Introduction to Commutative Algebra

M. F. ATIYAH I. G. MACDONALD

October 23, 2022

Contents

Introduction

Notation and Terminology

Chapter 1

Rings and Ideals

Chapter 2

Modules

Chapter 3

Rings and Modules of Fractions

Chapter 4

Primary Decomposition

The decomposition of an ideal into primary ideals is a traditional pillar of ideal theory. It provides the algebraic foundation for decomposing an algebraic variety into its irreducible components—although it is only fair to point out that the algebraic picture is more complicated than naïve geometry would suggest. From another point of view primary decomposition provides a generalization of the factorization of an integer as a product of prime-powers. In the modern treatment, with its emphasis on localization, primary decomposition is no longer such a central tool in the theory. It is still, however, of interest in itself and in this chapter we establish the classical uniqueness theorems.

The prototypes of commutative rings are $\mathbf{Z}$ and the ring of polynomials $k[x_1, \cdots, x_n]$ where k is a field; both these are unique factorization domains. This is not true of arbitrary commutative rings, even if they are integral domains (the classical example is the ring $\mathbf{Z}[\sqrt{-5}]$, in which the element 6 has two essentially distinct factorizations, $2 \cdot 3$ and $(1+\sqrt{-5})(1-\sqrt{-5})$. However, there is a generalized form of "unique factorization" of *ideals* (not of elements) in a wide class of rings (the Noetherian rings).

A prime ideal in a ring A is in some sense a generalization of a prime number. The corresponding generalization of a power of a prime number is a primary ideal. An ideal $\mathfrak{q}$ in a ring A is *primary* if $\mathfrak{q} \neq A$ and if

$$xy \in \mathfrak{q} \implies \text{either } x \in \mathfrak{q} \text{ or } y^n \in \mathfrak{q} \text{ for some } n > 0.$$

In other words,

$$\mathfrak{q} \text{ is primary} \iff A/\mathfrak{q} \neq 0 \text{ and every zero-divisor in } A/\mathfrak{q} \text{ is nilpotent.}$$

Clearly every prime ideal is primary. Also the contraction of a primary ideal is primary, for if $f : A \to B$ and if $\mathfrak{q}$ is a primary ideal in B , then $A/\mathfrak{q}^c$ is isomorphic to a subring of $B/\mathfrak{q}$.

Proposition 4.1. *Let $\mathfrak{q}$ be a primary ideal in a ring A. Then $r(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proof. By (1.8) it is enough to show that $\mathfrak{p} = r(\mathfrak{q})$ is prime. Let $xy \in r(\mathfrak{q})$, then $(xy)^m \in \mathfrak{q}$ for some $m > 0$, and therefore either $x^m \in \mathfrak{q}$ or $y^{mn} \in \mathfrak{q}$ for some $n > 0$; i.e., either $x \in r(\mathfrak{q})$ or $y \in r(\mathfrak{q})$. □

If $\mathfrak{p} = r(\mathfrak{q})$, then $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary.

Example. 1. *The primary ideals in* $\mathbf{Z}$ *are* (0) *and* (p^n), *where* p *is prime. For these are the only ideals in* $\mathbf{Z}$ *with prime radical, and it is immediately checked that they are primary.*

2. *Let* $A = k[x, y]$, $\mathfrak{q} = (x, y^2)$. *Then* $A/\mathfrak{q} \simeq k[y]/(y^2)$, *in which the zero-divisors are all the multiples of* y, *hence are nilpotent. Hence* $\mathfrak{q}$ *is primary, and its radical* $\mathfrak{p}$ *is* (x, y). *We have* $\mathfrak{p}^2 \subset \mathfrak{q} \subset \mathfrak{p}$ *(strict inclusions), so that a primary ideal is not necessarily a prime-power.*

3. *Conversely, a prime power* $\mathfrak{p}^n$ *is not necessarily primary, although its radical is the prime ideal* $\mathfrak{p}$. *For example, let* $A = k[x, y, z]/(xy - z^2)$ *and let* $\bar{x}, \bar{y}, \bar{z}$ *denote the images of* x, y, z *respectively in* A. *Then* $\mathfrak{p} = (\bar{x}, \bar{z})$ *is prime (since* $A/\mathfrak{p} \simeq k[y]$, *an integral domain); we have* $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$ *but* $x \notin \mathfrak{p}^2$ *and* $y \notin r(\mathfrak{p}^2) = \mathfrak{p}$; *hence* $\mathfrak{p}^2$ *is not primary. However, there is the following result:*

Proposition 4.2. *If* $r(\mathfrak{a})$ *is maximal, then* $\mathfrak{a}$ *is primary. In particular, the powers of a maximal ideal* $\mathfrak{m}$ *are* $\mathfrak{m}$*-primary.*

Proof. Let $r(\mathfrak{a}) = \mathfrak{m}$. The image of $\mathfrak{m}$ in $A/\mathfrak{a}$ is the nilradical of $A/\mathfrak{a}$, hence $A/\mathfrak{a}$ has only one prime ideal, by (1.8). Hence every element of $A/\mathfrak{a}$ is either a unit or nilpotent, and so every zero-divisor in $A/\mathfrak{a}$ is nilpotent. □

We are going to study presentations of an ideal as an intersection of primary ideals. First, a couple of lemmas:

lemma 4.3. *If* $\mathfrak{q}_i$, $(1 \leq i \leq n)$ *are* $\mathfrak{p}$*-primary, then* $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ *is* $\mathfrak{p}$*-primary.*

Proof. $r(\mathfrak{q}) = r(\bigcap_{i=1}^n \mathfrak{q}_i) = \cap r(\mathfrak{q}_i) = \mathfrak{p}$. Let $xy \in \mathfrak{q}$, $y \notin \mathfrak{q}$. Then for some i we have $xy \in \mathfrak{q}_i$ and $y \notin \mathfrak{q}_i$ hence $x \in \mathfrak{p}$. since $\mathfrak{q}_i$ is primary. □

lemma 4.4. *Let* q *be a* $\mathfrak{p}$*-primary ideal,* x *an element of* A. *Then*

1. *if* $x \in \mathfrak{q}$ *then* $(\mathfrak{q} : x) = (1)$;

2. *if* $x \notin \mathfrak{q}$ *then* $(\mathfrak{q} : x)$ *is* $\mathfrak{p}$*-primary, and therefore* $r(\mathfrak{q} : x) = \mathfrak{p}$;

3. *if* $x \notin \mathfrak{p}$ *then* $(\mathfrak{q} : x) = \mathfrak{q}$.

Proof. i) and iii) follow immediately from the definitions. ii): if $y \in (\mathfrak{q} : x)$ then $xy \in \mathfrak{q}$, hence (as $x \notin \mathfrak{q}$) we have $y \in \mathfrak{p}$ Hence $\mathfrak{q} \subseteq (\mathfrak{q} : x) \subseteq \mathfrak{p}$;taking radicals, we get $r(\mathfrak{q} : x) = \mathfrak{p}$. Let $yz \in (\mathfrak{q} : x)$ with $y \notin \mathfrak{p}$; then $xyz \in \mathfrak{q}$, hence $xz \in \mathfrak{q}$, hence $z \in (\mathfrak{q} : x)$. □

A primary decomposition of an ideal a in A is an expression of a as a finite intersection of primary ideals, say

$$a = \bigcap_{i=1}^{n} \mathfrak{q}_i \tag{4.1}$$

(In general such a primary decomposition need not exist; in this chapter we shall restrict our attention to ideals which have a primary decomposition.) If more