Hilbert's Grand Hotel

When an Infinite Hotel ran out of rooms!

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1 Introduction

The Infinite Hotel Paradox, proposed by the German mathematician David Hilbert in the early 20th century, stands as one of the most perplexing concepts in the realm of mathematics. Initially, the concept of an infinite hotel might appear fantastical, but its significance reaches far into the core of set theory and the fundamental essence of infinity.

2 Setup

Imagine a hotel with an infinite number of rooms, each numbered sequentially with numbers from $\mathbb{N}$: room 1, room 2, room 3, and so on, stretching to infinity. Now, suppose this hotel is fully occupied, with each room containing a guest. Despite being at full capacity, a new guest arrives, seeking accommodation. Intuitively, one might assume that the hotel cannot accommodate any more guests. However, the Infinite Hotel Paradox challenges this notion, revealing the fascinating properties of infinite sets.

3 Assumptions

1. Each room can be occupied by one and only one person.
2. There are no hotel rooms numbered using non-Natural numbers.
3. Occupants are always Natural Numbered. That is, there can not be 102.5 passengers in a bus.

4 Allocation Rooms

4.1 A Single Person

A man walks into the hotel and asks for a room. Instead of turn him down, the manager decides to make room for him. The manager requests every person in room 'n' to move to room 'n+1'. Hence, the person in room 1 moves to room 2, the person in room 2 moves to room 3 and so on. This makes the room 1 available for our new customer.

4.2 A finite number of people

Let's say 'm' people wants to move in to our hotel. The Manager asks every person in room 'n' to move to room 'n+m', thereby opening up the first 'm' rooms.

4.3 A countable Infinite number of people

Now an infinitely large bus with a countably infinite number of passengers pulls up to our hotel. countably infinite is the key. Now, the manager asks every person in room 'n' to move to room '2n'. The person in room 1 moves to room 2, the one in room 2 moves to room 4 and so on. As a result, every even numbered room is now occupied with previous people while the odd numbered room (countably infinite in number!) are available for the new passengers to occupy.

4.4 An Infinite number of Infinitely many people

As the last challenge to our manager, a infinite line of infinitely large buses, each with a countably infinite number of passengers visits the hotel, looking for accommodation. Since it is known that there are an infinite number of prime numbers, the Manager uses powers of prime numbers as room numbers. He asks all the current occupants of the hotel to move from their room 'n' to room '2^n'. So, the current occupant of room 5 goes to room number 2^5 which is room 32.

Similarly, he asks passengers of bus 1 to move to rooms '3^n where 'n' is their seat number; passengers of bus 2 to move to rooms '5^n' and so on. Hence , the person on the 6th seat of 4th bus will reside in room 9^6, i.e., room 531441.

4.5 An uncountable infinite number of people?

Imagine a situation where an infinitely long bus pulls up to our hotel but inside the bus there are no seats. Instead, the passengers are uniquely identified by their names, which are an infinitely long sequence of 0s and 1s. For example, there's a person named '1000010100010010...', '111111101010101...' and so on.

Now all the strategies of our manager are in vain, since there's no particular way to map each of the passengers using a set of Natural Numbers. The passengers are, therefore, uncountably infinite in number and hence cannot be accommodated inside our hotel containing a countably infinite number of rooms.

5 Conclusion

The manager's strategies are only feasible due to the nature of infinity. The Infinite Hotel only deals with the lowest level of infinity, mainly, the countable infinity of the natural numbers, 1, 2, 3, 4, and so on. We use natural numbers for the room numbers as well as the seat numbers on the buses.
If we were dealing with higher orders of infinity, such as that of the real numbers, these strategies would no longer be possible as we have no way to systematically include every number.

6 Sources

1. The Infinite Hotel Paradox - Jeff Dekofsky
2. How An Infinite Hotel Ran Out Of Room
3. Wikipedia Article