REMARK OF TAYLOR THEOREM ON DECEMBER 21

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(i) If $f \in C^m$ near $x = a$, then $R_m(h) = o(h^m)$. In fact, remainder term can be written as

$$R_m(h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1}[f^{(m)}(a+th) - f^{(m)}(a)]dt$$

(ii) If $f \in C^m$ near $x = a$ and f is $(m+1)$-times differentiable near $x = a$, then the remainder term of order m is OBVIOUSLY little o with order m and the remainder term is

$$R_m(h) = \frac{f^{(m+1)}(\xi)}{(m+1)!}h^{m+1}$$

(iii) Continuing the statement above, $f \in C^m$ near $x = a$ and f is $(m+1)$-times differentiable at $x = a$, the remainder term of order $m+1$ is actually little o with order $m+1$, *i.e.* $R_{m+1}(h) = o(h^{m+1})$, but we could not explicitly write down $R_{m+1}(x-a)$.

(iv) If $f \in C^m$ near $x = a$ and moreover $f \in C^{m+1}$, then we could write down remainder term $R_m(x-a)$ more precisely, not just exist ξ. That is,

$$f(x) - f(a) = f'(\xi)(x-a)$$

$$f(x) - f(a) = \int_a^x f'(t)dt$$

(v) Let $f \in C^m$ near $x = a$ and f can be written as

$$f(x) = a_0 + a_1(x-a) + \cdots + a_m(x-a)^m + R_m(x-a).$$

Make sure you could prove if $R_m(x-a) = o((x-a)^m)$ then $a_i = \frac{f^{(i)}(a)}{i!}$ uniquely.

(vi) Make sure you could tell the difference between T_n converge on which interval I as $n \to \infty$ and T_n converge to original f as $n \to \infty$. *Hint:* Consider the following function

$$f(x) = e^{-\frac{1}{x^2}}, \quad x = 0.$$

(vii) By above example, the function and Taylor polynomial is not bijection. For instance, both $f(x)$ and $f(x) + e^{-\frac{1}{x^2}}$ have same Taylor polynomial.

(viii) Inverse the Taylor theorem is not true. That is, $f(x) = T_m(x-a) + R_m(x-a)$ does not imply $f \in C^m$ near $x = a$. *Hint:* Consider the following function

$$f(x) = \sin\left(\frac{1}{x^4}\right)e^{-\frac{1}{x^2}}, \quad x = 0.$$

For any m, $T_m(x) = 0$ and $R_m(x) = f(x) \in o(x^k)$, for all k. However, $f(x) \in C^1$ is just differentiable at $x = 0$