BRIEFLY SOLUTION OF INTRODUCTION TO MATHEMATICAL ANALYSIS CHAPTER 3

TA: SINGYUAN YEH

1. **HW5 Problem 4**

Let sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$. Show that the following definition is equivalent.

(a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.

(b) This set E contains all subsequential limits. Define $\limsup_{n\to\infty} := \sup E$.

Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.

First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n. Since y_n is supreme of $\{x_k : k \geq n\}$ for all n, there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$\begin{aligned} y_1 - 1 &< x_{n_1} < y_1 \\ y_2 - \frac{1}{2} &< x_{n_2} < y_2 \\ &\vdots \end{aligned}$$

where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, x_{n_i} bounded above by y_i, so $\alpha \geq \beta$.

Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than r. So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\begin{aligned} \alpha - \epsilon < r &< x_{n_1} < y_1 \\ \alpha - \epsilon < r &< x_{n_2} < y_2 \\ &\vdots \end{aligned}$$

by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by r and y_1, exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, y_i decreasing to α, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself[1].

Remark: You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

[1] Please refer to G. Folland, *Advanced Calculus*.

2. **HW6 Problem 2** Determine whether each of the following conditions implies the convergence of the sequence $\{x_n\}$ in metric space X. Here a subsequence x_{n_j} of x_n is called proper if $|\mathbb{N} \setminus \{n_j, j = 1, 2, \cdots\}| = \infty$.

   (a) Every proper subsequence of $\{x_n\}$ converges.

   (b) Suppose $X \subset \mathbb{R}$ and $\{x_n\}$ is a monotonic Cauchy sequence.

   Remark: Note the definition of proper subsequence. The subsequence $\{x_{n_i} : n_i = 2, 3, \cdots\}$ is not proper subsequence, because $|\mathbb{N} \setminus \{2, 3, \cdots\}|$ is finite.

   Hint:

   (a) Construct two proper subsequences which union is equal to origin sequence. We may assume two subsequence

   $$\{x_{n_i} : n_i = 2i\} \quad \text{and} \quad \{x_{m_i} : m_i = 2i - 1\}$$

   Note that above two sequence are proper subsequences. Assume they converge to x and y respectively. Suppose that $x \neq y$. Let another proper subsequence $\{x_{k_i} : k_i = 3i\}$. Let $\epsilon = \frac{d(x,y)}{4}$. If i, j sufficient large, $d(x_{n_i}, x) < \epsilon$ and $d(x_{m_j}, y) < \epsilon$. However,

   $$d(x_{k_i}, x_{k_{i+1}}) \geq d(x, y) - d(x_{k_i}, x) - d(x_{k_{i+1}}, y) > d(x, y) - \frac{d(x, y)}{4} - \frac{d(x, y)}{4}$$

   where for every i, one of $\{k_i, k_{i+1}\}$ belongs to the set $\{n_j = 2j : j \in \mathbb{N}\}$ and the other belongs to the set $\{m_j = 2j - 1 : j \in \mathbb{N}\}$, *i.e.* one is odd and the other is even. Now, we have $\lim_{i\to\infty} d(x_{k_i}, x_{k_{i+1}}) > 0$, which leads a contradiction to the proper subsequence $\{x_{k_i} : k_i = 3i\}$ converge. Therefore, $\{x_{n_i} : n_i = 2i\}, \{x_{m_i} : m_i = 2i - 1\}$ converge to the same point so the origin sequence converge, which is because of $\{x_i\} = \{x_{n_i}\} \cup \{x_{m_i}\}$.

   (b) Let $X = (0, 1)$ and $\{x_n = \frac{1}{n}\}$. Verify the sequence $\{x_n\}$ satisfy Cauchy sequence by yourself but x_n doesn't converge in X.

   Remark:

   - This is because of completeness of the space. Thus, we also can construct a rational sequence converge to irrational number, *e.g.* $a_n = \left(1 + \frac{1}{n}\right)^n$ converge to e.
   - Besides rational number, every compact metric space is complete. One can use above theorem to construct incomplete space.

3. **HW6 Problem 3**

   Deduce that

   $$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

   Hint: Since s_n is monotonically increasing, we have

   $$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

4. **HW6 Problem 6**
We will find all the possible $p \in \mathbb{R}$ such that $\sum_n (\sqrt[n]{n} - 1)^p$ converges.
Hint: First, we can use Taylor expansion of $\sqrt[n]{n} - 1$, so we gauss that we compare with $\left(\frac{\log n}{n}\right)^p$. By limit comparison test,

$$\lim_{n\to\infty} \frac{\sqrt[n]{n} - 1}{\left(\frac{\log n}{n}\right)^p} = 1\,.$$

Find above limit by yourself. Hence, $\sum_n \sqrt[n]{n} - 1$ converge if and only if $\sum_n \left(\frac{\log n}{n}\right)^p$ converge. Second, by Cauchy condensation test, $\sum_n \left(\frac{\log n}{n}\right)^p$ converge if and only if

$$\sum_n 2^n \left(\frac{\log 2^n}{2^n}\right)^p = (\log n)^p \sum_n \frac{n^p}{2^{n(p-1)}}$$

converge. By root test, we can know convergence of series, $p \neq 1$. For $p = 1$ case, just back to the first equation, compare $\sum \frac{\log n}{n}$ with $\sum \frac{1}{n}$, so the series diverge.

5. **HW6 Problem 7**
Show that if $a_n > 0$ then $\lim_{n\to\infty}(na_n) = l$ with $l \neq 0$ then series $\sum a_n$ diverge.
Hint: Since $\lim_{n\to\infty}(na_n) = l$, exist $N > 0$ if $n > N$ then $|na_n - l| < \frac{l}{2}$. For such N, if $n > N$, $na_n > \frac{l}{2}$. Hence, we can rewrite

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n > \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} \frac{l}{2n}$$

where the first term is finite and the second term is diverge by comparing with $\sum \frac{1}{n}$. Therefore, the series is diverge.

6. **HW7 Problem 1**
Solve $\sum_n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{\sin nx}{n}$
Hint: Let $a_n = \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$. Claim that $\sum_{n=1}^{m} \sin nx$ has uniform bound for all m. Do it by yourself.

7. **HW7 Problem 2**
If $\sum a_n$ is converge if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converge.
Hint: Let

$$A_n = \sum_{k=1}^{n} a_k$$

where $n \geq 1$. WLOG, assume $\{b_n\}$ is increasing. Since $\sum a_n$ converges, we know that the sequence A_n also converges. Hence, the series is bounded and for some M_1, we have $|A_n| < M_1$ for all $n \in \mathbb{N}$. On the other hands, since $\{b_n\}$ is increasing and bounded, the sequence converges, and hence there exists M_2 such that $|b_n| < M_2$ for all $n \in \mathbb{N}$.

Since $\{b_n\}$ converges, it is also a Cauchy sequence. Thus, there exists $N_1 > 0$ such that whenever $m, n > N_1$, we have

$$|b_m - b_n| < \frac{\epsilon}{M}\,,$$

where $M = \max\{M_1, M_2\}$. By summation by part,

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)\,,$$

where note that $A_{-1} = 0$. First, consider the first term. Since $A_n b_{n+1}$ is the product of two converging sequences, the limit $\lim_{n\to\infty} A_n b_{n+1}$ exists. Second, consider the second term. We claim that $\sum_{k=1}^{n} A_k(b_{k+1} - b_k)$ also converges as $n \to \infty$. If $n, m > N_1$, we have

$$\begin{aligned}
\left|\sum_{k=m}^{n} A_k\left(b_{k+1} - b_k\right)\right| &< M \sum_{k=m}^{n} |b_{k+1} - b_k| \\
&= M \sum_{k=m}^{n} \left(b_{k+1} - b_k\right) \qquad \because \{b_n\} \text{ increasing} \\
&= M\left(b_n - b_m\right) \\
&< M \cdot \frac{\epsilon}{M} = \epsilon
\end{aligned}$$

Hence, as $n \to \infty$, the series exists. By the similar, we can prove b_n decreasing.