Here are some more examples of long-form proofs. They illustrate proper numbering of steps and the use of the four types of blocks. We also give some of the proofs in regular (short-form) for comparison. Some of the "theorems" are false. For those, find the error in the proof.

Three short examples: primes; squares are positive; knights and knaves

Theorem 1. *There are infinitely many primes.*

The proof uses that, if a is a common factor of integers b and c, then a is a factor of $c - b$.

Proof (long form).

1. Suppose for contradiction that there are finitely many primes. *block: contradiction*
2. Let $p_1, p_2, \ldots, p_n$ be a list of all the primes (larger than 1).
3. Let $P = \prod_{i=1}^{n} p_i$ be their product.
4. Fix p_i to be any prime in the prime factorization of $P + 1$.
5. Then p_i evenly divides P, and also evenly divides $P + 1$.
6. So p_i evenly divides their difference, $(P + 1) - P$.
7. But this is a contradiction, as $(P + 1) - P = 1$ has no factor other than 1. □

Step 4 defines p_i to be a specific number (one that we know exists at that point) for use later in the proof. Step 4 does not open a new "for all" block. We would only use a "for all" block in order to prove something about *all* primes in the prime factorization of $P + 1$.

Here's the same proof in short form.

Proof (short form). Suppose for contradiction that there are finitely many primes: $p_1, p_2, \ldots, p_n$. Consider the product P of all the primes. Let $p_i > 1$ be any prime in the prime factorization of $P+1$. Then p_i is a factor of both P and $P+1$, so it is a factor of $(P+1) - P = 1$, which is impossible. □

"Positive" means "strictly greater than zero." "Negative" means "strictly less than zero."

Theorem 2 (False!)**.** *For every real number, its square is positive. That is,* $\forall x \in \mathbb{R},\ x^2 > 0$.

Proof (long form). Find the error!

1. Let $x \in \mathbb{R}$ be an arbitrary real number. *block: ∀*

2.1. Case 1, when x is positive ($x > 0$). *block: cases*

2.2. Then x^2 is a product of two positive numbers, so x^2 is positive.

3.1. Case 2, when x is negative ($x < 0$). *block: cases*

3.2. Then x^2 is a product of two negative numbers, so x^2 is positive.

4. In either case, x^2 is positive, so x^2 is positive. □

Proof (short form). The proof is by cases. If x is positive, then x^2 is a product of two positive numbers, so is positive. If x is negative, then x^2 is a product of two negative numbers, so is positive. In either case, x^2 is positive. So x^2 is positive. □

Here's the setup for the next example:

P1. Every native of Platonia is either a knight or a knave (and not both).

P2. Every statement a knight makes is true.

P3. Every statement a knave makes is false.

Theorem 3. *Given P1–P3, any person who states "I am a knave" is not a native of Platonia.*

Proof (long form).

1. Let X be an arbitrary person who states "I am a knave". *block:* $\forall$
2.1. Suppose for contradiction that X is a native of Platonia. *block: contradiction*
2.2. By P1, X is either a knight or a knave. We consider each case separately.
2.3.1. Case 1. First consider the case that X is a knight. *block: cases*
2.3.2. By P2, X's statement "I am a knave" is true, so X is a knave.
2.3.3. By P1, X is not a knight.
2.3.4. This is a contradiction (as X is a knight).
2.4.1. Case 2. Otherwise, by P1, X is a knave. *block: cases*
2.4.2. By P3, X's statement "I am a knave" is false.
2.4.3. That is, X is not a knave.
2.4.4. This is a contradiction (as X is a knave).
2.5. Either case leads to a contradiction, so X is not a native of Platonia. □

Proof (short form). Assume for contradiction that X is a native of Platonia. If X is a knight, then (by P2) X's statement "I am a knave" must be true, so X is a knight and a knave, contradicting P1. Otherwise (by P2) X is a knave, so (by P3) X's statement is false, so X is not a knave (contradicting that X is a knave). □

To note

– In these proofs, each named object (p_i, P, x, X, …) is defined. Each definition holds just within its block.

– Note the use of the four types of blocks (though only three are used so far). Note the numbering of steps.

– Each proof assumes the reader understands basic properties of the objects it works with (graphs, numbers, …), and uses those basic properties without further proof.

(continued)

All marbles are the same color!

Theorem 4. *In any finite set of marbles, all the marbles are the same color.*

Proof (long form). Where's the error?

1. Define $P(n) \equiv$ "in any set S of n marbles, the marbles are the same color."
2. We show by induction on n that $P(n)$ holds for all integers $n \geq 0$.
3. First we consider the base cases, $n \in \{0, 1\}$.

4.1. Let S be any set containing at most one marble. *block: ∀*
4.2. Since S has at most one marble, all marbles in S are the same color.
5. By Block 4, $P(0)$ and $P(1)$ hold.
6.1. Next we prove the induction step. Fix an arbitrary integer $n \geq 2$. *block: ∀*
6.2.1. Assume that $P(n-1)$ holds. We show that $P(n)$ holds. *block: if then*
6.2.2.1. Let $S = \{m_1, m_2, \ldots, m_n\}$ be an arbitrary set of n marbles, in any order. *block: ∀*
6.2.2.2. Let set $S_1 = \{m_1, m_2, \ldots, m_{n-1}\}$ contain the first $n-1$ marbles in S.
6.2.2.3. Let set $S_2 = \{m_2, m_3, \ldots, m_n\}$ contain the last $n-1$ marbles in S.
6.2.2.4. C_1 has size $n-1$, and $P(n-1)$ holds (Step 6.2.1), so the marbles in C_1 are one color.
6.2.2.5. Likewise all marbles in C_2 are the same color.
6.2.2.6. Next we show that every marble in S is the same color as marble m_2.
6.2.2.7.1. Let m_i be an arbitrary marble in S. *block: ∀*
6.2.2.7.2.1. Case 1. First consider the case that m_i is in S_1. *block: cases*
6.2.2.7.2.2. Since marbles m_i and m_2 are in S_1, by Step 6.2.2.4, m_i and m_2 are the same color.
6.2.2.7.3.1. Case 2. Otherwise (since $S = S_1 \cup S_2$) m_i is in S_2. *block: cases*
6.2.2.7.3.2. Since marbles m_i and m_2 are in S_2, by Step 6.2.2.5, m_i and m_2 are the same color.
6.2.2.7.4. Case 1 or Case 2 must hold, and in either case m_i and m_2 are the same color.
6.2.2.7.5. Therefore m_i and m_2 are the same color.
6.2.2.8. By Block 6.2.2.7, within S, all marbles are the same color as m_2.
6.2.2.9. So all marbles in S are all the same color.
6.2.3. By Block 6.2.2, $P(n+1)$ holds.
6.3. By Block 6.2, if $P(n-1)$ holds, then $P(n)$ holds.
7. By Block 6, for any $n \geq 2$, if $P(n-1)$ holds, then $P(n)$ holds.
8. By Step 5, $P(0)$ and $P(1)$ hold.
9. By the preceding two steps, and induction, $P(n)$ holds for all integers $n \geq 0$.
10. That is, within any finite set of marbles, all the marbles are the same color. □

Proof (short form). We show by induction on $n \geq 0$ that, within in any set S of n marbles, all of the marbles are the same color. For the base cases $n \in \{0, 1\}$ this is true because S contains at most one marble. Consider any set S of $n \geq 2$ marbles. Assume the inductive hypothesis holds for any set of $n-1$ marbles. Order the n marbles in S arbitrarily. Let S_1 contain the first $n-1$ marbles in S. Let S_2 contain the last $n-1$ marbles in S. By the inductive assumption, within S_1 all marbles are the same color. Likewise, within S_2 all marbles are the same color. Since S_1 and S_2 overlap, all marbles in $S = S_1 \cup S_2$ are the same color. This proves the theorem. □

Vertex weights equal. Let $G = (V, E)$ be any graph, where each vertex $v \in V$ has a numeric weight $w(v) \in \mathbb{R}$. Consider the following two properties:

> *Property A: For every vertex v in the graph, if v has neighbors, then the weight $w(v)$ of v equals the average of v's neighbor's weights.*
>
> *Property B: All vertices in G have the same weight.*

Theorem 5. *Any finite, connected graph G that has Property A also has Property B.*

Long-form proof 1.

1. Let G be an arbitrary finite, connected graph having Property A. *to prove* $\forall$

Before we give the proof, we prove the following utility lemma:

Lemma 1. *Let v be any vertex in G that has a neighbor x with larger weight ($w(x) > w(v)$). Then v has a neighbor y with a smaller weight ($w(y) < w(v)$).*

Proof of lemma.

2.1. Let v be an arbitrary vertex as in the lemma (so G has a neighbor x with larger weight).

2.2.1. Assume for contradiction that v's neighbors all have weight at least $w(v)$. *proof by contr.*

2.2.2. Then, since v also has at least one neighbor of weight strictly more than $w(v)$, the average of v's neighbors' weights exceeds v.

2.2.3. This is a contradiction (as G has Property A.)

2.3. By Block 2.2, vertex v has some neighbor with weight less than $w(v)$, proving the lemma. □

3.1. Now assume for contradiction that G does not have Property B. *proof by contradiction*

3.2. Let u and v be two vertices of different weight (they exist because G doesn't have Property B).

3.3. Consider any path from u to v in G (it exists as G is connected).

3.4. The path must have two adjacent vertices with different weights. (Otherwise, by transitivity, all vertices along the path would have the same weight, but u and v have different weights.)

3.5. Of these two adjacent vertices with different weights, let v_1 be the one with smaller weight.

3.6. Then v_1 has a neighbor with larger weight.

3.7. So, by Lemma 1 applied to v_1, v_1 has some neighbor v_2 with smaller weight ($w(v_1) > w(v_2)$).

3.8. So, by Lemma 1 applied to v_2, v_2 has some neighbor v_3 with smaller weight ($w(v_2) > w(v_3)$).

3.9. Inductively, there is an infinite path $v_1, v_2, v_3, \ldots$ with strictly decreasing vertex weights.

3.10. Since the weights strictly decrease along the path, no vertex in G occurs twice on the path.

3.11. So G has infinitely many vertices.

3.12. This is a contradiction (as G is finite).

4. By Block 3, G must have Property B. □

Proof (short form). Let G be any finite connected graph with Property A. We use the following observation: *If a vertex v in G has a neighbor x with larger weight ($w(x) > w(v)$) then v also has a neighbor y with smaller weight ($w(y) < w(v)$).* (Otherwise the weight of v would not equal the average of its neighbors' weights.) Suppose for contradiction that G doesn't have Property B. So it has two vertices with different weights. There is a path between these two vertices, along which there must be two neighbors whose weights differ. The smaller-weight neighbor, say v_1 (having a larger-weight neighbor) must also (by the observation) have a smaller-weight neighbor, say v_2. In turn, v_2 (with larger-weight neighbor v_1) has a smaller-weight neighbor v_3. Inductively, G contains an infinite path along which the vertex-weights strictly decrease, contradicting that G is finite. □

Theorem 1. *Any finite, connected graph G that has Property A also has Property B.*

Long-form proof 2.

1. Let G be an arbitrary finite, connected graph having Property A.
2. Let $W = \min_{v \in V} w(v)$ be the minimum vertex weight. (W is well-defined as G is finite.)

   We will use the following utility lemma:

Lemma 2. *If v is any vertex of weight W, then each of v's neighbors has weight W.*

Proof of lemma.

3.1. Let v be an arbitrary vertex with weight W.
3.2. Each of v's neighbors has weight *at least* W (by definition of W).
3.3.1. Assume for contradiction that v has a neighbor of weight strictly more than W.
3.3.2. Then the average of the neighbors' weights is strictly more than W.
3.3.3. This is a contradiction (as v has weight W and G has Property A).
3.4. By Block 3.3, each of v's neighbors has weight at most W.
3.5. By this and Line 3.2, each of v's neighbors has weight W. □

4. Let v_1 be any vertex of weight W. (Vertex v_1 exists by definition of W.)
5. By the lemma, all neighbors of v_1 have weight W.
6. Applying the lemma to each neighbor of v_1, all of the neighbors' neighbors also have weight W.
7. Continuing inductively, all vertices reachable from v_1 have weight W.
8. Since G is connected, all vertices are reachable from v_1, so all vertices have weight W.
9. So G has Property B. □

Long-form proof 3 (INCORRECT PROOF BY INDUCTION).

1. The proof is by induction on the size of G.
2. Base case: G has two vertices. Then it is easy to verify that Property A implies Property B.

3.1. For the induction step, for any $n \geq 2$, assume the theorem holds for all graphs with n vertices.
3.2.1. Let G be an arbitrary $(n+1)$-vertex graph having property A.
3.2.2. Let v_1 be any vertex in G, and let G' be obtained from G by deleting v_1 and its edges.
3.2.3. G' has Property A, because G does.
3.2.4. So, by the inductive assumption, G' has Property B.
3.2.5. That is, all vertices in G' have equal weight, say W.
3.2.6. Now consider v_1 in G. Each neighbor of v_1 is in G', so has weight W.
3.2.7. Since G has Property A, then, v_1 also has weight W.
3.2.8. So all vertices in G have weight W. That is, G has Property B.
3.3. By Block 3.2, the theorem holds for all graphs with $n+1$ vertices.

4. By Line 2, the theorem holds for any graph with two vertices. By Block 3, for every $n \geq 2$, it holds for any graph with $n+1$ vertices, as long as it holds for every graph with n vertices. Inductively, it holds for all graphs with $n \geq 2$ vertices. □

The error is in Line 3.2.3. G' does not have to have Property A, even though G does.

Bonus question. Suppose $G = (V, E)$ is an infinite connected graph with vertex weights and Property A. If all the vertex weights are non-negative, must G also have Property B? (The answer is no. Can you find an example?) What if G also has at least one vertex of weight zero?