

INTRODUCTION TO MATHEMATICAL ANALYSIS MIDTERM

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1. [Courant & John] Chapter 7.2

Let $a_n \geq 0$ for all n and fix $\epsilon > 0$. If

$$\frac{\log \frac{1}{a_n}}{\log n} > 1 + \epsilon,$$

show that $\sum a_n$ converge.

Hint: Compute directly,

$$\begin{aligned}\frac{\log \frac{1}{a_n}}{\log n} &> 1 + \epsilon \\ \log \frac{1}{a_n} &> \log n^{1+\epsilon} \\ a_n &< \frac{1}{n^{1+\epsilon}}\end{aligned}$$

By comparison test, you can prove it.

2. [Courant & John] Chapter 7.5

Let $a_k \in \mathbb{R}$ be sequence satisfy $\limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}} < 1$. Show that $\sum a_k$ converge absolutely.

Hint: Let $\limsup |a_k|^{\frac{1}{k}} = r < 1$, i.e.

$$\lim_{m \rightarrow \infty} \sup\{|a_k|^{\frac{1}{k}} : k \geq m\} = r.$$

Take $\epsilon = \frac{1-r}{2}$, i.e. $r + \epsilon < 1$. Exist M such that if $k > M$ then $|a_k|^{\frac{1}{k}} < r + \epsilon$. That is,

$$|a_k| < (r + \epsilon)^k.$$

Since $r + \epsilon < 1$, $\sum_{k=M}^{\infty} (r + \epsilon)^k$ converge, which implies $\sum_{k=1}^{\infty} (r + \epsilon)^k$ converge. By comparison test,

$$\sum_{k=1}^{\infty} a_k$$

converge.

3. [Courant & John] Chapter 3.15

For what values of s is the following integral convergent?

$$\int_0^{\infty} \frac{\sin x}{x^s} dx$$

Hint: Write down integral as

$$\int_0^\infty \frac{\sin x}{x^s} dx = \int_0^1 \frac{\sin x}{x^s} dx + \int_1^\infty \frac{\sin x}{x^s} dx.$$

Since $\frac{\sin x}{x} \geq 0$ for $x \in [0, 1]$, by ratio test,

$$\lim_{x \rightarrow 0} \frac{\sin x/x}{1/x^{s-1}} = 1 > 0$$

Hence, both $\int_0^1 \frac{\sin x}{x^s} dx$ and $\int_0^1 \frac{1}{x^{s-1}} dx$ have same convergent behavior. Thus, they converge when $s < 2$ and divergent when $s \geq 2$.

On the other hands, Since $\lim_{x \rightarrow \infty} \frac{\sin x}{x^s}$ doesn't exist when $s < 0$, $\int_1^\infty \frac{\sin x}{x^s} dx$ diverge if $s < 0$. Moreover,

$$\int_1^\infty \frac{\sin x}{x^s} dx = \left. \frac{-\cos x}{x^s} \right|_1^\infty - \int_1^\infty \frac{s \cos x}{x^{s+1}} dx$$

Focus on

$$\left| \int_1^\infty \frac{s \cos x}{x^{s+1}} dx \right| \leq s \int_1^\infty \frac{|\cos x|}{x^{s+1}} dx \leq \int_1^\infty \frac{1}{x^{s+1}} dx$$

which converge when $s + 1 > 1$, $s > 0$. Therefore,

$$\int_0^\infty \frac{\sin x}{x^s} dx$$

converge if $0 < s < 2$.

4. Marsden & Hoffman

Show the following series converge by integral test

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

5. [Folland] Chapter 2

Let sequence $\{x_n\}_{n=1}^\infty$ in \mathbb{R} . Show that the following definition is equivalent.

(a) Define $\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\}$.

(b) This set E contains all subsequential limits. Define $\limsup_{n \rightarrow \infty} x_n := \sup E$.

Hint: For convenience, let $y_n = \sup\{x_k : k \geq n\}$ and $\alpha = \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\}$, $\beta = \sup E$. WLOG, we only consider $\alpha, \beta < \infty$ here.

First, claim $\alpha \geq \beta$. We have to construct a subsequence bounded below by y_n . Since y_n is supreme of $\{x_k : k \geq n\}$ for all n , there exist x_n such that $y_n - \epsilon < x_n < y_n$. Choose $\epsilon = \frac{1}{i}$ for all $i \in \mathbb{N}$. We can construct subsequence $\{x_{n_i}\}$ by

$$y_1 - 1 < x_{n_1} < y_1$$

$$y_2 - \frac{1}{2} < x_{n_2} < y_2$$

⋮

where the index $n_i \neq n_j$ if $i \neq j$. By Sandwich theorem, $\{x_{n_i}\}$ converges to $\alpha = \lim_{i \rightarrow \infty} y_i$. However, x_{n_i} bounded above by y_i , so $\alpha \geq \beta$.

Second, claim $\alpha - \epsilon < \beta \leq \alpha$, for all ϵ . Take $r \in (\alpha - \epsilon, \alpha)$. Now, we hope to construct a subsequence converge to $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Now, claim that exist infinitely many x_i greater than r . So, we can construct the subsequence $\{x_{n_i}\}$ by

$$\alpha - \epsilon < r < x_{n_1} < y_1$$

$$\alpha - \epsilon < r < x_{n_2} < y_2$$

$$\vdots$$

by the claim, where the index $n_i \neq n_j$ if $i \neq j$. Since the subsequence $\{x_{n_i}\}$ bounded by r and y_1 , exist sub-subsequence of $\{x_{n_i}\}$ such that the sub-subsequence converges in $[r, y_1]$. However, y_i decreasing to α , so exist a subsequence converge in $[r, \alpha] \subset (\alpha - \epsilon, \alpha]$. Since ϵ is arbitrary chosen, we have $\alpha = \beta$, which the desired results follows. Finally, we have to prove the claim, do it by yourself¹.

Remark: You have to claim that there are infinitely many points to choose as subsequence, otherwise we cannot find $n_i \neq n_j$ for $i \neq j$.

6. [Courant & John] Chapter 1

Prove that the following principles are equivalent in the sense that any one can be derived as a consequence of any other.

- (a) Every nested sequence of intervals with real end points contains a real number.
- (b) Every bounded monotone sequence converges.
- (c) Every bounded infinite sequence has at least one accumulation or limit point.
- (d) Every Cauchy sequence converges.
- (e) Every bounded set of real numbers has an infimum and a supremum.

7. [Courant & John] Chapter 1

Determine the set the following function continuous and discontinuous

$$g(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{q}, & x = \frac{p}{q} \text{ rational in lowest terms} \end{cases}$$

8. [Lee] Chapter 2

Show the following space X is topological space.

- (a) Let $d(\cdot, \cdot)$ is discrete distance and \mathcal{T} is collection of all open set. Then, $X = (\mathbb{R}, \mathcal{T})$.
- (b) $X = (\mathbb{R}, \{\mathbb{R}, \emptyset\})$.

¹Please refer to G. FOLLAND, *Advanced Calculus*.

9. [Stewart]

Determine the convergence (absolute convergent/conditional convergent/divergent) of following series.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$

(b) $\sum_{n=1}^{\infty} (n^{\frac{1}{n}} - 1)$

(c) $\sum_{n=1}^{\infty} n e^{-n}$

(d) $\sum_{n=1}^{\infty} \sinh\left(\frac{1}{n^2}\right)$

(e) $\sum_{n=9}^{\infty} \frac{1}{n \ln(n) \cdot (\ln(\ln(n)))^2}$

Hint:

(a) **Absolute Convergence.**

By alternative series test and ratio test.

(b) **Divergence.**

Since $n^{\frac{1}{n}} = e^{\frac{1}{n} \log(n)} \approx 1 + \frac{1}{n} \log(n)$, try to compare with $\frac{1}{n}$.

(c) **Absolute Convergence.**

By ratio test or root test.

(d) **Absolute Convergence.**

Since $\sinh\left(\frac{1}{n^2}\right) = (e^{\frac{1}{n^2}} - e^{-\frac{1}{n^2}})/2 \approx [(1 + \frac{1}{n^2}) - (1 - \frac{1}{n^2})]/2 = \frac{1}{n^2}$, try to compare with $\frac{1}{n^2}$.

(e) **Absolute Convergence.**

By integral test.

10. [Lee] Chapter 2

Consider a metric space. Show that A is open if and only if it is union of open balls.

11. [Rudin]

Suppose $f \geq 0$, f is continuous on $[a, b]$. Exist $c \in [a, b]$ such that $f(c) \neq 0$. Prove that exist $d \neq c$ with $d \in [a, b]$ such that $f(d) \neq 0$.