Calculus 1

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Lecture notes from the second-year undergraduate course Calculus, given by professor Ross Kouzes at the Faculty of Mathematics, Science, and Technology. at Portland Community College in the academic year 2022. This course covers rates of change, limits and continuity, first derivative, anti derivatives, derivative formulas, chain rule, implicit differentiation, related rates, critical numbers, and graphing from formulas. Credit for the material in these notes is due to professor Ross, while the structure is loosely taken from the Calculus Lab Manual. The credit for the typesetting is my own.

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Lecture 1: Rates of Change

1.1 Velocity

Motion is frequently modeled using calculus. A building block for this application is the concept of **average velocity**.

Definition 1.1: Average Velocity

If p is a position function for something moving along a numbered line, then we define the **average velocity** over the time interval $[t_0, t_1]$ to be

$$\frac{p(t_1) - p(t_0)}{t_1 - t_0}.$$

1.2 Secant Line to a Curve

One of the building blocks in differential calculus is the secant line to a curve. The only requirement for a line to be considered a secant line to a curve is that the line must intersect the curve in at least two points.

The formula for f is $f(x) = 3 + 2x - x^2$. We can use this formula to come up with a generalized formula for the slope of a secant line on this curve. Specifically, the slope of the line connecting the poitn $(x_0, f(x_0))$ to the point $(x_1, f(x_1))$ is derived in the following example.

Example.

$$\begin{aligned}
m_{\text{sec}} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
&= \frac{(3 + 2x_1 - x_1^2) - (3 + 2x_0 - x_0^2)}{x_1 - x_0} \\
&= \frac{3 + 2x_1 - x_1^2 - 3 - 2x_0 + x_0^2}{x_1 - x_0} \\
&= \frac{(2x_1 - 2x_0) - (x_1^2 - x_0^2)}{x_1 - x_0} \\
&= \frac{2(x_1 - x_0) - (x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} \\
&= \frac{[2 - (x_1 + x_0)](x_1 - x_0)}{x_1 - x_0} \\
&= 2 - x_1 - x_0 \text{ where } x_1 \neq x_0.
\end{aligned}$$

1.3 The Difference Quotient

The algebra associated with secant lines (and average velocities) can sometimes be simplified if we designate the variable h to be the run between the two points. The expression on the right is called the **difference quotient** for f.

Definition 1.2: Difference Quotient

The difference quotient for the function $y = f(x)$ is the expression

$$\frac{f(x+h) - f(x)}{h}.$$

Example. The difference quotient for $f(x) = -16x^2 - 32x + 1008$ is as follows

$$\begin{aligned} f(x+h) &= -16(x+h)^2 - 32(x+h) + 1008 \\ &= -16(x^2 + 2xh + h^2) - 32(x+h) + 1008 \\ &= -16x^2 - 32xh - 16h^2 - 32x - 32h + 1008. \end{aligned}$$

Now, we plug and play

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{f(x+h) - [-16x^2 - 32x + 1008]}{h} \\ &= \frac{[-16x^2 - 32xh - 16h^2 - 32x - 32h + 1008] - [-16x^2 - 32x + 1008]}{h} \\ &= \frac{-16x^2 - 32xh - 16h^2 - 32x - 32h + 1008 + 16x^2 + 32x - 1008}{h} \\ &= \frac{-32xh - 16h^2 - 32h}{h} \\ &= \frac{16h(-2x - h - 2)}{h} \\ &= 16(-2x - h - 2) \\ &= -16h - 32x - 32 \text{ and } h \neq 0. \end{aligned}$$

1.4 Exercises

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Lecture 2: Limits and Continuity

2.1 Limits

Limits are used to explore the trend of a function as the input approaches a number, rather than the actual value of the function at that number. Mathematically, we describe these trends using **limits**.

Definition 2.1: Limit

The limit of $f(x)$, as x approaches a, equals L.
We denote this as follows

$$\lim_{x \to a} f(x).$$

We can specify where the limit is approaching a. Here's how we do it when it's approaching from the left

$$\lim_{x \to a^-} f(x).$$

And from the right

$$\lim_{x \to a^+} f(x).$$

Note.

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).$$

Put it simply, a limit approaching both sides exists if and only if ($\Leftrightarrow$) a limit exists approaching from the left and from the right.

Example.

$g(5) = 1$. There's a hole for the output of $g(5)$. Above it, there's a filled hole, meaning that's the actual output of $g(5)$. But, a limit doesn't care if the input for the function is defined. It just cares what the output is supposed to be, if it exists or not. $\lim_{x \to 5} g(x) = -1$. Here's where it gets tricky though. As mentioned above, a limit can't exist if the limit from the right and the limit from the left aren't equal. For example, $\lim_{x \to -2} g(x)$ ***does not exist***. It's because $\lim_{x \to -2^+} g(x) \approx 4.5 \neq \lim_{x \to -2^-} g(x) = 5$. So, the limit only exists if the limit from the left and right are equal. The limits from both ends exist, but the limit coming from both sides doesn't. To denote this, we just write $\lim_{x \to -2} g(x)$ DNE, where DNE means Does Not Exist.

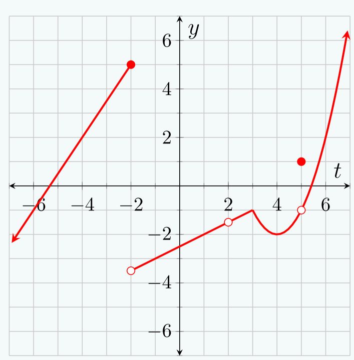


Figure 1: $y = g(t)$

2.2 Limit Laws

Theorem 2.1 (Replace Laws). The following laws allow you to replace a limit expression with an actual value. For example, the expression $\lim_{t\to 7^-} 6$ can be replaced with the number 6.

- R1: $\lim_{x\to a} x = \lim_{x\to a^-} x = \lim_{x\to a^+} x = a$.
- R2: $\lim_{x\to a} C = \lim_{x\to a^-} C = \lim_{x\to a^+} C = \lim_{x\to\infty} C = \lim_{x\to -\infty} C = C$.
- R3:

$$\lim_{x\to\infty} f(x) = \infty \implies \lim_{x\to\infty} \frac{C}{f(x)} = 0$$
$$\lim_{x\to\infty} f(x) = -\infty \implies \lim_{x\to\infty} \frac{C}{f(x)} = 0$$
$$\lim_{x\to-\infty} f(x) = \infty \implies \lim_{x\to-\infty} \frac{C}{f(x)} = 0$$
$$\lim_{x\to-\infty} f(x) = -\infty \implies \lim_{x\to-\infty} \frac{C}{f(x)} = 0.$$

Example.

- R1: $\lim_{x\to 4} x = 4$.
- R2: $\lim_{x\to a} 6 = 6$.
- R3: $\lim_{x\to\infty} \frac{10}{x} = 0$. Since a constant number over a huge positive number gives you a small number. For example

$$\frac{1}{10} = 0.1, \frac{1}{100} = 0.01, \frac{1}{1000} = 0.001\ldots.$$

Same thing for a constant over a negative huge number.

Many limit values don't exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases, we use the symbol ∞ and $-\infty$ to communicate the non-existence of the limits.

Theorem 2.2 (Algebraic Laws). The following laws allow you to replace a limit expression with equivalent limit expressions. For example, the expression

$$\lim_{x\to 5}(2 + x^2),$$

can be replaced with the expression

$$\lim_{x\to 5} 2 + \left(\lim_{x\to 5} x\right)^2.$$

Note. *Limit Laws A1-A6 are valid if and only if every limit in the equation exists. Meaning, any limit that approaches infinity doesn't exist, which implies that we can't use these limit laws on it.*

- A1: $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
- A2: $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.

- A3: $\lim_{x\to a}(Cf(x)) = C\lim_{x\to a} f(x)$.
- A4: $\lim_{x\to a}(f(x)\times g(x)) = \lim_{x\to a} f(x)\times\lim_{x\to a} g(x)$.
- A5: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \Leftrightarrow \lim_{x\to a} g(x)\neq 0$.
- A6: $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.
- A7: If there exists an open interval centered at a over which $f(x)=g(x)$ for $x\neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided that both limits exist.

  Additionally, if there exists an open interval centered at a over which $f(x)=g(x)$ for $x\neq a$, then $\lim_{x\to a}\infty \Leftrightarrow \lim_{x\to a} g(x)=\infty$, and similarly, $\lim_{x\to a} f(x) = -\infty \Leftrightarrow \lim_{x\to a} g(x) = -\infty$

Example.

- A1: $\lim_{x\to a} x^2+5 = \lim_{x\to a} x^2 + \lim_{x\to a} 5$.
- A2: $\lim_{x\to a} x^2-5 = \lim_{x\to a} x^2 - .\lim_{x\to a} 5$.
- A3: $\lim_{x\to a}(zx^2) = z\lim_{x\to a} x^2$.
- A4: $\lim_{x\to a} x^2\times x^x = \lim_{x\to a} x^2\times\lim_{x\to a} x^x$.
- A5: $\lim_{x\to a}\frac{x^2+y^x-55}{xy} = \frac{\lim_{x\to a} x^2+y^x-55}{\lim_{x\to a} xy}$.
- A6: $\lim_{x\to a} x^2 = \left(\lim_{x\to a} x\right)^2$.
- A7: $\lim_{x\to a}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x\to a} x+3$ where $x+3\neq 0$

Now, let's put everything together.

$$\begin{aligned}
\lim_{x\to 7}(4x^2+3) &= \lim_{x\to 7} 4x^2 + \lim_{x\to 7} 3 && \text{Limit Law A1}\\
&= 4\lim_{x\to 7} x^2 + \lim_{x\to 7} 3 && \text{Limit Law A3}\\
&= 4\left(\lim_{x\to 7} x\right)^2 + \lim_{x\to 7} 3 && \text{Limit Law A6}\\
&= 4\times 7^2+3 && \text{Limit Law R1, R2}\\
&= 199.
\end{aligned}$$

2.3 Indeterminate Limits

Definition 2.2: Indeterminate limits

Many limits have the form $\frac{0}{0}$, which means the expressions in both the numerator and denominator limit to zero. e.g.

$$\lim_{x\to 3}\frac{2x-6}{x-3}.$$

The form $\frac{0}{0}$ is called **indeterminate** because we don't know the value of the limit.

When we face an indeterminate limit, we must algebraically manipulate the expression so common factors causing the zeros in the numerator and denominator are isolated.

Example.

$$\begin{aligned}
\lim_{x\to 3}\frac{x^2-8x+15}{x-3} &= \lim_{x\to 3}\frac{(x-5)(x-3)}{x-3} \\
&= \lim_{x\to 3} x-5 && \text{Limit Law A7} \\
&= \lim_{x\to 3} x - \lim_{x\to 3} 5 && \text{Limit Law A2} \\
&= 3-5 && \text{Limit Law R1, R2} \\
&= -2.
\end{aligned}$$

2.4 Limits at Infinity

Most of the time, we're interested in the function's end behavior. That is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times, a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote $y = L$ represents the end behavior of the function f both as x increases without bound and as x decreases into negative without bound, we write $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$.

Note.

The entire point is that x is increasing without any bound on how large its value becomes. Secondly, there is no place on the real number ($\mathbb{R}$) line called **"infinity"**. Infinity is not a number. It's a concept. Hence, x certainly can't be approaching something that isn't even there. This just means that x is increasing without bound.

2.5 Limits at Infinity Tending to Zero

When $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = -\infty$, there isn't much that can be done here. Except, one thing, which is using Limit Law R3, which states that if a function is increasing or decreasing without any bound, then the value of a constant divided by that function must be approaching zero.

Example. Say the function $f(x) = x^2$.

Solve the following limit

$$\lim_{x\to\infty} \frac{5}{f(x)} = 0.$$

To solve this, I used the Limit Law R3, which states that *a small number over a huge number (like infinity) is going to zero.*

We can see this in the following demenstration

$$\frac{1}{10} = 0.1, \frac{1}{100} = 0.01, \frac{1}{1000} = 0.001\ldots.$$

2.6 Ratios of Infinities

Many limits have the form $\frac{\infty}{\infty}$, which we take to mean that the expressions in both the numerator and denominator are increasing or decreasing without bound. When we come across a limit with that form, we can resolve the limit if we multiply both the numerator and denominator by the inverse of the largest exponent in the denominator. The reason is because a small number of a big number goes to 0. This will make the limit in determinate form. The rest is basic algebra and limit manipulation.

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3t^2+5t}{3-5t^2} &= \lim_{x\to\infty} \left(\frac{3t^2+5t}{3-5t^2} \times \frac{\frac{1}{t^2}}{\frac{1}{t^2}}\right) \\
&= \lim_{x\to\infty} \frac{3+\frac{5}{t}}{\frac{3}{t^2}-5} && \text{No longer indeterminate form} \\
&= \frac{\lim_{x\to\infty}\left(3+\frac{5}{t}\right)}{\lim_{x\to\infty}\left(\frac{3}{t^2}-5\right)} && \text{Limit Law A5} \\
&= \frac{\lim_{t\to\infty} 3 + \lim_{t\to\infty}\frac{5}{t}}{\lim_{t\to\infty}\frac{3}{t^2} - \lim_{t\to\infty} 5} && \text{Limit Law A1, Limit Law A2} \\
&= \frac{3+0}{0-5} && \text{Limit Law R2, Limit Law R3} \\
&= -\frac{3}{5}.
\end{aligned}$$

Note. *Did you notice something? In the original limit, the highest power coefficient in the numerator over the highest power coefficient in the denominator was the answer.*

2.7 Non-existent Limits

Many limit values don't exist. Sometimes the non-existence is caused by the function value either increasing or decreasing without bound. These are called vertical asymptotes, when the function value is getting close to an x-value, but it never reaches it, which means that the original function is being divided by 0, where that x-value is making it divide by 0.

Example. Here's an example of a function with a vertical asymptote at $x = 2$

$$f(x) = \frac{5x}{x-2}.$$

When $x = 2$, then the output of $f(x) = \frac{10}{0}$, which creates a vertical asymptote since the function value for 2 isn't defined.

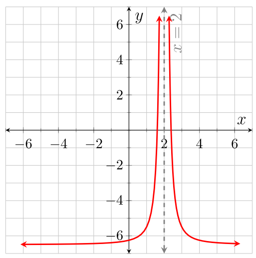


Figure 2: $y = X(a)$

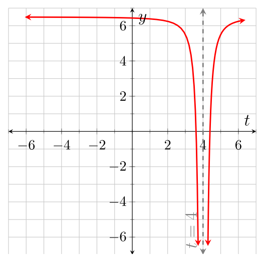


Figure 3: $y = Y(a)$

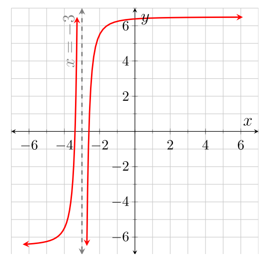


Figure 4: $y = Z(a)$

Here's how we denote that the point x doesn't exist on the given figures.

– Here's how we write it out for figure 2
$$\lim_{x \to 2} K(x) = \infty, \lim_{x \to 2^-} K(x) = \infty, \text{ and } \lim_{x \to 2^+} K(x)\infty.$$
– Here's how we write it out for figure 3
$$\lim_{x \to 4} Y(x) = -\infty, \lim_{x \to 4^-} Y(x) = -\infty, \text{ and } \lim_{x \to 4^+} Y(x) = -\infty.$$
– Here's how we write it out for figure 4
$$\lim_{x \to 3^-} Z(x) = \infty \text{ and } \lim_{x \to 3^+} Z(x) = -\infty.$$

2.8 Vertical Asymptotes

Definition 2.3: Vertical Asymptote

A vertical asymptote is a vertical line that guides the graph of the function but is not part of it. It can never be crossed by the graph because it occurs at the x-value that is not in the domain of the function. A function may have more than one vertical asymptote. Just like the line in Figure 2, when $x = 2$.

Whenever $\lim_{x \to a} f(x) \neq 0$, but $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ doesn't exist because from either side of a, the value of $\frac{f(x)}{g(x)}$ has an absolute value that will become arbitrarily large. In these situations, the line $x = a$ is a vertical asymptote.

2.9 Continuity

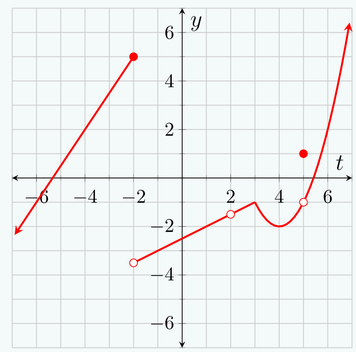
Definition 2.4: Continuity

The function f is continuous at a number $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$.
Here's how to determine if the given point is continuous

1. $f(a)$ must be defined.
2. $\lim_{x \to a} f(x)$ must exist (meaning $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$).
3. $\lim_{x \to a} f(x) = f(a)$.

Example.

For example, the graph is continuous at points $x = \{-4, 4, 6, \dots\}$.



2.10 Discontinuities

When a function has a discontinuity at a, the function is sometimes continuous only from the right and/or left at a.

Note.

If the limit from the right and left are both continuous, and the limit doesn't exist, that means the right and left limits aren't equal, since that's what makes the limit exist, if the limit from the right and left exist and are equal.

Definition 2.5: One-Sided Continuity

The function f is continuous from the left at a if and only if

$$\lim_{x \to a^-} f(x) = f(a).$$

The function f is continuous from the right at a if and only if

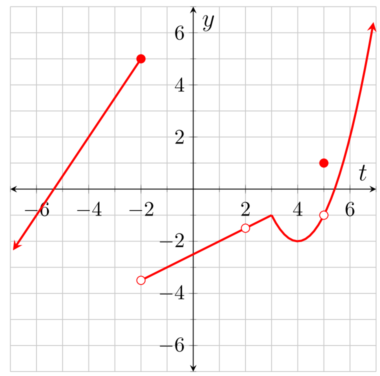
$$\lim_{x \to a^+} f(x) = f(a).$$

Some discontinuities are classified as **removable discontinuities**. Discontinuities that are holes or skips (holes with a secondary point) are removable.

Definition 2.6: Removable Discontinuity

We say that f has a removable discontinuity at a if f is discontinuous at a, but $\lim_{x \to a} f(x)$ exists.

For example, the point $x = -2$ is a removable discontinuity, because the limit for that point exists, but the function value doesn't. Same thing for points $x = \{-2, 2, 5\}$.



2.11 Continuity on an Interval

Now, let's go over being continuous over a given interval.

Definition 2.7: Continuity on an Interval

- Open Interval: (a, b).

  The function f is continuous over an open interval if and only if it is continuous at $\forall z \in (a, b)$. This just means that at every point within the interval (a, b), each and every point is continuous, except for a and b.

- Closed Interval: $[a, b]$.

  The function f is continuous over a closed interval if and only if it is continuous at $\forall z \in (a, b)$ and continuous from the right at a and from the left at b.

- Half-Open Interval: $(a, b]$ or $[a, b)$.

  The function f is continuous over a half-open interval if and only if it is continuous at $\forall z \in (a, b)$. Also

  - If the interval is $(a, b]$, then it must be continuous from the left at point b.
  - If the interval is $[a, b)$, then it must be continuous from the right at point a.

– The graph is continuous on the following open intervals

$$(-\infty, -2), (-2, 2), (2, 5), (5, \infty).$$

– The graph is continuous on the following closed interval

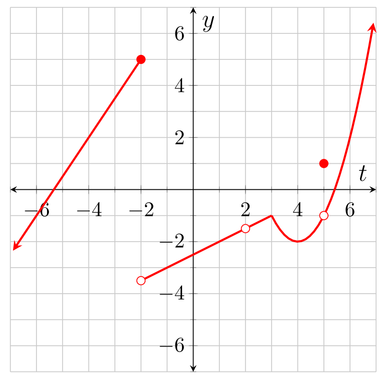
$$[-6, -2].$$

– The graph is continuous on the following half-open interval

$$(-\infty, -2].$$

Note.

When we use ∞, we always use parenthesis, because we don't include ∞, since it isn't an actual number, just a concept.



2.12 Exercises

Exercise 2.1. State the limit suggested by the values in the table and state whether or not the limit exists.

x	$f(x)$
$51,000$	-32×10^{-6}
$510,000$	-320×10^{-9}
$5,100,000$	-3.20×10^{-9}

t	$T(t)$
0.778	$29,990$
0.7778	$299,999$
0.77778	$2,999,999.9$

θ	$g(\theta)$
-0.9	$2,999,990$
-0.99	$2,999,999$
-0.99	$2,999,999.9$

t	$z(t)$
0.33	0.66
0.333	0.666
0.3333	0.6666

Exercise 2.2. Sketch onto Figure 5 a function f, with the following properties.

$$\lim_{x\to -4^+} f(x) = 1$$
$$\lim_{x\to -4^+} f(x) = -2$$
$$\lim_{x\to 3} f(x) = \lim_{x\to\infty} f(x) = -\infty.$$

The only discontinuities on f are at $-4, 3$. f has no x-intercepts. f is continuous from the right at -4. f has a constant slope -2 over $(-\infty, -4)$.

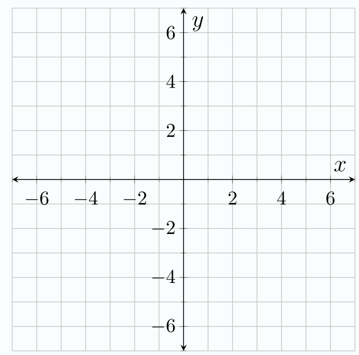


Figure 5: $y = f(x)$

Exercise 2.3. Determine the appropriate symbol to write after an equal sign following each of the given limits. In each case, the appropriate symbol is either a real number, ∞, or $-\infty$. Also, state whether or not each limit exists and if the limit exists prove its existence (and value) by applying the appropriate limit laws.

10 $\displaystyle\lim_{x\to 4^-}\left(5 - \frac{1}{x-4}\right).$

11 $\displaystyle\lim_{x\to\infty}\frac{e^{\frac{2}{x}}}{\frac{e}{\frac{1}{x}}}.$

12 $\displaystyle\lim_{x\to 2^+}\frac{x^2-4}{x^2+4}.$

13 $\displaystyle\lim_{x\to 2^+}\frac{x^2-4}{x^2-4x+4}.$

14 $\displaystyle\lim_{x\to\infty}\frac{\ln(x)+\ln(x^6)}{7\ln(x^2)}.$

15 $\displaystyle\lim_{x\to-\infty}\frac{3x^3+2x}{3x-2x^3}.$

16 $\displaystyle\lim_{x\to\infty}\sin\left(\frac{\pi e^{3x}}{2e^x+4e^{3x}}\right).$

17 $\displaystyle\lim_{x\to\infty}\frac{\ln\left(\frac{1}{x}\right)}{\ln\left(\frac{x}{x}\right)}.$

18 $\displaystyle\lim_{x\to 5}\sqrt{\frac{x^2-12x+35}{5-x}}.$

19 $\displaystyle\lim_{h\to 0}\frac{4(3+h)^2-5(3+h)-21}{h}.$

20 $\displaystyle\lim_{h\to 0}\frac{5h^2+3}{2-3h^2}.$

21 $\displaystyle\lim_{h\to 0}\frac{\sqrt{9-h}-3}{h}.$

22 $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\left(\theta+\frac{\pi}{2}\right)}{\sin(2\theta+\pi)}.$

23 $\displaystyle\lim_{x\to 0^+}\frac{\ln(x^e)}{\ln(e^x)}.$

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Lecture 3: Intro to The First Derivative

3.1 Instantaneous Velocity

The function f is defined as follows

$$f(x) = x^2 + 7\sin(x) - 5x,$$

and we're given the point $(x_1, f(x_1))$. We can find the slope between these two points using the slope equation like so

$$\frac{f(x_1) - f(a)}{x_1 - a}.$$

As the two points get closer and closer, the line gets smaller and smaller. But what happens when the $(x_1, f(x_1)) = (a, f(a))$? We'd be dividing by zero, since

$$\frac{f(x_1) - f(x_1)}{x_1 - x_1} = \frac{0}{0}.$$

This is where the limit comes into play. The actual function value doesn't exist, but like we've seen before, the limit value may exist. If it does, that's the slope of the line at the point $(x_1, f(x_1))$.

We call the slope at the point $(x_1, f(x_1))$ the **derivative**.

Here's a more formal definition of the derivative.

Definition 3.1: Derivative

The derivative is the slope of a line that lies tangent to the curve at the specific point. The limit of the instantaneous rate of change of the function as the time between measurements decreases to zero is an alternate derivative definition. We find the derivative using the limit as $h \to 0$ of the difference quotient, which is as so

$$\lim_{h \to 0} \frac{f(a+h) - f(1)}{h}.$$

Note. *If you haven't noticed already, but this is just simply the difference quotient, with a limit attached at the beginning.*

Notation (Derivative). We have multiple ways of notating derivatives.

- **Function notation**. It looks something like this

$$f^{'}(x),$$

which is read out as **f prime of x**. To denote multiple derivatives, we say

$$f^{''}(x),$$

which is read out as **f double-prime of x**.

- **Leibniz notation**. If $y = f(x)$, we say that the derivative of y with respect to x is equal to $f^{'}(x)$, which looks like this

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} f(x),$$

which is read out as **dy over dx**. To denote multiple derivatives, we say

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=a} f(x),$$

which is read out as **2 dy over dx**.

Example. Say the function f is defined as follows

$$f(x) = 40 + 40x - 16x^2.$$

Find the first order derivative of $f(x)$ at 2.

$$\begin{aligned}
\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=2} &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{[40 + 40(2+h) - 16(2+h)^2] - [40 + 40(2) - 16(2)^2]}{h} \\
&= \lim_{h\to 0} \frac{40 + 80 + 40h - 64 - 64h - 16h^2 - 56}{h} \\
&= \lim_{h\to 0} \frac{-24h - 16h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-24 - 16h)}{\cancel{h}} \\
&= -24 - 16 \times 0 \\
&= -24.
\end{aligned}$$

Note. *There's a difference between finding the derivative at a point, and finding the general derivative formula.*

3.2 Tangent Lines

Graphically, the difference quotient of a function f can be used to calculate the slope of secant lines to f, which is a line that connects two points on the graph. But what happens when we take the run of the secant line to 0? We're essentially connecting two points on the line that are really close to one another. When we send $h \to 0$, it goes from an average velocity to an instantaneous velocity, which turns a secant line to a tangent line, where the secant line is the average velocity and the tangent line is the instantaneous velocity.

Example. The tangent line at $x = 4$ to the function f, which is defined by $f(x) = \sqrt{x}$. Here's the calculation of the slope of this line.

$$
\begin{aligned}
\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=4} &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0}\frac{h}{h\sqrt{4+h}+2h} \\
&= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} \\
&= \frac{1}{\sqrt{4+0}+2} \\
&= \frac{1}{4}.
\end{aligned}
$$

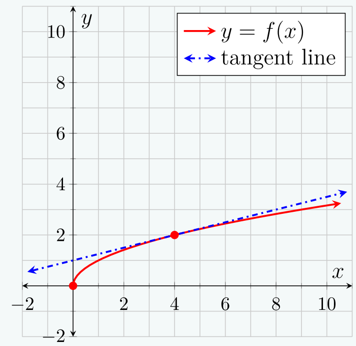

Figure 6: $y = g(t)$

To create the equation for the tangent line, we can use one of the following 2 equations for lines

$$
\begin{aligned}
y &= mx + b && \text{Slope-Intercept Form} \\
y &= m(x - x_1) + y_1 && \text{Point-Slope Form.}
\end{aligned}
$$

Here's the equation for the tangent line

$$
\begin{aligned}
y &= \frac{1}{4}x + 1 && \text{Slope-Intercept Form} \\
y &= \frac{1}{4}(x - 0) + 1 && \text{Point-Slope Form.}
\end{aligned}
$$

3.3 Exercises

Exercise 3.1. Find the first derivative formula for each of the following functions twice: first by evaluating $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and then by evaluating $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$.

1. $f(x) = x^2$.
2. $f(x) = \sqrt{x}$.
3. $f(x) = 7$.

Oct 17 2022 Mon (11:00:41)

Lecture 4: Functions, Derivatives, and Antiderivatives

Oct 24 2022 Mon (11:00:04)

Lecture 5: Derivative Formulas

Oct 31 2022 Mon (11:03:13)

Lecture 6: The Chain Rule

Nov 07 2022 Mon (11:01:44)

Lecture 7: Implicit Differentiation

Nov 14 2022 Mon (11:00:01)

Lecture 8: Related Rates

Nov 21 2022 Mon (11:04:52)

Lecture 9: Critical Numbers and Graphing from Formulas

Notes