 < 0 \equiv t < t_0$

$\quad = x(t-t_0) + x(t-t_0-2) \quad , \quad (t-t_0) \geq 0 \equiv t \geq t_0$

We see that $y_1(t) \neq y(t-t_0)$

iii) Linear ? <u>Yes</u>

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$\therefore y_3(t) = 0$ for $t < 0$

$\quad = x_3(t) + x_3(t-2) \quad , \quad t \geq 0$

$\quad = a_1 x_1(t) + a_2 x_2(t) + a_1 x_1(t-2) + a_2 x_2(t-2)$

$\quad = a_1 y_1(t) + a_2 y_2(t) \quad , \quad t \geq 0$

iv) Causal ? Yes. $y(t)$ depends on past values only, for $t \geq 0$

$= 0$ for $t < 0$.

v) Stable ? Yes If $|x(t)| \leq B \ \forall t$

$|y(t)| \leq 2B \ \forall t$.

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(e) $y(t) = 0 \quad , \quad x(t) < 0$

$= x(t) + x(t-2) \quad , \quad x(t) \geq 0$

i) Memoryless? No. $y(t)$ depends on $x(t-2)$.

ii) Time-invariant? Yes

Proof: Let $x_1(t) = x(t-t_0)$

$\Rightarrow \quad y_1(t) = 0 \quad , \quad x_1(t) < 0$

$= x_1(t) + x_1(t-2) \quad , \quad x_1(t) \geq 0$

$\equiv \quad y_1(t) = 0 \quad , \quad x(t-t_0) < 0$

$= x(t-t_0) + x(t-t_0-2) \quad , \quad x(t-t_0) \geq 0$

But $y(t-t_0)$

$= 0 \quad , \quad x(t-t_0) < 0$

$= x(t-t_0) + x(t-t_0-2) \quad , \quad x(t-t_0) \geq 0$

$\therefore \quad y(t-t_0) = y_1(t)$ ✓

iii) Linear?

Ex. $x_1(t) = -1 \quad \forall t \quad , \quad x_2(t) = 2 \quad \forall t$

$x_3(t) = x_1(t) + x_2(t)$

$= 1 \quad \forall t$

$y_3(t) = x_3(t) + x_3(t-2) \quad , \quad x_3(t) \geq 0$

$= 2 \quad \forall t$

But $y_1(t) = 0 \quad \forall t \quad \because x_1(t) < 0 \quad \forall t$

$y_2(t) = x_2(t) + x_2(t-2) = 4 \quad \forall t$

$y_3(t) \neq y_1(t) + y_2(t)$

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~~iv) Causal? Yes, depends on past~~

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iv) Causal? Yes, depends on past instants only

stable? Yes.

v) $|x(t)| \leq B \Rightarrow |y(t)| \leq 2B$

~~iv) Causal ? No. For t negative, y(t) depends on t-2~~

(f) $y(t) = x\left(\frac{t}{3}\right)$

i) Memoryless ? No Depends on input at diff instant $\left(\frac{t}{3}\right)$

ii) Time-invariant ? No

Let $x_1(t) = x(t - t_0)$

$\Rightarrow y_1(t) = x_1\left(\frac{t}{3}\right) = x\left(\frac{t}{3} - t_0\right)$

But $y(t - t_0) = x\left(\frac{t - t_0}{3}\right) \Rightarrow y_1(t) \neq y(t - t_0)$

iii) Linear ? Yes

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$y_3(t) = x_3\left(\frac{t}{3}\right) = a_1 x_1\left(\frac{t}{3}\right) + a_2 x_2\left(\frac{t}{3}\right)$

$= a_1 y_1(t) + a_2 y_2(t)$ ✓

iv) Causal ? No

Consider $t = -1 \Rightarrow y(-1) = x\left(-\frac{1}{3}\right) \leftarrow$ future instant

v) Stable ? Yes $|x(t)| \leq B \Rightarrow |y(t)| \leq B$

(g) $y(t) = \frac{d\,x(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t) - x(t - \Delta t)}{\Delta t}$

i) Memoryless ? No, derivative requires past instant

ii) Time-invariant ? Yes

Proof : Let $x_1(t) = x(t - t_0)$

$\Rightarrow y_1(t) = \lim_{\Delta t \to 0} \frac{x_1(t) - x_1(t - \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t - t_0) - x(t - t_0 -}{\Delta t}$

But $y(t - t_0) = \lim_{\Delta t \to 0} \frac{x(t - t_0) - x(t - t_0 - \Delta t)}{\Delta t} = y_1(t)$ ✓

iii) Linear ? Yes.

iv) Causal ? ~~No~~ Yes (assuming left limit is sufficient for a differentiable fn)

v) Stable ? Yes (assuming diff. ip)

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1.28. Determine ~~for each~~ which of the properties in P1.27 hold & which do not for the following discrete-time systems. Justify your answers: $y[n] \rightarrow$ output, $x[n] \rightarrow$ input.

(e) $y[n] = x[-n]$

Memoryless : X For any $n \neq 0$, $y[n]$ either depends on a past or a future value of $x[n]$

Time Invariance : X $x[-n-n_0]$

$\neq y[n-n_0] = x[-(n-n_0)]$

Linear : ✓ $x_3[n] = \alpha x_1[+n] + \beta x_2[+n] \longrightarrow$

$\alpha x_1[-n] + \beta x_2[-n] = \alpha y_1[n] + \beta y_2[n]$

Causal : X For -ve values of n, $y[n]$ would depend on a future value of $x[n]$

Stable : ✓ If $|x[n]| \leq B$ then $|y[n]| = |x[-n]| \leq B$, $\forall n \in \mathbb{Z}$

(g) $y[n] = x[4n+1]$

Memoryless : X ex: $y[1]$ depends on $x[5]$

Time Invariant : X $x[4n+1-n_0] \neq y[n-n_0] = x[4(n-n_0)+1]$

Linear : ✓ $x_3[n] = \alpha x_1[n] + \beta x_2[n] \longrightarrow \alpha x_1[4n+1] + \beta x_2[4n+1]$

$= \alpha y_1[n] + \beta y_2[n]$

Causal : X ~~For $n \geq 0$, $n \in \mathbb{Z}$~~

For ex: consider $n=1$, $y[1]$ depends on $x[5]$

Stable : ✓ If $|x[n]| \leq B$, then $|y[n]| \leq |x[4n+1]| \leq B$.

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