

# Truncated multirange percolation of words on the square lattice

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## Abstract

We study mixed long-range percolation on the square lattice. Each vertical edge of unit length is independently open with probability  $\varepsilon$ , and each horizontal edge of length  $i$  is independently open with probability  $p_i$ . Also, each vertex is assigned independently a random variable taking values 1 and 0 with probability  $p$  and  $1 - p$ , respectively. We prove that for a broad class of anisotropic long-range percolation models for which connection probabilities  $p_i$  satisfy some regularity conditions, all words (semi-infinite binary sequences) are seen simultaneously from the origin with positive probability, even if all edges with length larger than some constant (depending on  $\varepsilon$ ,  $p$ , and on the sequence  $(p_i)$ ) are suppressed.

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## 1 Introduction

We study long-range percolation on the square lattice. More precisely, let  $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}_V \cup \mathcal{E}_H)$ , where  $\mathcal{E}_V$  denotes the set of vertical edges of unit length, and  $\mathcal{E}_H$  denotes the set of horizontal edges of all lengths. Specifically, we define

$$\mathcal{E}_V = \left\{ \{(x_1, x_2), (y_1, y_2)\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |x_2 - y_2| = 1, x_1 = y_1 \right\}, \quad (1)$$

$$\mathcal{E}_{H,i} = \left\{ \{(x_1, x_2), (y_1, y_2)\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |x_1 - y_1| = i, x_2 = y_2 \right\}. \quad (2)$$

We then write  $\mathcal{E}_H = \bigcup_{i=1}^{\infty} \mathcal{E}_{H,i}$  and  $\mathcal{E} = \mathcal{E}_V \cup \mathcal{E}_H$ .

The percolation process is defined as follows: given a sequence  $(p_i)_{i \in \mathbb{N}}$ ,  $p_i \in [0, 1]$ , we consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega = \{0, 1\}^{\mathcal{E}}$  and  $\mathcal{F}$  is the  $\sigma$ -algebra generated by the cylinder sets

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of  $\Omega$ . A configuration  $\omega \in \{0, 1\}^{\mathcal{E}}$  is assigned to the edges, and for each  $e \in \mathcal{E}$ , we write  $\omega(e)$  for its value at  $e$ . If  $\omega(e) = 1$  ( $\omega(e) = 0$ ) we say  $e$  is open (closed). The probability measure is given by

$$\mathbb{P} = \prod_{\{x,y\} \in \mathcal{E}} \mu_{\{x,y\}},$$

where  $\mu_{\{x,y\}}(\omega(\{x,y\}) = 1) = p_{||x-y||}$  is a Bernoulli measure.

Given  $K \in \mathbb{N}$ , we introduce the  $K$ -truncated sequence  $\{p_{K,i}\}_{i \in \mathbb{N}}$ , defined by

$$p_{K,i} = \begin{cases} p_i & \text{if } i \leq K, \\ 0 & \text{if } i > K, \end{cases}$$

and the corresponding  $K$ -truncated percolation model  $(\Omega, \mathcal{F}, \mathbb{P}_K)$ , where

$$\mathbb{P}^K = \prod_{\{x,y\} \in \mathcal{E}} \mu_{K,\{x,y\}}, \quad (3)$$

with  $\mu_{K,\{x,y\}}(\omega(\{x,y\}) = 1) = p_{K,||x-y||}$ .

We say that two vertices  $u, v \in \mathbb{Z}^2$  are connected in the configuration  $\omega$  if there exists a sequence of distinct vertices  $(v_0 = u, v_1, \dots, v_n = v)$  in  $\mathbb{Z}^2$  such that  $\{v_i, v_{i+1}\} \in \mathcal{E}$  is open in  $\omega$  for all  $i = 0, \dots, n-1$ . We write  $\{u \leftrightarrow v\}$  for the event where  $u$  and  $v$  are connected. The cluster of  $v$  in the configuration  $\omega$  is defined as  $C_v(\omega) = \{u \in \mathbb{Z}^2 : v \leftrightarrow u \text{ in } \omega\}$  (when  $v$  is the origin we write simply  $C = C(\omega)$  instead of  $C_0(\omega)$ ). Percolation is said to occur at the vertex  $v$  if  $|C_v(\omega)|$  is infinite.

## 1.1 Background and motivation

The classical truncation question asks: if  $\mathbb{P}(|C| = \infty) > 0$ , is it true that  $\mathbb{P}^K(|C| = \infty) > 0$  for some sufficiently large  $K$ ? To the best of our knowledge, this problem was first considered in [15], where the authors provided an affirmative answer under the assumption that the sequence  $(p_i)_{i \in \mathbb{N}}$  decays exponentially.

A simpler (but still open) problem is the following: assuming the sum of the  $p_i$ 's diverges, does the truncation question on the square lattice have an affirmative answer? The first significant progress in this direction is found in [18], where the authors analyze a model with  $p_V = \varepsilon > 0$  and  $p_{H,i} = p_i$  for  $i \geq 1$ . They show that if  $p_i \geq (i \log i)^{-1}$  for all sufficiently large  $i$ , then the truncation question has an affirmative answer. Their proof reduces the problem to analyzing a dependent percolation process on a renormalized lattice. This process dominates an anisotropic nearest-neighbor Bernoulli percolation, where the probabilities of vertical and horizontal edges being open,  $p_V$  and  $p_H$ , respectively, satisfy  $p_V + p_H > 1$ . This condition ensures almost sure percolation, as detailed in page 54 of [10].

Removing the regularity assumption on the sequence  $(p_i)_{i \in \mathbb{N}}$ , in [7] the authors consider a long-range percolation model on  $\mathbb{Z}^d$ ,  $d \geq 2$ , where all edges of length  $i$  parallel to the axis are independently open with probability  $p_i$ . They prove that the truncation question has an affirmative answer under the condition that  $\limsup p_i > 0$ . Their proof involves constructing an isomorphism between a subgraph of

$\mathcal{G}$  and a  $s$ -dimensional slab of  $\mathbb{Z}^2$  for some  $s$  large, followed by an application of the classical Grimmett and Marstrand result [9]. For related results on other graphs under the same hypothesis, see [1].

The truncation question has also been explored under different assumptions on the  $p_i$ 's; see [14] and the more recent work [4]. For higher dimensions ( $d \geq 3$ ), affirmative answer have been provided in [5], [6], and more recently in [2].

## 1.2 Percolation of words

The problem of percolation of words was first introduced in a pioneering paper by I. Benjamini and H. Kesten [3]. The problem can be formulated as follows: consider a graph  $G = (V, E)$  with a countably infinite vertex set  $V$ . Each vertex  $v \in V$  is independently assigned a random variable  $\eta(v)$ , which takes the value 1 with probability  $p$  and 0 with probability  $1 - p$ . This induces the probability space  $(H, \mathcal{H}, \mathbb{P}_p)$ , where  $H = \{0, 1\}^V$ ,  $\mathcal{H}$  is the sigma-field generated by the cylinder sets of  $H$ , and  $\mathbb{P}_p$  is the product measure on  $\mathcal{H}$ . A typical element of  $H$  is denoted by  $\eta$ , and  $\eta(v)$  denotes the state of vertex  $v$  in the configuration  $\eta$ .

In percolation of words, one is interested in the existence or nonexistence of paths  $(v, v_1, v_2, \dots)$  with  $\eta(v_i) = \xi_i$ ,  $i \geq 1$ , for any prescribed sequence  $\xi = \{\xi_i\}_{i \geq 1} \in \{0, 1\}^{\mathbb{N}}$ . In this case, we say that  $\xi$  is seen from vertex  $v$  (note that the state of  $v$  plays no role here). The main goal is to understand when the collection of sequences  $\{\xi_i\}_{i \geq 1}$  that are seen is large. Formally, write

$$\Xi = \{0, 1\}^{\mathbb{N}}$$

and let  $\xi = \{\xi_n\}_{n \in \mathbb{N}}$  be an element of  $\Xi$ , which we refer to as a **word**. For  $v \in V$  and  $M \subset V$ , define

$$W_v = W_v(\eta) = \{\xi \in \Xi : \xi \text{ is seen from } v \text{ in } \eta\},$$

$$W_M = W_M(\eta) = \bigcup_{v \in M} \{\xi \in \Xi : \xi \text{ is seen from } v \text{ in } \eta\}.$$

Additionally, write

$$W_{\infty} = W_{\infty}(\eta) = \bigcup_{v \in V} W_v(\eta),$$

which denotes the set of words seen from some vertex in the graph  $G$ . Clearly, the largest possible set  $W_{\infty}$  can be is  $\Xi$ .

The study of percolation of words is particularly challenging due to its lack of monotonicity. Unlike traditional Bernoulli percolation (where the focus is on events such as "an infinite connected component of 1's exists"), the events considered in percolation of words are generally neither increasing nor decreasing.

Denote by  $\mathbb{L}^d$  the usual hypercubic lattice with nearest neighbors. In [3] the authors investigate the problem of percolation of words on  $\mathbb{L}_+^d$  and prove that the event  $\{W_{\infty} = \Xi\}$  occurs almost surely when  $p = 1/2$  and  $d \geq 10$ . Furthermore, the more restrict event  $\{\text{there exists } v \text{ such that } W_v = \Xi\}$  occurs almost surely when  $p = 1/2$  and  $d \geq 40$ . Thus, in  $\mathbb{L}_+^d$  with  $d \geq 40$  there is a vertex from which one sees

all words, and (by ergodicity) there is a strictly positive probability that one sees all words from the origin.

In [11], the authors show that **almost all** words (with respect to the product measure  $\nu_\lambda := ((1 - \lambda)\delta_0 + \lambda\delta_1)^{\otimes \mathbb{N}}$  on  $\Xi$ ,  $0 < \lambda < 1$ ) can be seen in critical site percolation on the triangular lattice, addressing Open Problem 1 in [3]. In a subsequent paper [12], the authors examine the problem of seeing all words in site percolation on the closed-packed graph of the square lattice, which is formed by adding diagonal edges to each face of  $\mathbb{L}^2$ . They show that for every fixed  $p \in (1 - p_c(\mathbb{L}^2), p_c(\mathbb{L}^2))$  the event  $\{\text{there exists } v \text{ such that } W_v = \Xi\}$  has probability 1. Here,  $p_c(\mathbb{L}^d)$  denotes the critical threshold for Bernoulli site percolation on  $\mathbb{L}^d$ .

In a recent work [17], the authors consider the problem of percolation of words on  $\mathbb{L}^d$ ,  $d \geq 3$ . They show that the event  $\{W_\infty = \Xi\}$  occurs almost surely for all  $d \geq 3$  and  $p \in (p_c(\mathbb{L}^d), 1 - p_c(\mathbb{L}^d))$ , answering the question posed in Open Problem 2 in [3]. Their proof is based on a coupling of a renormalized process with oriented percolation.

### 1.3 Results

All the papers discussed in the previous section address the problem of percolation of words on fixed graphs with all edges present. In contrast, we investigate a more general framework by studying this problem on a random graph, introducing two layers of randomness. First, a long-range bond configuration  $\omega$  is sampled according to  $\mathbb{P}^K$ . Then, the occurrence of a word  $\xi$  in  $\omega$  is determined by the conditional measure  $\mathbb{P}_p^\omega$ . Formally, let  $\mathbb{P}_p^K$  denote the probability measure on the space  $(\Omega \times H, \sigma(\mathcal{F} \times \mathcal{H}))$ . For every measurable rectangle  $R_1 \times R_2$ , we have

$$\mathbb{P}_p^K(R_1 \times R_2) = \int_{R_1} \mathbb{P}_p^\omega(R_2) d\mathbb{P}^K(\omega), \quad R_1 \in \mathcal{F}, R_2 \in \mathcal{H}. \quad (4)$$

Regarding the  $K$ -truncated long-range model, we say the word  $(\xi_1, \xi_2, \dots) \in \Xi$  is **seen from the vertex**  $v \in \mathbb{Z}^2$  in the configuration  $\omega \times \eta$  if there is a sequence  $(v = v_0, v_1, v_2, \dots)$ ,  $v_i \in \mathbb{Z}^2$ , such that  $v_i \neq v_j$ ,  $e_i = \{v_{i-1}, v_i\} \in \mathcal{E}$ ,  $\|v_{i-1}, v_i\| \leq K$ ,  $\eta(v_i) = \xi_i$ ,  $\omega(e_i) = 1$ , for all  $i, j$ . If this occurs, we say that  $\xi$  percolates in the  $K$ -truncated model.

In [8], the authors study a similar model on the  $d$ -dimensional hypercubic lattice,  $d \geq 3$ , incorporating one-dimensional long-range connections. They show that under the condition that the sum of  $p'_i$ 's diverges, the event  $\{W_0 = \Xi\}$  has probability arbitrarily close to 1, provided  $K$  is sufficiently large. For  $d = 2$ , the analogous result under the assumption that the sum of  $p'_i$ 's diverges remains an open problem. Indeed, as previously noted, it is still unresolved whether percolation occurs when relying solely on the divergence of the sum of  $p'_i$ 's.

In this work, we address the problem of percolation of words on a random subgraph of  $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E}_V \cup \mathcal{E}_H)$ . We will show (see Theorem 1) that the event  $\{W_0 = \Xi\}$  occurs with positive probability for a broad class of anisotropic truncated long-range percolation models for which connection probabilities  $p_i$  satisfy certain regularity conditions; see Remark 3. To streamline the presentation and avoid unnecessary

complications, we focus on the particular case where

$$p_i \geq (i \log i)^{-1}, \quad (5)$$

for all  $i$  sufficiently large. This choice not only simplifies the analysis but also allows us to establish a stronger result compared to [18], which shows that percolation occurs under the same hypotheses.

In the following, we consider the model in (4) with  $p_V = \varepsilon$  and  $p_{H,i} = p_i$  for all  $i \geq 1$ . We denote the relevant measure by  $\mathbb{P}_{p,\varepsilon}^K$ .

**Theorem 1.** *If  $p_i \geq (i \log i)^{-1}$  for all sufficiently large  $i$ , then for all  $p \in (0, 1)$  and for all  $\varepsilon \in (0, 1]$ , there exists a constant  $K = K(p, \varepsilon, \{p_i\}) \in \mathbb{N}$  such that*

$$\mathbb{P}_{p,\varepsilon}^K(W_0 = \Xi) > 0.$$

Consider Bernoulli site percolation on  $\mathcal{G}$  with parameter  $p \in (0, 1)$ . Suppose the sequence  $(p_i)_{i \geq 1}$  is such that the  $K$ -truncated model percolates at the origin with positive probability. With the coexistence of infinite clusters of 0's and 1's, it is reasonable to expect that all words can be seen, possibly with a suitably larger truncation constant. Motivated by this observation, we propose the following conjecture.

**Conjecture 1.** *Consider a supercritical mixed long-range percolation model on  $\mathcal{G}$  with  $p_V = \varepsilon$  and  $p_{H,i} = p_i$  for all  $i \geq 1$ . If the  $K$ -truncated process percolates for some  $K \in \mathbb{N}$ , then there exists  $K_1 \in \mathbb{N}$  such that all words can be seen from the origin with positive probability in the  $K_1$ -truncated model.*

The following theorem strengthens Conjecture 1. As noted in [7], the truncation question has an affirmative answer under the condition  $\limsup p_i > 0$  when considering the graph with vertex set  $\mathbb{Z}^2$  and all edges included. Here, we establish a stronger result under the same conditions on  $p_i$ , showing that all words can be seen from the origin with positive probability on  $\mathcal{G}$  when  $p_V = \varepsilon$  and  $p_{H,i} = p_i$  for all  $i \geq 1$ .

**Theorem 2.** *If  $\limsup p_i > 0$ , then for all  $p \in (0, 1)$ ,  $\varepsilon \in (0, 1]$ , and  $\alpha > 0$ , there exists  $K = K(p, \varepsilon, \alpha) \in \mathbb{N}$  large enough such that*

$$\mathbb{P}_{p,\varepsilon}^K(W_0 = \Xi) > 1 - \alpha.$$

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1. The proof is based on constructing a coupling between a certain exploration process and an independent highly supercritical oriented percolation process. In Section 3, we prove Theorem 2.

## 2 Proof of Theorem 1

The proof of Theorem 1 relies on constructing dynamically a renormalized process and coupling it with a highly supercritical oriented percolation model on  $\mathbb{L}_+^2$ . This construction is such that the letters of a word are seen at the good vertices of the corresponding renormalized coupled process. Following the approach in [8], the lattice  $\mathbb{L}_+^2$  is partitioned in slices that grow exponentially, with the occurrence of

good events in one slice implying, with high probability, the occurrence of good events in the subsequent slice.

We build a coupling for each  $\xi \in \Xi$ , with the resulting bounds been uniform across all words. The main difficulty lies in constructing the coupling such that the probability of a renormalized vertex being declared good is close to 1. Achieving this requires observing many letters of the word  $\xi$  during the process, which exponentially increases entropy. The strategy of creating couplings while controlling the entropy generated by the letters of the words is a fundamental concept from [17], also employed in [8].

The technique we use to establish the desired connections within good renormalized vertices is reminiscent of the method developed in [16]. In that work, a straightforward approach sufficed to demonstrate percolation. However, in our case, such an approach is not sufficient as it fails to provide the precise control over the entropy generated by words that our proof demands. To address this challenge, we develop a more intricate construction, which may involve performing multiple exploration processes for each renormalized vertex. We begin by introducing a general growth algorithm, described in detail below.

## 2.1 Growth algorithm

The vertices of the renormalized lattice will consist of unidimensional blocks of size  $n$ , for some suitable  $n \in \mathbb{N}$  (we describe these blocks in more detail in Section 2.2). With this in mind, let  $n \in \mathbb{N}$  and define  $[n] = \{1, \dots, n\}$ . Consider a realization of a random graph with vertex set  $[n]$ , where each edge  $\{i, j\}$  is independently open with probability  $p_{|j-i|}$  and each vertex is independently assigned the state 1 with probability  $p$  and 0 with probability  $1 - p$ . With some abuse of notation, we denote the law of this random graph by  $\mathbb{P}_p^n$ . The parameters of the algorithm are the natural numbers  $L, n$ , and  $0 < \alpha(n) < n$ , where the precise expression of  $\alpha(n)$  in terms of  $n$  will be given in Section 2.4.

Let  $\xi \in \Xi$  and consider the following growth algorithm in  $[n]$  with respect to  $\xi$ . Write  $A_1 = \{v_1\} \subset [n]$ ,  $B_1 \subset [n]$ ,  $t(v_1) \in \mathbb{N}$ , and  $Z_1 = 1$ . For some integer  $k \geq 1$ , assume that we have defined  $(A_k, B_k, Z_k)$  and  $t(v)$  for all  $v \in A_k$ , where  $A_k = \{v_1, \dots, v_{|A_k|}\}$  with

$$|A_k| = k + Z_k - 1. \quad (6)$$

If  $|A_k| \geq \alpha(n)$  or  $Z_k = 0$ , we stop the algorithm. Otherwise, we define  $(A_{k+1}, B_{k+1}, Z_{k+1})$  in two steps. Observe that  $|A_k| \geq k$ , implying that  $v_k \in A_k$  is well defined.

We say that Step 1 is successful if there are distinct  $u_1, u_2, \dots, u_L \in [n] \cap B_k^c$  such that  $\{v_k, u_i\}$  is open for all  $i = 1, \dots, L$ . If Step 1 fails, we set  $Z_{k+1} = Z_k - 1$ ,  $A_{k+1} = A_k$ ,  $B_{k+1} = B_k$ , and  $t(v)$  is unaltered for every  $v \in A_k$ . If Step 1 succeeds, we set  $B_{k+1} = B_k \cup \{u_1, \dots, u_L\}$  and go to Step 2.

We say that Step 2 is successful if there are distinct  $y_1, y_2 \in \{u_1, \dots, u_L\}$  such that  $\eta(y_1) = \eta(y_2) = \xi_{t(v_k)+1}$ . In this case, we write  $A_{k+1} = A_k \cup \{v_{|A_k|+1}, v_{|A_k|+2}\}$ , with  $v_{|A_k|+1} = y_1$ ,  $v_{|A_k|+2} = y_2$ ,  $t(v_{|A_k|+1}) = t(v_{|A_k|+2}) = t(v_k) + 1$ , and  $Z_{k+1} = Z_k + 1$ . If Step 2 fails, we write  $Z_{k+1} = Z_k - 1$ ,  $A_{k+1} = A_k$ , and  $t(v)$  is unaltered for every  $v \in A_k$ . Note that (6) still holds for  $k + 1$ .

As the reader may have noticed, in the next part of the proof we will think of  $t(v)$  as the number of

letters of  $\xi$  seen up to the time we reach vertex  $v$ .

Let  $T$  be the stopping time

$$T := \inf\{k : |A_k| \geq \alpha(n) \text{ or } Z_k = 0\}. \quad (7)$$

We say the algorithm is **successful** if  $Z_T > 0$ .

**Remark 1.** *We have the following observations.*

1. *If the algorithm is successful, the number of letters of  $\xi$  that are seen, starting from  $\xi_{t(v_1)}$ , is random and given by  $\max\{t(v) : v \in A_T\} - t(v_1) + 1$ . A straightforward upper bound for this quantity is  $\alpha(n)$ .*
2. *Note that  $|B_k| \leq |B_1| + (k-1)L$ . Since  $T \leq \alpha(n)$ , see (6) and (7), it holds that*

$$|B_T| \leq |B_1| + \alpha(n)L.$$

*Concerning the coupling, to increase our chances of obtaining good events on the renormalized vertices of each slice of  $\mathbb{L}_+^2$  without increasing the entropy, the algorithm may run multiple times for each renormalized vertex. With each iteration, the size of the corresponding set  $B_1$  grows. Consequently, the initial size of the set of unavailable vertices,  $B_1$ , must be carefully controlled to ensure that  $f(L, n) := |B_1| + \alpha(n)L$  remains bounded above by some suitable function.*

3. *Taking  $L$  large increases our chances of success in Step 2, while increasing  $n$  improves the chances of success in Step 1. If, for each  $L$ , the following condition holds:*

$$\sum_{i=f(L,n)}^{n/2} (i \log i)^{-1} \xrightarrow[n \rightarrow \infty]{} \infty, \quad (8)$$

*then by choosing  $L$  and  $n$  sufficiently large, one can show that the marginals of  $(Z_k)_{k=1}^T$  stochastically dominate a random walk on  $\mathbb{Z}_+$ , where the probability of a step to the right can be made arbitrarily high. This suffices to prove that  $Z_T$  is strictly positive with high probability, as we show now.*

**Lemma 1.** *Consider the growth algorithm in  $[n]$ , and assume that  $p_i \geq (i \log i)^{-1}$  for all sufficiently large  $i$ . Let  $f(L, n) = |B_1| + \alpha(n)L$  be such that, for each  $L \in \mathbb{N}$ , the limit in (8) holds. For every  $p \in (0, 1)$  and  $\delta > 0$ , there exist  $L_0$  and  $n_0 = n(L_0) \in \mathbb{N}$  such that*

$$\mathbb{P}_p^n(Z_T > 0) \geq 1 - \delta,$$

*for every  $n \geq n_0$ .*

*Proof.* We evaluate the growth algorithm at time  $k$ . We aim to show that, for any  $\rho > 0$ , there exist

$L(\rho)$  and  $n(L, \rho)$  sufficiently large such that

$$\mathbb{P}_p^n(Z_{k+1} = j+1 | Z_k = j) > 1 - \rho, \quad (9)$$

$k = 1, \dots, T-1$ , and  $j > 0$ .

On the event  $Z_k = j$ , we can write  $A_k = \{v_1, \dots, v_{k+j-1}\}$  and  $|B_k| \leq |B_1| + (k-1)L < f(L, n)$ . Let  $B_k^c$  be divided into  $L$  disjoint sets  $\Phi_1, \dots, \Phi_L$ . Writing,  $\Lambda_{k,i} = \{z \in [n] \cap \Phi_i : \{v_k, z\} \text{ is open}\}$ ,  $i = 1, \dots, L$ , it holds that

$$\mathbb{P}_p^n(|\Lambda_{k,i}| \geq 1) = \left[ 1 - \prod_{z \in \Phi_i} (1 - p_{|v_k-z|}) \right] \geq 1 - \exp \left( - \sum_{z \in \Phi_i} p_{|v_k-z|} \right).$$

On either the right or left-hand side of  $v_k$ , there are at least  $n/2 - |B_k|$  vertices in  $B_k^c$ . Using the fact that  $|B_k| < f(L, n)$  and the monotonicity of  $(i \log i)^{-1}$ , we obtain

$$\sum_{z \in B_k^c} p_{|v_k-z|} \geq \sum_{i=f(L,n)}^{n/2} (i \log i)^{-1} \xrightarrow[n \rightarrow \infty]{} \infty.$$

Hence, for every  $n \geq n(L)$ ,  $B_k^c$  can be partitioned in such way that  $\sum_{z \in \Phi_i} p_{|v_k-z|}$  is arbitrary large for each  $i = 1, \dots, L$ .

As discussed above, the probability of success in Step 1 can be made arbitrary large. To complete the proof of (9), it remains to show that, given success in Step 1, success in Step 2 can also be achieved with arbitrarily high probability by appropriately choosing  $L$  in terms of  $p$ . This conclusion follows directly from the observation that the probability of a binomial random variable  $\text{Bin}(L; p^*)$  exceeding 1 can be made arbitrarily large, where  $p^* = \min\{p, 1-p\}$ .

It is now immediate that, taking  $L$  and  $n$  large enough, the marginals of  $(Z_k)_{k=1}^T$  stochastically dominate a simple random walk on  $\mathbb{Z}_+$  whose probability of jumping to the right is arbitrarily large. Hence, for every  $\delta > 0$ , we can pick  $L_0$  and  $n_0 = n(L_0)$  large enough, to find  $\rho(\delta)$  small enough such that the statement of the lemma holds.  $\square$

## 2.2 Renormalization and coupling with oriented percolation

Fix  $\xi \in \Xi$  and consider the following oriented percolation process on a renormalized lattice  $\mathcal{L}_n$ . Fix  $n \in \mathbb{N}$  and, for each  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{Z}_+^2$ , define the one-dimensional horizontal box

$$S_{\mathbf{u}}^n = S_{(\mathbf{u}_1, \mathbf{u}_2)}^n := \{\mathbf{y} = (y_1, y_2) : y_1 \in [n\mathbf{u}_1, n(\mathbf{u}_1 + 1) - 1], y_2 = \mathbf{u}_2\},$$

which we refer to as a **renormalized vertex**. We say that two renormalized vertices  $S_{\mathbf{u}}^n$  and  $S_{\mathbf{v}}^n$  are adjacent if  $\|S_{\mathbf{u}}^n - S_{\mathbf{v}}^n\| := \inf_{x \in S_{\mathbf{u}}^n, y \in S_{\mathbf{v}}^n} \|x - y\| = 1$ . Clearly,  $\mathcal{L}_n$  is isomorphic to  $\mathbb{Z}_+^2$ .

We define the oriented boundary of a set  $A \subset \mathbb{Z}_+^2$  as

$$\partial_e A := \{\mathbf{u} \in A^c : \exists \mathbf{v} \in A \text{ such that } \mathbf{u} - \mathbf{v} = (1, 0) \text{ or } \mathbf{u} - \mathbf{v} = (0, 1)\}.$$

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots$  be a fixed ordering of the vertices of  $\mathcal{L}_n$ . In the following, we explore vertices on slices of  $\mathcal{L}_n$ , potentially repeating the exploration several times on each slice. In each attempt on a given slice, we check whether a certain event occurs. If the event occurs, we move on to explore the next slice. Otherwise, the exploration restarts on the same slice. The number of attempts allowed is bounded by

$$h = \alpha(n). \quad (10)$$

We construct inductively a sequence of ordered pairs  $\{C_m, D_m\}_{m \geq 0}$  of subsets of  $\mathcal{L}_n$ , induced by each of several exploration processes on slices of the renormalized lattice. More precisely, for  $\ell = 1, 2, \dots$ , write

$$F_\ell = \{(z_1, z_2) \in \mathbb{Z}_+^2 : z_1 + z_2 = 4^\ell - 1\} \quad (11)$$

and

$$T_\ell = \{(z_1, z_2) \in \mathbb{Z}_+^2 : 4^\ell \leq z_1 + z_2 < 4^{\ell+1}\}.$$

For a fixed  $\ell \in \mathbb{N}$ , let  $C_0^\ell \subset F_\ell$  and  $D_0 = \emptyset$ . For each  $\mathbf{v} \in C_0^\ell$ , there exist deterministic, disjoint sets  $H_{\mathbf{v}}^1, H_{\mathbf{v}}^2, \dots, H_{\mathbf{v}}^h \subset S_{\mathbf{v}}^n$ , with  $|H_{\mathbf{v}}^i| \geq \alpha(n)$ , for all  $i = 1, \dots, h$ . We construct several exploration processes starting from each of the sets  $H_{\mathbf{v}}^i$ ,  $\mathbf{v} \in C_0^\ell$ ,  $i = 1, \dots, \ell$ . We refer to the exploration of  $\xi$  starting at  $H_{\mathbf{v}}^i$ ,  $\mathbf{v} \in C_0^\ell$ , as the  $i$ -th  $\xi$ -exploration.

For each  $\mathbf{v} \in C_0^\ell$ , we set  $G_{\mathbf{v}}^1 = H_{\mathbf{v}}^1$ . We begin the 1-st  $\xi$ -exploration as follows: suppose that, for each  $\mathbf{v} \in C_0^\ell$ ,  $t(v)$  is well defined for all  $v \in G_{\mathbf{v}}^1$ , meaning  $\xi$  is seen up to the letter  $\xi_{t(v)}$  at  $v$ . Start with  $C_1^1 = C_0^\ell$  and  $D_1^1 = D_0$ . Assume  $(C_m^1, D_m^1)$  has been defined for some  $m \in \mathbb{N}$  and suppose that, for each  $\mathbf{v} \in C_m^1$ ,  $G_{\mathbf{v}}^1$  and  $\{t(v) : v \in G_{\mathbf{v}}^1\}$  are well defined, with  $|G_{\mathbf{v}}^1| \geq \alpha(n)$ . If  $\partial_e C_m^1 \cap D_m^1{}^c \cap T_\ell = \emptyset$ , stop the 1-st  $\xi$ -exploration and set  $(C_s^1, D_s^1) = (C_m^1, D_m^1)$  for all  $s \geq m$ . If  $\partial_e C_m^1 \cap D_m^1{}^c \cap T_\ell \neq \emptyset$ , let  $\mathbf{x}$  be the earliest vertex in the fixed ordering in  $\partial_e C_m^1 \cap D_m^1{}^c \cap T_\ell$ . We distinguish between two cases:

First, assume  $\mathbf{x} \notin F_{\ell+1}$ , and take  $\mathbf{y} \in C_m^1$  such that  $\mathbf{x} - \mathbf{y} = (1, 0)$  or  $\mathbf{x} - \mathbf{y} = (0, 1)$ . We say  $\mathbf{x}$  is **1-good** if the following conditions hold:

- C1. there are distinct  $x_1, \dots, x_L \in S_{\mathbf{x}}^n$  and  $y(x_1), \dots, y(x_L) \in G_{\mathbf{y}}^1$  (not necessarily distinct) such that  $\omega(\{y(x_k), x_k\}) = 1$ ;
- C2.  $\eta(x) = \xi_{t(y(x))} + 1$  for some  $x \in \{x_1, \dots, x_L\}$ ;
- C3. the growth algorithm in  $S_{\mathbf{x}}^n$ , starting with  $A_1 = \{x\}$ ,  $B_1 = \{x_1, \dots, x_L\}$ ,  $t(x) = t(y(x)) + 1$ , and  $Z_1 = 1$  is successful; see Figure 1.

**Remark 2.** In Condition C1, we first check the status of the edges connecting  $G_{\mathbf{y}}^1$  to  $S_{\mathbf{x}}^n$ . Then, in Condition C2 we check the status  $\eta(x)$  of only  $L$  vertices in  $S_{\mathbf{x}}^n$ . This is crucial to control the size of unavailable vertices in  $S_{\mathbf{x}}^n$  at each possible future evaluation of the growth algorithm during the several explorations; see also Remark 1.

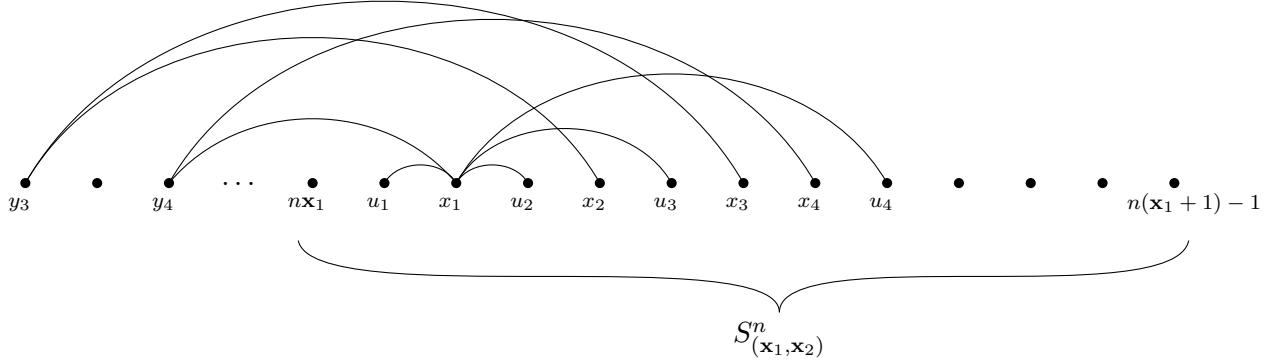


Figure 1: Let  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ . Starting from  $y(x) = y_4$ , once we reach  $x = x_1 \in S_{(\mathbf{x}_1, \mathbf{x}_2)}^n$  such that  $\eta(x) = \xi_{t(y(x)) + 1}$ , the growth algorithm finds, with high probability,  $\alpha(n)$  sites propagating the word  $\xi$ . In this figure,  $L = 4$ , and  $\eta(u_3) = \eta(u_4) = \xi_{t(x_1) + 1}$ .

If  $\mathbf{x}$  is 1-good, write

$$(C_{m+1}^1, D_{m+1}^1) = (C_m^1 \cup \{\mathbf{x}\}, D_{m+1}^1), G_{\mathbf{x}}^1 = A_T, \text{ and } R_{\mathbf{x}}^1 := B_T.$$

Note that  $t(x)$  is well defined for all  $x \in G_{\mathbf{x}}^1$  and also that  $|G_{\mathbf{x}}^1| \geq \alpha(n)$  since in Condition C3 the growth algorithm is successful.

If C1 fails, define

$$(C_{m+1}^1, D_{m+1}^1) = (C_m^1, D_m^1 \cup \{\mathbf{x}\}), G_{\mathbf{x}}^1 = \emptyset, \text{ and } R_{\mathbf{x}}^1 = \emptyset.$$

If C1 holds and C2 fails, write

$$(C_{m+1}^1, D_{m+1}^1) = (C_m^1, D_m^1 \cup \{\mathbf{x}\}), G_{\mathbf{x}}^1 = \emptyset, \text{ and } R_{\mathbf{x}}^1 := \{x_1, \dots, x_L\}.$$

Finally, if C1 and C2 hold but C3 fails, write

$$(C_{m+1}^1, D_{m+1}^1) = (C_m^1, D_m^1 \cup \{\mathbf{x}\}), G_{\mathbf{x}}^1 = \emptyset, \text{ and } R_{\mathbf{x}}^1 := B_T.$$

In the second case  $\mathbf{x} \in F_{\ell+1}$ , we simply replace the definition of  $T$  in (7) by

$$T := \inf\{k : |A_k| \geq \alpha(n)h \text{ or } Z_k = 0\}. \quad (12)$$

When the 1-st  $\xi$ -exploration stops, we write  $R_{\mathbf{u}}^1 = \emptyset$  if  $\mathbf{u} \in T_\ell$  was not evaluated at the 1-st  $\xi$ -exploration. This alternative definition in (12) is necessary to refresh all  $h$  exploration processes that start in  $F_{\ell+1}$ , as we discuss below.

Consider the set  $\Pi_\ell^i(\xi) = \cup_{m \geq 1} C_m^i$  of  $i$ -good vertices obtained during the  $i$ -th  $\xi$ -exploration in the slice  $T_\ell$ . Recall the definition of  $F_\ell$  in (11) and consider the line segment passing by the points in  $F_{\ell+1}$ .

We partition  $F_{\ell+1}$  into the sets

$$\begin{aligned} F_{\ell+1,1} &= \{(v_1, v_2) \in F_{\ell+1} : 0 \leq v_1 < 4^\ell\}, \\ F_{\ell+1,2} &= \{(v_1, v_2) \in F_{\ell+1} : 4^\ell \leq v_1 < 3 \times 4^\ell\}, \\ F_{\ell+1,3} &= \{(v_1, v_2) \in F_{\ell+1} : 3 \times 4^\ell \leq v_1 < 4^{\ell+1}\}. \end{aligned} \quad (13)$$

We say that the  $i$ -th  $\xi$ -exploration process in the slice  $T_\ell$  is successful if

$$|\Gamma_\ell^i(\xi)| := |\Pi_\ell^i(\xi) \cap F_{\ell+1,2}| \geq 4^\ell. \quad (14)$$

Suppose that the  $i$ -th  $\xi$ -exploration in  $T_\ell$  failed. In this case, we write  $(C_1^{i+1}, D_1^{i+1}) = (C_0^\ell, D_0)$ , and consider the  $(i+1)$ -th  $\xi$ -exploration of vertices in  $T_\ell$ . We follow the same ordering of the vertices of  $\mathcal{L}_n$ , with the difference that this time we start the exploration from the sets  $\{G_\mathbf{v}^{i+1} := H_\mathbf{v}^{i+1} : \mathbf{v} \in C_0^\ell\}$ , considering also that, for each  $\mathbf{v} \in C_0^\ell$ ,  $t(v)$  is well defined for all  $v \in G_\mathbf{v}^{i+1}$ .

Assume  $(C_m^{i+1}, D_m^{i+1})$  has been defined for some  $m \in \mathbb{N}$ . Assume also that, for each  $\mathbf{v} \in C_m^{i+1}$ ,  $G_\mathbf{v}^{i+1}$  and  $\{t(v) : v \in G_\mathbf{v}^{i+1}\}$  are defined, with  $|G_\mathbf{v}^{i+1}| \geq \alpha(n)$ . If  $\partial_e C_m^{i+1} \cap D_m^{i+1c} \cap T_\ell = \emptyset$ , stop the  $(i+1)$ -th  $\xi$ -exploration and set  $(C_s^{i+1}, D_s^{i+1}) = (C_m^{i+1}, D_m^{i+1})$  for all  $s \geq m$ . If  $\partial_e C_m^{i+1} \cap D_m^{i+1c} \cap T_\ell \neq \emptyset$ , let  $\mathbf{x}$  be the earliest vertex in the fixed ordering in  $\partial_e C_m^{i+1} \cap D_m^{i+1c} \cap T_\ell$ . Again, we distinguish between two cases:

Take  $\mathbf{y} \in C_m^{i+1}$  such that  $\mathbf{x} - \mathbf{y} = (1, 0)$  or  $\mathbf{x} - \mathbf{y} = (0, 1)$  and consider the case  $\mathbf{x} \notin F_{\ell+1}$ . We say  $\mathbf{x} \in$  is  $(i+1)$ -good if the following conditions hold:

C1'. there are distinct  $x_1, \dots, x_L \in S_\mathbf{x}^n \cap [\cup_{r=1}^i R_\mathbf{x}^r]^c$  and  $y(x_1), \dots, y(x_L) \in G_\mathbf{y}^{i+1}$  (not necessarily distinct) such that  $\omega(\{y(x_k), x_k\}) = 1$ ;

C2'.  $\eta(x) = \xi_{t(y(x)) + 1}$  for some  $x \in \{x_1, \dots, x_L\}$ ;

C3'. the growth algorithm in  $S_\mathbf{x}^n$  with  $A_1 = \{x\}$ ,  $B_1 = \{x_1, \dots, x_L\} \cup [\cup_{r=1}^i R_\mathbf{x}^r]$ ,  $t(x) = t(y(x)) + 1$  and  $Z_1 = 1$  is successful.

If  $\mathbf{x}$  is  $(i+1)$ -good, write

$$(C_{m+1}^{i+1}, D_{m+1}^{i+1}) = (C_m^{i+1} \cup \{\mathbf{x}\}, D_{m+1}^{i+1}), G_\mathbf{x}^{i+1} = A_T, \text{ and } R_\mathbf{x}^{i+1} := B_T.$$

Note that  $t(x)$  is well defined for all  $x \in G_\mathbf{x}^{i+1}$  and  $|G_\mathbf{x}^{i+1}| \geq \alpha(n)$ .

If Condition C1' fails, define

$$(C_{m+1}^{i+1}, D_{m+1}^{i+1}) = (C_m^{i+1}, D_m^{i+1} \cup \{\mathbf{x}\}), G_\mathbf{x}^{i+1} = \emptyset, \text{ and } R_\mathbf{x}^{i+1} := \emptyset.$$

If Condition C1' holds and Condition C2' fails, write

$$(C_{m+1}^{i+1}, D_{m+1}^{i+1}) = (C_m^{i+1}, D_m^{i+1} \cup \{\mathbf{x}\}), G_\mathbf{x}^{i+1} = \emptyset, \text{ and } R_\mathbf{x}^{i+1} := \{x_1, \dots, x_L\}.$$

Finally, if conditions C1' and C2' hold but Condition C3' fails, write

$$(C_{m+1}^{i+1}, D_{m+1}^{i+1}) = (C_m^{i+1}, D_m^{i+1} \cup \{\mathbf{x}\}), G_{\mathbf{x}}^{i+1} = \emptyset, \text{ and } R_{\mathbf{x}}^{i+1} := B_T.$$

If  $\mathbf{x} \in F_{\ell+1}$  we proceed as in (12).

When the  $(i+1)$ -th  $\xi$ -exploration stops, we write  $R_{\mathbf{u}}^{i+1} = \emptyset$  if  $\mathbf{u} \in T_\ell$  was not evaluated during this  $(i+1)$ -th  $\xi$ -exploration.

As described in (14), observe that if the  $i$ -th  $\xi$ -exploration is successful, then it reaches at least  $4^\ell$  renormalized vertices  $\mathbf{z} \in F_{\ell+1}$ , each with the property that  $|G_{\mathbf{z}}^i| \geq \alpha(n)h$ . The set of such vertices, more precisely  $\Gamma_\ell^i(\xi)$ , will serve as the starting set  $C_0^{\ell+1} \subset F_{\ell+1}$  for each subsequent  $\xi$ -exploration in the slice  $T_{\ell+1}$ . Thus, as in the beginning of the  $\xi$ -exploration in  $T_\ell$ , for each  $\mathbf{z} \in \Gamma_\ell^i(\xi)$ , there are sets  $H_{\mathbf{z}}^1, \dots, H_{\mathbf{z}}^h \subset G_{\mathbf{z}}^i$  with  $|H_{\mathbf{z}}^j| > \alpha(n)$ , for every  $j = 1, \dots, h$ , from which we continue the  $\xi$ -exploration in  $T_{\ell+1}$ . The 1-st  $\xi$ -exploration proceeds as before: we seek the earliest vertex, according to the fixed ordering, say vertex  $\mathbf{x} \in T_{\ell+1}$ , such that  $\mathbf{x}$  is a neighbor of some  $i$ -good vertex in  $H_{\mathbf{z}}^1$ , and we check if conditions C1, C2, and C3 hold. After that, we proceed as before.

Having defined the renormalization and all the  $h$  potential  $\xi$ -explorations in the slice  $T_\ell$ , write

$$\mathcal{B}_\ell^i(\xi) = \left\{ |\Gamma_\ell^i(\xi)| \geq 4^\ell \right\} \quad (15)$$

for the event where the  $i$ -th  $\xi$ -exploration is successful, and let

$$\mathcal{B}_\ell(\xi) = \bigcup_{i=1}^h \mathcal{B}_\ell^i(\xi) \quad (16)$$

denote the event where at least one of the  $h$  explorations is successful. Figure 2 illustrates the exploration in  $T_\ell$ .

### 2.3 Concatenation with finite words

To prove Theorem 1 we must show that all words are seen from the origin with positive probability. The main part of the argument consists in proving that all words are seen from the finite set  $F_\ell$ , for some large  $\ell \in \mathbb{N}$ . Let us explain how this argument can be improved allowing us to show the desired result.

Let us introduce one more parameter, namely  $\ell_0 \in \mathbb{N}$ . A deterministic result will be stated for this parameter. Before, we need to introduce further notation.

Consider the finite set

$$\Lambda = \{v \in \mathbb{Z}^2 : v \in S_{\mathbf{u}}^n \text{ for some } \mathbf{u} \in \mathbb{Z}_+^2 \text{ such that } \|\mathbf{u}\| < 4^{\ell_0}\},$$

and let  $(\omega \times \eta)_\Lambda$  denote the restriction of the configuration  $(\omega \times \eta)$  to  $\Lambda$ . We denote by  $\mathbb{P}_{p,\varepsilon|\Lambda}^K$  the probability measure  $\mathbb{P}_{p,\varepsilon}^K$  restricted to  $\Lambda$ .

Given a vertex  $v \in \Lambda$ , we say that at most  $m$  of the initial letters of  $\xi$  are seen from the origin up

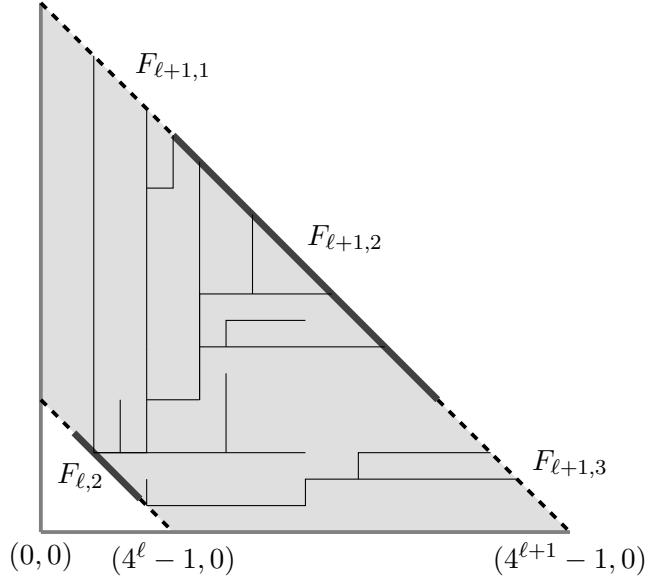


Figure 2: An illustration of the exploration in  $T_\ell$ . The exploration is successful if the number of vertices in  $F_{\ell+1,2}$  that can be reached from some vertex in  $F_{\ell,2}$  is at least  $4^\ell$ .

to vertex  $v$  in a configuration  $(\omega, \eta)_\Lambda$  if there exists a path  $(v_0, v_1, \dots, v_k)$  such that  $v_0$  is the origin,  $v_k = v$ ,  $k \leq m$ ,  $\omega(\{v_{i-1}, v_i\}) = 1$ , and  $\eta(v_i) = \xi_i$ , for  $i = 1, \dots, k$ .

Assume, with no loss of generality, that  $n$  is even. For each  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{Z}_+^2$ , we split the one-dimensional horizontal box  $S_{\mathbf{u}}^n$  into the sets

$$S_{\mathbf{u}}^{n,-} = S_{(\mathbf{u}_1, \mathbf{u}_2)}^{n,-} := \{y = (y_1, y_2) : y_1 \in [n\mathbf{u}_1, n\mathbf{u}_1 + n/2 - 1], y_2 = \mathbf{u}_2\}, \quad (17)$$

and

$$S_{\mathbf{u}}^{n,+} = S_{(\mathbf{u}_1, \mathbf{u}_2)}^{n,+} := \{y = (y_1, y_2) : y_1 \in [n\mathbf{u}_1 + n/2, n(\mathbf{u}_1 + 1) - 1], y_2 = \mathbf{u}_2\}. \quad (18)$$

We say that  $A^-(\xi)$  occurs if for every  $\mathbf{v} \in F_{\ell,2}$  and every  $v \in S_{\mathbf{v}}^{n,-}$ , at most  $2 \times 4^{\ell_0} + 1$  of the initial letters of  $\xi$  are seen up to  $v$ . The definition of  $A^+(\xi)$  is analogous.

**Lemma 2.** *Let  $m$  be such that  $p_i > 0$  for all  $i \geq m$ ,  $n \geq 2m + 6$ , and  $K \geq n4^{\ell_0}$ . There exists a configuration  $\omega_\Lambda^* = (\omega' \times \eta')_\Lambda$  with  $\mathbb{P}_{p,\varepsilon|\Lambda}^K(\omega_\Lambda^*) > 0$ , such that for every  $\xi \in \Xi$ , either  $A^-(\xi)$  or  $A^+(\xi)$  occurs in  $\omega_\Lambda^*$ .*

*Proof.* We build  $\omega_\Lambda^*$  as follows. Let  $\omega'$  be such that  $\omega'(\{u, u + (k, 0)\}) = 1$  for all pairs  $u, u + (k, 0) \in \Lambda$  such that  $p_k > 0$ . Given  $\ell_0 \in \mathbb{N}$ , let  $m - 1 = \sup\{i \geq 1 : p_i = 0\}$ . For each  $0 \leq y < 3 \cdot 4^{\ell_0-1}$  even, write

$$\eta'(x, y) = \begin{cases} 0, & x = m, m + 1 \\ 1, & x = m + 2, m + 3. \end{cases} \quad (19)$$

For each  $0 \leq y < 3 \cdot 4^{\ell_0-1}$  odd, write

$$\eta'(x, y) = \begin{cases} 0, & \text{if } x = m, m+2 \\ 1, & \text{if } x = m+1, m+3. \end{cases} \quad (20)$$

Also, define  $\eta'(x+m+3, y+1) = \eta'(x, y)$  for every  $0 \leq y < 3 \cdot 4^{\ell_0-1}$  and  $m \leq x \leq m+3$ .

For each  $\mathbf{u} \in F_{\ell_0,2}$ , set

$$\eta'(\mathbf{u}) = \begin{cases} 0, & \text{if } \mathbf{u} \in S_{\mathbf{u}}^{n,-} \\ 1, & \text{if } \mathbf{u} \in S_{\mathbf{u}}^{n,+}, \end{cases} \quad (21)$$

where the definitions of  $S_{\mathbf{u}}^{n,-}$  and  $S_{\mathbf{u}}^{n,+}$  are given in (17) and (18), respectively.

The status  $\eta'(\cdot)$  of the remaining vertices of  $\Lambda$  are irrelevant. Figure 3 illustrates why (19), (20), and (21) are sufficient for the occurrence of either  $A^+(\xi)$  or  $A^-(\xi)$  for every  $\xi$ . To complete the proof, observe that for each  $\xi \in \Xi$  and every  $y$  such that  $4^{\ell_0} \leq y < 3 \cdot 4^{\ell_0}$ , there exists at least one  $v \in S_{(0,y)}^n$  for which at most  $2y+1$  of the initial letters of  $\xi$  are seen from the origin up to vertex  $v$ . Since  $v$  is connected to every vertex  $u \in S_{(4^{\ell_0}-1-y,y)}^n$ , the result follows.  $\square$

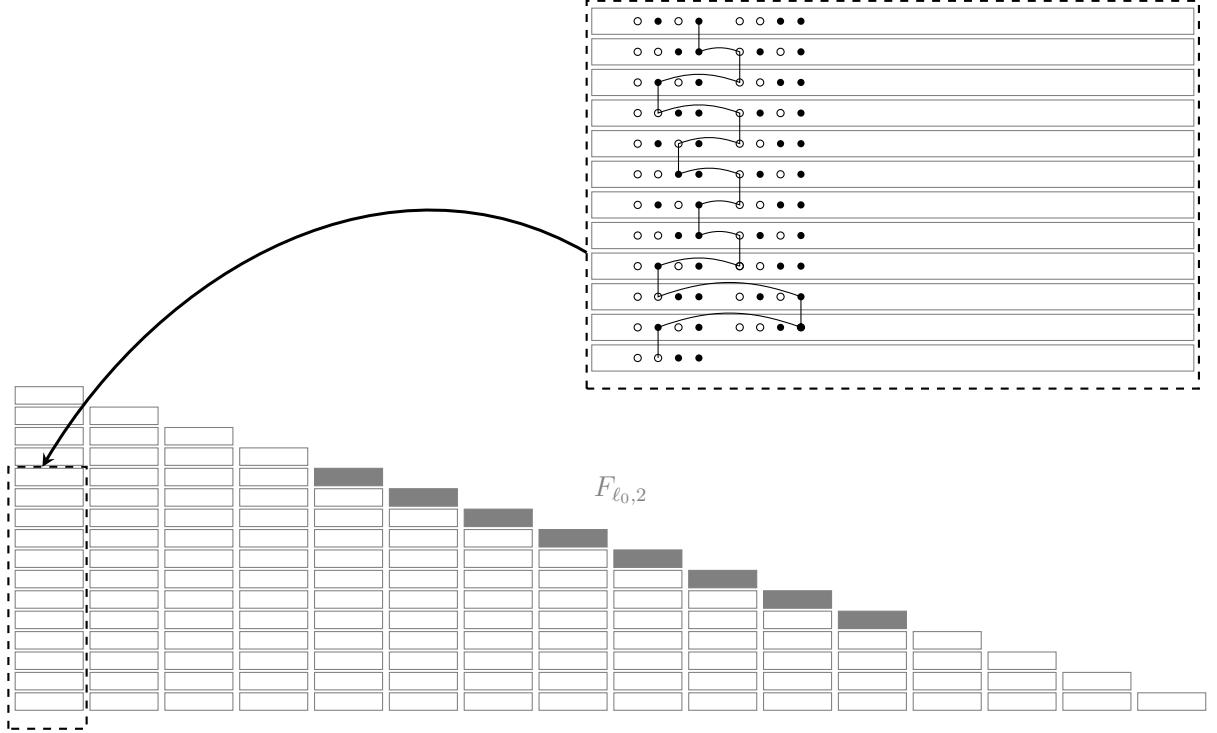


Figure 3: A configuration where the finite word  $\xi = (0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1)$  is seen from the origin. All the remaining letters of  $\xi$  are seen from some gray box in  $F_{\ell_0,2}$ .

Write  $r(\ell_0) = 2 \times 4^{\ell_0} + 1$ . Conditioning on the configuration  $\omega_\Lambda^*$  given in Lemma 2, we proceed with the exploration of renormalized vertices in the slices  $T_\ell$ ,  $\ell \geq \ell_0$ . Regarding the exploration in  $T_{\ell_0}$ , we take  $C_0^{\ell_0} = F_{\ell_0,2} \subset F_{\ell_0}$  as the starting set. If  $A^-(\xi)$  occurs, and since  $|S_{\mathbf{v}}^{n,-}| = n/2$  for every  $\mathbf{v} \in F_{\ell_0,2}$ ,

in case that  $\alpha(n)h < n/2$ , the set  $S_{\mathbf{v}}^{n,-}$  can be partitioned into disjoint sets  $H_{\mathbf{v}}^1, H_{\mathbf{v}}^2, \dots, H_{\mathbf{v}}^h \subset S_{\mathbf{v}}^{n,-}$ , with  $|H_{\mathbf{v}}^i| \geq \alpha(n)$ , for all  $i = 1, \dots, h$ . Also, the occurrence of  $A^-(\xi)$  yields the bound  $t(v) \leq r(\ell_0)$  for each  $v \in H_{\mathbf{v}}^i$ . If  $A^+(\xi)$  occurs, an analogous bound holds.

Suppose that we are at the  $i$ -th  $\xi$ -exploration and have just evaluated some vertex  $\mathbf{v} \in T_\ell$ , for some  $\ell \geq \ell_0$ . If  $\mathbf{v}$  is declared  $i$ -good, how many letters of  $\xi$  have we seen so far? To answer this question, note that since in each  $i$ -good renormalized vertex  $\mathbf{v}$ ,  $t(v)$  is defined for at most  $\alpha(n)$  vertices in  $\mathbf{v}$  when  $\mathbf{v} \notin F_{\ell+1}$ , and at most  $\alpha(n)h$  vertices in  $\mathbf{v}$  when  $\mathbf{v} \in F_{\ell+1}$ . The orientation of our percolation model yields the upper bound

$$\begin{aligned} r(\ell_0) + (4^{\ell+1} - 4^{\ell_0})\alpha(n) + (\ell + 1 - \ell_0)(h - 1)\alpha(n) \\ = 4^{\ell_0}(2 - \alpha(n)) + 1 + 4^{\ell+1}\alpha(n) + (\ell + 1 - \ell_0)(h - 1)\alpha(n) \\ \leq 4^{\ell+1}\alpha(n) + (\ell + 1)(h - 1)\alpha(n), \end{aligned} \quad (22)$$

which is uniform in  $i = 1, \dots, h$  and in  $\xi \in \Xi$ .

Taking  $\ell_0$  sufficiently large (as a function of  $n$ ; recall (10)) so that  $(\ell + 1)(h - 1) \leq 4^{\ell+1}$  for all  $\ell \geq \ell_0$ , the expression in (22) can be bounded from above by

$$s(\ell) := 2 \times 4^{\ell+1}\alpha(n), \quad \ell \geq \ell_0. \quad (23)$$

For each  $\ell \geq \ell_0$ , the upper bound  $s(\ell)$  tells us that the events  $\mathcal{B}_\ell^i(\xi)$  and  $\mathcal{B}_\ell(\xi)$  (see (15) and (16)) rely solely on the first  $s(\ell)$  letters of  $\xi$ . With this in mind, let us define

$$\Xi_\ell = \{0, 1\}^{s(\ell)},$$

the set of finite words  $(\xi_1, \xi_2, \dots, \xi_{s(\ell)})$  of length  $s(\ell)$ . We also define the projection map  $\sigma_\ell : \Xi \longrightarrow \Xi_\ell$ , where,  $\sigma_\ell(\xi_1, \xi_2, \dots) = (\xi_1, \dots, \xi_{s(\ell)})$ .

Let  $\phi \in \Xi_\ell$  and  $\xi_\phi \in \Xi$  such that  $\sigma(\xi_\phi) = \phi$ . For each  $\ell \geq \ell_0$  and  $\phi \in \Xi_\ell$  we define the events

$$\mathcal{B}_\ell^i(\phi) = \mathcal{B}_\ell^i(\xi_\phi), \quad i = 1, \dots, h,$$

and

$$\mathcal{B}_\ell(\phi) = \mathcal{B}_\ell(\xi_\phi).$$

Also, we write

$$\mathcal{D}_\ell = \bigcap_{\phi \in \Xi_\ell} \mathcal{B}_\ell(\phi).$$

For completeness, we define  $\mathcal{D}_{\ell_0-1}$  and  $\mathcal{B}_{\ell_0-1}(\xi)$ ,  $\xi \in \Xi$ , as the event where the configuration  $\omega_\Lambda^*$  given in Lemma 2 occurs when restricted to  $\Lambda$ .

With this notation,  $\mathcal{D}_\ell$  is the event that a certain random number of digits (bounded by  $s(\ell)$ ) of every  $\phi \in \Xi_\ell$  is seen from the origin with the respective renormalized processes reaching the set  $F_{\ell,2}$  with at least half of its vertices.

Recall that  $W_0$  denotes the set of words seen from the origin. Under the occurrence of  $\mathcal{D}_{\ell_0-1}$ , if  $W_0 \neq \{0, 1\}^{\mathbb{N}}$ , then there exists  $\ell \geq \ell_0$  and a word  $\phi^* \in \{0, 1\}^{s(\ell)}$  such that all  $h$  possible  $\xi^*$ -explorations (where  $\sigma_\ell(\xi^*) = \phi^*$ ) in  $T_\ell$  fail. That is,

$$\bigcap_{\ell \geq \ell_0-1} \mathcal{D}_\ell \subset \{W_0 = \Xi\}.$$

Therefore,

$$\mathbb{P}_{p,\varepsilon}^K(W_0 = \Xi) \geq \left[ \prod_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell | \mathcal{D}_{\ell-1}) \right] \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_{\ell_0-1}) = \left[ \prod_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell | \mathcal{D}_{\ell-1}) \right] \mathbb{P}_{p,\varepsilon|_\Lambda}^K(\omega_\Lambda^*).$$

By Lemma 2, it suffices to show that  $\prod_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell | \mathcal{D}_{\ell-1})$  is positive for  $K$  sufficiently large. This is equivalent to show that

$$\sum_{\ell \geq \ell_0} [1 - \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell | \mathcal{D}_{\ell-1})] = \sum_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell^c | \mathcal{D}_{\ell-1}) < \infty.$$

We have

$$\begin{aligned} \sum_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell^c | \mathcal{D}_{\ell-1}) &= \sum_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K \left( \bigcup_{\phi \in \Xi_\ell} \mathcal{B}_\ell^c(\phi) \middle| \mathcal{D}_{\ell-1} \right) \\ &\leq \sum_{\ell \geq \ell_0} \sum_{\phi \in \Xi_\ell} \mathbb{P}_{p,\varepsilon}^K \left( \bigcap_{i=1}^h (\mathcal{B}_\ell^i(\phi))^c \middle| \mathcal{B}_{\ell-1}(\xi_\phi) \right) \\ &\leq \sum_{\ell \geq \ell_0} \sum_{\phi \in \Xi_\ell} \prod_{i=1}^h \mathbb{P}_{p,\varepsilon}^K \left( (\mathcal{B}_\ell^i(\phi))^c \middle| [\bigcap_{j < i} (\mathcal{B}_\ell^j(\phi))^c] \bigcap \mathcal{B}_{\ell-1}(\xi_\phi) \right), \end{aligned} \quad (24)$$

where, for completeness,  $[\bigcap_{j < i} (\mathcal{B}_\ell^j(\phi))^c] \bigcap \mathcal{B}_{\ell-1}(\xi_\phi) = \mathcal{B}_{\ell-1}(\xi_\phi)$ , for  $i = 1$ .

The next section is dedicated to proving that, under the conditions of Theorem 1, for any  $a > 0$ , and with  $K = K(a)$  chosen sufficiently large, the bounds obtained through our couplings with oriented percolation imply that, for  $i = 1, \dots, h$ ,

$$\mathbb{P}_{p,\varepsilon}^K \left( (\mathcal{B}_\ell^i(\phi))^c \middle| [\bigcap_{j < i} (\mathcal{B}_\ell^j(\phi))^c] \bigcap \mathcal{B}_{\ell-1}(\xi_\phi) \right) \leq a^{4^\ell}, \quad (25)$$

for all  $\ell > \ell_0$ , and for all  $\phi \in \Xi_\ell$ . Assume for a moment that Inequality (25) is true and recall the bound in (23). The Inequality (24) becomes

$$\sum_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell^c | \mathcal{D}_{\ell-1}) \leq \sum_{\ell \geq \ell_0} \sum_{\phi \in \Xi_\ell} a^{4^\ell h} = \sum_{\ell \geq \ell_0} 2^{s(\ell)} a^{4^\ell h} = \sum_{\ell \geq \ell_0} 4^{4^{\ell+1}\alpha(n)} a^{4^\ell h}. \quad (26)$$

Choosing  $a < 1/4^4$ , and since  $h = \alpha(n)$  as defined in (10), we obtain the desired convergence of  $\sum_{\ell \geq \ell_0} \mathbb{P}_{p,\varepsilon}^K(\mathcal{D}_\ell^c | \mathcal{D}_{\ell-1})$ .

## 2.4 Oriented percolation

Consider Bernoulli oriented site percolation on  $\mathbb{L}^2$  with parameter  $\gamma \in [0, 1]$ . In this model, each vertex is independently open with probability  $\gamma$  or closed with probability  $1 - \gamma$ . Denote by  $P_\gamma$  the law of this process.

Let  $S \subset \mathbb{Z}_+^2$ . The open cluster of  $S$ , denoted by  $C_S$ , is defined as the set of vertices  $v \in \mathbb{Z}_+^2$  that can be reached from some vertex  $u \in S$  via a sequence of distinct vertices  $(v_0 = u, v_1, \dots, v_k = v)$  in  $\mathbb{Z}_+^2$ , such that for each consecutive pair  $(v_i, v_{i+1})$  in the sequence, the difference  $v_{i+1} - v_i$  is either  $(0, 1)$  or  $(1, 0)$ . Furthermore, each vertex in the sequence  $v_1, v_2, \dots, v_k$  is declared open.

For each  $\ell \geq \ell_0$ , recall the definitions of  $F_\ell$  and  $F_{\ell,2}$  given in (11) and (13). The following auxiliary proposition is reminiscent from [8] (see also Section 6 of [17]). We omit its proof, as it closely follows the argument used for Proposition 1 in [8].

**Proposition 1.** *Consider oriented Bernoulli site percolation on  $\mathbb{L}^2$  with parameter  $\gamma$ . For all  $a > 0$ , there exists  $\gamma(a) < 1$  such that for every  $\gamma \geq \gamma(a)$  and every  $\ell \geq \ell_0$ ,*

$$P_\gamma \left( |C_S \cap F_{\ell+1,2}| < 4^\ell \right) \leq a^{4^\ell},$$

for all  $S \subset F_{\ell,2}$  satisfying  $|S| \geq 4^{\ell-1}$ .

In terms of our coupling on the renormalized vertices, we choose  $L$  and  $n$  appropriately such that the probability of a renormalized vertex being declared  $i$ -good is uniformly large (independently of all steps in the current and previous explorations). In this case, Proposition 1 states that, conditionally on the event  $\mathcal{B}_\ell(\phi)$  - which ensures that at least half of the vertices in  $F_{\ell,2}$  are declared  $i$ -good (for a fixed word  $\phi$  and some  $i$  such that the  $i$ -th exploration is successful in  $T_{\ell-1}$ ) - it is highly unlikely (for a suitably chosen truncation constant) that the fraction of  $i$ -good vertices in  $F_{\ell+1,2}$  falls below  $1/2$ .

The observation above is crucial for balancing the probability of seeing a word with the entropy generated by the words, as used in (26). We achieve this balance combining Lemma 3 below and Proposition 1, with  $S$  as the initial set of the  $\xi$ -explorations in the slice  $T_\ell$ , which allows us to conclude that Inequality (25) holds true.

Suppose that at some point during the  $i$ -th  $\xi$ -exploration of vertices in  $T_\ell$ , we reach the renormalized vertex  $\mathbf{x}$ . Let  $\mathcal{F}(\mathbf{x}, i)$  denote the history of the process up to this step, meaning the sigma-algebra containing all the information about renormalized edges checked so far, as well as the checked bonds of the original graph, including all  $(i-1)$  previous  $\xi$ -exploration processes.

The next lemma shows that, conditional on the past, the probability that a new vertex  $\mathbf{x}$  is added to the component of  $i$ -good vertices can be made arbitrarily large by choosing  $L$  and  $n$  appropriately. To prove this lemma, we set  $\alpha(n) = \lfloor C \log n \rfloor$ , for some constant  $C > 0$ .

**Lemma 3.** Assume the hypotheses of Theorem 1, and let  $\alpha(n) = \lfloor C \log n \rfloor$ . Suppose that at some step of the  $i$ -th  $\xi$ -exploration we reach the renormalized vertex  $\mathbf{x} \in \mathcal{L}_n$ . For all  $p \in (0, 1)$ ,  $\varepsilon > 0$ , and  $\delta > 0$ , if  $L$  and  $C \geqslant C(L)$  are sufficiently large, then there exists  $n(L) \in \mathbb{N}$  large enough such that

$$\mathbb{P}_{p,\varepsilon}^{2n}(\mathbf{x} \text{ is } i\text{-good} | \mathcal{F}(\mathbf{x}, i)) > 1 - \delta, \text{ for all } n \geqslant n(L).$$

*Proof.* Recall the ordered pairs of sets defined in Section 2.2. Assume  $(C_{m-1}^i, D_{m-1}^i)$  has been constructed for some  $m > 1$ , and that  $G_y^j$  and  $R_y^j$  are defined for all  $\mathbf{y} \in C_{m-1}^i$  and for all  $1 \leqslant j \leqslant i-1$ .

We now bound the number of unavailable vertices, denoted by  $\mathcal{U}_{\mathbf{x}}^i := \cup_{r=1}^{i-1} R_{\mathbf{x}}^r$ , with  $\mathcal{U}_{\mathbf{x}}^1 = \emptyset$ . If  $\mathbf{x} \notin F_{\ell+1}$ , then in each exploration at  $\mathbf{x}$ , the statuses of at most  $L + L\alpha(n)$  new vertices are checked, yielding

$$|\mathcal{U}_{\mathbf{x}}^i| \leqslant (i-1)(\alpha(n)+1)L < h(\alpha(n)+1)L < c_1(\log n)^2, \text{ for all } i = 1, \dots, h,$$

for some constant  $c_1 > 0$ .

If  $\mathbf{x} \in F_{\ell+1}$ , then in each exploration at  $\mathbf{x}$ , the statuses of at most  $L + Lh\alpha(n)$  new vertices are checked, giving

$$|\mathcal{U}_{\mathbf{x}}^i| \leqslant (i-1)(h\alpha(n)+1)L < h(h\alpha(n)+1)L < c_2(\log n)^3, \text{ for all } i = 1, \dots, h,$$

for some constant  $c_2 > 0$ . In any case, for every  $\mathbf{x} \in \mathcal{L}_n$ , we have

$$|\mathcal{U}_{\mathbf{x}}^i| \leqslant c_3(\log n)^3, \text{ for all } i = 1, \dots, h, \quad (27)$$

for some constant  $c_3 > 0$ .

If the  $i$ -th  $\xi$ -exploration reaches  $\mathbf{x}$ , there must be some  $\mathbf{y} \in C_{m-1}^i$  with  $\mathbf{x} = \mathbf{y} + (1, 0)$  or  $\mathbf{x} = \mathbf{y} + (0, 1)$ . We begin by considering the case  $\mathbf{x} = \mathbf{y} + (1, 0)$ . Since  $\mathbf{y} \in C_{m-1}^i$ , we have  $|G_{\mathbf{y}}^i| \geqslant \alpha(n)$ . Consider the set

$$\Lambda_{\mathbf{x}}^i = \{x \in S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c : \{y, x\} \text{ is open for some } y \in G_{\mathbf{y}}^i\}.$$

Given  $x \in S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c$  and  $y \in G_{\mathbf{y}}^i$ , it is clear that  $|y - x| \leqslant 2n$ . By hypothesis,  $p_i \geqslant (i \log i)^{-1}$  for all  $i$  sufficiently large, thus we can assume that (except by a finite quantity of  $x$ 's near the border of  $S_{\mathbf{x}}^n$ , where it may be the case that  $p_{|x-y|} < [|x-y| \log(|x-y|)]^{-1}$ ) the probability that the edge  $\{y, x\}$  is declared open is bounded below by  $(2n \log(2n))^{-1}$ . Thus,

$$\mathbb{P}_{p,\varepsilon}^{2n}(x \in \Lambda_{\mathbf{x}}^i) \geqslant 1 - \left[1 - \frac{1}{2n \log(2n)}\right]^{\alpha(n)} \geqslant \frac{c_4}{n}, \quad (28)$$

where the constant  $c_4 > 0$  can be made arbitrary large as a function of  $C$ .

Recall (10) and also that  $\alpha(n) = \lfloor C \log n \rfloor$ . The bounds obtained for  $\mathcal{U}_{\mathbf{x}}^i$  imply that  $|S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c| \geqslant c_5 n$ , for some  $c_5 > 0$ . Hence, choosing  $C$  (and consequently  $c_4$ ) large enough, we obtain

$$\mathbb{P}_{p,\varepsilon}^{2n}(|\Lambda_{\mathbf{x}}^i| \geqslant L) \geqslant P\left(\text{Bin}\left(c_5 n, \frac{c_4}{n}\right) \geqslant L\right) \geqslant (1 - \delta)^{1/3}, \quad (29)$$

for every  $n$  large.

Next, we consider the case where  $\mathbf{x} = \mathbf{y} + (0, 1)$ . To connect  $G_{\mathbf{y}}^i$  to  $S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c$ , we have to consider only vertical edges  $\{y, y + (0, 1)\}$ ,  $y \in G_{\mathbf{y}}^i$ . Each of these edges is declared open with probability  $\varepsilon > 0$ . To ensure the connection, it suffices to identify  $m$  open edges, where  $m$  is chosen such that  $P(\text{Bin}(m, \varepsilon) \geq L) \geq (1 - \delta)^{1/3}$ .

If  $|\{y \in G_{\mathbf{y}}^i : y + (0, 1) \in S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c\}| < m$ , we adopt an alternative approach. Specifically, we connect the vertices in  $G_{\mathbf{y}}^i$  to the set

$$\{x - (0, 1) : x \in S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c\},$$

and then establish a vertical connection to  $S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c$ . To achieve this, we make a slight adjustment to conditions C1' and C2', replacing it with the following conditions:

- C1''. there are distinct vertices  $x_1, \dots, x_L \in S_{\mathbf{x}}^n \cap (\mathcal{U}_{\mathbf{x}}^i)^c$ , where  $x_1 - (0, 1), \dots, x_L - (0, 1) \notin \mathcal{U}_{\mathbf{y}}^{i+1}$ , and  $y(x_1), \dots, y(x_L) \in G_{\mathbf{y}}^1$  (not necessarily distinct) such that  $\omega(\{y(x_k), x_k - (0, 1)\}) = 1$  and  $\omega(\{x_k - (0, 1), x_k\}) = 1$ ;
- C2''.  $\eta(x - (0, 1)) = \xi_{t(y(x)) + 1}$  and  $\eta(x) = \xi_{t(y(x)) + 2}$ , for some  $x \in \{x_1, \dots, x_L\}$ .

Using similar reasoning as in (28) and (29), we can ensure that the probability that C1'' holds is also bigger than  $(1 - \delta)^{1/3}$  for every  $n$  large.

We observe that the status of the edge  $\{y(x_k), x_k - (0, 1)\}$  may have already been checked during the growth process of the  $i$ -th  $\xi$ -exploration at  $S_{\mathbf{y}}^n$ . To address this, we give an additional chance for the edge  $\{y(x_k), x_k - (0, 1)\}$  to be declared open, if necessary. By coupling with independent uniform random variables  $U(f) \sim U[0, 1]$  assigned to each edge  $f$ , we adjust the hypothesis to  $p_i/2 \geq (i \log i)^{-1}$  for all large  $i$ . Specifically, an edge  $f$  of size  $i$  is declared open during the first check if  $U(f) \in [0, p_i/2]$ ; otherwise, in a potential second check, it is declared open if  $U(f) \in [p_i/2, p_i]$ . See Remark 3 for ensuring that Theorem 1 remains valid under the original hypotheses.

We also note that when C2'' is verified, the statuses of the vertices  $x - (0, 1)$  in  $S_{\mathbf{y}}^n$  ( $x \in \{x_1, \dots, x_L\}$ ) are checked, and hence we declare  $x - (0, 1) \in R_{\mathbf{y}}^i$ . Finally, the bound in (27) still holds, possibly with a different constant.

We have established that Condition C1' (or C1'', when necessary) holds with arbitrary high probability, independently of the past  $\mathcal{U}_{\mathbf{x}}^i \in \mathcal{F}(\mathbf{x}, i)$ . To complete the proof, it remains to show the same for C2' (or C2'') and C3'. This follows because, assuming C1' (or C1'') holds, the probability that C2' (or C2'') also holds is larger than  $[1 - (\min\{p, 1 - p\})^2]^L \geq (1 - \delta)^{1/3}$ , provided  $L$  is chosen sufficiently large.

Finally, assuming that C1' and C2' hold (or C1'' and C2''), it suffices to show that the probability of the growth algorithm succeeding is also arbitrarily high. This follows directly from Lemma 1, as we explain now.

Conditioned on the past and that C1' and C2' hold, the growth algorithm in  $\mathbf{x}$  starts with a set  $B_1$  such that  $|B_1| \leq L + |\mathcal{U}_{\mathbf{x}}^i|$ . Using the bound in (27), we obtain  $|B_1| \leq c_6(\log n)^3$ , for some constant  $c_6 > 0$ .

Consider  $f(L, n) = |B_1| + \alpha(n)L$ , as defined in the statement of Lemma 1. Note that if  $\mathbf{x} \in F_{\ell+1}$ , we replace  $\alpha(n)$  with  $h\alpha(n)$  in the definition of  $f(L, n)$ . In either case, we have  $f(L, n) \leq c(\log n)^3$  for some  $c \geq 0$ . Since

$$\sum_{i>c(\log n)^3}^{n/2} (i \log i)^{-1} \xrightarrow[n \rightarrow \infty]{} \infty,$$

the condition in (8) is satisfied. Consequently, Lemma 1 ensures that for suitable choice of  $L$  and for every sufficiently large  $n$ , C3' is satisfied with probability larger than  $(1 - \delta)^{1/3}$ . This completes the proof.  $\square$

Let us prove (25). Given  $\ell > \ell_0$  and  $\phi \in \Xi_\ell$ , fix some  $\xi_\phi \in \Xi$  such that  $\sigma_\ell(\xi_\phi) = \phi$ . Given  $i = 1, \dots, h$ , recall that  $\mathcal{U}_\mathbf{x}^i := \cup_{r=1}^{i-1} R_\mathbf{x}^r$ , with  $\mathcal{U}_\mathbf{x}^1 = \emptyset$ . We denote  $\widetilde{\mathcal{U}}^i = (\mathcal{U}_\mathbf{x}^i)_{\mathbf{x} \in T_\ell}$ . Consider the set

$$\Delta' = \left\{ (\mathcal{U}_\mathbf{x})_{\mathbf{x} \in T_\ell} : \mathcal{U}_\mathbf{x} \subset S_\mathbf{x}^n \text{ and } |\mathcal{U}_\mathbf{x}| \leq c_3(\log n)^3 \right\}.$$

Write

$$\Psi = [\cap_{j < i} (\mathcal{B}_\ell^j(\phi))^c] \cap \mathcal{B}_{\ell-1}(\xi_\phi) = [\cap_{j < i} (\mathcal{B}_\ell^j(\xi_\phi))^c] \cap \mathcal{B}_{\ell-1}(\xi_\phi),$$

and define the probability measure

$$\mathbb{Q}_{p,\varepsilon}^K(A) = \mathbb{P}_{p,\varepsilon}^K(A|\Psi),$$

for any measurable set  $A$ .

Next, consider the random vector  $Y = (C_0^\ell, \widetilde{\mathcal{U}}^i)$ . Let

$$\Delta = \{(S, X) : |S| \geq 4^{\ell-1}, X \in \Delta'\}$$

and note that  $\mathbb{Q}_{p,\varepsilon}^K(\Delta) = 1$ . Given  $a > 0$ , let  $\gamma(a)$  be as obtained in Proposition 1. Applying Lemma 3 with  $1 - \delta > \gamma(a)$ , and setting  $K = K(a) = 2n$ , we have for every  $n$  sufficiently large that

$$\begin{aligned} \mathbb{P}_{p,\varepsilon}^K \left( (\mathcal{B}_\ell^i(\phi))^c \middle| \bigcap_{j < i} (\mathcal{B}_\ell^j(\phi))^c \cap \mathcal{B}_{\ell-1}(\xi_\phi) \right) &= \int_{\Delta} \mathbb{Q}_{p,\varepsilon}^K \left( (\mathcal{B}_\ell^i(\xi_\phi))^c \middle| Y = (S, X) \right) dF_Y(S, X) \\ &= \int_{\Delta} \mathbb{Q}_{p,\varepsilon}^K \left( |\Gamma_\ell^i(\xi_\phi)| < 4^\ell \middle| Y = (S, X) \right) dF_Y(S, X) \\ &\leq \int_{\Delta} P_\gamma \left( |C_S \cap F_{\ell+1,2}| < 4^\ell \right) dF_Y(S, X) \\ &\leq a^{4^\ell}. \end{aligned}$$

**Remark 3.** The conclusion of Theorem 1 remains valid if the condition in (5) is relaxed. Specifically, this holds if the following conditions are satisfied:

- for some  $i_0 \in \mathbb{N}$ ,  $p_i \geq q_i$  for all  $i \geq i_0$ , where  $\{q_i\}_{i \in \mathbb{N}}$  is a non-increasing sequence;

- for every  $C \geq 0$ , the sum

$$\sum_{i=C\beta(n)}^{n/2} q_i \longrightarrow \infty$$

as  $n$  goes to infinity, where  $\beta(n) = (nq_{2n})^{-3}$ .

Under these conditions, for all  $p \in (0, 1)$  and  $\epsilon > 0$ , there exists  $K = K(\epsilon, p, \{q_i\}) \in \mathbb{N}$  such that

$$\mathbb{P}_{p,\varepsilon}^K(W_0 = \Xi) > 0.$$

Indeed, it suffices to set  $\alpha(n) = c_1\beta(n)$  for a suitably large  $c_1$  depending on  $L$ , where  $L$  is chosen in terms of  $p$  and  $\varepsilon$ , as detailed earlier in this section. This choice ensures an analogous property to those obtained in (28) and (29). Additionally, the bound in (27) implies that  $|B_1| \leq c(\alpha(n))^3 < C(\beta(n))$ , for some positive constants  $c$  and  $C$ . Consequently, the second assumption in this remark leads to a result analogous to Lemma 3.

To illustrate, let  $\log_{(1)}(x) = \log x$  and  $\log_{(m+1)}(x) = \log \log_{(m)}(x)$  for  $m \in \mathbb{N}$ . For any  $m \in \mathbb{N}$  and  $C > 0$ , the conditions above are satisfied in the example

$$q_i = \frac{C}{i \log_{(1)}(i) \log_{(2)}(i) \dots \log_{(m)}(i)}.$$

In this case, the existence of percolation in the truncated lattice was established in [16].

### 3 Proof of Theorem 2

In this section, we prove Theorem 2. The idea of the proof is the following: first, we show our graph is isomorphic to a three-dimensional slab with one-dimensional long-range connections (see also [13]). Second, we construct (similarly as in [8]) a dynamical coupling between a percolation process on such a slab and an independent nearest neighbor oriented percolation process with parameter  $\gamma \in (0, 1)$ , which will be large, in general. This coupling will enable us to show that all words are seen in the slab, and consequently, all words are seen in our original graph, concluding the proof.

#### 3.1 The isomorphism

Let  $\mathbb{S}_k$  denote a three-dimensional slab of thickness  $k$  with one-dimensional long-range connections. More precisely, let  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  be the canonical basis of  $\mathbb{R}^3$ , and define the sets

$$\mathcal{E}_{V,n}^{sl} = \{\{u, u + n\mathbf{e}_3\} : u \in \mathbb{Z}^3\}, \quad 1 \leq n \leq k-1,$$

$$\mathcal{E}_{H,i}^{sl} = \{\{u, u + \mathbf{e}_i\} : u \in \mathbb{Z}^3\}, \quad i = 1, 2,$$

$$\mathcal{E}^{sl} = \mathcal{E}_V^{sl} \cup \mathcal{E}_{H,1}^{sl} \cup \mathcal{E}_{H,2}^{sl}.$$

Write  $\mathbb{V} = \mathbb{Z}^2 \times \{0, \dots, k\}$  and denote  $\mathbb{S}_k = (\mathbb{V}, \mathcal{E}^{sl})$ .

Let  $G_K = \left(\mathbb{Z}^2, \left(\mathcal{E}_V \cup \left(\bigcup_{i=1}^K \mathcal{E}_{H,i}\right)\right)\right)$  be a  $K$ -truncation of the graph introduced in Section 1, see (1) and (2). Consider the following subset of edges of  $G_K$

$$\mathcal{V}(K) = \left\{ \{(u_1, u_2), (v_1, v_2)\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : u_2 = v_2, \lfloor u_1/K \rfloor = \lfloor v_1/K \rfloor \right\} \cup \mathcal{E}_{H,K} \cup \mathcal{E}_V,$$

and define the graph  $F_K = (\mathbb{Z}^2, \mathcal{V}(K))$ .

We claim that  $F_K$  and  $\mathbb{S}_K$  are isomorphic (see also Section 3 of [13]). To see this, consider the function

$$\varphi : \mathbb{Z} \rightarrow \mathbb{Z} \times \{0, 1, \dots, K-1\},$$

with

$$\varphi(u) := \left( \left\lfloor \frac{u}{K} \right\rfloor, u \pmod{K} \right).$$

Indeed, the function

$$\Phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^2 \times \{0, 1, \dots, K-1\},$$

with  $\Phi(u, v) := (\varphi(u), v)$ , is a graph isomorphism between  $F_K$  and  $\mathbb{S}_K$ ; see Figure 4. We observe that edges in  $\mathcal{E}_{H,K}$  are identified with edges in  $\mathcal{E}_{H,1}^{sl}$ , edges in  $\mathcal{E}_{H,j}$  are identified with edges in  $\mathcal{E}_{V,j}^{sl}$ ,  $1 \leq j \leq K-1$ , and edges in  $\mathcal{E}_V$  are identified with edges in  $\mathcal{E}_{H,2}^{sl}$ .

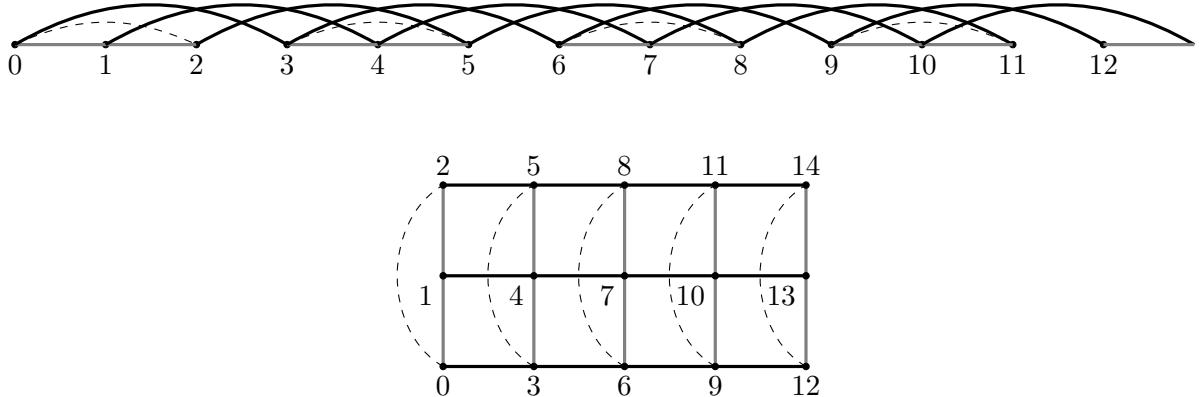


Figure 4: The map  $\Phi$  for  $K = 3$ .

Note that the  $K$ -truncated percolation process introduced in (3) induces a percolation model on  $\mathbb{S}_K$  to which we refer as the  $SLAB(K, \varepsilon)$  model.

### 3.2 The coupling

As shown in Section 3.1, the graphs  $F_K$  and  $\mathbb{S}_K$  are isomorphic. Since  $F_K$  is a subgraph of  $G_K$ , it is enough to show that all words are seen simultaneously on  $\mathbb{S}_K$ , for some  $K$  large enough.

To show that all words are seen on the slab  $\mathbb{S}_K$ , we construct a coupling between the long-range process on the slab and an independent, highly supercritical, nearest-neighbor oriented percolation process. This approach is reminiscent of the method used in [8], where the authors coupled a long-range percolation process on  $\mathbb{L}^3$  (considering the entire space) with two-dimensional oriented percolation.

However, in our case, percolation is confined to a slab, requiring a slight modification of the original argument. We present the necessary adjustments here for clarity and completeness.

Given a set  $A \subset \mathbb{Z}_+^2$ , write  $\partial_e A$  for the external vertex boundary of  $A$ , that is,

$$\partial_e A = \{u \in A^c : \exists v \in A \text{ such that } \|u - v\| = 1\}.$$

Fix  $\xi \in \Xi$ , and let  $x_1, x_2, \dots$  be a fixed ordering of the vertices of  $\mathbb{L}_+^2$ . We shall construct inductively a sequence  $\{A_n, B_n\}_{n \geq 0}$  of ordered pairs of subsets of  $\mathbb{Z}_+^2$  and a function  $\psi : A_n \rightarrow \mathbb{Z}_+$ . Write  $\mathcal{O} = (0, 0)$  for the origin of  $\mathbb{L}_+^2$  and set  $A_0 = \{\mathcal{O}\}$ ,  $B_0 = \emptyset$ , and  $\psi(\mathcal{O}) = 0$ .

Assume  $\{A_n, B_n\}$  have been constructed for some  $n \in \mathbb{N}$  and that  $\psi(x)$  is known for all  $x \in A_n$ . If  $\partial_e A_n \cap B_n^c = \emptyset$ , stop the construction and set  $(A_\ell, B_\ell) = (A_n, B_n)$  for all  $\ell \geq n$ . If  $\partial_e A_n \cap B_n^c \neq \emptyset$ , let  $x_n$  be the earliest vertex in the fixed ordering in  $\partial_e A_n \cap B_n^c$ , and define  $y_n$  as the vertex in  $A_n$  such that  $x_n = y_n + (1, 0)$  or  $x_n = y_n + (0, 1)$ . First, assume that  $x_n = y_n + (0, 1)$ . Fixed  $N, M \in \mathbb{N}$ , we distinguish between two cases:

1.  $\psi(y_n) \leq N + M$

We say  $x_n$  is *black* if, for some  $i \in \{1, \dots, N\}$ , the following conditions hold:

- $X(y_n, \psi(y_n) + i) = \xi_{2||y_n||+1}$ ,  $(y_n, \psi(y_n) + i) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ ,
- $X(x_n, \psi(y_n) + i) = \xi_{2||y_n||+2}$ ,  $(x_n, \psi(y_n) + i) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ ,
- $\{(y_n, \psi(y_n)), (y_n, \psi(y_n) + i)\} \in \mathcal{E}_V^{sl}$ , and  $\{(y_n, \psi(y_n) + i), (x_n, \psi(y_n) + i)\} \in \mathcal{E}_H^{sl}$  are open.

In this case, write  $\psi(x_n) = \psi(y_n) + i$ .

2.  $\psi(y_n) > N + M$

We say  $x_n$  is *black* if, for some  $i \in \{1, \dots, N\}$ , the following conditions hold:

- $X(y_n, \psi(y_n) - i) = \xi_{2||y_n||+1}$ ,  $(y_n, \psi(y_n) - i) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ ,
- $X(x_n, \psi(y_n) - i) = \xi_{2||y_n||+2}$ ,  $(x_n, \psi(y_n) - i) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ ,
- $\{(y_n, \psi(y_n)), (y_n, \psi(y_n) - i)\} \in \mathcal{E}_V$ , and  $\{(y_n, \psi(y_n) - i), (x_n, \psi(y_n) - i)\} \in \mathcal{E}_H$  are open.

In this case, write  $\psi(x_n) = \psi(y_n) - i$ .

Define

$$(A_{n+1}, B_{n+1}) = \begin{cases} (A_n \cup \{x_n\}, B_n) & \text{if } x_n \text{ is black,} \\ (A_n, B_n \cup \{x_n\}) & \text{otherwise.} \end{cases}$$

If  $x_n = y_n + (0, 1)$ , we proceed analogously with  $i \in \{N + 1, \dots, N + M\}$ . Such distinction is useful to avoid dependence issues. Writing

$$A(\xi) = \bigcup_{n \in \mathbb{N}} A_n,$$

it follows that if  $|A(\xi)| = \infty$ , then  $\xi$  is seen in the  $2(N + M)$ -truncated model. Note that

$$\mathbb{P}_{p,\varepsilon}^{2N+2M}(x_n \text{ is not black}) \leq \begin{cases} \prod_{i=1}^N [1 - \varepsilon p_i (\min\{p, 1-p\})^2] & \text{if } x_n = y_n + (1, 0), \\ \prod_{i=N+1}^{N+M} [1 - \varepsilon p_i (\min\{p, 1-p\})^2] & \text{if } x_n = y_n + (0, 1). \end{cases}$$

In any case, since  $\limsup p_i > 0$ , the sum of the  $p'_i$ s diverges and we get

$$\mathbb{P}_{p,\varepsilon}^{2N+2M}(x_n \text{ is black}) \rightarrow 1, \text{ when } N, M \rightarrow \infty, \quad (30)$$

for all  $n \in \mathbb{N}$ . With (30), Lemma 1 and Lemma 2 in [8] follows in a rather analogous way, allowing us to prove that for all  $p \in (0, 1)$ ,  $\varepsilon > 0$ , and  $\alpha > 0$ , there exists  $N = N(p, \varepsilon, \alpha)$  and  $M = M(p, \varepsilon, \alpha)$  such that all words are seen simultaneously from the origin on the  $SLAB(2N + 2M, \varepsilon)$  model with probability larger than  $1 - \alpha$ .

*Proof of Theorem 2:* Fix  $\varepsilon$ ,  $\alpha$ , and  $p$ . By hypothesis, there exists  $\delta > 0$  such that  $\limsup p_i > \delta$ . By the discussion in Sections 3.1 and 3.2, there exists  $K_1 = K_1(p, \delta, \alpha)$  such that all words are seen simultaneously on the  $SLAB(K_1, \delta \wedge \varepsilon)$  model, almost surely. Next, pick  $K_2 \geq K_1$  such that  $p_{K_2} > \delta$ , noting that this choice is possible since  $\limsup p_i > \delta$ . Clearly, since  $\mathbb{S}_{K_1}$  is a subgraph of  $\mathbb{S}_{K_2}$ , all words are seen on the  $SLAB(K_2, \delta \wedge \varepsilon)$  model, almost surely. The result follows by observing that  $\mathbb{S}_{K_2}$  is isomorphic to  $F_{K_2}$ , which is a subgraph of  $G_{K_2}$ .

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