## All Pairs Shortest Paths / Floyd-Warshall algorithm

**input:** An edge-weighted digraph $G = (V, E)$ with no negative-weight cycles. To ease notation assume $V = \{1, 2, \ldots, n\}$.

**output:** An array of shortest-path distances $d$ such that for all $u, w \in V$, $d[u, w] = \mathsf{dist}(u, w)$ is the minimum shortest-path distance in $G$ from $u$ to $w$.

Here is the algorithm. Fix any input instance $G = (V, E)$.

**subproblems.** For each $u, w \in V$ and $i \in \{0, 1, 2, \ldots, n\}$ define $M[u, w, i]$ to be the minimum weight of any $u$-to-$w$ path in $G$ whose interior vertices are all in $\{1, 2, \ldots, i\}$. (If there is no such path, then $M[u, w, i] = \infty$.) The final answer is given by an array $d$ where $d[u, w] = M[u, w, n]$ for all $u, w \in V$.

**recurrence.** For any $u, w \in V$ and $i \in \{0, 1, \ldots, n\}$,

$$M[u,w,i] = \begin{cases} 0 & \text{if } u = w \\ \mathsf{wt}(u,w) & \text{if } u \neq w,\ i = 0, \text{ and } (u,w) \in E \\ \infty & \text{if } u \neq w,\ i = 0, \text{ and } (u,w) \notin E \\ \min(M[u,w,i-1], M[u,i,i-1] + M[i,w,i-1]) & \text{if } u \neq w \text{ and } i \neq 0 \end{cases}$$

The intuition for the main case of the recurrence (when $i \neq 0$ and $u \neq w$) is as follows. Say a path $p$ from $u$ to $w$ in $G$ is *feasible for* $M[u, w, i]$ if the interior vertices of $p$ (the vertices in $p$ other than the endpoints $u$ and $w$) are all in $\{1, 2, \ldots, i\}$. Then $M[u, w, i]$ is the minimum weight of any feasible path for $M[u, w, i]$. We partition the feasible paths for $M[u, w, i]$ into three sets: (i) those that don't have $i$ as an interior vertex, (ii) those on which $i$ occurs once as an interior vertex, and (iii) those on which $i$ occurs more than once as an interior vertex.

The paths in the first set (i) are just those that are feasible for $M[u, w, i-1]$. The minimum weight of any path in set (i) is $M[u, w, i-1]$.

The paths in the second set (ii) are those paths $p$ that can be obtained as follows. Let $p_1$ be any feasible path for $M[u, i, i-1]$. Let $p_2$ be any feasible path for $M[i, w, i-1]$. Obtain path $p = p_1 \cup p_2$ by appending $p_2$ to $p_1$, so $\mathsf{wt}(p) = \mathsf{wt}(p_1) + \mathsf{wt}(p_2)$. A minimum weight path $p$ in set (ii) can be obtained by taking $p_1$ to be a minimum-weight feasible path for $M[u, i, i-1]$, and taking $p_2$ to be a minimum-weight feasible path for $M[i, w, i-1]$, so $p$ has weight $M[u, i, i-1] + M[i, w, i-1]$.

We can ignore any path $p$ in the third set (iii), because by removing cycles (from $i$ to $i$) from $p$, we can find a path $p'$ in set (ii) with $\mathsf{wt}(p') \leq \mathsf{wt}(p)$.

**time.** There are $O(n^3)$ subproblems. For each, the right-hand side of the recurrence can be evaluated in time $O(1)$. So the total time to compute $d[u, w]$ for all $u, w \in V$ is $O(n^3)$.

**correctness.** Here's a long-form proof that the recurrence is correct.

**Lemma 1.** *The recurrence is correct.*

*Proof (long form).*

1. Following the explanation, for any $u, w \in V$ and $i \in \{0, \ldots, n\}$, say a path $p$ from $u$ to $w$ is *feasible for* $M[u, w, i]$ if all interior vertices of $p$ are in $\{1, 2, \ldots, i\}$. Then $M[u, w, i]$ is the minimum weight of any feasible path for $M[u, w, i]$.

2.1. Consider any $u, w \in V$ and $i \in \{0, 1, \ldots, n\}$.

2.2. <u>First we consider the boundary cases of the recurrence.</u>

2.3.1. <u>Case 1.</u> Suppose $u = w$.

2.3.2. The empty path (of weight zero) is a $u$-to-$w$ path.

2.3.3. There is no path of less weight (as the graph has no negative cycles).

2.3.4. So $M[u, w, i] = 0$, and the recurrence holds in Case 1.

2.4.1. <u>Case 2.</u> Suppose $u \neq w$, $i = 0$, and $(u, w) \in E$.

2.4.2. As $i = 0$, the feasible paths for $M[u, w, i]$ are the $u$-to-$w$ paths with no interior vertices.

2.4.3. The only such path consists of just the edge $(u, w)$, with weight $\mathsf{wt}(u, w)$.

2.4.4. So $M[u, w, i] = \mathsf{wt}(u, w)$, and the recurrence holds in Case 2.

2.5.1. <u>Case 3.</u> Suppose $u \neq w$, $i = 0$, and $(u, w) \notin E$.

2.5.2. As $i = 0$, the feasible paths for $M[u, w, i]$ are the $u$-to-$w$ paths with no interior vertices.

2.5.3. Since $(u, w)$ is not an edge, there is no such feasible path.

2.5.4. So $M[u, w, i] = \infty$, and the recurrence holds in Case 3.

2.6. <u>Next we consider the main case of the recurrence.</u>

2.7.1. <u>Case 4.</u> In the remaining case, $u \neq w$, and $i \neq 0$.

2.7.2. We show that $M[u, w, i] = \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.

2.7.3. Every feasible path for $M[u, w, i-1]$ is also feasible for $M[u, w, i]$.

2.7.4. So $M[u, w, i] \leq M[u, w, i-1]$.

2.7.5. Next we show $M[u, w, i] \leq M[u, i, i-1] + M[i, w, i-1]$.

2.7.6. If $M[u, i, i-1] + M[i, w, i-1] = \infty$, then the above inequality holds, so assume otherwise.

2.7.7. Let $p_1$ be a feasible path for $M[u, i, i-1]$ with $\mathsf{wt}(p_1) = M[u, i, i-1]$.
(It exists, as $M[u, i, i-1]$ is finite.)

2.7.8. Let $p_2$ be a feasible path for $M[i, w, i-1]$ with $\mathsf{wt}(p_2) = M[i, w, i-1]$.
(It exists, as $M[i, w, i-1]$ is finite.)

2.7.9. Then $p = p_1 \cup p_2$ is a path from $u$ to $w$ whose interior vertices are all in $\{1, \ldots, i\}$.

2.7.10. That is, $p$ is a feasible path for $M[u, w, i]$.

2.7.11. So $M[u, w, i] \leq \mathsf{wt}(p) = \mathsf{wt}(p_1) + \mathsf{wt}(p_2) = M[u, i, i-1] + M[i, w, i-1]$.

2.7.12. By this and Step 2.7.4, $M[u, w, i] \leq \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.

2.7.13. <u>Next we show $M[u, w, i] \geq \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.</u>

2.7.14. If $M[u, w, i]$ is infinite, then the inequality above holds, so assume otherwise.

2.7.15. Let $p$ be an *acyclic* feasible path for $M[u, w, i]$ with $\mathsf{wt}(p) = M[u, w, i]$.
(It exists, because $M[u, w, i]$ is finite, and the graph has no negative-weight cycles.)

2.7.16.1. <u>Case 4.1</u> First consider the case that $i$ is not an interior vertex of $p$.

2.7.16.2. Then $p$ is feasible for $M[u, w, i-1]$, so $M[u, w, i-1] \leq \mathsf{wt}(p) = M[u, w, i]$.

2.7.16.3. So $M[u, w, i] \geq \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.

2.7.17.1. <u>Case 4.2</u> Otherwise $i$ occurs as an interior vertex of $p$, just once since $p$ is acyclic.

2.7.17.2. Let $p_1$ be the prefix of $p$ ending at $i$. Let $p_2$ be the suffix of $p$ starting at $i$.

2.7.17.3. Since $i$ occurs just once as an interior vertex of $p$, $i$ is not an interior vertex of $p_1$ or $p_2$.

2.7.17.4. So $p_1$ is a feasible path for $M[u, i, i-1]$, and $p_2$ is a feasible path for $M[i, w, i-1]$.

2.7.17.5. So $\mathsf{wt}(p_1) \geq M[u, i, i-1]$, and $\mathsf{wt}(p_2) \geq M[i, w, i-1]$.

2.7.17.6. So $M[u, w, i] = \mathsf{wt}(p) = \mathsf{wt}(p_1) + \mathsf{wt}(p_2) \geq M[u, i, i-1] + M[i, w, i-1]$.
2.7.17.7. So $M[u, w, i] \geq \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.
2.7.18. By Blocks 2.7.16 and 2.7.17, $M[u, w, i] \geq \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.
2.7.19. By this and Step 2.7.12, $M[u, w, i] = \min(M[u, w, i-1], M[u, i, i-1] + M[i, w, i-1])$.
2.7.20. So the recurrence holds in Case 4.
2.8. By Blocks 2.3-2.7, the recurrence holds for $M[u, w, i]$. $\square$