Pre-calculus 2

Hashem A. Damrah

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Contents

Lecture 1: Sets and Numbers

Apr 04 2022 (10:01:23)

Definition 1 (Set). A set is a collection of objects specified in a manner that enables one to determine if a given object is or isn't in the set.

(Solution ??) Which of the following represent a set?

1. The students registered for MTH 112 at PCC this quarter.
2. The good students registered for MTH 112 at PCC this quarter.

Notation. Roster Notation involves listing the elements in a set within curly brackets: "$\{\}$" like the following: "$\{1, 2, 3, 4\}$"

Definition 2 (Element). An object in a set is called an **element** of the set. (symbol: "$\in$")

Example. 5 is an element of the set $\{4, 5, 6, 7, 8, 9\}$. We can express this symbolically:

$$5 \in \{4, 5, 6, 7, 8, 9\}$$

.

Definition 3 (Subset). A set S of a set T, denoted $S \subseteq T$, if all elements of S are also elements of T.

If S and T are sets and $S = T$, then $S \subseteq T$. Sometimes it's useful to consider a subset S of a set T that isn't equal to T. In such case, we write $S \subset T$ and say that S is a proper subset of T.

Example. $\{4, 7, 8\}$ is a subset of the set $\{4, 5, 6, 7, 8, 9\}$.

We can express this fact symbolically by $\{4, 7, 8\} \subseteq \{4, 5, 6, 7, 8, 9\}$

Since these two sets aren't equal, $\{4, 7, 8\}$ is a proper subset of $\{4, 5, 6, 7, 8, 9\}$, so can write:

$$\{4, 7, 8\} \subset \{4, 5, 6, 7, 8, 9\}$$

.

We can use the other symbol as follows:

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}$$

.

Definition 4 (Empty Set)**.** The empty set, denoted $\emptyset$, is the set with no elements

$$\emptyset = \{\}$$

.

Definition 5 (Union)**.** The union of two sets A and B, denoted $A \cup B$, is the set containing all of the elements in either A or B (or both A and B).

Example. Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then . . .

- $\{4, 7, 8\} \cup \{1, 3, 5, 7\} = \{1, 3, 4, 5, 7, 8\}$
- $\{4, 7, 8\} \cup \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 7, 8\}$
- $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Definition 6 (Intersection)**.** The intersection of two sets A and B, denoted $A \cap B$, is the set containing all the elements in both A and B.

Example. Consider the sets $\{4, 7, 8\}$, $\{0, 2, 4, 6, 8\}$, and $\{1, 3, 5, 7\}$. Then . . .

- $\{4, 7, 8\} \cap \{0, 2, 4, 6, 8\} = \{4, 8\}$
- $\{4, 7, 8\} \cap \{1, 3, 5, 7\} = \{7\}$
- $\{0, 2, 4, 6, 8\} \cap \{1, 3, 5, 7\} = \emptyset$

Notation. Set Builder Notation.
"All the whole numbers between 3 and 10" $= \{x | x \in \mathbb{Z} \text{ and } 3 < x < 10\}$

Definition 7 (Important Sets of Numbers)**.** The set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

.

The set of integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

.

The set of rational numbers:

$$\mathbb{Q} = \left\{ x | x = \frac{p}{q} \text{ and } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

.

The set of real numbers: $\mathbb{R}$

(All the numbers on the number line)

.

The set of complex numbers:

$$\mathbb{C} = \left\{ x | x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1} \right\}$$

.

Note. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, the set of natural numbers ($\mathbb{N}$) is a subset of the set of integers ($\mathbb{Z}$) which is a subset of the set of rational numbers ($\mathbb{Q}$) which is a subset of the set of real numbers ($\mathbb{R}$) which is a subset of the set of complex numbers ($\mathbb{C}$).

Notation. Since we use the real numbers so often, we have special notation for subsets of the real numbers. **Interval Notation**. Interval Notation involves square or round brackets.

Example. Quick demo of Interval Notation

- $\{x | x \in \mathbb{R} \text{ and } -2 \leq x \leq 3\} = [-2, 3]$
- $\{x | x \in \mathbb{R} \text{ and } -2 < x < 3\} = (-2, 3)$
- $\{x | x \in \mathbb{R} \text{ and } -2 < x \leq 3\} = (-2, 3]$
- $\{x | x \in \mathbb{R} \text{ and } -2 \leq x < 3\} = [-2, 3)$

When the interval has no upper or lower bound, we use the infinity symbol (∞ or $-\infty$)

- $\{x | x \in \mathbb{R} \text{ and } x \leq 4\} = (-\infty, 4]$
- $\{x | x \in \mathbb{R} \text{ and } x \geq 4\} = [4, -\infty)$

(Solution ??) Simplify the following expressions:

- $(-4, \infty) \cup [-8, 3]$
- $(-4, \infty) \cup (-\infty, 2]$

- $(-4, \infty) \cap (-\infty, 2]$
- $(-4, \infty) \cap [-10, -5]$

Lecture 2: Angles and Arc-Length

Apr 06 2022 (07:25:34)

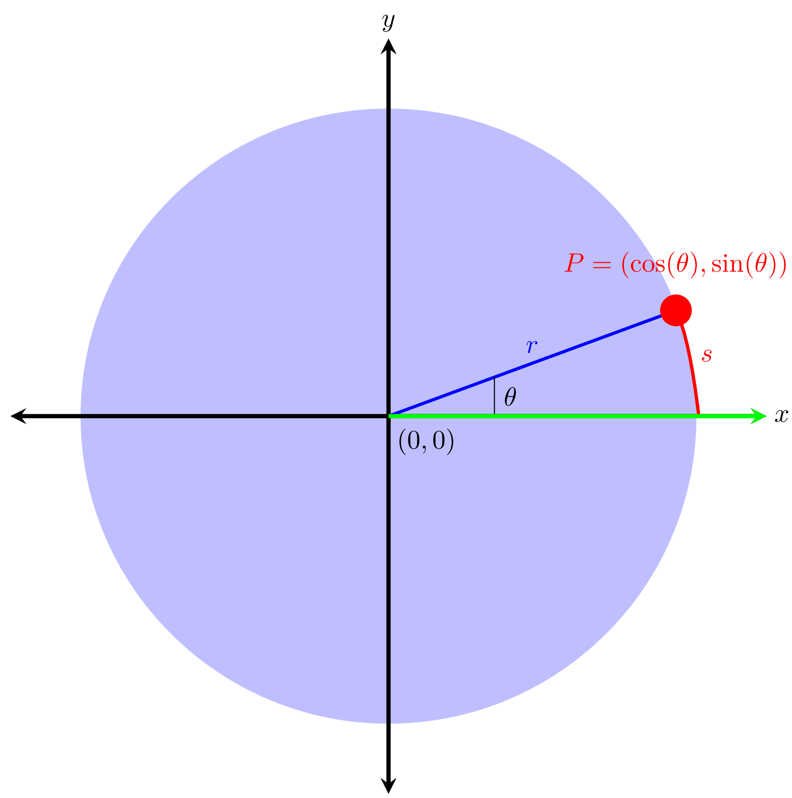

Figure 1: Diagram of a Circle

- The standard way that we put an angle in a circle we start the angle at the positive x-axis and go **counterclockwise**. We measure negative angles **clockwise**.
- When put an angle in standard position and you rotate, it ends someplace. Where it ends is called the **Terminal Side**.
- The point P on the circumference of the circle is **specified by the angle** θ.
- Angle θ corresponds with a portion of the circumference of the circle called the **arc spanned by** θ
- Two angles with the same terminal side are called **co-terminal angles**.

Three hundred and sixty degrees ($360°$) represents a complete trip around a circle, which is a full rotation. So, $1°$ corresponds to $\frac{1}{360°}$ of a full rotation.

Degrees are more like percentages, where they represent concepts, and not numbers. But, just like how you can transform a percentage to a number ($10\% = \frac{10}{100}$), you can do the same with degrees ($10° = \frac{10}{360}$).

Since $360°$ represents a full rotation around the circle, if we add any integer multiple of $360°$ to an angle θ_1, we'll obtain an angle co-terminal to θ_1.

Example. So $45°$ and $405°$ are co-terminal.

Definition 8 (π)**.** The number π represents the ratio of the circumference of a circle to the diameter of the circle. $\pi = \frac{c}{d}$, where c is the circumference and d is the diameter.

$\pi \approx 3.14$, but π is an **"irrational"** number.

Definition 9 (Radian)**.** The **radian** measure of an angle is the ratio of the length of the arc on the circumference of the circle spanned by the angle, s, and the radius, r, of the circle.

$$\begin{aligned} \theta &= \frac{s}{r} \\ \pi &= \frac{c}{d} \\ &= \frac{2}{2r} && d = 2r \\ 2\pi &= \frac{c}{r} && \text{Multiply both sides by 2} \end{aligned}$$

From this, we can conclude that: $2\pi = 360°$. From that, we can get $\pi = 180°$ since we just divided both sides by 2. From that, we get $\frac{\pi}{2} = 90°$. So on, and so forth.

Here's a table with all of the radians:

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Table 1: Degrees into Radians

(Solution ??) Convert 8 radians into degrees and 8 degrees into radians:

Definition 10 (Arc Length)**.** The **arc length** s spanned in a circle of radius r by an angle θ measured in radians is given by:

$$s = r \times \mid \theta \mid$$

.

Note. This formula only applies if θ is measured in radians.

(Solution ??) What is the arc length spanned by a $40°$ angle on a circle radius 30 meters?

Lecture 3: Introduction to Periodic Functions

Apr 07 2022 (18:43:36)

Any activity that repeats on a regular time interval can be described as *periodic*.

Definition 11 (Periodic Function)**.** A **periodic function** whose values repeat on regular intervals. Hence, f is a periodic if there exists some constant c such that:

$$f(x+c) = f(x)$$

.

for all x in the domain of f such that $f(x+c)$ is defined.

Recall that this means that if the graph $y = f(x)$ is shifted horizontally c units then it will appear unaffected.

Definition 12 (Period)**.** The **period** of a function f is the smallest value $\mid c \mid$ such that $f(x+t)$ for all x in the domain of f such that $f(x+c)$ is defined.

(Solution ??) Find the period of the function graphed below:

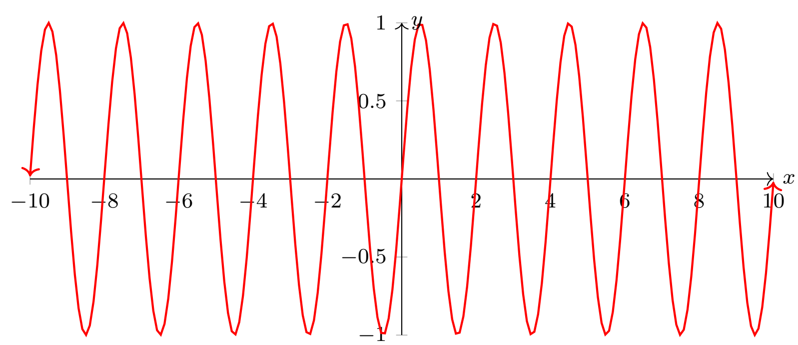

Figure 2

Definition 13 (Midline)**.** The **midline** of a periodic function is the horizontal line midway between the function's minimum and maximum values.

If $y = f(t)$ is periodic and f_{max} and f_{min} are the maximum and minimum values of f, then the equation of the midline is:

$$y = \frac{f_{max} - f_m in}{2}$$

.

Definition 14 (Amplitude). The **amplitude** of a period function is the distance between the function's maximum value and the midline (or the function's minimum value and the midline).

(Solution ??) Find the midline and amplitude of the function graphed below:

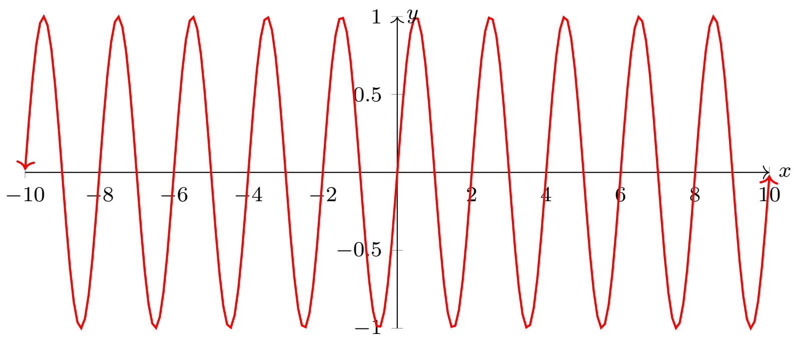

Figure 3

Lecture 4: Part 1: Intro to Trig Functions

Apr 12 2022 (16:27:32)

Definition 15 (Unit Circle). A **unit circle** is a circle with a radius of 1 unit.

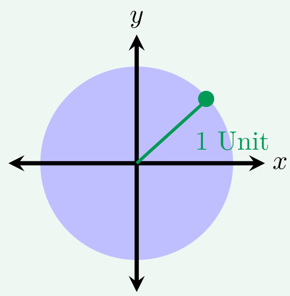

Figure 4: Unit Circle

Now we can define the sin and cos functions.

Note. There are 4 other trigonometric functions that can be defined in terms of sin and cos functions so first we'll get familiar with sin and cos and then later in the next lecture we'll define the 4 other trigonometric functions.

Definition 16 (Sine and Cosine). The **sine function**, denoted $\sin(\theta)$, associates each angle θ with the vertical coordinate of the point P specified by θ on the circumference of a unit circle.

The **cosine function**, denoted $\cos(\theta)$, associates each angle θ with the horizontal coordinate of the point P specified by θ on the circumference of a unit circle.

So the point P in the figure below has coordinates:

$$(x, y) = (\cos(\theta), \sin(\theta))$$

.

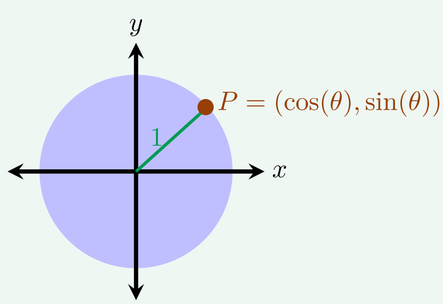

Figure 5: Sin and Cos on Graph

(Solution ??) The angle of θ specifies the point $P = \left(-\frac{3}{5}, \frac{4}{5}\right)$ on the circumference of a unit circle. Find $\sin(\theta)$ and $\cos(\theta)$.

Let's determine the signs of the sin and cos functions in the four quadrants:

- When the terminal side of angle θ is in Quadrant I, both the x and y coordinates of point P are positive. Therefore, **if θ is in Quadrant I,** $\cos(\theta) > 0$ **and** $\sin(\theta) > 0$
- When the terminal side of angle θ is in Quadrant II, the y coordinate of point P is positive but the x coordinate is negative. Therefore, **if θ is in Quadrant II,** $\cos(\theta) < 0$ **and** $\sin(\theta) > 0$
- When the terminal side of angle θ is in Quadrant III, both the x and y coordinates of point P are negative. Therefore, **if θ is in Quadrant IIII,** $\cos(\theta) < 0$ **and** $\sin(\theta) < 0$
- When the terminal side of angle θ is in Quadrant IIII, the x coordinate of point P is positive but the y coordinate is negative. Therefore, **if θ is in Quadrant IIII,** $\cos(\theta) > 0$ **and** $\sin(\theta) < 0$

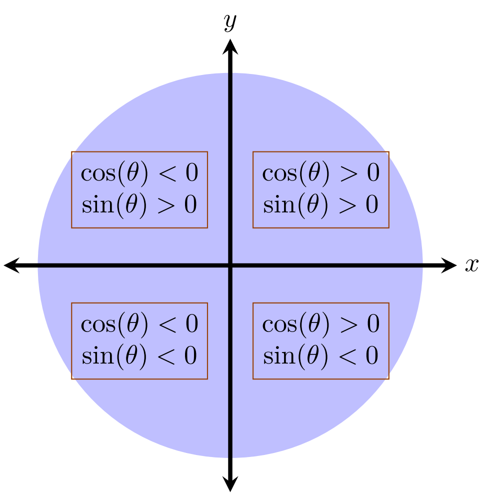

Figure 6: Quadrants of a Graph

Now let's find the sin and cos of a few particular angles. The easiest points for us to find on the unit circle are points where the circumference of the circle intersects the coordinate axes. Let's start by finding the corresponding sin and cos values.

Note. Keep in mind that cos represents the x-coordinate and sin represents the y-coordinate.

- The angle $\theta = 90°$ ($\theta = \frac{\pi}{2}$ radians), specifies the point $(0, 1)$ on the circumference of a unit circle. Thus ...

$$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$$

.

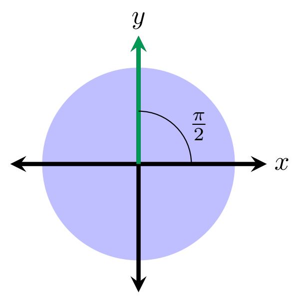

Figure 7

- The angle $\theta = 180°$ ($\theta = \pi$ radians), specifies the point $(-1, 0)$ on circumference of a unit circle. Thus ...

$$\cos(\pi) = -1 \qquad \sin(\pi) = 0$$

.

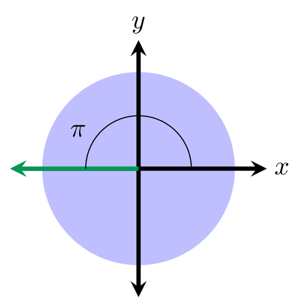

Figure 8

- The angle $\theta = 270°$ ($\theta = \frac{3\pi}{2}$ radians), specifies the point $(0, -1)$ on the circumference of a unit circle. Thus ...

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1$$

.

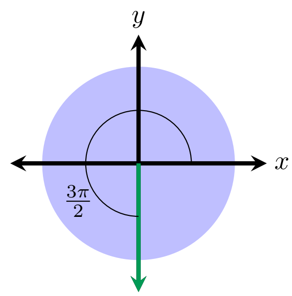

Figure 9

- The angle $\theta = 360°$ ($\theta = 2\pi$ radians), specifies the point $(1, 0)$ on the circumference of a unit circle. Thus ...

$$\cos(2\pi) = 1 \qquad \sin(2\pi) = 0$$

.

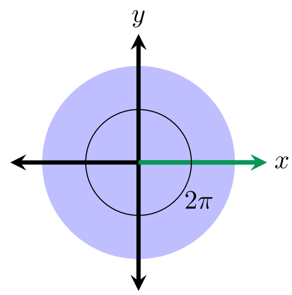

Figure 10

Notice that angles of measure 2π radians and 0 radians specify the same point: $(1, 0)$. Thus, the sin and cos values for 2π and 0 radians are the same:

$$\cos(2\pi) = \cos(0) = 1 \qquad \text{and} \qquad \sin(2\pi) = \sin(0) = 0$$

.

Since any angle θ and $\theta + 2\pi$ specify the same point on the unit circle, the sin and cos values of θ and $\theta + 2\pi$ are the same. Therefore, the period of the sin and cos function is 2π radians.

Theorem 1. For all θ, $\sin(\theta) = \sin(\theta + 2\pi)$ and $\cos(\theta) = \cos(\theta + 2\pi)$ so the period of both $s(\theta) = \sin(\theta)$ and $c(\theta) = \cos(\theta)$ is 2π radians.

Now, we'll sketch graphs of the sin and cos functions.

We first start by organizing the function values in a table:

θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$	$540°$	$630°$	$720°$
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$y = \cos(\theta)$	1	0	-1	0	1	0	-1	0	1
$y = \sin(\theta)$	0	1	0	-1	0	1	0	-1	0

Table 2: Values of Sin and Cos

Here's how it looks like when we graph:

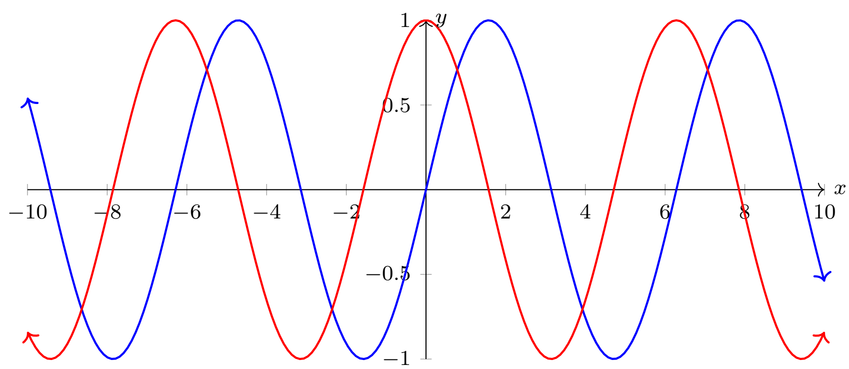

Figure 11: The graph of $y = \cos(\theta)$ and $y = \sin(\theta)$

Notice that the graphs of $y = \cos(\theta)$ and $y = \sin(\theta)$ are very similar. In fact, if we shift $y = \sin(\theta)$ to the left $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \cos(\theta)$. This means that:

$$\cos = \sin\left(\theta + \frac{\pi}{2}\right)$$

.

Similarly, if we shift $y = \cos(\theta)$ to the right $\frac{\pi}{2}$ units, we'll obtain the graph of $y = \sin(\theta)$. This means that:

$$\sin = \cos\left(\theta - \frac{\pi}{2}\right)$$

.

Definition 17 (Identity). An **identity** is an equation that is true for all values in the domains of the involved expressions.

Some important trig identities

- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$
- $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(\theta) = \cos(2\pi - \theta)$
- $\sin(\theta) = \cos(\pi - \theta)$

We can generalize the definitions of sin and cos functions so that they are applicable to circles of any size, rather than only for unit circles.

Definition 18 (More Applicable version of Sin and Cos)**.** If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius r then:

$$\cos(\theta) = \frac{x}{r} \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r}$$

.

Note. If $r = 1$, then this definition $\cos(\theta)$ and $\sin(\theta)$ are equivalent to what we saw at the beginning of this lecture.

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

.

If we solve the equations $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ for x and y, we can obtain the coordinates of a point on the circumference of a circle of any r :

$$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta) \qquad \text{and} \qquad \sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$$

.

If the point $T = (x, y)$ is specified by the angle θ on the circumference of a circle of radius, r, then:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$

.

(Solution ??) A circle with a radius of 6 units is given. The point Q is specified by the angle α. Use the sin and cos function to express the exact coordinates of point Q.

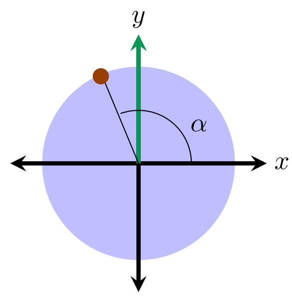

Figure 12

Recall that the cos and sin functions represent the horizontal and vertical coordinates of a point on the circumference of a unit circle. This situation creates a right triangle with hypotenuse of length 1 unit and side-lengths of $\cos(\theta)$ and $\sin(\theta)$ units.

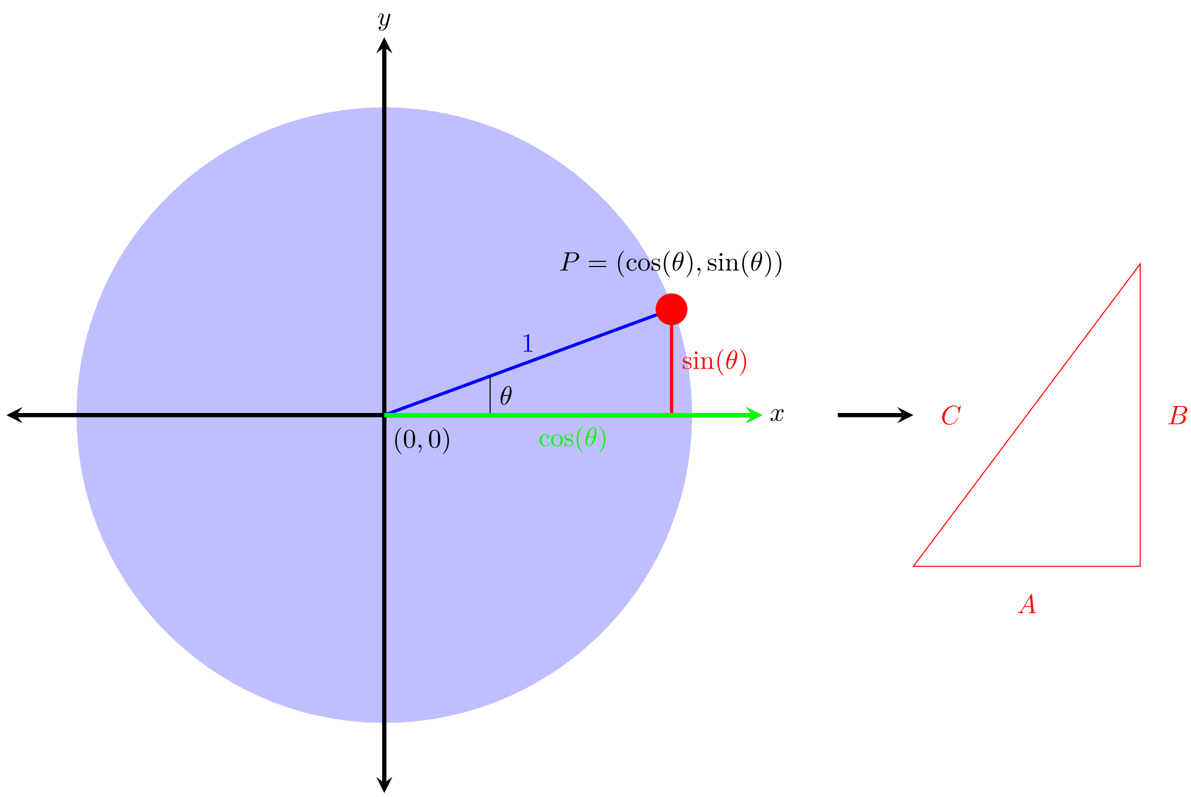

Figure 13: Diagram of a Circle

Now we can apply the Pythagorean Theorem to this right triangle. First let's review the Pythagorean Theorem:

Theorem 2 (Pythagorean Theorem)**.** If the sides of a right triangle are labeled. Then . . .

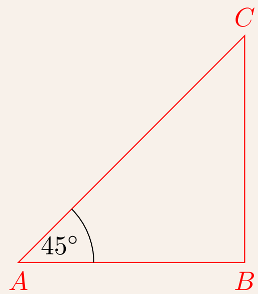

Figure 14: Right Triangle

$$a^2 + b^2 = c^2$$

.

Applying the Pythagorean Theorem to the right triangle we obtain what is called the Pythagorean Identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

. When we use exponents in trigonometric functions we can use unusual notation. Instead of using parentheses around the entire expression, we can put the exponents between the letters that name the function and the input for the function. Thus, we can write an expression like $(\sin(\theta))^2$ as $\sin^2(\theta)$. We can use this notation to express the Pythagorean Identity:

Notation.

$$\theta \in \mathbb{R},\ \sin^2(\theta) + \cos^2(\theta) = 1$$

.

(Solution ??) If $\sin(A) = \frac{1}{3}$ and $\frac{\pi}{2} < A < \pi$, which means A is in Quadrant II, find $\cos(A)$.

Lecture 5: Part 2: Intro to Trig Functions

Apr 16 2022 (17:10:40)

There are four other trigonometric functions besides sin and cos. These four functions are defined in terms of sin and cos functions:

Definition 19 (The Other 4 Trig Functions)**.** The **tangent function**, denoted $\tan(\theta)$, is defined by:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

.

The **cotangent function**, denoted $\cot(\theta)$, is defined by:

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

.

The **secant function**, denoted $\sec(\theta)$, is defined by:

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

.

The **cosecant function**, denoted $\csc(\theta)$, is defined by:

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

.

The tangent function is the ratio of the sin and cos functions so it provides truly different information than does sin or cos. So the tan function is important. But the cot, sec, and csc are just reciprocals of tan, cos, and sin. So they don't provide any new information, so they are arguably less important functions.

There are 2 other identities that can be obtained from the Pythagorean Identity.

One of these identities can be found by dividing both sides of the Pythagorean Identity by $\cos^2(\theta)$:

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow & \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \\ \Rightarrow & \tan^2(\theta) + 1 = \sec^2(\theta). \end{aligned}$$

Alternatively, we can divide both sides of the Pythagorean Identity by $\sin^2(\theta)$ and find another identity.

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow\ & \frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \\ \Rightarrow\ & \quad 1 + \cot^2(\theta) = \csc^2(\theta). \end{aligned}$$

This gives us three identities that are considered **"The Pythagorean Identities"**:

The Pythagorean Identities

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \tan^2(\theta) + 1 &= \sec^2(\theta) \\ 1 + \cot^2(\theta) &= \csc^2(\theta). \end{aligned}$$

Lecture 6: Part 3: Intro to Trig Functions

Apr 16 2022 (18:04:27)

Now let's determine the sin and cos of some important angles, namely, $30°$, $45°$, and $60°$ ($\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians). We focus on these angles since we can use some basic geometry to easily find their sin and cos values – but we cannot easily find the sin and cos of most other angles.

Let's start by finding the sin and cos of $30°$ ($\frac{\pi}{6}$ radians). The fact about triangles is the sum of all the angles is equal to $180°$. So, we get:

$$30° + 90° + c = 180° \Rightarrow c = 60$$

.

So, here's how our triangle will look like:

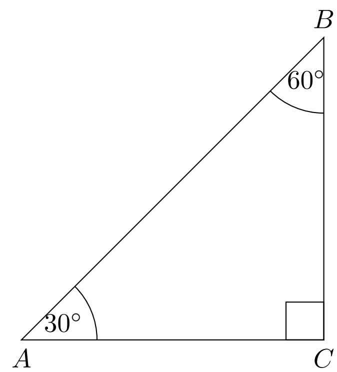

Figure 15: Our Original Triangle

Let's take this triangle, flip it, and put it right next to the original. Here's how it looks after we did this:

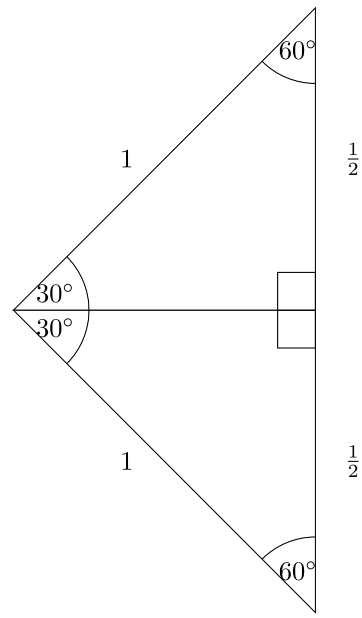

Figure 16: An isosceles triangle

This means that $b = \frac{1}{2}$ and from this, we can use the Pythagorean Theorem to find the value for a:

$$\left(\frac{1}{2}\right)^2 + a^2 = 1^2 \Rightarrow a = \frac{\sqrt{3}}{2}$$

.

Here's the final triangle:

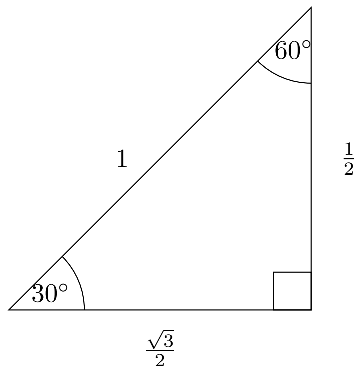

Figure 17: Triangle with all sides and angles

$$P = (\cos(30°), \sin(30°)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

.

So, using the definition of sin and cos, here's our final answer:

$$P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow \qquad \sin(30°) = \frac{1}{2},\ \cos(30°) = \frac{\sqrt{3}}{2}$$

.

Here's a table of the most common values for sin and cos:

θ (degrees)	$30°$	$45°$	$60°$
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

Table 3: Common Values for Sin and Cos

(Solution ??) Find $\tan\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, $\csc\left(\frac{\pi}{6}\right)$, and $\cot\left(\frac{\pi}{6}\right)$.

(Solution ??) Find $\tan\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, $\csc\left(\frac{\pi}{4}\right)$, and $\cot\left(\frac{\pi}{4}\right)$.

(Solution ??) Find $\tan\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, $\csc\left(\frac{\pi}{3}\right)$, and $\cot\left(\frac{\pi}{3}\right)$.

Lecture 7: Part 4: Intro to Trig Functions

Apr 16 2022 (21:49:19)

Recall that the sin and cos functions represent the coordinates of points in the circumference of a unit circle. We found the values for 30°, 45°, and 60° by finding the coordinates of the points on the circumference of the unit circle specified by these angles. The points we found were all in Quadrant I, but since a circle is symmetric about both the x and y axes, we can reflect these points about the coordinate axes to determine the coordinates of corresponding points in the other quadrants. This means we can use the sin and cos values of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ to find the sin and cos values of corresponding angles in other quadrants.

Because of the symmetry of a circle, we can take a point in Quadrant I and reflect it about the x-axis, the y-axis, and about both axes in order to obtain corresponding points, one in each of the three other quadrants; the absolute value of the coordinates of all four of these points is the same, i.e., they only differ by their signs.

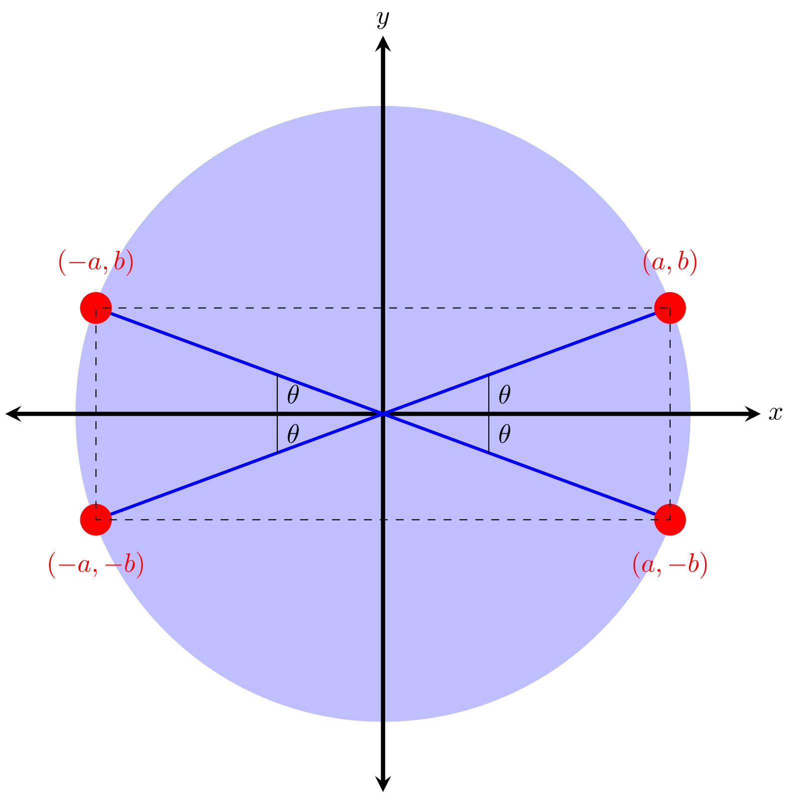

Figure 18: Plot of (a, b) specified by angle θ with the other points in all the quadrants.

Although all four of the points are specified by a different angle, all four of the angles share the same reference angle, θ.

Definition 20 (Reference Angle)**.** The **reference angle** for an angle is the acute angle between the terminal side of the angle and the x-axis.

Find the reference angle for $150°$ and $\frac{5\pi}{4}$.

Now let's find out how we can use reference angles to determine the sin and cos of any integer multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Lecture 8: Sinusoidal Functions

Apr 23 2022 (19:15:57)

Definition 21 (Sinusoidal Function)**.** A **Sinusoidal Function** is a function of the form:

$$y = A\sin(\omega(t-h)) + k \text{ or } y = A\cos(\omega(t-h)) + k$$

.

where $A, \omega, h, k \in \mathbb{R}$

Based on what we know about graph transformations, we should recognize that a sinusoidal function is a transformation of $y = \sin(t)$ or $y = \cos(t)$. Consequently, sinusoidal function are waves with the same curvy shape as the graphs of sin and cos but with different periods, midlines, and/or amplitudes.

Summary of Graph Transformation

Suppose that f and g are functions such that $g(t) = A \times f(\omega(t-h)) = k$ and $A, \omega, h, k \in \mathbb{R}$. In order to transform the graph of the function f into the graph of g ...

1. horizontally stretch/compress the graph of f by a factor of $\frac{1}{|\omega|}$ and, if $\omega < 0$, reflect it about the y-axis.
2. shift the graph horizontally h units (shift right if h is positive and left if h is negative).
3. vertically stretch/compress the graph by a factor of $\mid A \mid$ and, if $A < 0$, reflect it about the t-axis.
4. shift the graph vertically k units (shift up if k is positive and down if k is negative).

Note. The order in which these transformations are performed matters.

(Solution ??) Describe how we can transform the graph of $f(t) = \sin(t)$ into the graph of $g(t) = 2\sin(t) - 3$. State the period, midline, and amplitude of $y = g(t)$.

Here's a quick summary of sinusoidal functions:

Theorem 3. The graphs of the sinusoidal functions

$$y = A\sin(\omega(t-h)) + k \text{ and } y = A\cos(\omega(t-h)) + k$$

.

where $A, \omega, h, k \in \mathbb{R}$ have the following properties:

period: $\frac{2\pi}{|\omega|}$ units
midline: $y = k$
amplitude: $|\ A\ |$ units
horizontal shift: h units
angular frequency: ω radians per unit of t

Now let's try to sketch the graph of:

$$f(t) = 2\sin\left(\pi t - \frac{\pi}{4}\right) - 3$$

.

To do this, we need to find the midline, amplitude, horizontal shift, and period. We can determine the midline by noticing that this function has a -3, which causes the sin wave to be shifted down 3 units. The amplitude can be determined by the number outside the function, which will cause a vertical stretch. So, our amplitude is 2 units. To find the horizontal shift, we first need to factor out π from the function's input.

$$f(t) = 2\sin\left(\pi\left(t - \frac{1}{4}\right)\right) - 3$$

. The reason we did this is because we need to find out how much we're moving left or right, and $-\frac{1}{4}$ indicates that we're moving $\frac{1}{4}$ units to the right. The period comes from two features of the formula. The first one is we're using the trig function sin, which has a period of 2π units and the second feature is we have π multiplying all of the numbers in the input, which horizontally compresses the function.

$$2\pi \times \frac{1}{\pi} = 2 \text{ units}$$

.

Here's the final graph:

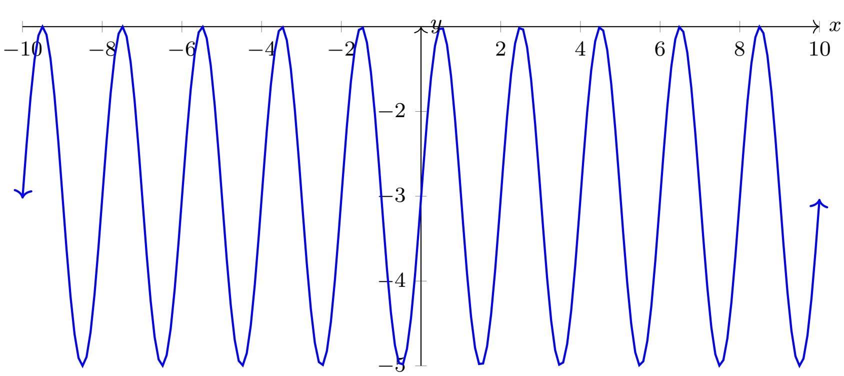

Figure 19

(Solution ??) Find two different algebraic rules (one involving sin and one involving cos) for the function $y = g(t)$ graphed below:

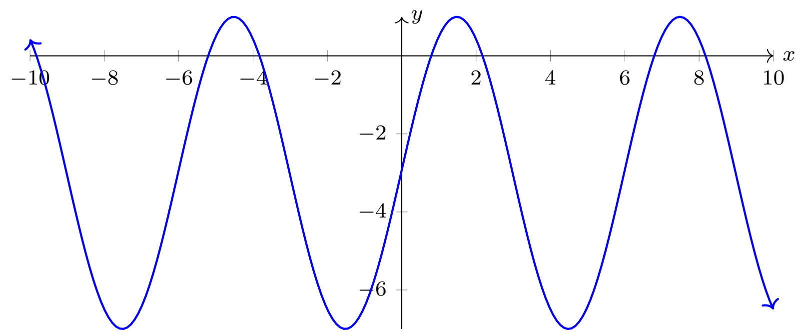

Figure 20

Lecture 9: Inverse Trig Functions

Apr 28 2022 (09:01:09)

For example, if f and f^{-1} are inverses of one another and if $f(a) = b$, then $f^{-1}(b) = a$. Inverse functions are extremely valuable since they "undo" one another and allow us to solve equations. For example, we can solve the equation $x^3 = 10$ by using the inverse of cubing function, namely the cube-root of the function, to "undo" the cubing involved in the equation:

$$\begin{aligned} x^3 &= 10 \\ \sqrt[3]{x^3} &= \sqrt[3]{10} \\ x &= \sqrt[3]{10} \end{aligned}$$

.

The cubing function has an inverse function because each output value corresponds to exactly one input value. This means that the cubing function is **one-to-one**, and it's only one-to-one functions whose inverses are also functions.

Unfortunately, the trig functions aren't one-to-one so, in their natural form, they don't have inverse functions. For example, consider the output $\frac{1}{2}$ for the sin function. This output corresponds to the inputs, $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$, etc.

Since inverse functions can be so valuable, we really want inverse trig functions, so we need to restrict the domains of the functions to intervals on which they are on-to-one, and then we can construct inverse functions. Let's start by constructing the inverse of the sin function.

In order to construct the inverse of the sin function, we need to restrict the domain to an interval on which the function is one-to-one, and we need to choose an interval of the domain that utilizes the entire range of the sin function, $[-1, 1]$. We'll choose the interval $-\frac{\pi}{2}, \frac{\pi}{2}$.

Definition 22 (Inverse Sin Function)**.** The inverse sin function, denoted $y = \sin^{-1}(t)$, is defined by the following:

$$\text{if } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } \sin(y) = t \text{then } y = \sin^{-1}(t)$$

.

By construction, the range of $y = \sin^{-1}(t)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and the domain is the same as the range of the sin function: $[-1, 1]$.

Note. The inverse sin function is often called the **arcsine function** and denoted $y = \arcsin(t)$

Now let's construct the inverse of the cos function. Like the sin function, the cos function isn't one-to-one, s we'll need to restrict its domain to construct the inverse cos function. The cos function is one-to-one on the interval $[0, \pi]$ and, on this interval, the graph utilizes the entire range of the cos function, $[-1, 1]$. So we can define the inverse cos function on this portion of the cos function.

Definition 23 (Inverse Cos Function). The inverse cos function, denoted $y = \cos^{-1}(t)$, is defined by the following:

$$\text{if } 0 \leq y \leq \pi \text{ and } \cos(y) = t \text{ then } y = \cos^{-1}(t)$$

.

By construction, the range of $y = \cos^{-1}(t)$ is $[0, \pi]$, and the domain is the same as the range of the cos function: $[-1, 1]$.

Note. The inverse cos function is often called the **arccosine function** and denoted $y = \arccos(t)$

Now let's define the inverse tan function. The tan function is one-to-one on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, since the period of tangent is π units, this interval represents a complete period of tangent. In order to construct the inverse tan function, we restrict the tan function to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Definition 24 (Inverse Tan Function). The inverse tan function, denoted $y = \tan^{-1}(t)$ is defined by the following:

$$\text{if } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } \tan(y) = t \text{ then } y = \tan^{-1}(t)$$

.

By construction, the range of $y = \tan^{-1}(t)$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the domain is the same as the range of the tan function: $\mathbb{R}$.

Note. The inverse tan function is often called the **arctangent function** and denoted $y = \arctan(t)$

(Solution ??) Evaluate the following expressions:

- $\sin^{-1}(-\frac{1}{2})$
- $\sin(\sin^{-1}(\frac{\sqrt{3}}{2}))$

(Solution ??) Evaluate the following expressions:

- $\cos^{-1}(0)$
- $\cos(\cos^{-1}(\frac{1}{2}))$

(Solution ??) Evaluate the following expressions:

- $\tan^{-1}(1)$
- $\tan(\tan^{-1}(-\sqrt{3}))$

Lecture 10: Solving Trig Equations

Apr 30 2022 (20:45:24)

The inverse functions we constructed in lecture 9 can be used to solve equations like $\sin(t) = \frac{1}{2}$ but the fraction $\frac{1}{2}$ is a "friendly" sin value so we don't need to use the inverse sin function. We know that $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so we know that $t = \frac{\pi}{6}$ is a solution to $\sin(t) = \frac{1}{2}$. We also know that the sin function is periodic with period 2π, so its values repeat every 2π units. We can represent multiples of the period with the expression $2k\pi$, where k is any integer, $k \in \mathbb{Z}$, so we can represent all the solutions that are "related" to $\frac{\pi}{6}$ with the expression $\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$. This expression represents infinitely many solutions, but it still doesn't represent all the solutions.

Recall the identity $\sin(t) = \sin(\pi - t)$. This identity tells us that angles t and $\pi - t$ always have the same sin value. This means that whenever we've found a solution, t, to an equation involving sin, we can find another solution by computing $\pi - t$. Now let's apply this observation to find the rest of the solutions to $\sin(t) = \frac{1}{2}$. Since we know that $t = \frac{\pi}{6}$ is a solution we know that $t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is another solution. And now we can again utilize the fact that the period of the sin function is 2π so we can express the rest of the solutions with $t = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$. So the complete solution to the equation $\sin(t) = \frac{1}{2}$ is:

$$t = \frac{\pi}{6} + 2k\pi \text{ or } t = \frac{5\pi}{6} + 2k\pi \text{ for all } k \in \mathbb{Z}$$

.

(Solution ??) Solve the equation $\sin(t) = -0.555$.

Lecture 11: Right Triangle Trigonometry

May 14 2022 (07:51:22)

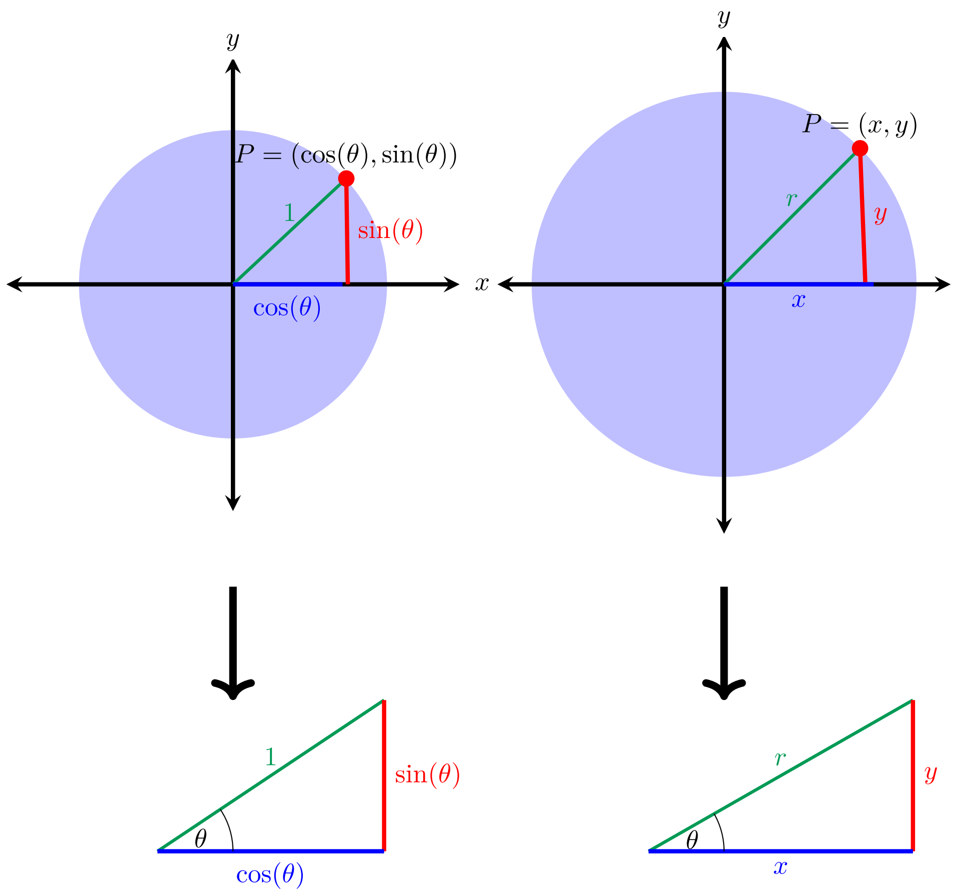

Figure 21: The angle of θ in both a unit circle and in a circle of radius r, including similar right triangles.

We can use these triangles to create ratios.

$$\begin{aligned} \frac{\sin(\theta)}{1} &= \frac{y}{r} \\ \Rightarrow \qquad \sin(\theta) &= \frac{y}{r} \end{aligned}$$

$$\begin{aligned} \frac{\cos(\theta)}{1} &= \frac{x}{r} \\ \Rightarrow \qquad \cos(\theta) &= \frac{x}{r} \end{aligned}$$

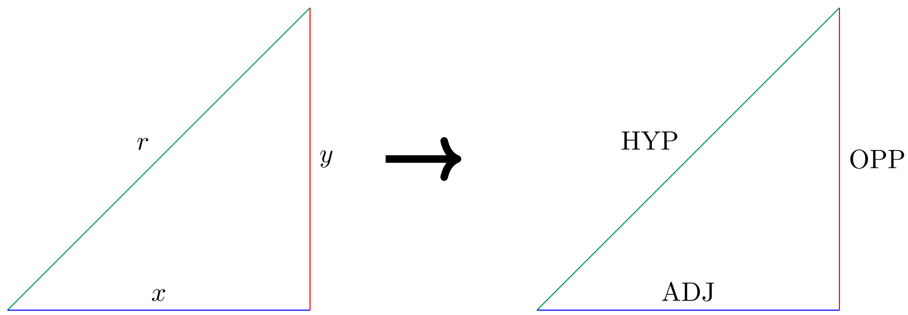

Figure 22: We use the terms **opposite** (or **OPP**), adjacent (or **ADJ**), and **hypotenuse** (or **HYP**) to refer to the sides of a right triangle.

Definition 25. If θ is the angle given in the right triangles in Figure 22, then

$$\sin(\theta) = \frac{y}{r} = \frac{\text{OPP}}{\text{HYP}} \qquad \cos(\theta) = \frac{x}{r} = \frac{\text{AJD}}{\text{HYP}} \qquad \tan(\theta) = \frac{y}{x} = \frac{\text{OPP}}{\text{ADJ}}$$

.

Consequently, the other trigonometric functions can be defined as follows:

$$\cot(\theta) = \frac{\text{ADJ}}{\text{OPP}} \qquad \sec(\theta) = \frac{\text{HYP}}{\text{ADJ}} \qquad \csc(\theta) = \frac{\text{HYP}}{\text{OPP}}$$

.

There's a mnemonic that you can use to remember this:

SOH CAH TOA

(Solution ??) Find the value for all six trigonometric functions of the angle α given in the right triangle in Figure 23. (The triangle isn't drawn to scale)

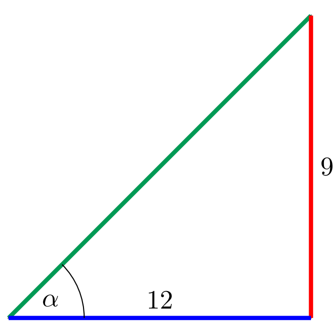

Figure 23

We can use the trig functions, along with the Pythagorean Theorem to **"solve a right triangle"**, i.e., find the missing side-lengths and missing angle-measures for a triangle.

(Solution ??) Solve the right triangle given in Figure 24 by finding A, b, and c. (The triangle may not be drawn to scale.)

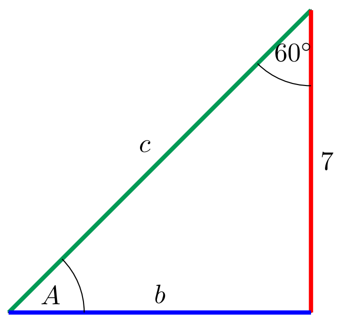

Figure 24

Lecture 12: The Laws of Sin and Cos

May 14 2022 (21:49:09)

The triangle in Figure 25 is a non-right triangle since none of its angles measure $90°$. We'll start by deriving the **Laws of Sin and Cos** so that we can study non-right triangles.

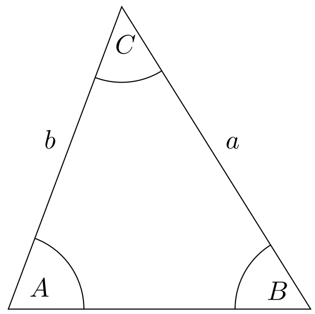

Figure 25

The Law of Sin

To derive the **Law of Sin**, let's construct a segment h in the triangle, which connects the vertex of angle C to the side c. This segment should be perpendicular to side c and is called a *height* of the triangle. See Figure 26.

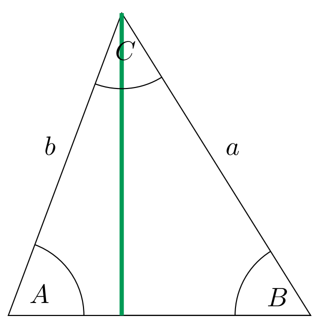

Figure 26

The segment h splits the triangle into two right triangles on which we can apply what we know about right triangle trigonometry. See Figure 27

We can use the 2 right triangles to obtain expressions for both $\sin(A)$ and $\sin(B)$.

$$\sin(A) = \frac{h}{b} \text{ and } \sin(B) = \frac{h}{a}$$

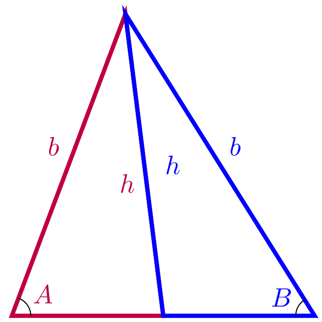

Figure 27

. We can now solve both of these equations for h:

$$\begin{aligned} & \sin(A) = \frac{h}{b} \text{ and } \sin(B) = \frac{h}{a} \\ \Rightarrow \qquad & h = b\sin(A) \text{ and } h = a\sin(B). \end{aligned}$$

Now, since both of the h's represent the length of the same segment, they are equal. By setting h's equal to each other we obtain the following:

$$b\sin(a) = a\sin(B)$$

. This equation provides us with what is known as the **Law of Sin**. Typically, the law is written in terms of ratios. If we divide both sides by $a \times b$, we obtain the following:

$$\begin{aligned} & b\sin(A) = a\sin(B) \\ \Rightarrow \qquad & \frac{b\sin(A)}{a \times b} = \frac{a\sin(B)}{a \times b} \\ \Rightarrow \qquad & \frac{\sin(A)}{a} = \frac{\sin(B)}{b}. \end{aligned}$$

Definition 26 (Law of Sines)**.** If a triangle's sides and angles are labeled like the triangle in Figure 28, then:

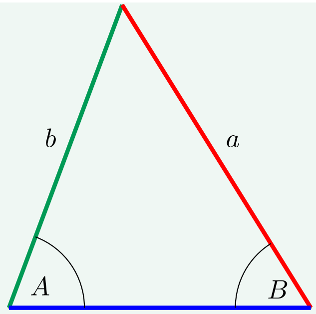

Figure 28

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

. This is an **identity** since it is true for *all* triangles.

(Solution ??) Find all the missing angles and side-lengths of the triangle given in Figure 29 (The triangle is not necessarily drawn to scale.)

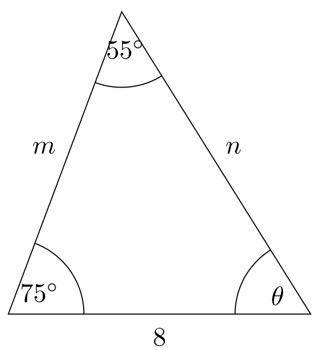

Figure 29

The Law of Cos

To derive the **Law of Cos**, let's start with a generic triangle and draw the height, h, just as we did when we derived the Law of Sin. See Figure 30.

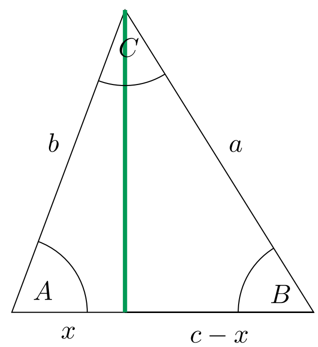

Figure 30

Again we want to consider the two right triangles induced by constructing the line h on the triangle given in Figure 30. This time, we want to use the two pieces that the side c is split into. Let's call the segment on the left x and then the segments on the right must be $c - x$ units long. We've emphasized the two right triangles and labeled the two pieces of side c in Figure 31

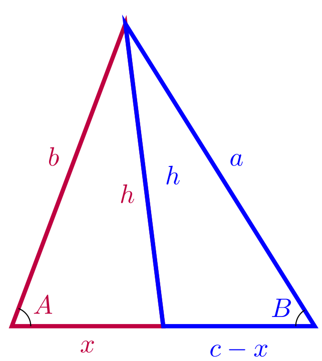

Figure 31

First, notice that the following is true:

$$
\begin{aligned}
& \cos(A) = \frac{x}{b} \\
\Rightarrow \qquad & x = b\cos(A).
\end{aligned}
$$

Now, let's apply the Pythagorean Theorem to each of the two triangles in Figure 31. The blue triangle on the left gives us:

$$x^2 + h^2 = b^2$$

.

and we can solve this for h^2 and obtain:

$$h^2 = b^2 - x^2$$

.

The blue triangle on the right gives us:

$$(c - x)^2 + h^2 = a^2$$

.

and we can use the fact that $h^2 = b^2 - x^2$ to eliminate h from this equation:

$$\begin{aligned} & (c - x)^2 + h^2 = a^2 \\ \Rightarrow \quad & (c - x)^2 + (b^2 - x^2) = a^2. \end{aligned}$$

Finally, we can simplify the left side of this equation and use the fact that $x = b\cos(A)$ to eliminate x:

$$\begin{aligned} & (c - x)^2 + (b^2 - x^2) = a^2 \\ \Rightarrow \quad & c^2 - 2cx + x^2 + b^2 - x^2 = a^2 \\ \Rightarrow \quad & c^2 - 2cx + b^2 = a^2 \\ \Rightarrow \quad & c^2 - 3c \times b\cos(A) + b^2 = a^2 \end{aligned}$$

.

This last equation is known as the **Law of Cosines**.

Definition 27 (Law of Cosines)**.** If a triangle's sides and angles are labeled like the triangle in Figure 33, then:

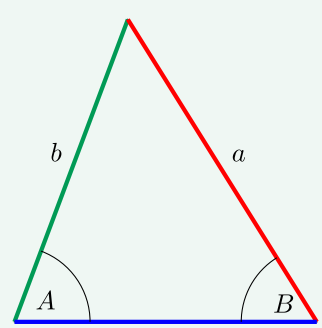

Figure 32

$$a^2 = b^2 + c^2 - 2bc \times \cos(A)$$

. This is an **identity**, since it is true for *all* triangles.

(Solution ??) Find all of the missing angles and side-lengths of the triangle given in Figure 33 (The triangle is not necessarily drawn to scale.)

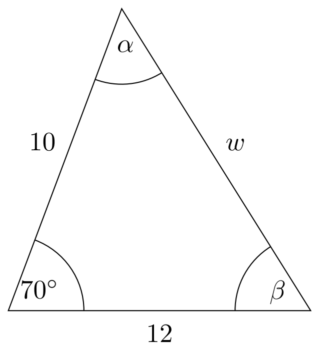

Figure 33

Lecture 13: Proving Trigonometric Identities

May 16 2022 (20:01:59)

Up until now, we've studied trigonometric identities that are useful. But there are many other identities that aren't particularly important but they exist and they offer us an opportunity to learn another skill, which is proving mathematical statements.

Let's prove the identity $\sin(x) = \frac{\tan(x)}{\sec(x)}$. We prove identities by manipulating the expression on one side of the equation until it looks like the expression on the other side of the equation. We can start with either side of the equation, but it's usually most common to start with the **"more complicated"** side since it will be easier to manipulate it:

Proof.

$$\begin{aligned}\frac{\tan(x)}{\sec(x)} &= \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} \\ &= \frac{\sin(x)}{\cos(x)} \times \frac{\cos(x)}{1} \\ \sin(x) &= \sin(x)\end{aligned}$$

.

Q.E.D.

Let's try to prove the identity $\cot(x) + \tan(x) = \csc(x)\sec(x)$. Here, both sides are equally complicated so it's not obvious which side we should start with. In such a case, just start with either side and see what happens. If you get stuck, start over using the other side. Let's start with the left side:

Proof.

$$\begin{aligned}\cot(x) + \tan(x) &= \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos(x)}{\sin(x)} \times \frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)} \times \frac{\sin(x)}{\sin(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)\cos(x)} \\ &= \frac{1}{\sin(x)\cos(x)} \\ \csc(x)\sec(x) &= \csc(x)\sec(x)\end{aligned}$$

.

Q.E.D.

Let's try to prove the identity $\frac{(1+\cos(t))(1-\cos(t))}{\sin(t)} = \sin(t)$. The left side is more complicated, so I'll start with that:

Proof.

$$\begin{aligned}\frac{(1+\cos(t))(1-\cos(t))}{\sin(t)} &= \frac{1-\cos^2(t)}{\sin(t)} \\ &= \frac{\sin^2(t)}{\sin(t)} \\ \sin(t) &= \sin(t) \\ &.\end{aligned}$$

Q.E.D.

Let's try to prove the identity function $\frac{\cos(\theta)}{1-\sin(\theta)} = \frac{1+\sin(\theta)}{\cos(\theta)}$. To prove identities like this, we use a *"trick"* called **conjugate**. A conjugate of an expression is the opposite. For example, the conjugate of $a+b$ is $a-b$.

Proof.

$$\begin{aligned}\frac{\cos(\theta)}{1-\sin(\theta)} &= \frac{\cos(\theta)}{1-\sin(\theta)} \times \frac{1+\sin(\theta)}{1+\sin(\theta)} \\ &= \frac{\cos(\theta)(1+\sin(\theta))}{(1-\sin(\theta))(1+\sin(\theta))} \\ &= \frac{\cos(\theta)(1+\sin(\theta))}{1-\sin^2(\theta)} \\ &= \frac{\cos(\theta)(1+\sin(\theta))}{\cos^2(\theta)} \\ \frac{1+\sin(\theta)}{\cos(\theta)} &= \frac{1+\sin(\theta)}{\cos(\theta)}.\end{aligned}$$

Q.E.D.

Lecture 14: Sum and Difference Identities

May 17 2022 (10:18:43)

Definition 28 (The Sum and Difference of Angles Identites).

$$\text{sin:}\quad \begin{cases} \sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A) \\ \sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A) \end{cases}$$

$$\text{cos:}\quad \begin{cases} \cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \\ \cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \end{cases}$$

(Solution ??) Use the sum-of-angles or the difference-of-angles identities to calculate the following:

- $\cos(75^\circ)$
- $\sin(-15^\circ)$
- $\sin(\frac{11\pi}{12})$

Lecture 15: Double-Angle and Half-Angle Identities

May 21 2022 (22:19:18)

Double-Angle Identities

When creating a right triangle, we typically use the $\angle\theta$ that's opposite of point P. Instead, we'll be using $\angle\alpha$.

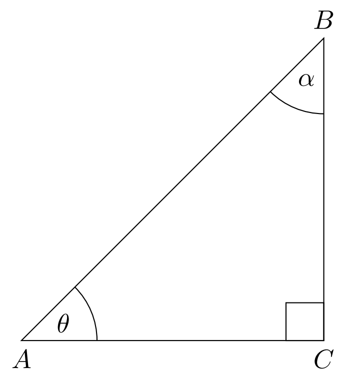

Figure 34

Notice that:

$$\begin{aligned}\sin(\alpha) &= \frac{\cos(\theta)}{1}\\ &= \cos(\theta).\end{aligned}$$

Now, let's construct the mirror-image of this triangle below the x-axis in Quadrant IV.

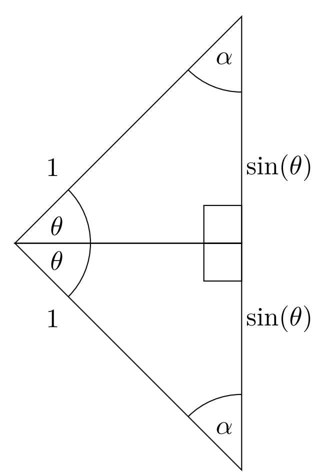

Figure 35

These two right triangles together form a larger non-right triangle that has an angle of measure 2θ. First, let's apply the **Law of Sines** to obtain the double-angle identity for sine.

$$\begin{aligned} & \frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\sin(\alpha)}{1} \\ \Rightarrow \quad & \frac{\sin(2\theta)}{2\sin(\theta)} = \frac{\cos(\theta)}{1} \\ \Rightarrow \quad & \sin(2\theta) = 2\sin(\theta)\cos(\theta). \end{aligned}$$

Next, let's apply the **Law of Cosine** to obtain the double-angle identity for cosine.

$$\begin{aligned} & (2\sin(\theta))^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(2\theta) \\ \Rightarrow \quad & 4\sin^2(\theta) = 1 + 1 - 2\cos(2\theta) \\ \Rightarrow \quad & 4\sin^2(\theta) = 2 - 2\cos(2\theta) \\ \Rightarrow \quad & 2\cos(2\theta) = 2 - 4\sin^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = 1 - 2\sin^2(\theta). \end{aligned}$$

We can use the Pythagorean Theorem identity to obtain two other forms of the double-angle identity for cosine.

$$\begin{aligned} & \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = 1 - 2(1 - \cos^2(\theta)) \\ \Rightarrow \quad & \cos(2\theta) = 1 - 2 + 2\cos^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - 1. \end{aligned}$$

We can obtain a third double-angle identity for substituting $\sin^2(\theta) + \cos^2(\theta)$ for 1.

$$\begin{aligned} & \cos(2\theta) = 2\cos^2(\theta) - 1 \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - (\sin^2(\theta) + \cos^2(\theta)) \\ \Rightarrow \quad & \cos(2\theta) = 2\cos^2(\theta) - \sin^2(\theta) - \cos^2(\theta) \\ \Rightarrow \quad & \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta). \end{aligned}$$

Identity (Double-Angle Identities).

sine	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
cosine	$\begin{cases} \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \cos(2\theta) = 2\cos^2(\theta) - 1 \\ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \end{cases}$

Exercise 1

Suppose that $\sin(\alpha) = \frac{1}{3}$ and that α is in Quadrant II. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

Solution 1

First, find $\cos(\theta)$.

$$\begin{aligned} & \sin^2(\theta) + \cos^2(\theta) = 1 \\ \Rightarrow \quad & \left(\frac{1}{3}\right)^2 + \cos^2(\theta) = 1 \\ \Rightarrow \quad & \frac{1}{9} + \cos^2(\theta) = 1 \\ \Rightarrow \quad & \cos^2(\theta) = \frac{8}{9} \\ \Rightarrow \quad & \cos(\theta) = -\frac{2\sqrt{2}}{3}. \end{aligned}$$

We take the negative square root of $\frac{8}{9}$ since α is in Quadrant II so that $\cos(\alpha)$ must be negative.

Now, let's find $\sin(2\alpha)$.

$$\begin{aligned} \sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\ &= 2\left(\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) \\ \sin(2\alpha) &= -\frac{4\sqrt{2}}{9}. \end{aligned}$$

Now, let's find $\cos(2\alpha)$.

$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2(\theta) \\ &= 1 - 2\left(\frac{1}{3}\right)^2 \\ &= 1 - 2\left(\frac{1}{9}\right) \\ &= 1 - \frac{2}{9} \\ \cos(2\alpha) &= \frac{7}{9}.\end{aligned}$$

Now, let's find $\tan(2\alpha)$.

$$\begin{aligned}\tan(2\alpha) &= \frac{\sin(2\alpha)}{\cos(2\alpha)} \\ &= \frac{-\frac{4\sqrt{2}}{9}}{\frac{7}{9}} \\ &= -\frac{4\sqrt{2}}{9} \times \frac{9}{7} \\ &= -\frac{36\sqrt{2}}{63} \\ \tan(2\alpha) &= -\frac{4\sqrt{2}}{7}.\end{aligned}$$

Half-Angle Identities

We can use the double-angle identities for cosine to derive the **half-angle identities**. Recall that $\cos(2\theta) = 1 - 2\sin^2(\theta)$. We can use this identity to find a half-angle identity for sine.

Let $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$.

$$\begin{aligned} & \cos(2\theta) = 1 - 2\sin^2(\theta) \\ \Rightarrow\quad & \cos(\alpha) = 1 - 2\sin^2(\frac{\alpha}{2}) \\ \Rightarrow\quad & 2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos(\alpha) \\ \Rightarrow\quad & \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos(\alpha)}{2} \\ \Rightarrow\quad & \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}.\end{aligned}$$

We can use $\cos(2\theta) = 2\cos^2(\theta) - 1$ to find a half-angle identity for cosine.

Let $\alpha = 2\theta$. Then $\theta = \frac{\alpha}{2}$.

$$\begin{aligned}
& \cos(2\theta) = 2\cos^2(\theta) - 1 \\
\Rightarrow \quad & \cos(\alpha) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \\
\Rightarrow \quad & 1 + \cos(\alpha) = 2\cos^2\left(\frac{\alpha}{2}\right) \\
\Rightarrow \quad & \frac{1 + \cos(\alpha)}{2} = \cos^2\left(\frac{\alpha}{2}\right) \\
\Rightarrow \quad & \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1 + \cos(\alpha)}{2}}.
\end{aligned}$$

Identity (Half-Angle Identities)**.**

sine	$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$
cosine	$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$

Exercise 2

Suppose that $\sin(\alpha) = \frac{1}{3}$ and that α is in Quadrant II. Find $\sin(\frac{\alpha}{2})$, $\cos(\frac{\alpha}{2})$, and $\tan(\frac{\alpha}{2})$.

Solution 2

First, find $\cos(\theta)$.

$$\begin{aligned}
& \sin^2(\theta) + \cos^2(\theta) = 1 \\
\Rightarrow \quad & \left(\frac{1}{3}\right)^2 + \cos^2(\theta) = 1 \\
\Rightarrow \quad & \frac{1}{9} + \cos^2(\theta) = 1 \\
\Rightarrow \quad & \cos^2(\theta) = \frac{8}{9} \\
\Rightarrow \quad & \cos(\theta) = -\frac{2\sqrt{2}}{3}.
\end{aligned}$$

We take the negative square root of $\frac{8}{9}$ since α is in Quadrant II so that $\cos(\alpha)$ must be negative.

Now, let's find $\sin(\frac{\alpha}{2})$.

Note. Since α is in Quadrant II, $\frac{\pi}{2} < \alpha < \pi$. Thus.

$$\begin{aligned} & \frac{\frac{\pi}{2}}{2} < \frac{\alpha}{2} < \frac{\pi}{2} \\ \Rightarrow \qquad & \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}. \end{aligned}$$

Now, let's find $\sin(\frac{\alpha}{2})$.

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right) &= \sqrt{\frac{1-\cos(\alpha)}{2}} \\ &= \sqrt{\frac{\left(\frac{3}{3} - \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}} \\ &= \sqrt{\frac{1}{2}\left(\frac{3+2\sqrt{2}}{2}\right)} \\ \sin\left(\frac{\alpha}{2}\right) &= \sqrt{\frac{3+2\sqrt{2}}{6}}. \end{aligned}$$

Now, let's find $\cos(\frac{\alpha}{2})$.

$$\begin{aligned} \cos\left(\frac{\alpha}{2}\right) &= \sqrt{\frac{1+\cos(\alpha)}{2}} \\ &= \sqrt{\frac{\left(\frac{3}{3} + \left(-\frac{2\sqrt{2}}{3}\right)\right)}{2}} \\ &= \sqrt{\frac{1}{2}\left(\frac{3-2\sqrt{2}}{2}\right)} \\ \cos\left(\frac{\alpha}{2}\right) &= \sqrt{\frac{3-2\sqrt{2}}{6}}. \end{aligned}$$

Now, let's find $\tan(\frac{\alpha}{2})$.

$$
\begin{aligned}
\tan\left(\frac{\alpha}{2}\right) &= \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} \\
&= \frac{\sqrt{\frac{3+2\sqrt{2}}{6}}}{\sqrt{\frac{3-2\sqrt{2}}{6}}} \\
\tan\left(\frac{\alpha}{2}\right) &= \frac{\sqrt{3+2\sqrt{2}}}{\sqrt{3-2\sqrt{2}}}.
\end{aligned}
$$

Lecture 16: Introduction to Polar Coordinates

May 23 2022 (14:00:25)

Instead of using regular rectangular coordinates, we can use a circular coordinate system to describe points on the plane.

Definition 29 (Polar Coordinates). Ordered pairs in polar coordinates have the following form

$$(r, \theta).$$

Exercise 1

Convert $(\sqrt{3}, 1)$ to polar form.

Solution 1

First, let's find r, the distance from point P to the origin

$$\begin{aligned} & r^2 = (\sqrt{3})^2 + 1^2 \\ \Rightarrow \quad & r^2 = 3 + 1 \\ \Rightarrow \quad & r^2 = 4 \\ \Rightarrow \quad & r = 2. \end{aligned}$$

Now, we need to find the angle between the positive x-axis and the segment labeled r. This is angle θ

$$\begin{aligned} & \sin(\theta) = \frac{1}{2} \\ \Rightarrow \quad & \theta = \frac{\pi}{6}. \end{aligned}$$

Thus, in polar coordinates, $P = (2, \frac{\pi}{6})$.

Exercise 2

Determine the rectangular coordinates of the point $(10, \frac{5\pi}{4})$.

Solution 2

Using this information, we can see that

$$
\begin{aligned}
&\cos\left(\frac{\pi}{4}\right) = \frac{|x|}{10} && && \sin\left(\frac{\pi}{4}\right) = \frac{|y|}{10} \\
\Rightarrow\quad &|x| = 10\cos\left(\frac{\pi}{4}\right) && \text{and} \qquad \Rightarrow\quad && |y| = 10\cos\left(\frac{\pi}{4}\right) \\
\Rightarrow\quad &|x| = 10\frac{\sqrt{2}}{2} && \qquad\quad \Rightarrow\quad && |y| = 10\frac{\sqrt{2}}{2} \\
\Rightarrow\quad &|x| = 5\sqrt{2}, && \qquad\quad \Rightarrow\quad && |y| = 5\sqrt{2}.
\end{aligned}
$$

Since point A is in Quadrant III, we know that both x and y are negative. Thus, the rectangular coordinates of point A are $(-5\sqrt{2}, -5\sqrt{2})$.

Lecture 17: Polar Equations and Functions

May 24 2022 (10:01:01)

Just as we can create equations in rectangular coordinates, we can create equations in polar coordinates.

Example.

$$r = 3\sin(\theta),$$

is an equation in polar coordinates since it's an equation that involves the polar coordinates r and θ.

When we have an equation in either polar or rectangular coordinates, we can convert them from one to another.

Exercise 1

Convert the polar equation $r = 3\sin(\theta)$ into an equivalent equation in rectangular coordinates.

Solution 1

The following identities can be used to convert from polar coordinates (r, θ) to rectangular coordinates (x, y)

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

Since the equation $r = 3\sin(\theta)$ involves r, we just replace it with $\sqrt{x^2 + y^2}$. Since the equation involves $\sin(\theta)$, we can replace it with the following

$$\begin{aligned} & y = r\sin(\theta) \\ \Rightarrow \quad & \sin(\theta) = \frac{y}{r} \\ \Rightarrow \quad & \sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}}. \end{aligned}$$

Thus

$$\begin{aligned} & r = 3\sin(\theta) \\ \Rightarrow \quad & \sqrt{x^2 + y^2} = 3 \times \frac{y}{\sqrt{x^2 + y^2}} \\ \Rightarrow \quad & x^2 + y^2 = 3y. \end{aligned}$$

So, the equation $r = 3\sin(\theta)$ in polar coordinates is equivalent to the equation $x^2 + y^2 = 3y$ in rectangular coordinates.

Exercise 2

Convert the rectangular equation $y = 4x - 3$ into an equivalent equation in polar coordinates.

Solution 2

We can use the following identities to convert from rectangular coordinates to polar coordinates

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}.$$

Thus

$$\begin{aligned} & y = 4x - 3 \\ \Rightarrow \quad & r\sin(\theta) = 4 \times r\cos(\theta) - 3 \\ \Rightarrow \quad & r\sin(\theta) - 4r\cos(\theta) = -3 \\ \Rightarrow \quad & r = -\frac{3}{\sin(\theta) - 4\cos(\theta)} \end{aligned}$$

So, the equation $y = 4x - 3$ in rectangular coordinates is equivalent to the equation $r = -\frac{3}{\sin(\theta) - 4\cos(\theta)}$ in polar coordinates.

Lecture 18: Complex Numbers

May 27 2022 (14:02:24)

Observe (Set of Complex Numbers)**.** Recall that

$$C = \{x | x = a + bi \text{ and } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}.$$

If a complex number has the form $a + bi$, we say that its **real part** is a and its **imaginary part** is b.

Because a complex number has two parts, we can use the two dimensional rectangular coordinate plane to plot complex numbers. We use the horizontal axis to represent the real part and the vertical axis to represent the complex part. The complex number $a + bi$ can be represented with (a, b) in rectangular form. Thus

$$\begin{aligned} a + bi &= r\cos(\theta) + r\sin(\theta) \times i \\ &= r(\cos(\theta) + \sin(\theta) \times i). \end{aligned}$$

Using this, we can derive the following identity

Identity (Euler's Formula)**.**

$$e^{i\theta} = \cos(\theta) + \sin(\theta) \times i.$$

To prove this, we need to understand advanced calculus, so we'll just skip that for now. By multiplying both sides of Euler's formula by r, we obtain the formula

$$re^{i\theta} = r\cos(\theta) + r\sin(\theta) \times i.$$

This allows us to write any complex number in polar form.

Identity (Polar Form of Complex Number)**.** The **polar form** of the complex number $z = a + bi$ is

$$z = re^{i\theta},$$

where $r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$.

Exercise 1

Express the $z = 6e^{i \times \frac{5\pi}{6}}$ in the form $z = a + bi$.

Solution 1

$$\begin{aligned}
z &= 6e^{i\times\frac{5\pi}{6}} \\
&= 6\cos\left(\frac{5\pi}{6}\right) + 6\sin\left(\frac{5\pi}{6}\right)\times i \\
&= 6\left(-\frac{\sqrt{3}}{2}\right) + 6\left(\frac{1}{2}\right)\times i \\
&= -3\sqrt{3} + 3i.
\end{aligned}$$

Therefore, $z = 6e^{i\times\frac{5\pi}{6}}$ can be expressed as $z = -3\sqrt{3} + 3i$.

Exercise 2

Express the complex number $z = 3 - 3i$ in polar form, $z = re^{i\theta}$.

Solution 2

Lecture 19: Intro to Vectors

Jun 01 2022 (10:33:28)

Vectors are mathematical objects used to represent physical quantities like velocity, force, and displacement.

Definition 30 (Vectors). A **vector** is a mathematical object that has both a magnitude and a direction.

In order to distinguish between vectors and scalers, we need to use a different notation to denote vectors.

Notation. To denote a vector, we use a small arrow on top.

$$\vec{\mathbf{v}}$$

.

A two-dimensional vector can be represented by an arrow on the coordinate plane. The length of the arrow represents the **magnitude** of the vector and the **direction** of the arrow represents the direction of the vector.

Example. The vector $\vec{\mathbf{v}}$ is depicted as an arrow on the coordinate plane.

TODO: Draw vector on coordinate plane (with tikz)

The **tip** of the vector is where the arrow ends and the **tail** of the vector is where the arrow begins. Therefore, the tip of $\vec{\mathbf{v}}$ is at the point $(4, 3)$ and the tail of the vector is at the origin, $(0, 0)$.

Notation. We denote the magnitude of vector $\vec{\mathbf{v}}$ by $\|\vec{\mathbf{v}}\|$.

To find the magnitude of $\vec{\mathbf{v}}$, we need to find the length of the arrow. We an do this by thinking of the arrow as being the hypotenuse of a right-triangle with side lengths of 4 and 3 and then use the Pythagorean Theorem to find $\|\vec{\mathbf{v}}\|$.

$$\begin{aligned} \|\vec{\mathbf{v}}\| &= \sqrt{a^2 + b^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= 5. \end{aligned}$$

We can find the angle between the positive x-axis and the arrow to describe the **direction** of the vector. We can use trig to find the angle θ.

$$\begin{aligned} \tan(\theta) &= \frac{3}{4} \\ \theta &= \tan^{-1}\left(\frac{3}{4}\right) \\ &\approx 36.9^\circ \end{aligned}$$

.

Although the magnitude and direction of the vector describe it completely, it is often useful to describe it by using its **horizontal** and **vertical** components. An example is:

$$\vec{\mathbf{v}} = \langle 4, 3 \rangle$$

.

When graphing vectors, you don't need to start anywhere specific. For example, all the arrows in Figure **??** represent $\vec{\mathbf{v}}$ since all of these vectors have a horizontal component of 4 units and a vertical component of 3 units.

Vector Operations

We can multiply any vector by a scalar and we can add or subtract any two vectors.

When we multiply a vector by a scalar, we multiply the components of the vector by the scalar. Thus, $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ and $k \in \mathbb{R}$, then $k\vec{\mathbf{a}} = \langle ka_1, ka_2 \rangle$

Theorem 4. If $\vec{\mathbf{a}} = \langle a_1, a_3 \rangle$ is a vector and $k \in \mathbb{R}$, then $k\vec{\mathbf{a}} = \langle ka_1, ka_2 \rangle$ has a magnitude $|k| \times \|\vec{\mathbf{a}}\|$. If $k > 0$, then $k \times \vec{\mathbf{a}}$ points in the same direction as $\vec{\mathbf{a}}$. If $k < 0$, then $k \times \vec{\mathbf{a}}$ points in the opposite direction as $\vec{\mathbf{a}}$.

(Solution ??) Let $\vec{\mathbf{v}} = \langle 4, 3 \rangle$. Find and draw vectors $\vec{\mathbf{m}} = 2\vec{\mathbf{v}}$ and $\vec{\mathbf{n}} = -2\vec{\mathbf{v}}$:

Lecture 20: The Dot Product

Jun 02 2022 (10:34:00)