INTRODUCTION TO MATHEMATICAL ANALYSIS MIDTERM

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1. $\boxed{\text{[Courant \& John] Chapter 7.2}}$
Let $a_n \geq 0$ for all n and fix $\epsilon > 0$. If
$$\frac{\log \frac{1}{a_n}}{\log n} > 1 + \epsilon \,,$$
show that $\sum a_n$ converge.
Hint: Compute directly,
$$\begin{aligned} \frac{\log \frac{1}{a_n}}{\log n} &> 1 + \epsilon \\ \log \frac{1}{a_n} &> \log n^{1+\epsilon} \\ a_n &< \frac{1}{n^{1+\epsilon}} \end{aligned}$$
By comparison test, you can prove it.

2. $\boxed{\text{[Courant \& John] Chapter 7.5}}$
Let $a_k \in \mathbb{R}$ be sequence satisfy $\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} < 1$. Show that $\sum a_k$ converge absolutely.
Hint: Let $\limsup |a_k|^{\frac{1}{k}} = r < 1$, i.e.
$$\lim_{m\to\infty} \sup\{|a_k|^{\frac{1}{k}} : k \geq m\} = r \,.$$
Take $\epsilon = \frac{1-r}{2}$, i.e. $r + \epsilon < 1$. Exist M such that if $k > M$ then $|a_k|^{\frac{1}{k}} < r + \epsilon$. That is,
$$|a_k| < (r+\epsilon)^k \,.$$
Since $r+\epsilon < 1$, $\sum_{k=M}^{\infty}(r+\epsilon)^k$ converge, which implies $\sum_{k=1}^{\infty}(r+\epsilon)^k$ converge. By comparison test,
$$\sum_{k=1}^{\infty} a_k$$
converge.

3. $\boxed{\text{[Courant \& John] Chapter 3.15}}$
For what values of s is the following integral convergent?
$$\int_0^{\infty} \frac{\sin x}{x^s} dx$$

Hint: Write down integral as

$$\int_0^\infty \frac{\sin x}{x^s} dx = \int_0^1 \frac{\sin x}{x^s} dx + \int_1^\infty \frac{\sin x}{x^s} dx\,.$$

Since $\frac{\sin x}{x} \geq 0$ for $x \in [0,1]$, by ratio test,

$$\lim_{x\to 0} \frac{\sin x/x}{1/x^{s-1}} = 1 > 0$$

Hence, both $\int_0^1 \frac{\sin x}{x^s} dx$ and $\int_0^1 \frac{1}{x^{s-1}} dx$ have same convergent behavior. Thus, they converge when $s < 2$ and divergent when $s \geq 2$.
On the other hands, Since $\lim_{x\to\infty} \frac{\sin x}{x^s}$ doesn't exist when $s < 0$, $\int_1^\infty \frac{\sin x}{x^s} dx$ diverge if $s < 0$. Moreover,

$$\int_1^\infty \frac{\sin x}{x^s} dx = \frac{-\cos x}{x^s}\bigg|_1^\infty - \int_1^\infty \frac{s\cos x}{x^{s+1}} dx$$

Focus on

$$\left| \int_1^\infty \frac{s\cos x}{x^{s+1}} dx \right| \leq s \int_1^\infty \frac{|\cos x|}{x^{s+1}} dx \leq \int_1^\infty \frac{1}{x^{s+1}} dx$$

which converge when $s + 1 > 1$, $s > 0$. Therefore,

$$\int_0^\infty \frac{\sin x}{x^s} dx$$

converge if $0 < s < 2$.

4. Marsden & Hoffman

Show the following series converge by integral test

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

5. [Folland] Chapter 2

Let sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$. Show that the following definition is equivalent.

(a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.

(b) This set E contains all subsequential limits. Define $\limsup_{n\to\infty} x_n := \sup E$.

Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.
First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n. Since y_n is supreme of $\{x_k : k \geq n\}$ for all n, there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$y_1 - 1 < x_{n_1} < y_1$$

$$y_2 - \frac{1}{2} < x_{n_2} < y_2$$

$$\vdots$$

where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, x_{n_i} bounded above by y_i, so $\alpha \geq \beta$.
Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than r. So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\begin{gathered} \alpha - \epsilon < r < x_{n_1} < y_1 \\ \alpha - \epsilon < r < x_{n_2} < y_2 \\ \vdots \end{gathered}$$

by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by r and y_1, exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, y_i decreasing to α, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself[1].
Remark: You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

6. [Courant & John] Chapter 1

Prove that the following principles are equivalent in the sense that any one can be derived as a consequence of any other.
(a) Every nested sequence of intervals with real end points contains a real number.
(b) Every bounded monotone sequence converges.
(c) Every bounded infinite sequence has at least one accumulation or limit point.
(d) Every Cauchy sequence converges.
(e) Every bounded set of real numbers has an infimum and a supremum.

7. [Courant & John] Chapter 1

Determine the set the following function continuous and discontinuous

$$g(x) = \begin{cases} 0, & x \quad \text{irrational} \\ \frac{1}{q}, & x = \frac{p}{q} \quad \text{rational in lowest terms} \end{cases}$$

8. [Lee] Chapter 2

Show the following space X is topological space.
(a) Let $d(\cdot,\cdot)$ is discrete distance and $\mathcal{T}$ is collection of all open set. Then, $X = (\mathbb{R}, \mathcal{T})$.
(b) $X = (\mathbb{R}, \{\mathbb{R}, \varnothing\})$.

[1]Please refer to G. Folland, *Advanced Calculus*.

9. $\boxed{\text{Stewart}}$

Determine the convergence (absolute convergent/conditional convergent/divergent) of following series.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

(b) $\sum_{n=1}^{\infty}(n^{\frac{1}{n}} - 1)$

(c) $\sum_{n=1}^{\infty} ne^{-n}$

(d) $\sum_{n=1}^{\infty} \sinh(\frac{1}{n^2})$

(e) $\sum_{n=9}^{\infty} \frac{1}{n\ln(n)\cdot(\ln(\ln(n)))^2}$