Blue Victoria 50 Combinatorics Problem Set

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Introduction

Welcome to Blue Victoria Problem Set: 50 Amazing Combinatorics Problems! This book is meticulously crafted for aspiring mathematicians aiming to elevate their problem-solving skills and conquer challenging math competitions like JEE Mains, JEE Advanced, ISI UGA/UGB, CMI, IOQM, AIME, BMO, RMO, and more.

Combinatorics is the heart of mathematical problem-solving, requiring a blend of logic, creativity, and strategic thinking. Each problem in this collection is meticulously designed to mimic the complexity and depth found in top-tier math competitions, following the pattern of AMC, AIME, and beyond.

As you progress through the book, you'll encounter a curated selection of 50 mind-bending problems, carefully structured to escalate in difficulty, preparing you for the tough mathematical challenges ahead. The problems are organized in a daywise pattern with 5 problems per day over 10 days, accompanied by motivational and inspirational quotes from famous personalities.

Beyond just testing your ability in mathematical problem-solving, these problems are crafted to assess your ability to dissect problems, formulate strategies, and think critically in real-world scenarios.

To support your journey, I have provided a solution for each problem, guiding you through the thought process and underlying concepts step by step, ensuring a deep understanding and mastery of the material.

Source of Questions:
The majority of the problems in this set are original creations, crafted to challenge and inspire. Some of the final questions draw inspiration from prestigious international Olympiads and tournaments. For the original problems, while I do not claim them to be entirely novel, their concepts are inspired by past Olympiad challenges.

I understand the rigorous journey of preparing for math competitions, and it is my sincere hope that this book serves as a valuable tool in your pursuit of excellence. Success in these competitions requires not only hard work but also strategic preparation. So, dive into the Blue Victoria 50 Combinatorics Problem Set, unlock the mysteries of combinatorics, and embark on a journey of mathematical discovery.

Best wishes for your mathematical endeavors.

HAPPY PROBLEM SOLVING...!!!

Thanks and regards,
Nrip Dave

Contents

Day 1

"LOGIC IS THE COMPASS THAT GUIDES US THROUGH THE UNKNOWN"

Problem 1

There are four families: Indian, American, French, and Japanese. The Indian and French families each consist of five members,
while the American family consists of four, and the Japanese family consists of three. The seating arrangement should ensure that members of the same family are not separated.
In how many ways can they be seated in a Round Table Conference?

Select One Correct Option-

A) 5!4!3!

B) $(5!)^2\ 3!$

C) $(5!)^2(4!)(3!)^2$

D) $(5!)(4!)^3$

Problem 2

In Tribeca neighborhood New York city,

There are 15 boys $(\mathrm{B}_1, B_2, ..., B_{15})$ and 7 girls $(G_1, G_2, ..., G_7)$.
They've decided to establish a chess club comprising 8 boys and 5 girls
However, they've made it a rule not to include both
G_1 and B_1 in the club together.

How many ways can they form such a chess club?

Problem 3

$$\sum_{n=3}^{2026}(n+1)(n)^{n-1}P_{n-2} \quad - \quad \sum_{n=2}^{2023}(n+2)(n+1)^{n}P_{n-1} = \quad \text{a!+b!}$$

where gcd(a,b)=1 such that $a > b$

Compute $\binom{a}{b}$

Problem 4

Dave, afflicted with fever, sought medical advice from a doctor who prescribed a remedy of consuming 9 pizzas for the next 20 days.

However, to ensure a balanced diet, Dave was instructed not to consume two pizzas on consecutive days. Therefore, Dave must plan his pizza consumption over the upcoming 20 days in a manner that adheres to this restriction.

How many distinct ways can Dave consume pizzas over the specified period?

Problem 5

On the eve of Christmas, George intends to distribute 23 cakes among 7 children. Among them, three children specifically request odd-numbered cakes in positive integers.

The remaining four children are open to receiving even-numbered cakes but prefer them in non-negative integers.

How many distinct ways can George distribute all the cakes among these children?

Select one correct Option-

A) $\binom{23}{7}$

B) $\binom{16}{7}$

C) $\binom{23}{6}$

D) $\binom{16}{6}$

Day 2

"IMAGINATION IS MORE IMPORTANT THAN KNOWLEDGE."

- Albert Einstein

Problem 6

Consider the equation $|x| + |y| + |z| + |w| = 16$,

where x, y,$z \in \mathbb{Z}$ and $w \in \mathbb{N}$.

The number of possible solutions are:

Problem 7

$N = \left\lfloor \frac{1}{5} + \frac{1}{108} \right\rfloor + \left\lfloor \frac{1}{5} + \frac{2}{108} \right\rfloor + \left\lfloor \frac{1}{5} + \frac{3}{108} \right\rfloor + \ldots + \left\lfloor \frac{1}{5} + \frac{200}{108} \right\rfloor,$

Then Find the exponent of 3 in $(N)!$

Problem 8

In a soccer match, there are 22 players available, divided into two teams, each consisting of 11 players. Among these 22 players, 2 are highly skilled professionals. We need to find the number of ways in which these 22 players can be divided into two teams of 11 each, ensuring that the two highly skilled professional players are always in different teams.

Select One Correct Option-

A) $\frac{20!}{10!^2}$

B) $\frac{22!}{2!11!}$

C) $\frac{20!}{2!11!}$

D) $\frac{22!}{11!^2}$

E) $\frac{20!}{11!^2}$

Problem 9

Find the number of 5-letter words that can be formed using elements from the set $N = \{a, b, c, \ldots, x, y, z\}$, where two distinct consonants from the English alphabet occupy the first and last positions respectively, and three distinct vowels occupy the remaining middle positions

Select One Correct Option-

A) 55125

B) 50000

C) 25200

D) 24000

Problem 10

Determine the number of words, whether meaningful or not, consisting of one vowel and four consonants formed from the letters of the word "STANFORD"

Select One Correct Option-

A) 1200

B) 3600

C) 1800

D) 2400

E) 3000

Day 3

"SUCCESS CAN COME TO YOU BY COURAGEOUS DEVOTION TO THE TASK LYING IN FRONT OF YOU"

- C.V Raman

Problem 11

A team of NASA scientists embarks on an ambitious mission to establish a research base, venturing into the vast expanse of space. As they commence their groundbreaking work on the Saturn's surface, an unforeseen encounter with an extraterrestrial spacecraft disrupts their plans, leaving them stranded and cut off from communication with Earth. The alien visitors, intrigued by humanity's presence on the Saturn, issue a challenge to the scientists to prove their problem-solving prowess. They present a high-tech digital lock securing a vital oxygen supply compartment, demanding the scientists to unlock it using a 5-digit numerical code N of the aliens' devising.
With the oxygen levels dwindling rapidly and their lifeline to Earth severed, the scientists are thrust into a race against time. They must decipher the cryptic clues left by the aliens, strategically placed throughout their mission, in order to unravel the code and restore the oxygen supply before it's too late.

How many unique N could the aliens have chosen for the oxygen supply compartment such that N is even and The digits occupying odd-numbered positions must be even numbers and all digits should be distinct and zero is not included in any digit

Problem 12

It is Defined that a sequence consisting of any one digit from the set $S = \{1, 6, 7\}$ is known as Dave's Sequence.
Calculate the number of Dave's sequences having n terms.

Select One Correct Option-

A) 7^n

B) $(3!)^n$

C) 3^n

D) 2^n

E) 5^n

Problem 13

Find the number of 7-Digit numbers divisible by 25 which can be formed using 0,1,2,3,4,5 when repetition of digits is allowed

Problem 14

George wrote integers from 1 to 100000 on Blackboard in order around a circle.
Starting at 1, every Thirtieth number is marked (that is 1,31,61,.....etc)
This process is continued until a number is reached which has already been marked, then find the all unmarked numbers
Justify your answer

Problem 15

Find the Number of times of the digit 7 will be written when listing the integer from 1 to 10000.

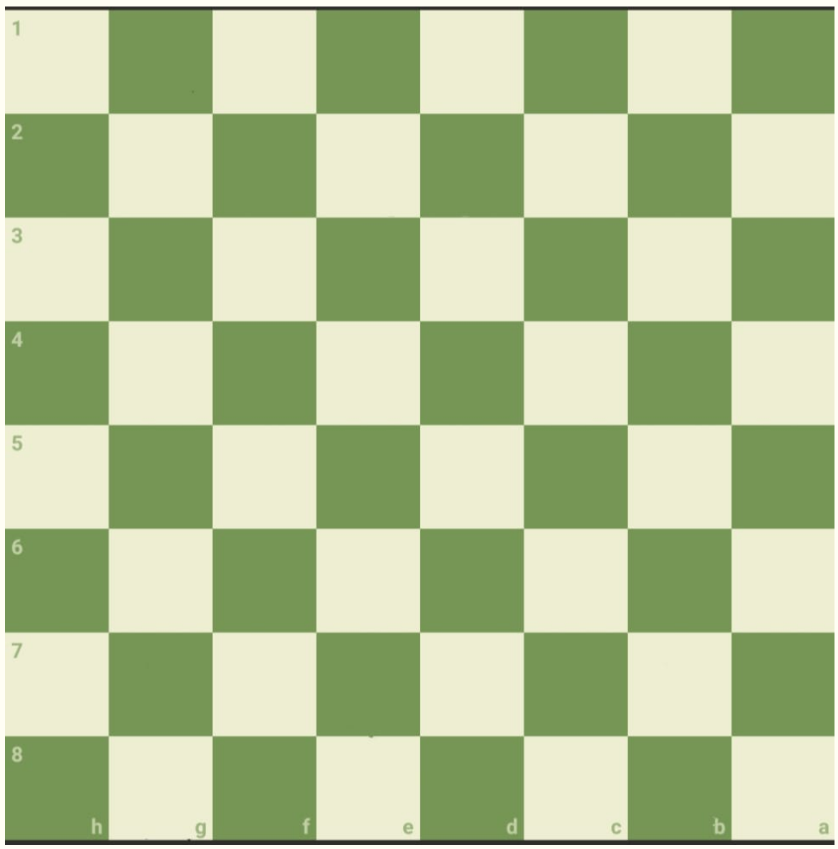
Day 4

"TELL ME AND I FORGET
TEACH ME AND I REMEMBER
INVOLVE ME AND I LEARN"

- Benjamin Franklin

Problem 16

In How many ways two Bishops can be put on chessboard shown below such that they are in the position to attack each other....!!!



Problem 17

Column A	Column B
(A) Find the number of integral solutions of $a+b+c = 16$ where a ≥ 1 and b,c ≥ 0	**(P)** 120
(B) Two families are gathered for round table conference named family N and family D where N consist of 5 people and D consist of 4 people In how many ways can they be seated in a Round Table Conference such that The seating arrangement should ensure that members of the same family are not separated.	**(Q)** 720
(C) Albert have 7 freinds. In how many ways he can invite one or more of them to his Birthday Party	**(R)** 136
(D) How many five-Digit numbers can be made using elements from the set $N \in$\{1,2,3,7,8,9\} when repetition of digits is not allowed	**(S)** $(24)^2 \times 5$

Select One Correct match-

1) (A-Q) ; (B-S) ; (C-P) ; (D-R)

2) (A-R) ; (B-S) ; (C-Q) ; (D-P)

3) (A-R) ; (B-S) ; (C-P) ; (D-Q)

4) (A-S) ; (B- Q) ; (C- P) ; (D-R)

5) (A-P) ; (B-Q) ; (C - R) ; (D-S)

Problem 18

Barron is attempting Blue Victoria 50 Combinatorics problems set which is divided into 10 days (same as sections) there are given 5 questions for each day . Barron has freedom to answer any number of questions attempting at least one from each day.
In how many ways can the problem set be attempted by Barron?

Problem 19

The Number of factors of the product of $a^{26}b^{25}c^{24}d^{23}e^{22}\ldots x^3y^2z$ (where $a, b, c, d, e, f, \ldots x, y, z$ are all prime numbers) is N

Compute the exponent of 2 in N

Problem 20

Nrip visits a library with the intention of procuring a selection of books. The library offers a curated collection, consisting of 5 identical books on topology, 4 identical books on linear algebra, 3 identical books on real analysis, 2 identical books on calculus, and 1 book on number theory. If Nrip intends to purchase exactly 5 books from this assortment,
How many distinct arrangements of books are possible?

Day 5

"AN EXPERT IS A PERSON WHO HAS MADE ALL THE MISTAKES THAT CAN BE MADE IN A VERY NARROW FIELD."

- Neils Bohr

Problem 21

In a famous restaurant, there are 70 types of pizzas available. Charles visits the restaurant and loves all the pizzas. He purchases 7 pizzas at a time, without selecting the same combination of 7 pizzas more than once. The number of ways he can purchase all the pizzas in such a manner is N.

Compute $\phi(N)$

(where ϕ denotes Euler's totient function).

Problem 22

A person, X, predicts the outcomes of 30 chess matches in a chess tournament.
Each match can result in a win, loss, or draw.
Find the total number of ways in which he can make predictions so that at least 16 predictions are correct.

Problem 23

Find the number of different words that can be formed from the letters of the word "CAMBRIDGE" under the following conditions:

- When all the letters are used, let the number of words under this condition be A.
- When all the letters are used but the middle letter is always "R", let the number of words under this condition be B.
- When all the letters are used and the letters "C" and "E" occupy the first and the last places respectively, let the number of words under this condition be C.
- When all the letters are used and the vowels are separated, let the number of words under this condition be D.

Compute $\frac{AB}{CD}$

Problem 24

Consider the word "BERKELEY":

- If the letters of this word are written in all possible orders and these words are arranged in a dictionary, find the rank of the word "BERKELEY". Let this rank be N.
- If the number of words that can be formed using all the letters of this word without changing the relative order of the vowels and consonants is D,

Compute $N + D$.

Problem 25

In How Many Ways You can spell "ASTRODYNAMICS" in a connected path

A
S S
T T T
R R R R
O O O O O
D D D D D D
Y Y Y Y Y Y Y
N N N N N N
A A A A A
M M M M
I I I
C C
S

Day 6

"STAY FOOLISH
STAY HUNGRY"

- Steve Jobs

Problem 26

The number two distinct integers from 1 to 10^5 such that their difference is at most 16 are N

Compute sum of digits of N

Problem 27

In how many distinct arrangements can Jack distribute $54n$ identical Calculus books into 3 identical boxes, where an empty box is permitted?

Select One Correct Option-

A) $247n^2 + 30n + 1$

B) $243n^2 + 27n + 1$

C) $247n^2 + 27n + 1$

D) $243n^2 + 30n + 1$

E) $540n^2 + 27n + 1$

Problem 28

In the video game XD, a child named X is trapped in a mine and can move in six directions: North, West, East, South, Upward, and Downward. To win, X needs to take exactly nine steps and end up one step away from his starting point in any direction.

How many possible ways are there for X to achieve this victory in the game XD?

Problem 29

- In N number of ways, we can distribute 7 distinguishable pastries in 3 identical boxes, where an empty box is allowed.
- There are D permutations of the letters of the word PENNSYLVANIA such that neither the pattern 'PENN' nor 'SYLV' nor 'ANIA' appears.

Compute $N + D$

Problem 30

Daniel, a fervent advocate for knowledge dissemination, faces the delightful task of allocating 28 Physics books among three Physics enthusiastic recipients: X, Y, and Z.
Each enthusiast has their unique preferences, presenting Daniel with an intriguing challenge.

Enthusiast X, renowned for their boundless curiosity, stipulates a minimum of 3 books to quench their thirst for understanding, while capping their request at 9 to ensure thorough exploration.

Y, with a more measured approach, seeks no more than 8 books, valuing depth over quantity in their academic pursuits.

Meanwhile, Z, the discerning scholar, desires a minimum of 7 books to delve into their studies, setting an upper limit of 17 to maintain focus and balance.

Amidst these varied preferences, Daniel must orchestrate the distribution of the Physics books. How many distinct arrangements exist for Daniel to satisfy each enthusiast's demands while ensuring fairness and joy in the pursuit of knowledge?

Day 7

"MATHEMATICS IS THE RESULT OF MYSTERIOUS POWERS WHICH NO ONE UNDERSTANDS, AND WHICH THE UNCONSCIOUS RECOGNITION OF BEAUTY MUST PLAY AN IMPORTANT PART"

- Richard Feynman

Problem 31

In the busy and exciting City of Mathematica, which is a big square measuring 18 miles by 18 miles, there are ten esteemed mathematical research laboratories. These labs are placed randomly within the city's 18 by 18 miles area.

Even though their locations are random, we can prove something interesting. We can show that at least two of these labs are at most $6\sqrt{2}$ miles apart.

Prove that among the ten research labs in the City of Mathematica, at least two of them are at most $6\sqrt{2}$ miles apart.

Problem 32

In the technologically advanced city of Numerica, where cutting-edge computing and digital innovation thrive, the inhabitants use a unique numeral system based on hexadecimal (base 16). In this bustling metropolis, data scientists and engineers often work with long hexadecimal numbers in their advanced algorithms and computations.

One day, three brilliant data scientists—Alice, Bob, and Charlie—were each tasked with generating an eleven-digit hexadecimal number. However, there was a catch: none of the digits in these numbers could be repeated. Each digit could only appear once in any given number.

As they compared their results, they stumbled upon an intriguing mathematical certainty. Despite the immense variety possible in an eleven-digit hexadecimal number, they realized that there was an inevitable repetition.
They sought to prove a curious claim:

Prove that among any three eleven-digit hexadecimal numbers, where repetition of digits is not allowed within each number, at least one digit must appear at least three times in total across the three numbers.

Problem 33

Let $A \subset \{1, 2, 3, \ldots, 175\}$ and $|A| = 12$. Prove that it is possible to choose two disjoint non-empty subsets X and Y of A
Such that

$$\sum_{a \in X} \mathrm{a} = \sum_{b \in Y} \mathrm{b}$$

Problem 34

In the whimsical land of Playtopia, where everyone has a quirky relationship with everyone else, a grand gathering of 120 people took place. In this gathering, each person had their unique way of interacting with others—some were friends, while others were considered "enemies." In Playtopia, the term "enemy" is used lightly and humorously; it simply means someone with whom one prefers playful disagreements or friendly rivalries.

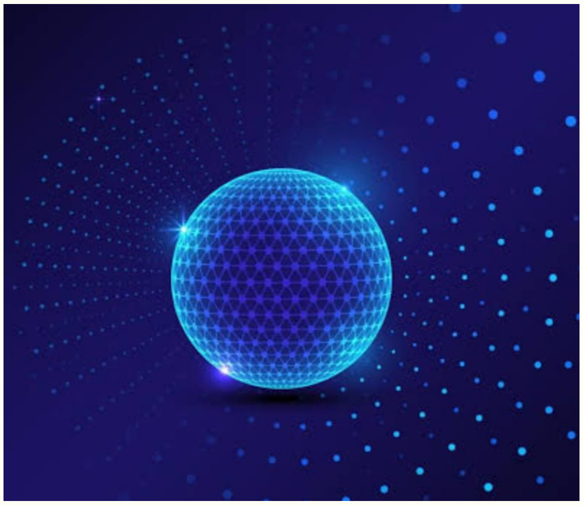
At this gathering, a curious mathematician named Dr. Dave decided to explore the dynamics of these playful rivalries. He pondered over the intriguing relationships and wondered about the number of "enemies" each person had. With 120 individuals, each having a different number of enemies, He was determined to find an interesting pattern.

After observing the group for a while, Dr. Dave had a hilarious realization. Despite the varied nature of friendships and playful rivalries, there was an inevitable conclusion:

Prove that in this lively group of 120 people, there are at least two individuals who have the same number of enemies.

Problem 35

Given any 81 points on this given sphere,
show that some seven of them must lie within one closed hexadecant (one-sixteenth of a sphere).



Day 8

"YOUR WORK IS GOING TO FILL A LARGE PART OF YOUR LIFE, AND THE ONLY WAY TO BE TRULY SATISFIED IS TO DO WHAT YOU BELIEVE IS GREAT WORK. AND THE ONLY WAY TO DO GREAT WORK IS TO LOVE WHAT YOU DO."

- Steve Jobs

Problem 36

Vedant and Aditya are playing a game. In this game, they use pieces of paper with 2014 positions, in which some permutation of the numbers 1, 2, ...2014 are to be written. (Each number will be written exactly once). Vedant fills in a piece of paper first.

How many pieces of paper must Aditya fill in to ensure that at least one of his pieces of paper will have a permutation that has the same number as Vedant's in at least one positon?

Problem 37

Eight all different Dosas are placed evenly on the edge of a round table, whose surface can rotate around the center. Eight people also evenly sit around the table, each with one Dosa in front. Each person has one favorite dosa among these eight, and they are all distinct. They find that no matter how they rotate the table, there are never more than three people who have their favorite dosa in front of them simultaneously.

By this requirement, how many different possible arrangements of the eight dosas are there? Two arrangements that differ by a rotation are considered the same

Problem 38

Louis has an orange 3-by-3-by-3 cube, which is comprised of 27 distinguishable, 1-by-1-by-1 cubes. Each small cube was initially orange, but Louis painted 10 of the small cubes completely black. In how many ways could he have chosen 10 of these smaller cubes to paint black such that every one of the 27 3-by-1-by-1 sub-blocks of the 3-by-3-by-3 cube contains at least one small black cube?

Problem 39

Charles, Lucas, Ellie, Harsh, George, and Yang sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time,Charles, Lucas, Ellie, Harsh, George, and Yang each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Charles, Lucas, Ellie, and Harsh say 19, 22, 21, and 23, respectively.
Compute the product of the dice rolls

Problem 40

There are 2017 Bald Eagles in a Huge Cage. Every second, two Bald Eagles are chosen uniformly at random and combined to form
one super-Bald Eagle. (Super-Bald Eagles are still Eagles.)
The probability that after 2015 seconds (meaning when there are only two Bald Eagles remaining) there is some Bald Eagles that has never been combined with another Eagle can be written in the form $\frac{N}{D}$ where N and D are relatively prime positive integers.

Compute N + D

Day 9

"MATHEMATICS HAS TWO GREAT BEAUTIES: THE TRUTHS IT TEACHES AND THE EFFORT IT DEMANDS."

- Blaise Pascal

Problem 41

An Industrialist Taro wishes to cover a circle to establish his innovative Factory with circumference 10 with six different types of colored arcs. Each type of arc has radius 5, has length either a or 2, and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:

- Any two arc are of different colors
- Any three adjacent arcs where the middle arc has length π are of three different colors

Find the number of distinct ways Industrialist Taro can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

Problem 42

A light pulse starts at a corner of a reflective square. It bounces around inside the square, reflecting off of the square's perimeter n times before ending in a different corner. The path of the light pulse, when traced, divides the square into exactly 2021 regions.

Compute the smallest possible value of n.

Problem 43

Random Sequences a_1,a_2,a_3 ... and b_1,b_2,b_3 ... are chosen so that every element in each sequence is chosen independently and uniformly from the set $\{1, 2, 3, \ldots, 100\}$

Compute the expected value of the smallest non-negative Integer s such that there exist positive integers m and n with

$$s = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Problem 44

Dmitry writes the number 147 on a blackboard. Then, at any point, if the number on the blackboard is n, he can perform one of the following three operations:

- If n is even, he can replace n with $\frac{n}{2}$
- If n is odd, he can replace n with $\frac{n+255}{2}$
- If $\mathrm{n} \geq 64, \mathrm{he} \quad \mathrm{can} \quad \mathrm{replace\ n\ with\ (n{-}64)}$

Compute the number of possible values that Dmitry can obtain by doing zero or more operations.

Problem 45

Compute the number of triples (f, g, h) of permutations on $\{1, 2, 3, 4, 5\}$ such that

$$f(g(h(x))) = h(g(f(x))) = g(x),$$
$$g(h(f(x))) = f(h(g(x))) = h(x),$$
$$h(f(g(x))) = g(f(h(x))) = f(x)$$

for all $x \in \{1, 2, 3, 4, 5\}$.

Day 10

"AN EQUATION FOR ME HAS NO MEANING UNLESS IT EXPRESS A THOUGHT OF GOD."

- Srinivasa Ramanujan

Problem 46

Xang has a permutation $(a_1, a_2, \ldots, a_{2019})$ of $S = \{1, 2, \ldots, 2019\}$, and Alexander wants to guess his permutation.
With each guess Alexander gives Xang an n-tuple $(y_1, y_2, \ldots, y_{2019})$ of integers in S, and then Xang gives the number of indices $i \in S$ such that $a_i = y_i$.

(a) Show that Alexander can always guess Xang's permutation with at most 1200000 guesses.

(b) Show that Alexander can always guess Xang's permutation with at most 24000 guesses.

Problem 47

A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.

Problem 48

Let n be a positive integer. Alice writes n real numbers a_1,a_2,a_3 $\dots$,a_n in a line (in that order). Every move, she picks one number and replaces it with the average of itself and its neighbors (a_n is not a neighbor of a_1, nor vice versa). A number changes sign if it changes from being nonnegative to negative or vice versa. In terms of n, determine the maximum number of times that a1 can change sign, across all possible values of a_1,a_2,a_3 $\dots$,a_n and all possible sequences of moves Alice may make

Problem 49

Call a simple graph G quasi-colorable if we can color each edge blue, red, green, and white such that

- for each vertex v of degree 3 in G, the three edges containing v as an endpoint are either colored blue, red, and green, or all three edges are white
- not all edges are white

A connected graph G has a vertices of degree 4, b vertices of degree 3, and no other vertices, where a and b are positive integers. Find the smallest real number c so that the following statement is true:
"If a/b > c, then G is quasi-colorable."

Problem 50

A spider is walking on the boundary of equilateral triangle $\triangle ABC$ (vertices labelled in counterclockwise order), starting at vertex A. Each second, she moves to one of her two adjacent vertices with equal probability. The *windiness* of a path that starts and ends at A is the net number of counterclockwise revolutions made. For example, the windiness of the path $ABCA$ is 1, and the windiness of the path $ABCACBACBA$ is -1. What is the remainder modulo 1000 of the sum of the squares of the windiness values taken over all possible paths that end back at vertex A after 2025 seconds?

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Solution to Day 1

Solution to Problem 1

- Indian Family with 5 members
- American family with 4 members
- French family with 5 members
- Japanese family with 3 members

As it is said same family members should not be separated then consider making them in a closed box

That is-

$5!5!4!3! \times 3!$ (As this is on Round Table)

Hence, $\boxed{C)(5!)^2(4!)(3!)^2}$

Solution to Problem 2

Since, we can take cases here but let's be a little smart and apply the complement method:

Total ways without any restriction − (ways in which B_1 and G_1 are included)

1. **Total ways without any restriction:**

$$\binom{15}{8} \times \binom{7}{5}$$

2. **Ways in which both B_1 and G_1 are included:**

$$\binom{14}{7} \times \binom{6}{4}$$

3. **Valid ways to form the chess club:**

$$\binom{15}{8} \times \binom{7}{5} - \binom{14}{7} \times \binom{6}{4}$$

So, the number of ways to form the chess club without including both B_1 and G_1 together is:

$$\boxed{\binom{15}{8} \times \binom{7}{5} - \binom{14}{7} \times \binom{6}{4}}$$

Solution to Problem 3

First of all,
$^{n-1}P_{n-2} = (n-1)!$

$$\sum_{n=3}^{2026} (n+1)(n)^{n-1}P_{n-2} = (n+1)!$$

$$\sum_{n=2}^{2023} (n+2)(n+1)^{n}P_{n-1} = (n+2)!$$

So, 4! + 5! + 6! + . . . + 2027! -(4! + 5! + 6! . . . + 2025!)
= 2026! + 2027!

$\binom{a}{b} = \binom{2027}{2026} = \boxed{2027}$

Solution to Problem 4

First thing first, Separate it
Consider Δ as the days when dave will eat pizza
Consider Δ' as the days when dave will not eat pizza

we have,
$\Delta\Delta\Delta\Delta\Delta\Delta\Delta\Delta\Delta \quad \Delta'\Delta'\Delta'\Delta'\Delta'\Delta'\Delta'\Delta'\Delta'\Delta'\Delta'$

We see 12 days gap in it and we have to choose 9 from it

$\binom{12}{9} = \boxed{220}$

Solution to Problem 5

That three children specifically requested odd-numbered cakes in positive integers would be in form of $2k + 1$ where 0 $\leq$k
and k being a non-negative integer

And The remaining four children who are open to receiving even-numbered cakes but prefer them in non-negative integers would be in form $2p$
where 0 $\leq$p
and p being a non-negative integer

Hence our final equation is
$2a + 1 + 2b + 1 + 2c + 1 + 2d + 2e + 2f + 2g = 23$
$\implies$ 2a+2b+2c+2d+2e+2f+2g= 20
$\implies$ a+b+c+d+e+f+g= 10

Applying stars and bars
$\binom{10+7-1}{7-1} = \binom{16}{6}$
Hence ,

$$\boxed{\binom{16}{6}}$$

Solution to Day 2

Solution to Problem 6

Notice that $w \in \mathbb{N}$.
So,
$|x| + |y| + |z| + w = 15,$

Case 1:
$x, y, z \neq 0$
Thus we get,

$|x|' + |y|' + |z|' + w = 12,$

Applying stars and bars
$\implies$
$\binom{12+4-1}{4-1} \times 2^3$ (Now, why i did" $\times 2^3$"? Think by yourself)

Case 2:
any one from $x, y, z = 0$
Let $x = 0$

$|y|' + |z|' + w = 13,$

Applying stars and bars
$\implies$
$\binom{13-3+1}{3-1} \times 2^2 \times 3$

Case 3:
any two from $x, y, z = 0$
$|x|' + w = 14$

Applying stars and bars
$\implies$
$\binom{14+2-1}{2-1} \times 2 \times 3$

Case 4:
all three $x = y = z = 0$

$w = 16$

Now, Add all Cases and Done

Solution to Problem 7

We have,

$$\sum_{n=1}^{200} \lfloor \frac{108+5(n)}{540} \rfloor$$

Now,
$n = 87, 88, \ldots, 194$ will give you 1
$n = 195, 196, \ldots, 200$ will give you 2

$N = 108 + 2(6) = 120$

Exponent of 3 in $(120)!$
$\implies$
$\lfloor \frac{120}{3} \rfloor + \lfloor \frac{120}{9} \rfloor + \lfloor \frac{120}{27} \rfloor + \lfloor \frac{120}{81} \rfloor = 40 + 13 + 4 + 1$

$\boxed{= 58}$

Solution to Problem 8

First, Separated them 2 highly skilled players,

Let them be A and B

A B $\boxed{\Delta\Delta\Delta\Delta\Delta \ldots \Delta\Delta\Delta\Delta}$ (20 Rest of Players)

We want to divide 20 into two groups of 10 and 10
$\implies$

$\binom{20}{10} = \frac{20!}{10!10!}$

$\boxed{\textbf{A)}\, \frac{20!}{10!^2}}$

Solution to Problem 9

- The First Place have 21 Options
- The Second place have 5 Options
- The Third Place have 4 Options
- The fourth Place have 3 Options
- The fifth Place have 20 Options

Thus we have,
$21 \times 5 \times 4 \times 3 \times 20$

$\boxed{C)25200}$

Solution to Problem 10

In word "STANFORD"
We have 2 vowels (A,O) and 6 consonants (S,T,N,F,R,D)

$\binom{2}{1} \cdot \binom{6}{4} = 2 \times 3 \times 5 = 30$

$30 \times 5! = \boxed{3600}$

Solution to Day 3

Solution to Problem 11

It is Said No Zero in it thus we have 4 Options for even numbers

- The First Place have 3 Options
- The Second place have 6 Options
- The Third Place have 2 Options
- The fourth Place have 5 Options
- The fifth Place have 4 Options

Thus the Aliens might have,
$3 \times 6 \times 2 \times 5 \times 4$

$\boxed{= 320}$

Solution to Problem 12

Since every term of the binary sequence has three options (1,6 or 7), therefore the number of binary sequences of n terms is:

$$\underbrace{3 \times 3 \times 3 \times \cdots \times 3}_{\text{n times}} = 3^n \quad \text{(using multiplication principle).}$$

$\boxed{C)\ \ 3^n}$

Solution to Problem 13

We have 3 Cases in Total
it must have last two digits as $00, 25, 50$

Last two Digits 00

- The First Place have 5 Options
- The Second place have 6 Options
- The Third Place have 6 Options
- The fourth Place have 6 Options
- The fifth Place have 6 Options

If you notice the scenario is same for 25 and 50 as Last Digit

$\implies$
$(5\ \times 6^4) \times 3$

$\boxed{= 19440}$

Solution to Problem 14

After Observing pattern
We get,

Except $5k+1$, for $k=0,1,2,\ldots,19998$

All numbers will be unmarked

Solution to Problem 15

The Scenario is-
$0 \leq x,y,z,w \leq 9$

<u>Case 1:</u> One Digit being 7

$\binom{4}{1} \times 9^3 = 729 \times 4 = 2916$

<u>Case 2:</u> Two Digits being 7

$\binom{4}{2} \times 9^2 = 81 \times 6 \times 2 = 972$

<u>Case 3:</u> Three Digits being 7

$\binom{4}{3} \times 9 \times 3 = 108$

<u>Case 4:</u> four Digits being 7

$\binom{4}{4} \times 4 = 4$

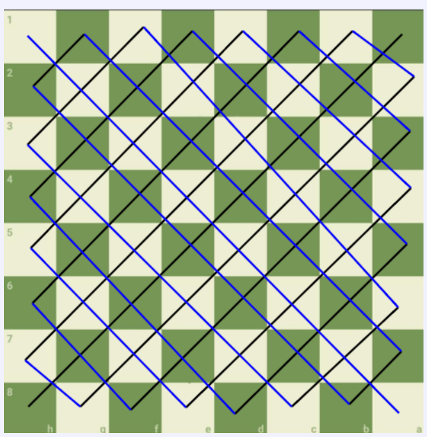
Adding all ,

$2916 + 972 + 108 + 4$

$\boxed{= 4000}$

Solution to Day 4

Solution to Problem 16



In Above Image i have Shown number of ways they can be arranged
We Can Place them like,

$2 \times (2 \times (\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2}) + \binom{8}{2})$
$\implies$
$2 \times (2 \times (1 + 3 + 6 + 10 + 15 + 21) + 28)$
$\implies$
$2 \times (136)$
$\implies$
$\boxed{= 272}$

Solution to Problem 17

Talking About **(A)**
$a+b+c = 16 \implies$ a+b+c= 15
Using Stars and Bars,

$\binom{15+3-1}{3-1} = \binom{17}{2}$

$\boxed{\textbf{(R)}136}$

Talking About **(B)**
$5! \times 4! \times (2-1)! =$

$\boxed{\textbf{(S)}(24)^2 \times 5}$

Talking About **(C)**
$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6}\ ) + \binom{7}{7} = 7 + 21 + 35 + 35 + 21 + 1$

$\boxed{\textbf{(P)}120}$

Talking About **(D)**

- The First Place have 6 Options
- The Second place have 5 Options
- The Third Place have 4 Options
- The fourth Place have 3 Options
- The fifth Place have 2 Options

$\boxed{\textbf{(Q)}720}$

$\boxed{\textbf{3)}(A-R);(B-S);(C-P);(D-Q)}$

Solution to Problem 18

The Total ways to select 5 problems is 2^5
But here we overcounted something
Remember Barron has to select atleast 1 problem from each day so subtract one case when he will leave all problems

$\boxed{= (2^5 - 1)^{10}}$

Solution to Problem 19

We know the the method to find the number of factors for any number and as it is given $a, b, c, d, e, f, \dots x, y, z$ are all prime numbers

$N = (26{+}1)(25{+}1)(24{+}1)(23{+}1)\dots(3)(2) = 27!$

Exponent of 2 in 27!

$\lfloor \frac{27}{2} \rfloor + \lfloor \frac{27}{4} \rfloor + \lfloor \frac{27}{8} \rfloor + \lfloor \frac{27}{16} \rfloor = 13 + 6 + 3 + 1$

$\boxed{= 23}$

Part 1 : Solution to Problem 20

Let us go case by case:

1. **All alike**:
    - There is one group of all alike books (5 topology books)
    - Number of ways to choose 1 group = $\binom{1}{1} = 1$
2. **4 alike and 1 distinct**:
    - There are 2 groups of 4 alike books (4 topology books, 4 linear algebra books) and after selecting one group, there are 4 distinct books left from where we require to choose one book.
    - Number of ways to select '4 alike and 1 distinct' = $\binom{2}{1} \times \binom{4}{1} = 8$
3. **3 alike and 2 alike**:
    - Select 3 alike books from 3 groups of 3 alike books (3 on topology,3 on linear algebra,3 on real analysis) in $\binom{3}{1}$ ways. Then select 2 alike books from remaining 3 groups of 2 alike books in $\binom{3}{1}$ ways.
    - Number of ways to select '3 alike and 2 alike' = $\binom{3}{1} \times \binom{3}{1} = 9$
4. **3 alike and 2 distinct**:
    - Select 3 alike books from 3 groups of 3 alike books in $\binom{3}{1}$ ways. Select 2 books from remaining 4 distinct books in $\binom{4}{2}$ ways.
    - Number of ways to select '3 alike and 2 distinct' = $\binom{3}{1} \times \binom{4}{2} = 18$

Part 2 : Solution to Problem 20

1. **2 alike, 2 alike and 1 distinct**:
   - Select 2 groups of 2 alike books from 4 groups of 2 alike books in $\binom{4}{2}$ ways. Further select 1 book from remaining 3 distinct books in $\binom{3}{1}$ ways.
   - Number of ways to select '2 alike, 2 alike and 1 distinct' = $\binom{4}{2} \times \binom{3}{1} = 18$
2. **2 alike and 3 distinct**:
   - Select one group of 2 alike books from 4 groups of 2 alike books in $\binom{4}{1}$ ways. Then select 3 books from remaining 4 distinct books in $\binom{4}{3}$ ways.
   - Number of ways to select '2 alike and 3 distinct' = $\binom{4}{1} \times \binom{4}{3} = 16$
3. **All distinct**:
   - Select 5 distinct books from 5 distinct books in $\binom{5}{5}$ ways.
   - Number of ways to select 'All distinct' = $\binom{5}{5} = 1$

Combining all above cases, total number of ways in which child can select 5 books is:

$$1 + 8 + 9 + 18 + 18 + 16 + 1 \boxed{= 71}$$

Solution to Day 5

Solution to Problem 21

Number of ways is $\binom{70}{7} = N$

Compute $\phi(N)$ on your own

Solution to Problem 22

Since it is said for "atleast" 16
We have,

$\binom{30}{16} \times 1 \times 3^{14}$

$$\boxed{= \binom{30}{16} \times 3^{14}}$$

Solution to Problem 23

There are 9 letters and all distinct

- A = 9!
- B = 8!
- C = 7!
- D = 6! $\times \binom{7}{3} \times$ 3!

$\frac{AB}{CD} = \frac{9! \times 8!}{7! \times 6! 3! \times 35}$

$\boxed{\dfrac{AB}{CD} = \frac{3 \times 4 \times 8}{5}}$

Solution to Problem 24

$N = \boxed{1467}$

In word "BERKELEY" there are 3 vowels and 5 consonants
and that 3 vowels are same (i.e. 3 times "e") and 5 different consonants

$D = \frac{3!}{3!} \times 5! = 120$

$N + D = 1467 + 120 = \boxed{1587}$

Solution to Problem 25

Tilt This Figure to 45°
We will end up getting a Rectangle
Consider this as a rectangle (i know the above figure is bit messy but consider it as a rectangle)
We have

$$\begin{array}{ccccccc} A & S & T & R & O & D & Y \\ S & T & R & O & D & Y & N \\ T & R & O & D & Y & N & A \\ R & O & D & Y & N & A & M \\ O & D & Y & N & A & M & I \\ D & Y & N & A & M & I & C \\ Y & N & A & M & I & C & S \end{array}$$

We want to go from A $\longrightarrow S$
Also we know one formula for $m\times$n rectangle

$$\binom{m+n}{n} = \binom{6+6}{6}$$

$$\boxed{\binom{12}{6}}$$

Solution to Day 6

Solution to Problem 26

1. **Counting pairs for each difference d from 1 to 16:**
   - For $d = 1$:
     $$\text{Number of pairs} = (1, 2), (2, 3), \ldots, (99999, 100000)$$
     This gives $10^5 - 1 = 99999$ pairs.
   - For $d = 2$:
     $$\text{Number of pairs} = (1, 3), (2, 4), \ldots, (99998, 100000)$$
     This gives $10^5 - 2 = 99998$ pairs.
   - Similarly, for any d, the number of pairs is $10^5 - d$.
2. **Summing up the pairs for all differences from 1 to 16:**
   - The total number of pairs N is the sum of pairs for $d = 1, 2, \ldots, 16$:
     $$N = \sum_{d=1}^{16} (10^5 - d)$$
3. **Calculating the sum:**
   - We can factor out 10^5 and use the sum of the first 16 natural numbers:
     $$N = \sum_{d=1}^{16} (10^5 - d) = 16 \times 10^5 - \sum_{d=1}^{16} d$$
   - The sum of the first 16 natural numbers is:
     $$\sum_{d=1}^{16} d = \frac{16 \times 17}{2} = 136$$
   - Thus:
     $$N = 16 \times 10^5 - 136 = 1600000 - 136 = 1599864$$

Conclusion The total number of pairs (a, b) such that $1 \leq a < b \leq 10^5$ and $b - a \leq 16$ is:

$$\boxed{1599864}$$

Solution to Problem 27

To determine the number of distinct arrangements in which Jack can distribute $54n$ identical Calculus books into 3 identical boxes (where empty boxes are allowed), we use the partition function $P_k(n)$ for $k = 1, 2, 3$.

1. **Partitions into 1 part**:
$$P_1(54n) = 1$$

2. **Partitions into 2 parts**:
$$P_2(54n) = \left\lfloor \frac{54n}{2} \right\rfloor = 27n$$

3. **Partitions into 3 parts**: Using the given approximation for $P_3(n)$:
$$P_3(n) \approx \frac{n^2}{12}$$
Substitute $54n$ for n:
$$P_3(54n) \approx \frac{(54n)^2}{12} = \frac{2916n^2}{12} = 243n^2$$

4. **Total Number of Partitions**: Sum the contributions from partitions into 1, 2, and 3 parts:
$$\sum_{k=1}^{3} P_k(54n) = P_1(54n) + P_2(54n) + P_3(54n)$$
$$\sum_{k=1}^{3} P_k(54n) = 1 + 27n + 243n^2$$

Hence, the total number of distinct arrangements for Jack to distribute $54n$ identical Calculus books into 3 identical boxes is:

$$\boxed{1 + 27n + 243n^2}$$

Solution to Problem 28

To solve this, note that X needs to end up one step away from the starting point after 9 steps. The possible net displacements are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, or $(0, 0, \pm 1)$.
Let x, x', y, y', z, z' be the number of steps in the East, West, North, South, Upward, and Downward directions, respectively. The steps must satisfy:

$$x + x' + y + y' + z + z' = 9$$

and one of $x - x', y - y', z - z'$ must be ± 1.
General Case Assume $x = x' + 1$ (or $x = x' - 1$), $y = y'$, and $z = z'$. Then:

$$(x' + 1) + x' + y + y + z + z = 9 \Rightarrow 2x' + 1 + 2y + 2z = 9 \Rightarrow x' + y + z = 4$$

The number of non-negative integer solutions to $x' + y + z = 4$ is:

$$\binom{4+2}{2} = 15$$

Since the same logic applies for displacements in the y and z directions as well, and each of these can be ± 1:

$$6 \times 15 = 90$$

Therefore, the total number of ways for X to win by ending up one step away from the starting point after 9 steps is:

$$\boxed{90}$$

part 1 ; Solution to Problem 29

Given that the boxes are identical, we need to consider partitions of the pastries. The formula involves summing over the Stirling numbers of the second kind:

$$N = \sum_{k=1}^{3} S(7,k)$$

Here, $S(7,k)$ represents the number of ways to partition 7 distinguishable pastries into k non-empty subsets.
Calculation of $S(7,k)$
1. **For $k = 1$:**

$$S(7,1) = 1$$

There is only one way to partition 7 objects into 1 subset (all objects in one subset).
2. **For $k = 2$:**

$$S(7,2) = \frac{1}{2!}\sum_{j=0}^{2}(-1)^j \binom{2}{j}(2-j)^7$$

$$S(7,2) = \frac{1}{2}\left[(2)^7 - \binom{2}{1}(1)^7 + \binom{2}{2}(0)^7\right]$$

$$S(7,2) = \frac{1}{2}[128 - 2\cdot 1 + 1\cdot 0] = \frac{1}{2}\cdot 126 = 63$$

3. **For $k = 3$:**

$$S(7,3) = \frac{1}{3!}\sum_{j=0}^{3}(-1)^j \binom{3}{j}(3-j)^7$$

$$S(7,3) = \frac{1}{6}\left[(3)^7 - \binom{3}{1}(2)^7 + \binom{3}{2}(1)^7 - \binom{3}{3}(0)^7\right]$$

$$S(7,3) = \frac{1}{6}[2187 - 3\cdot 128 + 3\cdot 1 - 1\cdot 0] = \frac{1}{6}[2187 - 384 + 3] = \frac{1806}{6} = 301$$

Summing up the results:

$$N = S(7,1) + S(7,2) + S(7,3)$$

$$N = 1 + 63 + 301 = 365$$

Therefore, the number of ways to distribute 7 distinguishable pastries into 3 identical boxes where empty boxes are allowed is: $\boxed{365}$

part 2 ; Solution to Problem 29

$$N = \frac{12!}{3! \cdot 2!} = 39916800$$

Using the principle of inclusion-exclusion:

$$|A_1' \cap A_2' \cap A_3'| = N - (|A_1| + |A_2| + |A_3|$$

- $|\ A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|)$

1. Permutations containing 'PENN' ($|A_1|$):

$$|A_1| = \frac{9!}{2!} = 181440$$

2. Permutations containing 'SYLV' ($|A_2|$):

$$|A_2| = \frac{9!}{3! \cdot 2!} = 30240$$

3. Permutations containing 'ANIA' ($|A_3|$):

$$|A_3| = \frac{9!}{2!} = 181440$$

4. Permutations containing 'PENN' and 'SYLV' ($|A_1 \cap A_2|$):

$$|A_1 \cap A_2| = \frac{6!}{2!} = 360$$

5. Permutations containing 'PENN' and 'ANIA' ($|A_1 \cap A_3|$):

$$|A_1 \cap A_3| = 6! = 720$$

6. Permutations containing 'SYLV' and 'ANIA' ($|A_2 \cap A_3|$):

$$|A_2 \cap A_3| = \frac{6!}{2!} = 360$$

7. Permutations containing 'PENN', 'SYLV', and 'ANIA' ($|A_1 \cap A_2 \cap A_3|$):

$$|A_1 \cap A_2 \cap A_3| = 3! = 6$$

Now combine all the terms:

$|A_1 \cup A_2 \cup A_3| = 181440 + 30240 + 181440 - 360 - 720 - 360 + 6 = 391686$

$|A_1' \cap A_2' \cap A_3'| = N - |A_1 \cup A_2 \cup A_3| = 39916800 - 391686 = 39525114$

Thus, Final Answer is: $\boxed{39525114}$

part 3 ; Solution to Problem 29

$39525114 + 365 = \boxed{39525479}$

Solution to Problem 30

We will be using principle of Inclusion-Exclusion here

$$\binom{18+3-1}{3-1} = \binom{20}{2} = 190.$$

Let
n_1 be the distribution where the X gets at least 7 books;
n_2, the distributions where the Y gets at least 9 books and
n_3, the distributions where the Z gets at least 11 books.

$$|n_1| = \binom{18-7+3-1}{3-1} = \binom{13}{2} = \frac{13 \times 12}{1 \times 2} = 78$$

$$|n_2| = \binom{18-9+3-1}{3-1} = \binom{11}{2} = \frac{11 \times 10}{1 \times 2} = 55$$

$$|n_3| = \binom{18-11+3-1}{3-1} = \binom{9}{2} = \frac{9 \times 8}{1 \times 2} = 36$$

$$|n_1 \cap n_2| = \binom{18-7-9+3-1}{3-1} = \binom{4}{2} = \frac{4 \times 3}{1 \times 2} = 6$$

$$|n_2 \cap n_3| = \binom{18-9-11+3-1}{3-1} = \binom{0}{2} = 0$$

$$|n_3 \cap n_1| = \binom{18-11-7+3-1}{3-1} = \binom{2}{2} = \frac{2 \times 1}{1 \times 2} = 1$$

$$|n_1 \cap n_2 \cap n_3| = 0,$$

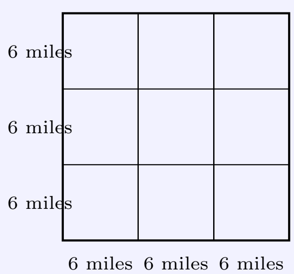
$$\Rightarrow |n_1 \cup n_2 \cup n_3| = 78 + 55 + 36 - 6 - 0 - 1 + 0 = 162.$$

So, the required number of solutions $= 190 - 162 = \boxed{28}$.

Solution to Day 7

Solution to Problem 31

City of Mathematica (side length 18 miles)



Now here we have 10 Labs to Accommodate in these 9 squares

From Pigeon Hole Principle (PHP)

We conclude that atleast one small square will have two labs in it and at least two of them will be at most $6\sqrt{2}$ miles apart.

Solution to Problem 32

1. **Total Digits:** Each number has 11 unique digits.
   - Total digits across three numbers: $3 \times 11 = 33$.
2. **Unique Hexadecimal Digits:** There are 16 unique hexadecimal digits: $\{0, 1, 2, \ldots, 9, A, B, C, D, E, F\}$.
3. **Pigeonhole Principle:** Distribute 33 digits among 16 unique digits.
   - By the pigeonhole principle, if n items are placed into m containers and $n > m$, at least one container must contain more than $\lceil \frac{n}{m} \rceil$ items.
   - Here, $n = 33$ and $m = 16$.
4. **Calculation:**
$$\left\lceil \frac{33}{16} \right\rceil = 3$$
5. **Conclusion:** At least one digit must appear at least 3 times among the total 33 digits.

This completes the proof

Solution to Problem 33

Since $\mid A \mid = 12$, the number of non-empty, proper subsets of X is $2^{12} - 2 = 4094$.
The sum of the elements of the proper subsets of A can possibly range from

$$1 \text{ to } \sum_{i=1}^{11}(164+i).$$

That is 1 to $(165 + 166 + \cdots + 175)$, i.e., 1 to 1870.
That is, the 1022 subsets can have sums from 1 to 1870.
By the Pigeonhole Principle, at least two distinct subsets B and C will have the same sum.
There are 1870 different sums, and so if we have more than 1870 subsets, then at least two of them have the same sum.
If B and C are not disjoint, then let

$$A = B - (B \cap C),$$

$$X = C - (B \cap C).$$

Clearly, A and X are disjoint and non-empty and have the same sum of their elements.
Define $s(A) =$ sum of the elements of A. We have B and C not necessarily disjoint such that $s(B) = s(C)$.
Now,

$$s(A) = s(B) - s(B \cap C)$$
$$s(X) = s(C) - s(B \cap C)$$
$$s(B) = s(C).$$

but,

$$s(X) = s(Y).$$

Hence, $s(X) = s(Y)$.
Also $X \neq \phi$. For if X is empty, then $B \subset C$ which implies $s(B) < s(C)$ (a contradiction). Thus, X and Y are non-empty and $s(X) = s(Y)$.

Solution to Problem 34

Consider the total number of people, which is 120. Let's denote the number of enemies a person has by k.
We will consider two cases:

Case 1: Suppose each of the 120 members of the group has at least 1 friend. In this scenario, the number of enemies each person can have ranges from 0 to 119.
However, if each person has at least 1 friend, the number of enemies each person can have ranges from 1 to 119. This gives us 119 possible distinct values.
Now, applying the Pigeonhole Principle: If we have 120 people but only 119 possible distinct values for the number of enemies, at least two people must have the same number of enemies.

Case 2: Suppose there exists a person who has no friends (thus having 119 enemies).
In this scenario, the number of enemies can range from 0 to 119. If one person has 0 enemies, then another person having 119 enemies is not possible because the person with 119 enemies would consider the person with 0 enemies as an enemy, which contradicts the definition. Therefore, the possible number of enemies ranges from 0 to 118, giving us 119 possible values in total.
Again, applying the Pigeonhole Principle: With 120 people and 119 possible distinct values, at least two people must have the same number of enemies.

In either case, we conclude that in this lively group of 120 people, there are at least two individuals who have the same number of enemies.

Solution to Problem 35

1. **Sphere and Rings**:
   - Consider a sphere centered at point O.
   - The sphere is divided into 8 great circles (rings), each dividing the sphere into 16 equal parts (hexadecants).
2. **Placing Points**:
   - Place 81 points $P_1, P_2, \ldots, P_{81}$ on the surface of the sphere.
   - Place 2 points on opposite sides of each of the 8 rings. This uses $8 \times 2 = 16$ points.
   - After placing these points, $81 - 16 = 65$ points remain.
3. **Applying the Pigeonhole Principle**:
   - Distribute the remaining 65 points among the 16 hexadecants formed by the sphere's division using the rings.
   - By the pigeonhole principle, if we distribute 65 points into 16 hexadecants, $65 = 16 \times 4 + 1$

     Now we have kn+1 objects are distributed among k boxes then atleast one of the box, must contain atleast (n+1) Objects
4. **Hence Proved**:
   - here $n = 16$ and $k = 4$ so atleast one of the hexadecant, must contain atleast 5 points
   - Already we have 2 points plotted on 8 great circles
   - 5 + 2 = 7

And we are done. □

Solution to Day 8

Solution to Problem 36

Aditya writes 1 to 1008 for the first 1008 spots and for the 1008 different pieces of paper, he cycle through the numbers such that the i^{th} paper will have $i, i+1, ..., 1008, 1, 2, ..., i-1$ for the first 1008 position. Hence if any of the numbers $\{1,\ 2,\ 3,\ \ldots,\ 1008\}$ appears in the first 1008 position on Vedant's paper, Aditya would have gotten it right. Otherwise, Vedant would have written the first 1008 numbers in the 1009^{th} to 2014^{th} position, which is clearly impossible as there are more numbers than positions. Thus Aditya can get it right in $\boxed{1008}$ tries

With any 1007 pieces written, we shall show there exist a piece that doesn't coincide with any of the 1007 at any position. Let s_n be the set of 1007 numbers that occurs at the n^{th} position. We see that for the i^{th} position 1 $\leq$i$\leq$ 1007we can put in some number x such that
x $\notin$s$_i$ and x is distinct from the $i-1$ entries before which.

We continue assigning numbers until it is impossible at the j^{th} position. Let the set of numbers in the first $j-1$ positions be s. Let S be the set of all 2008 numbers. We shall show it is possible to find $l < j$ such that we can switch the number at the l^{th} position to the l^{th} position and find another number for the l^{th} positon that avoids s_l . Assuming the contrary that no such l exist, let us consider s$/s_j$. Since s $\cup$s$_l$ $= S$,
we see that $|s/s_j| \geq 1007$
Also S/s $\in$s$_j$. For any positon l which has number in s$/s_j$, we need that s $\cup$s$_l = S$ which is equivalent to S/s $\in$s$_l$. Thus S/s $\in$s$_i$ for at least 1008 distinct i. this is clearly impossible. Hence we can always find a position l to swap. Continuing this algorithm, we see that we can always construct a piece of paper that is distinct at every position from any 1007 pieces of paper that Adiya picks.

Solution to Problem 37

Under rotation, there are totally $\frac{8!}{8}$ different arrangements. We find the answer by subtracting the cases where more than 3 people can match their favorite Dosa by rotation. Take an example of the case where maximum exactly 4 people can match their favorite Dosa by some rotation. (Cases of more than 4 people are even simpler.)

Suppose we have already rotated the table to the optimal position, when 4 people get matched. There are $\binom{8}{4}$ ways of choosing these 4 people. The other 4 people must be all mismatched at this moment. This is the number of "Error Permutation" and can be calculated with recursive relation $f(n) = (n-1)(f(n-1) + f(n-2))$, or simply by enumeration. It turns out to be 9. But some arrangements are counted twice. These are the cases when the mismatched 4 people can all get matched by another rotation.

If we number the Dosas from 1 to 8 by the order of positions around the table, suppose 4 people are matched to their favorite Dosa at one moment, and (a, b, c, d) $(a < b < c < d)$ are the ones that are mismatched. If under some rotation all of (a, b, c, d) get matched, then originally the favorite Dosa of the people in front of (a, b, c, d) must be (b, c, d, a), (c, d, a, b) or (d, a, b, c). So either

$$(b-a) \equiv (c-b) \equiv (d-c) \equiv (a-d) \pmod 8$$

or

$$\begin{cases} (b-a) \equiv (d-c) \pmod 8 \\ (c-b) \equiv (a-d) \pmod 8 \\ (b-a) \not\equiv (c-b) \pmod 8 \end{cases}$$

In the first case, $(a, b, c, d) = (1, 3, 5, 7)$ or $(2, 4, 6, 8)$, and 3 arrangements are counted twice.

In the second case, $(a, b, c, d) = (1, 2, 5, 6)$ or $(2, 3, 6, 7)$ or $(3, 4, 7, 8)$ or $(1, 4, 5, 8)$, and 2 arrangements are counted twice. So in total there are 5 extra counts.

The answer is

$$\frac{8!}{8} - \binom{8}{6} - \binom{8}{5} \times 2 - \binom{8}{4} \times 9 + 5 = 4274$$

Solution to Problem 38

Divide the 3-by-3-by-3 cube into 3 1-by-3-by-3 blocks. If 10 total smaller cubes are painted black, then two of these blocks must contain 3 black cube 4. Now, if a black does not have a diagonal of black cubes (allowing wrap-arounds), it must contain at least 4 cubes, so there are at least two blocks with diagonals and with exactly 3 cubes. We consider two cases

Case 1:
The diagonals of these two blocks are oriented in the same direction.
Clearly, the third block must contain a diagonal oriented in the same direction as well. The remaining black cube can be anywhere else in the block. There are $3 \cdot 6 \cdot 2 = 36$ ways to choose the first two blocks and their diagonals. There are $1 \cdot 6 = 6$ ways to choose black cubes in the remaining block. This gives a total of 216 colorings.

Case 2: They are oriented in opposite directions.
Then, the black cubes in the remaining block is determined (consider the projection of the blocks on top of one another; four squares are missing and the remaining block contains four black cubes). There are $3 \cdot 6 \cdot 3 = 54$ ways to choose the first to blocks and their diagonals. There is only 1 way to choose the black cubes in the remaining block. This gives a total of 54 colorings.

In total, then, there are $216 + 54 = \boxed{270}$ ways to choose 10 of the smaller cubes to paint black.

Solution to Problem 39

The sum of the two rolls each person sees is what they say minus 14 (the expected sum of the rolls they don't see). Since the stated numbers are all different, the sum of the two rolls each person sees is a different number, which means that no two people look at each other's dice, so everyone looks in the same direction. Assume that Charles looks at Lucas, Lucas looks at Ellie, and so on. (The other case is identical.) The sum of the two rolls that Charles, Lucas, Ellie, and Harsh see, respectively, is 5, 8, 7, and 9.

If we let c through y be the rolls of Charles through Yang, this means that $c + l = 5$, $l + e = 8$, $e + h = 7$, and $h + g = 9$. Adding the second and fourth equations and subtracting the first and third gives $gc = 5$, meaning that $g = 6$ and $c = 1$. Thus, $l = 4, c = 4$, and $h = 3$. It remains to determine y, and note that $g + y$ and $y + c$ do not belong to the set $\{5,7,8,9\}$, for this would violate the condition that everyone says a different number. This rules out everything but y = 5.

Thus, we have

clehgy $= 1 \cdot 4 \cdot 4 \cdot 3 \cdot 6 \cdot 5 = \boxed{1440}$

Solution to Problem 40

Fix a particular Bald Eagle. The probability it does not get combined with another Bald Eagle when there are n total Bald Eagles is $1 - \frac{2}{n}$. Therefore, the probability it is not combined in 2015 seconds is

$$\prod_{k=0}^{2014} \left(1 - \frac{2}{2017 - k}\right) = \prod_{k=0}^{2014} \frac{2015 - k}{2017 - k} = \frac{2}{2017 \cdot 2016}.$$

Since it is impossible for more than one Bald Eagle to never be combined, the desired probability is

$$\frac{2017 \cdot 2}{(2017)(2016)} = \frac{1}{1008}.$$

So our final answer is $1 + 1008 = \boxed{1009}$.

Solution to Day 9

Solution to Problem 41

Fix an orientation of the circle, and observe that the the problem is equivalent to finding the number of ways to color ten equal arcs of the circle such that each arc is one of three different colors, and any two arcs which are separated by exactly one arc are of different colors. We can consider every other arc, so we are trying to color just five arcs so that no two adjacent arcs are of the same color. This is independent from the coloring of the other five arcs.

Let a_i be the number of ways to color i arcs in three colors so that no two adjacent arcs are the same color. Note that $a_1 = 3$ and $a_2 = 6$. We claim that $a_i + a_{i+1} = 3 \cdot 2^i$
for $i \geq 2$.
To prove this, observe that a_i counts the number of ways to color i + 1 points in a line so that no two adjacent points are the same color, and the first and (i + 1)thpoints are the same color. Meanwhile,
a_{i+1} counts the number of ways to color i + 1 points in a line so that no two adjacent points are the same color, and the first and (i + 1)th points are different colors. Then
$a_i + a_{i+1}$ is the number of ways to color i + 1 points in a line so that no two adjacent points are the same color. There are clearly 3×2^i ways to do this, as we pick the colors from left to right, with 3 choices for the first color and 2 for the rest. We then compute $a_3 = 6$, $a_4 = 18$, $a_5 = 30$. Then we can color the whole original circle by picking one of the 30 possible colorings for each of the two sets of 5 alternating arcs, for $30^2 = 900$

Now, we must consider the rotational symmetry. If a configuration has no rotational symmetry, then we have counted it 10 times. If a configuration has 180° rotational symmetry, then we have counted it 5 times. This occurs exactly when we have picked the same coloring from our 30 for both choices, and in exactly one particular orientation, so there are 30 such cases. Having 72° or 36° rotational symmetry is impossible, as arcs with exactly one arc between them must be different colors. Then after we correct for overcounting our answer is

$$\frac{900 - 30}{10} + \frac{30}{5} = \boxed{93}$$

Solution to Problem 42

Answer is $\boxed{129}$

The main claim is that if the light pulse reflects vertically (on the left-/right edges) a times and horizontally b times, then $\gcd(a+1, b+1) = 1$, and the number of regions is $\frac{(a+2)(b+2)}{2}$. This claim can be conjectured by looking at small values of a and b; we give a full proof at the end.
Assuming the claim, we are trying to find the least possible value of $a + b$ when $(a + 2)(b + 2) = 2 \cdot 2021 = 2 \cdot 43 \cdot 47$. This happens when $(a + 2, b + 2) = (47, 86)$, which also satisfies $\gcd(a + 1, b + 1) = 1$, and gives $a + b = 47 + 86 - 4 = 129$.
We now prove the claim. Imagine that at each reflection, it is the square that gets reflected instead. Then the path p of the light pulse becomes a straight segment from $(0, 0)$ to $(a + 1, b + 1)$ of slope $+m = \frac{a+1}{b+1}$.

- The square starts as 1 region; the light pulse hitting a corner at the end creates 1 more region.
- Each reflection of the light pulse creates a region. These correspond to intersections of s with a line $x = n$ or $y = n$ for $x \in [a], y \in [b]$. There are $a + b$ such intersections.
- Each self-intersection of p creates a region. An intersection on p corresponds to two on s, and each intersection of s happens with a line of slope $-m$ passing through an even integral point, i.e. a line of the form $(b+1)x+(a+1)y = 2k$. The open segment s intersects these lines for $k \in [ab + a + b]$. However, the $a + b$ intersections that happens on a gridline $x \in \mathbb{Z}$ or $y \in \mathbb{Z}$ do not count, so here we have an additional $\frac{ab}{2}$ regions.

Therefore, the total number of regions is

$$2 + a + b + \frac{ab}{2} = \frac{(a + 2)(b + 2)}{2}.$$

Solution to Problem 43

Let's first solve the problem, ignoring the possibility that the a_i and b_i can be zero. Call a positive integer s an A-sum if $s = \sum_{i=1}^{m} a_i$ for some nonnegative integer m (in particular, 0 is always an A-sum). Define the term B-sum similarly. Let E be the expected value of the smallest positive integer that is both an A-sum and a B-sum.

The first key observation to make is that if s is both an A-sum and B-sum, then the distance to the next number that is both an A-sum and a B-sum is E. To see this, note that if

$$s = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_i,$$

the distance to the next number that is both an A-sum and a B-sum is the minimal positive integer t so that there exist m' and n' so that

$$t = \sum_{i=1}^{m'} a_{m+i} = \sum_{j=1}^{n'} b_{n+j}.$$

This is the same question of which we defined E to be the answer, but with renamed variables, so the expected value of t is E. As a result, we conclude that the expected density of numbers that are both A-sums and B-sums is $\frac{1}{E}$.

We now compute this density. Note that since the expected value of a_i is $\frac{101}{2}$, the density of A-sums is $\frac{2}{101}$. Also, the density of B-sums if $\frac{2}{101}$. Moreover, as n goes to infinity, the probability that n is an A-sum approaches $\frac{2}{101}$ and the probability that n is a B-sum approaches $\frac{2}{101}$. Thus, the density of numbers that are simultaneously A-sums and B-sums is $\frac{101}{2} \cdot \frac{2}{101} = \frac{4}{101^2}$, so $E = \frac{101^2}{4}$.

We now add the possibility that some of the a_i and b_i can be 0. The only way this changes our answer is that the s we seek can be 0, which happens if and only if $a_1 = b_1 = 0$. Thus our final answer is

$$\frac{1}{101^2} \cdot 0 + \frac{101^2 - 1}{101^2} \cdot \frac{101^2}{4} = \frac{101^2 - 1}{4} = 2550.$$

Solution to Problem 44

The Answer is $\boxed{163} = \sum_{i=0}^{4} \binom{8}{i}$
This is because we can obtain any integer less than 2^8
with less than or equal to 4 ones in its binary representation.
Note that $147 = 2^7 + 2^4 + 2^1 + 2^0$

We work in binary. Firstly, no operation can increase the number of ones in n's binary representation. The first two operations cycle the digits of n to the right, and the last operation can change a 11, 10, 01 at the front of n to 10, 01, 00, respectively. This provides an upper bound

To show we can obtain any of these integers, we'll show that given a number m_1 with base 2 sum of digits k, we can obtain every number with base 2 sum of digits k. Since we can, by cycling, change any 10 to an 01, we can move all of m_1's ones to the end, and then cycle so they're all at the front. From here, we can just perform a series of swaps to obtain any other integer with this same sum of digits. It's also easy to see that we can decrement the sum of digits of n, by cycling a 1 to the second digit of the number and then performing the third operation.
So this proves the claim

Solution to Problem 45

Let fg represent the composition of permutations f and g, where $(fg)(x) = f(g(x))$ for all $x \in \{1, 2, 3, 4, 5\}$.
Evaluating $fghfh$ in two ways, we get

$$f = gfh = (fgh)fh = fghfh = f(ghf)h = fhh,$$

so $hh = 1$. Similarly, we get f, g, and h are all involutions. Then

$$fgh = g \implies fg = gh,$$

so $fg = gh = hf$. Let $x := fg = gh = hf$. Then

$$x^3 = (fg)(gh)(hf) = 1.$$

We can also show that $fg = gh = hf$ along with f, g, h being involutions is enough to recover the initial conditions, so we focus on satisfying these new conditions.
If $x = 1$, then $f = g = h$ is an involution. There are $1+\binom{5}{2}+\frac{1}{2}\binom{5}{2,2,1} = 26$ involutions, so this case gives 26 solutions.
Suppose $x \neq 1$. Then since $x^3 = 1$, x is composed of a 3-cycle and two fixed points, of which there are 20 choices. WLOG $x = (123)$. It can be checked that $\{1, 2, 3\}$ must map to itself for all of f, g, h and also $\{4, 5\}$. We can either have all of f, g, h map 4 and 5 to themselves or each other. Restricted to $\{1, 2, 3\}$, they are some rotation of (123), (231), (312). Each of the 20 cases thus gives $2 \cdot 3 = 6$ triples, so overall we get $20 \cdot 6 = 120$.
The final answer is $26 + 120 = \boxed{146}$.

Solution to Day 10

Solution to Problem 46

Part (a) uses the idea that for $x \neq y$, the guess $(x, y, y, \ldots, y)$ returns 0 if the first number is y, 2 if the first number is x, and 1 if the number is something else. If he tests the pairs $(x, y) = (1, 2), \ldots, (2017, 2018)$, he will get a 0 or 2 (and therefore find out what the first number is) or get only 1 (so the first number is 2019). In general, with n numbers unknown, Alexander needs $\left\lfloor \frac{n}{2} \right\rfloor$ guesses to find out the next number in the permutation. Using this strategy, he can guess the permutation within at most

$$\sum_{n=1}^{2019} \left\lfloor \frac{n}{2} \right\rfloor + 1 = 1009 \cdot 1010 + 1 = 1019091$$

guesses.

Part (b) uses the idea that after the first number has been determined and n numbers remain undetermined, it takes $\lceil \log_2 n \rceil$ guesses to determine the index of another number, by doing a binary search to halve the candidate positions each time (using the first number as a filler). Using this Alexander can guess the permutation within

$$1009 + \sum_{n=1}^{2018} \lceil \log_2 n \rceil + 1 = 1009 + 1 \cdot 1 + 2 \cdot 2 + 4 \cdot 3 + \cdots + 512 \cdot 10 + 994 \cdot 11 + 1$$

$$= 21161$$

guesses.

Solution to Problem 47

Let Δ denote the triangle, $\circ$ denote the circle, and Δ' denote the reflection of the triangle in the center of the circle. Letting overline denote complement,

$$[\Delta \cap \circ] \leq \frac{1}{3}[\Delta] + \frac{1}{2}[\circ]$$

$$\iff [\Delta\cap\Delta'\cap\circ]+[\Delta\cap\overline{\Delta'}\cap\circ] \leq \frac{1}{3}[\Delta]+\frac{1}{2}[\circ\cap\Delta'\cap\Delta]+2[\circ\cap\Delta\cap\overline{\Delta'}]+[\circ\cap\overline{\Delta}\cap\Delta']$$

$$\iff \frac{1}{2}[\Delta \cap \Delta' \cap \circ] \leq \frac{1}{3}[\Delta] + \frac{1}{2}[\circ \cap \Delta \cap \Delta']$$

$$\iff [\Delta \cap \Delta' \cap \circ] \leq \frac{2}{3}[\Delta] + [\circ \cap \Delta \cap \Delta']$$

Since $\Delta\cap\Delta'\cap\circ$ is centrally symmetric, it's enough to show the following well-known lemma.

Lemma. *A centrally symmetric region R of a triangle can have area at most $\frac{2}{3}$ that of the triangle.*

Let the triangle be ABC with medial triangle DEF. We take two cases.

- **Case 1.** *The center of symmetry is in one of the outer triangles, say ΔAEF. Then the maximal possible R is a parallelogram with one vertex A and two other vertices on AB and AC. By enlarging the parallelogram, we may assume its fourth vertex lies on BC. If this vertex divides BC into an $a : b$ ratio with $a + b = 1$, then the fraction of the area taken up by the parallelogram is $1-a^2-b^2 \leq \frac{1}{2}$.*
- **Case 2.** *The center of symmetry is in ΔDEF. Then the maximal possible R is a centrally symmetric hexagon with two vertices on each side. Then there are three little triangles similar to ΔABC on the hexagon; by equal lengths, the similarity ratios a, b, c sum to 1. Then the fraction of the area taken up by the hexagon is $1 - a^2 - b^2 - c^2 \leq \frac{2}{3}$.* $\square$

Part 1: Solution to Problem 48

The maximum number is $\boxed{n-1.}$

We first prove the upper bound. For simplicity, color all negative numbers red, and all non-negative numbers blue. Let X be the number of color changes among adjacent elements (i.e. pairs of adjacent elements with different colors). It is clear that the following two statements are true:

1. When a_1 changes sign, X decreases by 1. If a_1 changes from negative (red) to non-negative (blue), a_2 must have been non-negative (blue), so the first two colors changed from RB to BB. The same applies when a_1 changes from non-negative to negative.

2. X cannot increase after a move. Suppose Alice picks a_i for her move where $1 \leq i \leq n$. If a_i does not change sign, then X clearly remains the same. Else, if a_i changes sign and $i = 1$ or n, then X decreases by 1 (from (1)). Finally, if a_i changes sign and $1 < i < n$, we have two cases:

Case 1: a_{i-1} and a_{i+1} are of the same color. If they are both negative (red), then if a_i changes color, it must be from non-negative to negative (blue to red). Thus, the colors change from RBR to RRR and X decreases by 2. The same holds if both a_{i-1} and a_{i+1} are non-negative.

Part 2: Solution to Problem 48

Case 2: a_{i-1} and a_{i+1} are of different colors. No matter what the color of a_i is, there is exactly one color change among the three numbers, so X will remain the same.

Now, since the initial value of X is at most $n-1$, it can decrease by 1 at most $n-1$ times. Hence, a_1 can change signs at most $n-1$ times.

Now we prove the lower bound by constructing such a sequence inductively. Specifically, we induct on the following statement:

For every $n \geq 2$, there exists a sequence $a_1, a_2, \ldots, a_n$ such that by picking

$$a_1, a_2, a_3, a_5, a_6, a_4, a_7, \ldots, a_{n-1}, a_{n-2}, a_n, \ldots, a_2, a_1$$

in that order, a_1 changes sign $n-1$ times.

When $n=2$, we can let a_1 change sign once by starting with the sequence $(1, -3)$, and picking a_1 to obtain $(1, -3)$, which satisfies the conditions in our statement.

Suppose we have proven the statement for $n-1$. For n, let $a_1, a_2, \ldots, a_{n-1}, a_n$ be as defined in our construction for $n-1$ (we shall fix the value of a_n later). After executing the steps $a_1, a_2, a_3, a_4, a_5, a_6, a_4, a_7, \ldots, a_2, a_1$, a_1 would have changed sign $n-2$ times.

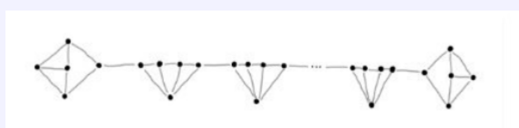
It now remains to pick $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1$ in order so that a_1 changes sign one more time. This always applies as long as a_n is sufficiently large in magnitude and of the opposite sign as a_1. Since the value of a_n has remained unchanged since the start (as we have not picked a_n at all), it suffices to let a_n be any number satisfying the above conditions at the start.

This completes our induction, and we conclude that the maximum number of times that a_1 can change sign is $n-1$.

Solution to Problem 49

Answer is $\boxed{1/4}$

Consider a graph G such that $\frac{a}{b} > \frac{1}{4}$. Note that the number of edges is $\frac{4a+3b}{2}$. Additionally, if any two vertices of degree 4 are adjacent, we can simply color that edge red and every other edge in G white to get a valid quasi-coloring. Thus, suppose no two vertices of degree 4 are adjacent.
Then, consider the subgraph G' resulting from removing all vertices of degree 4. Note that G' has $\frac{4a+3b-4a}{2} = \frac{3b}{2}$ edges. Thus, G' has fewer edges than vertices, which means that one of its connected components must be a tree T. In this tree T, we can just select some arbitrary vertex v, and perform a breadth-first search on T, greedily coloring the edges as we go along. What this means is that we partition the vertices of T into disjoint subsets $S_0 = \{v\}, S_1, S_2, \ldots$, where S_j contains all vertices of T whose distance to v is exactly j, and then we color edges between S_0 and S_1, and then greedily color the edges between S_1 and S_2, and so on. Once we color all the edges in T, we can color the edges between vertices in T and the vertices of degree 4 appropriately, and color the remaining edges white to get a valid quasi-coloring.
To show that $c = \frac{1}{4}$ is indeed the smallest possible solution, consider the graph shown below. This graph is not quasi-colorable (one can see that coloring the tails of this graph is impossible).



If there are N degree vertices of degree 4, there are $4N + 10$ vertices of degree 3. Since $\frac{N}{4N+10} = \frac{1}{4}$ as $N \to \infty$, we have that any valid value of $c < \frac{1}{4}$, can be found in this example. Hence, the smallest value for c is $\frac{1}{4}$.
In general, we can find N vertices of degree 4 and $b = 4N + 10$ vertices of degree 3 such that $\frac{a}{b} = \frac{N}{4N+10} = \frac{1}{4}$.

Solution to Problem 50

Answer: $\boxed{50}$

Let $3n = 2025$. We seek $S = \sum_{k=0}^{n} \binom{3n}{3k}(2k-n)^2$. Expanding, this equals

$$S = \sum_{k=0}^{n} \binom{3n}{3k} \left(4k^2 - 4nk + n^2\right)$$

Let $p(x) = (1+x)^{3n}$. Note that $p(x) = \sum_{\ell=0}^{3n} \binom{3n}{\ell} x^\ell$. Note that $xp'(x) = (3n)x(1+x)^{3n-1} = \sum_{\ell=0}^{3n} \binom{3n}{\ell} \ell x^\ell$.
Note that $x^2 p''(x) = (3n)(3n-1)x^2(1+x)^{3n-2} = \sum_{\ell=0}^{3n} \binom{3n}{\ell} \ell(\ell-1)x^\ell$. (Here, $q'(x)$ denotes the formal derivative of the polynomial q, defined by $q'(x) = ax^{a-1}$ for $q(x) = x^a$ and extended linearly.) Note that $4 \cdot \frac{1}{9}[\ell(\ell-1)+\ell] - 4n \cdot \frac{1}{3}\ell + n^2 = \frac{4}{9}\ell^2 - \frac{4n}{3}\ell + n^2$. For $\ell = 3k$, this equals $4k^2 - 4nk + n^2$. Let $\zeta = \exp(2\pi i/3)$. We apply the roots of unity filter to obtain

$$S = \frac{1}{3}\sum_{j=0}^{2} \left[\frac{4}{9}\left(x^2 p''(x) - xp'(x)\right) - \frac{4n}{3} xp'(x) + n^2 p(x)\right](\zeta^j)$$

where we evaluate the summand at $x = \zeta^j$. The summand simplifies to the expression

$$\frac{1}{3} n(1+x)^{3n-2}\left(3n(x-1)^2 + 4x\right)$$

For $x = 1$, the summand is clearly $\frac{1}{3}n2^{3n-2} \cdot 4 = \frac{1}{3}n8^n$. For $x = \zeta$, the summand is

$$\frac{1}{3} n(1+\zeta)^{3n-2}\left(3n(\zeta-1)^2 + 4\zeta\right) = \frac{1}{3}n(-1)^n(-9n+4)$$

using the fact that $1 + \zeta = -\zeta^2$ and $(\zeta - 1)^2 = -3\zeta$. By symmetry the same holds for $x = \zeta^2$. It follows that the sum is given by

$$S = \frac{1}{3}\left[\frac{1}{3}n8^n + \frac{2}{3}(-1)^n\left(-9n^2 + 4n\right)\right] = \frac{1}{9}n8^n + \frac{2}{9}(-1)^n n(4-9n)$$

We set $n = 2025/3 = 675$ and compute mod 1000 to obtain the desired answer.

Final Insights

Congratulations on completing this full book! Your dedication and hard work have brought you through 50 amazing combinatorics problems, each designed to challenge and enhance your mathematical prowess.

I hope this book has been a valuable tool in your mathematical journey, helping you to develop your problem-solving skills and prepare for various math competitions. Beyond mastering combinatorics, I hope you have also gained insights into logical thinking, creativity, and strategic problem-solving.

Your feedback is important to me, and I would love to hear your thoughts and suggestions. Please feel free to reach out to me on :-
Telegram: `@Nrip_Dave108`
or Discord: `nrip_dave`.

Best wishes for your mathematical endeavors. Remember, every problem you solve brings you one step closer to becoming a master of mathematics.

Thanks and regards,
Nrip Dave