Probability Assignment

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12.13.1.10
A black and a red dice are rolled.

1. Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

2. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution

1. Let $X_1 \in (1,2,3,4,5,6)$ be a random variable representing the outcomes of red die.
Let $X_2 \in (1,2,3,4,5,6)$ be a random variable representing the outcomes of black die.
We need the conditional probability of event $(X_1 + X_2 > 9)$ given that $(X_2 = 5)$ has occurred. $\Pr\left((X_1 + X_2 > 9) \mid (X_2 = 5)\right)$

$$= \frac{\Pr\left((X_1 + X_2 > 9)(X_2 = 5)\right)}{\Pr\left(X_2 = 5\right)} \tag{1}$$

We have that,

$$p_{X_i}(n) = \Pr\left(X_i = n\right) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{2}$$

Therefore using equation (2) we can write that,

$$\Pr\left(X_2 = 5\right) = \frac{1}{6} \tag{3}$$

From binomial distribution we can write ,

$$\Pr\left(X_1 + X_2 > 9\right) = \Pr\left(X_1 + X_2 = 10\right) + \Pr\left(X_1 + X_2 = 11\right) \tag{4}$$

$$= \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \binom{2}{2}\left(\frac{1}{6}\right)^2 \tag{5}$$

$$= \frac{11}{36} \tag{6}$$

The event $((X_1 + X_2 > 9)(X_2 = 5))$ is such that the sum is greater than 9 with a number 5.
There are only two possible cases {5,5} and {6,5} out of 36 possible cases.
Hence,

$$\Pr\left((X_1 + X_2 > 9)(X_2 = 5)\right) = \frac{2}{36} \tag{7}$$

Substituting equations (3), (7) in (1), we get

$$\begin{aligned} \Pr\left((X_1 + X_2 > 9) \mid (X_2 = 5)\right) &= \frac{\frac{2}{36}}{\frac{1}{6}} \\ &= \frac{1}{3} \end{aligned} \tag{8}$$

Hence the probability of obtaining a sum greater than 9, with black die in a 5 is $\frac{1}{3}$.

2. Let $X_1 \in (1,2,3,4,5,6)$ be a random variable representing the outcomes of black die.
Let $X_2 \in \{1,2,3,4,5,6\}$ be a random variable representing the outcomes of red die.
We need the conditional probability of event $(X_1 + X_2 = 8)$ given that $(X_2 < 4)$ has occurred. $\Pr\left((X_1 + X_2 = 8) \mid (X_2 < 4)\right)$

$$= \frac{\Pr\left((X_1 + X_2 = 8)((X_2 = 2) + (X_2 = 3))\right)}{\Pr\left((X_2 = 2) + (X_2 = 3)\right)} \tag{9}$$

We have that,

$$\Pr\left(X_1 + X_2 = n\right) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{10}$$

Therefore using equation (10) we can write that,

$$\Pr\left(X_1 + X_2 = 8\right) = \frac{5}{36} \tag{11}$$

From binomial distribution we can write ,

$$\Pr\left(X_2 < 4\right) = \Pr\left(X_2 = 2\right) + \Pr\left(X_2 = 3\right) \tag{12}$$

$$= \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \binom{2}{2}\left(\frac{1}{6}\right)^2 \tag{13}$$

$$= \frac{11}{36} \tag{14}$$

The event $((X_1 + X_2 = 8)((X_2 = 2) + (X_2 = 3)))$ is such that the sum is 8 with a number less than 4.
There are only two possible cases {5,3} and {6,2} out of 36 possible cases.
Hence,

$$\Pr\left((X_1 + X_2 = 8)((X_2 = 2) + (X_2 = 3))\right) = \frac{2}{36} \tag{15}$$

Substituting equations (15) in (1), we get

$$\begin{aligned} \Pr\left((X_1 + X_2 = 8) \mid (X_2 < 4)\right) &= \frac{\frac{2}{36}}{\frac{1}{2}} \\ &= \frac{1}{9} \end{aligned} \tag{16}$$

Hence the probability of obtaining the sum 8 when a number is less than 4 is $\frac{1}{9}$.