1. **Equivalent definition in Rudin**[1]

Let sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$. Show that the following definition is equivalent.

(a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.

(b) This set $E$ contains all subsequential limits. Define $\limsup_{n\to\infty} := \sup E$.

*Hint:* For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.

First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by $y_n$. Since $y_n$ is supreme of $\{x_k : k \geq n\}$ for all $n$, there exist $x_n$ such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$\begin{gathered} y_1 - 1 < x_{n_1} < y_1 \\ y_2 - \frac{1}{2} < x_{n_2} < y_2 \\ \vdots \end{gathered}$$

where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, $x_{n_i}$ bounded above by $y_i$, so $\alpha \geq \beta$.

Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all $\epsilon$. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many $x_i$ greater than $r$. So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\begin{gathered} \alpha - \epsilon < r < x_{n_1} < y_1 \\ \alpha - \epsilon < r < x_{n_2} < y_2 \\ \vdots \end{gathered}$$

by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by $r$ and $y_1$, exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, $y_i$ decreasing to $\alpha$, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since $\epsilon$ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself. Please refer to G. Folland, *Advanced Calculus*.

**Remark:** You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

[1] W. Rudin, *Principles of Mathematical Analysis*.