Pre-Calculus II: Midterm

Due on May 06, 2022 at 6:45pm

Professor P. Haberman

Hashem A. Damrah

Problem 1

Circle **T** for *true* or **F** for *false*. (You don't need to justify your answers.)

- **a.** **T** **F** $\sin^{-1}(\sin(\frac{2\pi}{3})) = \frac{2\pi}{3}$
- **b.** **T** **F** $\cos(-t) = \cos(t)$ for all t.
- **c.** **T** **F** An angle of measure $60°$ in a circle of radius 2 units is larger than an angle of measure $60°$ in a circle of radius 1 unit.
- **d.** **T** **F** An angle of measure 1 radian is larger than an angle of measure $1°$.

Solution 1

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Problem 2

Use the sin and cos functions to find the exact coordinates of point P specified by the angle $\frac{5\pi}{3}$ on the circumference of a circle of radius 4 units. Be sure to ***show your use of sin and cos*** and to provide completely simplified, exact numerical values.

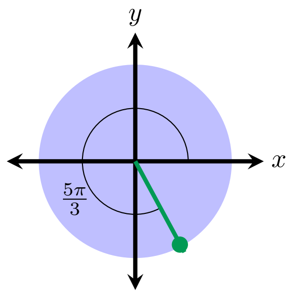

Figure 1

Solution 2

$$
\begin{aligned}
& P = \left(\cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right)\right) \\
\Rightarrow \quad & P = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)
\end{aligned}
$$

.

Since we aren't on the **Unit Circle**, meaning we don't have a radius of 1 units, we're going to scale up the length of the radius, which is 4 units.

$$
P = \left(2, -2\sqrt{3}\right)
$$

.

Problem 3

Find the **exact** value for each of the following expressions. Be sure to use proper notation to communicate your answer, i.e., link the given expression and your answer with an equal sign. If the given expression is undefined write, *"The expression is undefined."*

a. $\cos(0)$

b. $\sin(\frac{\pi}{6})$

c. $\cos(\frac{5\pi}{4})$

d. $\sin(\frac{4\pi}{3})$

e. $\cos(-\frac{2\pi}{3})$

f. $\sin(\frac{11\pi}{4})$

g. $\tan(\pi)$

h. $\sec(\frac{\pi}{2})$

i. $\tan(\frac{11\pi}{6})$

j. $\frac{10\pi}{3}$

Solution 3

a. $\cos(0) = 1$

b. $\sin(\frac{\pi}{6}) = \frac{1}{2}$

c. $\cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

d. $\sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$

e. $\cos(-\frac{2\pi}{3}) = \frac{1}{2}$

f. $\sin(\frac{11\pi}{4}) = \frac{\sqrt{2}}{2}$

g. $\tan(\pi) = 0$

h. $\sec(\frac{\pi}{2}) =$ *The expression is undefined*

i. $\tan(\frac{11\pi}{6}) = \sqrt{3}$

j. $\csc(\frac{10\pi}{3}) = -\frac{2\sqrt{3}}{3}$

Problem 4

Find the length of the arc spanned by an angle of 200° in a circle of radius 12 feet. [Provide a completely simplified, exact numerical value.]

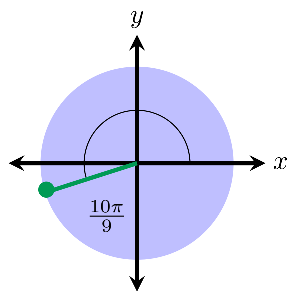

Figure 2

Solution 4

To find the arc spanned by the angle, we can use the following formula:

$$s = r \times \theta$$

. Here's the catch. We cannot use degrees. We must convert 200° to radians, which is pretty simple:

$$200^\circ \times \frac{\pi}{180^\circ} = \frac{10\pi}{9}$$

. Now we can use the formula:

$$s = 12 \times \frac{10\pi}{9} = \frac{40\pi}{3}$$

. So, the arc spanned by the angle 200° ($\frac{10\pi}{9}$ radians) is $\frac{40\pi}{3}$ feet.

Problem 5

If $\sin(\theta) = \frac{\sqrt{7}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact numerical values for the expressions given below. [Be sure to compose conclusions that directly communicate the values of the given expressions.]

a. $\cos(\theta)$

b. $\tan(\theta)$

c. $\csc(\theta)$

Solution 5

a. $\cos(\theta)$

$$
\begin{aligned}
& \sin^2(\theta) = \cos^2(\theta) = 1 \\
\Rightarrow \quad & \left(\frac{\sqrt{7}}{5}\right)^2 + \cos^2(\theta) = 1 \\
\Rightarrow \quad & \frac{7}{25} + \cos^2(\theta) = 1 \\
\Rightarrow \quad & \cos^2(\theta) = \frac{18}{25} \\
\Rightarrow \quad & \sqrt{\cos^2}(\theta) = \sqrt{\frac{18}{25}} \\
\Rightarrow \quad & \cos(\theta) = \frac{3\sqrt{2}}{5} \\
. &
\end{aligned}
$$

b. $\tan(\theta)$

$$
\begin{aligned}
& \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \\
\Rightarrow \quad & \tan(\theta) = \frac{\frac{\sqrt{7}}{5}}{\frac{3\sqrt{2}}{5}} \\
\Rightarrow \quad & \tan(\theta) = \frac{\sqrt{14}}{6} \\
. &
\end{aligned}
$$

c. $\csc(\theta)$

$$
\begin{aligned}
& \csc(\theta) = \frac{1}{\sin(\theta)} \\
\Rightarrow \quad & \csc(\theta) = \frac{1}{\frac{\sqrt{7}}{5}} \\
\Rightarrow \quad & \csc(\theta) = \frac{5}{\sqrt{7}} \\
\Rightarrow \quad & \csc(\theta) = \frac{5\sqrt{7}}{7}
\end{aligned}
$$

.

Problem 6

Draw a graph of at least two period of the function $g(t) = 3\cos(\frac{\pi}{2}t + \frac{\pi}{2}) - 2$ by

(a) plotting the points where the graph intersects the midline

(b) plotting the points where the graph achieves maximum and minimum values

(c) connecting these points with an appropriately curved sinusoidal wave.

List the period, midline, and amplitude of the function and **label the scale on the axes.**

Solution 6

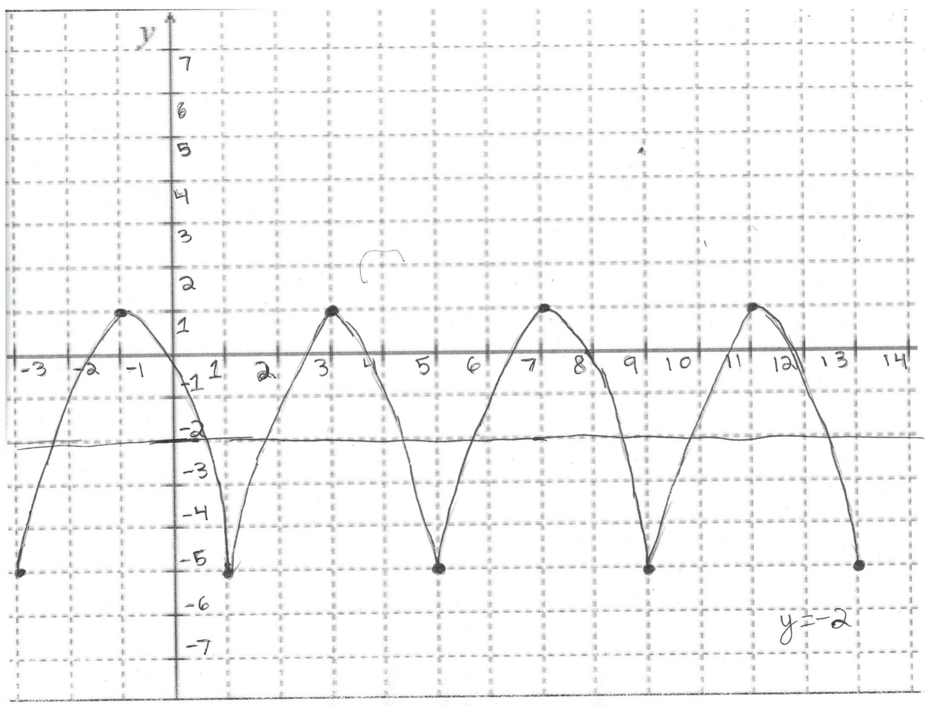

Figure 3: Sketch a graph of $g(t) = 3\cos(\frac{\pi}{2}t + \frac{\pi}{2}) - 2$

Problem 7

Find a possible algebraic rule for the sinusoidal function f graphed below. (You only need to provide one algebraic rule and you may utilize either the sin or cos function.)

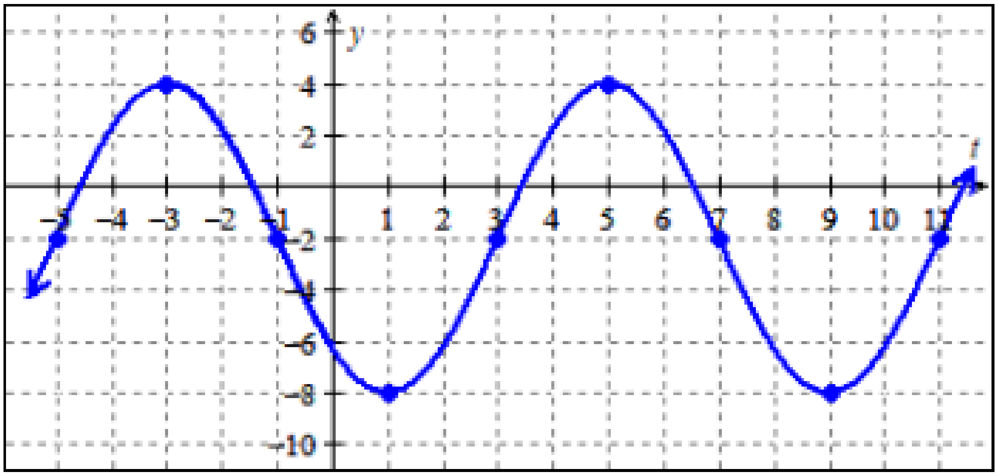

Figure 4: The graph of $y = f(t)$

Solution 7

To create an algebraic rule, we're going to use the following formula:

$$y = A\sin(\omega(t-h)) + k \text{ or } y = A\cos(\omega(t-h)) + k$$

.
Where:

1 A is the amplitude

2 ω is the period

3 h is the horizontal shift

4 k is the midline

Finding the amplitude is easy, but first we must find the midline. The midline for this graph is -2. To find the amplitude, we need to see how many units it takes to go from the maximum/minimum to the midline. In this case, that's 6 units. Now I'm going to find the **period**, which is how long it takes to repeat itself. The period is 8 units because it takes 8 units to go from one maximum/minimum to the next. We only found the period. To find ω, we know that sin and cos have a period of 2π.

$$\frac{2\pi}{\omega} = 8 \Rightarrow \omega = \frac{\pi}{4}$$

.
Since I'm using the sin function, $h = 3$.
Here's our final algebraic rule:

$$y = -2\left(\frac{\pi}{4}(t-3)\right) - 2$$

.

Problem 8

Evaluate the following expressions. [Be sure to compose conclusions that directly communicate what the given expressions equal and to provide completely simplified exact numerical values.]

a. $\sin(\cos^{-1}(-\frac{1}{2}))$

b. $\cos^{-1}(\cos(\frac{7\pi}{6}))$

Solution 8

a. $\sin(\cos^{-1}(-\frac{1}{2}))$

$$\begin{aligned} & \sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) \\ \Rightarrow \quad & \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \end{aligned}$$

.

b. $\cos^{-1}(\cos(\frac{7\pi}{6}))$

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \frac{7\pi}{6}$$

.

Problem 9

Find **all** of the solutions of the following trigonometric equation. [Provide completely simplified, exact numerical values.]

$$4\sin(2\theta) + 2 = 0$$

.

Solution 9

$$\begin{aligned}
& 4\sin(2\theta) + 2 = 0 \\
\Rightarrow \quad & 4\sin(2\theta) = -2 \\
\Rightarrow \quad & \sin(2\theta) = -\frac{1}{2} \\
\Rightarrow \quad & \sin^{-1}(\sin(2\theta)) = \sin^{-1}\left(-\frac{1}{2}\right) \\
\Rightarrow \quad & 2\theta = -\frac{\pi}{6} + 2k\pi \text{ or } 2\theta = \frac{7\pi}{6} + 2k\pi \\
\Rightarrow \quad & \frac{2\theta}{2} = \frac{-\frac{\pi}{6}}{2} + \frac{2k\pi}{2} \text{ or } \frac{2\theta}{2} = \frac{\frac{7\pi}{6}}{2} + \frac{2k\pi}{2} \\
\Rightarrow \quad & \theta = -\frac{\pi}{12} + k\pi \text{ or } \theta = \frac{7\pi}{12} + k\pi \forall k \in \mathbb{Z}.
\end{aligned}$$

Problem 10

Find all the solutions of the following trigonometric equation **on the interval** $[0, 2\pi]$. [Provide completely simplified, exact numerical values.]

$$6\sin(3x) + 4 = -2$$

.

Solution 10

$$\begin{aligned}
& 6\sin(3x) + 4 = -2 \\
\Rightarrow \quad & 6\sin(3x) = -6 \\
\Rightarrow \quad & \frac{6\sin(3x)}{6} = -\frac{6}{6} \\
\Rightarrow \quad & \sin(3x) = -1 \\
\Rightarrow \quad & \sin^{-1}(\sin(3x)) = \sin^{-1}(-1) \\
\Rightarrow \quad & 3x = -\frac{\pi}{2} \\
\Rightarrow \quad & 3x = -\frac{\pi}{2} + 2k\pi \text{ or } 3x = \frac{3\pi}{2} + 2k\pi \\
\Rightarrow \quad & \frac{3x}{3} = \frac{-\frac{\pi}{2}}{3} + \frac{2k\pi}{3} \text{ or } \frac{3x}{3} = \frac{\frac{3\pi}{2}}{3} + \frac{2k\pi}{3} \\
\Rightarrow \quad & x = -\frac{\pi}{6} + \frac{2k\pi}{3} \text{ or } x = \frac{\pi}{2} + \frac{2k\pi}{3} \forall k \in \mathbb{Z}.
\end{aligned}$$