BRIEFLY SOLUTION OF INTRODUCTION TO MATHEMATICAL ANALYSIS CHAPTER 2

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1. **HW2 Problem 4**
Let f be a real-valued function defined on $[0, 1]$. Suppose there exists $M > 0$ such that for every choice of a finite number of points, $x_j \in [0, 1]$ for all $1 \leq j \leq n$, the sum
$$|f(x_1)| + \cdots + |f(x_n)| \leq M$$
Show that the set $U = \{x \in [0, 1] : f(x) \neq 0\}$ is at most countable. Note that M depends on n.
Hint: Let $U_n = \{x_n \in [0, 1] : |f(x_n)| > \frac{1}{n}\}$ Claim that $U \subseteq \bigcup_n U_n$. In fact, the equality holds, but we only need one direction. Let $x \in U$ then exist some n such that $\frac{1}{n} < |f(x)|$. Hence, we proved the claim. Now, it is sufficient to show that $|U_n|$ is finite.
$$M \geq |f(x_1)| + \cdots + |f(x_{K_n})| > \frac{1}{n} \times K_n$$
where $|U_n| = K_n$ which depends on n. Hence, $K_n < nM$ should be finite. Now, U is subset of countable union of finite set, so U is countable infinite.

2. **HW2 Problem 6**
Let S be any non-empty set and $\mathcal{P}(S)$ the set of all subsets of S which is called the power set of S.
   (a) Show that the cardinality of $S <$ the cardinality of $\mathcal{P}(S)$.
   (b) Argue that "the set of all set" makes no sense.
   Hint:
   (a) First , we take $f(x) = x$, then this is a one-to-one mapping from S onto $P(S)$. Then, suppose cardinality of $S =$ the cardinality of $P(S)$, then there exists a 1-1 function f maps S onto $P(S)$. Next, consider the set
   $$A = \{x \in S : x \notin f(x)\}.$$
   Clear $A \subset S$ and $A \in \mathcal{P}(S)$. Claim that there doesn't exist $c \in S$ such that $f(c) = A$. Hence, f is not onto mapping. Now, we prove the claim. Suppose f is one-to-one map S onto $\mathcal{P}(A)$. If $c \in A$, then $c \notin A$, this is a contradiction . If $c \notin A$, then $A \neq f(c)$, this is a contradiction. By above two contradictions , we know that the set A does not have a map onto $P(S)$. Hence the cardinality of $S <$ the cardinality of $P(S)$.
   (b) Let U is the set contain all sets. Then, $\mathcal{P}(U) \subset U$. However, $|U| \leq |\mathcal{P}(U)|$.

3. **HW2 Problem 8**
Let nonempty sets S and T be given.
   (a) Show that if there exists a function $f : S \to T$ which is onto, then there exists a function $U : T \to S$ which is one-to-one.

(b) Suppose that there is a function f which maps S onto T, and there is also a function g which maps T onto S. Prove that S and T have the same cardinality.

4. **HW3 Problem 4 [R] Ex. 2.22 & Ex. 2.29**

(a) A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

(b) Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

Hint:

(a) Let $D = \mathbb{Q}^k$. Claim that D is dense in $\mathbb{R}^k$. Given $x \in \mathbb{R}^k$, for $r > 0$ arbitrary chosen, take $l = \sqrt{\frac{r^2}{k}}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, exist $q_i \in (x_i - l, x_i + l)$. Let $q = (q_1, \cdots, q_k) \in D$. We have

$$d(x, q) = \sqrt{\sum_i |q_i - x_i|} < \sqrt{\sum_i l^2} < r$$

Hence, $q \in B_r(x)$, for every r. Therefore, D is countable and dense in $\mathbb{R}^k$.

(b) Let E be an open set of $\mathbb{R}$. For each $x \in E$. let I_x be the largest interval containing x and $I_x \subset E$. Let

$$a_x = \inf\{a \mid (a, x) \subseteq E\}$$

and

$$b_x = \sup\{b \mid (x, b) \subseteq E\}$$

Then, $I_x = (a_x, b_x)$. Now, we have to claim that $I_x \cap I_y = \varnothing$ for $x \neq y$ and $I_x \neq I_y$. Suppose not, $I_x \cap I_y \neq \varnothing$. Since I_x is the largest interval containing x and contained in E, if $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$, then $I_x = I_y$. Finally, we have to claim they are at most countable. Since each open interval contains at least one rational number, by axiom of choice, each interval could label by a rational number. Thus, this open interval at most countable.

5. **HW3 Problem 5 [R] Ex. 2.23**

Prove that every separable metric space has a countable base.

Hint: Since X is separable, exist countable contains a countable dense subset, called $D = \{z_1, z_2, \cdots\}$. Claim

$$\mathcal{B} = \{B_q(z) : z \in D, q \in \mathbb{Q}\}$$

is countable base for X. Clearly, since z and r is belong to countable set, $\mathcal{B}$ is countable collection.

Now, we show that $\mathcal{B}$ is basis. For any G is open in X, there exist $B_r(x) \subset G$. If $x \in D$. Choose rational number q such that $0 < q < r$. If $x \neq D$. Since D is dense, exist $z \in D$ such that $z \in B_{\frac{r}{4}}(x)$. Choose rational number q such that $0 < q < \frac{r}{4}$. Now, we claim that $B_q(z) \subset B_r(x) \subset G$. For $y \in B_q(z)$,

$$d(y, x) \leq d(y, z) + d(z, x) < q + \frac{r}{4} < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$$

Hence, $B_q(z) \subset B_r(x) \subset G$. Therefore, $\mathcal{B}$ is countable base.

6. **HW4 Problem 1 [R] Ex. 2.24**

   Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

   Hint: Construct countable and dense subset $D \subset X$. Fix $\delta > 0$. Choose $x_1 \in X$ and choose $x_2 \in X$ satisfied $d(x_1, x_2) \geq \delta$. Continue this process, we have $\{x_1, x_2, \cdots\}$ with $d(x_{j+1}, d_i) \geq \delta$ for $j = 1, \cdots, j$. Claim that $\{x_1, x_2, \cdots\}$ must be finite. Suppose not, $\{x_i\}$ has limit point, called p. Hence, exist $N > 0$ such that if $m, n > N$

   $$d(x_n, x_m) < d(x_n, p) + d(x_m, p) < \delta$$

   which leads a contradiction to $d(x_n, x_m) \geq \delta$. Let $E^{(n)} = \{x_1^{(n)}, \cdots, x_{K_n}^{(n)}\}$ represent that let $\delta = \frac{1}{n}$ and do the above method to collect the point $x_i^{(n)} \in X$. Note that K_n is finite number which depends on n.

   It's sufficient to show $D = \cup_{n=1}^{\infty} E^{(n)}$ is countable dense subset of X. Since $E^{(n)}$ is finite for all n and union of countable $E^{(n)}$, D is countable. Claim that D is dense. Let $z \in X$ arbitrary chosen. If $z \in D$, done. If $z \neq D$, For every r, exist r such that $\frac{1}{n} < r$. For such n, exist $x_i^{(n)} \in E^{(n)}$ such that $d(x_i^{(n)}, z) < \frac{1}{n} < r$.

7. **HW4 Problem 2 [R] Ex. 2.25**

   Prove that every compact metric space K has a countable base, and that K is therefore separable.

8. **HW4 Problem 3 [R] Ex. 2.26**

   A metric space is compact if and only if every infinite subset has a limit point in it.

   Hint:

   $\Rightarrow$) Suppose not, exist infinite subset has no limit points. Do it by yourself.

   $\Leftarrow$) Since X is metric space, X has countable base, called $\{V_k\}_{k=1}^{\infty}$. Given any open cover $\{G_\alpha\}$ of X, there is some countable subcover $\{G_n\} \subset \{G_\alpha\}$ such that $V_n \subseteq G_n$ for all $n \in \mathbb{N}$. Now, $\{G_n\}$ is countable subcover of X. Suppose there's no finite subcollection of $\{G_n\}$ covers X, then $(\bigcup_{i=1}^{n} G_i)^c \neq \varnothing$.

   Let $F_n = (\bigcup_{i=1}^{n} G_i)^c$, F^n is closed, and note that $F_n \neq \varnothing$ for all n, and notice that $F_{n+1} \subset F_n$ for all n. We also have

   $$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (\bigcup_{i=1}^{n} G_i)^c = (\bigcup_{n=1}^{\infty} (\bigcup_{i=1}^{n} G_i))^c = (\bigcup_{i=1}^{\infty} G_i)^c = X^c = \varnothing$$

   Now create an infinite set $E = \{x_1, x_2, \cdots\}$, $x_i \in F_i$ distinct. E is a infinite set, thus E has a limit point p. Since $F_{n+1} \subset F_n$, for each n, claim that

   $$p \in \bigcap_{n=1}^{\infty} F_n$$

   We prove the claim. If not, $p \in F_M^c$ for some M, then x_i cannot approach to p. However, we have proved

   $$\bigcap_{n=1}^{\infty} F_n = \varnothing$$

which is contradicts to $p \in \bigcap_{n=1}^{\infty} F_n$. Therefore, there must exist finite subcover of $\{G_n\}$ covering X.

9. **HW5 Problem 1 [A] Ex. 4.37**

   A topological space S is connected if, and only if, the only subsets of S which are both open and closed in S are the empty set and S itself.

   Hint:

   $\Rightarrow$) Suppose not, exist $A \subset X$ with $A \neq X$ and $A \neq \varnothing$ such that A is open and closed. Let $B = S \setminus A$. Since A is closed,

   $$\bar{A} \cap B = A \cap B = \varnothing$$

   Since A is open, B is closed. Hence,

   $$\bar{B} \cap A = B \cap A = \varnothing\,.$$

   Therefore, S is not connected.

   $\Leftarrow$) Suppose not, S is not connected. exist $A, B \subset S$ with $S = A \cup B$ and $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. Claim A is open and closed. Suppose A is not open. exist $x \in A$ such that $B_r(x) \cap A^c \neq \varnothing$, for all r. Since r is arbitrary chosen, x is limit point of $A^c = B$. Then, $A \cap \bar{B} \neq \varnothing$, which contradicts to S is dis-connected. Thus, A should not be open. By the similar work, we could show that A should not be closed. Therefore, we find that A is closed and open but $A \neq X$ and $A \neq \varnothing$.

10. **HW5 Problem 2 [A] Ex. 4.39**

    Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subset Y \subset \bar{X}$, where $\bar{X}$ is closure of X. Prove that Y is also connected.