Physics Project 2: Spherical Model

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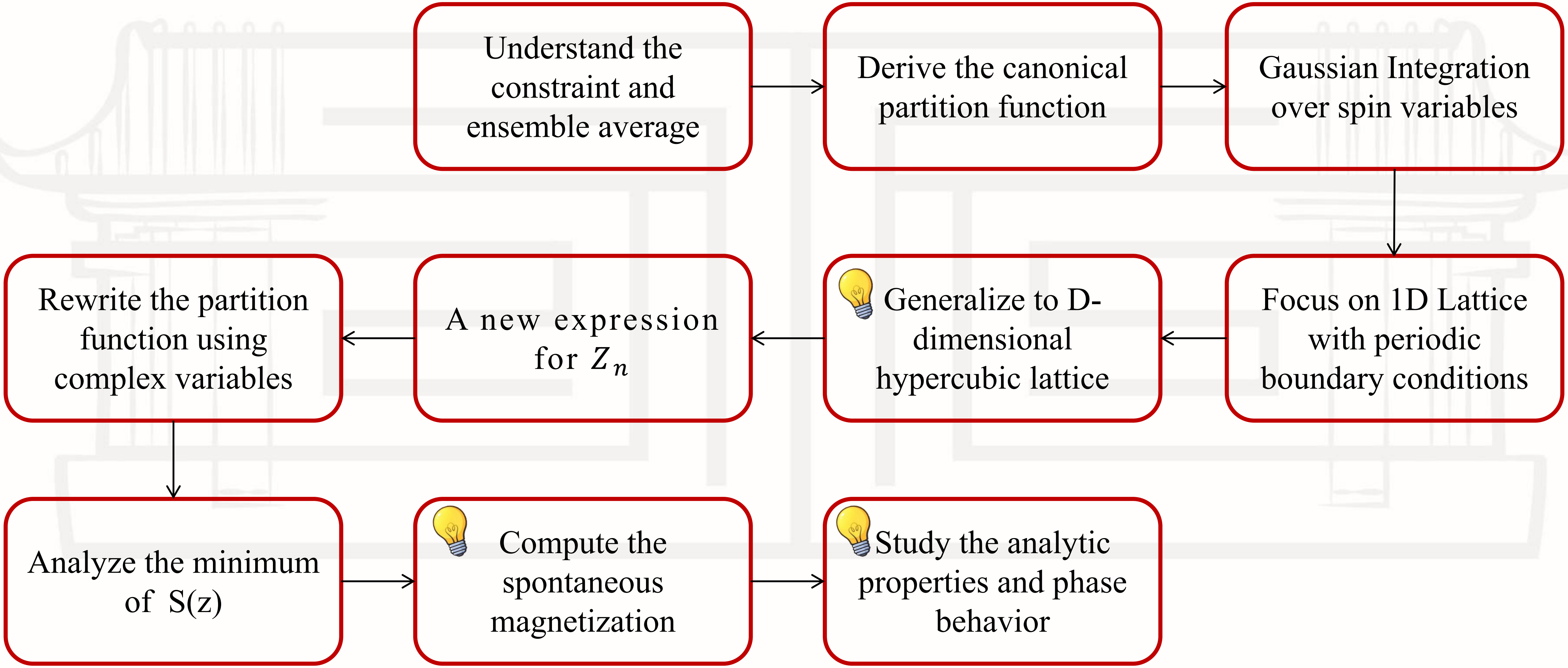
Research Framework



01 Understand the constraint and ensemble average

- Spherical Model: ferromagnetic model, not dissimilar from the Ising model; however, the spin degrees of freedom will take different values from the Ising model.

- System constraint is a continuazation of the constraint of Ising Model: $\sum_{j=1}^{N} \sigma_j^2 = N \quad \vec{\sigma} \in \Gamma$

- Cutting out a $(N-1)$- dimensional sphere Γ inside the space of all configurations $\mathbb{R}^N$.

- Energy is given by:

$$E(\vec{\sigma}) = -J \sum_{\langle j,l \rangle} \sigma_j \sigma_l - H \sum_{j=1}^{N} \sigma_j$$

- $\langle \sigma_j^2 \rangle$ is equal to 1 because
  - σ_1 *to* σ_N are symmetric
  - linearity of expectation
  - sum of expectation = expectation of sum = N
  - every $\langle \sigma_j^2 \rangle$ is equal to 1

02 Deriving partition function Z_n

- The partition function can be written as:

$$Z_N = \int_\Gamma e^{-\beta E(\vec{\sigma})}\, d\vec{\sigma} = \int_{\sum_{j=1}^N \sigma_j^2 = N} e^{K\sum_{\langle j,l\rangle}\sigma_j\sigma_l + h\sum_{j=1}^N \sigma_j}\, d\vec{\sigma}$$

- ***LEMMA***: For a smooth function $g\colon \mathbb{R}^N \to \mathbb{R}$, $a \in \mathbb{R}^+$, the following equation holds:

$$\int_{\sum_{j=1}^N \sigma_j^2 = N} g(\vec{\sigma})\, d\vec{\sigma} = \int_{-\infty}^{+\infty}\frac{ds}{2\pi}\int_{\mathbb{R}^N} d\sigma_1\cdots d\sigma_N\, e^{(a+\sqrt{-1}s)\left(N-\sum_{j=1}^N\sigma_j^2\right)} g(\vec{\sigma})$$

- ***PROOF***: Using the Dirac delta function, we have:

$$\delta(x) = \int_{-\infty}^{+\infty} e^{-ax-\sqrt{-1}sx}\,\frac{ds}{2\pi}$$

- From the Fourier transform representation of the Dirac delta function, and introducing a convergence factor $a > 0$, we have:

$$\text{LHS} = \int_{\mathbb{R}^N} g(\vec{\sigma})\,\delta(\|\vec{\sigma}\|^2 - N)\, d\vec{\sigma}$$

- Substituting into the above expression, we obtain:

$$\int_{\mathbb{R}^N} g(\vec{\sigma})\,\delta(\|\vec{\sigma}\|^2 - N)\, d\vec{\sigma} = \int_{\mathbb{R}^N} d\vec{\sigma}\int_{-\infty}^{+\infty}\frac{ds}{2\pi} g(\vec{\sigma})\, e^{(a+\sqrt{-1}s)\left(N-\sum_{j=1}^N\sigma_j^2\right)} \qquad \square$$

- By the ***Lemma***, we have:

$$Z_N = \int_{-\infty}^{+\infty}\frac{ds}{2\pi}\int_{\mathbb{R}^N} d\sigma_1\cdots d\sigma_N\,\exp\left[(a+\sqrt{-1}s)\left(N-\sum_{j=1}^N\sigma_j^2\right) + K\sum_{\langle j,l\rangle}\sigma_j\sigma_l + h\sum_{j=1}^N\sigma_j\right] = \boxed{\int_{-\infty}^{+\infty}\frac{ds}{2\pi}\int_{\mathbb{R}^N} d\sigma_1\cdots d\sigma_N\,\exp\left[-\sum_{j,l}V_{jl}(s)\sigma_j\sigma_l + h\sum_{j=1}^N\sigma_j + (a+\sqrt{-1}s)N\right]}$$

where

$$V_{jl}(s) = \begin{cases} -\frac{1}{2}K + (a+\sqrt{-1}s)\delta_{jl}, & \text{if } \langle j,l\rangle \text{ are neighbors,} \\ (a+\sqrt{-1}s)\delta_{jl}, & \text{otherwise.}\end{cases}$$

03 Gaussian Integration over spin variables

- ***LEMMA***: Let $a \in \mathbb{R}^+$ be sufficiently large, and let $B \in M_{n\times n}(\mathbb{C})$. Define $A = aI_n + B$. Then, for $\beta \in \mathbb{R}^N$, we have:
$$\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det(A)}}\, e^{\frac{1}{4}\beta^T A^{-1}\beta},$$
where the branch of $\sqrt{\det(A)}$ is chosen so that its real part is positive.
- ***PROOF***: There exists an invertible matrix P such that: $A = P^T P$
- Making change of the variables $\sigma \mapsto x = P\sigma$, then: $\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma = \frac{1}{\sqrt{\det(A)}}\int_{\mathbb{R}^N} e^{-x^T x+\beta^T P^{-1}x}\, dx$
- Completing the square: $-x^T x + \beta^T P^{-1}x = -\left\|x - \frac{1}{2}(P^{-1})^T\beta\right\|^2 + \frac{1}{4}\beta^T P^{-1}(P^{-1})^T\beta.$
- Since $(P^{-1})^T P^{-1} = A^{-1}$, this becomes: $-\left\|x - \frac{1}{2}(P^{-1})^T\beta\right\|^2 + \frac{1}{4}\beta^T A^{-1}\beta.$
- Applying the translation $x \mapsto x + \frac{1}{2}(P^{-1})^T\beta$:
$$\int_{\mathbb{R}^N} e^{-x^T x+\beta^T P^{-1}x}\, dx = e^{\frac{1}{4}\beta^T A^{-1}\beta}\,\pi^{N/2}$$
where Gaussian integral is used.
- Combining the factors gives: $\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det(A)}}\, e^{\frac{1}{4}\beta^T A^{-1}\beta}$ $\square$
- Hence, according to the ***Lemma***, denote $\beta = (h, h, \dots, h)^T \in \mathbb{R}^N$

$$Z_N = \int_{-\infty}^{+\infty}\frac{ds}{2\pi}\int_{\mathbb{R}^N} d\sigma\, e^{-\sigma^T V_{jl}(s)\sigma+\beta^T\sigma}\cdot e^{\left(a+\sqrt{-1}\,s\right)N} = \int_{-\infty}^{+\infty}\frac{\pi^{\frac{N}{2}}\, ds}{2\pi}\, e^{\frac{1}{4}\beta^T V_{jl}^{-1}(s)\beta+\left(a+\sqrt{-1}\,s\right)N-\frac{1}{2}\ln(\det V_{jl}(s))}$$

04 Focus on 1D Lattice with periodic boundary conditions

- Note that all the neighbors are $\langle 1,2\rangle, \langle 2,3\rangle, \dots, \langle N-1, N\rangle$, $\langle N, 1\rangle$, hence:

$$V_{jl}(s) = \begin{pmatrix} a+\sqrt{-1}s & -\frac{1}{2}K & & & -\frac{1}{2}K \\ -\frac{1}{2}K & a+\sqrt{-1}s & -\frac{1}{2}K & & \\ & -\frac{1}{2}K & \ddots & \ddots & \\ & & \ddots & a+\sqrt{-1}s & -\frac{1}{2}K \\ -\frac{1}{2}K & & & -\frac{1}{2}K & a+\sqrt{-1}s \end{pmatrix} \quad \text{(the blank space is zero)}$$

- Let $\zeta = e^{\frac{2\pi\sqrt{-1}}{N}}$. Note that for any $1 \le k \le N$:

$$\begin{pmatrix} a+\sqrt{-1}s & -\frac{1}{2}K & & & -\frac{1}{2}K \\ -\frac{1}{2}K & a+\sqrt{-1}s & -\frac{1}{2}K & & \\ & -\frac{1}{2}K & \ddots & \ddots & \\ & & \ddots & a+\sqrt{-1}s & -\frac{1}{2}K \\ -\frac{1}{2}K & & & -\frac{1}{2}K & a+\sqrt{-1}s \end{pmatrix} \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix} = \left(a+\sqrt{-1}s - K\cos\frac{2\pi k}{N}\right) \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}$$

  and the vectors are linearly independent

- Hence the eigenvalues and eigenvectors are:

$$a+\sqrt{-1}s - K\cos\frac{2\pi k}{N}, \quad (1 \le k \le N) \qquad \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}, \quad (1 \le k \le N), \quad \text{with } \zeta = e^{\frac{2\pi\sqrt{-1}}{N}}.$$

05-1 Generalize to D-dimensional hypercubic lattice

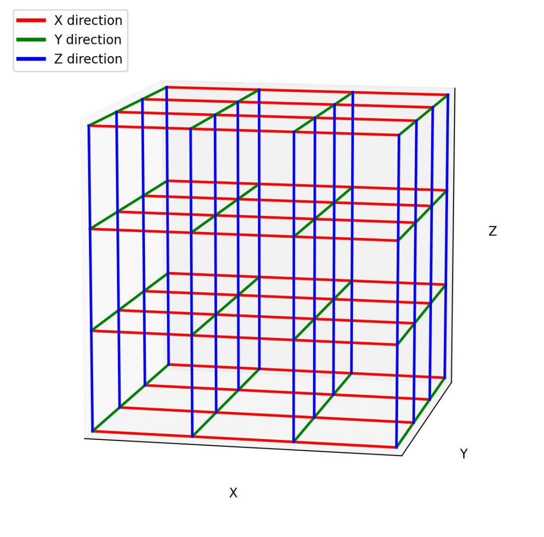
- ***LEMMA***: We define a graph over the Abelian group $(\mathbb{Z}/L\mathbb{Z})^D$, there exist an edge between a and b iff $b-a$ or $a-b$ is in $\{(\bar{1},\bar{0},\dots,\bar{0}),(\bar{0},\bar{1},\bar{0},\dots,\bar{0}),\dots,(\bar{0},\bar{0},\dots,\bar{0},\bar{1})\}$.
- Let $X \in M_{L^D \times L^D}(\mathbb{C})$ be the adjacency matrix of the graph. Then if we label

$(\overline{r_1},\overline{r_2},\dots,\overline{r_D})$ $(\overline{r_1},\overline{r_2},\dots,\overline{r_D} \in \{0,1,\dots,L-1\})$ with

$1+(r_1+r_2L+r_3L^2+\cdots+r_DL^{D-1})$, we will get

$X=\sum_{t=0}^{D-1} I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$, where the matrix in 1D case is:

$$A=\begin{pmatrix} 010\cdots01 \\ 101\cdots00 \\ 010\cdots00 \\ \vdots\vdots\vdots\ddots\vdots\vdots \\ 000\cdots01 \\ 100\cdots10 \end{pmatrix}_{L\times L}$$



- ***PROOF***: we only need to prove that $I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$ contains (only) the adjacent relations at the $t+1$-th dimension. In fact

$$I_{L^{D-1-t}} \otimes A \otimes I_{L^t} = \begin{pmatrix} A\otimes I_{L^t} & 0 & \cdots & 0 \\ 0 & A\otimes I_{L^t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A\otimes I_{L^t} \end{pmatrix} \quad (L^{D-1-t} \text{ copies of } A\otimes I_{L^t})$$

Where

$$A\otimes I_{L^t} = \begin{pmatrix} O_{L^t} & I_{L^t} & 0 & \cdots & 0 & I_{L^t} \\ I_{L^t} & O_{L^t} & I_{L^t} & \cdots & 0 & 0 \\ 0 & I_{L^t} & O_t^t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & O_{L^t} & I_{L^t} \\ I_{L^t} & 0 & 0 & \cdots & I_{L^t} & O_{L^t} \end{pmatrix}$$

- Hence, we know that the (i,j) element of $I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$ is 1 if and only if the $(t+1)$-th digit of $i-1$ and $j-1$ are adjacent and other digits are the same $(1 \le i,j \le L^D)$. Thus $I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$ contains all the adjacent relations at the $(t+1)$-th dimension. So

$$X=\sum_{t=0}^{D-1} I_{L^{D-1-t}} \otimes A \otimes I_{L^t}. \qquad \square$$

05-2 Generalize to D-dimensional hypercubic lattice

- ***CLAIM***: For $j = 1 + (j_1 + j_2 L + j_3 L^2 + \cdots + j_D L^{D-1})$, define a vector $v_j \in \mathbb{C}^{L^D}$ where $(v_j)_\alpha = e^{\frac{\pi 2\sqrt{-1}}{L}\sum_{u=1}^{D} j_u \alpha_u}$ for $\alpha = 1 + (\alpha_1 + \alpha_2 L + \alpha_3 L^2 + \cdots + \alpha_D L^{D-1})$. Then

$$(I_{L^{D-1-t}} \otimes A \otimes I_{L^t}) v_j = (2\cos\frac{2\pi j_{i+1}}{L}) v_j$$

- ***PROOF***: The M-th element of LHS $(1 \le M \le L^D)$ is $\sum_{\alpha=1}^{L^D} (I_{L^{D-1-t}} \otimes A \otimes I_{L^t})_{M\alpha} (v_j)_\alpha$

$$= \sum_{\substack{1\le\alpha\le L^D \\ \text{the } (t+1)\text{-th digit of } \alpha-1 \text{ and } M-1 \text{ are adjacent} \\ \text{and other digits are the same}}} (v_j)_\alpha$$

$$= e^{\frac{2\pi\sqrt{-1}}{L}\left(\sum_{u\neq t+1} j_u\alpha_u + j_{t+1}(\alpha_{t+1}+1)\right)} + e^{\frac{2\pi\sqrt{-1}}{L}\left(\sum_{u\neq t+1} j_u\alpha_u + j_{t+1}(\alpha_{t+1}-1)\right)}$$

$$= \left(2\cos\frac{2\pi j_{t+1}}{L}\right)(v_j)_M$$

which is equal to the M-th element of RHS. Hence LHS=RHS. □

- ***CLAIM***: $v_1 \sim v_n$ are $\mathbb{C}$-linearly independent. By proving they are orthogonal under the standard inner product

06 A new expression for Z_n

- Firstly we compute $\beta^T v_{jl}^{-1}\beta$ for $\beta = (h, h, \ldots, h)^T \in \mathbb{C}^N$, note that $v_1 = (1,1,\ldots,1)^T$ is an eigenvector of v_{jl}, the corresponding eigenvalue is $a + \sqrt{-1}s - KD$. Hence:
  - Thus: $v_{jl}\beta = (a+\sqrt{-1}s - KD)\beta \qquad v_{jl}^{-1}\beta = (a+\sqrt{-1}s-KD)^{-1}\beta$

$$\beta^T v_{jl}^{-1}\beta = \beta^T (a+\sqrt{-1}s-KD)^{-1}\beta = Nh^2(a+\sqrt{-1}s-KD)^{-1}$$

- So we have:

$$Z_N = \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[\frac{1}{4}\beta^T v_{jl}^{-1}\beta + (a+\sqrt{-1}s)N - \frac{1}{2}\ln(\det(v_{jl}))]}$$

$$= \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[\frac{L^D h^2}{4(a+\sqrt{-1}s-DK)} + (a+\sqrt{-1}s)L^D - \frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(a+\sqrt{-1}s - K\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})]}$$

$$\boxed{= \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[L^D(\frac{h^2}{4(a+\sqrt{-1}s-DK)} + (a+\sqrt{-1}s)) - \frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(a+\sqrt{-1}s - K\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})]}}$$

07 Rewrite the partition function using complex variables

i. $s = \frac{K(z+D)-a}{\sqrt{-1}}, \frac{ds}{dz} = \frac{K}{\sqrt{-1}}$ $\quad \mathcal{L} = \left\{ z_0 \in \zeta \mid \Re(z_0) = \frac{a}{K} - D \right\}$

$$Z_N = \pi^{\frac{N}{2}} \int_{\mathcal{L}} \frac{1}{2\pi} \cdot \left(\frac{ds}{dz}\right) \cdot dz \cdot e^{L^D\left(\frac{h^2}{4Kz}+Kz+KD\right)} \cdot e^{-\frac{1}{2}\sum_{i_1=0}^{L-1} \cdots \sum_{i_D=0}^{L-1} \ln\left(Kz+KD-K\sum_{\alpha=1}^{D} \cos\frac{2\pi i_\alpha}{L}\right)}$$

$$\boxed{C_K = Ke^{L^D\left(KD-\frac{1}{2}\ln k\right)}}$$

$$= \pi^{\frac{N}{2}} \boxed{C_K} \int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{L^D \boxed{S(z)}}$$

$$\boxed{S(z) = Kz + \frac{h^2}{4Kz} - \frac{1}{2L^D} \sum_{i_1=0}^{L-1} \cdots \sum_{i_D=0}^{L-1} \ln\left(z + D - \sum_{\alpha=1}^{D} \cos\frac{2\pi i_\alpha}{L}\right)}$$

ii. $A = K,\ B = \frac{h^2}{4K}$, when $L \gg 1$ we know that:

$$\frac{1}{L^D} \sum_{i_1=0}^{L-1} \cdots \sum_{i_D=0}^{L-1} \ln\left(z + D - \sum_{\alpha=1}^{D} \cos\frac{2\pi i_\alpha}{L}\right) \cong \int_{[0,2\pi]^D} \ln\left(z + D - \sum_{\alpha=1}^{D} \cos\gamma_\alpha\right) \frac{1}{(2\pi)^D} d\gamma_1 \cdots d\gamma_D$$

i. Thus $g(z) = \frac{1}{(2\pi)^D} \int_{[0,2\pi]^D} \ln\left(z + D - \sum_{\alpha=1}^{D} \cos\gamma_\alpha\right) d\gamma_1 \cdots d\gamma_D$

iii. $S'(z)$ is computed by:

$$S'(z)\,|_{L\gg 1} = K - \frac{h^2}{4Kz^2} - \frac{1}{2(2\pi)^D} \int_{[0,2\pi]^D} \frac{d\gamma_1 \cdots d\gamma_D}{z + D - \sum_{\alpha=1}^{D} \cos\gamma_\alpha}$$

$$\boxed{K - \frac{h^2}{4Kz^2} - \frac{1}{2(2\pi)^D} \int_{[0,2\pi]^D} \frac{d\gamma_1 \cdots d\gamma_D}{z + D - \sum_{\alpha=1}^{D} \cos\gamma_\alpha} = 0}$$

08 Analyze the minimum of S(z)

- Note that $S'(z)$ is increasing for $z \in R^+$ and $\lim_{z\in\mathbb{R}^+, z\to+\infty} S'(z) = K > 0$
- ***Case 1***: $S'(z) > 0, \forall z \in \mathbb{R}^+$. Then $S(z)$ is monotonously increasing, so it's minimum is $z_0 = 0$.
- ***Case 2***: $\exists S'(z) \leq 0$. Since $(0, +\infty)$ is an open set, $\exists S'(z) < 0$. Thus, in this case, $S(z)$ firstly decreases, and then increases, admitting only one minimum.
- Whatever H is non-zero or not.

09 Compute the spontaneous magnetization

- Extra work: We have already learned that $M = \lim_{N\to\infty} \langle \frac{\sum_{j=1}^{N} \sigma_j}{N} \rangle$

which is equal to:

$$\lim_{N\to\infty} \frac{1}{N} \int_{\sum_{j=1}^{N} \sigma_j^2 = N} (\sum_{j=1}^{N} \sigma_j) \frac{e^{K\sum_{\langle jl \rangle} \sigma_i \sigma_j + h\sum_{j=1}^{N} \sigma_j}}{Z_N} d\vec{\sigma}$$

$$= \lim_{N\to\infty} \frac{1}{N} \cdot \frac{\frac{d}{dh} Z_N}{Z_N} = \lim_{N\to\infty} \frac{1}{N} \cdot \frac{d}{dh} \ln Z_N$$

$$\ln Z_N = \ln\left(\pi^{N/2} C_K\right) + \ln\left(\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}\right)$$

- Note that $\pi^{N/2} C_K = \pi^{N/2} K e^{N(KD - \frac{1}{2}\ln K)}$ is h-independent, and $g(z)$ is also h-independent

$$M = \lim_{N\to+\infty} \frac{1}{N} \frac{d}{dh} \ln Z_N = \lim_{N\to+\infty} \frac{\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} \frac{h}{2Kz} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}{\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}.$$

- In problem 8 we have proved that $S(z)$ admits only one minimum for $z \in R_{\geq 0}$. Set $a = K(D + Z_0)$ such that $\mathcal{L}$ becomes $\{z_0 \in \mathbb{C} | \mathrm{Re}(z) = z_0\}$, passing through z_0. Using saddle point approximation:

$$M = \frac{h}{2Kz_0}$$

- Substitute into $K - \frac{h^2}{4Kz^2} - \frac{1}{2} g'(z) = 0$:

$$K - KM^2 - \frac{1}{2} g'(z) = 0$$

- Hence

$$2K(1 - M^2) = g'(z_0)$$

10-1 Study the analytic properties and phase behavior

(i) Original expression of $g'(z)$:

$$g'(z) = \int_{[0,1]^D} \frac{d\theta_1 \cdots d\theta_D}{z + D - \sum_{\alpha=1}^{D} \cos(2\pi\theta_\alpha)}$$

LEMMA: For any positive real number t, we have:

$$\int_0^{+\infty} e^{-tx}\, dx = \frac{1}{t}$$

Thus

$$\begin{aligned} g'(z) &= \int_{[0,1]^D} d\theta_1 \cdots d\theta_D \int_0^{+\infty} e^{-(z+D-\sum_{\alpha=1}^{D}\cos(2\pi\theta_\alpha))x}\, dx \\ &= \int_0^{+\infty} dx\, e^{-(z+D)x} \int_{[0,1]^D} e^{x\sum_{\alpha=1}^{D}\cos(2\pi\theta_\alpha)}\, d\theta_1 \cdots d\theta_D \\ &= \int_0^{+\infty} dx\, e^{-(z+D)x} \left(\int_0^1 e^{x\cos(2\pi\theta)}\, d\theta \right)^D \end{aligned}$$

is analytic about D

10-2 Study the analytic properties and phase behavior

D>2, H→0, K↑

- ***LEMMA:*** **Lemma 5.** $\lim_{z\in\mathbb{R}^+, z\to 0^+} g'(z)$ *is finite (i.e.*

$$\int_0^{+\infty} dx\, e^{-Dx}\left(\int_0^1 e^{x\cos(2\pi\theta)}\, d\theta\right)^D$$

converges) iff $D > 2$.

- ***PROOF:***

$$\int_0^{+\infty} dx\, e^{-Dx}\left(\int_0^1 e^{x\cos(2\pi\theta)}\, d\theta\right)^D$$

$$= \frac{1}{(2\pi)^D}\int_0^{+\infty} dx\left(\int_0^{2\pi} e^{x(\cos\theta-1)}\, d\theta\right)^D$$

The only singular point is $+\infty$, thus we're going to study the asymptotic property of

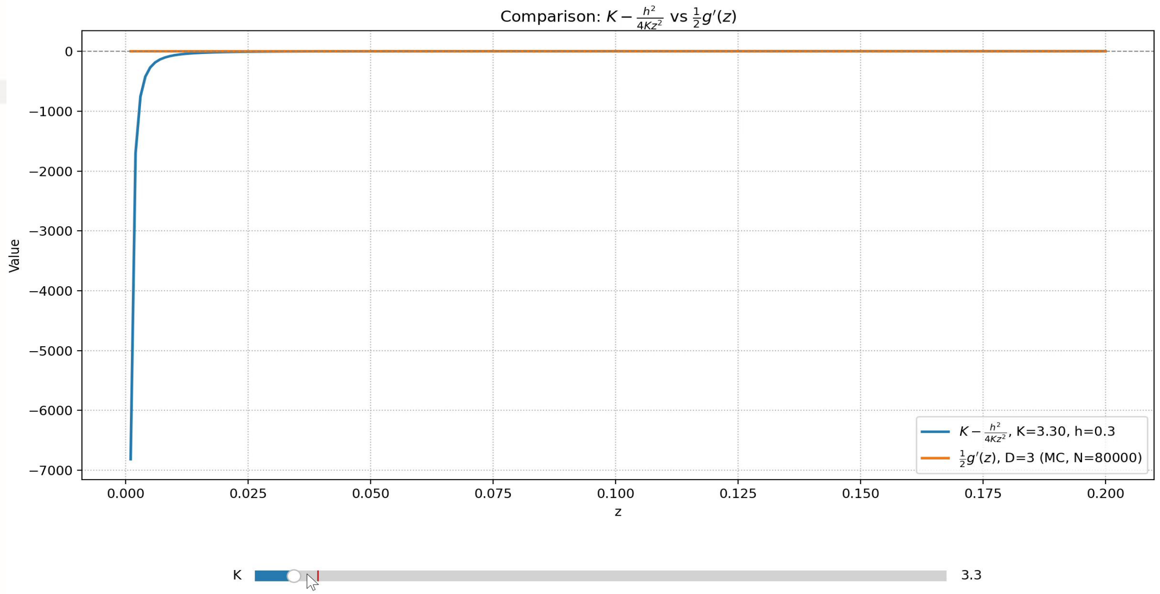
when x>>1, $$I(x) = \int_0^{\pi} e^{x(\cos\theta-1)}\, d\theta = \int_0^{\pi} e^{-\frac{1}{2}x\theta^2\left(1-\sum_{k=1}^{\infty}\frac{2(-1)^k}{(2k+2)!}\theta^{2k}\right)}\, d\theta \approx^{x\gg1} \int_0^{\pi} e^{-\frac{1}{2}x\theta^2}\, d\theta$$

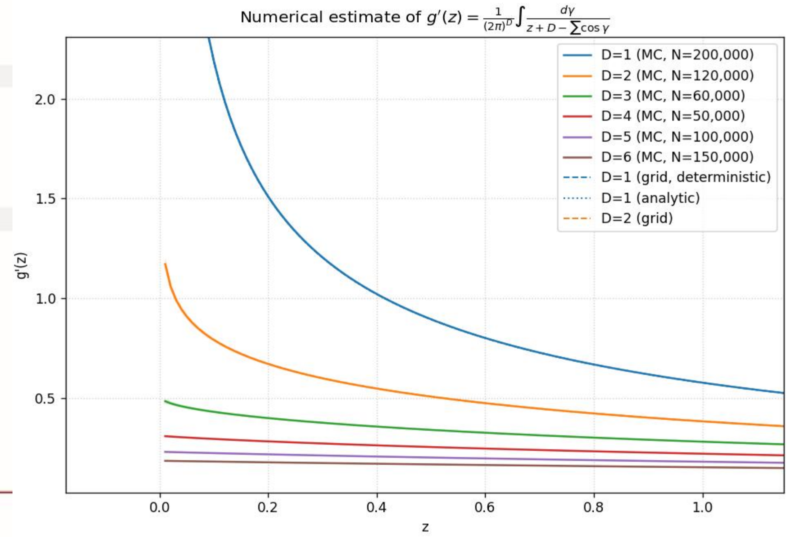
It is well known that $$\frac{\sqrt{\pi}}{2}\left(1-e^{-u^2}\right)^{1/2} < \int_0^u e^{-y^2}\, dy < \frac{\sqrt{\pi}}{2}\left(1-e^{-\frac{4u^2}{\pi}}\right)^{1/2}.$$

thus $$\sqrt{\frac{\pi}{2x}}\left(1-e^{-\pi^2}\right)^{1/2} < \int_0^{\pi} e^{-\frac{1}{2}x\theta^2}\, d\theta < \sqrt{\frac{\pi}{2x}}\left(1-e^{-4\pi}\right)^{1/2}.$$

Hence $I(x) \sim Cx^{-1/2} \quad (x \gg 1).$

Therefore, the integrand $\sim Cx^{-\frac{1}{2}D}(x >> 1)$ Hence the integral converges iff D > 2

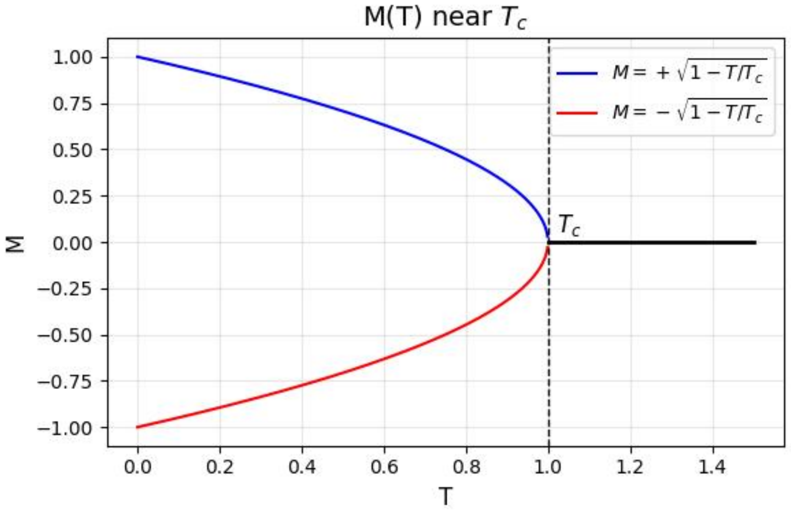




10-3 Study the analytic properties and phase behavior

- ***Case 1***: **$D \leq 2$**, then the integral diverges, $\lim_{z \to 0^+} g'(z) = +\infty$. Let z_0' be the intersection of $y = k$ and $y = \frac{1}{2} g'(z)$, is a positive real number independent on H and T, then $z_0 \geq z_0'$ always holds. Thus when $H \to 0$, $M = \frac{h}{2Kz_0} \to 0$ (denominator $\geq Kz_0'$ always holds), whatever T is.

- ***Case 2***: **$D > 2$**, then the integral converges (i.e. $g'(0)$ exists). Let $T_C = \frac{2J}{k_\beta g'(0)}$, then if $T > T_C$, define z_0'' the same way as z_0'. Thus we also have $\lim_{H \to 0} M = 0$.
  - If $T = T_C$, recall $2K(1 - M^2) = g'(z_0)$, $M^2 = 1 - \frac{g'(z_0)}{2K} = 1 - \frac{g'(z_0)}{g'(0)}$, and $z_0 \to 0^+$ when $H \to 0$.
  - If $T < T_C$, then $\lim_{H \to 0} M^2 = \lim_{H \to 0} 1 - \frac{g'(z_0)}{2K} = \lim_{H \to 0} 1 - \frac{g'(0)}{2K} = \lim_{H \to 0} 1 - \frac{2K_C}{2K} = \lim_{H \to 0} 1 - \frac{T}{T_C} = 1 - \frac{T}{T_C}$. Thus $M \sim \pm (1 - \frac{T}{T_C})^{\frac{1}{2}}$, for $T < T_C$.



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