

A quantum algorithm for estimating the determinant

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We present a quantum algorithm for estimating the matrix determinant based on quantum spectral sampling. The algorithm estimates the logarithm of the determinant of an $n \times n$ positive sparse matrix to an accuracy ϵ in time $\mathcal{O}(\log n/\epsilon^3)$, exponentially faster than previously existing classical or quantum algorithms that scale linearly in n . The quantum spectral sampling algorithm generalizes to estimating any quantity $\sum_j f(\lambda_j)$, where λ_j are the matrix eigenvalues. For example, the algorithm allows the efficient estimation of the partition function $Z(\beta) = \sum_j e^{-\beta E_j}$ of a Hamiltonian system with energy eigenvalues E_j , and of the entropy $S = -\sum_j p_j \log p_j$ of a density matrix with eigenvalues p_j .

Quantum computers can provide exponential speedups over classical computers for a variety of linear algebraic tasks, including fast Fourier transforms, finding eigenvectors and eigenvalues, and matrix inversion [1–4]. Here we provide a quantum spectral sampling algorithm, and apply it to a fundamental linear algebraic problem, that of estimating the determinant of a large sparse positive matrix. Estimating the value of the determinant of large matrices is a common procedure in linear algebra, with wide application in, e.g., data analysis and machine learning. Our algorithm is related to the power of a single qubit/qumode algorithm, as well as to quantum matrix inversion. Starting with a single quantum mode or a few quantum bits in a pure state, together with a quantum system in a fully mixed state, quantum phase estimation allows us to sample uniformly from the eigenvalues of Hermitian matrices. We use quantum phase estimation to sample from the eigenvalues λ_j of an $n \times n$ sparse, positive Hermitian matrix A , whose smallest eigenvalue is bounded below by λ_{\min} , and whose largest eigenvalue is bounded above by λ_{\max} , so that the condition number of the matrix is bounded by $\kappa = \lambda_{\max}/\lambda_{\min}$. For convenience, we rescale A so that $\lambda_{\max} = 1/2$ and $\lambda_{\min} = 1/2\kappa$ (see App. A). We then use Monte Carlo sampling to estimate its log-determinant, i.e. the quantity $\alpha := \log(\det A) = \sum_{j=1}^n \log \lambda_j$. Assuming A is encoded into a quantum random access memory [1, 5–11] or that it is efficiently simulable [12, 13], we show that α can be estimated with a relative error ϵ in a number of computational steps

$$\#_{\text{LgD}} \simeq \mathcal{O}\left(\frac{\kappa (\log \kappa)^2 s^2}{\epsilon^3} \log n \log^2(s^2/\epsilon)\right), \quad (1)$$

where s is the sparsity of the matrix A (the maximum number of nonzero elements for each row or column), while κ bounds its condition number (the ratio between its largest and smallest eigenvalues). The right-hand-side of (1) formally corresponds to the time complexity $T_{\text{LgD}}^{[\text{ora}]}$ of the procedure, as measured in an oracle model [14] that supplies the elements of A in response to appropriate queries (see App. B). Existing classical and quantum algorithms for estimating α all scale at least pro-

portionally to n , e.g. [15–23], whence our exponential-advantage claim. Our algorithm follows by merging the quantum spectral sampling with quantum phase estimation to give a Monte Carlo estimate of the determinant. Moreover, the quantum spectral sampling algorithm also allows one to estimate the sum of any function of the eigenvalues, $\sum_j f(\lambda_j)$, thereby allowing estimates of the partition function for Hamiltonian quantum systems and the entropy of a density matrix. Because it only requires a single purified quantum mode, or a small number of pure state qubits, quantum spectral sampling is particularly parsimonious in terms of the quantum resources used, and could be implemented effectively on near term noisy quantum information processors without quantum error correction.

We first describe the quantum spectral sampling algorithm in terms of estimating the determinant, and then generalize to sums of arbitrary functions of eigenvalues below. Suppose we are given a suitable description of the matrix A , for example, an algorithm for efficiently calculating its elements [12, 13], or a description of its elements stored in a quantum random access memory [1, 5], we can use standard techniques of quantum simulation [13, 14, 24–30] to enact the time evolution operator e^{-iAt} to accuracy η in a number of computational steps $N_{\text{qs}} = \mathcal{O}(\tau \log n \log^2(\tau/\eta)/\log \log(\tau/\eta))$ [14] (see also App. B), where $\tau = s^2 t \|A\|_m$ with s the maximum number of non-zero entries in each row/column of α and $\|A\|_m$ an upper bound to the modulus of the entries of α . In our case we need $t \simeq 1$ and with the rescaling of α , also $\|A\|_m$ is of order one. So, neglecting the $\log \log$ term, we can approximate $N_{\text{qs}} = \mathcal{O}(s^2 \log n \log^2(s^2/\eta))$.

We use this ability to apply the quantum phase estimation algorithm to an n -dimensional quantum system in the fully mixed state described by density matrix $\mathbb{1}/n$. The quantum phase algorithm [31, 32] is a digital analogue of von Neumann’s pointer variable model of measurement [33]. Von Neumann’s object in the pointer variable model was to construct a physical interaction between a system and a (continuous) pointer variable that correlates the energy eigenstates of α with a value of the pointer variable that contains an estimate of the corre-

sponding energy eigenvalue: a physical model of how to perform a von Neumann measurement. In particular, von Neumann considered the application of the Hamiltonian $A \otimes P$, where P is the momentum operator for the pointer variable, to the initial state $|\psi\rangle = \sum_j \psi_j |j\rangle |x=0\rangle$, where $A|j\rangle = \lambda_j |j\rangle$. Applying this Hamiltonian for time t results in the state $\sum_j \psi_j |j\rangle |x=\lambda_j t\rangle$. The power-of-a-single-quantum-mode algorithm encodes the pointer variable in the continuous variable state of a quantized harmonic oscillator, and then uses quantum simulation to enact the von Neumann pointer variable Hamiltonian.

The quantum phase estimation algorithm [31, 32] digitizes von Neumann's model by substituting an m -qubit register with 2^m states for the continuous pointer variable, and employing the discretized version of the momentum operator. Applying Hamiltonian quantum simulation for a time $t = O(1)$ [i.e., one iteration of the phase estimation that requires 2^m applications of a controlled e^{-iAt} plus a final quantum Fourier transform], then yields the state $\sum_j \psi_j |j\rangle |x=\tilde{\lambda}_j\rangle$, where $\tilde{\lambda}_j$ is an estimate of λ_j that is accurate to $\mathcal{O}(1/2^m)$, for a suitable choice of the initial state of the discretized pointer variable [32]. The quantum phase estimation algorithm takes $\mathcal{O}(2^m)$ steps to attain this accuracy. Applying the quantum phase estimation algorithm to the fully mixed state, and then measuring the pointer variable register allows us to sample uniformly from the eigenvalues λ_j of A , and to determine each eigenvalue to a root mean square error (rmse) accuracy $\delta\lambda_j \propto \mathcal{O}(1/2^m)$. The accuracy to which the quantum phase estimation algorithm determines $\log \lambda_j$ then goes as $\Delta(\log \lambda_j) = \delta\lambda_j |\frac{\partial \log \lambda_j}{\partial \lambda_j}| = \delta\lambda_j / \lambda_j \approx 1/(2^m \lambda_j) \leq \kappa/2^{m-1}$, where κ is the upper bound to the condition number as above, so that the quantum phase estimation rmse on α is $\Delta\alpha_{\text{qpe}} = \sqrt{\sum_j (\delta\lambda_j/\lambda_j)^2} \leq \sum_j |\delta\lambda_j/\lambda_j| \leq n\kappa/2^{m-1}$, where we used the triangle inequality (we are only interested in the order of magnitude). We then use Monte Carlo sampling to estimate the log-determinant α . If we take N_{mc} samples $\{\lambda_\ell\}$, then the estimate of α is $\alpha_{\text{est}} := \frac{n}{N_{\text{mc}}} \sum_{\ell=1}^{N_{\text{mc}}} \log \lambda_\ell$, which tends to α for large N_{mc} , with a statistical rmse $\Delta\alpha_{\text{mc}} = n\Delta/\sqrt{N_{\text{mc}}}$, where given $\mu := \frac{1}{n} \sum_{j=1}^n \log \lambda_j = \alpha/n$ the arithmetic mean of the logarithm of the eigenvalues of A , $\Delta^2 := \frac{1}{n} \sum_{j=1}^n (\log \lambda_j - \mu)^2$ is the associated variance (see App. A). The total rmse for α can finally be bounded by the sum of $\Delta\alpha_{\text{mc}}$ and $\Delta\alpha_{\text{qpe}}$, i.e. $\Delta\alpha_{\text{err}} \leq \Delta\alpha_{\text{qpe}} + \Delta\alpha_{\text{mc}}$ (see App. C), so the relative error is

$$\frac{\Delta\alpha_{\text{err}}}{|\alpha|} \leq \frac{n}{|\alpha|} \left(\frac{2\kappa}{2^m} + \frac{\Delta}{\sqrt{N_{\text{mc}}}} \right) = \frac{1}{|\mu|} \left(\frac{2\kappa}{2^m} + \frac{\Delta}{\sqrt{N_{\text{mc}}}} \right). \quad (2)$$

By setting $N_{\text{mc}} := (\frac{2\Delta}{|\mu|\epsilon})^2$ and $2^m = N_{\text{qpe}} := \frac{4\kappa}{|\mu|\epsilon}$, we ensure that $\frac{\Delta\alpha_{\text{err}}}{|\alpha|}$ is smaller than or equal to ϵ . With this choice the total number of steps to attain accuracy ϵ goes

as

$$\#_{\text{LgD}} \simeq N_{\text{qs}} N_{\text{qpe}} N_{\text{mc}} = \mathcal{O} \left(\frac{\kappa s^2 \Delta^2}{|\mu|^3 \epsilon^3} \log n \log^2(s^2/\eta) \right), \quad (3)$$

which reduces to Eq. (1) once we set $\eta \simeq \epsilon$ and link the factor $\Delta^2/|\mu|^3$ to the condition number κ via the inequalities $|\mu| \geq \log 2$ (since $\lambda_j \leq \frac{1}{2}$) and $\Delta \leq (\log \kappa)/2$ (since the largest Δ is when half the eigenvalues are $\lambda_{\min} = \frac{1}{2\kappa}$ and half are $\lambda_{\max} = \frac{1}{2}$), see App. A. We conclude by noticing that when running the algorithm we do not know *a priori* the values of μ and Δ which determine the relative error in Eq. (2), but as the Monte Carlo sampling progresses, we obtain estimates of μ and Δ from the sample mean and standard deviation. As we sample further (increasing N_{mc}), we obtain more accurate estimates of these quantities, and can increase the accuracy of the eigenvalue determination by increasing m , if needed, so that the error is not dominated neither by the statistical Monte Carlo nor by the systematic quantum phase estimation errors (see App. D).

Other applications

The quantum algorithm for estimating determinants shows the power of being able to sample from the eigen-spectrum of a matrix. The same method of supplementing the quantum phase estimation algorithm with Monte Carlo sampling applies to estimating $F := \sum_{j=1}^n f(\lambda_j)$ for any readily computable non-negative function f . For example, if we can implement the dynamics e^{-iAt} , where A is the Hamiltonian for a physical system, and $f(x) = e^{-\beta x}$, then these techniques allow us to estimate the partition function $Z(\beta) = \sum_j e^{-\beta E_j}$ of the Hamiltonian. Similarly, if we are given multiple copies of a density matrix ρ , we can use density matrix exponentiation [34] to perform transformations of the form $e^{-i\rho t}$ and to estimate the entropy $S = -\sum_j p_j \log p_j$, where p_j are the eigenvalues of ρ .

For estimating F for an arbitrary Eq. (2) is modified as

$$\frac{\Delta F_{\text{err}}}{|F|} \approx \frac{1}{\mu_f} \left(\frac{\sum_j |\dot{f}(\lambda_j)|}{2^m n} + \frac{\Delta_f}{\sqrt{N_{\text{mc}}}} \right) \leq \frac{1}{\mu_f} \left(\frac{|\dot{f}|_{\max}}{2^m} + \frac{\Delta_f}{\sqrt{N_{\text{mc}}}} \right), \quad (4)$$

where $\mu_f = (1/n) \sum_j f(\lambda_j)$ is the average of f , Δ_f^2 is its variance, $\dot{f} = df/dx$ is the derivative of f , and $|\dot{f}|_{\max}$ is the maximum magnitude of the derivative of f over the range of eigenvalues.

After the submission of this work we were made aware that a very similar algorithm appears as a subroutine for the quantum “supervised machine learning Gaussian processes” algorithm that was proposed in [35], see also [36].

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Appendix A: Notation

In this section we fix the notation and define the relevant quantities of the problem. Let A be a (strictly) positive $n \times n$ matrix which is s -sparse (i.e. in each row and column of A there are no more than s non-null matrix

elements). The log-determinant of A is defined as

$$\alpha := \log(\det[A]) = \sum_{j=1}^n \log \lambda_j , \quad (\text{A1})$$

with $\lambda_j \in \text{Sp}[A]$ being the j -th eigenvalue of A . We also introduce the mean value and the variance of the log spectrum of A ,

$$\mu := \frac{1}{n} \sum_{j=1}^n \log \lambda_j = \alpha/n , \quad (\text{A2})$$

$$\Delta^2 := \frac{1}{n} \sum_{j=1}^n (\log \lambda_j - \mu)^2 . \quad (\text{A3})$$

By proper rescaling we assume the spectrum $\text{Sp}[A]$ to be bounded as follows

$$\lambda_{\max} \geq \lambda_j \geq \lambda_{\min} , \quad \forall \lambda_j \in \text{Sp}[A] , \quad (\text{A4})$$

where we set

$$\lambda_{\max} := 1/2 , \quad \lambda_{\min} := 1/(2\kappa) , \quad (\text{A5})$$

the constant $\kappa \geq 1$ being an upper bound on the condition number $K(A)$ of the matrix, i.e.

$$\kappa \geq K(A) := \frac{\max_j \lambda_j}{\min_{j'} \lambda_{j'}} . \quad (\text{A6})$$

Under these conditions it hence follows that μ and Δ can be bounded via κ through the inequalities

$$\log 2 \leq |\mu| \leq \log(2\kappa) , \quad \Delta \leq \frac{\log \kappa}{2} . \quad (\text{A7})$$

Appendix B: Resource accounting

In this appendix, we detail the resource accounting of the quantum Hamiltonian simulation and of the quantum phase estimation algorithms.

The matrix A can be seen as an Hamiltonian generator acting on a quantum register M_0 of $n_Q = \log n$ qubits. Assume hence the possibility of coupling M_0 with a second register M_1 containing m qubits, via a series of control unitary operations $c - U(t)$ with $U(t)$ the unitary gate $U(t) = \exp[i t A]$. Under this premise we can use the quantum phase estimation algorithm (QPA) to determine the j -th eigenvalue of A via the following operations:

1. initialize M_0 and M_1 into the quantum state $|j\rangle \otimes |O\rangle_B$ with $|j\rangle$ the eigenvector of A associated with λ_j , and $|O\rangle = |0\rangle^{\otimes m}$;
2. act on M_1 via m Hadamard gates;
3. couple M_1 and M_0 via a sequence of 2^m , $c - U(t)$ transformations;

4. operate on M_1 with the inverse Quantum Fourier Transform (QFT^{-1}) and measure M_1 (no measurement on M_0 is needed).

This allows one to get the value of $t\lambda_j/(2\pi)$ with an accuracy $\simeq 2^{-m}$. In particular, by setting $t = 2\pi$, we see that 2^{-m} is exactly the error on the estimation of λ_j . Considering that the implementation of QFT^{-1} on a m qubit register requires $\mathcal{O}(m \log m)$ logical operations, the computational cost (number of logical operations required to complete the task) of the above procedure is

$$N_{\text{qpe}} \simeq 2^m \times \#_U + \mathcal{O}(m \log m) , \quad (\text{B1})$$

with $\#_U$ the computational cost of implementing a single $c - U(2\pi)$ operation (of course in case we have an oracle implementation of $c - U(2\pi)$ then we do not need to translate the 2^m calls in logical gates). The same procedure can be applied to a uniform random sampling of the spectrum of α by simply replacing the input state in step 1 with the density matrix $\mathbb{1}/n \otimes |O\rangle\langle O|$. Running the algorithm N times as needed by the log-determinant algorithm, will generate a random sequence $\tilde{\mathcal{S}} = (\lambda_{j_1}^{(1)}, \lambda_{j_2}^{(2)}, \dots, \lambda_{j_N}^{(N)})$, uniformly distributed, each result with an error $\delta \lambda \simeq 2^{-m}$. The total computational cost in this case is clearly given by N times the value (B1), i.e.

$$\#_{\text{LgD}} = N \times N_{\text{qpe}} \simeq N 2^m \times \#_U + \mathcal{O}(Nm \log m) . \quad (\text{B2})$$

From [14] we learn that in the case of an s -sparse Hamiltonian A one can implement the action of the $c - U(t = 2\pi)$ with an accuracy η on M_1 by means of

$$\#_{\text{query}(A)} \simeq \mathcal{O}\left(\tau \frac{\log(\tau/\eta)}{\log \log(\tau/\eta)}\right) , \quad (\text{B3})$$

queries of the Hamiltonian A and

$$\#_{2-\text{q}} \simeq \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right) , \quad (\text{B4})$$

extra 2-qubit operations, where given $\|A\|_m$ an upper bound on the absolute values of entries of the $n \times n$ matrix A , one has

$$\tau = 2\pi s^2 \|A\|_m . \quad (\text{B5})$$

We remark that Ref. [14] is based on an oracle model, where it is assumed that the user has access to a black-box that accepts a row index i and a number j between 1 and s , and returns the position and value of the j -th nonzero entry of A in row i . More precisely, Ref. [14] assumes the existence of a quantum circuit that, given the indexes i and j , gives a control-quantum phase gate whose value is proportional to the entry corresponding entry of A , i.e. something of the form

$$|i, j\rangle \otimes |b\rangle \longrightarrow (-1)^{b A_{ij} t} |i, j\rangle \otimes |b\rangle , \quad (\text{B6})$$

where $|i, j\rangle$ is a quantum register that enumerates all the non-zero entries of A , and $|b\rangle$ with $b \in \{0, 1\}$ is a control qubit. Each individual operation (B6) counts as a *query*¹; the *oracle complexity* of the algorithm [14] is given by $\#\text{query}(A)$ while the sum of this number and the number $\#\text{2-q}$ defines the *time complexity* of the algorithm, i.e.

$$\begin{aligned} T_{\text{U}}^{[\text{ora}]} &\simeq \#\text{query}(A) + \#\text{2-q} \\ &= \mathcal{O}\left(\tau \frac{\log(\tau/\eta)}{\log \log(\tau/\eta)}\right) + \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right) \\ &\simeq \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right), \end{aligned} \quad (\text{B7})$$

where in the final identity we dropped the smallest of the two contributions (i.e. $\#\text{query}(A)$). The global computational cost of implementing the action of $c - U(2\pi)$ can instead be estimated as

$$\begin{aligned} \#_{\text{U}} &\simeq \#\text{query}(A) \times \#_A + \#\text{2-q} \\ &= \mathcal{O}\left(\tau \frac{\log(\tau/\eta)}{\log \log(\tau/\eta)}\right) \times \#_A + \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right), \end{aligned} \quad (\text{B8})$$

where now $\#_A$ indicates the computational cost needed to implement a single query of A according to the above prescriptions, and the second contribution represents the extra cost in terms of two-qubit gates one has to pay in order to complete the task (see also Table 1 of Ref. [30]). From Eq. (B7) it immediately follows that within the oracle model of Ref. [14] the time-complexity of the log-determinant algorithm can be estimated as

$$\begin{aligned} T_{\text{LgD}}^{[\text{ora}]} &\simeq N 2^m T_{\text{U}}^{[\text{ora}]} + \mathcal{O}(Nm \log m) \\ &\simeq N 2^m \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right) + \mathcal{O}(Nm \log m) \\ &\simeq \mathcal{O}(N 2^m s^2 \log n \log^2(s^2/\eta)), \end{aligned} \quad (\text{B9})$$

where, in the first expression, we have omitted the order-one constant factor $2\pi \|A\|_m$ and a subleading $\log \log$ factor, while in the second line, the $\mathcal{O}(Nm \log m)$ contribution is dropped as it is clearly dominated by the leading term. Notice that if $\#_A$ does not grow faster than $\log n$, the first contribution in Eq. (B8) can be neglected. In this case, $\#_{\text{U}}$ is effectively determined by the second term alone and can be identified with the time complexity cost T_{U} given in Eq. (B7):

$$\#_{\text{U}} \simeq T_{\text{U}} \simeq \mathcal{O}\left(\tau \log n \frac{\log^2(\tau/\eta)}{\log \log(\tau/\eta)}\right), \quad (\text{B10})$$

so that Eq. (B2) yields

$$\#_{\text{LgD}} \simeq T_{\text{LgD}}^{[\text{ora}]} \simeq \mathcal{O}(N 2^m s^2 \log n \log^2(s^2/\eta)), \quad (\text{B11})$$

This condition can be met either by ensuring that the operation A is implemented efficiently [12, 13], or by assuming that the output of a prior quantum computation has been stored in a quantum random access memory [1, 5]. In the latter case, all relevant performance metrics – such as addressing time, average number of activated gates, and expected energy consumption – scale as $\log n$, even in the worst-case scenario where all memory entries are accessed in superposition.

Appendix C: Error budget

In this appendix, we give the details of the error budget calculated in the main text.

Consider the ideal case where the evaluation of the eigenvalues of the matrix A is affected solely by the statistical error associated with Monte Carlo sampling, with no additional errors introduced by the quantum phase estimation procedure. Under this condition, after N measurements, we shall obtain a string

$$\mathcal{S}_N := (\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_N}), \quad (\text{C1})$$

of random elements of $\text{Sp}[A]$. Since the process is ruled by a uniform probability distribution a good estimator of the log-determinant of A is obtained by taking the average of the logarithms of the elements of \mathcal{S}_N ,

$$\alpha_{\text{est}} = \frac{n}{N} \sum_{\ell=1}^N \log \lambda_{j_\ell} = \frac{n}{N} \sum_{j=1}^n N_j \log \lambda_j, \quad (\text{C2})$$

where N_j is the number of times that the j -th element of $\text{Sp}[A]$ (i.e. λ_j) appears in \mathcal{S}_N . Indeed, one can easily verify that, in the limit of large N , α_{est} approaches α with high probability, i.e.

$$\alpha_{\text{est}} \Big|_{N \gg 1} \longrightarrow \langle \alpha_{\text{est}} \rangle = \alpha, \quad (\text{C3})$$

with a statistical error (square root of the associated variance)

$$\Delta \alpha_{\text{mc}} := \sqrt{\langle (\alpha_{\text{est}} - \alpha)^2 \rangle} = n \frac{\Delta}{\sqrt{N}}, \quad (\text{C4})$$

with Δ defined in Eq. (A3) (we use the symbol $\langle \dots \rangle$ to represent the mean with respect to the stochastic process that generates the corresponding variable). We recall that the operational meaning of (C4) is provided by the Chebyshev inequality which states that the probability that the distance between the estimator α_{est} and α is larger than or equal to γ can be bounded by the inequality

$$P_{\text{rob}}(|\alpha_{\text{est}} - \alpha| \geq \gamma) \leq \frac{(\Delta \alpha_{\text{mc}})^2}{\gamma^2} = \frac{n^2 \Delta^2}{N \gamma^2}. \quad (\text{C5})$$

The relative error associated with (C4) is

$$\frac{\Delta \alpha_{\text{mc}}}{|\alpha|} = \frac{n \Delta}{|\alpha| \sqrt{N}} = \frac{1}{|\mu|} \frac{\Delta}{\sqrt{N}}, \quad (\text{C6})$$

¹ In a subsequent work [25] the same authors of Ref. [14] prove that a similar scaling can be achieved also when A is represented as sum of the form $\sum_k c_k V_k$ where V_k are unitary operations which can be physically implemented on the register either by some oracle or via some quantum circuit.

with μ defined in Eq. (A2). When the sampling of $\text{Sp}[A]$ is performed with some error, the procedure will produce N independent random outcomes which approximate the exact values λ_j 's up to an error $\delta\lambda$ which is fixed by the selected estimation procedure (e.g. the quantum phase estimation (QPA) algorithm). In the case of QPA, this is not a statistical error, but a systematic one: a truncation in the binary expansion. More precisely, during the sampling, each element $\lambda_j \in \text{Sp}[A]$, is replaced by an associated random variable $\lambda_j^{(k)}$ distributed according to some, for now unspecified, conditional probability function $P(k|j)$. We shall however assume that, for $j \in \{1, \dots, n\}$ the random values $\lambda_j^{(k)}$ does not differ from the j -th element of $\text{Sp}[A]$ more than a fixed threshold value $\delta\lambda$, i.e.

$$|\lambda_j^{(k)} - \lambda_j| < \delta\lambda, \quad \forall k. \quad (\text{C7})$$

To ensure the positivity of the variables $\lambda_j^{(k)}$, in what follows we shall also assume that $\delta\lambda$ is not larger than the lower bound λ_{\min} . In this scenario after N samplings, instead of getting the strings \mathcal{S}_N defined in the previous paragraphs, we actually get the noisy string

$$\tilde{\mathcal{S}}_N := (\lambda_{j_1}^{(k_1)}, \lambda_{j_2}^{(k_2)}, \dots, \lambda_{j_N}^{(k_N)}), \quad (\text{C8})$$

whose elements are produced with probabilities $P(j, k) := P(k|j)/n$ ($1/n$ being the probability of getting a j value from $\text{Sp}[A]$). For future purposes it is useful to establish a formal connection between (C1) and (C8): specifically we say that $\tilde{\mathcal{S}}_N$ is a noisy version of \mathcal{S}_N , if for all $\ell \in \{1, \dots, N\}$, the ℓ -th element of the former has the same j value of the corresponding ℓ -th element of the latter.

Using the string $\tilde{\mathcal{S}}_N$ we now wish to recover α using a simple averaging scheme. Specifically, we assume as

estimator the quantity

$$\tilde{\alpha}_{\text{est}} := \frac{n}{N} \sum_{\ell=1}^N \log \lambda_{j_\ell}^{(k_\ell)} = \frac{n}{N} \sum_{j=1}^n \sum_k N_{j,k} \log \lambda_j^{(k)}, \quad (\text{C9})$$

where $N_{j,k}$ counts the number of times the element $\lambda_j^{(k)}$ appears in the string $\tilde{\mathcal{S}}_N$. As in the case of (C3) for large N , $\tilde{\alpha}_{\text{est}}$ will approach its nominal expectation value, which in this case is given by

$$\begin{aligned} \tilde{\alpha}_{\text{est}} \Big|_{N \gg 1} \rightarrow \langle \tilde{\alpha}_{\text{est}} \rangle &= \tilde{\alpha} := n \sum_{j=1}^n \sum_k P(j, k) \log \lambda_j^{(k)} \\ &= \sum_{j=1}^n \sum_k P(k|j) \log \lambda_j^{(k)}. \end{aligned} \quad (\text{C10})$$

Of course in general $\tilde{\alpha}$ will differ from α , indicating that the estimator $\tilde{\alpha}_{\text{est}}$ is not un-biased (not even asymptotically). In this case, instead of considering the variance of $\tilde{\alpha}_{\text{est}}$ we hence need to focus on the corresponding mean square error, i.e. the quantity

$$\Delta^2 \alpha_{\text{err}} := \langle (\tilde{\alpha}_{\text{est}} - \alpha)^2 \rangle, \quad (\text{C11})$$

which by the Chebyshev inequality bounds the error probability of the estimation via the relation

$$P_{\text{rob}}(|\tilde{\alpha}_{\text{est}} - \alpha| \geq \gamma) \leq \frac{(\Delta \alpha_{\text{err}})^2}{\gamma^2}. \quad (\text{C12})$$

To evaluate $\Delta \alpha_{\text{err}}$ we relate it with the genuine Monte Carlo variance $\Delta \alpha_{\text{mc}}$ determined in Eq. (C4) and the statistical distance of the noise estimator (C9) from the estimator $a_{\text{est}}^{(N)}$ of a not-noisy string \mathcal{S}_N , i.e.

$$\Delta \alpha_{\text{qpe}} := \sqrt{\langle |\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)}|^2 \rangle}. \quad (\text{C13})$$

Specifically we write

$$\begin{aligned} \Delta^2 \alpha_{\text{err}} &= \langle (\tilde{\alpha}_{\text{est}} - \alpha)^2 \rangle = \langle (\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)} + a_{\text{est}}^{(N)} - \alpha)^2 \rangle \\ &= \langle (\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)})^2 \rangle + \langle (a_{\text{est}}^{(N)} - \alpha)^2 \rangle + 2 \langle (\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)}) (a_{\text{est}}^{(N)} - \alpha) \rangle \\ &\leq \langle |\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)}|^2 \rangle + \langle |a_{\text{est}}^{(N)} - \alpha|^2 \rangle + 2 \sqrt{\langle |\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)}|^2 \rangle} \sqrt{\langle |a_{\text{est}}^{(N)} - \alpha|^2 \rangle} \\ &= \left(\sqrt{\langle |\tilde{\alpha}_{\text{est}} - a_{\text{est}}^{(N)}|^2 \rangle} + \sqrt{\langle |a_{\text{est}}^{(N)} - \alpha|^2 \rangle} \right)^2 \implies \Delta \alpha_{\text{err}} \leq \Delta \alpha_{\text{qpe}} + \Delta \alpha_{\text{mc}}. \end{aligned} \quad (\text{C14})$$

Next we use the constraint (C7) to bound $\Delta \alpha_{\text{qpe}}$. As a first step, recalling the definition of κ , we notice that for $j \in \{1, \dots, n\}$ from (C7) we can write

$$\begin{aligned} |\log \lambda_j^{(k)} - \log \lambda_j| &\leq \begin{cases} \log(\lambda_j + \delta\lambda) - \log(\lambda_j) & \text{if } \lambda_j^{(k)} \geq \lambda_j, \\ \log(\lambda_j) - \log(\lambda_j - \delta\lambda) & \text{if } \lambda_j^{(k)} < \lambda_j, \end{cases} \leq \begin{cases} \delta\lambda/\lambda_j & \text{if } \lambda_j^{(k)} \geq \lambda_j, \\ \delta\lambda/(\lambda_j - \delta\lambda) & \text{if } \lambda_j^{(k)} < \lambda_j, \end{cases} \\ \frac{\delta\lambda}{\lambda_j - \delta\lambda} &\leq \frac{\delta\lambda}{\lambda_{\min} - \delta\lambda} = \frac{2\kappa\delta\lambda}{1 - 2\kappa\delta\lambda}, \end{aligned} \quad (\text{C15})$$

(the first inequality follows from the fact that $\log(x)$ is monotonically increasing in x ; the second instead follows from the fact that $\log(x)$ is concave with first derivative $1/x$; notice also that we explicitly made use of the fact that $\delta\lambda$ is smaller than λ_{\min} so that $\lambda_j - \delta\lambda \geq \lambda_{\min} - \delta\lambda > 0$). The condition $\delta\lambda \leq \lambda_{\min}$ can be enforced (possibly iteratively) through a suitable choice of m . From the above relation we can then establish the following inequality

$$\begin{aligned} |\tilde{\alpha}_{est} - \alpha_{est}| &= \frac{n}{N} \left| \sum_{\ell=1}^N \log \lambda_{j_\ell}^{(k_\ell)} - \log \lambda_{j_\ell} \right| \quad (\text{C16}) \\ &\leq \frac{n}{N} \sum_{\ell=1}^N \left| \log \lambda_{j_\ell}^{(k_\ell)} - \log \lambda_{j_\ell} \right| \leq \frac{2n\kappa\delta\lambda}{1 - 2\kappa\delta\lambda}, \end{aligned}$$

which replaced into (C13) leads to

$$\Delta\alpha_{qpe} \leq \frac{2n\kappa\delta\lambda}{1 - 2\kappa\delta\lambda}. \quad (\text{C17})$$

From (C14) and (C4) it hence follows that the Root Mean Square Error (RMSE) of the noisy estimator can be bounded as

$$\Delta\alpha_{err} \leq n \left(\frac{2\kappa\delta\lambda}{1 - 2\kappa\delta\lambda} + \frac{\Delta}{\sqrt{N}} \right). \quad (\text{C18})$$

The relative error is then bounded by

$$\begin{aligned} \frac{\Delta\alpha_{err}}{|\alpha|} &\leq \frac{1}{|\mu|} \left(\frac{2\kappa\delta\lambda}{1 - 2\kappa\delta\lambda} + \frac{\Delta}{\sqrt{N}} \right) \\ &= \frac{1}{|\mu|} \left(\frac{2\kappa 2^{-m}}{1 - 2\kappa 2^{-m}} + \frac{\Delta}{\sqrt{N}} \right), \quad (\text{C19}) \end{aligned}$$

where in the last line we set $\delta\lambda \simeq 2^{-m}$. We now impose that this error to be no larger than a certain target value ϵ by requiring

$$\frac{\Delta}{|\mu|\sqrt{N}} \leq \epsilon/2, \quad \frac{2\kappa 2^{-m}}{|\mu|(1 - 2\kappa 2^{-m})} \leq \epsilon/2, \quad (\text{C20})$$

which can be satisfied e.g. by taking

$$\begin{cases} N = N_{mc} := \left(\frac{2\Delta}{|\mu|\epsilon} \right)^2, \\ 2^m = N_{qpe} := 2\kappa \left(\frac{2}{|\mu|\epsilon} + 1 \right), \end{cases} \Rightarrow N 2^m \simeq \frac{16\kappa\Delta^2}{|\mu|^3\epsilon^3}. \quad (\text{C21})$$

where in writing the last identity we assumed $|\mu|\epsilon \ll 1$. Replacing this into Eq. (B11) finally gives

$$\#_{LgD} \simeq \mathcal{O} \left(\frac{16 \kappa s^2 \Delta^2}{|\mu|^3\epsilon^3} \log n \log^2(s^2/\eta) \right), \quad (\text{C22})$$

which (apart from irrelevant multiplicative constants) corresponds to Eq. (3) of the main text, once we equate all the errors: the error from the sampling, the one from quantum phase estimation and the one from the Hamiltonian simulation, namely $\epsilon = \eta$. Equation (C22) is the computational cost required to get the log-determinant of A with a relative error ϵ , i.e. $\frac{\Delta\alpha_{err}}{|\alpha|} \leq \epsilon$ via quantum random sampling that employs quantum phase estimation, quantum Hamiltonian simulation and Monte Carlo sampling.

Appendix D: Implementation of the method

In this appendix, we detail how the algorithm presented in the main text can be implemented in practice.

The starting point is the choice of the relative accuracy ϵ with which we want to estimate α . This, given the sparsity s and the dimension n of the matrix A allows us to set up the Hamiltonian simulation algorithm. We start with seed values of N_{mc} and m and start to (randomly) estimate N_{mc} eigenvalues λ_j using the algorithm detailed in the main text to accuracy ϵ . We then check whether the seed m is sufficiently large by recalling that it should be larger than $\log_2[4\kappa/(\epsilon|\mu|)]$, where κ and μ can be estimated from the data. If m is too small, we must start from scratch with a larger m , which just introduces a multiplicative constant overhead. (Alternatively, we can use the quantum phase estimation only for the least significant digits.) Then, we repeat the whole procedure N_{mc} times, with $N_{mc} \simeq (2\Delta/|\mu|\epsilon)^2$, where again Δ and μ can be estimated from the data. Since the error $\Delta\alpha_{qpe}$ is a systematic (not statistical), the final estimate of $|\alpha|$ will be affected by an error bar which is skewed towards smaller values: the quantum phase estimation basically gives a truncation of the estimated phase in the binary representation.