The first homework will ask you to choose some problems from the list below. Each problem states a theorem and gives an incomplete proof. The problem is to complete the proof.

Problem 1. (Problem 1.13 of MCS)

Theorem 1. *Let*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + x^m$$

be an arbitrary polynomial with integer coefficients, whose highest-degree term has coefficient 1. Then any rational root of $p(x)$ is an integer.

Proof (long form).

1. Let p be an arbitrary polynomial as in the theorem, and let x be any rational root of p.
2. Fix integers i and j with $j > 0$ such that $x = i/j$.
3. . . .
4. So x is an integer. □

Problem 2. (red/blue/round/square)

Theorem 2. *In a set S of objects, each is either red or blue, and each is either round or square. S has at least one red object, at least one blue object, at least one round object, and at least one square object. Then S has two objects that differ both in color and in shape.*

Proof (long form).

1. Let S be the set of objects in question.

2.1. Case 1. Suppose all red objects in S are the same shape.

2.2. Assume that the red objects are all square (the case when they are all round is symmetric).

2.3. . . .

2.4. So, in Case 1, there exist two objects that differ both in color and in shape.

3.1. Case 2. Otherwise S has a red-round object and a red-square object.

3.2. . . .

3.3. So, in Case 2, there exist two objects that differ both in color and in shape.

4. Case 1 or Case 2 must hold, and in either case the theorem holds, so the theorem holds. □

Problem 3. (partition the integers)

The positive integers are partitioned into several subsets $A_1, A_2, \ldots, A_n$ such that, for $i = 1, 2, \ldots, n$, for each x in A_i, $2x$ is not in A_i. What is the minimum possible value of n?

Theorem 3. *The minimum possible value of n is 2.*

Proof (long form).

1. For any such partition n is at least 2. (Otherwise 1 and 2 would be in A_1, violating the claimed property.)
2. Define $A_1 = \ldots$ and $A_2 = \ldots$. To finish we show that A_1, A_2 is a partition with the claimed property.
3. . . . □

Problem 4. (handshakes after the party) ⋆ $N+1$ couples (where $N \geq 0$) attend a dinner. At the end of the dinner, one person (Alice) asks each other person to write down the number of distinct people with whom he or she shook hands at the dinner. Surprisingly, all numbers $0, 1, 2, \ldots, 2N$ are written down. Assuming that no person shook hands with their partner, how many people did Alice shake hands with at the dinner?

Theorem 4. *Alice shook hands with N people.*

Proof (long form).

1. We prove by induction on N that, in any such scenario, Alice shakes hands with N people.

2. In the base case, $N = 0$, ...
3.1. For the induction step, consider any $n \geq 1$, and assume that the statement holds for any such scenario with $N = n - 1$ couples. We show that the statement holds for any such party scenario with $N = n$ couples.
3.2. ...
4. ... □

Hint: one person shook hands with $2N$ people. What about that person's partner?

Problem 5. (Inclusion exclusion for reals) ⋆

Theorem 5. *Every triple x, y, z of real numbers satisfies the inequality $|x| + |y| + |z| - |x + y| - |x + z| - |y + z| + |x + y + z| \geq 0$.*

Proof (long form).

1. Define $f(x, y, z) = |x| + |y| + |z| - |x + y| - |x + z| - |y + z| + |x + y + z|$.
2. To prove the theorem, we show that $f(x, y, z) \geq 0$ for all $x, y, z \in \mathbb{R}$.
3. ...
4. So $f(x, y, z) \geq 0$ for all $x, y, z \in \mathbb{R}$. □

Problem 6. (Hanoi lower bound) ⋆

Check out the towers of hanoi puzzle. (http://en.wikipedia.org/wiki/Tower_of_Hanoi.)

Theorem 6. *Any legal sequence of moves (which never puts any disc on top of a smaller disk, and moves N discs from one peg to another) takes at least $2^N - 1$ moves.*

Proof (long form).

1. The proof is by induction on $N \in \{1, 2, 3, \ldots\}$.
2. The base case is $N = 1$.
3.1. For the induction step, consider any $n \geq 1$, and assume that the theorem holds for $N = n$.
3.2. We show that the theorem holds for $N = n + 1$.
3.3. ...
4. ... □

Problem 7. (boys and girls around a table) ⋆

Theorem 7. *Fifty chairs are arranged, evenly spaced, around a round table. For any seating arrangement of 25 boys and 25 girls (placing one boy or girl in each chair), at least one person has a girl on each side.*

Proof (long form).

1. Suppose for contradiction that there is a seating arrangement, say X, where nobody sits between two girls.
2. ...
3. This is a contradiction, so some person sits between two girls. □

Hint: make sure your reasoning is complete and covers all possible arrangements. It's not enough to observe that one particular kind of arrangement fails.

Problem 8. (all pairs with different sums) ⋆⋆

Theorem 8. *A subset S of $\{1, 2, \ldots, 100\}$ has the following property: for every quadruple u, v, w, x of distinct numbers in S, the sum of u and v differs from the sum of w and x. Then the size of S is at most fifteen.*

Proof (long form).

1. Consider any such subset S.
2. ...
3. Therefore the size of S is at most fifteen. □