Real Analysis 1

Hashem A. Damrah

April 10, 2022

Lecture 1: Introduction and Proofs

Feb 20 2022 (10:16:58)

Logic

Let's say we have proposition P and Q. They can be joined by "and", "or", "implies", or "if and only if".

For example, if P is a proposition, then "not P" is a new propostion that is true whenever P is false and vice versa. The symbolic representation for "not P" is $\neg P$ or $\overline{P}$

Two propositions, P and Q, can be joined by "and", "or", "implies", or "if and only if" to form a new proposition. The truth of this new proposition is determined by the truth of P and Q according to the **Truth Table**.

P	Q	"P and Q" ($P \wedge Q$)	"P or Q" ($P \Rightarrow Q$)	"P implies Q or" "if P, then Q" ($P \Rightarrow Q$)	"P if and only if Q" or "P iff Q" ($P \Leftrightarrow Q$)
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

Here are a few hidden features within this table:

- The phrase "P or Q" is true if P is true, Q, or both are true.
- The phrase "P implies Q" is true when P is false or Q is true.

There are two more important phrases in mathematical writing: "for all" (symbolized by $\forall$) and "there exists" (symbolized by $\exists$). These are called ***quantifiers***.

Definition 1 (Quantifiers). A quantifier is always followed by a variable (and perhaps an indication of the range of that variable) and then a predicate, which typically involves that variable. Here are a couple of examples:

$$\forall x \in \mathbb{R}^+ \quad e^x < (1+x)^{1+x} \tag{1}$$
$$\exists n \in \mathbb{N} \quad 2^n > (100n)^{100}. \tag{2}$$

The first statement says that e^x is less than $(1+x)^{1+x}$ for every positive real number x. The second statement says that there exists a natural number n such that $2^n > (100n)^{100}$.

The special symbols such as $\forall, \exists, \neg$, and $\wedge$ are useful to logicians who are trying to express mathematicians ideas without resorting to the English language at all. Also, other mathematicians use these symbols as shorthands.

Proving an Implication

Let's try to prove the following theorem:

Theorem 1 (Let $P(a, b)$ be any predicate defined for all $a \in \mathbb{A}$ and $b \in (B)$ Then:).

$$(\exists a \in \mathbb{A} \quad \forall b \in \mathbb{B} \quad P(a, b)) \quad \Rightarrow \quad (\forall b \in \mathbb{B} \quad \exists a \in \mathbb{A} \quad P(a, b)). \tag{3}$$

Let's impose a specific interpretation in order to give concrete meaning:

$$\begin{aligned} \mathbb{A} &= \{6.042 \quad \text{students}\} \\ \mathbb{B} &= \{6.042 \quad \text{lectures}\}. \\ P(a, b) &= \text{"student } a \text{ falls asleep during lecture } b\text{"} \end{aligned} \tag{4}$$

Interpreting the left side in these terms give:

$$\begin{aligned} &\exists a \in \mathbb{A} \quad \forall b \in \mathbb{B} \quad P(a, b) \\ &= \text{"there exists a student that falls asleep in every lecture"}. \end{aligned} \tag{5}$$

So, this side states that some particular student always falls asleep. Let's call him Snoozer. Now, here's the right side:

$$\begin{aligned} &\forall b \in \mathbb{B} \quad \exists a \in \mathbb{A} \quad P(a, b) \\ &= \text{"in every lecture, some student falls asleep"}. \end{aligned} \tag{6}$$

This is slightly different than the left side because there might be a different sleeper in each lecture. The left side should imply the right. If Snoozer sleeps in every lecture, then in every lecture some student is surely asleep.

Lecture 1: Induction

Feb 20 2022 (16:53:18)