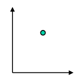
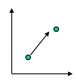
Representing Points and Vectors

- A 2D point: $p = \begin{bmatrix} x \\ y \end{bmatrix}$



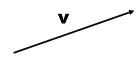
  - Represents a location with respect to some coordinate system
- A 2D vector: $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$



  - Represents a displacement from a position

Vectors

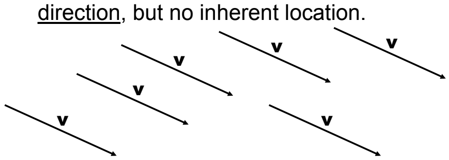
- Objects with <u>length</u> and <u>direction</u>.
- Represent motion between points in space.



- We will think of a vector as a <u>displacement</u> from one point to another.
- We use bold typeface to indicate vectors.

Vectors as Displacements

- As a displacement, a vector has <u>length</u> and <u>direction</u>, but no inherent location.



- It doesn't matter if you step one foot forward in Boston or in New York.

Vector Notation

- We represent vectors as a list of <u>components</u>.
- By convention, we write vectors as a <u>column matrix</u>:

$\mathbf{v} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ or $\mathbf{w} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$

- We can also write them as <u>row matrix</u>, indicated by $\mathbf{v}^T$:

$\mathbf{v}^T = [x_0, x_1]$ or $\mathbf{w}^T = [x_0, x_1, x_2]$

- All components of the <u>zero vector</u> **0** are 0.

Vector Operations

- Addition and multiplication by a scalar:
  - $\mathbf{a} + \mathbf{b}$
  - $s * \mathbf{a}$
- Examples:

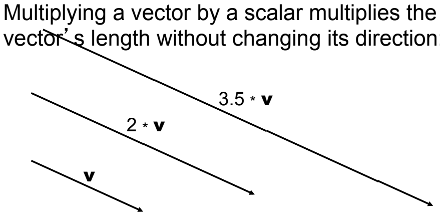
$\mathbf{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$6 * \mathbf{a} = \begin{bmatrix} 12 \\ 30 \end{bmatrix}$

Vector Scaling

- Multiplying a vector by a scalar multiplies the vector's length without changing its direction:



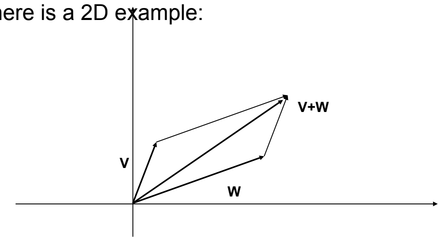
Vector scalar Multiplication

- Given $\mathbf{w}^T = [x_0, x_1, x_2]$ and scalars s and t
  $s * \mathbf{w}^T = [s * x_0, s * x_1, s * x_2]$
- Properties of Vector multiplication
  - Associative: (st)V = s(tV)
  - Multiplicative Identity: 1V = V
  - Scalar Distribution: (s+t)V = sV+tV
  - Vector Distribution: s(V+W) = sV+sW

Vector addition

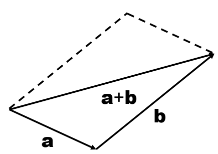
Paralleogram rule

- To visualize what vector addition is doing, here is a 2D example:



Vector Addition

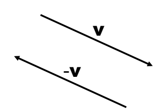
- Head to tail: At the end of **a**, place the start of **b**.



- **a + b = b + a**
- Components: $\mathbf{a} + \mathbf{b} = [a_0 + b_0, a_1 + b_1]^T$

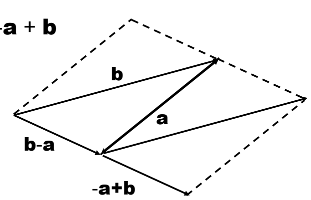
Unary Minus

- Algebraically, $-\mathbf{v} = (-1)\,\mathbf{v}$.
- The geometric interpretation of $-\mathbf{v}$ is a vector in the opposite direction:



Vector Subtraction

- **b - a = -a + b**



- Components: $\mathbf{b} - \mathbf{a} = [b_0 - a_0, b_1 - a_1]^T$

Linear Combinations

- A <u>linear combination</u> of m vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a vector of the form:
  $\boxed{\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m,}$
  where $a_1, a_2, \ldots, a_m$ are scalars.
- Example:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

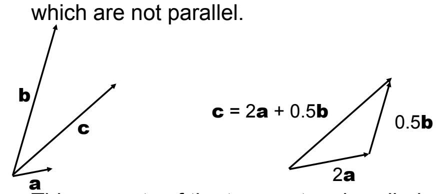
$$2\mathbf{v}_1 + 6\mathbf{v}_2 = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Vector Addition

- Given $\mathbf{v}^T = [x_0, x_1, x_2]$ and $\mathbf{w}^T = [y_0, y_1, y_2]$
  - $V+W = [x_0+y_0, x_1+y_1, x_2+y_2]$
- Properties of Vector addition
  - Commutative: V+W=W+V
  - Associative (U+V)+W = U+(V+W)
  - Additive Identity: V+0 = V
  - Additive Inverse: V+W = 0, W=-V

Linear Independence

- A 2D vector can be written as a linear combination of any two non-zero vectors which are not parallel.



- This property of the two vectors is called <u>linear independence</u> (i.e., not parallel)

Vector Basis

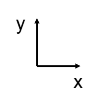
- Vectors which are linearly independent form a <u>basis</u>.
- The vectors are called <u>basis vectors</u>.
- Any vector can be expressed as linear combination of the basis vectors:

$$\mathbf{c} = a_c\mathbf{a} + b_c\mathbf{b}$$

- Note that the coefficients a_c and b_c are unique.

Orthogonal Basis

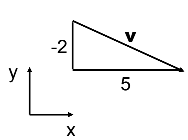
- If the two vectors are at right angles to each other, we call the it an <u>orthogonal basis</u>.
- If they are also vectors with length one, we call it an <u>orthonormal basis</u>.
- In 2D, we can use two such special vectors **x** and **y** to define the <u>Cartesian basis vectors</u>
- **x** = [1 0] **y** = [0 1]



The Basis Matters

- When we describe a vector, we need to agree on which basis we use.

$$\mathbf{v_1} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$



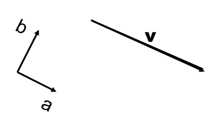
$$\mathbf{v}_1 = x_a\mathbf{x} + y_a\mathbf{y}$$

$$\mathbf{v}_1 = 5\mathbf{x} + (-2)\mathbf{y}$$

The Basis Matters

- Here is the same vector, expressed using a different vector basis.

$$\mathbf{v_2} = \begin{bmatrix} 5.39 \\ 0 \end{bmatrix}$$

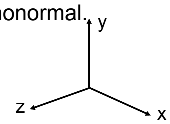


$$\mathbf{v}_2 = a_c\mathbf{a} + b_c\mathbf{b}$$

$$\mathbf{v}_2 = (5.39)\mathbf{a} + 0\mathbf{b}$$

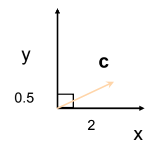
3D Cartesian Basis Vectors

- The Cartesian basis vectors in 3D are:
  - $\mathbf{x}$ = [1 0 0] $\mathbf{y}$ = [0 1 0] $\mathbf{z}$ = [0 0 1]
- They are orthonormal.



Cartisian

- Descarté
- One reason why cartesian coordinates are nice



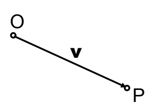
$$\mathbf{c} = 2\mathbf{x} + (0.5)\mathbf{y}$$

By Pythagorean theorem: length of **c** is:

$$\|\mathbf{c}\| = \sqrt{x_a^2 + y_a^2}$$

Vectors to Specify Locations

- Locations can be represented as displacements from another location.
- There is some understood origin location O from which all other locations are stored as offsets.



- Note that locations are not vectors!
- Locations are described using points.

Points and Vectors

- We normally say $\mathbf{v} = [3\ 2\ 7]^T$.
- Similarly, for a point we say $P = [3\ 2\ 7]^T$.
- But points and vectors are very different:
  - Points: have a location, but no direction.
  - Vectors: have direction, but no location.
- Point: A fixed location in a geometric world.
- Vector: Displacement between points in the world.

Operations on Vectors and Points

- Linear operations on vectors make sense, but the same operations do not make sense for points:
- Addition:
  - Vectors: concatenation of displacements.
  - Points: ?? (SLO + LA = ?)
- Multiplication by a scalar:
  - Vectors: increases displacement by a factor.
  - Points: ?? (What is 7 times the North Pole?)
- Zero element:
  - Vectors: The zero vector represents no displacement.
  - Points: ?? (Is there a zero point?)

Operations on Points

- However, some operations with points make sense.
- We can subtract two points. The result is the motion from one point to the other:

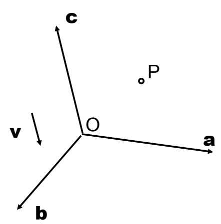
$$P - Q = \mathbf{v}$$

- We can also start with a point and move to another point:

$$Q + \mathbf{v} = P$$

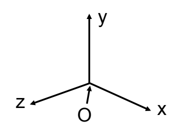
Coordinate Frames

- A coordinate system (also called a coordinate frame) consist of an origin point O and a basis of three linearly independent vectors **a**, **b**, **c**.



Standard Coordinate System

- The standard coordinate system in 3D is:
  - **x** = $[1\ 0\ 0]^T$ **y** = $[0\ 1\ 0]^T$ **z** = $[0\ 0\ 1]^T$ O = $[0\ 0\ 0]^T$



- Any 3D point $[a, b, c]^T$ can be written as:
  $[a, b, c]^T = O + a\,\mathbf{x} + b\,\mathbf{y} + c\,\mathbf{z}$.