Calculus 3: Takehome Final Exam

Due on June 11, 2023 at 11:59 PM

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Problem 1

Consider estimating the value of $e^{0.5}$. Answer the following.

a.) Using a known Maclaurin series, find the Taylor Polynomial of degree 4, $T_4(x)$, for $f(x) = e^x$ centered at $a = 0$. No work is needed. [***4pts***]

b.) Use this 4^{th} degree Taylor Polynomial to estimate the value of $e^{0.5}$, rounding to six decimal places. [***4pts***]

c.) Use **Taylor's Inequality** to determine the degree, n, of the Taylor Polynomial, $T_n(x)$, sufficient to approximate the value of $e^{0.5}$ accurate to six decimal places, and then use this n^{th} Taylor polynomial to approximate the value of $e^{0.5}$ accurate to six decimal places. Show all relevant work in an organized manner. [***4pts***]

Solution 1

a.) $f(x) = e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

b.) Using this, we get

$$\begin{aligned} f(0.5) &= e^{0.5} \\ &= \sum_{k=0}^{\infty} \frac{0.5^k}{k!} \\ &\approx 1 + 0.5 + \frac{0.25}{2} + \frac{0.125}{6} + \frac{0.0625}{24}. \end{aligned}$$

The 4^{th} degree polynomial rounds to 1.648438.

c.) To perform **Taylor's Inequality**, we need to find an M, such that $\left|f^{(k+1)}(x)\right| \leq M$, $\forall\, |x-a| \leq d$, where d is the interval. We know that the sum is centered at 0 and we need the interval to include 0.5. So, I'll let $d = [-0.5, 0.5]$. We need M to be bigger than $e^{0.5}$. Let $M = 2$.

We can now use Taylor's Inequality formula

$$|R_n| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

Plugging in all the values gives us

$$10^{-6} \leq \frac{2}{(n+1)!} |0.5|^{n+1}.$$

The value that makes this inequality true is $n = 7$, with an approximate error 1.937624×10^{-7}, which is smaller than 10^{-6}.

Problem 2

Consider the separable differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3yx^2 + y$.

a.) Solve the separable differential equation, simplifying as much as possible. [***10pts***]

b.) Solve the initial value problem if $y(0) = 2$ for the solution you got in part **a**. [***5pts***]

Solution 2

a.) To solve this separable differential equation, we first need to move all the y's on the left side and all the x's on the right

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1.$$

We now just integrate both sides

$$\begin{aligned}\int \frac{1}{y}\mathrm{d}y &= \int 3x^2 + 1\mathrm{d}x \\ \ln(|y|) &= \int 3x^2\mathrm{d}x + \int 1\mathrm{d}x \\ \ln(|y|) &= x^3 + x + C \\ &.\end{aligned}$$

b.) To solve for the specific point $(0, 2)$, we plug in 0 for all the x's and 2 for all the y's

$$\begin{aligned}\ln(|y|) &= x^3 + x + C \\ \ln(2) &= 0^3 + 0 + C \\ C &= \ln(2).\end{aligned}$$

$\therefore$ our specific solution to the separable differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2y + y$ that passes through the point $(0, 2)$ is

$$\ln(|y|) = x^3 + x + \ln(2).$$

Problem 3

Use the Remainder Estimation Theorem for the Integral Test (RETIT) and a calculator decide the maximum possible error that the 100$^{\text{th}}$ partial sum of $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^4}$ is to it's true value (which we can consider unknown). Show integration by hand. **[*5pts*]**

Solution 3

Let $a_n = \dfrac{3}{n^4}$. To apply RETIT on the sum, it first must pass all the criteria of the Integral Test (IT), which are

1. $\forall n \in a_n,\ n > 0$? ✓
2. a_n is continuous from $[1, \infty)$? ✓
3. Is $\{|a_n|\}$ strictly decreasing? ✓

Since all the above criteria were met, we can perform the IT, which states that

$$\int_N^\infty \frac{3}{n^4}\mathrm{d}n \text{ and } \sum_{n=N}^{\infty} \frac{3}{n^4},$$

share the same convergence status.

We can go ahead and perform the integral

$$\begin{aligned}
\int_1^\infty \frac{3}{n^4}\mathrm{d}n &= 3 \lim_{a\to\infty} f(x) \int_1^a \frac{1}{n^4}\mathrm{d}n \\
&= 3 \lim_{a\to\infty} \int_1^a n^{-4}\mathrm{d}n \\
&= 3 \lim_{a\to\infty} \left.\frac{n^{-4+1}}{-4+1}\right|_1^a \\
&= \lim_{a\to\infty} -\left.\frac{\cancel{3}}{\cancel{3}n^3}\right|_1^a \\
&= \lim_{a\to\infty} \left(\left[-\frac{1}{a^3}\right] - \left[-\frac{1}{1^4}\right]\right) \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

Since the improper integral converges, that means the original series must also converge.

According to the RETIT, we can approximate the error of an nth partial sum to the actual value of the infinite series. Let $R_n = S - S_n$, where S is the value of the infinite series and S_n is the nth partial sum. Another way of defining R_n is

$$\int_{n+1}^\infty a_n \mathrm{d}n \le R_n \le \int_n^\infty a_n \mathrm{d}n.$$

We can now perform the integrals and evaluate to get our maximum possible error at the 100th partial sum

$$\int_{n+1}^\infty \frac{3}{n^4}\mathrm{d}n \le R_n \le \int_n^\infty \frac{3}{n^4}\mathrm{d}n$$

$$
\begin{aligned}
3 \lim_{a\to\infty} \int_{n+1}^{a} \frac{1}{n^4}\mathrm{d}n &\le R_n \le 3 \lim_{a\to\infty} \int_{n}^{a} \frac{1}{n^4}\mathrm{d}n \\
3 \lim_{a\to\infty} \int_{n+1}^{a} n^{-4}\mathrm{d}n &\le R_n \le 3 \lim_{a\to\infty} \int_{n}^{a} n^{-4}\mathrm{d}n \\
3 \lim_{a\to\infty} \left.\frac{n^{-4+1}}{-4+1}\right|_{n+1}^{a} &\le R_n \le 3 \lim_{a\to\infty} \left.\frac{n^{-4+1}}{-4+1}\right|_{n}^{a} \\
\lim_{a\to\infty} \left[-\frac{\cancel{3}n^{-3}}{\cancel{3}}\right]_{n+1}^{a} &\le R_n \le \lim_{a\to\infty} \left[-\frac{\cancel{3}n^{-3}}{\cancel{3}}\right]_{n}^{a} \\
\lim_{a\to\infty} \left[-\frac{1}{n^3}\right]_{n+1}^{a} &\le R_n \le \lim_{a\to\infty} \left[-\frac{1}{n^3}\right]_{n}^{a} \\
\lim_{a\to\infty} \left[-\frac{1}{a^3} - -\frac{1}{(n+1)^3}\right] &\le R_n \le \lim_{a\to\infty} \left[-\frac{1}{a^3} - -\frac{1}{n^3}\right] \\
\left[0 + \frac{1}{(n+1)^3}\right] &\le R_n \le \left[0 + \frac{1}{n^3}\right] \\
\frac{1}{(n+1)^3} &\le R_n \le \frac{1}{n^3} \\
\frac{1}{(100+1)^3} &\le R_{100} \le \frac{1}{100^3} \\
9.705 \times 10^{-7} &\le R_{100} \le 10^{-6}.
\end{aligned}
$$

According to RETIT, the 100^{th} partial sum is estimated to have a maximum error of 10^{-6} and a minimum error of 9.705×10^{-7}.

Problem 4

Consider the integral of $\int \frac{1}{1+x^3}\mathrm{d}x$. Recall that $\sum_{n=0}^{\infty} u^n = \frac{1}{1-u}$ on $|u| < 1$.

a.) Create a power series representation for $\int \frac{1}{1+x^3}\mathrm{d}x$. [***10pts***]

b.) Represent $\int_0^{0.5} \frac{1}{1+x^3}\mathrm{d}x$ as an infinite series using your answer to part **a**. [***10pts***]

c.) Use the Alternating Series Estimation Theorem and a calculator to determine the number of term sufficient to approximate the definite integral $\int_0^{0.5} \frac{1}{1+x^3}\mathrm{d}x$ accurate to within 0.000001, showing all work. Then approximate the definite integral within that accuracy. [***5pts***]

Solution 4

a.) We can perform some simple algebra to get the integrand into the required form

$$\begin{aligned}
\int \frac{1}{1+x^3}\mathrm{d}x &= \int \frac{1}{1-(-x^3)}\mathrm{d}x \\
&= \int \sum_{n=0}^{\infty} \left(-x^3\right)^n \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \left((-1)^n \cdot \int x^{3n}\mathrm{d}x \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{3n+1}}{3n+1}.
\end{aligned}$$

b.) We can perform the same algebra as above, and perform the integral with bounds

$$\begin{aligned}
\int_0^{0.5} \frac{1}{1+x^3}\mathrm{d}x &= \int_0^{0.5} \frac{1}{1-(-x^3)}\mathrm{d}x \\
&= \int_0^{0.5} \sum_{n=0}^{\infty} \left(-x^3\right)^n \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \left((-1)^n \cdot \int_0^{0.5} x^{3n}\mathrm{d}x \right) \\
&= \sum_{n=0}^{\infty} \left((-1)^n \cdot \left[\frac{x^{3n+1}}{3n+1} \right]_0^{0.5} \right) \\
&= \sum_{n=0}^{\infty} \left((-1)^n \cdot \left[\frac{0.5^{3n+1}}{3n+1} - \frac{0^{3n+1}}{3n+1} \right] \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n \cdot 0.5^{3n+1}}{3n+1} \right).
\end{aligned}$$

c.) The Alternating Series Estimation Theorem (ASET) states that $|S - S_n| = |R_n| \leq a_{n+1}$, which in this case turns out to be

$$|S - S_n| = 10^{-7} \leq \frac{(-1)^{n+1} \cdot 0.5^{3n+4}}{3n+4}.$$

Solving for n gives us $n = 5$, with the value being 0.116784384679, with an approximate error of $7.8117405669 \times 10^{-8}$, which is less than 10^{-7}.

Problem 5

Use Euler's Method for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$ to generate an approximation for the value of $y(2)$ if you know that $y(0) = 2$, with a change-in- x value of 0.5. A graph of the slopefield of $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$ and the specific solution that passes through $(0, 2)$ are shown for confirmation of your estimations. Show your calculations: use fractions for all 4 values. [***10pts***]

Solution 5

To use Euler's method to approximate the value of $y(2)$ for the given differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$, we will use the provided initial condition $y(0) = 2$ and a step size of $\Delta x = 0.5$. The general formula for Euler's method is:

$$y_{n+1} = y_n + f(x_n, y_n) \cdot \Delta x,$$

where y_n and x_n are the current values of y and x, y_{n+1} is the next approximation of y, $f(x_n, y_n)$ is the value of the differential equation at (x_n, y_n), and Δx is the step size.

n	x_n	y_n
0	0	2
1	0.5	$2 + \frac{1}{2} \cdot \frac{0}{2} = 2$
2	1	$2 + \frac{1}{2} \cdot \frac{0.5}{2} = \frac{17}{8}$
3	1.5	$\frac{17}{8} + \frac{1}{2} \cdot \frac{1}{\frac{17}{8}} = \frac{321}{136}$
4	2	$\frac{321}{136} + \frac{1}{2} \cdot \frac{1.5}{\frac{321}{136}} = \frac{38971}{14552}$

Problem 6

Extra Credit: Use series to solve the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$. [Hints: the solutions are $y = C_1\cos(x) + C_2\sin(x)$, but don't use that to "solve", just check your answer at the end. The even and odd powered terms will be generating two different series (one for cosine and one for sine), so create a table large enough to see BOTH patterns.] [+***5pts***]

Solution 6

Let y be a power series where $x_0 = 0$. This gives us

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

To solve the ordinary differential equation (ODE), we must take the first and second derivative of y

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sum_{n=1}^{\infty} a_n n x^{n-1} &&= \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} &&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n.
\end{aligned}$$

We can use these summations to substitute into the original ODE

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \left[(a_{n+2}(n+2)(n+1) + a_n)\cdot x^n\right]
\end{aligned}$$

.

Since the whole sum equals 0 and since x^n is only 0 once, that means all the constants to its left must equal 0. We can setup an equation and solve for a_{n+2}

$$\begin{aligned}
0 &= a_{n+2}(n+2)(n+1) + a_n \\
a_{n+2} &= -\frac{a_n}{(n+2)(n+1)}.
\end{aligned}$$

This is a recursion relation, where the $(n+2)$nd term is using the nth term. This means that all the even integers will be mapped together and all the odd terms will be mapped together, making a total of two sums in our final answer. We can find the pattern for each one and then create the sum at the end.

Even	Odd
$a_2 = -\frac{a_0}{2\cdot 1}$	$a_3 = -\frac{a_1}{3\cdot 2}$
$a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!}$	$a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}$
$a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}$	$a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_0}{7!}$
$a_{2k} = -\frac{(-1)^k\cdot a_0}{(2k)!}$	$a_{2k+1} = -\frac{(-1)^k\cdot a_1}{(2k+1)!}$

Problem 6 continued on next page ...

Let's expand y to see the pattern and then start to put all the pieces together

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot a_0}{(2n)!} \cdot x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot a_1}{(2n+1)!} \cdot x^{2n+1}. \end{aligned}$$

The last two sums are the power series expansion of $\cos(x)$ and $\sin(x)$. So, our final answer is

$$y = a_0 \cos(x) + a_1 \sin(x).$$