Physics Project 2: Spherical Model

Jiaming Song, Xupeng Bai, Zhaoji Wang

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1 Question 1

Trivial

2 Question 2

According to the definition,

$$
\begin{aligned}
Z_N &= \int_\Gamma e^{-\beta E(\vec{\sigma})}\, d\vec{\sigma} \\
&= \int_{\sum_{j=1}^N \sigma_j^2 = N} e^{-\beta\left(-J\sum_{\langle j,l\rangle}\sigma_j\sigma_l - H\sum_{j=1}^N \sigma_j\right)}\, d\vec{\sigma} \\
&= \int_{\sum_{j=1}^N \sigma_j^2 = N} e^{K\sum_{\langle j,l\rangle}\sigma_j\sigma_l + h\sum_{j=1}^N \sigma_j}\, d\vec{\sigma}
\end{aligned}
$$

Denote $f(\vec{\sigma}) = e^{K\sum_{\langle j,l\rangle}\sigma_j\sigma_l + h\sum_{j=1}^N \sigma_j}$.

Lemma 1. *For a smooth function $g: \mathbb{R}^N \to \mathbb{R}$, $a \in \mathbb{R}^+$, the following equation holds:*

$$
\int_{\sum_{j=1}^N \sigma_j^2 = N} g(\vec{\sigma})\, d\vec{\sigma} = \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \int_{\mathbb{R}^N} d\sigma_1 \cdots d\sigma_N\, e^{(a+\sqrt{-1}s)\left(N - \sum_{j=1}^N \sigma_j^2\right)} g(\vec{\sigma})
$$

Proof. Using the Dirac delta function, we have

$$
\text{LHS} = \int_{\mathbb{R}^N} g(\vec{\sigma})\, \delta(\|\vec{\sigma}\|^2 - N)\, d\vec{\sigma}
$$

From the Fourier transform representation of the Dirac delta function, and introducing a convergence factor $a > 0$, we have

$$
\delta(x) = \int_{-\infty}^{+\infty} e^{-ax} \cdot e^{-\sqrt{-1}sx}\, \frac{ds}{2\pi}
$$

Substituting into the above expression, we obtain

$$
\int_{\mathbb{R}^N} g(\vec{\sigma})\, \delta(\|\vec{\sigma}\|^2 - N)\, d\vec{\sigma} = \int_{\mathbb{R}^N} d\vec{\sigma} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} g(\vec{\sigma})\, e^{(a+\sqrt{-1}s)\left(N - \sum_{j=1}^N \sigma_j^2\right)}
$$

Exchanging the order of integration gives the RHS. □

By the lemma, we have

$$
\begin{aligned}
Z_N &= \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \int_{\mathbb{R}^N} d\sigma_1 \cdots d\sigma_N\, \exp\left[(a+\sqrt{-1}s)\left(N - \sum_{j=1}^N \sigma_j^2\right) + K\sum_{\langle j,l\rangle}\sigma_j\sigma_l + h\sum_{j=1}^N \sigma_j\right] \\
&= \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \int_{\mathbb{R}^N} d\sigma_1 \cdots d\sigma_N\, \exp\left[(a+\sqrt{-1}s)N + h\sum_{j=1}^N \sigma_j - (a+\sqrt{-1}s)\sum_{j=1}^N \sigma_j^2 + K\sum_{\langle j,l\rangle}\sigma_j\sigma_l\right] \\
&= \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \int_{\mathbb{R}^N} d\sigma_1 \cdots d\sigma_N\, \exp\left[-\sum_{j,l} V_{jl}(s)\sigma_j\sigma_l + h\sum_{j=1}^N \sigma_j + (a+\sqrt{-1}s)N\right],
\end{aligned}
$$

where

$$V_{jl}(s) = \begin{cases} -\frac{1}{2}K + (a + \sqrt{-1}s)\delta_{jl}, & \text{if } \langle j, l \rangle \text{ are neighbors,} \\ (a + \sqrt{-1}s)\delta_{jl}, & \text{otherwise.} \end{cases}$$

The factor $\frac{1}{2}$ appears because both $V_{jl}(s)$ and $V_{lj}(s)$ are counted in the sum, while the original sum counts each pair only once.

3 Question 3

Lemma 2. *Let $a \in \mathbb{R}^+$ be sufficiently large, and let $B \in M_{n\times n}(\mathbb{C})$. Define $A = aI_n + B$. Then, for $\beta \in \mathbb{R}^N$, we have*

$$\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det(A)}}\, e^{\frac{1}{4}\beta^T A^{-1}\beta},$$

where the branch of $\sqrt{\det(A)}$ is chosen so that its real part is positive.

Proof. For a large enough, the real part of A is positive definite. Thus, there exists an invertible matrix P such that

$$A = P^T P.$$

Make the change of variables $\sigma \mapsto x = P\sigma$ (so $x \in \mathbb{R}^N$). Then

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma &= \int_{\mathbb{R}^N} e^{-\sigma^T P^T P\sigma+\beta^T\sigma}\, d\sigma \\
&= \int_{\mathbb{R}^N} e^{-x^T x+\beta^T P^{-1}x}\, |\det(P^{-1})|\, dx \\
&= \frac{1}{\sqrt{\det(A)}} \int_{\mathbb{R}^N} e^{-x^T x+\beta^T P^{-1}x}\, dx.
\end{aligned}$$

Completing the square:

$$-x^T x + \beta^T P^{-1}x = -\left\|x - \frac{1}{2}(P^{-1})^T\beta\right\|^2 + \frac{1}{4}\beta^T P^{-1}(P^{-1})^T\beta.$$

Since $(P^{-1})^T P^{-1} = A^{-1}$, this becomes

$$-\left\|x - \frac{1}{2}(P^{-1})^T\beta\right\|^2 + \frac{1}{4}\beta^T A^{-1}\beta.$$

Applying the translation $x \mapsto x + \frac{1}{2}(P^{-1})^T\beta$:

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{-x^T x+\beta^T P^{-1}x}\, dx &= e^{\frac{1}{4}\beta^T A^{-1}\beta} \int_{\mathbb{R}^N} e^{-\|x\|^2}\, dx \\
&= e^{\frac{1}{4}\beta^T A^{-1}\beta}\, \pi^{N/2},
\end{aligned}$$

where we used the standard Gaussian integral.

Combining the factors gives

$$\int_{\mathbb{R}^N} e^{-\sigma^T A\sigma+\beta^T\sigma}\, d\sigma = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det(A)}}\, e^{\frac{1}{4}\beta^T A^{-1}\beta}.$$

$\square$

Hence, according to the lemma, denote $\beta = (h, h, \ldots, h)^T \in \mathbb{R}^N$,

$$Z_N = \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \int_{\mathbb{R}^N} d\sigma \, e^{-\sigma^T V_{jl}(s)\sigma + \beta^T \sigma} \cdot e^{\left(a + \sqrt{-1}\, s\right) N}$$

$$= \int_{-\infty}^{+\infty} \frac{ds}{2\pi} \cdot \frac{\pi^{\frac{N}{2}}}{\sqrt{\det(V_{jl}(s))}} e^{\frac{1}{4}\beta^T V_{jl}^{-1}(s)\beta + \left(a + \sqrt{-1}\, s\right) N}$$

$$= \int_{-\infty}^{+\infty} \frac{\pi^{\frac{N}{2}} \, ds}{2\pi} \, e^{\frac{1}{4}\beta^T V_{jl}(s)\beta + \left(a + \sqrt{-1}\, s\right) N - \frac{1}{2} \ln(\det V_{jl}(s))}.$$

4 Question 4

Note that all the neighbors are $\langle 1,2\rangle, \langle 2,3\rangle, \dots, \langle N-1,N\rangle, \langle N,1\rangle$ (regardless of order).

Hence,

$$V_{jl}(s) = \begin{pmatrix} a+\sqrt{-1}s & -\frac{1}{2}K & & & -\frac{1}{2}K \\ -\frac{1}{2}K & a+\sqrt{-1}s & -\frac{1}{2}K & & \\ & -\frac{1}{2}K & \ddots & \ddots & \\ & & \ddots & a+\sqrt{-1}s & -\frac{1}{2}K \\ -\frac{1}{2}K & & & -\frac{1}{2}K & a+\sqrt{-1}s \end{pmatrix} \quad \text{(the blank space is zero)}$$

Let $\zeta = e^{\frac{2\pi\sqrt{-1}}{N}}$.

Note that for any $1 \le k \le N$,

$$\begin{pmatrix} a+\sqrt{-1}s & -\frac{1}{2}K & & & -\frac{1}{2}K \\ -\frac{1}{2}K & a+\sqrt{-1}s & -\frac{1}{2}K & & \\ & -\frac{1}{2}K & \ddots & \ddots & \\ & & \ddots & a+\sqrt{-1}s & -\frac{1}{2}K \\ -\frac{1}{2}K & & & -\frac{1}{2}K & a+\sqrt{-1}s \end{pmatrix} \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}$$

$$= \left[a+\sqrt{-1}s - \frac{1}{2}K\left(\zeta^k + \zeta^{-k}\right)\right] \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}$$

$$= \left(a+\sqrt{-1}s - K\cos\frac{2\pi k}{N}\right) \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}$$

and

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta & \zeta^2 & \cdots & \zeta^N \\ \zeta^2 & \zeta^4 & \cdots & \zeta^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{N-1} & \zeta^{2(N-1)} & \cdots & \zeta^{(N-1)N} \end{pmatrix} = \prod_{1\le i<j\le N} (\zeta^j - \zeta^i) \neq 0$$

thus the vectors

$$\begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{N-1} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \zeta^2 \\ \zeta^4 \\ \vdots \\ \zeta^{2(N-1)} \end{pmatrix}, \quad \cdots \quad, \quad \begin{pmatrix} 1 \\ \zeta^N \\ \zeta^{2N} \\ \vdots \\ \zeta^{(N-1)N} \end{pmatrix}$$

are N linearly independent vectors, spanning the whole space.

Hence the eigenvalues of V_{jl} are

$$a + \sqrt{-1}s - K \cos \frac{2\pi k}{N}, \quad (1 \leq k \leq N)$$

and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{(N-1)k} \end{pmatrix}, \quad (1 \leq k \leq N), \quad \text{with } \zeta = e^{\frac{2\pi\sqrt{-1}}{N}}.$$

5 Question 5

Lemma 3. *We define a graph over the Abelian group*

$$(\mathbb{Z}/L\mathbb{Z})^D,$$

there exists an edge (undirected) between a and b ($a, b \in (\mathbb{Z}/L\mathbb{Z})^D$) if and only if $b-a$ or $a-b$ is in

$$\{(\overline{1}, \overline{0}, \ldots, \overline{0}), (\overline{0}, \overline{1}, \overline{0}, \ldots, \overline{0}), \ldots, (\overline{0}, \overline{0}, \ldots, \overline{0}, \overline{1})\}.$$

Let the adjacency matrix of the graph be

$$X \in M_{L^D \times L^D}(\mathbb{C}).$$

Then if we label the vertices in this way:

$$\textit{label}(\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_D) \quad (r_1, r_2, \ldots, r_D \in \{0, 1, \ldots, L-1\})$$

with

$$1 + (r_1 + r_2 L + r_3 L^2 + \cdots + r_D L^{D-1})$$

we will get

$$X = \sum_{t=0}^{D-1} I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$$

(where $\otimes$ is the Kronecker product / tensor product)

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{L \times L}$$

is the matrix in $1D$ case.

Proof. we only need to prove that

$$I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$$

refers to the adjacent relations at the $(t+1)$-th dimension.

In fact,

$$I_{L^{D-1-t}} \otimes A \otimes I_{L^t} = \begin{pmatrix} A \otimes I_{L^t} & 0 & \cdots & 0 \\ 0 & A \otimes I_{L^t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \otimes I_{L^t} \end{pmatrix} \quad (L^{D-1-t} \text{ copies of } A \otimes I_{L^t})$$

and

$$A \otimes I_{L^t} = \begin{pmatrix} a_{11}I_{L^t} & a_{12}I_{L^t} & \cdots & a_{1L}I_{L^t} \\ a_{21}I_{L^t} & a_{22}I_{L^t} & \cdots & a_{2L}I_{L^t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{L1}I_{L^t} & a_{L2}I_{L^t} & \cdots & a_{LL}I_{L^t} \end{pmatrix}$$

with each a_{uv} $(1 \leq u, v \leq L)$ multiplied by I_{L^t} copies.

Moreover,

$$A \otimes I_{L^t} = \begin{pmatrix} O_{L^t} & I_{L^t} & 0 & \cdots & 0 & I_{L^t} \\ I_{L^t} & O_{L^t} & I_{L^t} & \cdots & 0 & 0 \\ 0 & I_{L^t} & O_t^t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & O_{L^t} & I_{L^t} \\ I_{L^t} & 0 & 0 & \cdots & I_{L^t} & O_{L^t} \end{pmatrix}$$

□

Hence, we know that the (i, j) element of $I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$ is 1 if and only if the $(t + 1)$-th digit of $i - 1$ and $j - 1$ are adjacent and other digits are the same $(1 \leq i, j \leq L^D)$, and others are 0. Thus $I_{L^{D-1-t}} \otimes A \otimes I_{L^t}$ contains all the adjacent relations at the $(t + 1)$-th dimension.

So

$$X = \sum_{t=0}^{D-1} I_{L^{D-1-t}} \otimes A \otimes I_{L^t}.$$

From

$$V_{jl} = (a + \sqrt{-1}s)I_N - \frac{1}{2}KX,$$

we know that

$$V_{jL} = (a + \sqrt{-1}s)I_N - \frac{1}{2}K \sum_{t=0}^{D-1} I_{L^{D-1-t}} \otimes A \otimes I_{L^t},$$

and the eigenvalues of A are

$$2\cos\frac{2\pi K}{L} \quad (1 \leq K \leq L).$$

Claim 1. *For*

$$j = 1 + (j_1 + j_2 L + \cdots + j_D L^{D-1}),$$

define a vector $v_j \in \mathbb{C}^{L^D}$ *where for*

$$\alpha = 1 + (\alpha_1 + \alpha_2 L + \cdots + \alpha_D L^{D-1}),$$

$$(v_j)_\alpha = e^{\frac{2\pi\sqrt{-1}}{L}\sum_{u=1}^{D} j_u \alpha_u}.$$

Then

$$(I_{L^{D-1-t}} \otimes A \otimes I_{L^t})\, v_j = \left(2\cos\frac{2\pi j_{t+1}}{L}\right) v_j.$$

Proof. The M-th element of LHS ($1 \le M \le L^D$) is

$$\sum_{\alpha=1}^{L^D} (I_{L^{D-1-t}} \otimes A \otimes I_{L^t})_{M,\alpha} (v_j)_\alpha.$$

$$= \sum_{\substack{1 \le \alpha \le L^D \\ \text{the } (t+1)\text{-th digit of } \alpha-1 \text{ and } M-1 \text{ are adjacent} \\ \text{and other digits are the same}}} (v_j)_\alpha$$

$$= e^{\frac{2\pi\sqrt{-1}}{L}\left(\sum_{u \ne t+1} j_u \alpha_u + j_{t+1}(\alpha_{t+1}+1)\right)} + e^{\frac{2\pi\sqrt{-1}}{L}\left(\sum_{u \ne t+1} j_u \alpha_u + j_{t+1}(\alpha_{t+1}-1)\right)},$$

$$= \left(2\cos\frac{2\pi j_{t+1}}{L}\right)(v_j)_M,$$

which equals the M-th element of RHS.

Hence LHS = RHS. □

Claim 2. *$v_1 \sim v_n$ are $\mathbb{C}$-linearly independent.*

Proof. we define the inner product over $\mathbb{C}^N$

$$\langle f, g\rangle = \sum_{u=1}^{N} f(u)\overline{g(u)}, \quad f(u), g(u) \text{ denotes the } u\text{-th element of } f \text{ and } g.$$

Then it is obvious that $\langle v_{M_1}, v_{M_2}\rangle = \sum_{u=1}^{N} e^{\sum_{k=1}^{D}(M_1)_k u_k - (M_2)_k u_k \cdot \frac{2\pi\sqrt{-1}}{L}}$

$$= \sum_{u=1}^{N} e^{\sum_{k=1}^{D}(M_1-M_2)_k u_k \cdot \frac{2\pi\sqrt{-1}}{L}}$$

$$= \sum_{u_1=0}^{L-1}\sum_{u_2=0}^{L-1}\cdots\sum_{u_D=0}^{L-1} e^{\sum_{k=1}^{D}(M_1-M_2)_k u_k \cdot \frac{2\pi\sqrt{-1}}{L}}$$

$$= \sum_{u_1=0}^{L-1}\sum_{u_2=0}^{L-1}\cdots\sum_{u_D=0}^{L-1}\prod_{k=1}^{D} e^{(M_1-M_2)_k u_k \cdot \frac{2\pi\sqrt{-1}}{L}}$$

$$= L^D \delta_{(M_1-M_2)_1,0}\delta_{(M_1-M_2)_2,0}\cdots\delta_{(M_1-M_2)_D,0}$$

$$= L^D \delta_{M_1,M_2} \text{ for any } 1 \le M_1, M_2 \le N.$$

Hence $\frac{v_1}{L^{D/2}}, \ldots, \frac{v_N}{L^{D/2}}$ forms an orthonormal basis of $(\mathbb{C}^N, \langle\cdot,\cdot\rangle)$, in particular, they are $\mathbb{C}$-linearly independent. □

Combining claim 1 and 2, we get to know that

$$V_{jl}v_M = \left(a + \sqrt{-1}s - K\sum_{t=0}^{D-1}\cos\frac{2\pi M_{t+1}}{L}\right)v_M$$

$v_1 \sim v_N$ are eigenvectors of V_{jl}, corresponding to the eigenvalues

$$\left(a + \sqrt{-1}s - K\sum_{t=0}^{D-1}\cos\frac{2\pi M_{t+1}}{L}\right)$$

In particular, the eigenvalues of V_{jl} are:

$$a + \sqrt{-1}s - K\sum_{t=1}^{D}\cos\frac{2\pi r_t}{L}, \text{ for } r_1, r_2, \ldots, r_D \in \{0, 1, \ldots, L-1\}.$$

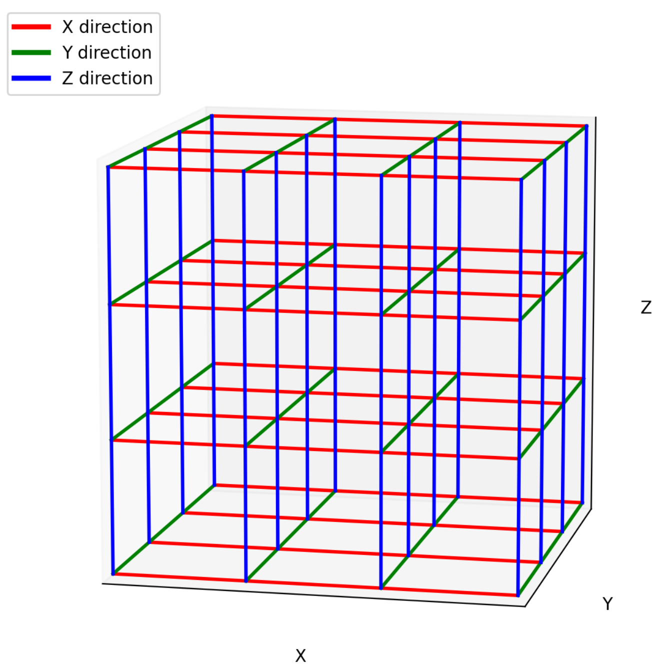

Figure 1: cube

6 Question 6

Firstly we compute $\beta^T V_{jl}^{-1}\beta$, for $\beta = (h, h, \cdots, h)^T \in \mathbb{C}^N$. Note that $v_1 = (1, 1, \cdots, 1)^T$ is an eigenvector of V_{jl} (which is proved in problem 5), the corresponding eigenvalue is $a + \sqrt{-1}s - KD$. Hence,

$$V_{jl}\beta = (a + \sqrt{-1}s - KD)\beta$$

$$V_{jl}^{-1}\beta = (a + \sqrt{-1}s - KD)^{-1}\beta$$

Thus,

$$\beta^T V_{jl}^{-1}\beta = \beta^T (a + \sqrt{-1}s - KD)^{-1}\beta = Nh^2(a + \sqrt{-1}s - KD)^{-1}$$

So we have

$$Z_N = \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[\frac{1}{4}\beta^T V_{jl}^{-1}\beta + (a+\sqrt{-1}s)N - \frac{1}{2}\ln(\det(V_{jl}))]}$$

$$= \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[\frac{L^D h^2}{4(a+\sqrt{-1}s-DK)} + (a+\sqrt{-1}s)L^D - \frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(a+\sqrt{-1}s - K\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})]}$$

$$= \pi^{\frac{N}{2}} \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{[L^D(\frac{h^2}{4(a+\sqrt{-1}s-DK)} + (a+\sqrt{-1}s)) - \frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(a+\sqrt{-1}s - K\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})]}$$

7 Question 7

(i).
$s = \frac{K(z+D)-a}{\sqrt{-1}}$, $\frac{ds}{dz} = \frac{K}{\sqrt{-1}}$

$$\mathcal{L} = \left\{ z_0 \in \mathbb{C} \mid \Re(z_0) = \frac{a}{K} - D \right\}$$

Hence,

$$Z_N = \pi^{\frac{N}{2}} \int_{\mathcal{L}} \frac{1}{2\pi} \cdot (\frac{ds}{dz}) \cdot dz \cdot e^{L^D(\frac{h^2}{4Kz}+Kz+KD)} \cdot e^{-\frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(Kz+KD-K\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})}$$

$$= \pi^{\frac{N}{2}} \int_{\mathcal{L}} \frac{Kdz}{2\pi\sqrt{-1}} e^{L^D(\frac{h^2}{4Kz}+Kz)} \cdot e^{L^D KD} \cdot e^{-\frac{1}{2}L^D \ln K} \cdot e^{-\frac{1}{2}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(z+D-\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})}$$

$$= \pi^{\frac{N}{2}} \cdot K e^{DL^D K - \frac{1}{2}L^D \ln k} \int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{L^D(\frac{h^2}{4Kz}+Kz-\frac{1}{2L^D}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(z+D-\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})}$$

$$= \pi^{\frac{N}{2}} C_K \int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{L^D S(z)}$$

where $C_k = Ke^{L^D(KD-\frac{1}{2}\ln k)}$.

$$S(z) = Kz + \frac{h^2}{4Kz} - \frac{1}{2L^D}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(z+D-\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})$$

(ii).
$A = K$, $B = \frac{h^2}{4K}$, and when $L \gg 1$ we know that (by definition of the definite integral):

$$\frac{1}{L^D}\sum_{i_1=0}^{L-1}\cdots\sum_{i_D=0}^{L-1}\ln(z+D-\sum_{\alpha=1}^{D}\cos\frac{2\pi i_\alpha}{L})$$

$$\cong \int_{[0,1]^D}\ln(z+D-\sum_{\alpha=1}^{D}\cos(2\pi\theta_\alpha))d\theta_1 d\theta_2\cdots d\theta_D$$

$$= \int_{[0,2\pi]^D}\ln(z+D-\sum_{\alpha=1}^{D}\cos\gamma_\alpha)\frac{1}{(2\pi)^D}d\gamma_1\cdots d\gamma_D$$

Thus,

$$g(z) = \frac{1}{(2\pi)^D}\int_{[0,2\pi]^D}\ln(z+D-\sum_{\alpha=1}^{D}\cos\gamma_\alpha)d\gamma_1\cdots d\gamma_D$$

(iii).

$$S'(z)\mid_{L\gg1} = K - \frac{h^2}{4Kz^2} - \frac{1}{2(2\pi)^D}\int_{[0,2\pi]^D}\frac{d\gamma_1\cdots d\gamma_D}{z+D-\sum_{\alpha=1}^{D}\cos\gamma_\alpha}$$

Thus the equation of state is:

$$K - \frac{h^2}{4Kz^2} - \frac{1}{2(2\pi)^D}\int_{[0,2\pi]^D}\frac{d\gamma_1\cdots d\gamma_D}{z+D-\sum_{\alpha=1}^{D}\cos\gamma_\alpha} = 0$$

8 Question 8

Note that $S'(z)$ is increasing for $z \in \mathbb{R}^+$

$$\lim_{z\in\mathbb{R}^+, z\to 0^+} S'(z) = -\infty$$

$$\lim_{z\in\mathbb{R}^+, z\to +\infty} S'(z) = K > 0$$

Hence $S'(z)$ admits only one zero for $z \in \mathbb{R}^+$, denote it as $z_0 > 0$, then $S(z)$ is decreasing for $z \in (0, z_0)$, and increasing for $z \in (z_0, +\infty)$. Thus, $S(z)$ admits only one minimum (and only exists one) for $z \in \mathbb{R}$, $z > 0$, say z_0.

9 Question 9

$$------EXTRAWORKBELOW------$$

We have already learned that

$$M = \lim_{N\to\infty} \left\langle \frac{\sum_{j=1}^{N} \sigma_j}{N} \right\rangle$$

$$\lim_{N\to\infty} \frac{1}{N} \int_{\sum_{j=1}^{N} \sigma_j^2 = N} (\sum_{j=1}^{N} \sigma_j) \frac{e^{K\sum_{\langle jl\rangle} \sigma_i\sigma_j + h\sum_{j=1}^{N} \sigma_j}}{Z_N} d\overrightarrow{\sigma}$$

$$= \lim_{N\to\infty} \frac{1}{N} \cdot \frac{1}{Z_N} \int_{\sum_{j=1}^{N} \sigma_j^2 = N} \frac{d}{dh} (e^{K\sum_{\langle jl\rangle} \sigma_i\sigma_j + h\sum_{j=1}^{N} \sigma_j}) d\overrightarrow{\sigma}$$

$$= \lim_{N\to\infty} \frac{1}{N} \cdot \frac{1}{Z_N} \cdot \frac{d}{dh} \int_{\sum_{j=1}^{N} \sigma_j^2 = N} e^{K\sum_{\langle jl\rangle} \sigma_i\sigma_j + h\sum_{j=1}^{N} \sigma_j} d\overrightarrow{\sigma}$$

(Using the original expression of Z_n)

$$= \lim_{N\to\infty} \frac{1}{N} \cdot \frac{\frac{d}{dh} Z_N}{Z_N} = \lim_{N\to\infty} \frac{1}{N} \cdot \frac{d}{dh} \ln Z_N$$

$$------EXTRAWORKABOVE------$$

$$\ln Z_N = \ln\left(\pi^{N/2} C_K\right) + \ln\left(\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}}\, e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}\right)$$

Note that $\pi^{N/2} C_K = \pi^{N/2} K\, e^{N(KD - \frac{1}{2}\ln K)}$ is h-independent, and $g(z)$ is also h-independent.

$$\frac{d}{dh} \ln Z_N = \frac{d}{dh} \ln\left(\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}\right)$$
$$= \frac{\displaystyle\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} \frac{Nh}{2Kz}\, e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}{\displaystyle\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}.$$

Hence

$$M = \lim_{N\to+\infty} \frac{1}{N} \frac{d}{dh} \ln Z_N = \lim_{N\to+\infty} \frac{\displaystyle\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} \frac{h}{2Kz} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}{\displaystyle\int_{\mathcal{L}} \frac{dz}{2\pi\sqrt{-1}} e^{N\left(Kz + \frac{h^2}{4Kz} - \frac{1}{2}g(z)\right)}}.$$

In problem 8 we have proved that $S(z)$ admits only one minimum for $z \in \mathbb{R}_{\geq 0}$ (Recall $\mathcal{L} : \left\{z_0 \in \mathbb{C} \mid \Re(z) = \frac{a}{K} - D\right\}$). Now we simply set $a = K(D + Z_0)$ such that $\mathcal{L}$ becomes $\{z_0 \in \mathbb{C} \mid \Re(z) = z_0\}$, which passes through z_0. Using saddle point approximation, we get to know that

$$M = \frac{h}{2Kz_0}$$

(when $N \gg 1$, the integrals are led mostly by z_0). Recall that $K - \frac{h^2}{4Kz^2} - \frac{1}{2}g'(z) = 0$. Substituting $M = \frac{h}{2Kz_0}$, we get

$$K - KM^2 - \frac{1}{2}g'(z) = 0$$

Thus

$$2K(1 - M^2) = g'(z_0)$$

$$------EXTRAWORKBELOW------$$

$$------EXTRAWORKABOVE------$$

10 Question 10

(i). Recall that the original expression we get of $g'(z)$ is:

$$g'(z) = \int_{[0,1]^D} \frac{d\theta_1 \cdots d\theta_D}{z + D - \sum_{\alpha=1}^{D} \cos(2\pi\theta_\alpha)}.$$

Lemma 4. *For any positive real number t, we have*

$$\int_0^{+\infty} e^{-tx}\, dx = \frac{1}{t}.$$

(or more generally we can let t be a complex number with positive real part)

Proof.

$$\int_0^{+\infty} e^{-tx}\, dx = -\frac{1}{t} e^{-tx}\Big|_{x=0}^{x=+\infty} = \frac{1}{t}.$$

Thus

$$g'(z) = \int_{[0,1]^D} d\theta_1 \cdots d\theta_D \int_0^{+\infty} e^{-(z+D-\sum_{\alpha=1}^{D}\cos(2\pi\theta_\alpha))x}\, dx$$

(exchanging the order of integration)

$$= \int_0^{+\infty} dx\, e^{-(z+D)x} \int_{[0,1]^D} e^{x\sum_{\alpha=1}^{D}\cos(2\pi\theta_\alpha)}\, d\theta_1 \cdots d\theta_D$$

$$= \int_0^{+\infty} dx\, e^{-(z+D)x} \left(\int_0^1 e^{x\cos(2\pi\theta)}\, d\theta \right)^D$$

is obviously an analytic function of D.

□

(ii). by plot

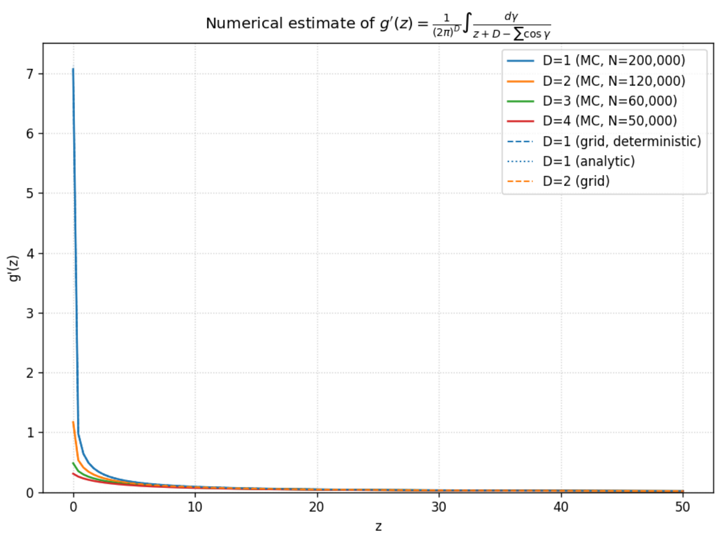

Figure 2: numerical estimation of $g'(z)$

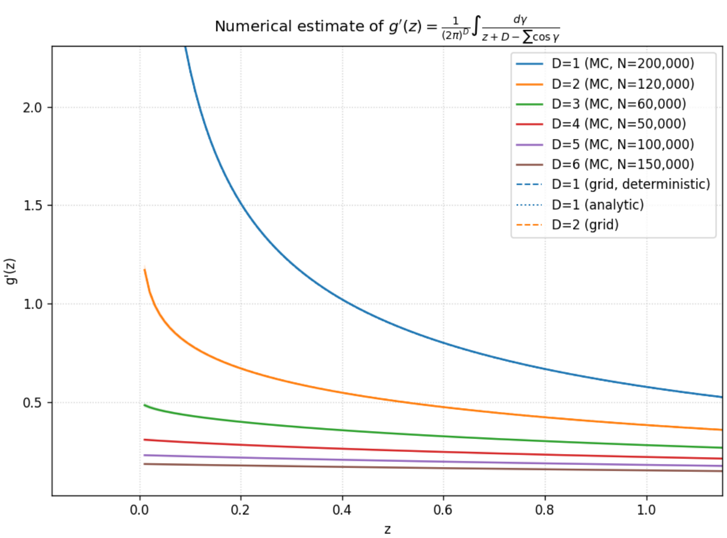

Figure 3: numerical estimation of $g'(z)$, zoom in

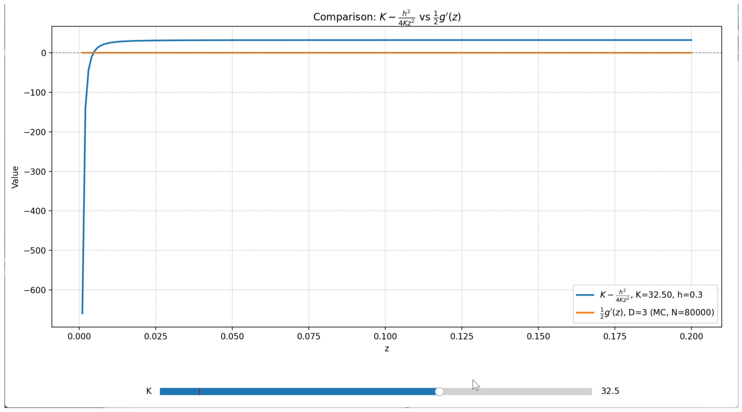

Figure 4: plot of $K - \frac{h^2}{4Kz^2}$ and $\frac{1}{2}g'(z)$, $K = 32.5$

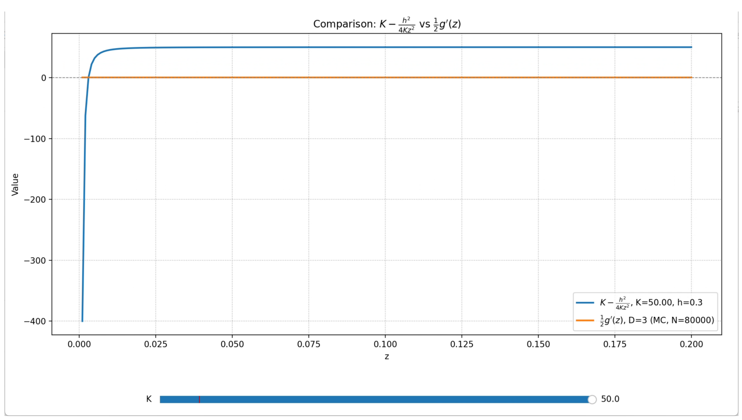

Figure 5: plot of $K - \frac{h^2}{4Kz^2}$ and $\frac{1}{2}g'(z)$,$K = 50$

(iii).

Lemma 5. $\lim_{z\in\mathbb{R}^+, z\to 0^+} g'(z)$ *is finite (i.e.*

$$\int_0^{+\infty} dx\, e^{-Dx} \left(\int_0^{2\pi} e^{x\cos(2\pi\theta)}\, d\theta\right)^D$$

converges) iff $D > 2$.

Proof.

$$\begin{aligned}
&\int_0^{+\infty} dx\, e^{-Dx} \left(\int_0^{1} e^{x\cos(2\pi\theta)}\, d\theta\right)^D \\
= &\int_0^{+\infty} dx\, e^{-Dx} \frac{1}{(2\pi)^D} \left(\int_0^{2\pi} e^{x\cos\theta}\, d\theta\right)^D \\
= &\frac{1}{(2\pi)^D} \int_0^{+\infty} dx \left(\int_0^{2\pi} e^{x(\cos\theta - 1)}\, d\theta\right)^D
\end{aligned}$$

The only singular point is $+\infty$, thus we're going to study the asymptotic property of

$$I(x) = \int_0^{\pi} e^{x(\cos\theta-1)}\, d\theta \quad \text{when } x \gg 1.$$

In fact, when $x \to \infty$, what really matters is when θ is near 0. Note that we have

$$\cos\theta - 1 = -\sum_{k=1}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}, \quad \forall \theta \in \mathbb{R}.$$

Thus

$$\begin{aligned}
I(x) &= \int_0^{\pi} e^{-\frac{1}{2}x\theta^2\left(1 - \sum_{k=1}^{\infty} \frac{2(-1)^k}{(2k+2)!}\theta^{2k}\right)}\, d\theta \\
&\approx^{x\gg 1} \int_0^{\pi} e^{-\frac{1}{2}x\theta^2}\, d\theta
\end{aligned}$$

It is well known that

$$\frac{\sqrt{\pi}}{2}\left(1 - e^{-u^2}\right)^{1/2} < \int_0^{u} e^{-y^2}\, dy < \frac{\sqrt{\pi}}{2}\left(1 - e^{-\frac{4u^2}{\pi}}\right)^{1/2}.$$

Thus

$$\sqrt{\frac{\pi}{2x}}\left(1 - e^{-\pi^2}\right)^{1/2} < \int_0^{\pi} e^{-\frac{1}{2}x\theta^2}\, d\theta < \sqrt{\frac{\pi}{2x}}\left(1 - e^{-4\pi}\right)^{1/2}.$$

Hence

$$I(x) \sim x^{-1/2} \quad (x \gg 1).$$

(The inequality above is easy to prove by multiple integral.)

Therefore, the integrand $\sim x^{-\frac{1}{2}D} (x >> 1)$ Hence the integral converges iff $D > 2$

□