STATISTICS

1) Measures of Central Tendency
- ↳* mean
- ↳ median
- ↳ mode
- ↳ Range

2) Measures of Deviation
- ↳ Mean Deviation
- ↳ Median Deviation
- ↳ Quartile Deviation
- ↳* Standard Deviation
- ↳* Variance Deviation

Mean ($\bar{x}$) =

$$= \frac{\text{sum of observations}}{\text{number of observations}} \rightarrow \boxed{\bar{x} = \frac{\sum_{i=1}^{n} x}{n}}$$

Eg. Find the mean of first n natural numbers

$$\bar{x} = \frac{1+2+3+4+\cdots+n}{n} \rightarrow \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

$$\bar{x} = \frac{n+1}{2}$$

Eg. Find the mean of binomial coefficients

$$\bar{x} = \frac{{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n}{n+1}$$

$$\bar{x} = \frac{2^n}{n+1}$$

Q) The mean of 20 observations is 15. If it is found that three observations are wrongly copied as 3,6 instead of 4,8 find the correct mean.

$$15 = \frac{S}{20} \Rightarrow S = 300$$

$$\bar{x} = \frac{S - 3 - 6 + 4 + 8}{20} = \frac{303}{200} = 15.15$$

Q) In a class of 30 students, average marks of 20 boys is 30 and average marks of 10 girls is 30.5. Find the avg mark

$$30 = \frac{S_b}{20} \Rightarrow S_b = 600$$

$$30.5 = \frac{S_g}{10} \cdot S_g = 305$$

$$\bar{x} = \frac{S_b + S_g}{30} = \frac{600 + 305}{30} = \frac{905}{30} = 30.16$$

Q) The average marks of boys in a class is 52 and girls is 42. If the average marks of the class is 50. Find % of boys

$$50 = \frac{52x + 42y}{x+y}$$

$$50x + 50y = 52x + 42y$$

$$2x = 8y \quad x = 4y \Rightarrow 80\%$$

$$\frac{x}{x + \frac{x}{4}} \times 100$$

Weighted AM.

x_1	x_2	x_3	x_4		.	x_n
w_1	w_2	w_3	w_4			w_n

weighted mean = $\dfrac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1+w_2+\cdots+w_n}$

Find the mean of the data

$1, 2, 2, 3, 3, 3, \ldots, \underbrace{10, 10, 10, \ldots 10}_{10\text{ times}}$

$\bar{x} = \dfrac{1\times1 + 2\times2 + 3\times3 + \cdots + 10\times10}{1+2+3+\cdots+10}$

$\bar{x} = \dfrac{\frac{10\times11\times21}{6}}{\frac{10\times11}{2}} = 7$

Q). $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots, \underbrace{\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10}}_{10\text{ times}}$

$\bar{x} = \dfrac{1\times1 + 2\times\frac{1}{2} + 3\times\frac{1}{3} + \cdots + 10\times\frac{1}{10}}{1+2+3+\cdots+10}$

$= \dfrac{10}{55} = \dfrac{2}{11}$

Q) $1, 1, 2, 2, 2, 3, 3, 3, 3, \ldots, \underbrace{10, 10, 10 \ldots 10}_{11\text{ times}}$

$\dfrac{1\times2 + 2\times3 + 3\times4 + \cdots + 10\times11}{2+3+4+\cdots+11}$

$\rightarrow \dfrac{1(1+1) + 2(2+1) + 3(3+1) + \cdots + 10(10+1)}{2+3+4+\cdots+11}$

$$\frac{(1+2+3+\cdots+10)(1^2+2^2+3^2+\cdots+10^2)}{65}$$

$$\frac{1}{\cancel{65}_{13}} \times \frac{\cancel{10}\times 11}{\cancel{2}} \times \frac{10\times 11\times 21}{6} = \frac{23410}{78}$$

Median

$x_1 < x_2 < x_3 < \cdots < x_n$.

n is odd.

median $= \frac{n+1}{2}^{th}$ observation. $\rightarrow x_{\frac{n+1}{2}}$

n is even

median $= \frac{n}{2}^{th}$ observation

Mode

1 mode $\rightarrow$ unimodal data

2 mode $\rightarrow$ bi modal data

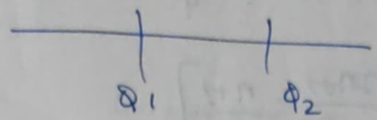
Emphuical formula.

Mean Deviation

$$\frac{\sum |x_i - \bar{x}|}{n}.$$

Quartile Deviation

$x_1 < x_2 < x_3 \ldots\ldots < x_n.$



cut the total no. of values into three parts.

$$Q.D = \frac{Q_2 - Q_1}{2}$$

Standard Deviation - Variance.

$\sigma \qquad \sigma^2$

$$\sigma^2 = \frac{\Sigma (x_i - \bar{x})^2}{n} \quad \text{or} \quad \sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2$$

1, 2, 3, 4, 5

Find variance, SD.

$$\bar{x} = \frac{\frac{5\times 6}{2}}{5} = 3$$

$$\sigma^2 = \frac{(3-1)^2 + (3-2)^2 + (3-3)^2 + (3-4)^2 + (3-5)^2}{5}$$

$$= \frac{4+1+0+1+4}{5} = 2$$

variance = 2 $\quad$ SD = $\sqrt{2}$

Find the variance of first 'n' natural number

$$\mu = \bar{x} = \frac{n+1}{2} \qquad \sigma^2 = \frac{\Sigma x_i^2}{n} - \mu^2$$

$$= \frac{1^2+2^2+3^2+\ldots+n^2}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2(n+1)(2n+1)}{6n} - \frac{(n+1)^2}{4}$$

$$= (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right]$$

$$= (n+1)\frac{(8n+4-6n-6)}{24}$$

$$\sigma^2 = \frac{n^2-1}{12} \Rightarrow SD = \sqrt{\frac{n^2-1}{12}}$$

#

Data : $x_1, x_2, \dots x_n$

	Mean	Variance	SD
X	μ	σ^2	σ
$X+k$	$\mu+k$	σ^2	σ
aX	$a\mu$	$a^2\sigma^2$	$a\sigma$
$aX+b$	$a\mu+b$	$a^2\sigma$	$a\sigma$

Variance of first n natural numbers $= \frac{n^2-1}{12}$

The variance of first 10 natural no.s $= \frac{99}{12} = \frac{33}{4}$

The variance of 2021, 2022, 2023, --- 2030

1+2020, 2+2020, 3+2020, --- +10+2020

Variance = $33/4$. $SD = \frac{\sqrt{33}}{2}$

The variance of first n odd numbers $= 2^2 \times \sigma^2$

$= 2^2 \times \frac{n^2-1}{12}$

$\Rightarrow \frac{n^2-1}{3}$

Mean of a, b, 8, 5, 10 is 6
variance of a, b, 8, 5, 10 is 6.8. Find a, b.

$$\frac{a+b+8+5+10}{5} = 6 \Rightarrow a+b = 7. \quad (1)$$

$$\frac{a^2+b^2+64+25+100}{5} - 36 = 6.8$$

$$a^2+b^2+189 = 214$$

$$a^2+b^2 = 25 \quad (2)$$

$(a+b)^2 = 49$

$25 + 2ab = 49 \rightarrow ab = 12 \qquad b = \frac{12}{a}.$

$a + \frac{12}{a} = 7 \rightarrow a^2 - 7a + 12 = 0 \rightarrow a = 3$ or 4
$b = 4$ or $3.$

$\Sigma(x-5) = 3 \qquad \Sigma(x-5)^2 = 43 \qquad$ no. of observations = 18.
Find SD of $x-5$

$$\sigma^2 = \frac{43}{18} - \left(\frac{3}{18}\right)^2$$

$$= \frac{43}{18} - \frac{1}{36} = \frac{85}{36} \Rightarrow \sigma = \frac{\sqrt{85}}{6}$$

There are two sets of data of 20 observations each. SD = 5 both. The mean of the first data is 17. the mean of the second data is 22. Find SD of combined data

$$\text{Variance} = \frac{\Sigma x_i^2}{20} - 17^2 = 5^2 = 25.$$

(I) $\Sigma x_i^2 = (289+25) \times 20 = 6280$

(II) $\Sigma x_i^2 = (484+25) \times 20 = 10180$

$\Rightarrow \Sigma x_i^2 = 6280 + 10180 = 16460$

Combined variance $= \dfrac{\Sigma x_i^2}{40} - \mu^2 = \dfrac{16460}{40} - \dfrac{39^2}{4}$

$\Rightarrow \dfrac{16460}{40} - \dfrac{1521}{4} \Rightarrow \dfrac{1250}{40}$

Combined variance $\rightarrow \dfrac{125}{4} = \sigma^2 \Rightarrow SD = \dfrac{\sqrt{125}}{2}$

Q) For two datas, each of size 5, the variances are given to be 4 & 5. The means are given to be 2 & 4 resp.

$\dfrac{\Sigma x_i^2}{5} - 2^2 = 4$ $\qquad$ $\dfrac{\Sigma y_i^2}{5} - 16 = 5$

$\Sigma y_i^2 = 105$

$\Sigma x_i^2 = 40$

$\sigma^2 = \dfrac{\Sigma x_i^2}{10} - 3^2$

$= \dfrac{145}{10} - 9 = 14.5 - 9 = 5.5$

variation $= 5.5 \rightarrow SD = \sqrt{5.5}$

In weighted

$\sigma^2 = \dfrac{\Sigma x_i^2 \, w_i}{\Sigma w_i} - \mu^2$

Ma

Statem

Car

Negate

Compound

Properties

$P \& Q$ =

$P \wedge (q \wedge r)$

$P \wedge ($

$P \vee P$

$P \vee Q$ =

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$P \vee$

Mathematical Reasoning

Statement → definite
↓
can be true or false.

Negate of a statement → opposite of a statement like adding a NOT or It is false - - -

compound statement

Combination of 2 or more statements

$\sim P \to$ complement of P.

Properties

$P \& Q = q \& P$ (Commutative)

$P \wedge (q \wedge r) = (p \wedge q) \wedge r$ (Associative)

$P \wedge (\sim P)$ = Fallacy (Always wrong)

$P \vee P = P$

$P \vee Q = q \vee P$

$P \vee (q \vee r) = (P \vee q) \vee r$

$P \vee (\sim P) = \tau$ - Tautology

$P \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$P \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$$\left.\begin{array}{l} \sim(p \vee q) = \sim p \wedge \sim q \\ \sim(p \wedge q) = \sim p \vee \sim q \end{array}\right\} \text{Demorgan's Law}$$

$P \vee \tau = \tau$ $\quad$ $p \wedge \tau = P$

$p \wedge F = F$

$p \Rightarrow q$

then converse $\rightarrow q \Rightarrow p$

Inverse $\rightarrow \sim p \Rightarrow \sim q$

Contrapositive $\sim q \Rightarrow \sim p$

(Inverse of the converse)

Exact differential equations

1) $y\,dx - (x+y^2)\,dy = 0$

$y\,dx - x\,dy - 2y^2\,dy = 0$

divide by y^2

$$\frac{y\,dx - x\,dy}{y^2} - \frac{2y^2\,dy}{y^2} = 0$$

$$\int d\left(\frac{x}{y}\right) - \int 2\,dy = 0$$

$$\frac{x}{y} - 2y = c$$

2) $$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$$

$$\frac{x\,dx + y\,dy}{y\,dx - x\,dy} = \frac{(x^2+y^2)^2}{x^2}$$

$$\Rightarrow \frac{1}{2}\,\frac{2x\,dx + 2y\,dy}{(x^2+y^2)^2} = \frac{y\,dx - x\,dy}{x^2}$$

$$\frac{1}{2}\int \frac{d(x^2+y^2)}{(x^2+y^2)^2} = \int d\left(\frac{y}{x}\right)$$

$$\frac{1}{2}\cdot\frac{-1}{x^2+y^2} = -\frac{y}{x} + c$$

$$\frac{y}{x} - \frac{1}{2(x^2+y^2)} = c$$

1) $\left(\frac{1}{x} - \frac{y^2}{(x-y)^2}\right)dx + \left(\frac{x^2}{(x-y)^2} - \frac{1}{y}\right)dy$

$$\frac{dx}{x} - \frac{dy}{y} + \frac{x^2 dy - y^2 dx}{(x-y)^2}$$

$$\cancel{x^2 \frac{(dy-dx)}{(x-y)^2} + \frac{(x^2-y^2)}{(x-y)^2}dx}$$

2) $\cancel{\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}}$

Linear D. Eqn (LDE)

$$\frac{dy}{dx} + P(x)y = Q(x) \rightarrow \text{LDE in } x$$

Integral factor $IF = e^{\int P(x)dx}$

solution $y(IF) = \int (IF)\, Q(x)\, dx$

Q) $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$$\frac{dy}{dx} + y P(x) = Q(x)$$

$$P(x) = \frac{1}{1+x^2} \Rightarrow IF = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

$$y(IF) = \int IF \cdot Q(x) \cdot dx$$

$$y \cdot e^{\tan^{-1}x} = \int e^{\tan^{-1}x} \cdot \frac{e^{\tan^{-1}x}}{1+x^2} dx$$

$$\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$$

$$\int e^{2t}\,dt = \frac{e^{2t}}{2} + c = \frac{e^{2\tan^{-1}x}}{2} + c$$

$$y \cdot e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c$$

$$\Rightarrow y = \frac{e^{\tan^{-1}x}}{2} + c$$

$$\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$$

$$IF = e^{\int -\cot x\,dx} = e^{-\log|\sin x|} = \operatorname{cosec} x.$$

$$y\operatorname{cosec} x = \int \operatorname{cosec}^2 x\,dx.$$

$$= -\cot x + c.$$

$$(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\cdot y = \frac{x}{\sqrt{1-x^2}}$$

$$P(x) = \frac{2x}{1-x^2}, \quad IF = e^{-\int \frac{-2x}{1-x^2}dx}$$

$$= e^{-\log|1-x^2|}$$

$$= \frac{1}{1-x^2}$$

$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}\,dx.$$

$$= -\frac{1}{2}\int \frac{-2x\,dx}{(1-x^2)^{3/2}} = \frac{-1}{2} \times \frac{(1-x^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}$$

$$= \frac{1}{\sqrt{1-x^2}} + c$$

$$\boxed{y = \sqrt{1-x^2} + c(1-x^2)}$$

Q) $x \log x \frac{dy}{dx} + y = x^2$

$\cancel{\frac{dy}{dx}}$ $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{x}{\log x}$

$\frac{dy}{dx} + y \cdot P(x) = Q(x)$

$P(x) = \frac{1}{x \log x}$ $\qquad Q(x) = \frac{x}{\log x}$

$IF = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

$y(IF) = \int IF \cdot Q(x)\, dx$

$y\, \cancel{e^{\log(\log x)}} \log x = \int \cancel{e^{\log \log x}} \log x \cdot \frac{x}{\log x} dx.$

~~$\Rightarrow \log x = t \quad dt = \frac{1}{x} dx \quad \frac{1}{x} dx$~~

$= \int \cancel{e^{\log \log x}} \cdot \frac{\cancel{\frac{1}{\log x}}}{\cancel{\frac{1}{x dx}}}$

$y \log x = \frac{x^2}{2} + c$

LDE in y $\frac{dx}{dy} + P(y) \cdot x = Q(y)$

$IF = e^{\int P(y)\, dy}$

$x\,(IF) = \int (IF) \cdot Q(y)\, dy$

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2}dx} = e^{\tan^{-1}y}$$

Q) $y' - y\tan x = \sec^3 x$.

$$\frac{dy}{dx} - \tan x \cdot y = \sec^3 x.$$

$P(x) = -\tan x$. $Q(x) = \sec^3 x$.

$$IF = e^{\int -\tan x\,dx} = e^{\log|\sec x|^{-1}} = \cos x.$$

$$\cancel{y \cdot e^{\sec^2 x} = \int e^{\sec^2 x}\sec^3 x\,dx.}$$

$$\cancel{y\,e^{-\sec^2 x} = \int e^{-\sec^2 x}\sec^3 x\,dx.}$$

$$\cancel{f(x) = -\sec^2 x \quad f'(x) = 2\sec x \cdot \sec x \tan x.}$$

$$\cancel{g(x) = \frac{\tan x}{2} = \frac{1}{2\tan x}}$$

$$y \cdot \cos x = \int \cos x \cdot \sec^3 x\,dx = \int \sec^2 x\,dx$$

$$y \cdot \cos x = \tan x + c$$

$$y = \tan x \cdot \sec x + c$$

Q) $y\,dx - x\,dy + \ln x\,dx = 0$

$(y + \ln x)dx = x\,dy$

$\frac{dy}{dx} = \frac{y}{x} + \frac{\ln x}{x} \rightarrow \frac{dy}{dx} - \frac{y}{x} - \frac{\ln x}{x} = 0$

$P(x) = -1/x \quad Q(x) = -\frac{\ln x}{x}$

$IF = e^{-\int 1/x\,dx} = e^{-\log|x|} = \cancel{x}\ \frac{1}{x}$.

$y \cdot \cancel{e^{\log|x|}}$

$y \cdot \frac{1}{x} = \int \frac{1}{x} \cdot -\frac{\ln x}{x}\,dx$.

$y \cdot \frac{1}{x} = \int -\frac{\ln x\,dx}{x^2}$

$\cancel{y \cdot \frac{1}{x}} = -\left(\ln x \cdot \left(-\frac{1}{x}\right) - \int \frac{1}{x} \cdot \frac{-1}{x}\,dx\right)$

$= \frac{\ln x}{x} + \frac{1}{x}$.

$y \cdot \frac{1}{x} = \frac{\ln x + 1}{x} + c$

$y = \ln x + 1 + cx$

Bernouli's D. Eqn.

$$\frac{dy}{dx} + P(x)\,y = Q(x)\cdot y^n$$ → Convert into linear differential equation.

$$\frac{1}{y^n}\frac{dy}{dx} + \underbrace{\frac{1}{y^{n-1}}}_{\to t} P(x) = Q(x)$$

$$y^{1-n} = t$$

$$(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-n}\cdot\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dt}{dx}$$

$$\frac{1}{1-n}\cdot\frac{dt}{dx} + t\cdot P(x) = Q(x)$$

$$\cancel{y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}}$$

$$\cancel{\frac{1}{1-n}\cdot\frac{dt}{dx} + t\cdot P(x) = Q(x)}$$

$$\boxed{\frac{dt}{dx} + (1-n)\cdot P(x)\cdot t = (1-n)\,Q(x)}$$

↳ solve by LDE

Q). $$\frac{dy}{dx} = x^2y^2 + xy$$

$$\frac{dy}{dx} - xy = x^2y^2$$

$$\frac{1}{y^2}\cdot\frac{dy}{dx} - \frac{1}{y}\cdot x = x^2$$

$$\frac{-1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + t \cdot x = x^2 \qquad IF = e^{\int x\,dx} = e^{x}$$

$$t \cdot e^{x} = \int e^{x} \cdot x^2\,dx$$

$$= e^{x}(x^2 - 2x + 2) + c$$

$$-\frac{1}{y} \cdot e^{x} = e^{x}(x^2 - 2x + 2) + c$$

$$\frac{1}{y} = -(x^2 - 2x + 2) + c \cdot e^{-x}$$

2) $(xy^3(1+\cos x) - y)\,dx + x\,dy = 0$

$$\frac{1}{y^3} x \cdot \frac{dy}{dx} - \frac{y}{y^3 x} = (1+\cos x) \cdot \cancel{y^3}$$

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2} \cdot \frac{1}{x} = (1+\cos x)$$

let $t = \frac{-1}{y^2} \Rightarrow \frac{2}{y^3} \cdot \frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{1}{2}\frac{dt}{dx} + t \cdot \frac{1}{x} = 1 + \cos x$$

$$\frac{dt}{dx} = t \cdot \frac{2}{x} = 2(1+\cos x)$$

$$IF = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$$

$$t \cdot x^2 = \int x^2 \cdot 2(1+\cos x)\,dx.$$

$$\frac{-x^2}{y^2} = 2\int x^2 + x^2\cos x\,dx$$

$$= 2\cdot\frac{x^3}{3} + 2\left[x^2\sin x + 2x\cos x \cdot 2\sin x\right] + c$$

$$2\frac{dy}{dx} = \cancel{2}xy\,(y^2\sin^{2}x^2+1)$$

$$2\frac{dy}{dx} = 2xy^3\sin^{2}x^2 + xy$$

$$\frac{2}{y^3}\frac{dy}{dx} - \frac{2}{y^2}x = 2x\sin^{2}x^2.$$

$$t = -\frac{1}{y^2} = -y^{-2} \qquad \frac{dt}{dx} = \frac{2}{y^3}\frac{dy}{dx}$$

$$\cancel{\frac{1}{2}}\cdot\frac{dt}{dx} + t\cdot 2x = 2x\sin^{2}x^2.$$

$$IF = e^{\int x\,dx} = e^{x^2}$$

$$t\cdot e^{x^2} = \int e^{x^2}\cdot 2x\sin^{2}x^2\,dx.$$

$$x^2 = K$$

$$t\cdot e^K = \int e^K \cdot \sin K\,dK$$

$$= \frac{e^K}{2}\int(K\sin K - \cos K) + c$$

$$\frac{e^{x^2}}{y^2} = \frac{e^{x^2}}{2}\left[x\sin^{2}x^2 - \cos x^2\right] + c$$

The population $p(t)$ at a time 't' of a certain mouse species satisfy the d.eqn

$$\frac{dP}{dt} = 0.5\, P(t) - 450$$

If $P(0) = 850$, Then the time at which the population of the mouse in zero.

$$\frac{dP}{dt} = \frac{P(t) - 900}{2}$$

$$\frac{dP}{P(t) - 900} = \frac{dt}{2}$$

$$\ln |P(t) - 900| = \frac{t}{2} + c$$

at $t = 0$, $P(t) = 850$

$$\ln |850 - 900| = c \rightarrow c = \ln 50$$

$$\frac{P(t) - 900}{50} = e^{t/2}$$

$$P(t) = -50 e^{t/2} + 900$$

$$\ln |900 - P(t)| = \frac{t}{2} + \ln$$

$t \Rightarrow P(t) = 50$

$$\ln |900| = \frac{t}{2} + \ln 50$$

$$\frac{t}{2} = \ln \frac{900}{50} = \ln 18$$

$$t = 2 \ln (18)$$

The population of a country doubles in 50 years

If the rate of increase in the population is proportional to the no. of inhabitants

If x_0 be the no. of inhabitants, at time $t=0$.

$x(t)$ is the population at time t.

$$\frac{dx}{dt} \propto x(t)$$

$$\Rightarrow \frac{dx}{dt} = k \cdot x(t)$$

$$\frac{dx}{x} = k \cdot dt. \rightarrow \ln x = kt + c$$

$$x = c \cdot e^{kt}.$$

at $t = 0$, $x = x_0$

$$x_0 = ce^0 = c.$$

$$\boxed{x(t) = x_0 \cdot e^{kt}}$$

at $t = 50$

$$x(t) = 2x_0$$

$$2x_0 = x_0 \cdot e^{50k}.$$

$$\ln 2 = 50k$$

$$k = \frac{\ln 2}{50}$$

$$x(t) = x_0 \cdot e^{\frac{\ln 2}{50} t}$$

for $x(t) = 3x_0$

$$3x_0 = x_0 \cdot e^{\frac{\ln 2}{50} \cdot t}.$$

$$\ln 3 = \frac{\ln 2}{50} t \rightarrow t = 50 \cdot \frac{\ln 3}{\ln 2} \approx 80 \text{ years}$$

Orthogonal Trajectories

1) To form D.Eqn m/s $y = f(x, c)$

$$\frac{dy}{dx} = \phi(x, y)$$

2) Replace by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \phi(x, y)$$

↳ solve this equation to find out orthogonal curve

Find OT of $y^2 = 4ax$

$$2y \cdot y' = 4a$$

$$y^2 = 2y \cdot \frac{dy}{dx} x.$$

$$\rightarrow y = 2 \frac{dy}{dx} \cdot x$$

For OT $\frac{dy}{dx} \rightarrow \frac{-dx}{dy}$

$$y = 2x \cdot \frac{dx}{dy} x$$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = x^2 + c$$

$$\boxed{x - \frac{y^2}{2} = c}$$

Find the OT of $xy = c^2$

$$x\frac{dy}{dx} + y = 0$$

For OT, $\frac{dy}{dx} \to -\frac{dx}{dy}$

$$x \times -\frac{dx}{dy} + y = 0$$

$$-\int x\,dx + \int y\,dy = 0$$

$$\frac{-x^2}{2} + \frac{y^2}{2} = c$$

$$y^2 - x^2 = c$$

Find the OT of $x^2 + y^2 - 2ax = 0$.

$$2x + 2y \cdot \frac{dy}{dx} - 2a = 0.$$

$$x + y \cdot \frac{dy}{dx} = a$$

For OT, $\frac{dy}{dx} \to -\frac{dx}{dy}$

$$x + y \cdot \left(-\frac{dx}{dy}\right) = a$$

$$\frac{x\,dy - y\,dx}{y^2} = \frac{a\,dy}{y^2}$$

$$d\left(\frac{x}{y}\right) = a\frac{dy}{y^2}$$

$$\int d\left(\frac{x}{y}\right) = a\int \frac{dy}{y^2}$$

$$\frac{x}{y} = \frac{-2a}{y^3} \qquad xy^2 = -2a$$

$$x^2+y^2-2ax=0$$

$$2x+2yy'=2a$$

$$x^2+y^2=x(2x+2y\cdot y')$$

$$x^2+y^2=2x^2+2xy\cdot y'$$

$$y^2-x^2=2xy\cdot\frac{dy}{dx}$$

For OT, $\frac{dy}{dx}\rightarrow\frac{-dx}{dy}$

$$y^2-x^2=2xy\cdot\frac{-dx}{dy}$$

$$\frac{dx}{dy}=\frac{x^2-y^2}{2xy}$$

$$x=vy$$

$$v+y\frac{dv}{dy}=\frac{v^2y^2-y^2}{2vy^2}=\frac{v^2-1}{2v}$$

$$y\cdot\frac{dv}{dy}=\frac{v^2-1}{2v}-v$$

$$=\frac{-v^2-1}{2v}$$

$$\frac{2v}{v^2+1}=-\frac{dy}{y}$$

$$\log|v^2+1|=-\log|y|+c$$

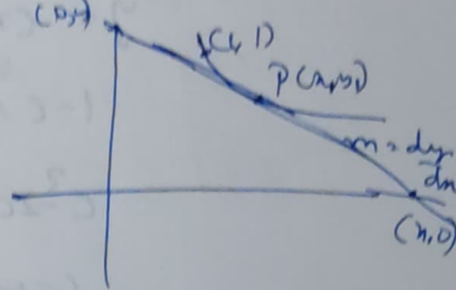
$$y(v^2+1)=c.$$

$$y\left(\frac{x^2}{y^2}+1\right)=c$$

$$x^2+y^2=yc$$

Let y = f(x) be a curve passing through (1,1) such that the triangle formed by the coordinate axis & the tangent at any point in the curve lies in the first quadrant has area 2 sq. units. Find the curve.

$y = f(x)$ at $(1,1)$



$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$y - y_1 = m(x - x_1)$$

at A, $y = 0$

$$\Rightarrow -y_1 = m(x - x_1)$$

$$-\frac{y_1}{m} + x_1 = x$$

at B, $x = 0$

$$y - y_1 = m(-x_1)$$

$$y = -y_1 - mx_1$$

Area of ΔOAB = 2 sq units

$$\frac{1}{2}\left|\left(x_1 - \frac{y_1}{m}\right)(+y_1 - mx_1)\right| = 2$$

$$x_1y_1 - mx_1^2 - \frac{y_1^2}{m} + x_1y = 4$$

$$m^2x_1^2 - (2x_1y_1 - 4)m + y_1^2 = 0$$

$$m^2x_1^2 - 2x_1my + y_1^2 = -4m$$

$$(mx_1 - y_1)^2 = -4m$$

$$mx_1 - y_1 = \pm\sqrt{-4m}$$

$$y_1 - mx_1 = \pm\sqrt{4m}$$

$$y_1 = mx_1 \pm \sqrt{4m}$$

By claiut's DEqn, passing through $(1,1)$

$$y = cx \pm \sqrt{-4c}$$

$$1 - c = \sqrt{-4c}$$

$$c^2 - 2c + 1 = -4c$$

$$(c+1)^2 = 0.$$

$$c = -1$$

Clairut's deqn.

$$y = mx + \sqrt{f(m)}$$

$$m = \frac{dy}{dx} \; ; \quad m \to c$$

$$y = cx \pm \sqrt{f(c)}$$

A curve is such that the length of ⊥r from origin onto the tangent at any point P of the curve is equal to abscissa of P. Find the curve.

let $P(x_1, y_1)$ be a point on the curve

The tangent at P. $y - y_1 = m(x - x_1)$

$$mx - y + y_1 - mx_1 = 0$$

distance from the origin $= x_1$

$$\frac{|y_1 - mx_1|}{\sqrt{m^2+1}} = x_1$$

$$(y - mx)^2 = x^2(m^2+1)$$

$$y^2 + \cancel{m^2x^2} - 2xym = \cancel{m^2x^2} + x^2$$

$$y^2 - x^2 = 2xy \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$$y = vx$$

$$\frac{dy}{dx} = \frac{v^2\cancel{x^2} - \cancel{x^2}}{2 \cdot \cancel{x} \cdot v \cancel{x}} = \frac{v^2-1}{2v}$$

$$v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v} = \frac{-(v^2+1)}{2v}$$

$$\frac{dx}{x} = \frac{-2v}{v^2+1}\, dv$$

$$\log x = -\log|v^2+1| \;{+c}$$

$$\log x + \log v^2+1 = c$$

$$x(v^2+1) = c$$

$$x\left(\frac{y^2}{x^2}+1\right) = c$$

$$x^2+y^2 = xc$$

$$y = (c_1+c_2)\cos(x+c_3) - (c_4+c_5)\sin(x+c_6) + c_7 e^{x+c_8}$$

$c_1, \dots, c_8$ are arb constant

Find the order of the differential eq'n

$\Rightarrow$

$$\cos x\left[(c_1+c_2)\cos c_3 - (c_4+c_5)\sin c_6\right] +$$
$$\sin x\left[-(c_1+c_2)\sin c_3 - (c_4+c_5)\cos c_6\right] +$$
$$c_7 \cdot e^{?} \cdot c e^{x}$$

$\rightarrow$ 3 constants $\rightarrow$ order - 3

$$(1+x^2y^2)y\,dx - (1-x^2y^2)\cdot x\,dy = 0$$

$$y\,dx - x\,dy + (x^2y^2)y\,dx + (x^2y^2)x\,dy = 0$$

$$y\,dx - x\,dy + x^2y^2(y\,dx + x\,dy) = 0$$

$$\frac{y\,dx - x\,dy}{xy} + \frac{x^2y^2(y\,dx + x\,dy)}{xy} = 0$$

$$\int d\log\left(\frac{x}{y}\right) + \int xy \cdot d(xy) = 0$$

$$\log\frac{x}{y} + \frac{x^2y^2}{2} = c$$

Q) $(xy^4+y)\,dx - x\,dy = 0$

$$\frac{xy^4\,dx + y\,dx - x\,dy}{y^2} = 0$$

$$\frac{y\,dx - x\,dy}{y^2} = -\frac{xy^2\,dx}{x^2} \times x^2 \rightarrow \int\left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = -x^3 dx$$

$$\rightarrow \frac{\left(\frac{x}{y}\right)^3}{3} = -\frac{x^4}{4} + c$$

Q) $(2x\cos y + y^2\cos x)\,dx + (2y\sin x - x^2\sin y)\,dy = 0$

$2x\,dx\,\cos y + x^2(-\sin y)\,dy + y^2(\cos x\,dx) + (2y\,dy)\sin x = 0$

$\int d(x^2)\cos y + x^2 d(\cos y) + \int y^2(d\sin x) + d y^2 \sin x$ -

$x^2\cos y + y^2\sin x = 0$

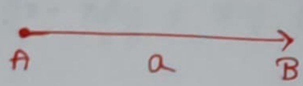
VECTOR ALGEBRA

- ↳ Algebra of vectors
- ↳ Scalar product
- ↳ vector product
- ↳ Scalar triple product
- ↳ Vector triple product
- ↳ Multiple products

Physical quantities

- Scalar
  - No direction
  - only magnitude
- Vector
  - Both direction and magnitude

Directed ~~the~~ line segment



Vector $\overrightarrow{AB}$, maybe denoted by $\overrightarrow{a}$

$A \rightarrow$ initial point

$B \rightarrow$ terminal point

$AB \rightarrow$ Length of the vector

$$AB = |\overrightarrow{AB}| = |\vec{a}|$$

Types of Vectors

- ↳ Zero vector — Arbitrary direction
- ↳ unit vector
- ↳ Parallel or collinear vector
- ↳ Like - unlike vectors
- ↳ Equal - Unequal vectors
- ↳ Co-initial vectors
- ↳ Co-terminal vectors

↳ Coplanar vector
↳

Two vectors are said to be parallel or collinear if they have same line of support or parallel line of support.

CONDITION

If $\bar{a}, \bar{b}$ are parallel then $\bar{b} = \lambda \bar{a}$ for some scalar λ

If $\bar{a}, \bar{b}$ are collinear, then $\bar{b} = \lambda \bar{a}$ for some scalar λ

a, b are equal if

↳ Line of supports are equal/ parallel.
↳ directions are equal
↳ magnitudes are equal.

a, b are opposite if

↳ Line of supports are equal / parallel
↳ opposite directions
↳ magnitudes are equal

Algebra of vectors

↳ Scalar multiplication
↳ Vector addition

Scalar multiplication

Let λ be a scalar & $\bar{a}$ be a vector

Then $\lambda \cdot \bar{a}$ is called scalar multiplication.

* $\vec{a}, \lambda\vec{a}$ are always parallel or collinear
* $\vec{a}, \lambda\vec{a}$ are like vectors if $\lambda > 0$
* $\vec{a}, \lambda\vec{a}$ are unlike vectors if $\lambda < 0$

* $|\lambda \vec{a}| = |\lambda| \, |\vec{a}|$

* $0 \cdot \vec{a} = \vec{0}$

* $k \cdot \vec{0} = \vec{0}$

* $\lambda \vec{a} = \vec{0} \Rightarrow$ either $\lambda = 0$ or $\vec{a} = \vec{0}$

Addition of Vectors

The resultant of a, b is the diagonal of the parallelogram who adjacent sides are a, b.

The diagonals are $\vec{a}+\vec{b}$, $\vec{b}-\vec{a}$

Properties

* $\vec{a}+\vec{b} = \vec{b}+\vec{a}$
* $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$
* $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$
* $\vec{a}+(-\vec{a}) = \vec{0}$
* $\lambda(\vec{a}+\vec{b}) = \lambda\vec{a}+\lambda\vec{b}$
* $(\lambda_1+\lambda_2)\vec{a} = \lambda_1\vec{a}+\lambda_2\vec{a}$

* The vector in the direction of the angular bisector of the angle between $\vec{a}, \vec{b}$ is $\lambda(\hat{a}+\hat{b})$

The unit vector in the direction of angle bisector of angle between $\vec{a}$ & $\vec{b}$ is

$$\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

3-D vector

$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{x}| = \sqrt{x^2+y^2+z^2}$ (or) $|x\hat{i}+y\hat{j}+z\hat{k}| = \sqrt{x^2+y^2+z^2}$

$\vec{a}, \vec{b}$ are collinear if $\vec{b} = \lambda\vec{a}$

$\vec{a}, \vec{b}, \vec{c}$ are coplanar iff $\underbrace{\vec{c} = x\vec{a} + y\vec{b}}_{\text{Linear combination of } a, b.}$ for some x, y

Linear Combination of Vectors

Let $\vec{a_1}, \vec{a_2}, \vec{a_3}, \ldots, \vec{a_n}$ be n-vectors in a n-dimensional vector space.

Then $x_1 \cdot \vec{a_1} + x_2 \cdot \vec{a_2} + x_3 \vec{a_3} + \ldots + x_n \cdot \vec{a_n}$ is called Linear combination of the given vectors

eg. $\vec{a} = \hat{i} + \hat{j}$ $\vec{b} = 3\hat{i} - \hat{j}$ $\vec{c} = 2\hat{i} + 3\hat{j}$

$\vec{c} = x\vec{a} + y\vec{b}$

$2\hat{i} + 3\hat{j} = x(\hat{i}+\hat{j}) + y(3\hat{i}-\hat{j})$

$x + 3y = 2$
$x - y = 3$ $\rightarrow$ $y = -1/4$ $x = 13/4$

$\vec{c} = \dfrac{13\vec{a} - \vec{b}}{4} = (x\vec{a} + y\vec{b})$

Linear Independent vectors

Let $\vec{a_1}, \vec{a_2}, \vec{a_3}, \dots, \vec{a_n}$ be n-vectors of a n-dimensional vector space

$$x_1\bar{a_1} + x_2\bar{a_2} + x_3\bar{a_3} + \dots + x_n.\bar{a_n} = \bar{0}$$ for some sc $x_1, x_2, \dots$

The given vectors are said to be

1) Linearly independent vectors if
$x_1 = x_2 = x_3 = \dots = x_n = 0$

2) Linearly dependent vectors if, at least one of
$x_1, x_2, \dots, x_n$ is non zero.

~~$0x + 0y = 0 \quad x + y = 0$~~

eg. $2i, i+j$

$\bar{a} = 2i, \quad \bar{b} = i+j$

$x\bar{a} + y\bar{b} = 0$

$x(2i) + y(i+j) = 0$

$2x + y = 0 \quad y = 0 \quad \rightarrow x = 0 \; y = 0.$

The vectors $i+j, \; j-k, \; i-j+2k$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 1 - 1(1) = 0$$

Linear dependent

{ $\neq 0 \rightarrow$ independent }

If $\bar{0}$ is one of the vectors then they will always be linear dependent eg) $i+j-k$, $i-j+2k$ are

$$x(i+j-k) + y(i-j+2k) = \bar{0}$$

$$x+y=0 \quad x-y=0 \quad -x+2y=0.$$

$$x=y=0 \Rightarrow \text{Linearly independent}$$

If the number of dimensions is ~~grea~~ lesser than the number of vectors, dependent

$(1-a)i + j + k$, $i + (1-b)j + k$, $i + j + (1-c)k$ are L.D then.

$$\begin{vmatrix} 1-a & 1 & 1 \\ 1 & 1-b & 1 \\ 1 & 1 & 1-c \end{vmatrix} = 0$$

$$(1-a)(1-b-c+bc-1) = 0$$

$$a=1 \quad \text{or} \quad b+c=bc.$$

#

1) If $\bar{a}, \bar{b}$ are collinear vectors then $\bar{a}, \bar{b}$ are L.D.

2) If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors then $\bar{a}, \bar{b}, c$ are L.D

~~Po~~ Position Vector

Let P be a point on a plane. The position vector of a point P is called $\vec{r}$ if $\overrightarrow{OP} = \bar{r}$.

$$\text{If } P(\bar{r}) \Rightarrow \overrightarrow{OP} = \vec{r}$$

Position vector is a point, not a line segment

Colinear points

Let $A(\bar{a}), B(\bar{b}), C(\bar{c})$ be 3 points. ~~A,B~~

A, B, C are collinear if $\overrightarrow{AB}$ & $\overrightarrow{AC}$ are collinear

$$\overrightarrow{AC} = \lambda \overrightarrow{AB}$$

$$\overline{c-a} = \lambda \cdot (\bar{b} - \bar{a})$$

$$(\lambda - 1)\bar{a} - \lambda \bar{b} + \bar{c} = \vec{0}$$

↳ $\bar{a}, \bar{b}, \bar{c}$ are LD then matrix = 0.

↳ $x\bar{a} + y\bar{b} + c\bar{y} = 0$ then $x + y + z = 0$

If $\bar{a}, \bar{b}, \bar{c}$ are 3 collinear points $\Rightarrow x\bar{a} + y\bar{b} + 10\bar{c} = 0$

$x, y \in N$, No. of possible (x, y) is

$$x + y - 10 = 0$$

$$x + y = 10 \quad \rightarrow 9 \text{ values.}$$

$\bar{a}, \bar{b}, \bar{c}$ are collinear points

$$(\lambda^2 - 1)\bar{a} + (2 - 3\lambda)\bar{b} + (\lambda)\bar{c} = \bar{0}$$

$$\lambda^2 - 1 + 2 - 3\lambda + \cancel{\lambda} = 0$$

$$\lambda^2 - 3\lambda + \cancel{1}\, 2 = 0$$

$$\lambda = 1, 2$$

Coplanar points

Given $A(\bar{a}), B(\bar{b}), C(\bar{c}), D(\bar{d})$ be 4 points.

A, B, C, D are coplanar if ~~AB~~

$\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar

$\overrightarrow{AD} = x \cdot \overrightarrow{AB} + y \cdot \overrightarrow{AC}$

$AD = x \cdot (\bar{b} - \bar{a}) + y \cdot (\bar{c} - \bar{a})$

$(x+y-1)\bar{a} - x\bar{b} - y\bar{c} + \bar{d} = 0$

$\hookrightarrow \bar{a}, \bar{b}, \bar{c}, \bar{d}$ are LD

$\hookrightarrow x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0} \rightarrow x+y+z+w=0$

* If the vector $x\hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} - 3\hat{j} + \hat{k}$, $-\hat{i} - \hat{j} + 2\hat{k}$ are coplanar then x, —

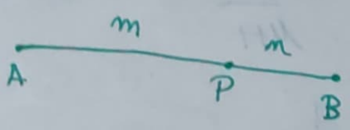
$x(x\hat{i} + \hat{j} + 2\hat{k}) + y(i - j + k) + z(2i - 3j + k) +$

$w(-i - j + 2k) = 0$

$x^2 + y + 2z - w = 0$

$x + y +$

Section formula



$\dfrac{m \cdot \bar{b} + n \cdot \bar{a}}{m+n}$ if internal division

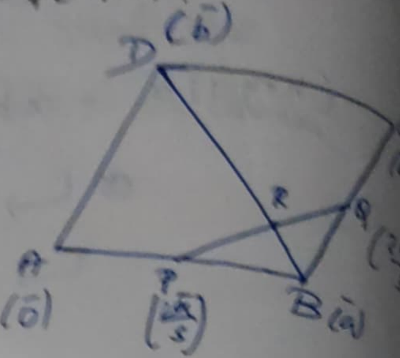
$\dfrac{m\bar{b} - n\bar{a}}{m+n}$ if external division

Q) let ABCD be a parallelogram & P, Q be points of $\overline{AB}, \overline{BC}$ $\ni$ $AP:PB = 2:3$, $BQ:QC = 1:2$

If $\overline{PQ}$ divides $\overline{BD}$ at R.

Then i) $BR:RD$

2) $PR:RQ$



P_v of $P = \frac{2\bar{a}}{5}$

P_v of $Q = \frac{2\bar{a} + 1(\bar{a}+\bar{b})}{3} = \frac{3\bar{a}+\bar{b}}{3}$

P_v of $R =$ P_v of R.

Along $\overline{PQ}$ Along $\overline{BD}$

$$\frac{\lambda\left(\frac{2\bar{a}}{5}\right) + 1\left(\frac{3\bar{a}+\bar{b}}{3}\right)}{1+\lambda} = \frac{1\cdot\bar{b} + \mu\bar{a}}{\mu+1}$$

$PR:RQ = 1:\lambda = 1:\frac{5}{9} = 9:5$

Compare $\bar{a}$ components

$$\frac{\frac{2\lambda}{5}+1}{1+\lambda} = \frac{\mu}{\mu+1}$$

$$\frac{1}{3(1+\lambda)} = \frac{1}{\mu+1}$$

$$\frac{2+3\lambda}{3+3\lambda} = \frac{\mu}{\mu+1}$$

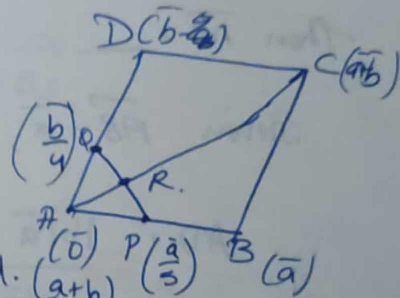
$$\frac{\frac{2\lambda+1}{5}}{\cancel{1+\lambda}} = \frac{2+3\lambda}{3(\cancel{1+\lambda})}$$

$$\frac{6\lambda}{5} + 3 = 2 + 3\lambda.$$

$$3\lambda - \frac{6\lambda}{5} = 1 \qquad \frac{9\lambda}{5} = 1 \Rightarrow \boxed{\lambda = 5/9}$$

let ABCD be a parallelogram & P, Q be points of $\overline{AB}$, $\overline{AD}$ ∋ $AP:PB = 1:4$, $AQ:QD = 1:3$ If $\overline{AC}$ intersects PQ at R, then $AR:RC$ is $\to \mu$ $\quad \lambda \to PR:RQ$.

ABCD is a parallelogram



$$\frac{\lambda\left(\frac{a}{5}\right) + 1\left(\frac{b}{4}\right)}{1+\lambda} = \frac{1.0 + \mu.(a+b)}{\mu+1}$$

$$\frac{\frac{\lambda a}{5} + \frac{b}{4}}{1+\lambda} = \frac{\mu.a + \mu.b}{\mu+1.}$$

$$\frac{\lambda}{5(1+\lambda)} = \frac{\mu}{\mu+1} \qquad \frac{b \cdot 1}{4(1+\lambda)} = \frac{\mu}{\mu+1.}$$

$$\frac{\lambda}{5(\cancel{1+\lambda})} = \frac{1}{4(\cancel{1+\lambda})} \qquad \frac{1}{4\left(1+\frac{5}{4}\right)} = \frac{\mu}{\mu+1}$$

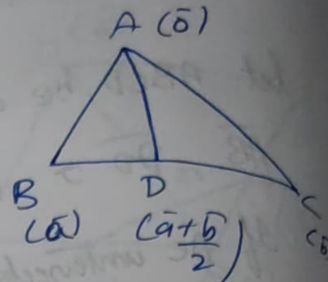
$$4\lambda = 5 \qquad \frac{\mu}{\mu+1} = \frac{1}{9}$$

$$\boxed{\lambda = 5/4} \qquad 9\mu = \mu+1$$

$$\boxed{\mu = 1/8}$$

In ΔABC, If $\overline{AB} = 3\hat{i} + 4\hat{j}$, $\overline{AC} = 5\hat{i} - 12\hat{j}$ and D is mid points of $\overline{BC}$, find $\overline{AB}$

$$\overrightarrow{AD} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}$$

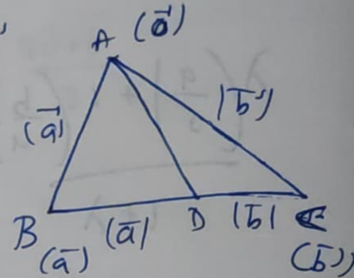


In ΔABD, AD is then angular bisector of $\angle A$.

Then $\overline{AD} =$ ____

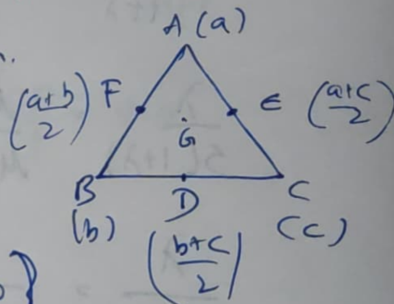
Given $\overrightarrow{AB} = \bar{a}$, $\overrightarrow{AC} = \bar{b}'$

$$AD = \frac{|\bar{b}|\bar{a} + |\bar{a}|\bar{b}}{|\bar{a}| + |\bar{b}|}$$



Centeroid, circumcenter, orthocenter

Let ABC be a Δ^{le} then.



1) $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$$\left\{ \bar{b} - \bar{a} + \bar{c}' - \bar{b} + \bar{a} - \bar{c} = \bar{0} \right\}$$

2) $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}' = 0$

$$\left\{ \frac{b+c}{2} - \bar{a} + \frac{c+a}{2} - \bar{b} + \frac{a+b}{2} - c \right\} = 0$$

3) $\overrightarrow{DE} + \overrightarrow{EF}' + \overrightarrow{FD}' = 0$

$$\frac{1}{2}\overrightarrow{BA}' + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{AC}'$$

$$= \frac{1}{2}(\bar{a} - \bar{b} + \bar{b} - \bar{c} + \bar{c} - \bar{a}) = 0$$

4) $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$

$\vec{a} - \vec{OG} + \vec{b} - \vec{OG} + \vec{c} - \vec{OG}$

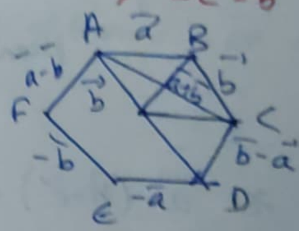
$a+b+c-3\vec{OG} = a+b+c-3\left(\frac{a+b+c}{3}\right) = \vec{0}$

5) $\vec{GD} + \vec{GE} + \vec{GF} = \vec{0}$

6) $\vec{SA} + \vec{SB} + \vec{SC} = \vec{SH}$

$\vec{OA} - \vec{OS} + \vec{OB} - \vec{OS} + \vec{OC} - \vec{OS}$

ABCDEF is a regular hexagon $\vec{AB} = \vec{a}$, $\vec{BC} = \vec{b}$ then $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$.



$\vec{AB} = \vec{a}$ $\vec{AC} = \cancel{\vec{BC}}\ \vec{a} + \vec{b}$

$AD = \vec{a} + \vec{b} + \vec{b} - \vec{a} = 2\vec{b}$

$AE = 2\vec{b} - \vec{a}$

$\vec{AF} = \vec{b} - \vec{a}$

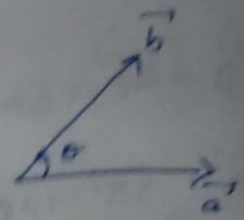
$\Rightarrow$ $\cancel{\vec{a}} + \vec{\cancel{a}+b} + 2\vec{b} + 2\vec{b} - \cancel{\vec{a}} + \vec{b} - \cancel{\vec{a}}$

$= 6\vec{b} = 6\vec{AB} = 6 \cdot \vec{AO}$

Dot Product of vectors (Scalar products)

$\bar{a}, \bar{b} \rightarrow$ two vectors

$$\boxed{\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta}$$



Properties

* If $\bar{a} \perp \bar{b}$, $\bar{a} \cdot \bar{b} = 0$
* $\bar{a} \cdot \bar{a} = |\bar{a}|^2$
* $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
* If θ is the angle ~~bisector~~ between the vectors $\bar{a}, \bar{b}$ then $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$
* $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
* $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2 \cdot \vec{a} \cdot \vec{b}$
* $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2 \cdot \vec{a} \cdot \vec{b}$
* $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$
* $|x\vec{a} + y\vec{b}|^2 + |x\vec{b} - y\vec{a}|^2 = (x^2 + y^2)(|\vec{a}|^2 + |\vec{b}|^2)$

* If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and θ = angle b/w $\vec{a}$ & $\vec{b}$

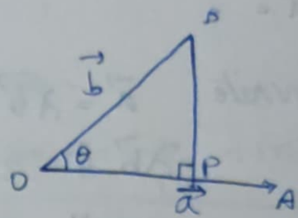
  $i \cdot i = j \cdot j = k \cdot k = 1$

  $i \cdot j = j \cdot i = j \cdot k = k \cdot j = i \cdot k = k \cdot i = 0$

* $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
* $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$

Geometrical Applications

$\cos\theta = \frac{OP}{\vec{b}}$



$OP = \frac{|\vec{b}|\cos\theta \cdot |\vec{a}|}{|\vec{a}|}$

$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

The projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Illy The projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

The vector component of the projection of $\vec{b}$ on $\vec{a} = \vec{OP} = OP\cdot\hat{a}$

$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$

$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\cdot\vec{a}$

Illy The vector component of the projection of $\vec{a}$ on $\vec{b}$

$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\cdot\vec{b}$

The vector component of the projection of $\vec{b}$ on $\vec{a}$ in the direction $\perp$ to $\vec{a}$

$\vec{PB} = \vec{OB} - \vec{OP}$

$= \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\cdot\vec{a}$

If $\vec{a}$ is collinear with $\vec{b} = 3i + 6j + 6\hat{k}$ and $\vec{a} \cdot \vec{b} = 27$

Then $\vec{a} =$

write $\vec{a} = \lambda \vec{b}$

$\lambda \vec{b} \cdot \vec{b} = 27$

$\lambda |\vec{b}| = 27$

$\lambda (9 + 36 + 36) = 27 \quad \lambda = 1/3$

$\vec{a} = \frac{\vec{b}}{3} = 1 + 2\hat{j} + 2\hat{k}$

If $\vec{a}, \vec{b}, \vec{c}$ are $\ni \vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 3$

Then $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{0}|^2$

$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$1 + 4 + 9 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -7$

If $\vec{a} \cdot \vec{b}$ are unit vectors then $|3\vec{a} + 4\vec{b}|^2 + |4\vec{a} + 3\vec{b}|^2$

$|3^2 + 4^2| (|a|^2 + |\vec{b}|^2)$

$25(1+1) = 50$

If $|\vec{a}| = |\vec{b}| = |\vec{a} - \vec{b}|$ then the angle between $\vec{a}, \vec{b}$ is

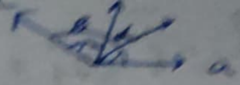
$a = b = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

$b^2 = 2ab \quad b = 2a\cos\theta. \quad a = 2b\cos\theta$

$b \neq 4b\cos\theta \quad \cos\theta = 0 \Rightarrow \theta = \pi/2$

Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors $\ni \vec{a} + \vec{b} + \vec{c}$ is a unit vector

If $a \perp \vec{b}$ and the angles made by $\vec{c}$ with $\vec{a}, \vec{b}$ are α, β respectively. Then $\cos\alpha + \cos\beta$



$R_{AB} = \sqrt{a^2+b^2}$

$R_{ABC} = \sqrt{a^2+b^2+c^2+2\sqrt{a^2+b^2}\cdot c\cos\theta} = 1$ $\alpha = 90+\beta.$

$a^2+b^2+c^2+2\sqrt{a^2+b^2}\cdot c\cos\theta = 1.$

$3+2\sqrt{2}\cos\theta = 1$ $\cos\theta = -1/\sqrt{2} \Rightarrow 135° = \alpha.$

$\beta = 45$

$\vec{a}\cdot\vec{b} = 0$ $\vec{c}\cdot\vec{a} = \cos\alpha$ $\vec{c}\cdot\vec{b} = \cos\beta.$

$|\vec{a}+\vec{b}+\vec{c}|^2 = 1$

$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(0+\cos\alpha+\cos\beta) = 1.$

$\Rightarrow \cos\alpha+\cos\beta = -1.$

~~Find the range of $\vec{a}$ for which $\vec{a} = 2a^2\hat{i} + 4a\hat{j} + k$ $\vec{b} = 7\hat{i} - 2\hat{j}$~~

If a, b, c are p^{th}, q^{th}, r^{th} terms of a positive G.P. Then find the angle between the vectors

$\vec{a} = (\log a^3)\hat{i} + (\log b^3)\hat{j} + (\log c^3)\hat{k}$

$\vec{b} = (q-r)\hat{i} + (r-p)\hat{j} + (p-q)\hat{k}$

$\vec{a}\cdot\vec{b} = 3(q-r)\log a + 3(r-p)\log b + 3(p-q)\log c$

$a = AR^{p-1}$ $b = AR^{q-1}$ $c = AR^{r-1}$

$= 3[(q-r)(\log A + (p-1)\log R) + (r-p)(\log A + (q-1)\log R) + (p-q)(\log A + (r-1)\log R)]$

$= 3[\log A(q-r+r-p+p-q) + \log R[(q-r)(p-1) + (r-p)(q-1) + (r-1)(p-q)]]$

$= 0$

∴ angle between $= \pi/2$

Q) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$

Find 1) Projection of $\vec{a}$ on $\vec{b}$

2) " $\vec{b}$ on $\vec{a}$

3) The vector component of the projection of $\vec{a}$ on $\vec{b}$ in the direction of $\vec{b}$

4) The vector component of the projection of $\vec{a}$ on $\vec{b}$ in the direction $\perp$r to $\vec{b}$

$\vec{a} \cdot \vec{b} = 1 - 2 + 3 = 2$

$|\vec{a}| = \sqrt{14}$ $|\vec{b}| = \sqrt{3}$

1) $OP = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{2}{\sqrt{3}}$

2) $OP = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{2}{\sqrt{14}} = \sqrt{\frac{2}{7}}$

3) $\vec{OP} = \frac{2}{\sqrt{3}} \hat{b} = \frac{2}{\sqrt{3}} \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} = \frac{2}{3}(\hat{i} + \hat{j} + \hat{k})$

4) $\vec{PA} = \vec{OA} - \vec{OP} = \hat{i} - 2\hat{j} + 3\hat{k} - \frac{2(\hat{i} + \hat{j} + \hat{k})}{3}$

$= \frac{\hat{i} - 8\hat{j} + 7\hat{k}}{3}$

$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ $c = \hat{i} + \hat{j} - 2\hat{k}$

Find the vector in the plane of $\vec{b}, \vec{c}$ whose projection on $\vec{a}$ is $\sqrt{\frac{2}{3}}$ units

Let $x\vec{b}+y\vec{c}$ be the vector in the place of $\vec{b}, \vec{c}$

Given $\frac{(x\vec{b}+y\vec{c})\cdot\vec{a}}{|\vec{a}|} = \sqrt{\frac{2}{3}}$

$$\frac{x(\vec{b}\cdot\vec{a}) + y(\vec{c}\cdot\vec{a})}{\sqrt{6}\sqrt{2}} = \sqrt{\frac{2}{3}}$$

$$x(-1) + y(-1) = 2$$

$$x+y = -2$$

$$x\vec{b} + (-2-x)\vec{c}$$

$$= x(\hat{i}+2\hat{j}-\hat{k}) + (-2-x)(\hat{i}+\hat{j}-2\hat{k})$$

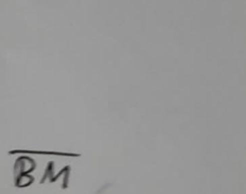
$$= -2\hat{i} + (x-2)\hat{j} + (x+4)\hat{k}$$

* A$(\hat{i}-2\hat{j}+2\hat{k})$, B$(\hat{i}+4\hat{j})$, C$(-4\hat{i}+\hat{j}+\hat{k})$

Let m be the foot of ⊥ from B to $\overline{AC}$. Find $\overline{BM}$

$\overrightarrow{AB} = 6\hat{j}-2\hat{k}$ $\overrightarrow{AC} = -5\hat{i}+3\hat{j}-\hat{k}$

$$AM = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AC}|} = \frac{16}{\sqrt{35}}$$



$$\overrightarrow{AM} = AM \cdot \hat{AC}$$

$$\frac{-16}{\sqrt{35}} \; \frac{-5\hat{i}+3\hat{j}+\hat{k}}{\sqrt{35}} = \frac{16(-5\hat{i}+3\hat{j}+\hat{k})}{35}$$

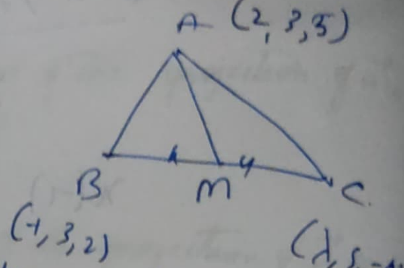
$$\overrightarrow{MB} = \overrightarrow{AB} - \overrightarrow{AM}$$

$$\overrightarrow{BM} = AM - \overrightarrow{AB} = \frac{16(-5\hat{i}+3\hat{j}+\hat{k})}{35} - 6\hat{j} - 2\hat{k}$$

Q) $A(2\hat{i}+3\hat{j}+5\hat{k})$, $B(-\hat{i}+3\hat{j}+2\hat{k})$ $C(\lambda\hat{i}+5\hat{j}+\mu\hat{k})$

are the vertical of a triangle.

If the median M is equally inclined to the coordinate axis then $\lambda =$ _ $\mu =$



$$\overline{AM} = \left(\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}\right)$$

$\bar{a}$ is equally inclined to the coordinate axis

$$\bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{a}\cdot i = \bar{a}\cdot j = \bar{a}\cdot k$$

$$x = y = z$$

$$\frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2}$$

$$\boxed{\mu = 10} \quad \boxed{\lambda = 7}$$

Q) $\bar{a}, \bar{b}, \bar{c}$ are unit vectors, Then the max value of $|\bar{a}-\bar{b}|^2 + |\bar{b}-\bar{c}|^2 + |\bar{c}-\bar{a}|^2$ is

$$|\bar{a}-\bar{b}|^2 + |\bar{b}-\bar{c}|^2 + |\bar{c}-\bar{a}|^2$$

$$= |\bar{a}|^2 + |\bar{b}|^2 - 2(\vec{a}\cdot\vec{b}) + |\bar{b}|^2 + |\bar{c}|^2 - 2(\bar{b}\cdot\bar{c}) + |\vec{c}|^2 + |\vec{a}|^2 - 2(\bar{c}\cdot\bar{a})$$

$$= 6 - 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) \to (1)$$

Consider $|\vec{a}+\vec{b}+\vec{c}|^2 \geq 0$

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) \geq 0$$

suppose angle b/w $\vec{a}$ & $\vec{b}$ is α, $\vec{b}$ & $\vec{c}$ is β, $\vec{c}$ & $\vec{a}$ is γ.

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} \geq -3/2$$

$$\cos\alpha+\cos\beta+\cos\gamma \geq -3/2$$

① $\rightarrow$ $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2 \leq 6-2\times -\frac{3}{2} = 9$

$$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2 \leq 9$$

If $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2 = 9$, then

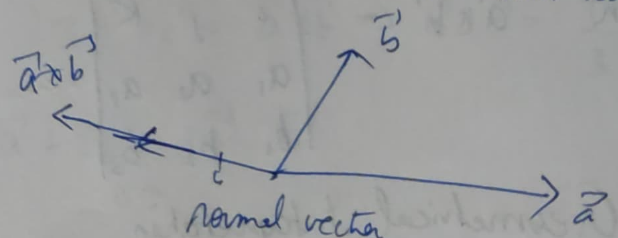
$\cos\alpha = \cos\beta = \cos\gamma = -1/2$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = -1/2$

Vector Product (cross product)

If $\vec{a}$, $\vec{b}$ are two of vectors

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta \cdot \hat{n}$$

here $\hat{n}$ = The unit vector $\perp^{ar}$ to $\vec{a}$, $\vec{b}$



* $\vec{a}\times\vec{b}$ is always a vector
* $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$
* If $\vec{a} \parallel \vec{b}$, $\vec{a}\times\vec{b} = \vec{0}$
* $(x\vec{a})\times(y\vec{b}) = (xy)(\vec{a}\times\vec{b})$
* $\vec{a}\times\vec{0} = \vec{0}\times\vec{a} = \vec{0}$

* If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$

$\rightarrow \vec{a}\times(\vec{b}-\vec{c}) = 0$

$\rightarrow \vec{a} = \vec{0}$ or $\vec{b}-\vec{c} = 0$ or $\vec{a} \parallel \vec{b}-\vec{c}$

$\vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \parallel (\vec{b}-\vec{c})$

* $\vec{a}(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

* $|\vec{a}\times\vec{b}| = -|\vec{b}\times\vec{a}|$

* <u>Lagrange's Identity</u>

$$(\vec{a}\cdot\vec{b})^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$$

* The unit vector ⊥r to both $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

* $\vec{a} \perp \vec{a}\times\vec{b}$, $\vec{b} \perp \vec{a}\times\vec{b}$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ & $\vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

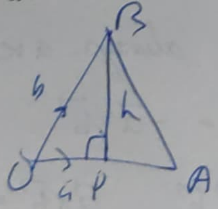
* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{a} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

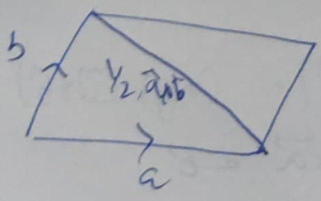
Then $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Geometrical Interpretation

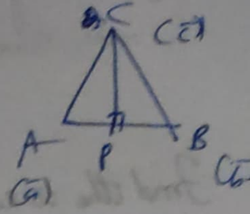
$\Delta = \frac{1}{2}|\vec{a}\times\vec{b}|$



area of parallelogram $= |\vec{a}\times\vec{b}|$



The area of ΔABC whose vertices are $A(\vec{a}), B(\vec{b}), C(\vec{c})$ given



$$\Delta = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} |(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})|$$

$$\boxed{\Delta = \frac{1}{2} |\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|}$$

If $\vec{a}, \vec{b}, \vec{c}$ are 3 collinear points, then,

$\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

Q) If $3\vec{a} + 7\vec{b} - 10\vec{c} = \vec{0}$ Then $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = ?$

$3+7-10 = 0 \Rightarrow$ collinear

$\Rightarrow \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

Q) If $\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$

$\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \lambda(\vec{a}\times\vec{b}) = \mu(\vec{b}\times\vec{c}) = \nu(\vec{c}\times\vec{a})$

$\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$
$\times\vec{b}$

$\vec{a}\times\vec{b} + \vec{0} + 3\cdot\vec{c}\times\vec{b} = \vec{0}$

$3\vec{b}\times\vec{c} = \vec{a}\times\vec{b} \Rightarrow \vec{b}\times\vec{c} = \frac{1}{3}(\vec{a}\times\vec{b})$ ✓

$\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$
$\times a$

$\vec{0} + 2\vec{b}\times\vec{a} + 3\vec{c}\times\vec{a} = \vec{0} \Rightarrow 3\vec{c}\times a = 2(\vec{a}\times\vec{b})$

$\Rightarrow \vec{c}\times\vec{a} = \frac{2}{3}(\vec{a}\times\vec{b})$

Now, $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$

$\vec{a}\times\vec{b} + \frac{1}{3}(\vec{a}\times\vec{b}) + \frac{2}{3}(\vec{a}\times\vec{b})$

$= 2(\vec{a}\times\vec{b}) \rightarrow \lambda = 2$

$= 2(3(\vec{b}\times\vec{c})) = 6(\vec{b}\times c) \rightarrow \boxed{\mu = 6}$

$2\left(\frac{3}{2}\cdot\vec{c}\times\vec{a}\right)$

$= 3 \boxed{\nu = 3}$

diagonal of a parallelogram are $\vec{d_1}, \vec{d_2}$

Area $= \frac{1}{2} |\vec{d_1} \times \vec{d_2}|$

Find the area of the Δ^{le} formed by the P.Vs

$A(\vec{a}) = 2\hat{i}+3\hat{j}+4\hat{k}$, $B(\vec{b}) = 3\hat{i}+4\hat{j}+2\hat{k}$, $C(\vec{c}) = 4\hat{i}+3\hat{j}+3\hat{k}$

Area $= \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$\vec{AB} = \hat{i}+\hat{j}-2\hat{k}$ $\quad \vec{AC} = 2\hat{i} - \hat{j} - \hat{k}$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} = i(-3) + -j(3) + k(-3)$$

$$= -3(\hat{i}+\hat{j}+\hat{k})$$

Area $= \frac{1}{2} |-3(\hat{i}+\hat{j}+\hat{k})| = \frac{3}{2} \times \sqrt{3}$ sq units

The length of the altitude through A.

$\frac{1}{2} h_a |\hat{i}-2\hat{j}+\hat{k}| = \frac{3\sqrt{3}}{\cancel{2}}$

$h_a \sqrt{6} = 3\sqrt{3}$ $\quad \therefore h_a = 3/\sqrt{2}$ units

If $|\vec{a}| = 1$ $|\vec{b}| = 2$ $(\vec{a}, \vec{b}) = 2\pi/3$ Then the value of

$|(\vec{a}+3\vec{b}) \times (3\vec{a}-\vec{b})|^2$

λ_0

If the area of the parallelogram whose adjacent sides are $3\hat{i} - 2\hat{j} + \lambda\hat{k}$ and $2\hat{j} - \hat{k}$ is $\sqrt{436}$ sq units then λ

$$|\vec{a} \times \vec{b}| = \sqrt{436}.$$

$$|(16 - 2\lambda)\hat{i} + \hat{j}(12) + \hat{k}(6)| = \sqrt{436}$$

$$(16 - 2\lambda)^2 + 144 + 36 = 436$$

$$(16 - 2\lambda)^2 = 256$$

$16 - 2\lambda = 16$ $\qquad$ $16 - 2\lambda = -16$

$\boxed{\lambda = 0}$ $\qquad$ $\boxed{\lambda = 16}$

If $\vec{\alpha} = \bar{a} \times (\bar{b} + \bar{c})$ $\quad$ $\vec{\beta} = \bar{b} \times (\bar{c} + \bar{a})$, $\vec{\gamma} = \bar{c} \times (\vec{a} + \bar{b})$, $\quad \vec{\alpha} + \vec{\beta} + \vec{\gamma} = 0$

Then $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are Linear dependent & coplanar → sk

Q) If $\bar{a}, \bar{b}, \bar{c}$ are non planar vectors such that $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$, $\vec{c} \times \vec{a} = \vec{b}$ then $|\vec{a} + \vec{b} + \vec{c}| = ?$

$$\vec{a} \perp \vec{b} \quad \vec{b} \perp \vec{c} \quad \vec{c} \perp \vec{a}$$

$$|\vec{a}||\vec{b}| = |\vec{c}| \qquad |\vec{b}||\vec{c}| = |\vec{a}| \qquad |\vec{c}||\vec{a}| = |\vec{b}|$$

$$\Rightarrow |\vec{a}||\vec{b}||\vec{c}| = 1 \Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 1 + 1 + 1 = 3.$$

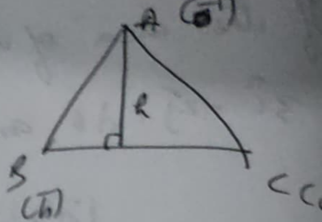
$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}$$

Q) Let $\vec{A}, \vec{B}, \vec{C}$ be 3 unit vectors such that $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C} = 0$ and $\bar{A} = k(\vec{B} \times \vec{C})$ then $k =$ $\qquad (\vec{B}, \vec{C}) = 2\pi/3.$

$$|A|^2 = k^2 (|B|^2 |C|^2 \sin^2\theta)$$

$$1 = k^2 \cdot 1 \cdot 1 \cdot 3/4 = \quad k = \pm 2/\sqrt{3}$$

Q) If $\vec{AB} = \vec{b}$, $\vec{AC} = \vec{c}$. find h.



$[\Delta AOB] = \frac{1}{2}|\vec{b}\times\vec{c}|$

$\frac{1}{2}|\vec{b}-\vec{c}|\,h = \frac{1}{2}|\vec{b}\times\vec{c}|$

$$\boxed{h = \frac{|\vec{b}\times\vec{c}|}{|\vec{b}-\vec{c}|}}$$

Q) $\bar{a}, \bar{b}, \bar{c}$ are three vectors $\ni$ $\vec{a}\times\vec{b} = 3(\vec{a}\times\vec{c})$

Also $|\vec{a}| = |\vec{b}| = 1$, $|\vec{c}| = 1/3$. $(\bar{b}, \bar{c}) = \pi/6$, $\vec{b} =$.

$\vec{a}\times\vec{b} = \vec{a}\times 3\vec{c}$

$\vec{a}\times(\vec{b}-3\vec{c}) = 0$ $\quad \vec{b}-3\vec{c} \parallel \vec{a}$

$\Rightarrow \vec{b}-3\vec{c} = \lambda\vec{a}$

$|\vec{b}-3\vec{c}|^2 = \lambda^2|\vec{a}|^2$

$|\vec{b}| + 9|\vec{c}|^2 - 6|\vec{b}||\vec{c}|\cos\theta = \lambda^2|\vec{a}|^2$

$1 + 9\times\frac{1}{9} - 6\times1\times\frac{1}{3}\times\frac{1}{2} = \lambda^2$

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\Rightarrow \bar{b} - 3\bar{c} = \pm\bar{a}$

$\boxed{\vec{b} = 3\vec{c}+\vec{a}}$ or $\boxed{\vec{b} = 3\vec{c}-\vec{a}}$

Q) Let $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that

$|\vec{a}| = \frac{1}{\sqrt{2}}$, $|\vec{b}| = \frac{1}{\sqrt{3}}$, $|\vec{c}| = \frac{1}{\sqrt{6}}$, $|\vec{a}+\vec{b}+\vec{c}| = 1$, $c = \lambda(\vec{a}\times\vec{b})$

find the angle b/w $(\vec{a}, \vec{b})$

$\frac{\sqrt{3}}{2}$ 1+

$\cancel{|\vec{c}|^2 = \lambda^2}$ $\quad \vec{c} = \lambda(\vec{a}\times\vec{b}) \Rightarrow$ angle with a, c & $a, b = 0$

$\cos = 0$

$|\vec{a}+\vec{b}+\vec{c}|^2 = 1$

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6} + 2\times\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}\cos\alpha = 1$

$\cos\alpha = 0$

$\boxed{\alpha = \pi/2}$

Q) $|\vec{a}| = |\vec{b}| = 1$, $|\vec{a}+\vec{b}| = \sqrt{3}$, $\vec{c}$ is a vector $\ni$ $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a} \times \vec{b})$

Then $\vec{c} \cdot \vec{b} =$

$$a^2 + b^2 + 2ab\cos\theta = 3 \Rightarrow 1 + 1 + 2\cos\theta = 3 \qquad \cos\theta = 1/2$$

$$\theta = \pi/3$$

$$|\vec{c} - \vec{a} - 2\vec{b}| = 3|1 \times 1 \times \sqrt{3}/2|$$

$$|\vec{c} - \vec{a} - 2\vec{b}| = 3\sqrt{3}/2.$$

$$\vec{c} = \vec{a} + 2\vec{b} + 3(\vec{a} \times \vec{b})$$

$$\vec{c} \cdot \vec{b} = \vec{a} \cdot \vec{b} + 2|\vec{b}|^2 + 3(\underbrace{\vec{b} \cdot \vec{a} \times \vec{b}}_{0})$$

$$= \vec{a} \cdot \vec{b} + 2 = \frac{1}{2} + 2 = 5/2.$$

$$\boxed{\vec{c} \cdot \vec{b} = 5/2}$$

Scalar triple Product $[\bar{a}\ \bar{b}\ \bar{c}]$

Let $\bar{a}, \bar{b}, \bar{c}$ be 3 vectors, then $\bar{a} \cdot \bar{b} \times \vec{c}$ — box product

Scalar triple product

* The system $(\vec{a}, \vec{b}, \vec{c})$ is RHS if $[\vec{a}\ \vec{b}\ \vec{c}] > 0$
* The system $(\vec{a}, \vec{b}, \vec{c})$ is LHS if $[\vec{a} \cdot \vec{b} \cdot \vec{c}] < 0$
* $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar

Q) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ $\qquad a \cdot \vec{b} \times c$

$$\vec{b} \times \vec{c} \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\bar{a} \cdot \bar{b} \times c = \bar{a} \times \bar{b} \cdot c$

PROPERTIES

* $[\hat{i}\ \hat{j}\ \hat{k}] = 1$
* $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}]$
* $[\bar{a}\ \bar{a}\ \bar{b}] = 0$
* $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$
* $[x\bar{a}\ \ y\bar{b}\ \ z\bar{c}] = (xyz)[\bar{a}\ \bar{b}\ \bar{c}]$
* $[a_1\bar{a} + b_1\bar{b} + c_1\bar{c}\quad a_2\bar{a} + b_2\bar{b} + c_2\bar{c}\quad a_3\bar{a} + b_3\bar{b} + c_3\bar{c}]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} [\bar{a}\ \bar{b}\ \bar{c}]$$

Q) If $[\bar{a}\ \bar{b}\ \bar{c}] = 3$ then

$[2\bar{a} + 3\bar{b} + \bar{c}\quad \bar{a} - 2\bar{b} + 5\bar{c}\quad \bar{a} + 2\bar{b} - 3\bar{c}]$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 5 \\ 1 & 2 & -3 \end{vmatrix} [\bar{a}\ \bar{b}\ \bar{c}]$$

$$= (2(-4) - 3(-8) + 1(4))\ (3) = 60$$

~~Q) If $\bar{a}, \bar{b}, \bar{c}$ are coplanar, then any 3 vectors which~~ are LC of $\bar{a}, \bar{b}$

* If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then any 3 vectors which are LC of $\bar{a}, \bar{b}, \bar{c}$ are also coplanar

Q) $\dfrac{[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}]}{[\bar{a}\,\bar{b}\,\bar{c}]} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \phi & 0 & 1 \end{vmatrix}$

$= 1((1-0)-1(0-1)$

$1(1)-1(-1) = 2$

$$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = [\bar{a}\,\bar{b}\,\bar{c}]^2$$

$= [\bar{a}\,\bar{b}\,\bar{c}][\bar{l}\,\bar{m}\,\bar{n}]$

Q) If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors & $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 1/2$

Then $[\bar{a}\,\bar{b}\,\bar{c}]$ ~~is~~

$$[\bar{a}\,\bar{b}\,\bar{c}]^2 = \begin{vmatrix} \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} & \bar{a}\cdot\bar{c} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} & \bar{b}\cdot\bar{c} \\ \bar{c}\cdot\bar{a} & \bar{c}\cdot\bar{b} & \bar{c}\cdot\bar{c} \end{vmatrix}$$

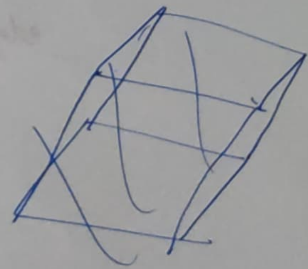
$$\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = 1(1-1/4) - 1/2(1/2-1/4) + 1/2(1/4-1/2)$$

$= 3/4 - 1/8 - 1/8 = 1/2$

$[\bar{a}\,\bar{b}\,\bar{c}] = \pm 1/\sqrt{2}$

Geometrical Interpretation

Let $\theta = (\bar{a}\times\bar{b}, \vec{c})$



Volume of parallelopipe

= Area of the base × height

$= |\bar{a}\times\bar{b}|\cdot h$

$$\cos\theta = \frac{h}{|\bar{c}|} \Rightarrow h = |\bar{c}|\cos\theta$$

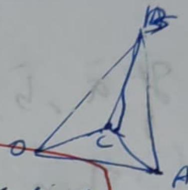
$$= |\bar{a}\times\bar{b}|\cdot|\bar{c}|\cos(\bar{a}\times\bar{b}, c)$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]$$

The volume of the parallelopipe whose co terminating edges are $\bar{a}, \bar{b}, \bar{c}$ = $[\bar{a}\ \bar{b}\ \bar{c}]$

Tetrahedron

C doesn't lie on the plane $\triangle OAB$



$\boxed{\text{Volume of tetrahedron} = \frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}] \cdot \text{cu. units}}$

Proof

$$V = \frac{1}{3}\cdot \text{Area of the base} \cdot \text{height}$$

$$= \frac{1}{3}\cdot\frac{1}{2}|\bar{a}\times\bar{b}|\cdot h$$

$$= \frac{1}{6}|\bar{a}\times\bar{b}|\cdot|\bar{c}|\cos(\bar{a}\times\bar{b}, \bar{c})$$

$$= \frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}] \text{ cu. units.}$$

$$\cos\theta = h/|\bar{c}| \quad h = |\bar{c}|\cos\theta$$

The volume of the tetrahedron whose coterminals edges are $\bar{a}, \bar{b}, \bar{c}$ is $\frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$ cu. units

Q) $(\bar{a}+2\bar{b})\cdot\left((3\bar{a}+2\bar{b}+\bar{c})\times(\bar{a}-2\bar{b}+\bar{c})\right)$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} |\bar{a}\ \bar{b}\ \bar{c}|$$

$$\left(1(2+2)-2(3-1)+0\right)[\bar{a}\bar{b}\bar{c}] = 0$$

$$\boxed{[\bar{a}\ \bar{b}\ \bar{c}] \le |\bar{a}||\bar{b}||\bar{c}| \quad \{\text{equals if they are mutually } \perp\}}$$

Q) If $(\bar{a},\bar{b}) = \pi/6$ $\quad c \perp \bar{a}$, $\bar{c} \perp \bar{b}$ Then

$|\bar{a}| = 3 \quad |\bar{b}| = 4 \quad |\bar{c}| = 6$ Then $[a\ b\ c]$ may be

$[\bar{a}\ \bar{b}\ \bar{c}) = a\cdot(\bar{b}\times\bar{c})$

$\bar{a}\cdot(24) = 3\times 24\times\frac{1}{2} = \pm 36$

~~Q) If $\bar{a}, \bar{b}, \bar{c}$ makes a~~

Q) $\bar{a} = -\hat{i}+\hat{j}+\hat{k}$ $\quad \bar{b} = 2\hat{i}+\hat{k}$ $\quad \bar{c}$ is a vector $\ni$

c is coplanar with a, b, $c \perp b$ $\quad \bar{a}\cdot\bar{c} = 7$.

Then c is

$\bar{c} = x\bar{a}+y\bar{b}$

$\bar{c}\cdot\bar{b} = x(\bar{a}\cdot\bar{b}) + y(\bar{b}\cdot\bar{b})$

$0 = x(-1)+y(5)$

$x = 5y \quad \cancel{x = 5/2}$

$\bar{a}\cdot\bar{c} = x\cdot(\bar{a}\cdot\bar{a}) + y(\bar{a}\cdot\bar{b}) = 7 = 3x - y$

$y = 1/2 \Rightarrow x = 5/2 \quad \Rightarrow 14y = 7$

$y =$

$$\bar{c} = \frac{3\bar{a}+\bar{b}}{2} = \frac{-3\hat{i}+5\hat{j}+6\hat{k}}{2}$$

Q) $\bar{\alpha} = a\hat{i}+b\hat{j}+c\hat{k}$, $\bar{\beta} = b\hat{i}+c\hat{j}+a\hat{k}$, $\gamma = c\hat{i}+a\hat{j}+b\hat{k}$

If $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are coplanar, then $(a+b+c)$

$\bar{r} = \hat{i}+\hat{j}+\hat{k}$ in h ro.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \rightarrow a=b=c \text{ or}$$

$a+b+c=0$

y

$\hat{i} \perp \alpha, \beta, \gamma$

Q) $\bar{u}, \bar{v}, \bar{w}$ are non coplanar vectors, $p, q, \lambda \ni$

$$[3\bar{u} \;\; p\bar{v} \;\; p\bar{w}] - [p\bar{v} \;\; \bar{w} \;\; q\bar{u}] - [2\bar{w} \;\; q\bar{v} \;\; q\bar{u}] = 0$$

holds for $3p^2[\bar{u}\;\bar{v}\;\bar{w}] - pq[\bar{u}\;\bar{v}\;\bar{w}] + 2q^2[\bar{u}\;\bar{v}\;\bar{w}] = 0$

$$\rightarrow [\bar{u}\;\bar{v}\;\bar{w}](3p^2 - pq + 2q^2) = 0$$

$$3p^2 - pq + 2q^2 = 0 \rightarrow p = 0 = q$$

$$3\left(\frac{p}{q}\right)^2 - \frac{p}{q} + 2 = 0$$

$\Delta < 0 \rightarrow$ No other solution

$(0, 0)$ only solution

Find the volume of parallelopiped whose co terminal edges are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\hat{j}$, $\hat{i}+2\hat{j}-\hat{k}$

So

Volume $= |[\bar{a}\ \bar{b}\ \bar{c}]|$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix} = 1-1(-1)+1(3) = 5 \text{ cu.units}$$

Find the volume of the tetrahedron whose co-terminal edges are $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}$, $\hat{i}+\hat{j}+\hat{k}$:

Volume $= \frac{1}{6}|[\bar{a}\ \bar{b}\ \bar{c}]|$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\left(1\times 3 - 1(2) - 1(-1)\right) = \frac{1}{3} \text{ cu.units}$$

If $A(\hat{i}+\hat{j}+\hat{k})$ $B(2\hat{i}-3\hat{j})$ $C(\hat{i}-\hat{j}+2\hat{k})$ $D(3\hat{i}+\hat{j}-2\hat{k})$ are the vertices of a tetrahedron. Find its volume.

$\vec{AB} = \hat{i}-4\hat{j}-\hat{k}$ $\vec{AC} = -2\hat{j}+\hat{k}$ $\vec{AD} = 2\hat{i}-3\hat{k}$

Volume $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}] = \frac{1}{6}\begin{vmatrix} 1 & -4 & -1 \\ 0 & -2 & 1 \\ 2 & 0 & -3 \end{vmatrix}$

$$= \frac{1}{6}\left(6+4(-2)-1(4)\right)$$

$$= 1 \text{ unit}^3$$

volume $= \frac{1}{6} |[\bar{a}\ \bar{b}\ \bar{c}]|$

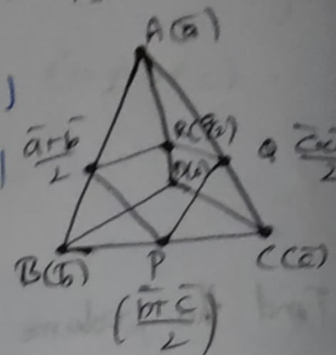
$= \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{6} (1 \times 3 - 1(2) - 1(-1))$

$= 1/3$ cu units

Vol. of Tetr ABCD $= V_1$ $\frac{1}{6} [\bar{a}\ \bar{b}\ \bar{c}]$

Vol. of Tetr PQRS $= V_2$, $\frac{1}{48} |\bar{a}\ \bar{b}\ \bar{c}|$

Then $\frac{V_1}{V_2} = 8$



let $\bar{a}\ \bar{b}\ \bar{c}$ be the P.vs of A B C. & α, β, γ be the inclination between $(\bar{b}, \bar{c})$, $(\bar{a}\ \bar{b})$ & $(\bar{a}, \bar{c})$ resp. If v = vol. of the tetrahedron OABC, then v^2 ____

$v = \frac{1}{6} [\bar{a}\ \bar{b}\ \bar{c}]$

$v^2 = \frac{1}{36} [\bar{a}\ \bar{b}\ \bar{c}]^2 = 1/36 \begin{vmatrix} a\cdot\bar{a} & \bar{a}\cdot\bar{b} & \bar{a}\cdot\bar{c} \\ \bar{b}\ \bar{a} & \bar{b}\cdot\bar{b} & \bar{b}\cdot\bar{c} \\ \bar{c}\cdot\bar{a} & \bar{c}\cdot\bar{b} & \bar{c}\cdot\bar{c} \end{vmatrix}$

$= 1/36 \begin{vmatrix} a^2 & ab\cos\beta & ac\cos\gamma \\ b\cos\beta & b^2 & b\cdot c\cos\alpha \\ ca\cos\gamma & cb\cos\alpha & c^2 \end{vmatrix}$

$= \frac{a^2b^2c^2}{36} \begin{vmatrix} 1 & \cos\beta & \cos\gamma \\ \cos\beta & 1 & \cos\alpha \\ \cos\gamma & \cos\alpha & 1 \end{vmatrix}$

Vector triple product

Let $\bar{a}, \bar{b}, \bar{c}$ be 3 vectors

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$$

* $(\bar{a} \times \bar{b}) \times c = -[\bar{c} \times [\bar{a} \times \bar{b}]]$

$$= -[(\bar{c} \cdot \bar{b})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b}]$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{c} \cdot \bar{a})\bar{b} - (\bar{c} \cdot \bar{b})\bar{a}$$

* $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

* $\bar{a} \times (\bar{b} \times \bar{c})$ = The vector coplanar with $\bar{b}, \bar{c}$ and orthogonal to $\bar{a}$.

* $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) =$

$= (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} + (\bar{b} \cdot \bar{a})\bar{c} - (\bar{b} \cdot \bar{c})\bar{a} +$
$(\bar{c} \cdot \bar{b})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b} = \bar{0}$

8) Let $\bar{a}, \bar{b}, \bar{c}$ are 3 vectors $\ni |\bar{a}| = 1$ $|\bar{b}| = 1$ $|\bar{c}| = 2$ and $\bar{a} \times (\bar{a} \times \bar{c}) + \bar{b} = \bar{0}$. Then the angle between $\bar{a}, \bar{c}$ is

$$\bar{a} \times (\bar{a} \times \bar{c}) = -\bar{b}$$

$$(\bar{a} \cdot \bar{c})\bar{a} - (\bar{a} \cdot \bar{a})\bar{c} = -\bar{b}$$

$$|(\bar{a} \cdot \bar{c})\bar{a} - \bar{c}|^2 = |-\bar{b}|^2$$

$$(\bar{a} \cdot \bar{c})^2 |\bar{a}|^2 + |\bar{c}|^2 - 2(\bar{a} \cdot \bar{c})^2 = |\bar{b}|^2$$

$$(\bar{a} \cdot \bar{c})^2 + 4 - 2(\bar{a} \cdot \bar{c})^2 = 1$$

$$(\bar{a} \cdot \bar{c})^2 = 3 \Rightarrow \bar{a} \cdot \bar{c} = \pm\sqrt{3}$$

$|\bar{a}| \cdot |\bar{c}| \cos\alpha = \pm\sqrt{3}$

$1 \cdot 2 \cos\alpha = \pm\sqrt{3}$

$\cos\alpha = \pm\frac{\sqrt{3}}{2}$

$\alpha = 60^\circ$ & 150°

Q) If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors $\ni$

$\bar{a} \times (\bar{b} \times \bar{c}) = \frac{\bar{b}+\bar{c}}{2}$ Find $(\bar{a}, \bar{b}), (\bar{a}, \bar{c})$

$\cancel{(\bar{a} \cdot \bar{c})\bar{b}}$

$(\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = \frac{\bar{b}+\bar{c}}{2}$

$\Rightarrow \bar{a} \cdot \bar{c} = 1/2$ & $\bar{a} \cdot \bar{b} = -1/2$

$\cos(\bar{a}, \bar{c}) = 1/2$ & $\cos(\bar{a}, \bar{b}) = -1/2$

$(\bar{a}, \bar{c}) = 60^\circ$ & $(\bar{a}, \bar{b}) = 120^\circ$

Q) Find the unit vector coplanar with $\bar{b}, \bar{c}$ and orthogonal to $\bar{a}$

$\bar{a} = \hat{i} + \hat{j} - \hat{k}$ $\bar{b} = \hat{i} + \hat{j} - 2\hat{k}$ $\bar{c} = 2\hat{i} - \hat{j} + \hat{k}$

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = (-\bar{b}) - (4)\bar{c}$

$= -9\hat{i} + 3\hat{j} - 10\hat{k}$

The unit vector coplanar with $\bar{b}, \bar{c}$ and orthogonal to $\bar{a}$ $= \frac{-9\hat{i} + 3\hat{j} \cancel{+} 10\hat{k}}{\sqrt{81 + 9 + 100}} = \frac{-9\hat{i} + 3\hat{j} - 10\hat{k}}{\sqrt{190}}$

Find an unit vector to plane with $\bar{a}, \bar{b}$ & $\perp$ to $\bar{a}$

$\bar{a} = 2\hat{i} + \hat{j} + \hat{k}$ $\bar{b} = \hat{i} + 2\hat{j} + \hat{k}$

$\hookrightarrow$ $\dfrac{\bar{a} \times (\bar{a} \times \bar{b})}{|\bar{a} \times (\bar{a} \times \bar{b})|} = (\bar{a}\cdot\bar{b})\bar{a} - (\bar{a}\cdot\bar{a})\bar{b}$

$\perp$ coplanar

2 2 1   1 1 1

$\Rightarrow$ ~~$\hat{i} + 2\hat{j} + \hat{k} - 4\hat{i} - \hat{j} - \hat{k} = 0$~~

$3\bar{a} - 6\bar{b} \Rightarrow \dfrac{-9\hat{j} - 3\hat{k}}{\sqrt{90}} = \dfrac{3\hat{j} - \hat{k}}{\sqrt{10}}$

Multiple products

* $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d} \end{vmatrix}$

* $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

* $[\bar{a}\times\bar{b} \quad \bar{b}\times\bar{c} \quad \bar{c}\times\bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]^2$

$\hookrightarrow$ Note $\bar{a}\times\bar{b}$, $\bar{b}\times\bar{c}$, $\bar{c}\times\bar{a}$ are coplanar then $\bar{a}, \bar{b}, \bar{c}$ are coplanar also

Vector equations

Line equation

The vector equation of the line passing through $A(\bar{a})$ and parallel to ~~parall~~ a vector $\bar{b}$ is

$\overline{AP} \parallel \bar{b}$

$\overline{AP} = \lambda \cdot \bar{b}$

$\bar{r} - \bar{a} = \lambda\bar{b} \quad \Rightarrow \bar{r} = \bar{a} + \lambda\bar{b}$

2) The vector equation of the line passing through $A(\bar{a})$ $B(\bar{b})$

$\overline{AP} \parallel \overline{PB}$

$\overline{AP} = \lambda \overline{AB}$

$\bar{r} - \bar{a} = \lambda(\bar{b} - \bar{a})$ $\bar{r} = (1-\lambda)\bar{a} + \lambda\bar{b}$

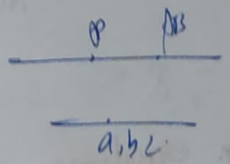
Q) The v. eqn of the line passing through $A(i+j+k)$ and parallel to $i - 2j + 3k$ is

$$\bar{r} = i + j + k + \lambda(i - 2j + 3k)$$

Q) The vector eqn of the line passing through $A(i - j + k)$ & $B(i + j - 2k)$ is

$$\bar{r} = (1-\lambda)\, i - j + k + \lambda(i + j - 2k)$$

Cartesian form

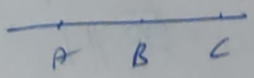


$\overline{AP} \parallel (a, b, c)$

$(x - x_1, y - y_1, z - z_1) \parallel (a, b, c)$

$$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}}$$

$\overline{AP} \parallel \overrightarrow{AB}$



$(x - x_1, y - y_1, z - z_1) \parallel (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

Point of intersection of two lines.

↳ by comparing corresponding coefficients.

Q) $P(3i - 4j + 5k)$ $Q(4k)$ $R(-4i + 5j + k)$ $S(-3i + 4j + 4k)$

Find the POI of $\overline{PQ}$, $\overline{RS}$

PQ eqn is

$\bar{r} = (1-\lambda)(3i - 4j + 5k) + \lambda(4k)$

$= (3-3\lambda)\hat{i} + (-4+4\lambda)\hat{j} + (5-\lambda)\hat{k}$ — ①

$\bar{RS}$ eqn is

$\bar{r} = (1-\mu)(-4i + 5j + k) + \mu(-3i + 4j + 4k)$

$= (-4+\mu)\hat{i} + (5-\mu)\hat{j} + (1+3\mu)\hat{k}$

at the POI ① = ②

$3-3\lambda = -4+\mu$ $\qquad -4+4\lambda = 5-\mu$

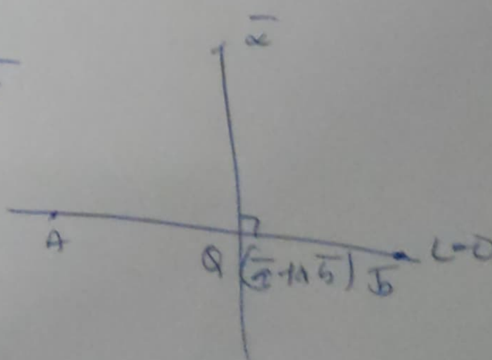
$-3\lambda - \mu = -7$ $\qquad 4\lambda + \mu = 9$

$\Rightarrow \boxed{\lambda = 2}$

POI = $-3i + 4j + 3k$

* $\bar{r} = \bar{a} + \lambda\bar{b}$

now $\overline{PQ} = a + \lambda\bar{b} - \bar{\alpha}$



$\overline{PQ} \perp L$

$\overline{PQ} \perp \bar{b}$

$\overline{PQ} \cdot \bar{b} = 0$

$(a + \lambda\bar{b} - \bar{\alpha}) \cdot \bar{b} = 0$

solve to get λ.

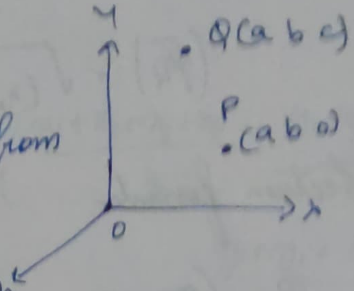
* Foot of $\perp$ = $\bar{a}+\lambda\bar{b}$

* Image of $\bar{\alpha}$ wrt $L=0$ is $2(\bar{a}+\lambda\bar{b})-\bar{\alpha}$

* $\perp$ distance = $|\overline{PQ}|$ = $|\bar{a}+\lambda\bar{b}-\bar{\alpha}|$

3-D Geometry

↳ 3-D Co ordinate geometry
↳ DLs & DRs
↳ Lines
↳ Planes



Let $P(a,b,c)$ be a point

a = The ⊥r distance of P from YZ- plane

b = The ⊥r distance of P from ZX plane

c = The ⊥r distance of P from XY-plane

Foot of ⊥r from $P(a\,b\,c)$ on to

XY plane - $P'(a, b, 0)$

YZ plane - $P'(0, b, c)$

ZX plane - $P'(a, 0, c)$

Image of $P(a,b,c)$ w.r.t

XY plane - $P'(a, b, -c)$

YZ plane - $P'(-a, b, c)$

ZX plane - $P'(a, -b, c)$

Distance of $P(a,b,c)$ from

XY plane - $|c|$

YZ plane - $|a|$

ZX plane - $|b|$

X - axis - $\sqrt{b^2+c^2}$

Y - axis - $\sqrt{c^2+a^2}$

Z - axis - $\sqrt{a^2+b^2}$

* from origin . $\sqrt{a^2+b^2+c^2}$

Distance formula

$P(x_1, y_1, z_1)$ $Q(x_2\ y_2\ z_2)$

$$|\overline{PQ}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Section formula

$R_1 = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ if internally divided

$R_1 = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$ if externally divided

Mid point

$$\left(\frac{x_1+x_2}{2}\ \frac{y_1+y_2}{2}\ \frac{z_1+z_2}{2}\right)$$

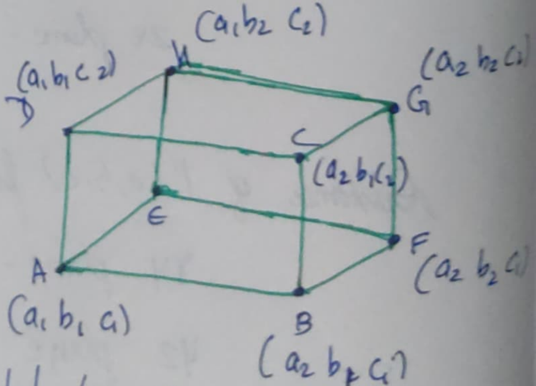
Centroid

$$\left(\frac{x_1+x_2+x_3}{3}\ \frac{y_1+y_2+y_3}{3}\ \frac{z_1+z_2+z_3}{3}\right)$$

length = $|a_2-a_1|$

breadth $|b_2-b_1|$

height - $|c_2-c_1|$



volume - $|a_2-a_1|\ |b_2-b_1|\ |c_2-c_1|$

parallelopiped - (2 3 5) & (5 7 10) find volume

$l = 5-2 = 3 \quad b = 7-3 = 4 \quad h = 10-5 = 5$

$V = 3 \times 4 \times 5 = 60$ cubic units

Centroid of the tetrahedron = $\left(\dfrac{x_1+x_2+x_3+x_4}{4}, \dfrac{y_1+y_2+y_3+y_4}{4}, \dfrac{z_1+z_2+z_3+z_4}{4}\right)$

Median of tetrahedron

The line segment joining each vertex to the centroid of the opposite face is median.

The point of concurrence of the medians is called centroid G.

G divides each median in the ratio 3:1

Let $A = (x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

The ratio in which

1) AB divided by x-y plane = $-z_1 : z_2$
2) AB divided by yz plane = $-x_1 : x_2$
3) AB divided by zx plane = $-y_1 : y_2$
4) AB divided by $ax+by+cz+d=0$ = $-(ax_1+by_1+cz_1+d) : (ax_2+by_2+cz_2+d)$

Q) If $(1,2,1)$ $(3,\alpha,5)$ $(4,6,\beta)$ are collinear.

then $\alpha, \beta = ?$

$$\frac{2}{1} = \frac{\alpha-2}{6-\alpha} = \frac{4}{\beta-5}$$

$$\beta-5=2 \rightarrow \boxed{\beta=7}$$

$$12-2\alpha = \alpha-2$$

$$\boxed{\alpha = 14/3}$$

Q) $A(1\,2\,1)$ $B(2\,{-3}\,5)$, C are collinear.

Find C as C lies on y-z plane

$$C \rightarrow (0, \alpha, \beta)$$

$$\frac{2-1}{0-2} = \frac{-3-\cancel{2}}{\alpha+3} \quad \frac{5-1}{\beta-5}$$

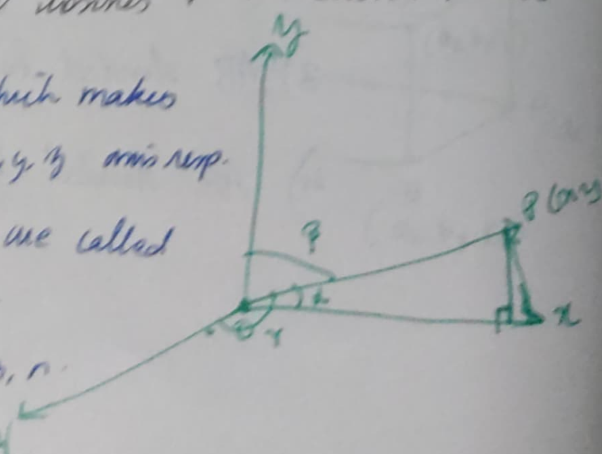
$$-\frac{1}{2} = \frac{-5}{\alpha+3} = \frac{4}{\beta-5}$$

$$-\alpha-3=-10 \qquad -\cancel{\beta+10} = -\beta+5 = 8$$

$$\alpha = 7 \qquad \boxed{\beta = -3}$$

$$C - (0,7,-3)$$

Directional cosines & Directional Ratios



Let $\overrightarrow{OP}$ be a line which makes α, β, γ angles with x, y, z axis resp.

Then $\cos\alpha, \cos\beta, \cos\gamma$ are called directional cosines of $\overrightarrow{OP}$.

generally denoted by l, m, n

(ie) $l = \cos\alpha$ $\quad m = \cos\beta$ $\quad n = \cos\gamma$

Let $P = (x, y, z)$

$\cos\alpha = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $\quad$ Illy $\quad$ $\cos\beta = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $\quad$ $\cos\gamma = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

dcs of x, y, z are

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

* If α, β, γ are the angles made by $\overrightarrow{OP}$ with x, y, z axis resp. Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ (ie) $l^2+m^2+n^2 = 1$

Let a line make angle θ with x, y, β with z axis such that $\sin\beta = \sqrt{3}\sin\theta$. Then $\cos^2\theta$?

$\sin^2\beta = 3\sin^2\theta$ $\qquad$ $2\cos^2\theta + \cos^2\beta = 1$

$1-\cos^2\beta = 3 - 3\cos^2\theta$ $\quad \Rightarrow \quad 2\cos^2\theta + 3\cos^2\theta - 2 = 1$

$\cos^2\beta = 3\cos^2\theta - 2$

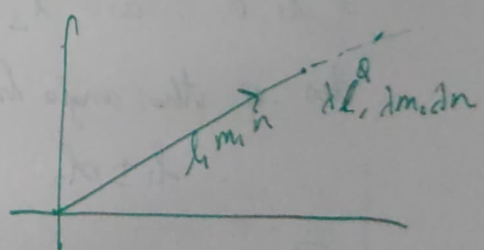
$5\cos^2\theta = 3$

$\cos^2\theta = 3/5$

Find dcs of a line inclined equally to co-ordinate axis

Let l, m, n be dcs of a line $\overrightarrow{OP}$

Then $\lambda l, \lambda m, \lambda n$ are called directional ratios of any point Q on $\overrightarrow{OP}$



If $P(a_1\ b_1\ c_1)$, $Q(a_2\ b_2\ c_2)$

$\overrightarrow{PQ}$ drs - $a_2-a_1, b_2-b_1, c_2-c_1$

PQ dcs. $\dfrac{a_2-a_1}{PQ}, \dfrac{b_2-b_1}{PQ}, \dfrac{c_2-c_1}{PQ}$

Q) If $D = (3\ -4\ 12)$

Find dcs of $\vec{OP}$

$\vec{OP}$ dcs $= \dfrac{3, -4, 12}{\sqrt{3^2+4^2+12^2}}$

$\vec{OP}$ dcs $\dfrac{3}{13}, \dfrac{-4}{13}, \dfrac{12}{13}$

Angle between two lines

Let $a_1\ b_1\ c_1$ & $a_2\ b_2\ c_2$ are drs of the two lines λ_1, λ_2 resp. Then θ be the angle b/w them

then $\boxed{\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}}$

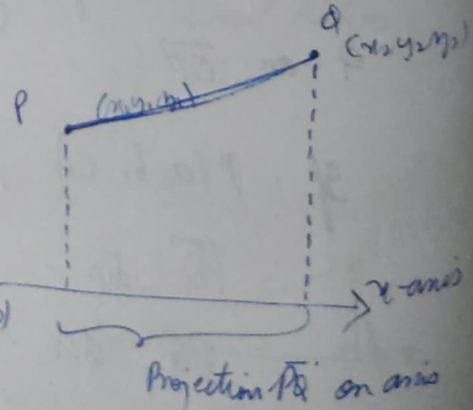
- If $\lambda_1 \perp \lambda_2$ Then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

If $\lambda_1 \parallel \lambda_2$ Then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

If l_1, m_1, n_1 and l_2, m_2, n_2 are dcs of two lines. Find the drs of the angle bisector of angle between the lines.

$l_1 \pm l_2$ & $m_1 \pm m_2$ $n_1 \pm n_2$

Projection



RS = projection of $\overline{PQ}$ on x-axis

$= \dfrac{(x_2-x_1, y_2-y_1, z_2-z_1)\cdot(1,0,0)}{1}$

$= x_2 - x_1$

Illy projection of $\vec{PQ}$ on y-axis = $y_2 - y_1$
projection of $\vec{PQ}$ on z axis = $z_2 - z_1$

$$RS = \frac{\vec{PQ} \cdot \vec{RS}}{|RS|} = \frac{(x_2-x_1)(x_4-x_3) + (y_2-y_1)(y_4-y_3) + (z_2-z_1)(z_4-z_3)}{\sqrt{(x_4-x_3)^2 + (y_4-y_3)^2 + (z_4-z_3)^2}}$$

P (1 2 -2) Q (8 10 11) R (1 2 3) S (3 5 7)

Find the length of projection $\vec{PQ}$ on $\vec{RS}$

$\vec{PQ}$ drs = 7, 8, 13

$\vec{RS}$ drs 2, 3, 4.

$$\frac{7\times 2 + 8\times 3 + 13\times 4}{\sqrt{2^2+3^2+4^2}} = \frac{14+24+52}{\sqrt{29}} = \frac{90}{\sqrt{29}}$$

* Let $l_1\, m_1\, n_1$ be dio of L_1 $\quad l_2\, m_2\, n_2$ be dio of L_2

Find the dio of the line which is ⊥r to both L_1, L_2.

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad L_1 \times L_2$$

$$\boxed{m_1 n_2 - m_2 n_1,\ l_2 n_1 - l_1 n_2,\ l_1 m_2 - l_2 m_1} \qquad \begin{vmatrix} \bar{l_1} & \bar{m_1} & \bar{n_1} \\ l_2 & m_2 & n_2 \end{vmatrix}$$

LIDRS

$$l l_1 + m m_1 + n n_1 = 0$$
$$l \cdot l_2 + m \cdot m_2 + n\, n_2 = 0$$

Find dcs of the lines satisfying $l+m+n=0$, & $lm=0$

$m=0$

$l=0$

$l+n=0$

$m+n=0$

$l=-n$

$m=-n$

$\frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$

$\frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$

L_2 drs = $1, 0, -1$

L_1 drs = $0, 1, -1$

$$\cos\theta = \frac{0\times1 + 1\times0 + (-1)(-1)}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2+0^2+(-1)^2}} = \frac{1}{2}$$

$\theta = 60°$

Find dcs of lines satisfying $l+m+n=0$ $2lm-mn+2ln=0$

$\Rightarrow n=-l-m$

$2lm - m(-l-m) + 2l(-l-m) = 0$

$2lm + lm + m^2 - 2lm - 2l^2 = 0$

$m^2 + lm - 2l^2 = 0$

$(m-l)(m+2l) = 0$

$m=l$ or $m=-2l$

$l=m$

$m=-2l$

$2m+n=0$

$\frac{l}{1}=\frac{m}{-2}$

$2m=-n$ $\frac{m}{1}=\frac{n}{-2}$

$l-2l+n=0$

$\frac{l}{1}=\frac{m}{1}=\frac{n}{-2}$

$l=n$

L_1 drs $1, 1, -2$

$\frac{l}{1}=\frac{m}{-2}=\frac{n}{1}$

L_2 drs $1, -2, 1$

$2l - m + 2n = 0$ $\qquad$ $mn + ln + lm = 0$

$m = 2l + 2n$

$(2l+2n)n + ln + l(2l+2n) = 0$

$2ln + 2n^2 + ln + 2l^2 + 2ln = 0$

$2n^2 + 5ln + 2l^2 = 0$

$2n^2 + 4ln + ln + 2l^2 = 0$

$2n(n+2l) + l(n+2l) = 0$ $\qquad$ $(2n+l)(l+2n) = 0$

$l = -2n$ $\qquad$ $n = -2l$

$l = -2n$

$\frac{l}{-2} = \frac{n}{1}$

$m = 2l - l = l$

$\frac{l}{-2} = \frac{m}{-2} = \frac{n}{1}$

$-2, -2, 1$

$m = -2l$

$\frac{l}{1} = \frac{m}{-2}$

$m = 2n - n = n$

$\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2}$

$1, -2, -2$

3-D lines

equation of the line passing through $A(x_1, y_1, z_1)$ and parallel to a line whose drs are a, b, c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

equation of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

* equation of the line passing through $O(0,0,0)$ and parallel to a line whose drs are $1,2,3$, is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

* equation of line passing through $A(1,2,3)$ & $B(3,2,1)$ is

$$\frac{x-1}{3-1} = \frac{y-2}{2-2} = \frac{z-3}{1-3} \Rightarrow \frac{x-1}{2} = \frac{y-2}{0} = \frac{z-3}{-2}$$

$P(\alpha, \beta, \gamma)$ be a point

$L = \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be a line.

at Q_1 . $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = t$

$x = at + x_1$ $y = bt + y_1$ $c = zt + z_1$

Angle between the lines

$$L_1 - \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2 = \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

$$(L_1, L_2) = \theta$$

Then $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$

If $L_1 \parallel L_2 = \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

If $L_1 \perp L_2 = 1 \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

Point of intersection of two lines

$$L_1 = \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} = t \quad - P(t)$$

$$L_2 \cdot \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} = s \quad - Q(s)$$

the point of intersection is $P(t) = Q(s)$
solve for t, s

Eg- $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = t$ &

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} = s.$$

$P(t) = \{2t+1 \quad 3t+2 \quad 4t+3\}$

$Q(s) = \{3s+2 \quad 4s+5 \quad 5s+4\}$

$\rightarrow t = s = -1$

~~P(t)~~ POI $\leftarrow \{-1, -1, -1\}$

Distance between parallel lines

$L_1 \cdot \bar{r} = \bar{a_1} + \lambda \bar{b}$

$L_2 \cdot \bar{r} = \bar{a_2} + \mu \bar{b}$

$$\sin\theta = \frac{d}{AB} = \frac{d}{|\bar{a_2} - \bar{a_1}|} = \frac{\sin\theta \cdot |\bar{a_2} - \bar{a_1}| \cdot |\bar{b}|}{|\bar{b}|}$$

$$\boxed{d = \frac{|\bar{b} \times (\bar{a_2} - \bar{a_1})|}{|\bar{b}|}}$$

Q) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$ & $\frac{x-3}{2} = \frac{y-3}{3} = \frac{z+5}{6}$

find distance between them

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} = t \qquad \frac{x-3}{2} = \frac{y-3}{3} = \frac{z+5}{6} = s$

$(1, 2, -4) \qquad (3, 3, 5)$

$\bar{a_1} = \hat{i} + 2\hat{j} - 4\hat{k} \qquad \bar{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$

$b = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$d = \frac{|2i + 3j + 6k \times (2i + 3j - k)|}{\sqrt{2^2 + 3^2 + 6^2}}$$

$$d = \frac{\cancel{|4 + 3 - 6|}}{\cancel{\sqrt{4+9+36}}} = \frac{4}{8}$$

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$-9\hat{i} + 14\hat{j} + -4\hat{k}$

$$d = \frac{\sqrt{81 + 196 + 16}}{7} = \frac{\sqrt{293}}{7}$$

Skew distance

$L_1 = \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$

$L_2 = \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$

Lines which are neither parallel nor intersecting are called skew lines

SD = shortest distance

= Projection of $\bar{a_2}-\bar{a_1}$ on $\bar{b_1} \times \bar{b_2}$

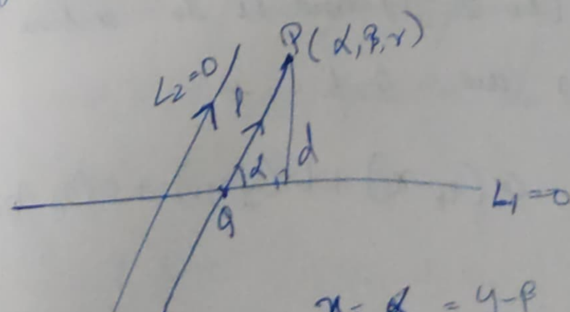
$= \frac{(\bar{a_2}-\bar{a_1}) \cdot \bar{b_1} \times \bar{b_2}}{|\bar{b_1} \times b_2|} = \frac{[\bar{a_2}-\bar{a_1} \ \bar{b_1} \ \bar{b_2}]}{|\bar{b_1} \times b_2|}$

Image of a line wrt another line

$L_1 = \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$

$L_2 = \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$

Reflection of line $L_2=0$ wrt $L_1=0$



$\frac{x-\alpha}{c_2} = \frac{y-\beta}{b_2} = \frac{z-\gamma}{c_2}$

Find POI & distance also $l = d\cos\theta$

3-D planes

Equation of planes

i) vector equation of plane in $\hat{n}$

$$\frac{\vec{r}\cdot\vec{n}}{|\vec{n}|} = p$$

$$\bar{r}\cdot\hat{n} = p \rightarrow \text{normal form}$$

Cartesian plane

$$\hat{n}(a, b, c) \quad \bar{r} - (x\ y\ z)$$

$$ax + by + cz = p$$

* The vector equation of the plane passing through $\bar{a}$ and perpendicular to $\bar{n}$ is –

$$\overline{AP} \perp \bar{n}$$

$$(\bar{r} - \bar{a})\bar{n} = 0$$

Cartesian form

Equation of the plane passing through $A(x_1, y_1, z_1)$ and $\perp$ to a line whose drs are a, b, c is:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

* Vector equation of the plane passing through $\vec{a}$ and parallel to $\bar{b}, \bar{c}$ is

$\overline{AP}, \bar{b}, \bar{c}$ are coplanar.

$[\overline{AP}\ \ \bar{b}\ \ \bar{c}] = 0 \qquad [\bar{r}-\bar{a}\ \ \bar{b}\ \ \bar{c}] = 0$

Cartesian plane

Equation of the plane passing through $A(x_1, y_1, z_1)$ and parallel to two line whose drs are a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

* Vector equation of the plane passing through $\bar{a}, \bar{b}$, and parallel to $\bar{c}$ is

$\overline{AP}, \overline{AB}, \bar{c}$ are coplanar

$[\bar{r}-a\ \ \bar{b}-a\ \ \bar{c}]$

Cartesian plane

Equation of the plane passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ parallel to two line whose drs are a, b, c is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ a & b & c \end{vmatrix} = 0$$

* vector equation of the plane passing through $\bar{a}, \bar{b}, \bar{c}$

$\overline{AP}$ $\overline{AB}$ $\overline{AC}$ are coplanar

$$[\bar{r}-\bar{a} \quad \bar{b}-\bar{a} \quad \bar{c}-\bar{a}] = 0$$

Cartesian form

equation of the plane passing through $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Q) Find the equation of the plane passing through the mid point of $\overline{AB}$. $A(1,2,3)$ $B(3,2,1)$ and ⊥ to $\overline{AB}$

$\bar{n}$ drs $(2, 0, -2)$

M -- mid point of $\overline{AB}$ = $(2,2,2)$

equation of plane =

$$2(x-2) + 0(x-2) + 2(x-2) = 0$$

$$x - z = 0 \quad \boxed{x = z}$$

Q) Find the eqn of the plane passing through $A(1,2,3)$ and parallel to $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \dots$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$(x-1)(-1) - (y-2)(2) + (z-3)(-1)$

$x - 2y + z = 0$

Q Find the equation of the plane passing through $A(-1,0,1)$ $B(1,2,3)$ and parallel to $\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+1}{2}$

$$\begin{vmatrix} x+1 & y-0 & z-1 \\ 2 & 2 & 2 \\ 1 & -3 & 2 \end{vmatrix}$$

$(x+1)(10) - y(2) +$

$(z-1)(-8) = 0$

$10x - 2y - 8z + 18 = 0$

$5x - y - 4z + 9 = 0$

Q) Find the equation of the plane passing through $A(1,2,3)$ $B(3,2,-1)$ $C(3,1,2)$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 0 & -4 \\ 2 & -1 & -5 \end{vmatrix} = 0$$

$(x-1)(-4) - (y-2)(-2) + (z-3)(-2) = 0$

$2x - y + z - 3 = 0$

Q passing through $A(3,2,1)$ & $\perp$ to y-axis

$0(x-3) + 1(y-2) + 0(z-1) = 0$

$y = 2$

Point and a plane

Π: $ax + by + cz + d = 0$

* perpendicular distance from P to $\pi = 0$

$$P = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

* $Q(\alpha, \beta, \gamma)$ is foot of $\perp$ from P to $\pi = 0$ then

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2+b^2+c^2}$$

* $R(\alpha, \beta, \gamma)$ is image (reflection) of $P(x_1, y_1, z_1)$ w.r.t $\pi = 0$ then

$$\frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2+b^2+c^2} = \frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = \cancel{2(ax_1}$$

* $\perp$ distance from $(0, 0, 0)$ to the plane $ax + by + cz + d = 0$ is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$

Line and a plane

$$L = \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\pi = a_2 x + b_2 y + c_2 z + d_2 = 0$$

* point of intersection of $L = 0$ & $\pi = 0$ is

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} = t$$

$$P(t) = (a_1 t + x_1, \ b_1 t + y_1, \ c_1 t + z_1) \in \pi = 0$$

$$a_2(a_1 t + x_1) + b_2(b_1 t + y_1) + c_2(c_1 t + z_1) + d_2 = 0$$

Solve for t

* Angle between line and plane

$$\sin\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

* Distance between the line and the plane (parallel)

$$P = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

* Image of a line wrt a plane

case-1) $L \parallel \pi$

let $P'(\alpha, \beta, \gamma)$ be the image of $P(x_1, y_1, z_1)$ wrt $\pi = 0$

$$L' = \frac{x-\alpha}{a_1} = \frac{y-\beta}{b_1} = \frac{z-\gamma}{c_1}$$

Case-2. $L \not\parallel \pi$ $\quad L = 0$.

The distance of $P(x_2\ y_2\ z_2)$ from $\pi = 0$ along L_1. PQ eqn is

$$\frac{x-x_2}{a_1} = \frac{y-y_2}{b_1} = \frac{z-z_2}{c_1} = t$$

$Q(t) \in \pi$ solve for t, find PQ.

Q) $L: \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+3}{2}$

$\Pi: x+2y+3z = 6$

1) Point of intersection of line with plane

$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+3}{2} = t$

$x = 2t+1 \quad y = -t+2 \quad z = 2t-3$

~~2t+1=2t~~ $\quad$ ~~-t+2=2 $\quad$ 2t-3=3~~

$2t+1 + 2(-t+2) + 3(2t-3) = 6$

$\cancel{2t}+1 - \cancel{2t} + 4 + 6t - 9 = 6$

$6t = 10 \quad t = 5/3$

$POI = \left(\frac{13}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Q2) Angle between the line & plane

$L_1 \cdot 2, -1, 2 \qquad \Pi_1 = 1, 2, 3.$

plane dirs $\perp$ to plane

$\sin\theta = \frac{2(1) + (-1)(2) + 2(3)}{\sqrt{2^2+1^2+2^2} \cdot \sqrt{1^2+2^2+3^2}}$

$\sin\theta = \frac{6}{3 \cdot \sqrt{14}} = \frac{2}{\sqrt{14}} = \sqrt{\frac{2}{7}}$

$\theta = \sin^{-1}\sqrt{\frac{2}{7}}$

Q3) Reflection of $L=0$ wrt the plane

Let α, β, γ be the image of P wrt $x+2y+3z=6$

$P(1, 2, -3)$

~~$\frac{\alpha-1}{1} = \frac{\beta-2}{2} = \frac{\gamma+3}{3}$~~

$$\frac{\alpha-1}{1} = \frac{\beta-2}{2} = \frac{\gamma+3}{3} = \frac{-2(1+4+9-6)}{14} = \frac{20}{14} = \frac{10}{7}$$

$$\alpha = \frac{17}{7} \quad \beta = \frac{34}{7} \quad \gamma = \frac{30}{7} - 3 = \frac{9}{7} \qquad \left(\frac{17}{7}, \frac{34}{7}, \frac{9}{7}\right)$$

Q) If the line is $\frac{x-1}{2} = \frac{y+1}{K} = \frac{z-1}{-3}$ is parallel to the plane $x - 2y + 2z = 5$. Then $K =$

drs are perpendicular

$(2, K, -3)$ $(1, -2, 2)$ dot product $= 0$.

$2 - 2K - 6 = 0$ ~~$2K = 7$ $K = 7/2$.~~

$2K = -4$ $\boxed{K = -2}$

Q) If the lines $\frac{x-1}{a} = \frac{y+2}{b} = \frac{z-3}{c}$ is ⊥ to the $2x + 3y + 4z = 5$ $(a, b, c \in N)$. Then the least value of $2a + 3b + 4c$ is

~~$\sin\theta = 1$~~ $\frac{2a + 3b + 4c}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{2^2+3^2+4^2}} = 1$

drs are corresponding

$\frac{a}{2} = \frac{b}{3} = \frac{c}{4} \rightarrow a = 2 \quad b = 3 \quad c = 4$

$2a + 3b + 4c \geq 29$.

Q) Projection of a ~~plan~~ line onto a plane.

get drs of both, find angle, length of line × $\cos\theta$.