

Probability Assignment

Sinkona Chinthamalla

January 10, 2023

12.13.1.10

A black and a red dice are rolled.

1. Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.
2. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution

1. *The Uniform Distribution:* Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, $i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The desired outcome is

$$X = X_1 + X_2, \quad (2)$$

$$\implies X \in \{1, 2, \dots, 12\} \quad (3)$$

Convolution:

From (2)

$$p_X(n) = \Pr(X_1 + X_2 > n) = \Pr(X_1 > n - X_2) \quad (4)$$

$$= \sum_k \Pr(X_1 > n - k | X_2 = k) p_{X_2}(k) \quad (5)$$

after unconditioning. $\because X_1$ and X_2 are independent. Then,

$$p_X(n) = \Pr(X_1 + X_2 > 9) = \Pr(X_1 > 9 - X_2) \quad (6)$$

$$= \Pr(X_1 > 9 - k | X_2 = k) p_{X_2}(k) \quad (7)$$

$$= \frac{1}{6} \Pr(X_1 > 9 - 5 | X_2 = 5) \quad (8)$$

$$= \frac{1}{6} \Pr(X_1 > 4) \quad (9)$$

$$= \frac{1}{6} (\Pr(X_1 = 5) + \Pr(X_1 = 6)) \quad (10)$$

$$= \frac{2}{36} \quad (11)$$

From (1) and (11), Conditional probability of event $(X_1 + X_2 > 9)$ given that $(X_2 = 5)$ has occurred is,

$$\begin{aligned} \Pr((X_1 + X_2 > 9) | (X_2 = 5)) &= \frac{\frac{2}{36}}{\frac{1}{6}} \\ &= \frac{1}{3} \end{aligned} \quad (12)$$

Hence the probability of obtaining a sum greater than 9, when black die resulted in a 5 is $\frac{1}{3}$.

2. *The Uniform Distribution:* Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, $i = 1, 2$, be the random variables representing the outcome for each die.

Convolution:

From (2)

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \quad (13)$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) p_{X_2}(k) \quad (14)$$

after unconditioning. $\because X_1$ and X_2 are independent. Then,

$$p_X(n) = \Pr(X_1 + X_2 = 8) = \Pr(X_1 = 8 - X_2) \quad (15)$$

$$= \Pr(X_1 = 8 - k | X_2 = k) p_{X_2}(k) \quad (16)$$

$$= \frac{1}{6} \Pr(X_1 = 8 - k | X_2 < 4) \quad (17)$$

$$= \frac{1}{6} (\Pr(X_1 = 5) + \Pr(X_1 = 6)) \quad (18)$$

$$= \frac{2}{36} \quad (19)$$

From (1) and (19), Conditional probability of event $(X_1 + X_2 = 8)$ given that $(X_2 < 4)$ has occurred is,

$$\begin{aligned} \Pr((X_1 + X_2 = 8) | (X_2 < 4)) &= \frac{\frac{2}{36}}{\frac{3}{6}} \\ &= \frac{1}{9} \end{aligned} \quad (20)$$

Hence the probability of obtaining the sum 8 when a number is less than 4 is $\frac{1}{9}$.