Lecture Notes | 2023

Calculus 3

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Introduction

Lectures notes from the course Calculus 3, given by professor Ross Kouzes at the Faculty of Mathematics at Portland Community College in the academic year 2023, in the Spring Term. This course covers infinite sequences and series and an introduction to differential equations and modeling. Credit for the material in these notes is due to professor Ross, while the structure is loosely taken from the Single Variable Calculus: Concepts and Contexts, 4th Edition textbook. The credit for the typesetting is my own.

Disclaimer: This document will inevitably contain some mistakes–both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material himself, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, Arabic, Spanish or Dutch, at singularisartt@gmail.com.

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Contents

Sections 3.9 and 4.7 Apr 04 2023 Mon (11:00:53)

Lecture 1: Approximation and Differentials

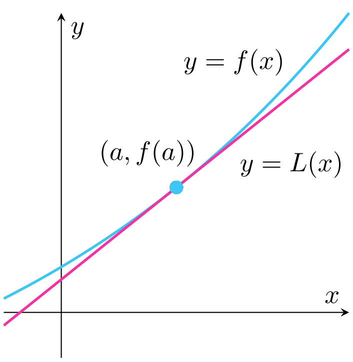
Lecture Note Overview

There are numerous ways to approximate a function, but I'll go over two ways:

① Linearization.

② Newton's Method.

1.1 Linearization



We know that the actual curve, $y = f(x)$, lies very close to its tangent line $(a, f(a))$ over some interval. The closer you are to that tangent line, the more accurate the approximation will be (as you can see from figure ??). The main idea is that it might be easy to calculate a value $f(a)$, but almost impossible to calculate the value $f(a+1)$. But, it is easy to calculate the value of $L(a+1)$. Because of this, we can use the tangent line at $(a, f(a))$ as an approximation of the curve $y = f(x)$ when x is near a. The linear approximation equation is

$$L(x) = f(a) + f'(a)(x-a). \tag{1}$$

QUESTION 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

SOLUTION Using the equation (1) from above, we can find the linearization of the function $f(x)$ by following a couple of steps.

① Find the derivative. In this case, the derivative of the function is

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(x+3)^{\frac{1}{2}}\right] \\ &= \frac{1}{2}(x+3)^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x+3}}. \end{aligned}$$

② Find the values of $f(a)$ and $f'(a)$. In this case, those values are $f(1) = 2$ and $f'(1) = \frac{1}{4}$.

③ Put everything together. We can now start to plug numbers in

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}.$$

The corresponding linear approximation of $f(x) = \sqrt{x+3}$ when x is near 1 is

$$\frac{7}{4} + \frac{x}{4}.$$

Note: When we find the $\sqrt{3.98}$ using $L(x)$, we don't plug in $L(3.98)$. We instead set $3.98 = x + 3$ and then solve for x, which gives us $x = 0.98$. We use that value when we plug it in for $L(x)$.

Using our linear approximation, we can approximation the given values as

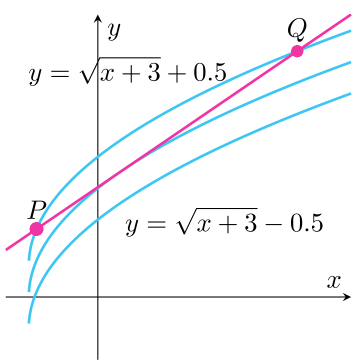
$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \quad \text{and} \quad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

This linear approximation is accurate up to a certain degree. For example, we shouldn't use this approximation for when $x = 36$. When we do so, we get

$$\sqrt{36} \approx \frac{7}{4} + \frac{33}{4} \approx 10.$$

But we all know that $\sqrt{36} = 6$.

1.1.1 Accuracy of a linear approximation



Sometimes, we may want to find what values of x is the linear approximation accurate to within 0.5. This means that the difference between $f(x)$ and $L(x)$ should be less than 0.5

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5.$$

Or we could write it as

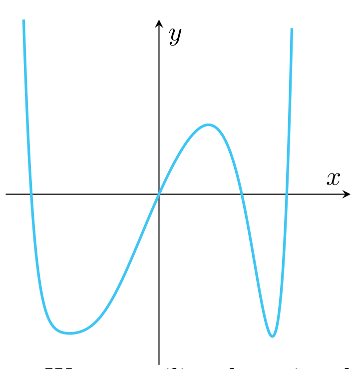
$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5.$$

We could also represent this graphically as seen in figure ??. The x-value of P is -2.66 and the x-value of Q is 8.66. Ergo, we see from the graph that the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4},$$

is accurate to within 0.5 when $-2.6 < x < 8.6$.

1.2 Newton's Method



Let's say you would like to find the roots of the polynomial

$$\left(2x^5 + x^2 - x - 1\right)^2 - 1.$$

There is no such formula to find the roots of any polynomial with a degree higher than 4. However, we can approximate those roots using **Newton's Method**. We first start with an approximation x_1. We either guess what x_1 is, or we can approximate it from a graph. The main idea behind Newton's Method is that the tangent line is close to the curve as well as its x-intercept, x_2 is close to the x-intercept of the curve, x_1. Since the tangent line is just a line, we can easily find its x-intercept

We can utilize the point-slope form equation

$$y - y_1 = m\left(x - x_1\right).$$

In this case, we already know what y_1, x_1, and m are. We can just plug in those values and we get

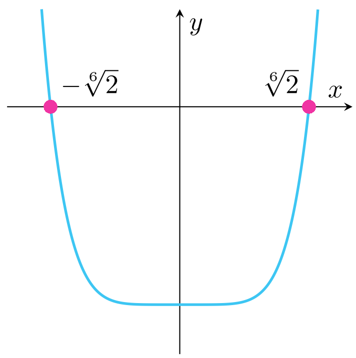
$$y - f\left(x_1\right) = f'\left(x_1\right)\left(x - x_1\right).$$

We can set $y = 0$, since the x-intercept is x_2 and when we do some rearranging, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We can keep repeating this process to get a better and better approximation of the root in the original function.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad \boxed{2}$$



Note: Let's say you want to find the value of $\sqrt[6]{2}$. How can we convert it to a polynomial that has $\sqrt[6]{2}$ as a root? To do so, we need to perform some algebra

$$\begin{aligned} \sqrt[6]{2} &= x \\ 0 &= x - \sqrt[6]{2} \\ 0 &= x^6 - 2. \end{aligned}$$

When you find the positive x-intercept of the polynomial, it should be $\sqrt[6]{2}$. You can now go ahead and perform Newton's Method to approximate that root.

1.2.1 When Newton's Method Fails

Newton's Method may fail for several reasons.

① Zero Slope: If $f'(a_1) = 0$, then we are stuck and cannot continue the process because the tangent line will have no x-intercept.

> **Example** Let's consider the function $f(x) = x^3$. If we start Newton's method with an initial approximation of $a_1 = 0$, we encounter a zero slope. The derivative of $f(x)$ is $f'(x) = 3x^2$. At $x = 0$, the derivative is $f'(0) = 0$, which means the tangent line will have no x-intercept. As a result, Newton's method fails to proceed further.

② Near-Zero Slope: Even if a_1 results in just a near-zero slope, it can cause a_2 to be a worse approximation than a_1.

> **Example** Consider the function $f(x) = x^3 - 1$. If we choose an initial approximation of $a_1 = 0.9$, the derivative at $x = 0.9$ is $f'(0.9) = 2.43$. Although this slope is near zero, it can cause Newton's method to produce a worse approximation in the next iteration. Let's assume the next approximation is $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$. Evaluating $f(a_2)$ gives $f(0.9 - \frac{f(0.9)}{2.43}) = f(0.5337) \approx -0.0881$. In this case, a_2 is a worse approximation than a_1 as it deviates further from the actual root, which is $x \approx 1$.

③ Non-differentiable: If f is not differentiable at its x-intercept, then Newton's Method will continue to fluctuate and not converge at the root.

> **Example** Let's consider the function $f(x) = |x|$. The derivative of $f(x)$ is not defined at $x = 0$ because the function is not differentiable at its x-intercept. When Newton's method is applied to this function, it will fluctuate and fail to converge at the root since the derivative is undefined at that point.

④ Domain Restrictions: If the domain of f is not $\mathbb{R}$, then Newton's Method could result in the x-intercept of the linearization being outside the domain of f.

> **Example** Consider the function $f(x) = \sqrt{x}$. The domain of $f(x)$ is restricted to non-negative real numbers, i.e., $x \geq 0$. If we start Newton's method with an initial approximation of $a_1 = -1$, the linearization's x-intercept will be outside the domain of $f(x)$. Newton's method will not be applicable in this case, as it requires the linearization's x-intercept to be within the domain of the function.

1.2.2 Find the root to a given accuracy

Let's say we want to find the root of the polynomial with an accuracy of 8 digits, or 10^{-8} with Newton's Method. To do this, we typically setup the equation

$$\left|\left(x_{n+1} - \frac{f(x_{n+1})}{f'(x_{n+1})}\right) - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right| < 10^{-8}.$$

In other words, we say the nth and the $(n+1)$th term have the same 6 digits beyond the decimal place.

QUESTION 2 Using Newton's Method, find $\sqrt[6]{2}$ correct to 8 decimal places.

SOLUTION We know that the $\sqrt[6]{2}$ is equal to

$$x^6 - 2 = 0.$$

We must take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^6 - 2\right] = 6x^5.$$

Using Newton's Method, we get

$$\begin{aligned}\sqrt[6]{2} &\approx x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^6 - 2}{6x_n^5}.\end{aligned}$$

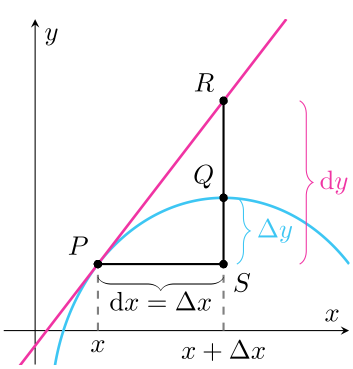
If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$\begin{aligned}x_2 &\approx 1.16666667 \\ x_3 &\approx 1.12644368 \\ x_4 &\approx 1.12249707 \\ x_5 &\approx 1.12246205 \\ x_6 &\approx 1.12246205.\end{aligned}$$

Since $x_6 - x_5 = 0$, which is less than 10^{-8}, then we can conclude that

$$\sqrt[6]{2} \approx 1.12246205.$$

1.3 Differentials



If $y = f(x)$, where f is a differentiable function, then the differential $\mathrm{d}x$ is an independent variable. Meaning, $\mathrm{d}x$ can be given the value of any real number. The differential $\mathrm{d}y$ is then defined in terms of $\mathrm{d}x$ by the equation

$$\mathrm{d}y = f'(x)\mathrm{d}x. \quad \boxed{3}$$

This means that $\mathrm{d}y$ depends on the values of x and $\mathrm{d}x$. We can visualize this graphically as shown in figure **??**. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of $f(x)$. Let $\mathrm{d}x = \Delta x$. This means that the corresponding change in y is given by the equation

$$\Delta y = f(x + \Delta x) - f(x). \quad \boxed{4}$$

The slope of the tangent line PR is the derivative $f'(x)$. Meaning, the distance between S and R is $f'(x)\mathrm{d}x = \mathrm{d}y$. This means that $\mathrm{d}y$ represents the amount the curve rises or falls, or the slope of the function, while Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount. This means that as Δx gets smaller, $\Delta y \approx \Delta x$ becomes more accurate.

If we let $\mathrm{d}x = x - a$, then $x = a + \mathrm{d}x$. We can rewrite the linear approximation in the notation of differentials

$$f(a + \mathrm{d}x) \approx f(a) + \mathrm{d}y.$$

For instance, in question 1.1, we have

$$\mathrm{d}y = f'(x)\mathrm{d}x = \frac{1}{2\sqrt{x+3}}\mathrm{d}x.$$

If $a = 1$ and $\mathrm{d}x = \Delta x = 0.05$, then

$$\mathrm{d}y = f'(x)\mathrm{d}x = \frac{1}{2\sqrt{1+3}}0.05 = 0.0125,$$

and

$$\sqrt{4.05} = f(1.05) \approx f(1) + \mathrm{d}y = 2.0125.$$

Sections 7.1, 7.2, and 7.3 Apr 11 2023 Mon (11:01:03)

Lecture 2: Slope Fields and DE/SDE

Lecture Note Overview

2.1 Intro to Differential Equations

Remark *I won't go into great detail about differential equations and their solutions because it's a vast subject covered in dedicated courses. However, it's important to grasp the basics, such as understanding what a differential equation is and how to check if a solution is correct. Towards the end of these notes, I will include an additional section on using power series, which we will explore in more depth later, to solve differential equations.*

1 Definition (Differential Equation)

A **Differential Equation** is an equation that contains one or more derivatives of an unknown function.

A function is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives into the equation.

The **order** to a differential equation is the set of all curves or formulas that satisfy the equation and usually have a constant C in the equation somewhere.

A **specific solution** or **particular solution** to a differential equation is one specific curve or formula that satisfies the equation and goes through a specific point. They are generally considered when solving **initial value problems**, which are problems that are presented with a differential equation and one point (an **initial condition**) on the specific solution you must find.

Note: While it is generally not recommended, you can still use the notation y' instead of $\frac{\mathrm{d}y}{\mathrm{d}x}$ primarily because it offers a shorter and more convenient way to write the derivative.

2.1.1 Modeling Population with Differential Equations

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population.

The rate at which a population grows is represented by the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$, where P denotes the population and t represents time, typically in hours. To express our assumption that the rate of population growth is proportional to the population size, we get

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP, \qquad \boxed{5}$$

where k is the proportionality constant

The solution to this differential equation is

$$P(t) = Ce^{kt},$$

where C is the starting population. I'll go over how to find this solution in section 2.3.

Equation 5 is good when modeling population growth under ideal conditions, but that's not always the case. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its carrying capacity M. We need to make two assumptions to make an accurate equation to model population growth:

① $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$ if P is small.

② $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ if $P > M$, meaning P decreases if it ever exceeds M.

Putting those things together, we can create the following expression

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right). \tag{6}$$

Equation 6 is called the **logistic differential equation**.

① If $P(0) \in (0, M)$, then $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$, which means the population is increasing.

② If $P < M$, then $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means the population is decreasing.

2.1.2 Verifying General Solution to a Differential Equation

To verify a solution to a differential equation, we follow a simple process. Given a differential equation and a proposed solution, we begin by calculating the nth derivative, where n is the order of the differential equation. Next, we substitute these derivatives and the solution into the original differential equation. Finally, we simplify the equation and check if it holds true for all values of the independent variable.

It's important to check that the suggested solution meets the provided differential equation throughout the process. The solution to the differential equation is valid if the equation can be reduced to an identity, such as $0 = 0$. On the other hand, if the equation is false, it implies that the suggested solution to the differential equation is invalid.

QUESTION 3 Let's say we're given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0,$$

and we're told that the solution to this differential equation is

$$y = 3e^{-2x}.$$

Verify that this is a solution to the differential equation.

SOLUTION To verify if the given solution, $y = 3e^{-2x}$, satisfies the given differential equation, we can substitute it into the equation and check if the equation holds true.

First, calculate the derivatives of y:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -6e^{-2x}\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 12e^{-2x}.\end{aligned}$$

We then just substitute them into the differential equation

$$\begin{aligned}0 &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y\\ &= 12e^{-2x} + 5\left(-6e^{-2x}\right) + 6(e^{-2x})\\ &= 12e^{-2x} - 30e^{-2x} + 18e^{-2x}\\ 0 &= 0.\end{aligned}$$

As we can see, the equation simplifies to $0 = 0$, which is always true. Therefore, the given solution $y = 3e^{-2x}$ does indeed satisfy the original differential equation.

2.1.3 Finding Specific Solution to a Differential Equation

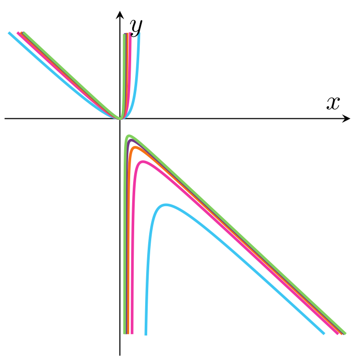
QUESTION 4 Verify that the function $y = \dfrac{cx^2}{1-cx}$, where c is an arbitrary constant, is a solution to the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + 2xy}{x^2}.$$

If it is indeed a solution to the first order differential equation, solve for c when you're given the point $(2, 1)$.

SOLUTION We first must take the derivative of y

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= c \cdot \left(\frac{(1-cx) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] - x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[1-cx\right]}{(1-cx)^2} \right) \\
&= c \cdot \frac{(1-cx)(2x) - x^2(-c)}{(1-cx)^2} \\
&= c \cdot \frac{2x - 2cx^2 + cx^2}{(1-cx)^2} \\
&= c \cdot \frac{2x - cx^2}{(1-cx)^2} \\
&= \frac{2cx - c^2x^2}{(1-cx)^2}.
\end{aligned}$$



We now just plug the value of y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ into the differential equation and see if it's true

$$\begin{aligned}
& \frac{\mathrm{d}y}{\mathrm{d}x} && = \frac{y^2 + 2xy}{x^2} \\
\Rightarrow\ & \frac{2cx - c^2x^2}{(1-cx)^2} && = \frac{\left(\frac{cx^2}{1-cx}\right)^2 + 2x\left(\frac{cx^2}{1-cx}\right)}{x^2} \\
\Rightarrow\ & x^2 \cdot \left(2cx - c^2x^2\right) && = (1-cx)^2 \left(\left(\frac{cx^2}{1-cx}\right)^2 + 2x\left(\frac{cx^2}{1-cx}\right) \right) \\
\Rightarrow\ & 2cx^3 - c^2x^4 && = (1-cx)^2 \left(\frac{c^2x^4}{(1-cx)^2} + \frac{2cx^3}{1-cx} \right) \\
\Rightarrow\ & 2cx^3 - c^2x^4 && = c^2x^4 + 2cx^3(1-cx) \\
\Rightarrow\ & 2cx^3 - c^2x^4 && = c^2x^4 + 2cx^3 - 2c^2x^4 \\
\Rightarrow\ & 2cx^3 - c^2x^4 && = 2cx^3 - c^2x^4.
\end{aligned}$$

To find the specific solution that passes through the point $(2, 1)$, we just plug in $x = 2$ and $y = 1$ and solve for the constant c

$$\begin{aligned}
y &= \frac{cx^2}{1-cx} \\
1 &= \frac{(2)^2c}{1-2c} \\
1 &= \frac{4c}{1-2c} \\
1 - 2c &= 4c \\
1 &= 6c \\
c &= \frac{1}{6}.
\end{aligned}$$

Now that we have our specific c value that makes the solution go through the point $(2, 1)$, we just plug it back into y and simplify, which gives us our final answer of

$$y = \frac{x^2}{6 - x}.$$

2.2 Slope Fields and Euler's Method

2.2.1 Slope Fields

2 **Definition** (Slope Field) A **slope field** is a graphical representation of a differential equation that shows the slope of the solution curve at various points.

Suppose we have the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y.$$

We don't have the solution to this differential equation, but we can use the slope field to see what the solution may look like. The equation tells us that the slope at any point (x, y) on the graph is equal to the sum of the x and y coordinates.

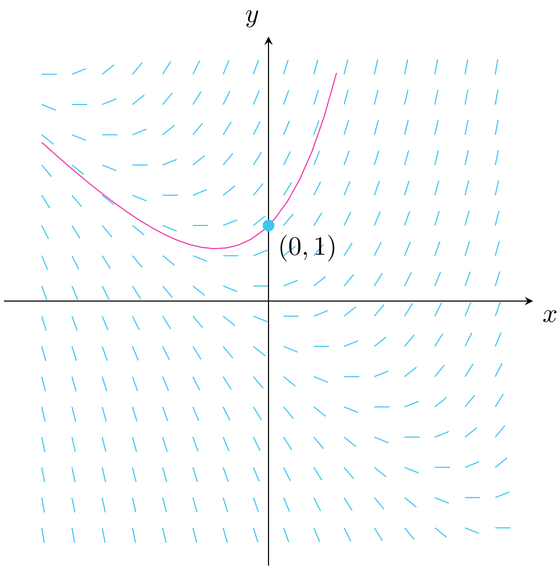


Figure 2.1: Slope field for $\frac{dy}{dx} = x + y$

x	y	$\frac{dy}{dx} = x + y$
-2	-2	$-2 + -2 = -4$
-2	-1	$-2 + -1 = -3$
-2	0	$-2 + 0 = -2$
-2	1	$-2 + 1 = -1$
-2	2	$-2 + 2 = 0$
-1	-2	$-1 + -2 = -3$
-1	-1	$-1 + -1 = -2$
-1	0	$-1 + 0 = -1$
-1	1	$-1 + 1 = 0$
-1	2	$-1 + 2 = 1$
0	-2	$0 + -2 = -2$
0	-1	$0 + -1 = -1$
0	0	$0 + 0 = 0$

0	1	$0 + 1 = 1$
0	2	$0 + 2 = 2$
1	-2	$1 + -2 = -1$
1	-1	$1 + -1 = 0$
1	0	$1 + 0 = 1$
1	1	$1 + 1 = 2$
1	2	$1 + 2 = 3$
2	-2	$2 + -2 = 0$
2	-1	$2 + -1 = 1$
2	0	$2 + 0 = 2$
2	1	$2 + 1 = 3$
2	2	$2 + 2 = 4$

Table I: Slope field values for $\frac{dy}{dx} = x + y$

2.2.2 Euler's Method

2.3 Separable Differential Equations

3 **Definition** (Separable Differential Equation) A **separable differential equation** is an equation that can be written in the form

$$g(y)\frac{\mathrm{d}}{\mathrm{d}x} = f(x) \tag{7}$$

The general solution to this equation is found by solving the equation

$$\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x \text{ for } y \tag{8}$$

Note: When solving separable differential equations, never add or subtract any terms containing y or x to the other side of the equation. This makes sure the preservation of separability during the solution process and helps in obtaining the correct solution.

QUESTION 5

① Solve for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y^2}$.

② Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SOLUTION

① We first separate the variables and integrate both sides

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{y^2}\\ y^2\mathrm{d}y &= x^2\mathrm{d}x\\ \int y^2\mathrm{d}y &= \int x^2\mathrm{d}x\\ \frac{1}{3}y^3 &= \frac{1}{3}x^3 + C.\end{aligned}$$

Solving for y gives us

$$y = \sqrt[3]{x^3 + K},$$

where $K = 3C$ since they're both just constants.

② Plugging in $x = 0$ and $y = 2$ gives us $y(0) = \sqrt[3]{K} = 2$. Therefore, $K = 8$, giving us our final answer of

$$y = \sqrt[3]{x^3 + 8}.$$

Sections 7.4 and 7.5 Apr 18 2023 Mon (11:02:00)

Lecture 3: Modeling with DE and Growth/Decay

Lecture Note Overview

Sections 8.1 and 8.2 Apr 25 2023 Mon (11:00:00)

Lecture 4: Sequences and Series

Lecture Note Overview

Sections 8.3 and 8.4 May 09 2023 Mon (11:00:00)

Lecture 6: Convergence Tests

Lecture Note Overview

Section 8.5 May 16 2023 Fri (11:05:10)

Lecture 7: Power Series

Lecture Note Overview

Section 8.6 May 23 2023 Fri (11:13:00)

Lecture 8: Functions to Power Series

Lecture Note Overview

Section 8.7 May 30 2023 Fri (11:01:20)

Lecture 9: Taylor Series

Lecture Note Overview

Jun 06 2023 Tue (11:00:57)

Lecture 10: Using Power Series to solve DE

Lecture Note Overview

Notes