A Bunch of Random Proofs

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1.2 Derivative Laws/Rules

1.2.1 Product Rule

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Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}.$$

Proof. *In order to prove our desired claim, we let $f(x) = x^n$, where $f(x)$ is a differentiable function.*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x^n) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0}\frac{(a+h)^n - a^n}{h} \\
&= \lim_{h\to 0}\frac{\displaystyle\sum_{k=0}^{n}\binom{n}{k}a^k h^{n-k} - (a^n)}{h} \\
&= \lim_{h\to 0}\frac{\left(\binom{n}{0}h^0 + \binom{n}{1}a^{n-1}h^1 + \cdots + \binom{n}{n}a^0 h^n\right)}{h} \\
&= \lim_{h\to 0}\frac{\left(na^{n-1}h + \frac{n(n-1)}{2}a^{n-2}h^2 + \cdots + h^n\right)}{h} \\
&= \lim_{h\to 0}\frac{\cancel{h}\left(na^{n-1} + \frac{n(n-1)}{2}a^{n-2}h + \cdots + h^{n-1}\right)}{\cancel{h}} \\
&= \lim_{h\to 0}\left(na^{n-1} + \frac{n(n-1)}{2}a^{n-2}h + \cdots + h^{n-1}\right) \\
&= \left(na^{n-1} + \frac{n(n-1)}{2}a^{n-2}0 + \cdots + 0^{n-1}\right) \\
&= na^{n-1}.
\end{aligned}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}.$$

□

1.2.6 Square Root Rule

Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{1}{s\sqrt{x}}.$$

Proof. □

1.2.7 Sine Rule

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1.2.19 Factoring a Constant

Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(cf(x)\right) = c\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right).$$

Proof. *In order to prove our desired claim, we let $g(x) = cf(x)$, where $g(x)$ and $f(x)$ are differentiable functions and $c \in \mathbb{R}$.*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right) &= \lim_{h\to 0}\frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0}\frac{cf(a+h)-c(f(a))}{h} \\
&= \lim_{h\to 0}\frac{c[f(a+h)-f(a)]}{h} \\
&= c\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.
\end{aligned}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(cf(x)\right) = c\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right).$$

$\square$

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1.2.22 Logarithm Rule

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CHAPTER 2

REAL ANALYSIS

2.1 Cardinality

2.1.1 Bijection Between Two Sets

Theorem. Let A and B be sets. If $|A| = |B|$, then $|B| = |A|$.

Proof. *Suppose* $|A| = |B|$. *Then,* $\exists$ *bijective function* $f : A \to B$. *Then* $f^{-1} : B \to A$ *is a bijection. So,* $|B| = |A|$. $\square$

2.1.2 Bijection Between Three Sets

Theorem. Let A, B, and C be sets. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof. *Since* $|A| = |B|$ *and* $|B| = |C|$, *then* $\exists$ *bijection from* $A \to B$ *and bijection from* $B \to C$. $\square$

CHAPTER 3

SET THEORY

3.1 Operations on sets

3.1.1 Distributing the union over the intersection

Theorem. Let A, B, and C be sets. Then, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. *Let's make a couple of definitions before we prove anything:*
def: $X \cap Y = \{a | a \in X \wedge a \in Y\}$
def: $X \cup Y = \{a | a \in X \vee a \in Y\}$
Let $D = B \cap C$. We can re-write the statement on the left using the definitions from above as follows:

$$\begin{aligned} A \cup D &= \{x | x \in A \vee x \in D\} \\ &= \{x | x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\}. \end{aligned}$$

Using the definitions from above

$$\{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} = (A \cap B) \cup (A \cap C). \qquad \square$$