Trigonometry

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Contents

Part I

The Basic Tools

Chapter 1

Geometry Tools Needed to Study Trigonometry

1.1 Right Triangles

There are three main types of angles:

- Acute Angles
- Right Angles
- Obtuse Angles

Definition 1 (Angles).
An **acute angle** is an angle with a measure less than 90°.
An **right angle** is an angle with a measure of 90°.
An **obtuse angle** is an angle with a measure greater than 90°.

Recall from geometry that you need three known measures to describe a triangle – the lengths of two sides and the measure of an angle, the lengths of three sides, or three other measures. For a right triangle, you need only two additional measures, because you already know that one of the angles' measure is 90°.

The following four cases describe all possible ways to describe a right triangle:

1. The lengths of two legs
2. The lengths of one leg and the hypotenuse
3. The lengths of one leg and the measure of an acute angle
4. The length of the hypotenuse and the measure of one acute angle

Suppose you need to calculate the lengths of all three sides. For cases 1 and 2, you only need geometry, specifically, the Pythagorean Theorem. But for cases 3 and 4, you need trig to get the remaining sides.

1.1.1 The Pythagorean Theorem

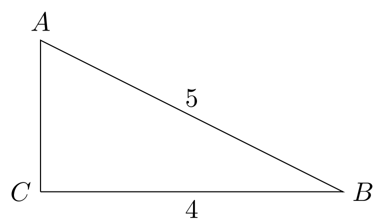
Let's take a look at the famous geometry tool *Pythagorean Theorem*, which helps us solve for cases 1 and 2.

Theorem 1 (Pythagorean Theorem)**.** The **Pythagorean Theorem** states that if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then:

$$a^2 + b^2 = c^2.$$

Exercise. Let's illustrate how cases 1 and 2 can be solved with the help of the Pythagorean Theorem.

Two legs of a right triangle measure 5 and 12 inches. Find the hypotenuse.



Since you know the length of the two legs, you can substitute those into the theorem's equation:

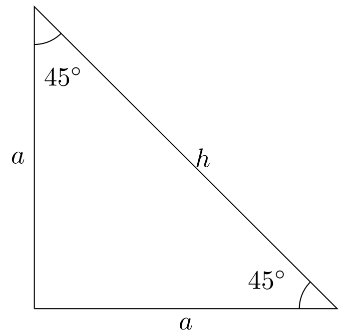
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 25 + 144 &= c^2 \\ 169 &= c^2 \\ \sqrt{169} &= c \\ c &= 13. \end{aligned}$$

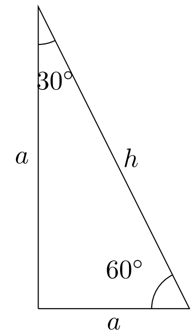
Solution.

The length of the hypotenuse is 13 inches long.

1.1.2 Special Right Triangles

In this section, we'll take a look at two special kinds of right triangles. For these special triangles, if you know the length of any one side of the triangle, you can find the length of the other two sides.

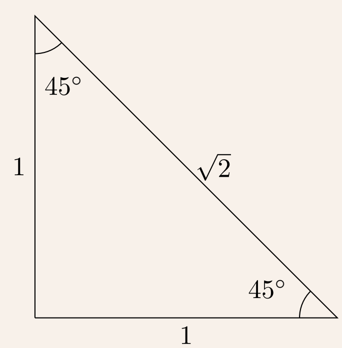




The triangle on the left is called a $45° - 45° - 90°$ triangle, or an isosceles right triangle. The triangle on the right is called a $30° - 60° - 90°$ triangle. The Pythagorean Theorem is used to prove the following relationship that exist in these two special right triangles.

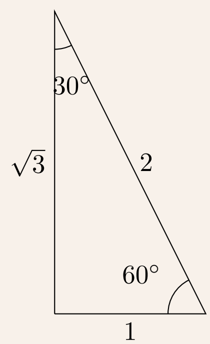
Definition 2 (Isosceles Right Triangle)**.** An **isosceles right triangle** is a triangle that has two equal sides and two equal angles, as well as a third angle that is a right angle. The only triangle that satisfies all these conditions is a $45° - 45° - 90°$ triangle.

Theorem 2. If each acute angle of a right triangle measures $45°$, the hypotenuse is $\sqrt{2}$ times as long as a leg.



Example. If one leg in a $45° - 45° - 90°$ triangle is 10 units long, then the hypotenuse is $10\sqrt{2}$ units long.

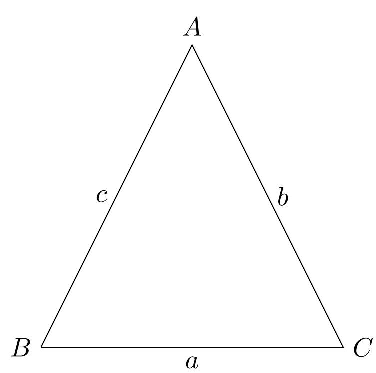
Theorem 3. If acute angles of a right triangle measure 30° and 60°, then the hypotenuse is twice as long as the shortest leg and the longer leg is $\sqrt{3}$ times as long as the shortest leg.



Example. If the shortest leg in a $30° - 60° - 90°$ triangle is 6 units long, then the hypotenuse is $2 \times 6 = 12$ units long, and the longest leg is $6\sqrt{3}$ units long.

1.1.3 Classifying Triangles

Any triangle is six elements: three sides and three angles. Throughout this book, I use capital letters (A, B, C) to denote measures of angles and lowercase letters. I also use lowercase letters (a, b, c) to denote sides of the triangles I use the small letter corresponding to the name of the angle opposite to this side. I also use the symbol $\triangle$ to identity a triangle. Therefore, an expression $\triangle ABC$ means "the triangle ABC".



When given the lengths of three sides, you can construct a triangle that is determined by these sides; therefore, its angles are determined as well. However, you need to be careful thinking this way, because not all arbitrary side lengths enable us to construct a triangle. Before constructing a triangle given its sides, you need to refer to the *Triangle Inequality*.

Definition 3 (Triangle Inequality)**.** The **triangle inequality** states that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Example. Let's state whether it's possible for a triangle to exist with sides of the given lengths:

Sample 1: The given sides are 4, 5, and 6 units long. You need to compare the sums of any two sides with the third side. That means you need to exhaust all possibilities:

$$4 + 5 = 9 > 6$$
$$4 + 6 = 10 > 5$$
$$5 + 6 = 11 > 4.$$

Sample 2: The given sides are 1, 2, and 3 units long. Let's try all possibilities:

$$2 + 3 = 5 > 1$$
$$3 + 1 = 4 > 2$$
$$1 + 2 = 3 = 3.$$

The sum of these two sides (1 and 2) are not greater than the third side (3). Therefore, this triangle doesn't exist.

A triangle can be classified as acute (having three acute angles), right (having one right angle), and obtuse (having one obtuse angle). As we discussed, the lengths of the sides of a triangle determine its angles. Given the lengths of the sides, can you tell whether the triangle is acute, right, or obtuse?

The pythagorean theorem gives us a partial answer. We know that if the lengths satisfy the relationship $a^2 + b^2 = c^2$, then we know it's a right triangle. If it doesn't, then it's either an acute or obtuse triangle.

$$\text{If } \angle C \text{ of } \triangle ABC \text{ is acute, then } c^2 < a^2 + b^2$$
$$\text{If } \angle C \text{ of } \triangle ABC \text{ is obtuse, then } c^2 > a^2 + b^2.$$

Using this information and given the sides' lengths, you can identify the type of triangle you are dealing with.

Example. If the sides are 6, 7, 8, by substituting those values, you get:

$$8^2 = 64 < 6^2 + 7^2 = 36 + 49 = 85.$$

So, this triangle is an acute triangle.

Note. In a triangle, the hypotenuse is always bigger in length than either of the sides. Hence, we always know which number should be substituted for c since it's the largest one of the three.

If the sides are 6, 8, and 10, then:

$$10^2 = 100 = 6^2 + 8^2 = 36 + 64 = 100.$$

So the triangle is a right triangle.

If the sides are 5, 12, and 14, then:

$$14^2 = 196 > 5^2 + 12^2 = 25 + 144 = 169.$$

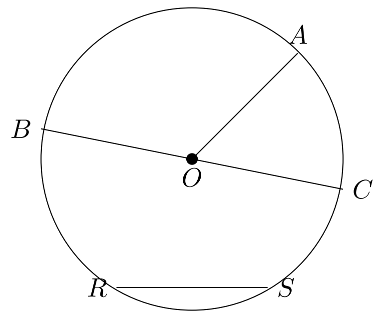
So the triangle is an obtuse triangle.

1.2 Circles, Arcs, and Chords

Another basic shape talked about in geometry is a circle. Let's review some basic elements of a circle because you use them a lot throughout trigonometry.

1.2.1 Chords and Arcs

Let's analyze the circle below and take note of its key elements. $\overline{OA}$ is a radius. $\overline{RS}$ is a chord. $\overline{BC}$ is a diameter.

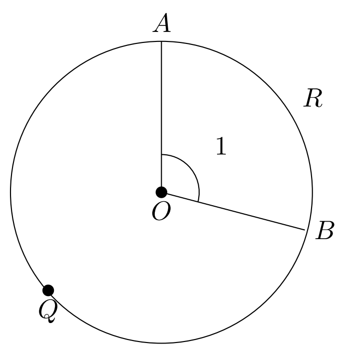


Definition 4 (Radius)**.** A **radius** of a circle is a segment that joins the center of a circle to a point on the circle.

Definition 5 (Chord)**.** A **chord** is a straight line segment joining two points that lie on the circumference of a circle. A chord whose endpoints lie opposite to each other on the circle and whose center passes through the circle's center is referred to as the diameter.

Definition 6 (Diameter)**.** A **diameter** is a chord that contains the center.

Now, let's discuss central angles and arcs of a circle.



$\angle 1$ is the central angle of the circle O. R is a minor arc between the points A and B, which is denoted as arc AB. The part that isn't between points A and B are referred to as the major arc AQB.

Note. To name a major arc, three letters must be used.

The measure of a minor arc is defined to be the measure of its central angle. If the measure of $\angle 1 = 40°$, you can write the measure of $AB = 40°$ as well. The measure of a major arc is calculated by subtracting $40°$ from $360°$:

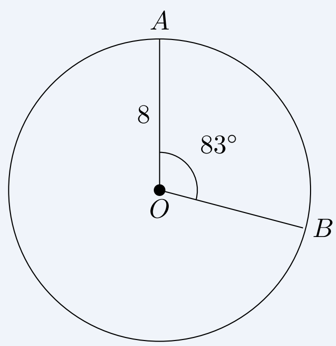
$$360° - 40° = 320°.$$

Definition 7 (Central Angle)**.** A **central angle** of a circle is an angle with a vertex at the center of the circle.

Definition 8 (Minor Arcj)**.** A **minor arc** of a circle is the union of two points on the circle and all the points of the circle that lie on the interior of the central angle with sides that contain the two points. Minor arcs measure less than 180°.

Definition 9 (Major Arc)**.** A **major arc** is an arc of a circle having measure greater than or equal to 180°.

Example.



In a circle of radius 8, how long is the chord of an arc of 83°? It turns out again that geometry doesn't have the tools to solve this problem. This problem provides a glimpse of types of problems that trigonometry will help solve.

1.2.2 Area of Circles

To find the area of a circle, you need to know its radius. You can then use the following formula to calculate its area.

$$A = \pi r^2.$$

Example. If the radius is 7 inches, the area of the circle can be calculated by substituting this value into the formula:

$$A = \pi r^2 \Rightarrow \pi 7^2 = 49\pi.$$

Chapter 2

Algebra Review

Basic algebra and trigonometry deal with two different areas of mathematics. Many people believe that trigonometry is a completely different discipline than algebra. But the truth is that isn't really the truth. You cannot fully comprehend trigonometry without knowing the basis of algebra, making algebra a prerequisite to trigonometry.

Algebra deals with finding the unknown variables and understanding functions – whereas trigonometry explores aspects of triangles, which are their sides and angles and finding a correlation between them. Algebra seeks to solve equations composed of multiple terms and to find their roots, whereas trigonometry focuses mainly on sin, cos, tan, degrees, radians, and polar coordinates.

It's helpful to imagine trigonometry as an extension to algebra because we often treat trigonometric concepts in algebraic terms. For example, solving a trigonometric equation is similar to solving an algebraic equation. You also learn the basic rules of vital topics like polynomials, exponents, graphing and much more in algebra. However, despite their similarities, there are still a number of differences. They are still both prerequisites to more advanced mathematical topics like calculus and differential equations.

2.1 Cartesian Coordinates

In algebra, you use the standard Cartesian coordinate system when graphing various functions. One of the most common methods of solving systems of equations is graphing them, where the solution is the point of intersection in all the lines.

> **Definition 10** (Cartesian Coordinate System). The **cartesian coordinate system** specifies each point on a plane by a pair of numerical coordinates, which are distances from the point to two fixed perpendicular lines, commonly referred to as the x-axis and y-axis.

The simplest coordinate system is a number line on which you represent numbers. Next, we represent ordered pairs of numbers in a plane on which two of these lines are located.

In plane geometry, there are two axes at right angles that are usually called x-axis and y-axis. The position of any point in the plane can be given by its two coordinates (x, y). These coordinates give the point's distance in the x and y directions from the origin, which is the point of intersection of the two axes. The origin is labeled with the pair of numbers both at zero – $(0, 0)$.

2.1.1 Writing Coordinates

The following drawing illustrates the coordinate system with four labeled points.

Note. Similar to the number line, you also can have negative values for x and y.

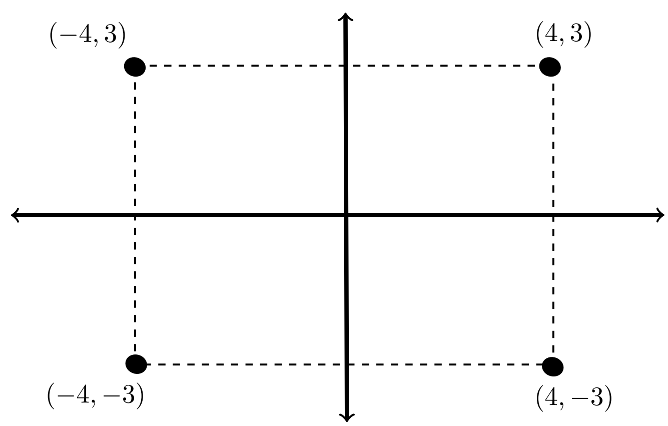


Figure 2.1: Coordinate System

The origin $(0, 0)$ is the starting point to determine distances when you need to graph other points. Two coordinates (x, y) are also called an ordered pair because the order is important. The value for the x-axis always comes first and the value for the y-axis always comes second.

2.1.2 Four Quadrants

The x-axis and y-axis of the Cartesian coordinate system divide the plane into four parts, which are called quadrants. They are numbered in a counterclockwise direction using Roman numerals.

Definition 11 (Quadrants). **Quadrants** are the four regions, which are divided by the coordinate plane.

Note. The values for the x-axis and y-axis are different for various coordinates in terms of their signs. For example, the values for the x-coordinates are positive in quadrants I and IV but negative in quadrants II and II. And the values for the y-coordinates are positive in quadrants II and II and negative in quadrants I and IV.

This information is useful when considering the signs of trigonometric functions in different quadrants.

Quadrant	x	y	**Example**
I	Positive	Positive	$(4, 6)$
II	Negative	Positive	$(-4, 6)$
III	Negative	Negative	$(-4, -6)$
IV	Positive	Negative	$(4, -6)$

When you need to locate a spot in the real world, you have to use three-dimensional coordinate system. In three dimensions, you have three axes at right angles. x-axis, y-axis, and z-axis.

Coordinates can be continued into four and more dimensions, which help mathematicians solve complex problems, because the higher you go, the easier the problem becomes.

2.2 What Are Functions

One of the most important tools in mathematics is a function. The term *function* first appeared in Gotta fried Leibniz's mathematics manuscript in 1673, which was later taken on by Leonhard Euler, who was one of the greatest mathematicians ever. He broadened the concept of a function and made it a core part of modern mathematics.

Definition 12 (Function). A **function** is a rule that assignments to every element in a set D exactly one element in a set R. The set D is called the domain of the function, and the set R is called the range.

Example. A is a set that contains all the names of your families and friends and their birthdays. The pairing of names and birthdays form a relation. In this relation, as with functions, the pairs of names and birthdays are ordered. This would imply that the name will always be the first bit of information in the pair and birthday will always be the second bit of information in the pair. The set of all the starting data, which are the names is called the **domain** and the set of all the ending data, which are the birthdays is called the **range**. The domain is what you start with and the range is what you end up with.

2.2.1 The Basics of Functions

A function is a relation represented by graphs, tables, and equations.

Definition 13. A **relation** is a relationship between two sets of information.

Entities are **ordered** if the order that they are presented in matters. Thus, if there are two entities, one comes first and the other comes second; this order should not be switched.

A **proper relation** is a type of relation in which given an x, you get only and exactly one y.

But a function is not just any relation but only a *proper relation*. This means that, although all functions are relations because functions pair two sets of information, not all relations are functions. This means that functions are sub classification of relations.

Example. As opposed to the previous case, let's flip things around now and assume that the domain is the set of everybody's birthdays. Let's imagine now that all family and friends have gathered for some event and someone ordered pizza, and she tells him only the birthday of the recipient. Too whom does the pizza guy deliver it to? What if nobody has the given birthday, or multiple people have a given birthday.

This means that the relation of "birthday indicates name" is not a proper relation, which implies that it isn't a function. Given the relationship $(x, y) = (\text{name}, \text{birthday})$, if there are four people with the same birthday, there will be four different possibilities for $y = \text{birthday}$. For a relation to be a function, there must be only and exactly one y that corresponds to a given x.

2.2.2 Inverse Functions

Recall that you can measure temperature using two scales: the Fahrenheit and the Celsius scales. The relationship between measurements in these two scales is as follows:

$$F^\circ = \frac{9}{5}C^\circ + 32$$
$$C^\circ = \frac{5}{9}(F^\circ - 32).$$

The first formula gives us a temperature in F° as a function of C°, and the second one gives us a temperature in C° as a function of F°. Because each formula undoes what the other one does, they are examples of *inverse functions*.

Definition 14 (Inverse Functions)**.** Two functions f and g are called **inverse functions** if the following statement is true:

- $g(f(x)) = x$ for all x in the domain of f
- $f(g(x)) = x$ for all x in the domain of g

An inverse function is denoted by f^{-1}.

The inverse of a function has all the same points as the original function, but the x's and y's are reversed.

Example. If the original function has points:

$$\{(2,3),(4,1),(6,-2)\}.$$

Then the inverse function has points:

$$\{(3,2),(1,4),(-2,6)\}.$$

In general, the graph of the inverse function f^{-1} can be obtained from the graph of f by changing every point (x, y) to the point (y, x). This means that the graph of f^{-1} is the reflection of the graph f in the line $y = x$.

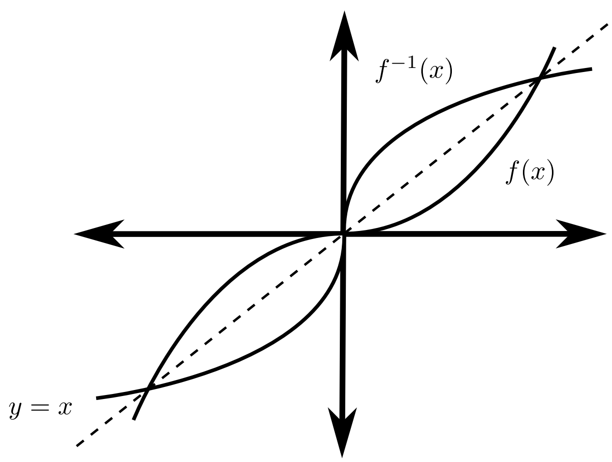


Figure 2.2: Graph of Inverse Function

The horizontal line test is used to determine whether a function has an inverse that is also a function.

Definition 15 (Horizontal Line Test)**.** If no horizontal line intersects a graph of a given function in more than one point, then this function has an inverse. This is called the **horizontal line test**.

For some functions, it is easy to find their inverses using basic algebraic operations.

Example. Let's find the inverse function of the following function:

$$\begin{aligned} f(x) &= \frac{x-1}{2} \\ y &= \frac{x-1}{2} \\ x &= \frac{y-1}{2} \\ 2x &= y-1 \\ y &= 2x+1. \end{aligned}$$

2.3 Complex Numbers

Definition 16 (Complex Numbers)**.** **Complex numbers** are numbers in the form $a+bi$, where a and b are real numbers, and i is the imaginary unit. a is the real part of such number and bi is the imaginary part of a complex number.

Definition 17 (Imaginary Number)**.** An **imaginary number** is a number with square that is negative. Imaginary numbers have the form bi, where b is a nonzero real number and i is the imaginary unit, defined such that $i^2 = -1$.

2.3.1 Operations with Complex Numbers

You can add or multiply two complex numbers by treating i as if it were a variable and using algebraic distribute laws:

$$\begin{aligned} &\text{Sample 1:} \qquad (5+41)+(9-2i) = 14+2i \\ &\text{Sample 2:} \qquad (2-3i)(5+4i) \quad = 10+8i-15i-12i^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad = 10-7i+12 \\ &\qquad\qquad\qquad\qquad\qquad\qquad = 22-7i \end{aligned}.$$

To perform complex numbers division, we must first be introduced to *complex conjugates*.

Definition 18 (Complex Conjugtes). The complex number $a + bi$ and $a - bi$ are called **complex conjugates**. Their sum is a real number, and their product is a non-negative real number. The conjugate of the complex number $z = a + bi$ is denoted by $\bar{z} = a - bi$.

Note. When dividing complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

$$
\begin{aligned}
\frac{3+4i}{2-3i} &= \frac{(3+4i)(2+3i)}{(2-3i)(2+3i)} \\
&= \frac{6+9i+8i+12i^2}{4+6i-6i-9i^2} \\
&= \frac{6+17i-12}{4+9} = \frac{-6+17i}{13} = -\frac{6}{13} + \frac{17}{13}i.
\end{aligned}
$$

Part II

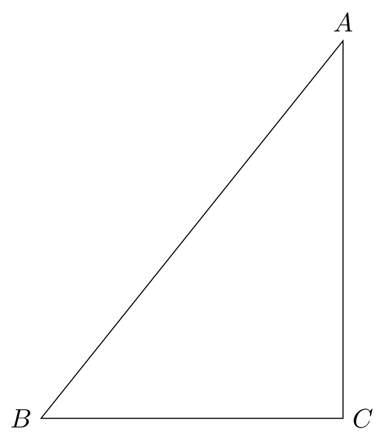
Triangle Trigonometry

Chapter 3

Trig Functions and Right Triangles

3.1 The Tangent Ratio

In the following figure, can we measure the distance of AC?



If $\triangle ABC$ is one of the special right triangles, then we can find AC.

For the $45° - 45° - 90°$ triangle, the ratio of its legs is $\frac{AC}{BC} = 1$.

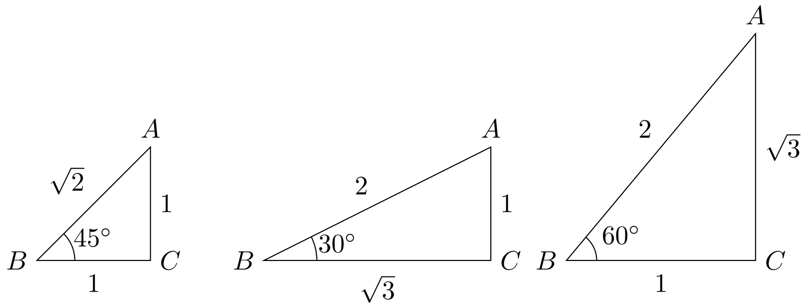
For the $30° - 60° - 90°$ triangle, the ratio of its legs is $\frac{AC}{BC} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

For the $60° - 30° - 90°$, the ratio of its legs is $\frac{AC}{BC} = \frac{\sqrt{3}}{1} = \sqrt{3}$.

If you denote all three acute angles as $\angle\alpha$, then you can state that the ratio of the length of the leg opposite to the $\angle\alpha$ to the length of the leg adjacent to the $\angle\alpha$ is equal to some number that is constant for each three values of $\angle\alpha$.

For $30°$, this value is always $\frac{\sqrt{3}}{3}$.

For $45°$, this value is always 1.



For $60°$, this value is always $\sqrt{3}$.

Example. For $\triangle ABC$, let the length of $BC = 80$yards and $\angle B = 30°$. Then, according to what we just discovered, you can derive the following:

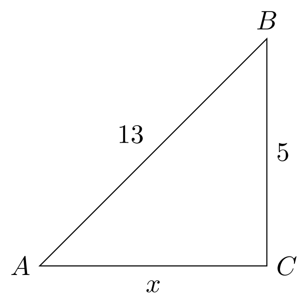
$$\frac{AC}{BC} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\frac{AC}{80} = \frac{\sqrt{3}}{3}$$
$$3 \times AC = 80\sqrt{3} \approx 80 \times (1.73) \approx 138.56$$
$$AC \approx 46 \text{ yards.}$$

These ratios for all acute angles have been calculated already and put into a special table called "Table of Trigonometric Ratios". Let's define an important term.

Definition 19 (Tangent)**.** The **tangent** (tan) of an acute angle of a right triangle is the ratio of the length of the leg opposite to the acute to the length of the leg adjacent to the acute angle.

$$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{\text{opp}}{\text{adj}}.$$

Exercise. Find $\tan(A)$:



Solution.

Step 1: According to the definition of $\tan(A)$

$$\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{5}{x}$$

Step 2: Find x by using the Pythagorean Theorem $\tan(A)$

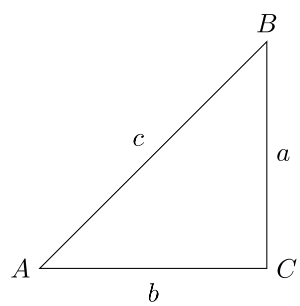
$$\begin{aligned}
c^2 &= a^2 + b^2 \\
13^2 &= 5^2 + x^2 \\
169 &= 25 + x^2 \\
169 - 25 &= 25 - 25 + x^2 \\
144 &= x^2 \\
x &= \sqrt{144} = 12
\end{aligned}$$

Step 3: Substitute

$$\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{5}{x} = \frac{5}{12} \approx 0.4167.$$

Solution: $\tan(A) = \frac{5}{12} \approx 0.4167$

3.2 Introducing the Sine and Cosine Ratios



In the right triangle $\triangle ABC$, $\sin(A) = \frac{a}{c}$. This value depends only on the measure of the angle, and not on the lengths of the sides of the particular triangle used.

In the right triangle, $\triangle ABC$, $\cos(A) = \frac{b}{c}$. This value depends only on the measure of the angle, and not on the lengths of the sides of the particular triangle used.

Definition 20 (Sine). The **sine** (sin) of an acute angle of a right triangle is the ratio of the length of the leg opposite the acute angle to the length of the hypotenuse.

$$\sin(\theta) = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{\text{opp}}{\text{hyp}}.$$

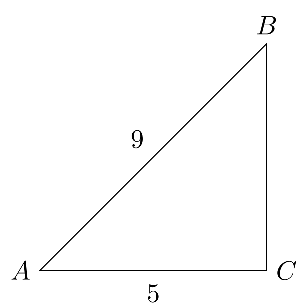
Definition 21 (Cosine). The **cosine** (cos) of an acute angle of a right triangle is the ratio of the length of the leg adjacent the acute angle to the length of the hypotenuse.

$$\cos(\theta) = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{\text{adj}}{\text{hyp}}.$$

As with tangent, the sine and cosine ratios are functions of an angle. The values of $\sin(\theta)$ and $\cos(\theta)$ do not depend on the particular triangle that contains this angle; they depend only on the value for θ.

3.2.1 Co-function Identity Between the Sine and Cosine

In the following figure, $\cos(A) = \frac{5}{9}$ and $\sin(A) = \frac{5}{9}$. If $\angle A$ and $\angle B$ are acute angles of the same right triangle, then



$$\sin(A) = \cos(B) \text{ and } \cos(A) = \sin(B).$$

It doesn't matter what the lengths of the sides are.

The sum of two acute angles in any right triangle is always 90°, so you can express this fact with the following mathematical statement: $A° + B° = 90°$, which leads to the following conclusions.

$$A^\circ + B^\circ = 90^\circ$$
$$B^\circ = 90^\circ - A^\circ$$

Now, you can rewrite the equalities found previously

$$\sin(A) = \cos(B) = \cos(90^\circ - A^\circ)$$
$$\cos(A) = \sin(B) = \sin(90^\circ - A^\circ).$$

These equations are called *co-function identities*.

Identity (Co-Function). **Co-function identities** for sine and cosine can be written as:

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right).$$

3.2.2 Pythagorean Identity with Sine and Cosine

Step 1: According to the Pythagorean Theorem, you write:

$$a^2 + b^2 = c^2$$
$$\frac{a^2 + b^2}{c^2} = 1$$

Step 2: According to the definition of sine and cosine:

$$\sin^2(A) = \frac{a}{c}, \text{ then } \sin^2(A) = \left(\frac{a}{c}\right)^2$$
$$\cos^2(A) = \frac{b}{c}, \text{ then } \cos^2(A) = \left(\frac{b}{c}\right)^2$$

Step 3: Add the squares of the sine and the cosine:

$$\sin^2(A) + \cos^2(A) = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2}$$

Step 4: Because they are equal, you can write:

$$\sin^2(A) + \cos^2(A) = \frac{a^2 + b^2}{c^2} = 1.$$

This expression is called the *Pythagorean Identity*.

Definition 22 (Pythagorean Identity). The **pythagorean identity** for the sine and cosine can be written as:

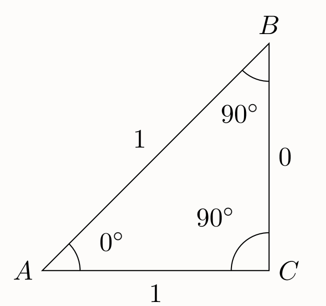
$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

3.2.3 Values of Trig Functions for $0°$ and $90°$ Angles

We've been working with sine and cosine with only acute angles. But can we define sine and cosine for $\sin(90°)$ or $\cos(0°)$ for example?

It's hard to explain the value of the trig functions for $0°$ and $90°$ using triangles to explain it because using triangles is limiting. But, for now, here's the proof.

Proof. Make a triangle with two $90°$ angles and 1 $0°$ angle. It would look something like this



Using the definitions of sine and cosine, we get the following

$$\begin{aligned}
\sin(0°) &= \frac{0}{1} = 0 \\
\cos(0°) &= \frac{1}{1} = 1 \\
\sin(90°) &= \frac{1}{1} = 1 \\
\cos(90°) &= \frac{0}{1} = 0.
\end{aligned}$$

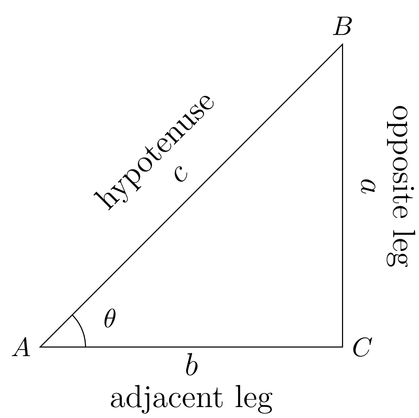
□

Chapter 4

Relations Among Trigonometric Ratios

4.1 Six Trigonometric Relatives

So far, we've only seen the following trig functions: *sine, cosine,* and *tan.* Now, it's time to see the rest of the trig functions.



In any right triangle, we have a total of 6 ratios. We've already named 3 of the 6

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}, \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}, \qquad \tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}.$$

Now, let's define the next 3.

Definition 23 (Cotangent). **Cotangent** is a trigonometric function of an angle that calculates the ratio of the adjacent side divided by the opposite side of the angle as follows

$$\cot(\theta) = \frac{\text{adj}}{\text{opp}} = \frac{b}{a}.$$

Definition 24 (Secant). **Secant** is a trigonometric function of an angle that calculates the ratio of the hypotenuse divided by the adjacent side of the angle as follows

$$\cot(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{c}{b}.$$

Definition 25 (Cosecant). **Cosecant** is a trigonometric function of an angle that calculates the ratio of the hypotenuse divided by the opposite side of the angle as follows

$$\cot(\theta) = \frac{\text{hyp}}{\text{opp}} = \frac{c}{a}.$$

4.1.1 Reciprocal Functions

Definition 26 (Reciprocal Fractions). **Reciprocal fractions** are fractions that represent $\frac{1}{x}$ of a given quantity of x. When multiplied by x, they give the product of 1.

Example. The reciprocal fraction of $\frac{1}{5}$ is 5 because $\frac{1}{5} \times 5 = 1$

Let's look at all 6 trigonometric functions, where a and b are the lengths of the triangle's legs and c is the length of the hypotenuse

$$\sin(\theta) = \frac{a}{c}, \cos(\theta) = \frac{b}{c}, \tan(\theta) = \frac{a}{b}, \cot(\theta) = \frac{b}{a}, \sec(\theta) = \frac{c}{b}, \csc(\theta) = \frac{c}{a}.$$

The cotangent is the reciprocal of the trangent, the cosecant is the reciprocal of the sine, and the secant is the reciprocal of the cosine.

You can summarize these facts by expressiong them as

$$\cot(\theta) = \frac{1}{\tan(\theta)}, \csc(\theta) = \frac{1}{\sin(\theta)}, \csc(\theta) = \frac{1}{\cos(\theta)}.$$

These formulas are helpful when proving trigonometric identites.

Definition 27 (Trigonometric Idententes). A **trigonometric identity** is an equation that is true for all values of the variable for which each side is defined.

4.1.2 Three Fundamental Identities

From the formulas for the reciprocal functions, you can derive many more identites. Fortunately, you don't have to remember all of them, except three fundamental ones, and all others can be derived from these three.

Identity (Pythagorean). The first fundamental identity is the Pythagorean relationship between the sine and the cosine

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Identity (Tangent). Consider the following

$$\begin{aligned} \frac{\sin(\theta)}{\cos(\theta)} &= \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \div \frac{b}{c} = \frac{a}{c} * \frac{c}{b} = \frac{a}{b} = \tan(\theta) \\ \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}. \end{aligned}$$

Identity (Cotangent). Recall that the cotangent is the reciprocal of the tangent, so you can write

$$\begin{aligned} \cot(\theta) &= \frac{1}{\tan(\theta)} \\ &= \frac{1}{\tan(\theta)} = \frac{1}{\frac{\sin(\theta)}{\cos(\theta)}} = \frac{\cos(\theta)}{\sin(\theta)} \\ \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}. \end{aligned}$$

4.1.3 Other Pythagorean Relationships

We can perform operations on the Pythagorean identity in order to obtain other Pythagorean identities.

Note. The other Pythagorean identities aren't new. They're just a rephrased version of the Pythagorean identity.

Step 1: The base of the Pythagorean identity

$\Rightarrow$ $\sin^2(\theta) + \cos^2(\theta) = 1$

Step 2: Divide each term by $\cos^2(\theta)$

$$\Rightarrow \quad \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Step 3: Because $\dfrac{x^2}{y^2} = \left(\dfrac{x}{y}\right)^2$, you can rewrite as

$$\Rightarrow \quad \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 + 1 = \frac{1}{\cos^2(\theta)}$$

Step 4: The term in the parentheses is a fundamental identity

$$\Rightarrow \quad \tan^2(\theta) + 1 = \frac{1}{\cos^2(\theta)}$$

Step 5: Using the reciprocal function $\sec(\theta) = \dfrac{1}{\cos(\theta)}$

$$\Rightarrow \quad \tan^2(\theta) + 1 = \sec^2(\theta).$$

This is the second Pythagorean identity for trigonometric functions. Now, let's deal with the last Pythagorean identity.

Step 1: The base of the Pythagorean identity

$\Rightarrow$ $\sin^2(\theta) + \cos^2(\theta) = 1$

Step 2: Divide each term by $\sin^2(\theta)$

$$\Rightarrow \quad \frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = 1$$

Step 3: You obtain

$$\Rightarrow \quad 1 + \cot^2(\theta) = \frac{1}{\sin^2(\theta)}$$

Step 4: Using the reciprocal function $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

$$\Rightarrow \quad 1 + \cot^2(\theta) = \csc^2(\theta).$$

Identity (The three Pythagorean identities)**.** The three Pythagorean identities are

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$1 + \cot^2(\theta) = \csc^2(\theta).$$

Exercise. Prove that $\cot(\theta)\sin(\theta) = \cos(\theta)$.
Solution.

Step 1:	It's useful to express all ratios in terms of sine and cosine
$\Rightarrow$	$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
Step 2:	Plug in the value for the cotangent into original identity
$\Rightarrow$	$\dfrac{\cos(\theta)}{\sin(\theta)} \times \sin(\theta)$
$\Rightarrow$	$\dfrac{\cos(\theta) \times \sin(\theta)}{\sin(\theta)}$
$\Rightarrow$	$\cos(\theta)$
$\Rightarrow$	$\cos(\theta) = \cos(\theta)$.

Exercise. Prove that $\dfrac{1-\sin^2(\theta)}{1-\cos^2(\theta)} = \dfrac{1}{\tan^2(\theta)}$.

Solution.

Step 1:	From the Pythagorean identity, you obtain
$\Rightarrow$	$\sin^2(\theta) + \cos^2(\theta) = 1$
$\Rightarrow$	$1 - \sin^2(\theta) = \cos^2(\theta)$
Step 2:	From the Pythagorean identity, you obtain
$\Rightarrow$	$1 - \cos^2(\theta) = \sin^2(\theta)$
Step 3:	Plug-in-play
$\Rightarrow$	$\dfrac{1 - \sin^2(\theta)}{1 - \cos^2(\theta)} = \dfrac{\cos^2(\theta)}{\sin^2(\theta)}$
$\Rightarrow$	$\dfrac{\cos^2(\theta)}{\sin^2(\theta)} = \left(\dfrac{\cos(\theta)}{\sin(\theta)}\right)^2$
$\Rightarrow$	$\left(\dfrac{\cos(\theta)}{\sin(\theta)}\right)^2 = \cot^2(\theta)$
$\Rightarrow$	$\cot^2(\theta) = \dfrac{1}{\tan^2(\theta)}.$

4.2 Determining Values of Trigonometric Functions

Let's construct a table with the values for each of the trigonometric function with the most commonly used angles. To do this, you're going to need some information, which you already have.

The first piece of information that you need is the relations between the sides in special right triangles.

	Opposite Leg	**Adjacent Leg**	**Hypotenuse**
$\triangle 45^\circ - 45^\circ - 90^\circ$	1	1	$\sqrt{2}$
$\triangle 30^\circ - 60^\circ - 90^\circ$ for $\angle 30^\circ$	1	$\sqrt{3}$	2
$\triangle 30^\circ - 60^\circ - 90^\circ$ for $\angle 60^\circ$	$\sqrt{3}$	1	2

Table 4.1: Special Right Triangle Ratios

The second piece of information is the values for the sine and cosine of 0° and 90° angles.

$$\sin(0°) = 0 \quad \cos(0°) = 1$$
$$\sin(90°) = 1 \quad \cos(90°) = 0$$

Table 4.2: Values for 0° and 90° for sine and cosine

And the last piece of information you need for the constructing the table is the definitions of the trigonometric functions.

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \quad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \quad \tan(\theta) = \frac{\text{opp}}{\text{hyp}}$$
$$\csc(\theta) = \frac{\text{hyp}}{\text{opp}} \quad \sec(\theta) = \frac{\text{hyp}}{\text{adj}} \quad \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

Table 4.3: The definitions of the trigonometric functions

Example. Now, using our information, let's find sin(60)

$$\sin(60°) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

Let's find csc(60°)

$$\csc(60°) = \frac{\text{hyp}}{\text{opp}} = \frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Now, let's setup a table with 6 columns and 7 rows. The first column shows the type of function. Columns $2 - 6$ show the values of functions for the five most commonly used angles.

Angle	0°	30°	45°	60°	90°
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
$\cot(\theta)$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
$\csc(\theta)$	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec(\theta)$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined

Table 4.4: Special Right Triangle Ratios

4.3 Solving Right Triangles

Definition 28 (Solving a Right Triangle). **Solving a Right Triangle** is the process of finding the unknown elements of a triangle using the trigonometric relationships.

4.3.1 Finding the Sides

Let's do some problems.

Exercise. If you have a tree that is 20 feet tall, and the rays of the sun make a 24° angle with the ground, how long will the shadow be? Here's a diagram to help you out.

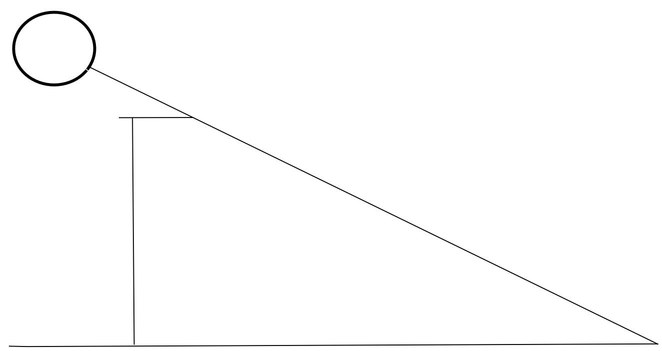


Figure 4.1

Solution.

Step 1: It's useful to express all ratios in terms of sine and cosine

$$\Rightarrow \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Step 2: Plug in the value for the cotangent into original identity

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