BRIEFLY SOLUTION OF INTRODUCTION TO MATHEMATICAL ANALYSIS CHAPTER 3

TA: SINGYUAN YEH

1. **HW5 Problem 4**
Let sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$. Show that the following definition is equivalent.
(a) Define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\}$.
(b) This set E contains all subsequential limits. Define $\limsup_{n\to\infty} := \sup E$.
Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.
First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n. Since y_n is supreme of $\{x_k : k \geq n\}$ for all n, there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by
$$y_1 - 1 < x_{n_1} < y_1$$
$$y_2 - \frac{1}{2} < x_{n_2} < y_2$$
$$\vdots$$
where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i\to\infty} y_i$. However, x_{n_i} bounded above by y_i, so $\alpha \geq \beta$.
Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ. Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than $\alpha - \epsilon$. So, we can construct the subsequence $\{x_{n_i}\}$ by
$$\alpha - \epsilon < r < x_{n_1} < y_1$$
$$\alpha - \epsilon < r < x_{n_2} < y_2$$
$$\vdots$$
by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the sequence $\{x_{n_i}\}$ bounded by r and y_1, exist subsequence of $\{x_{n_i}\}$ such that the subsequence converges in $[r, y_1]$. However, y_i decreasing to α, so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself.

2. **HW6 Problem 2** Determine whether each of the following conditions implies the convergence of the sequence $\{x_n\}$ in metric space X.
(a) Every proper subsequence of $\{x_n\}$ converges.
(b) Suppose $X \subset \mathbb{R}$ and $\{x_n\}$ is a monotonic Cauchy sequence.
(c) $\{x_n\}$ is a Cauchy sequence and some subsequence of $\{x_n\}$ converges.
Hint:

(a) Construct two proper subsequences which union is equal to origin sequence. We may assume two subsequence

$$\{x_{n_i} : 2i\} \quad \text{and} \quad \{x_{m_i} : 2i-1\}$$

Note that above two sequence are proper subsequences. Assume they converge to x and y respectively. Suppose that $x \neq y$. Let another proper subsequence $\{x_{k_i} : 3i\}$. Let $\epsilon = \frac{d(x,y)}{4}$. If i, j sufficient large, $d(x_{n_i}, x) < \epsilon$ and $d(x_{m_j}, y) < \epsilon$. However,

$$d(x_{k_i}, x_{k_{i+1}}) \geq d(x,y) - d(x_{k_i}, x) - d(x_{k_{i+1}}, y) > d(x,y) - \frac{d(x,y)}{4} - \frac{d(x,y)}{4}$$

where for every i, one of $\{k_i, k_{i+1}\}$ belongs to $\{n_j\}$ and the other belongs to $\{m_j\}$. Now, we have $\lim_{i\to\infty} d(x_{k_i}, x_{k_{i+1}}) > 0$, which leads a contradiction to the proper subsequence $\{x_{k_i} : 3i\}$ converge. Therefore, $\{x_{n_i} : 2i\}, \{x_{m_i} : 2i-1\}$ converge to the same point so the origin sequence converge.

(b) Let $X = (0,1)$ and $\{x_n = \frac{1}{n}\}$. Verify the sequence $\{x_n\}$ satisfy Cauchy sequence by yourself but x_n doesn't converge in X.

(c)

3. **HW6 Problem 6**
We will find all the possible $p \in \mathbb{R}$ such that $\sum_n(\sqrt[n]{n}-1)^p$ converges.
Hint: First, we can use Taylor expansion of $\sqrt[n]{n}-1$, so we gauss that we compare with $\left(\frac{\log n}{n}\right)^p$. By limit comparison test,

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\left(\frac{\log n}{n}\right)^p} = 1\,.$$

Find above limit by yourself. Hence, $\sum_n \sqrt[n]{n}-1$ converge if and only if $\sum_n \left(\frac{\log n}{n}\right)^p$ converge. Second, by Cauchy condensation test, $\sum_n \left(\frac{\log n}{n}\right)^p$ converge if and only if

$$\sum_n 2^n \left(\frac{\log 2^n}{2^n}\right)^p = (\log n)^p \sum_n \frac{n^p}{2^{n(p-1)}}$$

converge. By root test, we can know convergence of series, $p \neq 1$. For $p = 1$ case, just back to the first equation, compare $\sum \frac{\log n}{n}$ with $\sum \frac{1}{n}$, so the series diverge.

4. **HW6 Problem 7**
Show that if $a_n > 0$ then $lim_{n\to\infty}(na_n) = l$ with $l \neq 0$ then series $\sum a_n$ diverge.

5. **HW7 Problem 1**
Solve $\left(\sum \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{\sin nx}{n}\right)$

6. **HW7 Problem 2**
If $\sum a_n$ is converge if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converge.