

A Bunch of Random Proofs

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CONTENTS

1	Calculus	5
1.1	Limit Laws/Rules	5
1.1.1	Limit Law R1	5
1.1.2	Limit Law R2	5
1.1.3	Limit Law R3	5
1.1.4	Limit Law A1	5
1.1.5	Limit Law A2	5
1.1.6	Limit Law A3	5
1.1.7	Limit Law A4	5
1.1.8	Limit Law A5	5
1.1.9	Limit Law A6	5
1.1.10	Limit Law A7	5
1.2	Derivative Laws/Rules	6
1.2.1	Product Rule	6
1.2.2	Quotient Rule	6
1.2.3	Chain Rule	6
1.2.4	Constant Rule	6
1.2.5	Power Rule	6
1.2.6	Square Root Rule	7
1.2.7	Sine Rule	7
1.2.8	Cosine Rule	7
1.2.9	Tangent Rule	7
1.2.10	Cotangent Rule	7
1.2.11	Secant Rule	7
1.2.12	Cosecant Rule	7
1.2.13	Sine Inverse Rule	7
1.2.14	Cosine Inverse Rule	7
1.2.15	Tangent Inverse Rule	7

1.2.16	Cotangent Inverse Rule	7
1.2.17	Secant Inverse Rule	7
1.2.18	Cosecant Inverse Rule	7
1.2.19	Factoring a Constant	7
1.2.20	Variable Power Rule	8
1.2.21	e to the x Rule	8
1.2.22	Logarithm Rule	8
1.2.23	Absolute Value Rule	8
2	Real Analysis	9
2.1	Cardinality	9
2.1.1	Bijection Between Two Sets	9
2.1.2	Bijection Between Three Sets	9
3	Set Theory	11
3.1	Operations on sets	11
3.1.1	Distributing the union over the intersection	11

CHAPTER 1

CALCULUS

1.1 Limit Laws/Rules

1.1.1 Limit Law R1

1.1.2 Limit Law R2

1.1.3 Limit Law R3

1.1.4 Limit Law A1

1.1.5 Limit Law A2

1.1.6 Limit Law A3

1.1.7 Limit Law A4

1.1.8 Limit Law A5

1.1.9 Limit Law A6

1.1.10 Limit Law A7

1.2 Derivative Laws/Rules

1.2.1 Product Rule

1.2.2 Quotient Rule

1.2.3 Chain Rule

1.2.4 Constant Rule

1.2.5 Power Rule

Show that

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof. In order to prove our desired claim, we let $f(x) = x^n$, where $f(x)$ is a differentiable function.

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h)^n - a^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sum_{k=0}^n \binom{n}{k} a^k h^{n-k} - (a^n)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(\binom{n}{0} h^0 + \binom{n}{1} a^{n-1} h^1 + \cdots + \binom{n}{n} a^0 h^n \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(na^{n-1} h + \frac{n(n-1)}{2} a^{n-2} h^2 + \cdots + h^n \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h} \left(na^{n-1} + \frac{n(n-1)}{2} a^{n-2} h + \cdots + h^{n-1} \right)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} \left(na^{n-1} + \frac{n(n-1)}{2} a^{n-2} h + \cdots + h^{n-1} \right) \\ &= \left(na^{n-1} + \frac{n(n-1)}{2} a^{n-2} 0 + \cdots + 0^{n-1} \right) \\ &= na^{n-1}. \end{aligned}$$

$$\therefore \frac{d}{dx}(x^n) = nx^{n-1}.$$

□

1.2.6 Square Root Rule

Show that

$$\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Proof.

□

1.2.7 Sine Rule**1.2.8 Cosine Rule****1.2.9 Tangent Rule****1.2.10 Cotangent Rule****1.2.11 Secant Rule****1.2.12 Cosecant Rule****1.2.13 Sine Inverse Rule****1.2.14 Cosine Inverse Rule****1.2.15 Tangent Inverse Rule****1.2.16 Cotangent Inverse Rule****1.2.17 Secant Inverse Rule****1.2.18 Cosecant Inverse Rule****1.2.19 Factoring a Constant**

Show that

$$\frac{d}{dx} (cf(x)) = c \frac{d}{dx} (f(x)).$$

Proof. In order to prove our desired claim, we let $g(x) = cf(x)$, where $g(x)$ and $f(x)$ are differentiable functions and $c \in \mathbb{R}$.

$$\begin{aligned} \frac{d}{dx} (g(x)) &= \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{cf(a+h) - cf(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c[f(a+h) - f(a)]}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}. \end{aligned}$$

$$\therefore \frac{d}{dx} (cf(x)) = c \frac{d}{dx} (f(x)).$$

□

1.2.20 Variable Power Rule**1.2.21 e to the x Rule****1.2.22 Logarithm Rule****1.2.23 Absolute Value Rule**

CHAPTER 2

REAL ANALYSIS

2.1 Cardinality

2.1.1 Bijection Between Two Sets

Theorem. Let A and B be sets. If $|A| = |B|$, then $|B| = |A|$.

Proof. Suppose $|A| = |B|$. Then, \exists bijective function $f : A \rightarrow B$. Then $f^{-1} : B \rightarrow A$ is a bijection. So, $|B| = |A|$. \square

2.1.2 Bijection Between Three Sets

Theorem. Let A , B , and C be sets. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof. Since $|A| = |B|$ and $|B| = |C|$, then \exists bijection from $A \rightarrow B$ and bijection from $B \rightarrow C$. \square

CHAPTER 3

SET THEORY

3.1 Operations on sets

3.1.1 Distributing the union over the intersection

Theorem. Let A , B , and C be sets. Then, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. *Let's make a couple of definitions before we prove anything:*

def: $X \cap Y = \{a | a \in X \wedge a \in Y\}$

def: $X \cup Y = \{a | a \in X \vee a \in Y\}$

Let $D = B \cap C$. We can re-write the statement on the left using the definitions from above as follows:

$$\begin{aligned} A \cup D &= \{x | x \in A \vee x \in D\} \\ &= \{x | x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}. \end{aligned}$$

Using the definitions from above

$$\{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} = (A \cap B) \cup (A \cap C). \quad \square$$