

# Calculus II TA Session

December 14, 2023

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1. **(Separable Equations)** 1111 (01-05) Final Problem 5

In country A, a booster vaccine has been invented to conquer a mutated virus X. Let  $x$  (in thousands) be the total number of people in country A at time  $t$  (in months) and  $y$  be the those who has received the booster shot at time  $t$ . It is known that  $0 < x < 100$  and  $x, y$  satisfies the following equations.

$$(1) \quad \frac{dx}{dt} = 0.2x \left(1 - \frac{x}{100}\right), \quad (2) \quad \frac{dy}{dt} = 0.4y \left(1 - \frac{y}{x}\right)$$

It is known that  $x(0) = 10$  and  $y(0) = 1$ .

- (a) By solving (1), find  $x$  in terms of  $t$ . Express your answer in the form  $x(t) = \frac{100}{f_1(t)}$ .
- (b) By letting  $u = \frac{1}{y}$  in (2), find  $y$  in terms of  $t$ . Express your answer in the form  $y(t) = \frac{100}{f_2(t)}$ .
- (c) Hence, determine how long will it take for 80% of the population to have received the booster vaccine.

2. **(ODE)** 1071 A1 Final Problem 4

- 1. Show that  $y = 0$  is an orthogonal trajectory of the family of curves  $x^2 + \frac{y^2}{k} = 1$ , where  $k > 0$  is an arbitrary constant. Find the orthogonal trajectories of the same family of curves when  $y \neq 0$ .
- 2. Find  $u(t)$  that satisfies the ordinary differential equation

$$u'(t) + \ln(t)u(t) = e^{-t \ln(t)}, \quad t > 0$$

and the condition

$$\lim_{t \rightarrow 0^+} u(t) = 2$$

3. **(ODE)** 1081 A1 Final Problem 6

- (a) Find the orthogonal trajectories of the family of curves  $y = C \tan x$ , where  $C$  is an arbitrary constant.
- (b) Solve the differential equation  $(\cos x) \cdot y' + (\sin x) \cdot y = \tan x, y(0) = 1$ .

4. **(Improper integral)** 1091 (01-05) Final Problem 1

Show that  $\int_0^{\frac{\pi}{2}} e^x \csc x dx$  diverges to infinity by the comparison test.

**Solution:** Use the Taylor expansion to find the test function.

- (i)  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- (ii)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- (iii)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- (iv)  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$  for  $|x| < 1$
- (v)  $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$  for  $|x| < 1$

5. **(Improper integral)** 1111 (01-05) Final Problem 3

- (a) Determine whether  $\int_1^\infty \frac{\sin x}{x} dx$  is convergent or divergent.
- (b) (option) Show the Riemann sum on  $[\epsilon, 1]$  of  $\int_0^1 \frac{\sin(1/x)}{x} dx$  will depend on  $\epsilon$  which is closed to 0.

**Solution:** Let  $f(x) = \frac{\sin(1/x)}{x}$  and partition  $\mathcal{P} = \{0 < \epsilon < x_1 < x_2 < \cdots < x_n = 1\}$

$$\epsilon \cdot f(\epsilon) + (x_1 - \epsilon) f(x_1) + \sum_{k=2}^n f(x_k) (x_k - x_{k-1}) = \epsilon (f(\epsilon) - f(x_1)) + \sum_{k=1}^n f(x_k) (x_k - x_{k-1}).$$

6. **(FTOC)** 1051 A1 Final Problem 8 Let  $f(x)$  be a differentiable and increasing function on  $[a, b]$ , where  $a > 0$ . Find a horizontal line  $y = L$  that will minimize the function  $F(L) = \int_a^b x|f(x) - L|dx = \int_a^{f^{-1}(L)} x(L - f(x))dx + \int_{f^{-1}(L)}^b x(f(x) - L)dx$ .