Indian Institute of Technology Mandi
भारतीय प्रौद्योगिकी संस्थान मण्डी

IC-252
Theory Assignment - 3

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1. Consider two random variables X and Y with joint PMF given by

	$Y=0$	$Y=1$	$Y=2$
$X=0$	1/6	1/4	1/8
$X=1$	1/8	1/6	1/6

   (a) Find $P(X=0, Y \leq 1)$.
   (b) Find the marginal PMFs of X and Y.
   (c) Find $P(Y=1|X=0)$.
   (d) Are X and Y independent?

2. Let $X \sim Bernoulli(p)$ and $Y \sim Bernoulli(q)$ be independent, where $0 < p, q < 1$. Find the joint PMF and joint CDF for X and Y.

3. If you roll a fair die. Let X be the observed number. Find the conditional PMF of X given that you know the observed number was less than 5.

4. For two discrete random variables X and Y, show that $E[X+Y] = E[X] + E[Y]$.

5. Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

   (a) the joint probability function $f(x, y)$.
   (b) $P[(X, Y) \in A]$, where A is the region $\{(x, y)|x + y \leq 1\}$.

6. Suppose $X_1, X_2, \ldots, X_n$ are a random sample from the Poisson distribution with mean λ. Find the joint p.m.f. of $X_1, X_2, \ldots, X_n$.

7. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find constant c.
   (b) Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.
   (c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
   (d) Find the joint CDF for X and Y.

8. Let the joint pdf of two continuous random variables be

$$f_{XY}(x, y) = \begin{cases} a(e^{-x} + e^{-y}) & 0 \leq x, 0 \leq y \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find a.
   (b) Find $P(x + y \leq 1)$.

9. Let X and Y be two independent $Uniform(0, 1)$ random variables. Find $F_{XY}(x, y)$.

10. Let X, Y be two i.i.d random variables uniform in the interval $[a, b]$. Find $P(X + Y \leq a + b, Y \leq X)$.

11. Suppose

$$f_{XY}(x,y) = \begin{cases} 24x(1-x-y) & x,y \geq 0, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find

(a) $P(X > Y)$.

(b) $P(X > \frac{1}{2})$.

12. A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f_{XY}(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P[(X,Y) \in A]$, where $A = \{(x,y)|0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}$.

(b) Find the marginal distributions of X and Y.

(c) Find the conditional distribution of X, given that $Y = 1$, and use it to determine $P(X = 0|Y = 1)$.

13. Let the joint pdf of two continuous random variables be

$$f_{XY}(x,y) = \begin{cases} c(e^{-2x-3y}) & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find

(a) the value of c.

(b) $P(X < 1, Y < 2)$

(c) $P(1 < X < 2)$

(d) $P(Y > 3)$

(e) $E[X]$ and $E[Y]$

(f) $P(X|Y = 1)$

(g) $E[X|Y = 2]$

14. Let $X \sim Exponential(1)$.

(a) Find the conditional PDF and CDF of X given $X > 1$.

(b) Find $E[X|X > 1]$.

(c) Find $Var(X|X > 1)$.

15. Determine whether X and Y are independent:

(a)

$$f_{XY}(x,y) = \begin{cases} 2e^{-x-2y} & x,y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_{XY}(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

16. Let X, Y be two continuous random variables with the joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{1}{x} & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq 2Y)$.

17. The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f_{XY}(x,y) = \begin{cases} 10xy^2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal densities $g(x), h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

18. A theory of chemical reactions suggests that the variation in the quantities X and Y of two products C_1 and C_2 of a certain reaction is described by the joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{2}{(1+x+y)^3} & 0 \leq x, 0 \leq y, \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the probability that at least one unit of each product is produced?

(b) Determine the probability that quantity of C_1 produced is less than half that of C_2.

(c) Find the c.d.f. for the total quantity of C_1 and C_2.

19. Given the joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{x(1+3y^2)}{4} & 0 < x < 2, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $g(x), h(y)$, $f(x|y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2}|Y = \frac{1}{3})$.

20. Suppose that the shelf life, in years, of a certain perishable food product packaged in cardboard containers is a random variable whose probability density function is given by

$$f_{XY}(x,y) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let X_1, X_2, and X_3 represent the shelf lives for three of these containers selected independently and find $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.