Pre-calculus 2

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April 28, 2022

Lecture 1: Sets and Numbers

Apr 04 2022 (10:01:23)

Definition 1 (Set). A set is a collection of objects specified in a manner that enables one to determine if a given object is or isn't in the set.

Exercise 1 (Solution 1). Which of the following represent a set?

1 The students registered for MTH 112 at PCC this quarter.

2 The good students registered for MTH 112 at PCC this quarter.

Notation. Roster Notation involves listing the elements in a set within curly brackets: "$\{\}$" like the following: "$\{1, 2, 3, 4\}$"

Definition 2 (Element). An object in a set is called an **element** of the set. (symbol: "$\in$")

Example. 5 is an element of the set $\{4, 5, 6, 7, 8, 9\}$. We can express this symbolically:

$$5 \in \{4, 5, 6, 7, 8, 9\}$$

.

Definition 3 (Subset). A set S of a set T, denoted $S \subseteq T$, if all elements of S are also elements of T.

If S and T are sets and $S = T$, then $S \subseteq T$. Sometimes it's useful to consider a subset S of a set T that isn't equal to T. In such case, we write $S \subset T$ and say that S is a proper subset of T.

Example. $\{4, 7, 8\}$ is a subset of the set $\{4, 5, 6, 7, 8, 9\}$.

We can express this fact symbolically by $\{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\}$

Since these two sets aren't equal, $\{4, 7, 8\}$ is a proper subset of $\{4, 5, 6, 7, 8, 9\}$, so can write:

$$\{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\}$$

.

We can use the other symbol as follows:

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}$$

.

Definition 4 (Empty Set). The empty set, denoted $\emptyset$, is the set with no elements

$$\emptyset = \{\}$$

.

Definition 5 (Union)**.** The union of two sets A and B, denoted $A \cup B$, is the set containing all of the elements in either A or B (or both A and B).

Example. Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then ...

- $\{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\}$
- $\{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\}$
- $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Definition 6 (Intersection)**.** The intersection of two sets A and B, denoted $A \cap B$, is the set containing all the elements in both A and B.

Example. Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then ...

- $\{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\}$
- $\{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\}$
- $\{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \emptyset$

Notation. Set Builder Notation.
"All the whole numbers between 3 and 10" $= \{x | x \in \mathbb{Z} \text{ and } 3 < x < 10\}$

Definition 7 (Important Sets of Numbers)**.** The set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

.

The set of integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

.

The set of rational numbers:

$$\mathbb{Q} = \left\{ x | x = \frac{p}{q} \text{ and } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

.

The set of real numbers: $\mathbb{R}$

(All the numbers on the number line)

.

The set of complex numbers:

$$\mathbb{C} = \left\{x|x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\right\}$$

.

Note. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, the set of natural numbers ($\mathbb{N}$) is a subset of the set of integers ($\mathbb{Z}$) which is a subset of the set of rational numbers ($\mathbb{Q}$) which is a subset of the set of real numbers ($\mathbb{R}$) which is a subset of the set of complex numbers ($\mathbb{C}$).

Notation. Since we use the real numbers so often, we have special notation for subsets of the real numbers. **Interval Notation**. Interval Notation involves square or round brackets.

Example. Quick demo of Interval Notation

- $\{x|x \in \mathbb{R} \text{ and } -2 \leq x \leq 3\} = [-2, 3]$
- $\{x|x \in \mathbb{R} \text{ and } -2 < x < 3\} = (-2, 3)$
- $\{x|x \in \mathbb{R} \text{ and } -2 < x \leq 3\} = (-2, 3]$
- $\{x|x \in \mathbb{R} \text{ and } -2 \leq x < 3\} = [-2, 3)$

When the interval has no upper or lower bound, we use the infinity symbol (∞ or $-\infty$)

- $\{x|x \in \mathbb{R} \text{ and } x \leq 4\} = (-\infty, 4]$
- $\{x|x \in \mathbb{R} \text{ and } x \geq 4\} = [4, -\infty)$

Exercise 2 (Solution 2)**.** Simplify the following expressions:

- $(-4, \infty) \cup [-8, 3]$
- $(-4, \infty) \cup (-\infty, 2]$
- $(-4, \infty) \cap (-\infty, 2]$
- $(-4, \infty) \cap [-10, -5]$

Lecture 2: Angles and Arc-Length

Apr 06 2022 (07:25:34)

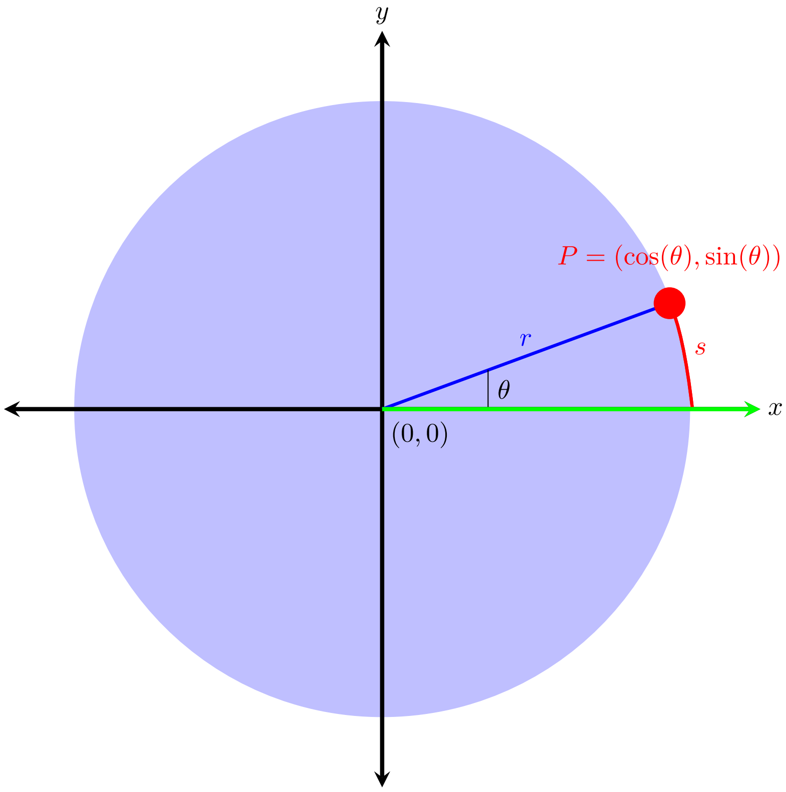

Figure 1: Diagram of a Circle

- The standard way that we put an angle in a circle we start the angle at the positive x-axis and go **counterclockwise**. We measure negative angles **clockwise**.
- When put an angle in standard position and you rotate, it ends someplace. Where it ends is called the **Terminal Side**.
- The point P on the circumference of the circle is **specified by the angle** θ.
- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by** θ
- Two angles with the same terminal side are called **co-terminal angles**.

Three hundred and sixty degrees ($360°$) represents a complete trip around a circle, which is a full rotation. So, $1°$ corresponds to $\frac{1}{360°}$ of a full rotation. Degrees are more like percentages, where they represent concepts, and not numbers. But, just like how you can transform a percentage to a number ($10\% = \frac{10}{100}$), you can do the same with degrees ($10° = \frac{10}{360}$).

Since $360°$ represents a full rotation around the circle, if we add any integer multiple of $360°$ to an angle θ_1, we'll obtain an angle co-terminal to θ_1.

Example. So 45° and 405° are co-terminal.

Definition 8 (π)**.** The number π represents the ratio of the circumference of a circle to the diameter of the circle. $\pi = \frac{c}{d}$, where c is the circumference and d is the diameter.

$\pi \approx 3.14$, but π is an **"irrational"** number.

Definition 9 (Radian)**.** The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle, s, and the radius, r, of the circle.

$$\theta = \frac{s}{r}$$

$$\begin{aligned} \pi &= \frac{c}{d} \\ &= \frac{2}{2r} && d = 2r \qquad . \\ 2\pi &= \frac{c}{r} && \text{Multiply both sides by 2} \end{aligned} \tag{1}$$

From this, we can conclude that: $2\pi = 360^\circ$. From that, we can get $\pi = 180^\circ$ since we just divided both sides by 2. From that, we get $\frac{\pi}{2} = 90^\circ$. So on, and so forth.

Here's a table with all of the radians:

θ (degrees)	0°	30°	45°	60°	90°	180°	270°	360°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Table 1: Degrees into Radians

Exercise 3. Convert 8 radians into degrees and 8 degrees into radians:

Definition 10 (Arc Length)**.** The **arc length** s spanned in a circle of radius r by an angle θ measured in radians is given by:

$$s = r \times \mid \theta \mid$$

.

Note. This formula only applies if θ is measured in radians.

Exercise 4. What is the arc length spanned by a 40° angle on a circle radius 30 meters?

Lecture 3: Introduction to Periodic Functions

Apr 07 2022 (18:43:36)

Any activity that repeats on a regular time interval can be described as *periodic*.

Definition 11 (Periodic Function). A **periodic function** whose values repeat on regular intervals. Hence, f is a periodic if there exists some constant c such that:

$$f(x+c) = f(x)$$

.

for all x in the domain of f such that $f(x+c)$ is defined.

Recall that this means that if the graph $y = f(x)$ is shifted horizontally c units then it will appear unaffected.

Definition 12 (Period). The **period** of a function f is the smallest value $|\ c\ |$ such that $f(x+t)$ for all x in the domain of f such that $f(x+c)$ is defined.

Exercise 5 (Solution 5). Find the period of the function graphed below:

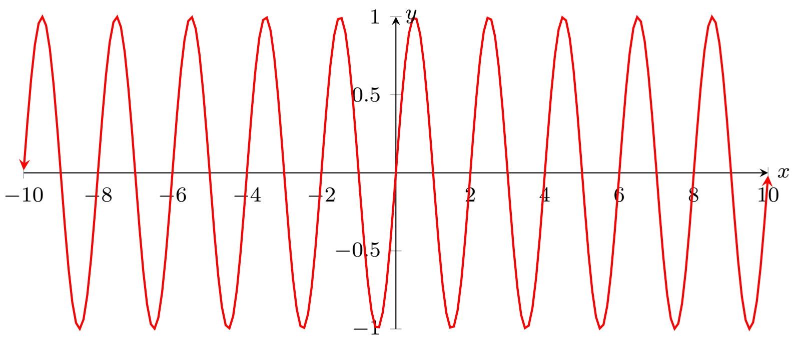

Figure 2

Definition 13 (Midline). The **midline** of a periodic function is the horizontal line midway between the function's minimum and maximum values.

If $y = f(t)$ is periodic and f_{max} and f_{min} are the maximum and minimum values of f, then the equation of the midline is:

$$y = \frac{f_{max} - f_m in}{2}$$

.

Definition 14 (Amplitude). The **amplitude** of a period function is the distance between the function's maximum value and the midline (or the function's minimum value and the midline).

Exercise 6 (Solution 6). Find the midline and amplitude of the function graphed below:

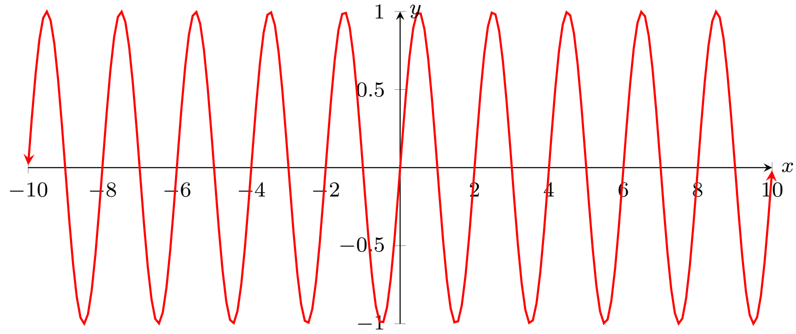

Figure 3

Lecture 4: Part 1: Intro to Trig Functions

Apr 12 2022 (16:27:32)

Definition 15 (Unit Circle)**.** A **unit circle** is a circle with a radius of 1 unit.

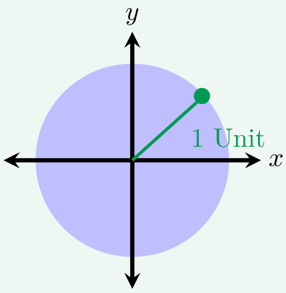

Figure 4: Unit Circle

Now we can define the sin and cos functions.

Note. There are 4 other trigonometric functions that can be defined in terms of sin and cos functions so first we'll get familiar with sin and cos and then later in the next lecture we'll define the 4 other trigonometric functions.

Definition 16 (Sine and Cosine)**.** The **sine function**, denoted $\sin(\theta)$, associates each angle θ with the vertical coordinate of the point P specified by θ on the circumference of a unit circle.

The **cosine function**, denoted $\cos(\theta)$, associates each angle θ with the horizontal coordinate of the point P specified by θ on the circumference of a unit circle.

So the point P in the figure below has coordinates:

$$(x, y) = (\cos(\theta), \sin(\theta))$$

.

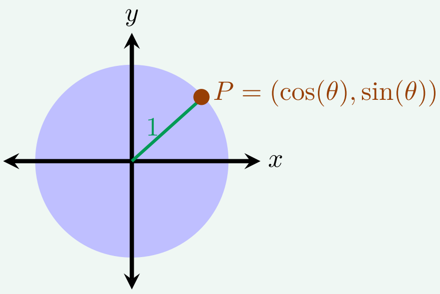

Figure 5: Sin and Cos on Graph

Exercise 7 (Solution 7)**.** The angle of θ specifies the point $P = \left(-\frac{3}{5}, \frac{4}{5}\right)$ on the circumference of a unit circle. Find $\sin(\theta)$ and $\cos(\theta)$.

Let's determine the signs of the sin and cos functions in the four quadrants:

- When the terminal side of angle θ is in Quadrant I, both the x and y coordinates of point P are positive. Therefore, **if θ is in Quadrant I,** $\cos(\theta) > 0$ **and** $\sin(\theta) > 0$
- When the terminal side of angle θ is in Quadrant II, the y coordinate of point P is positive but the x coordinate is negative. Therefore, **if θ is in Quadrant II,** $\cos(\theta) < 0$ **and** $\sin(\theta) > 0$
- When the terminal side of angle θ is in Quadrant III, both the x and y coordinates of point P are negative. Therefore, **if θ is in Quadrant IIII,** $\cos(\theta) < 0$ **and** $\sin(\theta) < 0$
- When the terminal side of angle θ is in Quadrant IIII, the x coordinate of point P is positive but the y coordinate is negative. Therefore, **if θ is in Quadrant IIII,** $\cos(\theta) > 0$ **and** $\sin(\theta) < 0$

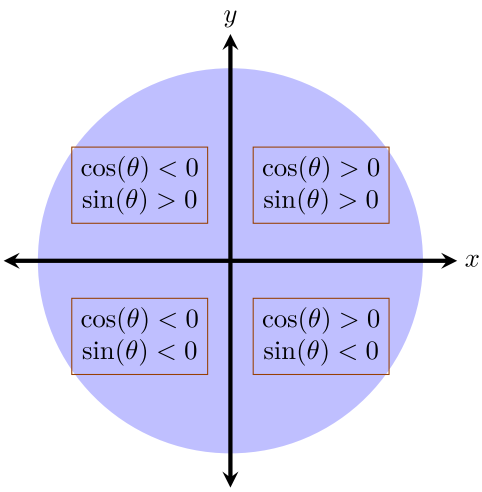

Figure 6: Quadrants of a Graph

Now let's find the sin and cos of a few particular angles. The easiest points for us to find on the unit circle are points where the circumference of the circle intersects the coordinate axes. Let's start by finding the corresponding sin and cos values.

Note. Keep in mind that cos represents the x-coordinate and sin represents the y-coordinate.

- The angle $\theta = 90°$ ($\theta = \frac{\pi}{2}$ radians), specifies the point $(0, 1)$ on the circumference of a unit circle. Thus . . .

$$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$$

.

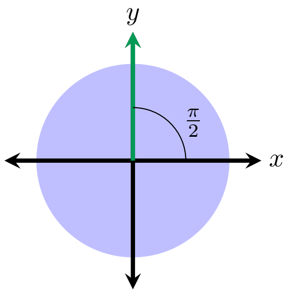

Figure 7

- The angle $\theta = 180°$ ($\theta = \pi$ radians), specifies the point $(-1, 0)$ on circumference of a unit circle. Thus . . .

$$\cos(\pi) = -1 \qquad \sin(\pi) = 0$$

.

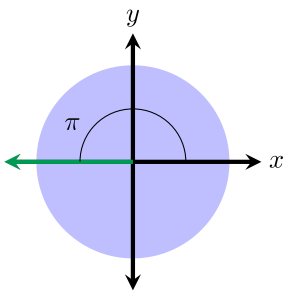

Figure 8

- The angle $\theta = 270°$ ($\theta = \frac{3\pi}{2}$ radians), specifies the point $(0, -1)$ on the circumference of a unit circle. Thus . . .

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1$$

.

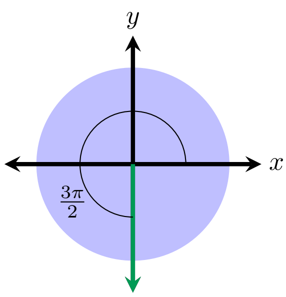

Figure 9

- The angle $\theta = 360°$ ($\theta = 2\pi$ radians), specifies the point $(1, 0)$ on the circumference of a unit circle. Thus ...

$$\cos(2\pi) = 1 \qquad \sin(2\pi) = 0$$

.

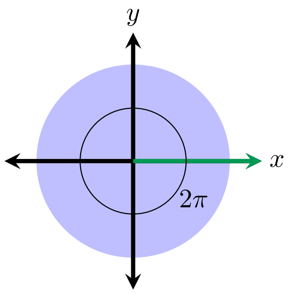

Figure 10

Notice that angles of measure 2π radians and 0 radians specify the same point: $(1, 0)$. Thus, the sin and cos values for 2π and 0 radians are the same:

$$\cos(2\pi) = \cos(0) = 1 \qquad \text{and} \qquad \sin(2\pi) = \sin(0) = 0$$

.

Since any angle θ and $\theta + 2\pi$ specify the same point on the unit circle, the sin and cos values of θ and $\theta + 2\pi$ are the same. Therefore, the period of the sin and cos function is 2π radians.

Theorem 1. For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$ so the period of both $s(\theta) = \sin(\theta)$ and $c(\theta) = \cos(\theta)$ is 2π radians.

Now, we'll sketch graphs of the sin and cos functions.

We first start by organizing the function values in a table:

θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$	$540°$	$630°$	$720°$
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$y = \cos(\theta)$	1	0	-1	0	1	0	-1	0	1
$y = \sin(\theta)$	0	1	0	-1	0	1	0	-1	0

Table 2: Values of Sin and Cos

Here's how it looks like when we graph:

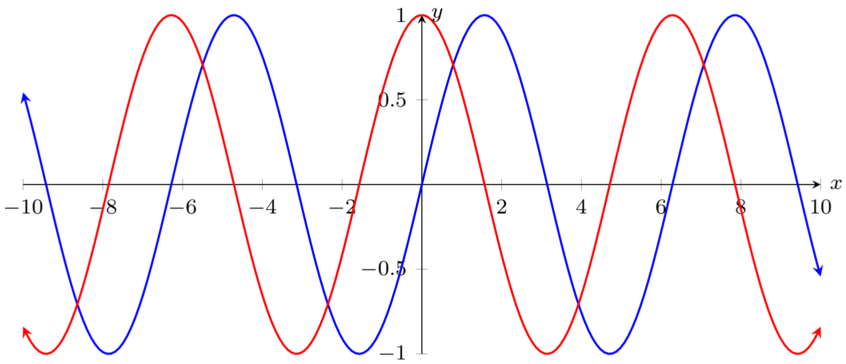

Figure 11: The graph of $y = \cos(\theta)$ and $y = \sin(\theta)$

Notice that the graphs of $y = \cos(\theta)$ and $y = \sin(\theta)$ are very similar. In fact, if we shift $y = \sin(\theta)$ to the left $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \cos(\theta)$. This means that:

$$\cos = \sin\left(\theta + \frac{\pi}{2}\right)$$

.

Similarly, if we shift $y = \cos(\theta)$ to the right $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \sin(\theta)$. This means that:

$$\sin = \cos\left(\theta - \frac{\pi}{2}\right)$$

.

Definition 17 (Identity). An **identity** is an equation that is true for all values in the domains of the involved expressions.

Some important trig identities

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(\theta) = \cos(2\pi - \theta)$
- $\sin(\theta) = \cos(\pi - \theta)$

We can generalize the definitions of sin and cos functions so that they are applicable to circles of any size, rather than only for unit circles.

Definition 18 (More Applicable version of Sin and Cos)**.** If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r then:

$$\cos(\theta) = \frac{x}{r} \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r}$$

.

Note. If $r = 1$, then this definition $\cos(\theta)$ and $\sin(\theta)$ are equivalent to what we saw at the beginning of this lecture.

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

.

If we solve the equations $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ for x and y, we can obtain the coordinates of a point on the circumference of a circle of any r :

$$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta) \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$$

.

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$

.

Exercise 8 (Solution 8)**.** A circle with a radius of 6 units is given. The point Q is specified by the angle α. Use the sin and cos function to express the exact coordinates of point Q.

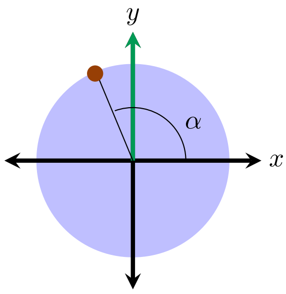

Figure 12

Recall that the cos and sin functions represent the horizontal and vertical coordinates of a point on the circumference of a unit circle. This situation creates a right triangle with hypotenuse of length 1 unit and side-lengths of $\cos(\theta)$ and $\sin(\theta)$ units.

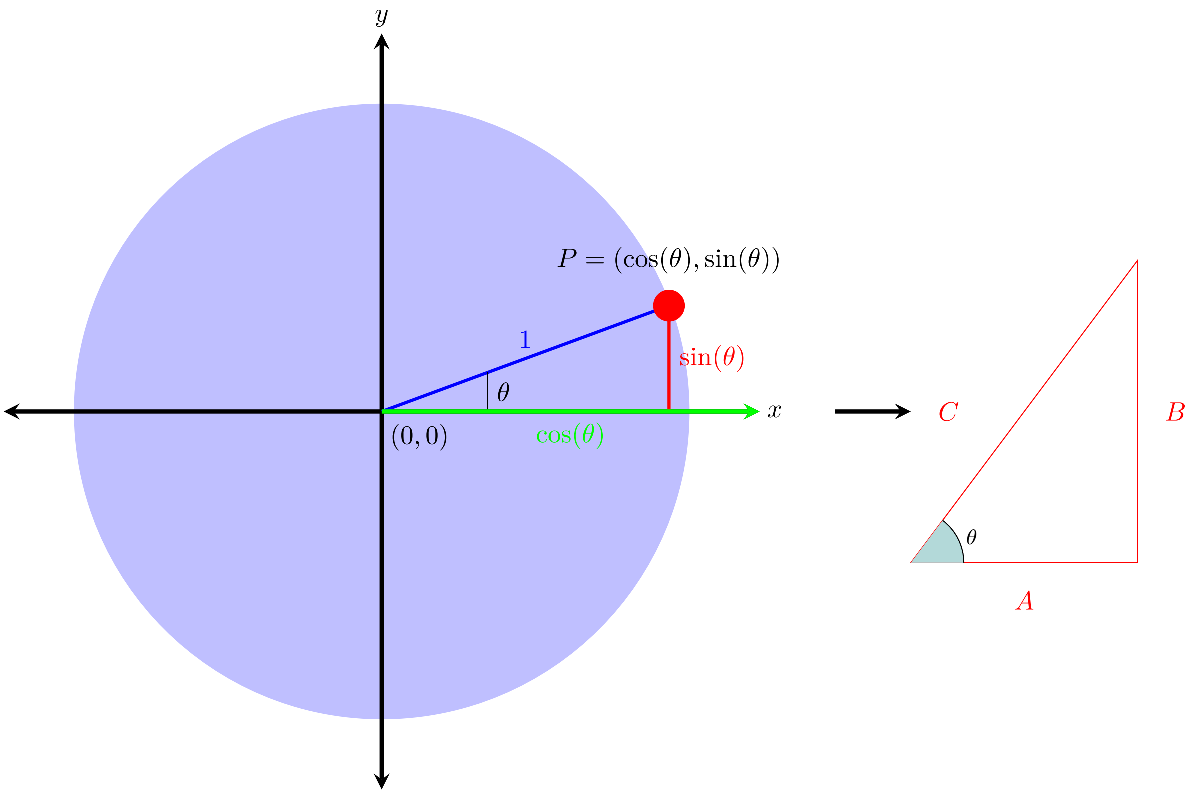

Figure 13: Diagram of a Circle

Now we can apply the Pythagorean Theorem to this right triangle. First let's review the Pythagorean Theorem:

Theorem 2 (Pythagorean Theorem)**.** If the sides of a right triangle are labeled. Then . . .

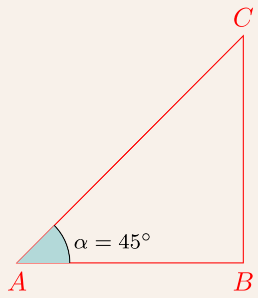

Figure 14: Right Triangle

$$a^2 + b^2 = c^2$$

.

Applying the Pythagorean Theorem to the right triangle we obtain what is called the Pythagorean Identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

. When we use exponents in trigonometric functions we can use unusual notation. Instead of using parentheses around the entire expression, we can put the exponents between the letters that name the function and the input for the function. Thus, we can write an expression like $(\sin(\theta))^2$ as $\sin^2(\theta)$. We can use this notation to express the Pythagorean Identity:

Notation.

$$\theta \in \mathbb{R},\ \sin^2(\theta) + \cos^2(\theta) = 1$$

.

Exercise 9 (Solution 9)**.** If $\sin(A) = \frac{1}{3}$ and $\frac{\pi}{2} < A < \pi$, which means A is in Quadrant II, find $\cos(A)$.

Lecture 5: Part 2: Intro to Trig Functions

Apr 16 2022 (17:10:40)

There are four other trigonometric functions besides sin and cos. These four functions are defined in terms of sin and cos functions:

Definition 19 (The Other 4 Trig Functions). The **tangent function**, denoted $\tan(\theta)$, is defined by:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

.

The **cotangent function**, denoted $\cot(\theta)$, is defined by:

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

.

The **secant function**, denoted $\sec(\theta)$, is defined by:

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

.

The **cosecant function**, denoted $\csc(\theta)$, is defined by:

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

.

The tangent function is the ratio of the sin and cos functions so it provides truly different information than does sin or cos. So the tan function is important. But the cot, sec, and csc are just reciprocals of tan, cos, and sin. So they don't provide any new information, so they are arguably less important functions.

There are 2 other identities that can be obtained from the Pythagorean Identity.

One of these identities can be found by dividing both sides of the Pythagorean Identity by $\cos^2(\theta)$:

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow & \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \\ \Rightarrow & \qquad \tan^2(\theta) + 1 = \sec^2(\theta). \end{aligned}$$

Alternatively, we can divide both sides of the Pythagorean Identity by $\sin^2(\theta)$ and find another identity.

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow & \frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \\ \Rightarrow & \qquad 1 + \cot^2(\theta) = \csc^2(\theta). \end{aligned}$$

This gives us three identities that are considered **"The Pythagorean Identities"**:

The Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$1 + \cot^2(\theta) = \csc^2(\theta).$$

Lecture 6: Part 3: Intro to Trig Functions

Apr 16 2022 (18:04:27)

Now let's determine the sin and cos of some important angles, namely, 30°, 45°, and 60° ($\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians). We focus on these angles since we can use some basic geometry to easily find their sin and cos values – but we cannot easily find the sin and cos of most other angles.

Let's start by finding the sin and cos of 30° ($\frac{\pi}{6}$ radians). The fact about triangles is the sum of all the angles is equal to 180°. So, we get:

$$30^\circ + 90^\circ + c = 180^\circ \Rightarrow c = 60$$

.

So, here's how our triangle will look like:

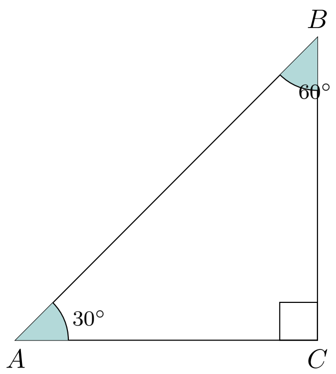

Figure 15: Our Original Triangle

Let's take this triangle, flip it, and put it right next to the original. Here's how it looks after we did this:

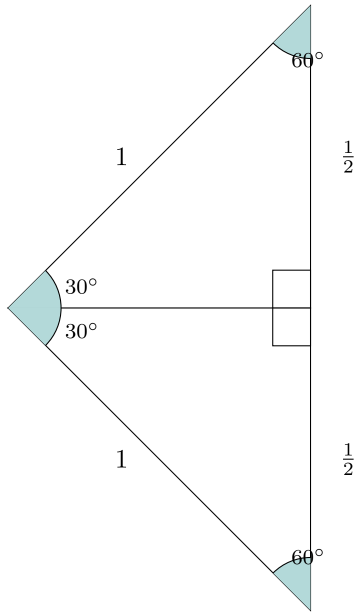

Figure 16: An isosceles triangle

This means that $b = \frac{1}{2}$ and from this, we can use the Pythagorean Theorem to find the value for a:

$$\left(\frac{1}{2}\right)^2 + a^2 = 1^2 \Rightarrow a = \frac{\sqrt{3}}{2}$$

.

Here's the final triangle:

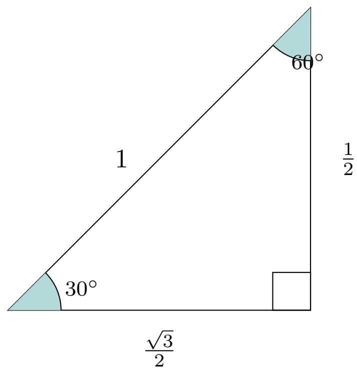

Figure 17: Triangle with all sides and angles

$$P = (\cos(30^\circ), \sin(30^\circ)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

.

So, using the definition of sin and cos, here's our final answer:

$$P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow \qquad \sin(30^\circ) = \frac{1}{2}, \ \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

.

Here's a table of the most common values for sin and cos:

θ (degrees)	30°	45°	60°
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

Table 3: Common Values for Sin and Cos

Exercise 10 (Solution 10)**.** Find $\tan\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, and $\cot\left(\frac{\pi}{6}\right)$.

Exercise 11 (Solution 11)**.** Find $\tan\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, and $\cot\left(\frac{\pi}{4}\right)$.

Exercise 12 (Solution 12)**.** Find $\tan\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, and $\cot\left(\frac{\pi}{3}\right)$.

Lecture 7: Part 4: Intro to Trig Functions

Apr 16 2022 (21:49:19)

Recall that the sin and cos functions represent the coordinates of points in the circumference of a unit circle. We found the values for 30°, 45°, and 60° by finding the coordinates of the points on the circumference of the unit circle specified by these angles. The points we found were all in Quadrant I, but since a circle is symmetric about both the x and y axes, we can reflect these points about the coordinate axes to determine the coordinates of corresponding points in the other quadrants. This means we can use the sin and cos values of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ to find the sin and cos values of corresponding angles in other quadrants.

Because of the symmetry of a circle, we can take a point in Quadrant I and reflect it about the x-axis, the y-axis, and about both axes in order to obtain corresponding points, one in each of the three other quadrants; the absolute value of the coordinates of all four of these points is the same, i.e., they only differ by their signs.

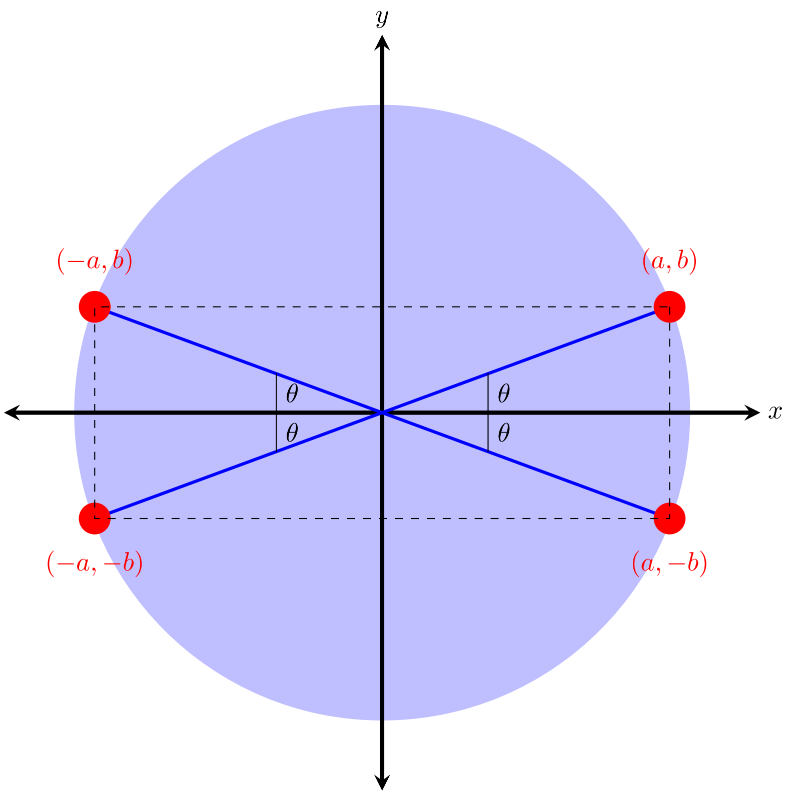

Figure 18: Plot of (a, b) specified by angle θ with the other points in all the quadrants.

Although all four of the points are specified by a different angle, all four of the angles share the same reference angle, θ.

Definition 20 (Reference Angle)**.** The **reference angle** for an angle is the acute angle between the terminal side of the angle and the x-axis.

Exercise 13. Find the reference angle for $150°$ and $\frac{5\pi}{4}$.

Now let's find out how we can use reference angles to determine the sin and cos of any integer multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Lecture 8: Sinusoidal Functions

Apr 23 2022 (19:15:57)

Definition 21 (Sinusoidal Function). A **Sinusoidal Function** is a function of the form:

$$y = A\sin(\omega(t-h)) + k \text{ or } y = A\cos(\omega(t-h)) + k$$

.

where $A, \omega, h, k \in \mathbb{R}$

Based on what we know about graph transformations, we should recognize that a sinusoidal function is a transformation of $y = \sin(t)$ or $y = \cos(t)$. Consequently, sinusoidal function are waves with the same curvy shape as the graphs of sin and cos but with different periods, midlines, and/or amplitudes.

Summary of Graph Transformation

Suppose that f and g are functions such that $g(t) = A \times f(\omega(t-h)) = k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g ...

1 horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis.

2 shift the graph horizontally h units (shift right if h is positive and left if h is negative).

3 vertically stretch/compress the graph by a factor of $|A|$ and, if $A < 0$, reflect it about the t-axis.

4 shift the graph vertically k units (shift up if k is positive and down if k is negative).

Note. The order in which these transformations are performed matters.

Exercise 14 (Solution 14). Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $g(t) = 2\sin(t) - 3$. State the period, midline, and amplitude of $y = g(t)$.

Here's a quick summary of sinusoidal functions:

The graphs of the sinusoidal functions

$$y = A\sin(\omega(t-h)) + k \text{ and } y = A\cos(\omega(t-h)) + k$$

. where $A, \omega, h, k \in \mathbb{R}$ have the following properties:

period: $\frac{2\pi}{|\omega|}$ units
midline: $y = k$
amplitude: $|A|$ units
horizontal shift: h units
angular frequency: ω radians per unit of t

Now let's try to sketch the graph of:

$$f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$$

.

To do this, we need to find the midline, amplitude, horizontal shift, and period. We can determine the midline by noticing that this function has a -3, which causes the sin wave to be shifted down 3 units. The amplitude can be determined by the number outside the function, which will cause a vertical stretch. So, our amplitude is 2 units. To find the horizontal shift, we first need to factor out π from the function's input.

$$f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$$

. The reason we did this is because we need to find out how much we're moving left or right, and $-\frac{1}{4}$ indicates that we're moving $\frac{1}{4}$ units to the right. The period comes from two features of the formula. The first one is we're using the trig function sin, which has a period of 2π units and the second feature is we have π multiplying all of the numbers in the input, which horizontally compresses the function.

$$2\pi \times \frac{1}{\pi} = 2 \text{ units}$$

.

Here's the final graph:

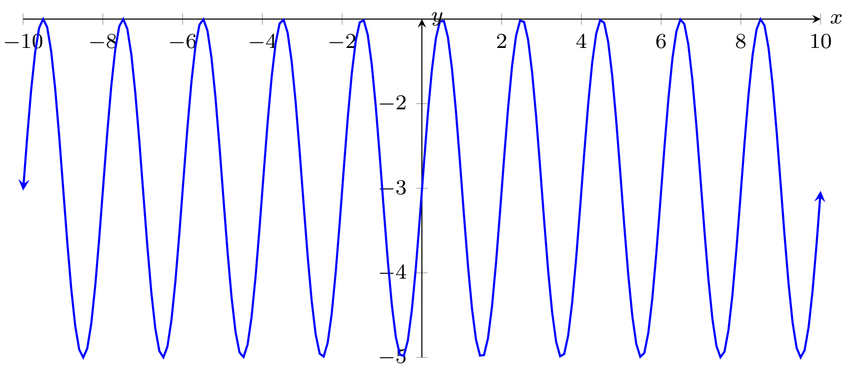

Figure 19

Exercise 15 (Solution 15). Find two different algebraic rules (one involving sin and one involving cos) for the function $y = g(t)$ graphed below:

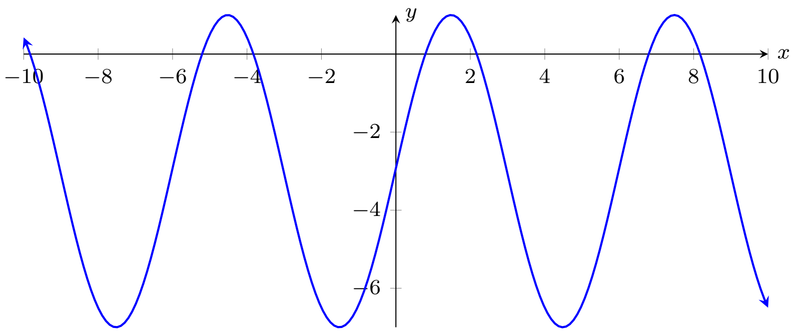

Figure 20

Solutions

Solution 1 (Exercise 1).

1 This represents a set since it's well defined. We all know what it means to be registered for a class.

2 This does not represent a set since it's not well defined. There are many different interpretations of what it means to be a good student (get an **A** or pass the class or attend class or avoid falling asleep in class or don't cause trouble in class)

Solution 2 (Exercise 2).

- $(-4, \infty) \cup [-8, 3] = [-8, \infty)$
- $(-4, \infty) \cup (-\infty, 2] = (-\infty, \infty) = \mathbb{R}$
- $(-4, \infty) \cap (-\infty, 2] = (-4, 2]$
- $(-4, \infty) \cap [-10, -5] = \emptyset$

Solution 3 (Exercise 3). Since $2\pi = 360°$ $\quad \frac{2\pi}{360°} = 1 = \frac{360°}{2\pi}$. We're trying to cancel out the radians. So, to do this, we will need to have the radians on the bottom, and the degrees on the top. After we do the multiplication, we will be left with the degrees.

$$\begin{aligned} 8\,\text{rad} \times \left(\frac{360°}{2\pi\,\text{rad}}\right) &= \frac{8 \times 360°}{2\pi} \\ &= \frac{1440°}{\pi} \\ &\approx 458.4. \end{aligned}$$

Now, let's convert $8°$ into 8 radians.

$$\begin{aligned} 8° \left(\frac{\pi\,\text{rad}}{180°}\right) &= \frac{8\pi}{180}\,\text{rad} \\ &= \frac{\pi}{45}\,\text{rad} \end{aligned}$$

.

Solution 4 (Exercise 4). Convert $40°$ into radians:

$$
\begin{aligned}
40^\circ \times \left(\frac{\pi}{180^\circ}\right) &= \frac{40\pi}{180} \text{ rad} \\
&= \frac{4\pi}{18} \\
&= \frac{2\pi}{9} \\
&\approx 0.7
\end{aligned}
$$

.

Now we're ready to compute the arc length:

$$
\begin{aligned}
s &= r\times \mid \theta \mid \\
&= (30m) \times (\frac{2\pi}{9}) \\
&= \frac{20\pi}{3} m.
\end{aligned}
$$

Solution 5 (Exercise 5). The **period** for the graph is 2. The reason is because if you shift the graph 2 units either left or right, you will get the same graph.

Another way to find the period is the take an x-intercept and the x-intercept to the right of it and subtract the two: $10 - 8 = 2$

Solution 6 (Exercise 6). The **midline** of the graph is 0.

$$
\frac{1 + (-1)}{2} = \frac{0}{2} = 0
$$

.

The **amplitude** of the graph is 1, which is the distance from the maximum or minimum to the midline.

Solution 7 (Exercise 7).

$$
\begin{aligned}
P &= (\cos(\theta), \sin(\theta)) \\
&= \cos(\theta) = -\frac{3}{5} \\
&= \sin(\theta) = \frac{4}{5}.
\end{aligned}
$$

Solution 8 (Exercise 8). The point Q is specified by α on the circumference of a circle of radius 6 units. Thus ...

$$
Q = (6\cos(\alpha), 6\sin(\alpha))
$$

.

Solution 9 (Exercise 9)**.** Since the Pythagorean Identity gives us an equation involving sin and cos, we can use it to find one of the values when we know the other value. In this case, we know the value of $\sin(A)$, so we can use the Pythagorean Identity to find $\cos(A)$:

$$\begin{aligned}
&\sin^2(A) + \cos^2(A) = 1 \\
\Rightarrow &\left(\frac{1}{2}\right)^2 + \cos^2(A) = 1 \\
\Rightarrow &\cos^2(A) = 1 - \left(\frac{1}{3}\right)^2 \\
\Rightarrow &\cos(A) = -\sqrt{\frac{8}{9}} \\
\Rightarrow &\cos(A) = -\frac{2\sqrt{2}}{3}.
\end{aligned}$$

The reason I made the answer negative is because the value of cos in Quadrant II is negative. If cos was in Quadrant I, then the answer would be positive.

Solution 10 (Exercise 10)**.**

$$\begin{aligned}
\tan\left(\frac{\pi}{6}\right) &= \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\
&= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec\left(\frac{\pi}{6}\right) &= \frac{1}{\cos\left(\frac{\pi}{6}\right)} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} \\
&= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc\left(\frac{\pi}{6}\right) &= \frac{1}{\sin\left(\frac{\pi}{6}\right)} \\
&= \frac{1}{\frac{1}{2}} \\
&= 2 \\
\cot\left(\frac{\pi}{6}\right) &= \frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= \sqrt{3}
\end{aligned}$$

.

Solution 11 (Exercise 11).

$$
\begin{aligned}
\tan\left(\frac{\pi}{4}\right) &= \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} \\
&= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\
&= 1 \\
\sec\left(\frac{\pi}{4}\right) &= \frac{1}{\cos\left(\frac{\pi}{4}\right)} \\
&= \frac{1}{\frac{\sqrt{2}}{2}} \\
&= \frac{2}{\sqrt{2}} = \sqrt{2} \\
\csc\left(\frac{\pi}{4}\right) &= \frac{1}{\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{1}{\frac{\sqrt{2}}{2}} \\
&= \frac{2}{\sqrt{2}} = \sqrt{2} \\
\cot\left(\frac{\pi}{4}\right) &= \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} \\
&= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\
&= 1
\end{aligned}
$$

.

Solution 12 (Exercise 12).

$$\begin{aligned}
\tan\left(\frac{\pi}{3}\right) &= \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= \sqrt{3} \\
\sec\left(\frac{\pi}{3}\right) &= \frac{1}{\cos\left(\frac{\pi}{3}\right)} \\
&= \frac{1}{\frac{1}{2}} \\
&= 2 \\
\csc\left(\frac{\pi}{3}\right) &= \frac{1}{\sin\left(\frac{\pi}{3}\right)} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} \\
&= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\end{aligned}$$

.

Solution 13 (Exercise 13). The reference angle for $150°$ is $30°$.
The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$.

Solution 14 (Exercise 14). It should be clear that function g is a sinusoidal function of the form $y = A\sin(\omega(t-h)) + k$ where $A = 2, \omega = 1, h = 0, k = -3$.

After inspecting the rules for the functions f and g, we should notice that we could construct the function $g(t) = 2\sin(t) - 3$ by multiplying the outputs of the function $f(t) = \sin(t)$ by 2 and then subtracting 3 from the result. We can express this algebraically with the equation below:

$$g(t) = 2f(t) - 3$$

.

Based on what we know about the graph transformations, we can conclude that we can obtain graph of g by starting with the graph of f and first stretching it vertically by a factor of 2 and then shifting it down 3 units. Since $f(t) = \sin(t)$ has amplitude of 1 unit, if we stretch it vertically by a factor of 2, then we'll double the amplitude, so we should expect that the amplitude of g to be 2 units. Also, since $f(t) = \sin(t)$ has midline $y = 0$, when we shift it down 3 units, the resulting midline for the function g will be $y = -3$.

period: 2π units
midline: $y = -3$
amplitude: 2 units
horizontal shift: 0 units

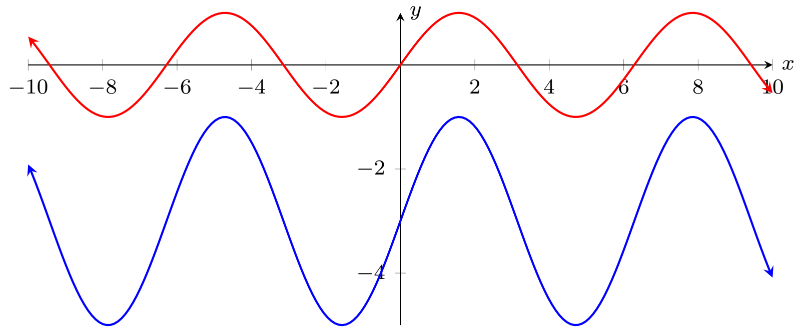

Figure 21: The graph of $f(t) = \sin(t)$ and $g(t) = 2\sin(t) - 3$

Solution 15 (Exercise 15)**.** Let's start by finding the values of A, ω, k. To do this, we need to find the midline, amplitude, and period.

The midline is $y = -3$, which dictates what the value of k is.
The amplitude is the distance from the maximum/minimum to the midline. The amplitude is 4 units.
The period is the distance from each maximum/minimum. $6.5 - 0.5 = 6$ units. The period doesn't actually appear as a number in the algebraic rule for the function. But the period is connected with the value that we use of ω. ω causes a horizontal stretch which gives us the desired period.

$$6 = 2\pi \times \frac{1}{\omega} = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

. When we're trying to find a formula that involves sin, we want to think in terms of sin waves, which start at the midline and come up out of the midline. So, we'll want to find a place in this function where the function is crossing through the midline and coming up out of the midline. That would be at the point: $(0, -1)$. So, for the sin function, $h = -1$. For cos, it's the opposite. Cos starts at the maximum, and goes down to the midline. That would be at the point: $(0, 0.5)$. So, for the cos function, $h = 0.5$.

$$g(t) = 4\sin\left(\frac{\pi}{3}(t+1)\right) - 3 \qquad g(t) = 4\cos\left(\frac{\pi}{3}\left(t - \frac{1}{2}\right)\right) - 3$$

.