Module 3

Partial Differential Equation and its application

Unit: 3

Subject Name: Mathematics-III

Subject Code: AAS0301A

B Tech 3rd Sem

Dr. Kunti Mishra

Department of Mathematics

Brief Introduction of Faculty

Dr. Kunti Mishra
Assistant Professor
Department of Mathematics

Qualifications :
M.Sc.(Maths), M. Tech.(Gold Medalist) in Applied and Computational Mathematics, Ph.D

Ph.D. Thesis : Some Investigations in Fractal Theory
International Journal Publications: 7
International Conference Papers: 7
Area of Interests: Fixed Point Theory, Fractals

Evaluation Scheme

NOIDA INSTITUTE OF ENGINEERING & TECHNOLOGY, GREATER NOIDA
(An Autonomous Institute)

B. TECH (CSE)
EVALUATION SCHEME
SEMESTER-III

Sl. No.	Subject Codes	Subject Name	Periods			Evaluation Schemes				End Semester		Total	Credit
			L	T	P	CT	TA	TOTAL	PS	TE	PE		
WEEKS COMPULSORY INDUCTION PROGRAM													
1	AAS0301A	Engineering Mathematics III	3	1	0	30	20	50		100		150	4
2	ACSE0304	Discrete Structures	3	0	0	30	20	50		100		150	3
3	ACSE0306	Digital Logic & Circuit Design	3	0	0	30	20	50		100		150	3
4	ACSE0301	Data Structures	3	1	0	30	20	50		100		150	4
5	ACSE0302	Object Oriented Techniques using Java	3	0	0	30	20	50		100		150	3
6	ACSE0305	Computer Organization & Architecture	3	0	0	30	20	50		100		150	3
7	ACSE0353	Digital Logic & Circuit Design Lab	0	0	2				25		25	50	1
8	ACSE0351	Data Structures Lab	0	0	2				25		25	50	1
9	ACSE0352	Object Oriented Techniques using Java Lab	0	0	2				25		25	50	1
10	ACSE0354	Internship Assessment-I	0	0	2				50			50	1
11	ANC0301 / ANC0302	Cyber Security*/ Environmental Science*(Non Credit)	2	0	0	30	20	50		50		100	0
12		MOOCs (For B.Tech. Hons. Degree)											
		GRAND TOTAL										1100	24

Syllabus

Unit-1 (Complex Variable: Differentiation)

Limit, Continuity and differentiability, Functions of complex variable, Analytic functions, Cauchy- Riemann equations (Cartesian and Polar form), Harmonic function, Method to find Analytic functions, Conformal mapping, Mobius transformation and their properties.

Unit-2 (Complex Variable: Integration)

Complex integrals, Contour integrals, Cauchy- Goursat theorem, Cauchy integral formula, Taylor's Series, Laurent series, Liouville's Theorem, Singularities, zero of analytic function, Residues, Method of finding residues, Cauchy Residue's theorem, Evaluation of real integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$ and $\int_{-\infty}^{\infty} f(x)\, dx$

Syllabus

Unit-3 (Partial Differential Equation and its Applications)

Introduction of partial differential equations, Second order linear partial differential equations with constant coefficients. Classification of second order partial differential equations, Method of separation of variables for solving partial differential equations, Solution of one and two dimensional wave and heat conduction equations.

Unit-4 (Numerical Techniques)

Error analysis, Zeroes of transcendental and polynomial equations using Bisection method, Regula-falsi method and Newton-Raphson method, Interpolation: Finite differences, Newton's forward and backward interpolation, Lagrange's and Newton's divided difference formula for unequal intervals. Solution of system of linear equations, Crout's method, Gauss- Seidel method. Numerical integration: Trapezoidal rule, Simpson's one third and three-eight rules, Solution of 1^{st} order ordinary differential equations by fourth-order Runge- Kutta methods.

Syllabus

Unit-5 (Aptitude-III)

Time & Work, Pipe & Cistern, Time, Speed & Distance, Boat & Stream, Sitting Arrangement, Clock & Calendar.

Branch wise Application

- **Computer graphics & Vision**
- **Image analysis**

Course Objective

The objective of this course is to familiarize the engineers with concept of function of complex variables, Partial differential equations & their applications, Numerical techniques for various mathematical tasks and numerical aptitude. It aims to show case the students with standard concepts and tools from B. Tech to deal with advanced level of mathematics and applications that would be essential for their disciplines. The students will learn:

- The idea of function of complex variables and analytic functions.
- The idea of concepts of complex functions for evaluation of definite integrals
- The concept of partial differential equation to solve partial differential and its applications.
- The concept of finding roots by numerical method, interpolation and numerical methods for system of linear equations, definite integral and 1st order ordinary differential equations.
- The concept of problems based on Time & Work, Pipe & Cistern, Time, Speed & Distance, Boat & Stream, Sitting Arrangement, Clock & Calendar.

Course Outcome(COs)

CO1: Apply the working methods of complex functions for finding analytic functions.

CO2: Apply the concepts of complex functions for finding Taylor's series, Laurent's series and evaluation of definite integrals.

CO3: Apply the concept of partial differential equation to solve partial differential Equations and problems concerned with partial differential equations

CO4: Apply the concept of numerical techniques to evaluate the zeroes of the Equation, concept of interpolation and numerical methods for various mathematical operations and tasks, such as integration, the solution of linear system of equations and the solution of differential equation.

CO5: Solve the problems of Time & Work, Pipe & Cistern, Time, Speed & Distance, Boat & Stream, Sitting Arrangement , Clock & Calendar.

Program Outcomes(POs)

S.No	Program Outcomes (POs)
PO 1	Engineering Knowledge
PO 2	Problem Analysis
PO 3	Design/Development of Solutions
PO 4	Conduct Investigations of Complex Problems
PO 5	Modern Tool Usage
PO 6	The Engineer & Society
PO 7	Environment and Sustainability
PO 8	Ethics
PO 9	Individual & Team Work
PO 10	Communication
PO 11	Project Management & Finance
PO 12	Lifelong Learning

CO-PO Mapping(CO3)

Sr. No	Course Outcome	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
1	CO1	H	H	H	H	L	L	L	L	L	L	L	M
2	CO2	H	H	H	H	L	L	L	L	L	L	M	M
3	CO3	H	H	H	H	L	L	L	L	L	L	M	M
4	CO4	H	H	H	H	L	L	L	L	L	L	L	M
5	CO5	H	H	H	H	L	L	L	L	L	L	M	M

*L= Low *M= Medium *H= High

PSO

PSO	Program Specific Outcomes (PSOs)
PSO 1	The ability to identify, analyze real world problems and design their ethical solutions using artificial intelligence, robotics, virtual/augmented reality, data analytics, block chain technology, and cloud computing.
PSO 2	The ability to design and develop the hardware sensor devices and related interfacing software systems for solving complex engineering problems.
PSO 3	:The ability to understand inter disciplinary computing techniques and to apply them in the design of advanced computing.

CO-PSO Mapping(CO3)

CO	PSO 1	PSO 2	PSO 3
CO1	H	L	M
CO2	L	M	L
CO3	M	M	M
CO4	H	M	M
CO5	H	M	M

*L= Low *M= Medium *H= High

Program Educational Objectives(PEOs)

PEO-1: To have an excellent scientific and engineering breadth so as to comprehend, analyze, design and provide sustainable solutions for real-life problems using state-of-the-art technologies.

PEO-2: To have a successful career in industries, to pursue higher studies or to support entrepreneurial endeavors and to face the global challenges.

PEO-3: To have an effective communication skills, professional attitude, ethical values and a desire to learn specific knowledge in emerging trends, technologies for research, innovation and product development and contribution to society.

PEO-4: To have life-long learning for up-skilling and re-skilling for successful professional career as engineer, scientist, entrepreneur and bureaucrat for betterment of society.

Result Analysis

Branch: CS, IT

Result: 100%

End Semester Question Paper Template

Printed page: .... **Subject Code: ....**

Roll No:

<u>NOIDA INSTITUTE OF ENGINEERING AND TECHNOLOGY ,GREATER NOIDA</u>

(An Autonomous Institute Affiliated to AKTU, Lucknow)

B.Tech/B.Voc./MBA/MCA/M.Tech (Integrated)

(SEM: ..... THEORY EXAMINATION (2020-2021)

Subject ....

Time: 3 Hours **Max. Marks:100**

<u>General Instructions:</u>

- All questions are compulsory. Answers should be brief and to the point.
- This Question paper consists of ....pages & ...8.........questions.
- It comprises of three Sections, A, B, and C. You are to attempt all the sections.
- <u>Section A</u> -Question No- 1 is objective type questions carrying 1 mark each, Question No- 2 is very short answer type carrying 2 mark each. You are expected to answer them as directed.
- <u>Section B</u> - Question No-3 is Long answer type -I questions with external choice carrying 6 marks each. You need to attempt any five out of seven questions given.
- Section C - Question No. 4-8 are Long answer type –II (within unit choice) questions carrying 10 marks each. You need to attempt any one part <u>*a or b.*</u>
- Students are instructed to cross the blank sheets before handing over the answer sheet to the invigilator.
- No sheet should be left blank. Any written material after a blank sheet will not be evaluated/checked.

End Semester Question Paper Template

		SECTION – A		CO
1.	Attempt all parts-		[10×1=10]	
	1-a.	Question-	(1)	
	1-b.	Question-	(1)	
	1-c.	Question-	(1)	
	1-d.	Question-	(1)	
	1-e.	Question-	(1)	
	1-f.	Question-	(1)	
	1-g.	Question-	(1)	
	1-h.	Question-	(1)	
	1-i.	Question-	(1)	
	1-j.	Question-	(1)	

End Semester Question Paper Template

2.	**Attempt all parts-**		**[5×2=10]**	**CO**
	2-a.	Question-	(2)	
	2-b.	Question-	(2)	
	2-c.	Question-	(2)	
	2-d.	Question-	(2)	
	2-e.	Question-	(2)	
SECTION – B				CO
3.	Answer any five of the following-		[5×6=30]	
	3-a.	Question-	(6)	
	3-b.	Question-	(6)	
	3-c.	Question-	(6)	
	3-d.	Question-	(6)	
	3-e.	Question-	(6)	
	3-f.	Question-	(6)	
	3-g.	Question-	(6)	
SECTION – C				CO

End Semester Question Paper Template

4	**Answer any one of the following-**		**[5×10=50]**	
	4-a.	Question-	(10)	
	4-b.	Question-	(10)	
5.	Answer any one of the following-			
	5-a.	Question-	(10)	
	5-b.	Question-	(10)	
6.	Answer any one of the following-			
	6-a.	Question-	(10)	
	6-b.	Question-	(10)	
7.	Answer any one of the following-			
	7-a.	Question-	(10)	
	7-b.	Question-	(10)	
8.	Answer any one of the following-			
	8-a.	Question-	(10)	
	8-b.	Question-	(10)	

Prerequisite and Recap(CO3)

- Knowledge of Maths 1 B.Tech.
- Knowledge of Maths 2 B.Tech

Brief Introduction about the subject

- We will discuss properties of complex function (limits, continuity, differentiability, Analytic, integration)
- In 3rd module we will discuss application of partial differential equations
- In 4th module we will discuss numerical methods for solving algebraic equations, system of linear equations, definite integral and 1st order ordinary differential equation.
- In 5th module we will discuss aptitude part.

Unit Content

- Introduction to Partial Differential Equations
- Solution of Second Order Linear Partial Differential Equation with constant coefficients
- Classification of second order partial differential equations
- Method of separation of variables for solving partial differential equations
- Solution of one and two dimensional wave and heat conduction equations.

Unit Objective(CO3)

- The objective of this module is to find the roots by numerical method, interpolation and numerical methods for system of linear equations, definite integral and 1^{st} order ordinary differential equations.

Topic Objective (CO3)

Partial Differential Equations

- Complex problems by applying the knowledge acquired to areas that are different to the original ones.
- Solving real problems by identifying them appropriately from the perspective of partial derivative equations.
- Use appropriate methods to study phenomena modeled with partial derivative equations.

Partial differential equations (CO3)

PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations (CO3)

- A differential equation containing dependent variables and independent variables and partial derivatives of dependent variable with respect to two or more independent variables is called a partial differential equation.

For example-

1. $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$
2. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

Partial differential equations (CO3)

- Partial derivatives denoted by some alphabets or notation which is denoted as follows:

$$\frac{\partial z}{\partial x} = p \text{ or } z_x \qquad \frac{\partial z}{\partial y} = \text{q or } z_y$$

$$\frac{\partial^2 z}{\partial x^2} = r \text{ or } z_{xx} \qquad \frac{\partial^2 z}{\partial y^2} = t \text{ or } z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = s \text{ or } z_{xy}$$

Partial differential equations (CO3)

Some important partial differential equations are as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

This equation is known as **Laplace equation.**

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

This equation is known as **wave equation.**

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

This equation is known as **heat conducting equation.**

Above equations are generally occurs in the problems of physics and engineering.

Partial differential equations (CO3)

❖ **Order of partial differential equations:**

The order of the **highest ordered derivative** present in the equation is called the order of partial differential equation.

Example:$y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0$

In this equation highest order of derivative is 1.

❖ **Degree of partial differential equation:**

Degree of partial differential equation is the **power of the highest ordered derivative** should be free from fraction powers and radical sign.

Example:$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$ degree is 1 and order is also 1.

Daily Quiz(CO3)

Discuss the order & degree of the following equations

Q1. $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

Q2. $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

Recap(CO3)

- ✓ PDE
- ✓ Order and degree of partial differential equation

Topic objective (CO3)

Linear Partial Differential Equation

- This topic present's the main results in the context of partial differential equations that allow learning about these models and to study numerical methods for the approximation of their solution. Analyze, synthesize, organize and plan projects in the field of study.

Linear Partial differential equation(CO3)

- **Linear Homogeneous Partial Differential Equation:**

A partial differential equation, is said to be linear if it is of the **first degree** in the dependent variable and its partial derivatives and also they are not multiplied together.

if , in addition, every term of the equation contains the dependent variable or its derivative, it is called a homogeneous equation.

Example: $p + 3q = 5z + tan(y - 3x)$ is a linear PDE of 1^{st} order.

- **QUASI LINEAR PARTIAL DIFFERENTIAL EQUATIONS:** A partial differential equation is said to be quasi linear if **degree of highest ordered derivative is one or no products of the partial derivatives** of the highest order are present.

Linear Partial differential equation(CO3)

Example 1: $\frac{y^2 z}{x} p + xzq = y^2$ is quasi linear p.d.e of 1st order.

Example 2: $z \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial y}\right)^2 = 0$ is quasi linear p.d.e of 2nd order.

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z)$$

This is known as general form of quasi linear p.d.e. of 1st order .

- **NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS:** A PDE which is neither linear nor quasi- linear is called non-linear PDE.

Example: $p^2 x + q^2 y = z^2$

PDE of higher order(CO3)

Definitions:

- **Complete solution:** The solution $f(x, y, z, a, b) = 0$ of a first ordered partial differential equation, which contains two arbitrary constants is called complete solution or complete integral.
- **Particular solution:** A solution obtained from the complete integral by giving particular values to the arbitrary constant is called particular solution or particular integral.
- **Boundary conditions:** The unique solution of a partial differential equation corresponding to the physical problem must satisfy certain other conditions at the boundary of the region R. these are known as boundary condition.
- **Initial condition:** If these conditions are given to the time $t = 0$, they are known as initial conditions.

Linear Partial differential equation(CO3)

Discuss the Type of Following Partial Differential equation:

Q1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Q2. $xp + yq = 3z$

Q3. $(3 - 2yz)p + x(2z - 1)q = 2x(y - 3)$

Q4. $yp + xq = xyz^2(x^2 - y^2)$

Q5. $(z^2 - 2zy - y^2)p + x(y + z)q = x(y - z)$

Q6. $\sqrt{p} + \sqrt{q} = 1$

Q7. $z = px + qy + \sqrt{1 + p^2 + q^2}$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Recap(CO3)

- ✓ Order and degree of partial differential equation
- ✓ Types of partial differential equation.

Topic objectives (CO3)

Homogeneous Partial differential equation

- Learn to solve systems of homogenous linear equations and application problems

PDE of higher order(CO3)

- **Linear partial differential equation with constant coefficients of higher order:**

A PDE in which the dependent variable and its derivatives appear only in the **<u>1st degree and are not multiplied together</u>** is called Linear partial differential equation with constant coefficients.

General form:

$$A_0 \frac{\partial^n u}{\partial x^n} + A_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \cdots + A_n \frac{\partial^n u}{\partial y^n} + B_0 \frac{\partial^{n-1} u}{\partial x^{n-1}} + B_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \cdots$$
$$+ B_n \frac{\partial^{n-1} u}{\partial y^{n-1}} + C_0 \frac{\partial u}{\partial x} + C_1 \frac{\partial u}{\partial y} + P_0 u = F(x, y)$$

PDE of higher order(CO3)

❖ **Partial differential equation of 2nd order:**

A PDE of the second order which include at least one of the partial derivatives $\mathrm{r} = \frac{\partial^2 z}{\partial x^2}$,$\mathrm{s} = \frac{\partial^2 z}{\partial x \partial y}$, $\mathrm{t} = \frac{\partial^2 z}{\partial y^2}$ but none of the higher order is said to be Partial differential equation of 2nd order.

Example: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$=0

❖ **Homogenous Linear Partial Differential Equation With Constant Coefficients** :

An equation of the of the form

$$a_0 \frac{\partial^n u}{\partial x^n} + a_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n u}{\partial y^n} = F(x, y) \text{ ......(1)}$$

Homogeneous linear PDE (CO3)

Where $a_0, a_1, a_2 \ldots a_n$ are constants, is called homogeneous partial differential of nth order with constant coefficients.

Taking $\frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$

Equation (1) can be written as $\emptyset(D, D')z = F(x, y)$ .....(2)

Complete solution of equation (2) is

$z =$ Complementary Function +Particular Integral= $\mathrm{C.F.} + \mathrm{P.I.}$

i. Complementary Function (C.F.): which is the complete solution of the equation $\emptyset(D, D')z = 0$

ii. Particular integral (P.I.): which is particular solution (free from arbitrary constants) of $\emptyset(D, D')z = F(x, y)$

Homogeneous linear PDE of (CO3)

- **RULES FOR FINDING C.F.:**

let us consider $\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

1. Taking $\frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D'$

So we get equation in this form

$(D^2 + a_1 \mathrm{DD}' + a_2 D'^2)z = 0$

2. Taking $\mathrm{D} = m, \& \ D' = 1$, we get

$(m^2 + a_1 m + a_2 = 0$ is called auxiliary equation.

3. We get roots m_1, m_2

Case 1. If the roots of A.E. are m_1, m_2(distinct roots),then

$\mathrm{C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x)$

Homogeneous linear PDE (CO3)

General form :If the roots of A.E. are $m_1, m_2, m_3 \ldots\ldots\ldots$(all distinct roots),then $\mathrm{C.F.} = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \cdots$

Case 2. If the roots of A.E. are m_1, m_1 (two equal roots) then

$\mathrm{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x)$

Note:

- If the roots of A.E. are m_1, m_1, m_2 then
  $$\mathrm{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x)$$
- If the roots of A.E. are m_1, m_1, m_1 (three equal roots),then
  $$\mathrm{C.F.} = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x)$$

Example 1. Solve $(D + 2D')(D - 3D')^2 = 0$

Solution. Auxiliary Equation is (Taking $D = m, D' = 1$)

Homogeneous linear PDE (CO3)

$(m+2)(m-3)^2 = 0$

$\Rightarrow m = -2,3,3$

$\text{C. F.} = f_1(y-2x) + f_2(y+3x) + xf_3(y+3x)$

$\text{P. I.} = 0$

Hence the solution is

$$z = \text{C. F.} + \text{P. I.} = f_1(y-2x) + f_2(y+3x) + xf_3(y+3x)$$

Where f_1, f_2, f_3 are arbitrary functions.

Example 2 Solve $\boldsymbol{\frac{\partial^3 z}{\partial x^3} - 3\,\frac{\partial^3 z}{\partial x^2 \partial y} + 2\,\frac{\partial^3 z}{\partial x \partial y^2} = 0}$

Solution: The given equation is

$(D^3 - 3D^2\mathrm{D}' + 2\mathrm{D}D'^2)z = 0$

The auxiliary equation is(Taking $D = m, D' = 1$)

$m^3 - 3m^2 + 2m = 0$

$\Rightarrow m(m-1)(m-2) = 0$

Homogeneous linear PDE (CO3)

$\Rightarrow m = 0,1,2$
$\text{C. F.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$
$\text{P. I.} = 0$

Hence the solution is

$$z = \text{C. F.} + \text{P. I.} = f_1(y) + f_2(y + x) + f_3(y + 2x)$$

Where f_1, f_2, f_3 are arbitrary functions.

Example 3.Solve $\mathbf{r} = \boldsymbol{a}^2\mathbf{t}$

Solution: The Given equation is $(D^2 - a^2 D'^2)z = 0$

The auxiliary equation is $m^2 - a^2 = 0$
$m = \pm a$
$\text{C. F.} = f_1(y + ax) + f_2(y - ax)$

$\text{P.I.} = 0$

Hence the solution is

$z = \text{C.F.} + \text{P.I.} = f_1(y + ax) + f_2(y - ax)$

Where $f_1, f_2,$ are arbitrary functions

Example 4. Solve $4r - 12s + 9t = 0$

Solution. The given equation $(4D^2 - 12DD' + 9D'^2)z = 0$

The auxiliary equation is $m^2 - 12\text{m} + 9 = 0 \Rightarrow m = \frac{3}{2}, \frac{3}{2}.$

$$\text{C.F.} = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$$

$\text{P.I.} = 0$

Hence the complete solution is

$z = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$ Where $f_1, and\ f_2$ are arbitrary functions.

Homogeneous linear PDE (CO3)

Example 5. Solve $\frac{\partial^4 z}{\partial x^4}+\frac{\partial^4 z}{\partial y^4}=0$

Solution: The Given equation is $(D^4+D'^4)z=0$

Auxiliary Equation is $m^4+1=0$

$m^4+1+2m^2=2m^2$

$\Rightarrow (m^2+1)^2-\left(m\sqrt{2}\right)^2=0$

$\Rightarrow (m^2+\sqrt{2}m+1)(m^2-\sqrt{2}m+1)=0$

$\Rightarrow m^2+\sqrt{2}m+1=0$ or $m^2-\sqrt{2}m+1=0$

$m=\frac{-1\pm i}{\sqrt{2}},\frac{1\pm i}{\sqrt{2}}$, $z_1=\frac{-1\pm i}{\sqrt{2}}$ and $z_2=\frac{1\pm i}{\sqrt{2}}$,$m=z_1,z_2,\overline{z_1},\overline{z_2}$

Here $\overline{z_1}$ and $\overline{z_2}$ are denote complex conjugate of z_1 and z_2 respectively.

$\mathrm{C.F.}=f_1(y+z_1x)+f_2(y+\overline{z_1}x)+f_3(y+z_2x)+f_4(y+\overline{z_2}x)$

Homogeneous linear PDE (CO3)

$\text{P.I.} = 0$

Hence the complete solution is

$z = f_1(y + z_1 x) + f_2(y + \overline{z_1} x) + f_3(y + z_2 x) + f_4(y + \overline{z_2} x)$

Where $f_1, f_2, f_{3,}$ and f_4 are arbitrary functions.

Daily Quiz(CO3)

Q1. Solve $(D^3 - 2D^2D')z = 0$

Ans:$z = f_1(y) + xf_2(y) + f_3(y + 2x)$

Q2. Solve $(D^2 - DD')z = 0$

Ans: $z = f_1(y) + f_2(y + x)$

Q3. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = 0$

Ans: $z = f_1(x) + f_2(y + x) + xf_3(y + x)$

Q4. Solve $r + s - 2t = 0$

Ans: $z = f_1(y + x) + f_2(y - 2x)$

Homogeneous linear PDE (CO3)

- **<u>Method's for finding P.I.</u>**

P.I of the equation $F(D, D') = \phi(x, y)$is given by $\frac{1}{F(D,D')}\phi(x, y)$

Short methods:

When $\phi(x, y)$ is a function of type $\phi(ax + by)$

To find out P.I of the equation

1. $F(D, D') = \phi(ax + by)$; $F(a, b\ ) \neq 0$ Where $F(D, D')$ is a homogeneous function of D and D' of degree n.
2. Replace D by a ,D' by b in $F(D, D')$ to get $F(a, b\ )$
3. Put $ax + by = u$ and integrate $\phi(u)$,n times w.r.t. u.
4. Then, $\mathrm{P.I.} = \frac{1}{F(a,b\ )}\int\int\int ......\int \phi(u)\ \mathrm{d}ududu.... du$ (n times).

Homogeneous linear PDE (CO3)

5. Replace u by $ax + by$ at last under the condition $F(a, b\ ) \neq 0$.

If $F(a, b\ ) = 0$ then the test fails.

Where $F(D, D')$ is a homogeneous function of D and D' of degree n.
then we apply following steps to get $\mathrm{P.I.}$

Steps 1.Differenciate $F(D, D')$ partially w.r.t D and simultaneously multiply the expression by x.

$$\mathrm{P.I.} = x\frac{1}{\frac{\partial}{\partial D}(F(D,D'))}\phi(ax + by) = \frac{x}{F'(D,D')}\phi(ax + by)$$

Steps 2 $put\ D = a, D' = b\ \&\ let\ u = ax + by$ Then

$$\mathrm{P.I.} = \frac{x}{F'(a,b)}\int\int\int \ldots\ldots\int \phi(u)\ dududu\ldots.\ du\ (n - 1 \text{ times}).$$

under the condition that $F'(a, b) \neq 0$

Step 3. This method fails if $F'(a, b) \neq 0$.Then we need to follow the same process again.

Homogeneous linear PDE (CO3)

Since $F'(a,b) = 0$ where $F'(D,D')$ is a homogenous functions of D and D' of degree $n-1$.

then we apply following steps to get $\mathrm{P.I.}$

Step 4. Repeat the procedure as again differentiate $F'(D,D')$ partially w.r.t D and simultaneously multiply the expression by x.

$$\mathrm{P.I.} = x^2 . \frac{1}{\frac{\partial}{\partial D}\{\mathrm{F'(D,D')}\}} \phi(ax+by) = \frac{x^2}{F''(D,D')} \phi(ax+by)$$

Step 5. $Put\ D = a, D' = b\ \&\ let\ u = ax + by$

Then $\mathrm{P.I.} = \frac{x^2}{F''(a,b)} \int \int \int \ldots\ldots \int \phi(u)\, dududu \ldots .\, du$ ($n-2$ times)

under the condition that $F''(a,b) \neq 0$.

Step 3. This method also fails if $F''(a,b) = 0$. Then we need to follow the same process again.

Homogeneous linear PDE (CO3)

Example 1. Solve the linear partial differential equation.

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}.$$

Solution: The given equation is

$(D^3 - 3D^2\mathrm{D}' + 4D'^3)z = e^{x+2y}$ where $\mathrm{D} = \frac{\partial}{\partial x}$ and $\mathrm{D}' = \frac{\partial}{\partial y}$

Auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$\Rightarrow (m-2)^2\,(\mathrm{m}+1) = 0$$

$$\Rightarrow m = 2,2,-1$$

two roots are equal and one is different so $\mathrm{C.F.}$

Homogeneous linear PDE (CO3)

$\text{C. F.} = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$ ....(1)

$\text{P. I.} = \frac{1}{D^3-3D^2D'+4D'^3} e^{x+2y}$

$put\ D = 1, D' = 2\ \&\ let\ u = x + 2y.$

$= \frac{1}{1^3-3(1)^2.2+4(2)^3} \int \int \int e^u dududu = \frac{1}{27} e^u$

Since $u = x + 2y$

$= \frac{1}{27} e^{x+2y}$ ....(2)

Hence the complete solution is (equation 1 & 2)

$$z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \frac{1}{27} e^{x+2y}$$

Where f_1, $f_2$and f_3 are arbitrary functions.

Homogeneous linear PDE (CO3)

Example 2: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} = sin\,x\,cos\,2y$

Solution: The given equation is

$(D^2 - 2\mathrm{DD}')z = \sin x \cos 2y$

Auxiliary equation is

$m^2 - 2\mathrm{m} = 0$

$m = 0{,}2$ roots are different

$\mathrm{C.F.} = f_1(y) + f_2(y + 2x)$ …………(1)

$\mathrm{P.I.} = \frac{1}{D^2 - 2DD'} sin\,x\,cos\,2y$

$= \frac{1}{2(D^2 - 2DD')}(sin\,(x + 2y) + sin(x - 2y))$

Homogeneous linear PDE (CO3)

$$= \frac{1}{2}\left[\frac{1}{D^2 - 2DD'} \sin(x+2y) + \frac{1}{D^2 - 2DD'} \sin(x-2y)\right]$$

$$= \frac{1}{2}(P_1 + P_2)$$

$$P_1 = \frac{1}{D^2 - 2DD'} \sin(x+2y)$$

$$= \frac{1}{1^2 - 2(1)(2)} \int \int Sin\, u\, du du \text{ where } u = x + 2y$$

$$= -\frac{1}{3}(-\sin u) = \frac{1}{3}\sin(x+2y)\text{........}(2)$$

$$P_2 = \frac{1}{D^2 - 2DD'} \sin(x-2y)$$

$$= \frac{1}{1^2 - 2(1)(-2)} \int \int Sin\, v\, dv dv \text{ where } v = x - 2y$$

Homogeneous linear PDE (CO3)

$= -\frac{1}{5} sin\, v = -\frac{1}{5} sin\,(x - 2y)$.............(3)

From (2 & 3),

$P.I. = \frac{1}{6} sin(x + 2y) - \frac{1}{10} sin\,(x - 2y)$..........(4)

Hence the complete solution is by equation (1 & 4)

$z = f_1(y) + f_2(y + 2x) + \frac{1}{6} sin(x + 2y) - \frac{1}{10} sin\,(x - 2y)$

where f_1 and f_2 are arbitrary functions.

Example 3: Solve $r + 2s + t = 2(y - x) + \sin(x - y)$

Solution: The given equation is

$(D^2 + 2DD' + D'^2)z = 2(y - x) + \sin(x - y)$

Auxiliary equation is

$m^2 + 2m + 1 = 0$

Homogeneous linear PDE (CO3)

$m = -1, -1$

Roots are equal so C.F.

$\text{C. F.} = f_1(y-x) + xf_2(y-x)$ ....(1)

$$P.I. = \frac{1}{(D+D')^2} 2(y-x) + \frac{1}{(D+D')^2} sin\,(x-y) = P_1 + P_2$$

$P_1 = \frac{1}{(D+D')^2} 2(y-x)$ because test fails $F(a,b) = 0$

$= 2x \frac{1}{2(D+D')}(y-x)$

$= x \frac{1}{D+D'}(y-x)$ again test fails $F'(a,b) = 0$

$= x^2(y-x)$ ....(2)

$P_2 = \frac{1}{(D+D')^2} sin\,(x-y)$ because test fails $F(a,b) = 0.$

Homogeneous linear PDE (CO3)

$= x\frac{1}{2(D+D')}sin(x-y)$ again test fails $F'(a,b) = 0$

$x^2.\frac{1}{2}\ sin(x-y)$..........(3)

By equation (2) & (3)

$P.I. = x^2(y-x) + \frac{x^2}{2}\sin(x-y)$...........(4)

By equation (1) & (4)

$$z = f_1(y-x) + xf_2(y-x) + x^2(y-x) + \frac{x^2}{2}\sin(x-y)$$

where f_1 and f_2 are arbitrary functions.

Daily Quiz(CO3)

Q1. $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$

Q2. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y + x)^{1/2}$

Homogeneous linear PDE (CO3)

- **When $\phi(x, y)$ is of the form $x^m y^n$ or A rational integral algebraic function of x & y.**

P. I. of the equation $F(D, D') = \phi(x, y)$ is given by

$$\text{P.I.} = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{1+W(D,D')}\phi(x,y)$$
$$= [1 + W(D, D')]^{-1}\phi(x, y)$$

Then expand $[1 + W(D, D')]^{-1}$ as a series and find the P. I.

Example 1. Solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

Solution: Auxiliary equation is

$m^2 - 6m + 9 = 0$

$m = 3,3$ roots are equal ,so

$\text{C.F.} = f_1(y + 3x) + xf_2(y + 3x)$………(1)

P.I.$= \frac{1}{(D^2-6DD'+9D'^2)}(12x^2 + 36xy)$

Homogeneous linear PDE (CO3)

$$= \frac{1}{(D^2 - 6DD' + 9D'^2)}(12x^2) + \frac{1}{(D^2 - 6DD' + 9D'^2)}(36xy)$$

$$= P_1 + P_2$$

$$P_1 = \frac{1}{(D^2 - 6DD' + 9D'^2)}(12x^2)$$

$$= \frac{1}{(D - 3D')^2}(12x^2)$$

$$= \frac{1}{D^2\left(1 - 3\frac{D'}{D}\right)^2}(12x^2)$$

$$= \frac{1}{D^2}\left(1 - 3\frac{D'}{D}\right)^{-2}(12x^2)$$

Neglecting higher power terms

$$= \frac{1}{D^2}(1 + 6\frac{D'}{D})(12x^2)$$

Homogeneous linear PDE (CO3)

$$= \frac{1}{D^2}\left(12x^2 + 6\frac{D'}{D}12x^2\right)$$

$$= \frac{1}{D^2}(12x^2 + 0)$$

integrating two times w.r.t. x

$$= \frac{12}{12}x^4$$

$$= x^4$$

$$P_2 = \frac{1}{(D^2 - 6DD' + 9D'^2)}(36xy)$$

$$= \frac{1}{(D - 3D')^2}(36xy)$$

$$= \frac{1}{D^2\left(1 - 3\frac{D'}{D}\right)^2}(36xy)$$

Homogeneous linear PDE (CO3)

$$= \frac{1}{D^2}\left(1 - 3\frac{D'}{D}\right)^{-2}(36xy)$$

Neglecting higher power terms

$$= \frac{1}{D^2}\left(1 + 6\frac{D'}{D}\right)(36xy)$$

$$= \frac{1}{D^2}(36xy + 6\frac{D'}{D}36xy)$$

$$= \frac{1}{D^2}(36xy + 6\frac{1}{D}36x)$$

$$= \frac{1}{D^2}(36xy + 6 \times \frac{36\,x^2}{2})$$

$$= 36\,(y\frac{1}{D^2}x + 3\frac{1}{D^2}x^2)$$

Homogeneous linear PDE (CO3)

integrating w.r.t. x

$$= 36\left(y\frac{x^3}{6} + 3\frac{x^4}{12}\right)$$

$$= (6x^3y + 9x^4)$$

P.I.$= P_1 + P_2$

$$= x^4 + 6x^3y + 9x^4$$

$$= 6x^3y + 10x^4 \text{........(2)}$$

By equation (1 & 2) General solution is

$$z = f_1(y + 3x) + xf_2(y + 3x) + 6x^3y + 10x^4$$

Homogeneous linear PDE (CO3)

Example2: Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$

Solution: $(D^2 + 3DD' + 2D'^2)z = 12xy$

Auxiliary equation is

$m^2 + 3m + 2 = 0$

$m = -1, -2$ roots are different , so

C.F. $= f_1(y-x) + f_2(y-2x)$.........(1)

P.I.$= \frac{1}{(D^2+3DD'+2D'^2)}(12xy)$

$= \frac{1}{D^2\left(1 + 3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)}(12xy)$

Homogeneous linear PDE (CO3)

$$= \frac{1}{D^2}\left(1 + 3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)^{-1} \; 12xy$$

Neglecting higher power terms

$$= \frac{1}{D^2}\left(1 - 3\frac{D'}{D} - 2\frac{D'^2}{D^2}\right) 12xy$$

$$= \frac{1}{D^2}(12xy - 3\frac{1}{D}12x - 2 \times 0)$$

$$= \frac{1}{D^2}(12xy - 3 \times 12\frac{x^2}{2})$$

$$= (12y\frac{1}{D^2}x - 3 \times 6\frac{1}{D^2}x^2)$$

$$= \left(12y\frac{x^3}{6} - 3 \times 6\frac{x^4}{12}\right)$$

Homogeneous linear PDE (CO3)

$= \left(2x^3y - \frac{3}{2}x^4\right)$................................(2)

By equation(1 & 2)

$$z = f_1(y-x) + f_2(y-2x) + 2x^3y - \frac{3}{2}x^4$$

- **General method of finding P.I. :**

$\frac{1}{D-mD\prime}\phi(x,y) = \int \phi(x, c-mx)dx,$ here $y \longrightarrow c - mx$

Q.1 Solve the linear partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$$

Homogeneous linear PDE (CO3)

Solution: $(D^2 + DD' - 6D'^2)z = y\cos x$

Auxiliary equation is

$m^2 + m - 6 = 0$

$m = 2, -3$ roots are different , so

$C.F. = f_1(y + 2x) + f_2(y - 3x)$………(1)

$$P.I. = \frac{1}{(D^2 + DD' - 6D'^2)} y\cos x$$

$$= \frac{1}{(D - 2D')(D + 3D')} y\cos x$$

$$= \frac{1}{(D-2D')} \int (c + 3x)\cos x \; dx \quad y \longrightarrow (c + 3x)$$

Homogeneous linear PDE (CO3)

$$= \frac{1}{(D-2D')}\left[\int c\cos x\,dx + \int 3x\,cosx\,dx\right]$$

$$= \frac{1}{(D-2D')}[c\sin x + 3\{x\sin x - \int 1.\sin x\,dx\}]$$

$$= \frac{1}{(D-2D')}[(c+3x)\sin x + 3\cos x]$$

$$= \frac{1}{(D-2D')}[y\sin x + 3\cos x] \qquad \text{where } c \longrightarrow y-3x$$

$$= \int[(b-2x)\sin x + 3\,cosx]dx \qquad y \longrightarrow (b-2x)$$

$$= (-b\cos x - 2\{x(-\cos x) - (-2)\int -\cos x\,dx\} + 3\sin x$$

Homogeneous linear PDE (CO3)

$= -b\cos x + 2x\cos x - 2\sin x + 3\sin x$

$= -(b-2x)\cos x + \sin x$

$= -y\cos x + \sin x$ where $y \longrightarrow b - 2x$

Hence the general solution is

$z = f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x$

Where f_1 and f_2 are arbitrary functions.

Daily Quiz(CO3)

Q1 $(D^2+(a+b)DD'+abD'^2)z = xy$

Ans: $f_1(y-ax)+f_2(y-bx)+\frac{1}{6}x^3y-\frac{(a+b)x^4}{24}$

Q2. $(D^2-DD'-2D'^2)z=(y-1)e^x$

Ans: $f_1(y-x)+f_2(y+2x)+e^xy$

Q3. $(D^2+2DD'+D'^2)z=2\cos y - x\sin y$

Ans: $f_1(y-x)+xf_2(y-x)+x\sin y$

Recap(CO3)

- ✓ Order and degree of partial differential equation.
- ✓ Homogenous Partial Differential equation
- ✓ C.F. for Homogenous Partial Differential equation.
- ✓ P.I. for Homogenous Partial Differential equation.

Topic objective (CO3)

Non homogenous linear partial differential equation

- It can lead to a shock wave solution and it bring our study of partial differential equations with some real time problems.

Non homogenous linear PDE(CO3)

❖ **Non homogenous linear PDE with constant coefficients:**

In the equation $\phi(D, D')z = F(x, y) \cdots\cdots (1)$

If the polynomial $\phi(D, D')$ is not homogenous then it is called non-homogenous linear PDE.

For e.g-

(i) $(D - D'^2)z = 0$

(ii) $(D^2 - D'^2 + D + 3D' - 2)z = 0$

Solution of (1) is given by $z =$ C.F. $+$ P.I.

Non homogenous linear PDE(CO3)

- **Method of Finding C.F.-**

Case-1: If we resolve $\phi(D, D')z = 0$ into linear factor of the form $D - mD' - a = 0$ then C.F. is given by-

C.F.= $e^{ax} f(y + mx)$

Note :

1. if $\phi(D, D') = (D - mD' - a)^2$

Then

C.F.= $e^{ax} f_1\,(y + mx) + xe^{ax} f_2\,(y + mx)$

2. if $\phi(D, D') = (D - mD' - a)^3$

Then

C.F.= $e^{ax} f_1\,(y + mx) + xe^{ax} f_2\,(y + mx) + x^2 e^{ax} f_3\,(y + mx)$

Non homogenous linear PDE(CO3)

Case-2: If we cannot resolve $\phi(D, D')z = 0$ into **linear** factor.

Let $\left(D - D'^2\right)z = 0 \cdots\cdots (1)$

Let the solution of (1) is

$z = Ae^{hx+ky}$

Then $Dz = Ahe^{hx+ky}$

$D'z = Ake^{hx+ky}$

$D'^2 z = Ak^2 e^{hx+ky}$

Then $Ahe^{hx+ky} - Ak^2 e^{hx+ky} = 0$

$\Rightarrow A(h - k^2)e^{hx+ky} = 0$

$\Rightarrow (h - k^2) = 0 \Rightarrow h = k^2$

So general solution is given by- $z = \sum Ae^{k^2 x+ky}$

Non homogenous linear PDE(CO3)

Example-1 Solve:$(D+4D'+5)^2\, z=0$

Solution: Here given PDE is non homogenous

So C.F.$=\; e^{-5x}f_1(y-4x)+xe^{-5x}f_2(y-4x)$

P.I.$=\;0$

$z=$ C.F. $+$ P.I.

$\Rightarrow z=e^{-5x}f_1(y-4x)+xe^{-5x}f_2(y-4x)$

Example-2 Solve: $r-t+p-q=0$

Solution: $\because r=\dfrac{\partial^2 z}{\partial x^2}=D^2z$

$$t=\frac{\partial^2 z}{\partial y^2}=D'^2z$$

Non homogenous linear PDE(CO3)

$$p = \frac{\partial z}{\partial x} = Dz$$

$$q = \frac{\partial z}{\partial y} = D'z$$

Put these values we get

$$\left(D^2 - D'^2 + D - D'\right)z = 0$$

$$\Rightarrow \left((D + D')(D - D') + (D - D')\right)z = 0$$

$$\Rightarrow (D - D')(D + D' + 1)z = 0$$

C.F.= $f_1(y + x) + e^{-x} f_2(y - x)$

P.I.= 0

$z =$ C.F. + P.I.

$$\Rightarrow z = f_1(y + x) + e^{-x} f_2(y - x)$$

Non homogenous linear PDE(CO3)

Example-3 Solve: $DD'(D-2D'-3)z=0$

Solution: Given equation is non homogenous

Here $Dz=0 \Rightarrow z=f_1(y)$

$D'z=0 \Rightarrow z=f_2(x)$

$(D-2D'-3)z=0 \Rightarrow z=e^{3x}f_3(y+2x)$

C.F.$= f_1(y)+f_2(x)+e^{3x}f_3(y+2x)$

P.I.$=0$

$z=$ C.F. $+$ P.I.

$\Rightarrow z=f_1(y)+f_2(x)+e^{3x}f_3(y+2x)$

Example-4 Solve: $\left(D^2-D'^2+D+3D'-2\right)z=0$

Solution: After factor

Non homogenous linear PDE(CO3)

$(D + D' - 1)(D - D' + 2)z = 0$

C.F.= $e^x f_1(y - x) + e^{-2x} f_2(y + x)$

P.I.= 0

$z =$ C.F. + P.I.

$\Rightarrow\ z = e^x f_1(y - x) + e^{-2x} f_2(y + x)$

$\Rightarrow z = e^x f_1(y - x) + e^{-2x} f_2(y + x)$

Example-5 Solve:$\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = 0$

Solution: Given PDE is non homogenous

To find C.F.

$\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = 0$

$\Rightarrow (D + D')(D - 2D' + 2)z = 0$

Non homogenous linear PDE(CO3)

C.F.= $f_1(y-x)+e^{-2x}f_2(y+2x)$

P.I. $=0$

$z=$ C.F. $+$ P.I.

$\Rightarrow\ z=f_1(y-x)+e^{-2x}f_2(y+2x)+0$

$\Rightarrow z=f_1(y-x)+e^{-2x}f_2(y+2x)$

Example-6 Solve: $(D^2-3DD'+D'+4)z=0$

Solution: Here **linear** factor is not possible

$(D^2-3DD'+D'+4)z=0\cdots\cdots(1)$

Let the solution of (1) is

$z=Ae^{hx+ky}$

Non homogenous linear PDE(CO3)

Then $Dz = Ahe^{hx+ky}$

$D^2z = Ah^2e^{hx+ky}$

$D'z = Ake^{hx+ky}$

$DD'z = Ahke^{hx+ky}$

Put these values in (1)

$\Rightarrow Ah^2e^{hx+ky} - 3Ahke^{hx+ky} + Ake^{hx+ky} + 4Ae^{hx+ky} = 0$

$\Rightarrow Ah^2e^{hx+ky} - 3Ahke^{hx+ky} + Ake^{hx+ky} + 4Ae^{hx+ky} = 0$

$\Rightarrow (h^2 - 3hk + k + 4)Ae^{hx+ky} = 0$

$\Rightarrow h^2 - 3hk + k + 4 = 0 \because Ae^{hx+ky} \neq 0$

So general solution is given by

$\Rightarrow z = \sum Ae^{hx+ky}$ where $h^2 - 3hk + k + 4 = 0$

Non homogenous linear PDE(CO3)

- **Method of finding P.I.-**

let $\phi(D,D')z = F(x,y) \cdots\cdots (1)$

Then P.I. $= \frac{1}{\phi(D,D')} F(x,y)$

Case-1: When $F(x,y) = e^{ax+by}$

Then P.I. $= \frac{1}{\phi(a,b\ )} F(x,y)$ where $\phi(a,b\ ) \neq 0.$

Case-2: When $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

Then P.I. $= \frac{1}{\phi(-a^2,-ab,-b^2)} F(x,y)$

where $\phi(-a^2,-ab,-b^2) \neq 0.$

i.e. put $D^2 = -a^2, DD' = -ab, D'^2 = -b^2$

Non homogenous linear PDE(CO3)

Case-3: When $F(x,y) = x^m y^n$

Then P.I. $= \frac{1}{\phi(D,D')} x^m y^n$

$\Rightarrow$ P.I. $= [1 + W(D,D')]^{-1} x^m y^n$

Case-4: When $F(x,y) = e^{ax+by}.V$ where V is any function of x, y.

Then P.I. $= \frac{1}{\phi(D,D')} e^{ax+by}.V$

$\Rightarrow$ P.I. $= e^{ax+by} \frac{1}{\phi(D+a,D'+b)} V$

Non homogenous linear PDE(CO3)

Example-7 Solve: $\left(D^2 - 4DD' + 4D'^2 - D + 2D'\right)z = e^{3x+4y}$

Solution: Given PDE is non homogenous

To find C.F.

$\left(D^2 - 4DD' + 4D'^2 - D + 2D'\right)z = 0$

$\Rightarrow \left((D - 2D')^2 - (D - 2D')\right)z = 0$

$\Rightarrow (D - 2D')(D - 2D' - 1)z = 0$

C.F.= $f_1(y + 2x) + e^x f_2(y + 2x)$

P.I. $= \dfrac{1}{D^2 - 4DD' + 4D'^2 - D + 2D'} e^{3x+4y}$

Here put $D = 3$ & $D' = 4$, $\phi(3,4) = 30 \neq 0$

Non homogenous linear PDE(CO3)

P.I. $= \dfrac{1}{3^2-4.3.4+4.4^2-3+2.4} e^{3x+4y}$

P.I. $= \dfrac{1}{30} e^{3x+4y}$

$z =$ C.F. + P.I.

$\Rightarrow z = f_1(y+2x) + e^x f_2(y+2x) + \dfrac{1}{30} e^{3x+4y}$

Example-8 Solve:$\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = \sin(2x+y)$

Sol: Given PDE is non homogenous

To find C.F.

$\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = 0$

$\Rightarrow (D+D')(D-2D'+2)z = 0$

Non homogenous linear PDE(CO3)

C.F.= $f_1(y-x)+e^{-2x}f_2(y+2x)$

P.I. $=\dfrac{1}{D^2-DD'-2D'^2+2D+2D'}\sin(2x+y)$

Here put $D^2=-2^2=-4$

$D'^2=-1^2=-1$

$DD'=-2.1=-2$

P.I.= $\dfrac{1}{-4+2-2\times-1+2D+2D'}\sin(2x+y)$

$=\dfrac{1}{2(D+D')}\sin(2x+y)$

$=\dfrac{1}{2(D+D')}\sin(2x+y)$

$=\dfrac{(D-D')}{2(D+D')(D-D')}\sin(2x+y)$

Non homogenous linear PDE(CO3)

$$= \frac{(D-D')}{2\left(D^2-D'^2\right)} \sin(2x+y)$$

$$= \frac{(D-D')}{2(-4+1)} \sin(2x+y)$$

$$= \frac{(2\cos(2x+y)-\cos(2x+y))}{-6}$$

$$= -\frac{1}{6}\cos(2x+y)$$

$z =$ C.F. + P.I.

$$\Rightarrow z = f_1(y-x) + e^{-2x} f_2(y+2x) - \frac{1}{6}\cos(2x+y)$$

Non homogenous linear PDE(CO3)

Example-9 Solve: $\left(D - D'^2\right)z = cos\ (x - 3y)$

Solution: Given PDE is non homogenous to find C.F.

Put $\left(D - D'^2\right)z = 0 \cdots\cdots (1)$

Here factor is not possible

Let the solution of (1) is

$z = Ae^{hx+ky}$

Then $Dz = Ahe^{hx+ky}$

$D'z = Ake^{hx+ky}$

$D'^2z = Ak^2e^{hx+ky}$

Then $Ahe^{hx+ky} - Ak^2e^{hx+ky} = 0$

Non homogenous linear PDE(CO3)

$\Rightarrow A(h-k^2)e^{hx+ky} = 0$

$\Rightarrow (h-k^2) = 0 \Rightarrow h = k^2$

So C.F. is given by- $\quad$ $\text{C. F.} = \sum Ae^{k^2x+ky}$

P.I. $= \frac{1}{\left(D-D'^2\right)} cos\ (x-3y)$

Put $D'^2 = -(-3)^2 = -9$

P.I. $= \frac{1}{(D+9)} cos\ (x-3y) = \frac{(D-9)}{D^2-81} \cos(x-3y)$

Put $D^2 = -(1)^2 = -1$

$= \frac{-\sin\,(x-3y)-9\cos(x-3y)}{-1-81}$

P.I.$= \frac{\sin\,(x-3y)+9\cos(x-3y)}{82}$

Non homogenous linear PDE(CO3)

$z=$ C.F. + P.I.

$\Rightarrow z=\sum Ae^{k^2x+ky}+\frac{1}{82}[\sin(x-3y)+9\cos(x-3y)]$

Example-10 Solve: $\left(D^2-D'^2-3D+3D'\right)z=xy+e^{x+2y}$

Solution: Given PDE is non homogenous, to find C.F. put

$\left(D^2-D'^2-3D+3D'\right)z=0$

$\Rightarrow \left((D+D')(D-D')-3(D-D')\right)z=0$

$\Rightarrow (D-D')(D+D'-3)z=0$

C.F.= $f_1(y+x)+e^{3x}f_2(y-x)$

P.I. $=\frac{1}{\left(D^2-D'^2-3D+3D'\right)}xy+e^{x+2y}$

Non homogenous linear PDE(CO3)

$$\text{P.I.} = \frac{1}{\left(D^2-D'^2-3D+3D'\right)}xy + \frac{1}{\left(D^2-D'^2-3D+3D'\right)}e^{x+2y}$$

$$= P_1 + P_2$$

$$\because P_1 = \frac{1}{\left(D^2-D'^2-3D+3D'\right)}xy$$

$$\Rightarrow P_1 = \frac{1}{(D-D')(D+D'-3)}xy$$

$$= \frac{1}{D\left[1-\frac{D'}{D}\right]\times(-3)\left[1-\frac{D+D'}{3}\right]}xy$$

$$= \frac{1}{-3D}\left[1-\frac{D'}{D}\right]^{-1}\left[1-\frac{D+D'}{3}\right]^{-1}xy$$

$$\because [1-x]^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots$$

Non homogenous linear PDE(CO3)

$$= -\frac{1}{3D}\left[1+\frac{D'}{D}+\left(\frac{D'}{D}\right)^2+\cdots\right]\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\right]xy$$

$$= -\frac{1}{3D}\left[1+\frac{D+D'}{3}+\frac{2DD'}{9}+\frac{D'}{D}+\frac{D'}{3}\right]xy$$

$$= -\frac{1}{3D}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}+\frac{1}{D}x+\frac{x}{3}\right]$$

$$= -\frac{1}{3D}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right]$$

$$= -\frac{1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{6}+\frac{2x}{9}+\frac{x^3}{6}+\frac{x^2}{6}\right]$$

$$P_1 = -\frac{1}{3}\left[\frac{x^2y}{3}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right]$$

Non homogenous linear PDE(CO3)

$$\because P_2 = \frac{1}{\left(D^2 - D'^2 - 3D + 3D'\right)} e^{2x+y}$$

Here put $D = 2$ & $D' = 1, \phi(2,1) = 0$

$$\Rightarrow P_2 = \frac{x}{\frac{\partial}{\partial D}(D^2 - D'^2 - 3D + 3D')} e^{2x+y}$$

$$\Rightarrow P_2 = \frac{x}{(2D-3)} e^{2x+y}$$

Here put $D = 2$ & $D' = 1, \phi(2,1) = 1 \neq 0$

$$\Rightarrow P_2 = \frac{x}{(2\times 2-3)} e^{2x+y}$$

$$\Rightarrow P_2 = xe^{2x+y}.$$

Non homogenous linear PDE(CO3)

$\Rightarrow P.I. = P_1 + P_2.$

$\Rightarrow P.I. = -\frac{1}{3}\left[\frac{x^2y}{3} + \frac{xy}{3} + \frac{x^2}{3} + \frac{2x}{9} + \frac{x^3}{6}\right] + xe^{2x+y}.$

So general solution is given by

$$z = f_1(y+x) + e^{3x}f_2(y-x) - \frac{1}{3}\left[\frac{x^2y}{3} + \frac{xy}{3} + \frac{x^2}{3} + \frac{2x}{9} + \frac{x^3}{6}\right] + xe^{2x+y}$$

Example-11 Solve: $s + p - q = z + xy$

Solution: Given PDE is non homogenous.

$$(DD' + D - D' - 1)z = xy$$
$$(D-1)(D'+1)z = xy$$

for find C.F.

Non homogenous linear PDE(CO3)

$$(D-1)(D'+1)z=0 \; or \; (D-0D'-1)(D'-0D+1)z=0$$

$\Rightarrow C.F. = e^{x} f_1(y+0x) + e^{-y} f_2(x+0y)$

$\Rightarrow C.F. = e^{x} f_1(y) + e^{-y} f_2(x)$

Now,

$P.I. = \dfrac{1}{(D-1)(D'+1)}(xy)$

$P.I. = -[(1-D)^{-1}(1+D')^{-1}](xy)$

$\Rightarrow P.I. = -[(1+D+D^2+\cdots)(1-D'+\cdots)](xy).$

$\Rightarrow P.I. = -[1+D-D'-DD'](xy).$

$\Rightarrow P.I. = -[xy+y-x-1]$

Hence the complete solution is

$z =$ C.F. $+$ P.I

$z = e^{x} f_1(y) + e^{-y} f_2(x) - [xy+y-x-1].$

Non homogenous linear PDE(CO3)

Example-12 Solve: $(D-3D'-2)^3\, z = 6e^{2x}\sin(3x+y)$

Solution: Given PDE is non homogenous, to find C.F. put

$(D-3D'-2)^3\, z = 0$

C.F.$= e^{2x}f_1(y+3x) + xe^{2x}f_2(y+3x) + x^2e^{2x}f_3(y+3x)$

P.I. $= \frac{1}{(D-3D'-2)^3} 6e^{2x}\sin(3x+y)$

Replace $D \to D+2, D' \to D'+0$, We get

$= 6e^{2x}\frac{1}{(D+2-3(D'+0)-2)^3}\sin(3x+y)$

$= 6e^{2x}\frac{1}{(D-3D')^3}\sin(3x+y)$

$= 6e^{2x}\frac{1}{(D-3D')^2}\left[\frac{1}{D-3D'}\sin(3x+y)\right]$

Non homogenous linear PDE(CO3)

$$= 6e^{2x}\frac{1}{(D-3D')^2}\int \sin(3x+c-3x)\,dx \qquad \because y \to c-3x$$

$$= 6e^{2x}\frac{1}{(D-3D')^2}\int \sin c\,dx$$

$$= 6e^{2x}\frac{1}{(D-3D')^2}x\sin c$$

$$= 6e^{2x}\frac{1}{(D-3D')^2}x\sin(3x+y) \qquad \because c \to 3x+y$$

$$= 6e^{2x}\frac{1}{(D-3D')}\left[\frac{1}{D-3D'}\ x\sin(3x+y)\right]$$

$$= 6e^{2x}\frac{1}{(D-3D')}\int x\sin(3x+c-3x)\,dx \qquad \because y \to c-3x$$

$$= 6e^{2x}\frac{1}{(D-3D')}\times\frac{x^2}{2}\sin c$$

Non homogenous linear PDE(CO3)

$$= 6e^{2x}\frac{1}{(D-3D')}\times\frac{x^2}{2}\sin(3x+y) \qquad \because c \to 3x+y$$

$$= 3e^{2x}\frac{1}{(D-3D')}x^2\sin(3x+y)$$

$$= 3e^{2x}\left[\frac{1}{D-3D'}x^2\sin(3x+y)\right]$$

$$= 3e^{2x}\int x^2\sin(3x+c-3x)\,dx \qquad \because y \to c-3x$$

$$= 3e^{2x}\int x^2\sin c\,dx$$

$$= 3e^{2x}\frac{x^3}{3}\sin c$$

$$= e^{2x}x^3\sin(3x+y) \qquad \because c \to 3x+y$$

General solution is given by-

$$\Rightarrow z = e^{2x}f_1(y+3x) + xe^{2x}f_2(y+3x) + x^2e^{2x}f_3(y+3x)$$
$$+ e^{2x}x^3\sin(3x+y)$$

Non homogenous linear PDE(CO3)

Example-13 Solve: $(D-3D'-2)^2\, z = 2e^{2x}\tan(y+3x)$

Solution: Given PDE is non homogenous, to find C.F. put

$$(D-3D'-2)^2\, z = 0$$

C.F.$= e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x)$

P.I. $= \frac{1}{(D-3D'-2)^2} 2e^{2x}\, tan(3x+y)$

Replace $D \to D+2, D' \to D'+0$, We get

$$= 2e^{2x}\frac{1}{(D+2-3(D'+0)-2)^2} tan(3x+y).$$

$$= 2e^{2x}\frac{1}{(D-3D')^2} tan(3x+y).$$

P.I. $= 2e^{2x}\frac{1}{(D-3D')}\left[\frac{1}{(D-3D')} tan(3x+y)\right]$

P.I. $= 2e^{2x}\frac{1}{(D-3D')}\int tan(3x+c-3x\,)dx \qquad \because y \to c-3x$

Non homogenous linear PDE(CO3)

$= 2e^{2x}\frac{1}{(D-3D')}\int tan\, cdx.$

$= 2e^{2x}\frac{1}{(D-3D')}xtanc.$

$= 2e^{2x}\frac{1}{(D-3D')}xtan(3x+y).$ $\qquad \because c \to 3x+y$

P.I. $= 2e^{2x}\int xtan(3x+c-3x\,)dx$ $\qquad \because y \to c-3x$

P.I. $= 2e^{2x}\int xtan(c\,)dx$

$= 2e^{2x}\frac{x^2}{2}tanc.$ $\qquad \because c \to 3x+y$

$= e^{2x}x^2\, tan(3x+y).$

General solution is given by-

$\Rightarrow z = e^{2x}f_1(y+3x) + xe^{2x}f_2(y+3x) + x^2e^{2x}\, tan(3x+y).$

Daily Quiz(CO3)

Q1 Solve: $(D^2 - DD' - 2D'^2 + 2D + 2D')Z = e^{2x+3y} + \sin(2x + y) + xy$

Weekly Assignment(CO3)

Q1 Solve $(2D^2 - 5DD' + 2D'^2)z = 0$

Ans: $\emptyset_1\left(y - \frac{x}{2}\right) + \emptyset_2(y - 2x)$

Q2. Solve $(D^2 + 3\,DD' + 2D'^2)\,z = x + y$

Ans:- $\emptyset_1\left(y - \frac{x}{2}\right) + \emptyset_2(y - 2x) + \frac{1}{36}(x + y)^3$

Q3. Solve $(D_x - D_y - 1)(D_x - D_y - 2)z = e^{2x-y} + x$

Ans:- $e^x\emptyset_1(y + x) + e^{2x}\emptyset_2(y + x) + \frac{1}{2}e^{2x-y} + \frac{x}{2} + \frac{3}{4}$

Recap(CO3)

- ✓ Order and degree of partial differential equation.
- ✓ Homogenous Partial Differential equation
- ✓ C.F. for Homogenous Partial Differential equation.
- ✓ P.I. for Homogenous Partial Differential equation.
- ✓ Non-Homogenous Partial Differential equation
- ✓ C.F. for Non-Homogenous Partial Differential equation.
- ✓ P.I. for Non-Homogenous Partial Differential equation.

Topic objective (CO3)

Classification of linear PDE

- To equip linear Partial Differential with different methods.
- Classify the fundamental principals of partial differential equations(PDEs).
- To learn mathematical formulations of phenomena in physics and engineering as well as biological processes among many other scenarios.

Classification of linear PDE(CO3)

Classification of linear partial differential equation of second order.

- When the differential equation of the 2nd order in two independent variables x and y

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0\ldots\ldots(1)$$

Where A,B, C are constants or continuous function of x and y possessing continuous partial derivatives and A is positive.

From equation (1)

i. Elliptic if $B^2-4AC<0$

ii. Hyperbolic if $B^2-4AC>0$

iii. Parabolic if $B^2-4AC=0$

Classification of linear PDE(CO3)

Examples:

Q 1. Classify the following PDE's:

i. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

Sol: $A = 1, B = 1, C = 1$

$B^2 - 4AC = (1)^2 - 4(1)(1) = -3 < 0$

So given operator is Elliptic.

ii. $z_{xx} = z_{tt}$

Sol: $z_{xx} - z_{tt} = 0$

$A = 1, B = 0, C = -1$

Classification of linear PDE(CO3)

$B^2 - 4AC = (0)^2 - 4(1)(-1) = 4 > 0$

So given operator is hyperbolic.

$iii. \quad z_{xx} + z_{tt} = 0$

Sol: $A = 1, B = 0, C = 1$
$B^2 - 4AC = (0)^2 - 4(1)(1) = -4 < 0$

So given operator is Elliptic.

$iv. \quad z_{xx} = z_y$

Sol: $z_{xx} - z_y = 0$
$A = 1, B = 0, C = 0$
$B^2 - 4AC = (0)^2 - 4(1)(0) = 0$

So given operator is parabolic.

Classification of linear PDE(CO3)

Q2. Classify the equation:

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0$$

Sol: $\mathrm{A} = (1-x^2), B = -2xy, C = (1-y^2)$

$$B^2 - 4AC = (-2xy)^2 - 4(1-x^2)(1-y^2)$$
$$= 4x^2y^2 - 4 + 4x^2 + 4y^2 - 4x^2y^2$$
$$= 4(x^2 + y^2 - 1)$$

If $x^2 + y^2 > 1$then operator is hyperbolic.

If $x^2 + y^2 < 1$then operator is elliptic.

If $x^2 + y^2 = 1$ then operator is parabolic.

Q3. show that the equation $u_{xx} + xu_{yy} + u_y = 0$ is elliptic for $x > 0$ and hyperbolic for $x < 0$.

Classification of linear PDE(CO3)

Sol: $\mathrm{A} = 1, B = 0, C = x$
$B^2 - 4AC = (0)^2 - 4(1)(x) = -4\mathrm{x}$

If $x < 0$ then operator is hyperbolic.

If $x >$ 0 then operator is elliptic.

Q4. show that the equation $z_{xx} + 2xz_{xy} + (1 - y^2)z_{yy} = 0$ is elliptic for values of x and y in the region $x^2 + y^2 < 1$, parabolic on the boundary and hyperbolic outside this region.

Sol: $\mathrm{A} = 1, B = 2x, C = (1 - y^2)$
$B^2 - 4AC = (2x)^2 - 4(1)(1 - y^2) = 4(x^2 + y^2 - 1)$

If $x^2 + y^2 > 1$then operator is hyperbolic outside this region.

If $x^2 + y^2 < 1$then operator is elliptic.

If $x^2 + y^2 = 1$ then operator is parabolic on the boundary.

Daily Quiz(CO3)

Q1. Classify the following operator

$$t\frac{\partial^2 u}{\partial t^2} + 3\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial^2 u}{\partial x^2} + 17\frac{\partial u}{\partial x}$$

Q2. Classify the equation: $y^2 r - 2xys + x^2 t = \frac{y^2}{z}p + \frac{x^2}{y}q$

Recap (CO3)

- ✓ Classification of PDE

Topic objective (CO3)

Method of separation of variables

- Familiarized techniques to solve partial differential equations and is based on the assumption that the solution of the equation is separable.
- The final solution can be represented as a product of several functions.

Method of separation of variables(CO3)

Method of separation of variables:

This method we separate the variables by assumes the solution of the partial differential equation to be the product of two functions which involves only one of the variable.

Q1. Solve by the method of separation of variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

Sol: Given equation is $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ ......(1)

In this equation u dependent on x and y so the give equation has the solution like

$u(x, y) = X(x)Y(y)$......(2)

Method of separation of variables(CO3)

Now by equation (2) $\frac{\partial u}{\partial x} = YX'$

$\frac{\partial u}{\partial y} = XY'$

Putting these values equation (1) we get

$YX' = XY'$

$\frac{X'}{X} = \frac{Y'}{Y} = k\ (says)$

Taking first and last part

$\Rightarrow \frac{X'}{X} = k$

Method of separation of variables(CO3)

$$\Rightarrow \frac{dX}{Xdx} = k$$

$$\Rightarrow \frac{dX}{X} = kdx$$

Now integrating

$$\Rightarrow \log X = kx + \log c_1$$

$$\Rightarrow \log X - \log c_1 \quad = kx$$

$$\Rightarrow \log \frac{X}{c_1} = kx$$

$$\Rightarrow \frac{X}{c_1} = e^{kx}$$

$$\Rightarrow X = c_1 e^{kx} \text{......(3)}$$

Method of separation of variables(CO3)

Now Taking second and last part

$$\frac{Y'}{Y} = k$$

$$\Longrightarrow \frac{dY}{Ydy} = k$$

$$\Longrightarrow \frac{dY}{Y} = kd\mathrm{y}$$

Now integrating

$$\Longrightarrow \log Y = ky + \log c_2$$

$$\Longrightarrow \log Y - \log c_2 \;\; = ky$$

$$\Longrightarrow \log \frac{y}{c_2} = ky$$

Method of separation of variables(CO3)

$$\Longrightarrow \frac{Y}{c_2} = e^{ky}$$

$\Longrightarrow Y = c_2 e^{ky}$……(4)

Now by equation (3) and (4) in equation (2)

$$u = XY = c_1 e^{kx} c_2 e^{ky} = c_1 c_2 e^{kx+ky}$$

Hence it is the solution of equation (1).

Q2. Solve by the method of separation of variables.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Sol: Given equation is $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$……(1)

Method of separation of variables(CO3)

In this equation u dependent on x and y so the give equation has the solution like

$u(x, y) = X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial u}{\partial x} = YX'$

$\frac{\partial u}{\partial y} = XY'$

Putting these values equation (1) we get

$xYX' + yXY' = 0$

$\frac{xX'}{X} = \frac{-yY'}{Y} = k\ (says)$

Method of separation of variables(CO3)

Taking first and last part

$$\Rightarrow \frac{xX'}{X} = k$$

$$\Rightarrow \frac{xdX}{Xdx} = k$$

$$\Rightarrow \frac{dX}{X} = k\frac{dx}{x}$$

Now integrating

$$\Rightarrow \log X = k \log x + \log c_1$$

$$\Rightarrow \log X - \log c_1 \;\; = k \log x$$

$$\Rightarrow \log \frac{X}{c_1} = k \log x$$

Method of separation of variables(CO3)

$$\Longrightarrow \frac{X}{c_1} = x^k$$

$$\Longrightarrow X = c_1 x^k \ldots\ldots(3)$$

Now Taking second and last part

$$\frac{-yY'}{Y} = k$$

$$\Longrightarrow \frac{-ydY}{Ydy} = k$$

$$\Longrightarrow \frac{dY}{Y} = -k\frac{dy}{y}$$

Now integrating

Method of separation of variables(CO3)

$$\Longrightarrow \log Y = -k \log y + \log c_2$$
$$\Longrightarrow \log Y - \log c_2 \;\; = -k \log y$$
$$\Longrightarrow \log \frac{y}{c_2} = -k \log y$$
$$\Longrightarrow \frac{Y}{c_2} = y^{-k}$$
$$\Longrightarrow Y = c_2 y^{-k} \text{......(4)}$$

Now by equation (3) and (4) in equation (2)

$$u = XY = c_1 x^k c_2 y^{-k} = c_1 c_2 \left(\frac{x}{y}\right)^k$$

Hence it is the solution of equation (1).

Method of separation of variables(CO3)

Q3. Solve by the method of separation of variables.

$$y^3\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}=0$$

Sol: Given equation is $y^3\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial y}=0$……(1)

In this equation u dependent on x and y so the give equation has the solution like

$u(x,y)=X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial u}{\partial x}=YX'$

$\frac{\partial u}{\partial y}=XY'$

Putting these values equation (1) we get

Method of separation of variables(CO3)

$$y^3YX' + x^2XY' = 0$$

$$\frac{X'}{x^2X} = \frac{-Y'}{y^3Y} = k\ (says)$$

Taking first and last part

$$\Longrightarrow \frac{X'}{x^2X} = k$$

$$\Longrightarrow \frac{dX}{x^2Xdx} = k$$

$$\Longrightarrow \frac{dX}{X} = kx^2\mathrm{dx}$$

Now integrating

Method of separation of variables(CO3)

$$\Longrightarrow \log X = k\frac{x^3}{3} + \log c_1$$

$$\Longrightarrow \log X - \log c_1 \quad = k\frac{x^3}{3}$$

$$\Longrightarrow \log\frac{X}{c_1} = k\frac{x^3}{3}$$

$$\Longrightarrow \frac{X}{c_1} = e^{k\frac{x^3}{3}}$$

$$\Longrightarrow X = c_1 e^{k\frac{x^3}{3}} \ldots\ldots(3)$$

Now Taking second and last part

$$\frac{-Y'}{y^3 Y} = k$$

Method of separation of variables(CO3)

$$\Longrightarrow \frac{-dY}{Yy^3dy} = k$$

$$\Longrightarrow \frac{dY}{Y} = -ky^3dy$$

Now integrating

$$\Longrightarrow \log Y = -k\frac{y^4}{4} + \log c_2$$

$$\Longrightarrow \log Y - \log c_2 \quad = -k\frac{y^4}{4}$$

$$\Longrightarrow \log \frac{Y}{c_2} = -k\frac{y^4}{4}$$

$$\Longrightarrow \frac{Y}{c_2} = e^{-k\frac{y^4}{4}}$$

$\Longrightarrow Y = c_2 e^{-k\frac{y^4}{4}}$......(4)

Now by equation (3) and (4) in equation (2)

$$u = XY = c_1 e^{k\frac{x^3}{3}} c_2 e^{-k\frac{y^4}{4}} = c_1 c_2 e^{k\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$$

Hence it is the solution of equation (1).

Q4. solve by the method of separation of variables.

$$x\frac{\partial^2 u}{\partial x \partial y} + 2yu = 0$$

Sol: Given equation is $x\ \frac{\partial^2 u}{\partial x \partial y} + 2yu = 0$......(1)

In this equation u dependent on x and y so the give equation

has the solution like
$u(x,y) = X(x)Y(y)$......(2)
Now by equation (2) $\frac{\partial u}{\partial x} = YX'$
$\frac{\partial^2 u}{\partial x \partial y} = X'Y'$
Putting these values equation (1) we get
$xX'Y' + 2yXY = 0$
$\frac{xX'}{X} = \frac{-2yY}{Y'} = k\ (says)$

Method of separation of variables(CO3)

has the solution like

$u(x,y) = X(x)Y(y)$......(2)

Now by equation (2) $\frac{\partial u}{\partial x} = YX'$

$$\frac{\partial^2 u}{\partial x \partial y} = X'Y'$$

Putting these values equation (1) we get

$$xX'Y' + 2yXY = 0$$

$$\frac{xX'}{X} = \frac{-2yY}{Y'} = k\ (says)$$

Method of separation of variables(CO3)

Taking first and last part

$$\Rightarrow \frac{xX'}{X} = k$$

$$\Rightarrow \frac{xdX}{Xdx} = k$$

$\Rightarrow \frac{dX}{X} = k\,\frac{dx}{x}$Now integrating

$$\Rightarrow \log X = k \log x + \log c_1$$

$$\Rightarrow \log X - \log c_1 \quad = k \log x$$

$$\Rightarrow \log \frac{X}{c_1} = k \log x$$

$$\Rightarrow \frac{X}{c_1} = x^k$$

Method of separation of variables(CO3)

$$\Rightarrow X = c_1 x^k ......(3)$$

Now Taking second and last part

$$\frac{-2yY}{Y'} = k$$

$$\Rightarrow \frac{-2Yydy}{dY} = k$$

$$\Rightarrow \frac{dY}{Y} = -\frac{2ydy}{k}$$

Now integrating

$$\Rightarrow \log Y = -\frac{y^2}{k} + \log c_2$$

Method of separation of variables(CO3)

$$\Rightarrow \log Y - \log c_2 \;\; = -\frac{y^2}{k}$$

$$\Rightarrow \log\frac{Y}{c_2} = -\frac{y^2}{k}$$

$$\Rightarrow \frac{Y}{c_2} = e^{-\frac{y^2}{k}}$$

$$\Rightarrow Y = c_2 e^{-\frac{y^2}{k}} \text{......(4)}$$

Now by equation (3) and (4) in equation (2)

$$u = XY = c_1 x^k c_2 e^{-\frac{y^2}{k}} = c_1 c_2 x^k e^{-\frac{y^2}{k}}$$

Hence it is the solution of equation (1).

Method of separation of variables(CO3)

Q5. Solve by the method of separation of variables.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

Sol: Given equation is $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$……(1)

In this equation u dependent on x and y so the give equation has the solution like

$u(x, y) = X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial u}{\partial x} = YX'$

$$\frac{\partial^2 u}{\partial x^2} = X''Y$$

Method of separation of variables(CO3)

$$\frac{\partial u}{\partial y} = XY'$$

Putting these values equation (1) we get

$$X''Y - XY' = 0$$

Now separating X variable with X' and Y variale with Y'

WE HAVE

$$\frac{X''}{X} = \frac{Y'}{Y} = k\ (says)$$

Taking first and last part

$$\Rightarrow \frac{X''}{X} = k$$

$$\Rightarrow X'' = Xk$$

$$\Rightarrow X'' - Xk = 0$$

Method of separation of variables(CO3)

Auxiliary equation

$m^2 - k = 0$

$\Rightarrow m = \pm\sqrt{k}$

$X = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right)$......(4)

Now Taking second and last part

$\frac{Y'}{Y} = k$

$\Longrightarrow \frac{dY}{Ydy} = k$

$\Longrightarrow \frac{dY}{Y} = kdy$

Now integrating

Method of separation of variables(CO3)

$\Longrightarrow \log Y \; = ky + \log c_3$

$\Longrightarrow \log Y - \log c_3 = ky$

$\Longrightarrow \log \frac{y}{c_3} = ky$

$\Longrightarrow \frac{Y}{c_3} = e^{ky}$

$\Longrightarrow Y = c_3 e^{ky}$……(4)

Now by equation (3) and (4) in equation (2)

$$u = XY = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right) c_3 e^{ky}$$

Hence it is the solution of equation (1).

Method of separation of variables(CO3)

Q6. Solve by the method of separation of variables.

$$3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0; u(x,0)=4e^{-x}$$

Sol: Given equation is $3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$ ……(1)

In this equation u dependent on x and y so the give equation has the solution like

$u(x,y)=X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial u}{\partial x}=YX'$

$\frac{\partial u}{\partial y}=XY'$

Putting these values equation (1) we get

Method of separation of variables(CO3)

$$3YX' + 2XY' = 0$$

$$\frac{X'}{X} = -\frac{2Y'}{3Y} = k\ (says)$$

Taking first and last part

$$\Rightarrow \frac{X'}{X} = k$$

$$\Rightarrow \frac{dX}{Xdx} = k$$

$$\Rightarrow \frac{dX}{X} = kdx$$

Method of separation of variables(CO3)

Now integrating

$$\Longrightarrow \log X = kx + \log c_1$$

$$\Longrightarrow \log X - \log c_1 \;\; = kx$$

$$\Longrightarrow \log \frac{X}{c_1} = kx$$

$$\Longrightarrow \frac{X}{c_1} = e^{kx}$$

$$\Longrightarrow X = c_1 e^{kx} \text{......(3)}$$

Now Taking second and last part

$$\frac{-2Y\prime}{3Y} = k$$

$$\Longrightarrow \frac{-2dY}{3Ydy} = k$$

Method of separation of variables(CO3)

$$\Rightarrow \frac{dY}{Y} = \frac{-3}{2} k d\mathrm{y}$$

Now integrating

$$\Rightarrow \log Y = \frac{-3}{2} ky + \log c_2$$

$$\Rightarrow \log Y - \log c_2 \;\; = \frac{-3}{2} ky$$

$$\Rightarrow \log \frac{Y}{c_2} = \frac{-3}{2} ky$$

$$\Rightarrow \frac{Y}{c_2} = e^{\frac{-3}{2} ky}$$

$$\Rightarrow Y = c_2 e^{\frac{-3}{2} ky} \text{......(4)}$$

Method of separation of variables(CO3)

Now by equation (3) and (4) in equation (2)

$$u(x,y) = XY = c_1 e^{kx} c_2 e^{-\frac{3}{2}ky} = c_1 c_2 e^{kx-\frac{3}{2}ky} \;\ldots\ldots(5)$$

Using condition $u(x,0) = 4e^{-x}$

$$u(x,0) = c_1 c_2 e^{kx}$$

$$4e^{-x} = c_1 c_2 e^{kx}$$

$$c_1 c_2 = 4, k = -1$$

Putting these values in equation (5),

$$u = 4e^{-(x-\frac{3}{2}y)} \;\ldots\ldots(5)$$

Hence it is the solution of equation (1).

Q7. Solve by the method of separation of variables.

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0; z(0,y) = 2e^{-y}$$

Method of separation of variables(CO3)

Sol: Given equation is $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0$……(1)

In this equation u dependent on x and y so the give equation has the solution like

$z(x, y) = X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial z}{\partial x} = YX'$

$$\frac{\partial z}{\partial y} = XY'$$

Putting these values equation (1) we get

$2YX' + 3XY' + 5XY = 0$

Method of separation of variables(CO3)

We have

$$\frac{2X\prime}{X} = -\frac{3Y'}{Y} - 5 = k\ (says)$$

Taking first and last part

$$\Longrightarrow \frac{2X'}{X} = k$$

$$\Longrightarrow \frac{2dX}{Xdx} = k$$

$$\Longrightarrow \frac{dX}{X} = \frac{1}{2}kdx$$

Now integrating

$$\Longrightarrow \log X = \frac{1}{2}kx + \log c_1$$

Method of separation of variables(CO3)

$$\Rightarrow log\, X = \frac{1}{2}kx + log\, c_1$$

$$\Rightarrow log\, X - log\, c_1 \ = \frac{1}{2}kx$$

$$\Rightarrow log\frac{X}{c_1} = \frac{1}{2}kx$$

$$\Rightarrow \frac{X}{c_1} = e^{\frac{1}{2}kx}$$

$$\Rightarrow X = c_1 e^{\frac{1}{2}kx} \ldots\ldots(3)$$

Now Taking second and last part

$$-\frac{3Y'}{Y} - 5 = k$$

Method of separation of variables(CO3)

$$\Rightarrow \frac{-3dY}{Ydy} = k + 5$$

$$\Rightarrow \frac{dY}{Y} = -\frac{1}{3}(k+5)dy$$

Now integrating

$$\Rightarrow \log Y = -\frac{1}{3}(k+5)y + \log c_2$$

$$\Rightarrow \log Y - \log c_2 = -\frac{1}{3}(k+5)y$$

$$\Rightarrow \log \frac{Y}{c_2} = -\frac{1}{3}(k+5)y$$

$$\Rightarrow \frac{Y}{c_2} = e^{-\frac{1}{3}(k+5)y}$$

Method of separation of variables(CO3)

$\Longrightarrow Y = c_2 e^{-\frac{1}{3}(k+5)y}$......(4)

Now by equation (3) and (4) in equation (2)

$z(x,y) = XY = c_1 e^{\frac{1}{2}kx} c_2 e^{-\frac{1}{3}(k+5)y} = c_1 c_2 e^{\frac{1}{2}kx-\frac{1}{3}(k+5)y} \ldots \ldots.(5)$

Using condition $z(0,y) = 2e^{-y}$

$z(0,y) = c_1 c_2 e^{-\frac{1}{3}(k+5)y}.$

$2e^{-y} = c_1 c_2 e^{-\frac{1}{3}(k+5)y}$

$c_1 c_2 = 2, -\frac{1}{3}(k+5) = -1 \; so \; k = -2$

Putting these values in equation (5),

$z = 2e^{-(x+y)} \ldots \ldots.(5)$

Hence it is the solution of equation (1).

Method of separation of variables(CO3)

Q8. Solve by the method of separation of variables.

$$4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u; u(0,y)=4e^{-y}-e^{-5y}$$

Sol: Given equation is $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$……(1)

$u(x,y)=X(x)Y(y)$……(2)

Now by equation (2) $\frac{\partial u}{\partial x}=YX'$

$\frac{\partial u}{\partial y}=XY'$

Putting these values equation (1) we get

Method of separation of variables(CO3)

$$4YX' + XY' = 3XY$$

$$\frac{4X'}{X} = 3 - \frac{Y'}{Y} = k\ (says)$$

Taking first and last part

$$\Rightarrow \frac{4X'}{X} = k$$

$$\Rightarrow \frac{4dX}{Xdx} = k$$

$$\Rightarrow \frac{dX}{X} = \frac{1}{4}kdx$$

Now integrating

Method of separation of variables(CO3)

$$\Longrightarrow \log X = \frac{1}{4}kx + \log c_1$$

$$\Longrightarrow \log X - \log c_1 \quad = \frac{1}{4}kx$$

$$\Longrightarrow \log \frac{X}{c_1} = \frac{1}{4}kx$$

$$\Longrightarrow \frac{X}{c_1} = e^{\frac{1}{4}kx}$$

$$\Longrightarrow X = c_1 e^{\frac{1}{4}kx} \text{......(3)}$$

Now Taking second and last part

$$3 - \frac{Y'}{Y} = k$$

Method of separation of variables(CO3)

$$\Rightarrow \frac{dY}{Ydy} = 3 - k$$

$$\Rightarrow \frac{dY}{Y} = (3 - k)dy$$

Now integrating

$$\Rightarrow \log Y = (3 - k)y + \log c_2$$

$$\Rightarrow \log Y - \log c_2 \quad = (3 - k)y$$

$$\Rightarrow \log \frac{y}{c_2} = (3 - k)y$$

$$\Rightarrow \frac{Y}{c_2} = e^{(3-k)y}$$

$$\Rightarrow Y = c_2 e^{(3-k)y} \text{......(4)}$$

Now by equation (3) and (4) in equation (2)

Method of separation of variables(CO3)

$$u = XY = c_1 e^{\frac{1}{4}kx} c_2 e^{(3-k)y} = c_1 c_2 e^{\frac{1}{4}kx+(3-k)y} \ldots \ldots (5)$$

General solution

$$u(x,y) = XY = \sum b_n e^{\frac{1}{4}kx+(3-k)y} \ldots \ldots (6)$$

Using condition $u(0,y) = 4e^{-y} - e^{-5y}$

$u(0,y) = \sum b_n e^{(3-k)y}$.

$$4e^{-y} - e^{-5y} = \sum b_n e^{(3-k)y}$$

$k = 4, b_1 = 4 \;\& k = 8, b_2 = -1$

Putting these values in equation (6),

$u = 4e^{-(x+y)} - e^{-(2x+5y)} \ldots \ldots (5)$

Hence it is the solution of equation (1).

Daily Quiz (CO3)

Solve the equation by method of separation of variables.

1. $\frac{\partial^2 u}{\partial x^2} = 2u + \frac{\partial u}{\partial y}$
2. $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0$
3. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}; u(0,y) = 8e^{-y}$ Ans: $u(x,y) = 8e^{-3y-12x}$
4. $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0; z(x,0) = 0; z(x,\pi) = 0; z(0,y) = 4\sin 3y$

Ans: $z(x,y) = 4e^{9x}\sin 3y$

Recap (CO3)

- ✓ Classification of PDE
- ✓ Variable separation Method

Topic objective (CO3)

Wave equation

- It tells us how the displacement u can change as a function of position and time and the function. The solutions to the wave equation (u(x,t)) are obtained by appropriate integration techniques.

Wave equation(CO3)

Solution of one dimensional wave equation:

1-d wave equation is given by-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

The solution of (1) is $u(x,t)$ which gives the displacement at any point x at any time t.

Using method of separation of variable

$$u(x,t) = X(x).T(t) \cdots\cdots (2)$$

Differentiate (2) partially w.r.t. x two times

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = X''\, T$$

Differentiate (2) partially w.r.t. t two times

Wave equation(CO3)

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = X\,T''$$

Now equation (1) becomes

$$\Rightarrow XT'' = c^2 X'' \text{ T}$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = k(constant)$$

Here k have 3 possibilities

Case-1: $\boldsymbol{k = 0}$

Case-2: $\boldsymbol{k = p^2}$

Case-3: $\boldsymbol{k = -p^2}$

Wave equation(CO3)

Case-1: When $k = 0$

Now $\dfrac{X''}{X} = 0$

$\Rightarrow X'' = 0$

$\Rightarrow \dfrac{d^2X}{dx^2} = 0$

$\Rightarrow \dfrac{dX}{dx} = c_1$

$\Rightarrow X(x) = c_1x + c_2$

Again

$\Rightarrow \dfrac{T''}{c^2T} = 0$

Wave equation(CO3)

$\Rightarrow \dfrac{d^2T}{dt^2} = 0$

$\Rightarrow T(t) = c_3 t + c_4$

$\because u(x,t) = XT$

$\Rightarrow u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

Case-2: When $k = p^2$

Now $\dfrac{X''}{X} = p^2$

$\Rightarrow X'' = p^2 X$

$\Rightarrow \dfrac{d^2X}{dx^2} = p^2 X$

$\Rightarrow \dfrac{d^2X}{dx^2} - p^2 X = 0$

Wave equation(CO3)

$\Rightarrow\ (D^2 - p^2)X = 0$

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

$\Rightarrow m^2 - p^2 = 0$

$\Rightarrow m^2 = p^2$

$\Rightarrow m = \pm p$

$\Rightarrow X(x) = c_1 e^{px} + c_2 e^{-px}$

Again

$\Rightarrow \frac{T''}{c^2 T} = p^2$

$\Rightarrow \frac{d^2T}{dt^2} = c^2 p^2 T$

Wave equation(CO3)

$$\Rightarrow \frac{d^2T}{dt^2} - c^2p^2T = 0$$

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

$$\Rightarrow m^2 - c^2p^2 = 0$$

$$\Rightarrow m = \pm cp$$

Then

$$\Rightarrow T(t) = (c_3e^{cpt} + c_4e^{-cpt})$$

$$\because u(x,t) = XT$$

$$\Rightarrow u(x,t) = \ (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt})$$

Wave equation(CO3)

Case-3: When $k = -p^2$

Now $\frac{X''}{X} = -p^2$

$\Rightarrow X'' = -p^2 X$

$\Rightarrow \frac{d^2X}{dx^2} = -p^2 X$

$\Rightarrow \frac{d^2X}{dx^2} + p^2 X = 0$

$\Rightarrow (D^2 + p^2)X = 0$

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

$\Rightarrow m^2 + p^2 = 0$

Wave equation(CO3)

$$\Rightarrow m^2 = -p^2$$

$$\Rightarrow m = \pm i\, p$$

$$\Rightarrow X(x) = c_1 \cos px + c_2 \sin px$$

Again

$$\Rightarrow \frac{T''}{c^2 T} = -p^2$$

$$\Rightarrow \frac{d^2T}{dt^2} = -c^2 p^2 \mathrm{T}$$

$$\Rightarrow \frac{d^2T}{dt^2} + c^2 p^2 T = 0$$

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

Wave equation(CO3)

$\Rightarrow m^2 + c^2p^2 = 0$

$\Rightarrow m = \pm icp$

$\Rightarrow T(t) = c_3 \cos cpt + c_4 \sin cpt$

$\because u(x,t) = XT$

$\Rightarrow u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

As we dealing with problem on vibration of string, So solution will be periodic then solution will be-

$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

Example-1: A string end fastened to two points l apart. Motion is started in the string in the form $u = a \sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t = 0$. Show that the displacement of any point at a

Wave equation(CO3)

distance x from one end at time t is given by-

$$u(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$$

Sol: 1-d wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

s.t.

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = a\sin\left(\frac{\pi x}{l}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

Wave equation(CO3)

Solution of (1) by method of separation variable

$\Rightarrow \quad u(x,t) = \ (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$\because u(0,t) = 0$

$\Rightarrow \ (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1 = 0 \qquad \because \ (c_3 \cos cpt + c_4 \sin cpt) \neq 0.$

Now solution becomes

$\Rightarrow \quad u(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)$

$\because u(l,t) = 0$

$\Rightarrow \quad c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$

Wave equation(CO3)

$\Rightarrow \quad c_2 \sin pl = 0$

$\Rightarrow \quad \sin pl = 0 \qquad\qquad$ from here $c_2 \neq 0$

$\Rightarrow \quad \sin pl = \sin n\pi$

$\Rightarrow \quad pl = n\pi$

$\Rightarrow \quad p = \frac{n\pi}{l}$

Now solution becomes

$$\Rightarrow \quad u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\left(\frac{n\pi ct}{l}\right) + c_4 \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$\because \quad \frac{\partial u}{\partial t}(x,0) = 0$

Now

Wave equation(CO3)

$$\frac{\partial u}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \sin\left(\frac{n\pi ct}{l}\right).\frac{n\pi c}{l} + c_4 \cos\left(\frac{n\pi ct}{l}\right).\frac{n\pi c}{l}\right]$$

Put $t = 0$

$$\Rightarrow 0 = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \times 0 \ + c_4 \cos(0).\frac{n\pi c}{l}\right]$$

$$\Rightarrow c_2 c_4 \sin\left(\frac{n\pi x}{l}\right)\frac{n\pi c}{l} = 0$$

$\Rightarrow c_4 = 0$ from here $c_2 \neq 0$

Now solution becomes

$$\Rightarrow \ u(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$\Rightarrow \ u(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$ where $b_n = c_2 c_3$

Wave equation(CO3)

Now complete solution is

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$\because u(x,0) = a \sin\left(\frac{\pi x}{l}\right)$$

Then $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$\Rightarrow a \sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow a \sin\left(\frac{\pi x}{l}\right) = b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots\cdots$$

On comparing

$$b_1 = a, b_2 = 0 \ldots\ldots\ldots\ldots$$

Put these value in complete solution we get required solution is

Wave equation(CO3)

$$\Rightarrow u(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$$

Example-2 A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 sin^3\left(\frac{\pi x}{l}\right)$.Find the displacement.

Sol: 1-d wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \cdots\cdots(1)$$

s.t.

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = 0$$

Wave equation(CO3)

$$\frac{\partial u}{\partial t}(x,0) = v_0 sin^3\left(\frac{\pi x}{l}\right)$$

Solution of (1) by method of separation variable

$$\Rightarrow \quad u(x,t) = \ (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\because u(0,t) = 0$$

$$\Rightarrow \ (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

$$\Rightarrow c_1 = 0 \qquad\qquad \because \ (c_3 \cos cpt + c_4 \sin cpt) \neq 0.$$

Now solution becomes

$$\Rightarrow \quad u(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)$$

Wave equation(CO3)

$\because u(l,t) = 0$

$\Rightarrow \quad c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow \quad c_2 \sin pl = 0$

$\Rightarrow \quad \sin pl = 0$ from here $c_2 \neq 0 \Rightarrow \quad \sin pl = \sin n\pi$

$\Rightarrow \quad pl = n\pi$

$\Rightarrow \quad p = \frac{n\pi}{l}$

Now solution becomes

$$\Rightarrow \quad u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\left(\frac{n\pi ct}{l}\right) + c_4 \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$\because u(x,0) = 0$

$$\Rightarrow \quad 0 = c_2 \sin\left(\frac{n\pi x}{l}\right)[c_3.1 + c_4.0]$$

Wave equation(CO3)

$$\Rightarrow c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) = 0$$

$\Rightarrow \; c_3 = 0 \quad$ from here $c_2 \neq 0.$

$$\Rightarrow \; u(x,t) = c_2 c_4 \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$$

The complete solution is given by-

$$\Rightarrow \; u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$$

Now $\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right) \frac{n\pi c}{l}$

$$\because \; \frac{\partial u}{\partial t}(x,0) = v_0 sin^3\left(\frac{\pi x}{l}\right)$$

So put $t = 0$

Wave equation(CO3)

Now

$\Rightarrow v_0 sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\frac{n\pi c}{l}$

$\because \sin 3A = 3\sin A - 4\, sin^3 A$

$\Rightarrow sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$

Now $\Rightarrow \frac{v_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] =$

$b_1.\frac{\pi c}{l}\sin\left(\frac{\pi x}{l}\right) + b_2.\frac{2\pi c}{l}\sin\left(\frac{2\pi x}{l}\right) + b_3.\frac{3\pi c}{l}\sin\left(\frac{3\pi x}{l}\right) + \cdots$

Comparing coefficient

$b_1 = \frac{3v_0 l}{4c\pi}$

$b_2 = 0$

Wave equation(CO3)

$b_3 = -\frac{lv_0}{12c\pi}$

$b_4 = 0$ ……….

Then solution is given by

$$u(x,t) = \frac{3v_0 l}{4c\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right) - \frac{lv_0}{12c\pi}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi ct}{l}\right)$$

Example-3. A tightly stretched flexible string has its end fixed at $x = 0$ and $x = l$. At time $t = 0$ the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is constant and then released. Find the displacement $\mathrm{u}(x,t)$ of any point x of the string at any time $t > 0$.

Wave equation(CO3)

Sol: 1-d wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots \cdots (1)$$

s.t.

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = \mu x(l-x),$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

Solution of (1) by method of separation variable

$$\Rightarrow \quad u(x,t) = \ (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

Wave equation(CO3)

$\because u(0,t) = 0$

$\Rightarrow \ (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\Rightarrow c_1 = 0 \qquad \because \ (c_3 \cos cpt + c_4 \sin cpt) \neq 0.$

Now solution becomes

$\Rightarrow \ u(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)$

$\because u(l,t) = 0$

$\Rightarrow \ c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$

Wave equation(CO3)

$\Rightarrow \quad c_2 \sin pl = 0$

$\Rightarrow \quad \sin pl = 0 \qquad\qquad \text{from here } \; c_2 \neq 0$

$\Rightarrow \quad \sin pl = \sin n\pi$

$\Rightarrow \quad pl = n\pi$

$\Rightarrow \quad p = \dfrac{n\pi}{l}$

Now solution becomes

$$\Rightarrow \quad u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\left(\frac{n\pi ct}{l}\right) + c_4 \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$$\because \quad \frac{\partial u}{\partial t}(x,0) = 0$$

Now

Wave equation(CO3)

$$\frac{\partial u}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \sin\left(\frac{n\pi ct}{l}\right).\frac{n\pi c}{l} + c_4 \cos\left(\frac{n\pi ct}{l}\right).\frac{n\pi c}{l}\right]$$

Put $t = 0$

$$\Rightarrow 0 = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \times 0 \; + c_4 \cos(0).\frac{n\pi c}{l}\right]$$

$$\Rightarrow c_2 c_4 \sin\left(\frac{n\pi x}{l}\right)\frac{n\pi c}{l} = 0$$

$\Rightarrow c_4 = 0$ from here $c_2 \neq 0$

Now solution becomes

$$\Rightarrow \; u(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$\Rightarrow \; u(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \text{ where } b_n = c_2 c_3$$

Wave equation(CO3)

Now complete solution is

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$\because u(x,0) = \mu x(l-x)$$

Then $u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$\Rightarrow \mu x(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \mu(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

This is half range Fourier sine series, so

$$b_n = \frac{2}{l}\int_0^l F(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\int_0^l \mu(lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{4\mu l^2}{n^3\pi^3}[1 - (-1)^n]$$

Hence the solution is given by

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \frac{4\mu l^2}{n^3\pi^3}[1 - (-1)^n]\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$\Rightarrow \quad u(x,t) = \frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{n^3}[1 - (-1)^n]\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right).$$

Daily Quiz (CO3)

Solve the equation by method of separation of variables.

1. A tightly stretched string with fix end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 sin^3 \left(\frac{\pi x}{l}\right)$.If it is released from rest from this position, find displacement $y(x, t)$.

2. The vibrations of an elastic string is given by the PDE: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$
The length of the string is π and ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(sinx + sin3x)$. Find the deflection $u(x, t)$ of the vibrating string at any time t.

Weekly Assignment(CO3)

1.Classify the PDE $4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

2. Solve the equation $2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} + 5u = 0; u(0, y) = 2e^{-y}$, by method of separation of variables.

3.Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u = 0; u(x, 0) = 10e^{-x} - 6e^{-4x}$, by method of separation of variables.

4.Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$, by method of separation of variables.

5. Classify the equation: $(1 - x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1 - y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0$.

Recap (CO3)

- ✓ Classification of PDE
- ✓ Variable separation Method
- ✓ Wave equation

Topic objective (CO3)

Heat equation

- Conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries. That is, we wish to know the temperature distribution, which represents how temperature varies with position in the medium.

Heat equation (CO3)

Solution of one dimensional heat equation:

1-d Heat equation is given by-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots \cdots (1)$$

The solution of (1) is $u(x, t)$ which gives the temperature at any point x at any time t.

Using method of separation of variable

$$u(x, t) = X(x).T(t) \cdots \cdots (2)$$

Differentiate (2) partially w.r.t. x two times

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = X''T$$

Differentiate (2) partially w.r.t. t

Heat equation (CO3)

$$\Rightarrow \frac{\partial u}{\partial t} = XT'$$

Now equation (1) becomes

$$\Rightarrow XT' = c^2 X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} = k(constant)$$

Here k have 3 possibilities

Case-1: $\boldsymbol{k = 0}$

Case-2: $\boldsymbol{k = p^2}$

Case-3: $\boldsymbol{k = -p^2}$

Case-1: When $k = 0$

Heat equation (CO3)

Now $\frac{X''}{X} = 0$

$\Rightarrow\ X''= 0$

$\Rightarrow\ \frac{d^2X}{dx^2} = 0$

$\Rightarrow\ \frac{dX}{dx} = c_1$

$\Rightarrow\ X(x) = c_1x + c_2$

Again

$\Rightarrow \frac{T'}{c^2T} = 0$

$\Rightarrow\ \frac{dT}{dt} = 0$

Heat equation (CO3)

$\Rightarrow\ T(t) = c_3$

$\because u(x,t) = XT$

$\Rightarrow u(x,t) = (c_1x + c_2)c_3$

Case-2: When $k = p^2$

Now $\dfrac{X''}{X} = p^2$

$\Rightarrow\ X'' = p^2X$

$\Rightarrow\ \dfrac{d^2X}{dx^2} = p^2X$

$\Rightarrow\ \dfrac{d^2X}{dx^2} - p^2X = 0$

$\Rightarrow\ (D^2 - p^2)X = 0$

Heat equation (CO3)

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

$$\Rightarrow m^2 - p^2 = 0$$

$$\Rightarrow m^2 = p^2$$

$$\Rightarrow m = \pm p$$

$$\Rightarrow X(x) = c_1 e^{px} + c_2 e^{-px}$$

Again

$$\Rightarrow \frac{T'}{c^2 T} = p^2$$

$$\Rightarrow \frac{dT}{dt} = c^2 p^2 T$$

$$\Rightarrow \frac{dT}{T} = c^2 p^2 dt$$

Heat equation (CO3)

Integrate both sides

$$\Rightarrow \log T = c^2p^2t + \log c_3$$

$$\Rightarrow \log T - \log c_3 = c^2p^2t$$

$$\Rightarrow \log \frac{T}{c_3} = c^2p^2t$$

$$\Rightarrow \frac{T}{c_3} = e^{c^2p^2t}$$

$$\Rightarrow T = c_3\, e^{c^2p^2t}$$

$$\because u(x,t) = XT$$

$$\Rightarrow u(x,t) = (c_1e^{px} + c_2e^{-px})\, c_3\, e^{c^2p^2t}$$

Case-3: When $k = -p^2$

Heat equation (CO3)

Now $\frac{X''}{X} = -p^2$

$\Rightarrow\ X'' = -p^2 X$

$\Rightarrow\ \frac{d^2X}{dx^2} = -p^2 X$

$\Rightarrow\ \frac{d^2X}{dx^2} + p^2 X = 0$

$\Rightarrow\ (D^2 + p^2)X = 0$

Which is linear differential equation with constant coefficient.

Auxiliary equation is given by-

$\Rightarrow m^2 + p^2 = 0$

Heat equation (CO3)

$$\Rightarrow m^2 = -p^2$$

$$\Rightarrow m = \pm i\, p$$

$$\Rightarrow X(x) = c_1 \cos px + c_2 \sin px$$

Again

$$\Rightarrow \frac{T'}{c^2 T} = -p^2$$

$$\Rightarrow \frac{dT}{dt} = -c^2 p^2 T$$

$$\Rightarrow \frac{dT}{T} = -c^2 p^2 dt$$

Integrate both sides

Heat equation (CO3)

$$\Rightarrow \log T = -c^2\, p^2 t + \log c_3$$

$$\Rightarrow \log T - \log c_3 = -c^2\, p^2 t$$

$$\Rightarrow \log \frac{T}{c_3} = -c^2\, p^2 t$$

$$\Rightarrow \frac{T}{c_3} = e^{-c^2 p^2 t}$$

$$\Rightarrow T = c_3\, e^{-c^2 p^2 t}$$

$$\because u(x,t) = XT$$

$$\Rightarrow u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2 p^2 t}$$

As we dealing with problem on heat conduction, it must be transient solution i.e. temperature u decrease with increase of time t.So solution is

Heat equation (CO3)

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2 p^2 t}$$

Example-1: Find the temperature in a bar of length 2 m whose end are kept at zero and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3sin\frac{5\pi x}{2}$.

Sol: 1-d Heat equation is given by

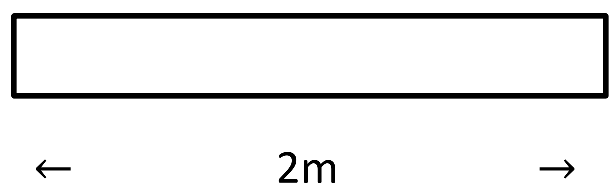
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

s.t.

$u(0,t) = 0$

$u(2,t) = 0$

$u(x,0) = \sin\frac{\pi x}{2} + 3sin\frac{5\pi x}{2}$



Heat equation (CO3)

Solution of (1) by method of separation variable

$\Rightarrow u(x,t) = \ (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t} \ \cdots (2)$

$\because u(0,t) = 0$

Put $x = 0$ in (2)

$\Rightarrow u(0,t) = \ (c_1 \cos 0 + c_2 \sin 0)\, c_3\, e^{-c^2p^2t}$

$\Rightarrow 0 = c_1 c_3 e^{-c^2p^2t}$

$\Rightarrow c_1 = 0$

from here $c_3 \neq 0$ & $e^{-c^2p^2t} \neq 0$

if $c_3 = 0$ then $u = 0.$

Now solution becomes if we put $c_1 = 0$

Heat equation (CO3)

$\Rightarrow u(x,t) = c_2\, c_3 \sin px\, e^{-c^2p^2t}$

Again $\because u(2,t) = 0$

Put $x = 2$ Then

$\Rightarrow u(2,t) = c_2\, c_3 \sin 2p\, e^{-c^2p^2t}$

$\Rightarrow c_2\, c_3 \sin 2p\, e^{-c^2p^2t} = 0$

$\Rightarrow \sin 2p = 0$ from here $c_2, c_3 \neq 0$ & $e^{-c^2p^2t} \neq 0$

if $c_3 = 0$ or $c_2 = 0$ then $u = 0$.

$\Rightarrow \sin 2p = \sin n\pi$

$\Rightarrow p = \frac{n\pi}{2}$

Now solution becomes

Heat equation (CO3)

$$\Rightarrow u(x,t) = c_2\, c_3 \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n^2\pi^2 c^2 t}{4}\right)}$$

$$\Rightarrow u(x,t) = b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n^2\pi^2 c^2 t}{4}\right)} \text{ where } b_n = c_2 c_3$$

Complete solution is given by-

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n^2\pi^2 c^2 t}{4}\right)}$$

$$\because u(x,0) = \sin\frac{\pi x}{2} + 3sin\frac{5\pi x}{2}$$

Put $t = 0$ we get

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Heat equation (CO3)

$$\Rightarrow \sin\frac{\pi x}{2} + 3sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} \; b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow \sin\frac{\pi x}{2} + 3sin\frac{5\pi x}{2}$$

$$= \; b_1 \sin\left(\frac{\pi x}{2}\right) + \; b_2 \sin\left(\frac{2\pi x}{2}\right) + \cdots + \; b_5 \sin\left(\frac{5\pi x}{2}\right) + \cdots$$

From here after comparing

$$b_1 = 1, b_2 = 0, b_3 = 0, b_4 = 0, b_5 = 3, b_6 = 0 \ldots$$

Now solution is $u(x,t) = \; \sin\left(\frac{\pi x}{2}\right) e^{-\left(\frac{\pi^2 c^2 t}{4}\right)} + 3\sin\left(\frac{5\pi x}{2}\right) e^{-\left(\frac{25\pi^2 c^2 t}{4}\right)}$

Heat equation (CO3)

Example-2: Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary condition $u(0,t) = 0, u(l,t) = 0 \;\&\; u(x,0) = 3\sin(n\pi x)\,, where\ 0 < x < l$

Sol: 1-d Heat equation is given by

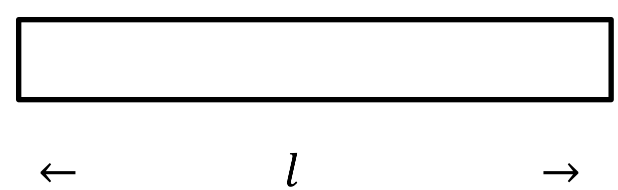
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

s.t.

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = 3\sin(n\pi x)$$



Heat equation (CO3)

Solution of (1) by method of separation variable

$\Rightarrow u(x,t) = \ (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-p^2 t} \ \cdots (2)$

$\because u(0,t) = 0$

Put $x = 0$ in (2)

$\Rightarrow u(0,t) = \ (c_1 \cos 0 + c_2 \sin 0)\, c_3\, e^{-p^2 t}$

$\Rightarrow 0 = c_1 c_3 e^{-p^2 t}$

$\Rightarrow c_1 = 0$

from here $c_3 \neq 0$ & $e^{-p^2 t} \neq 0$

if $c_3 = 0$ then $u = 0.$

Now solution becomes if we put $c_1 = 0$

Heat equation (CO3)

$\Rightarrow u(x,t) = \; c_2\, c_3 \sin px\, e^{-p^2 t}$

Again $\because u(l,t) = 0$

Put $x = l$ Then

$\Rightarrow u(l,t) = \; c_2\, c_3 \sin lp\, e^{-p^2 t}$

$\Rightarrow c_2\, c_3 \sin lp\, e^{-p^2 t} \; = 0$

$\Rightarrow \sin lp = 0$ from here $c_2, c_3 \neq 0$ & $e^{-p^2 t} \; \neq 0$

if $c_3 = 0$ or $c_2 = 0$ then $u = 0$.

$\Rightarrow \sin lp = \sin n\pi$

$\Rightarrow p = \dfrac{n\pi}{l}$

Now solution becomes

Heat equation (CO3)

$$\Rightarrow u(x,t) = c_2\, c_3 \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2 t}{l^2}\right)}$$

$$\Rightarrow u(x,t) = b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2 t}{l^2}\right)} \text{ where } b_n = c_2 c_3$$

Complete solution is given by-

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2 t}{l^2}\right)}$$

$\because u(x,0) = 3\sin(n\pi x)$

Put $t = 0$ we get

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow 3\sin(n\pi x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$\Rightarrow$ Comparison gives, $b_n = 3$ & $l = 1$

Now solution is $u(x,t) = \sum_{n=1}^{\infty} 3\sin\left(\frac{n\pi x}{1}\right) e^{-\left(\frac{n^2\pi^2 t}{1^2}\right)}$

$\Rightarrow u(x,t) = 3\sum_{n=1}^{\infty} \sin(n\pi x)\, e^{-(n^2\pi^2 t)}$.

Example-3: Solve the equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ under boundary condition

i. $u(0,t) = 0$

ii. $u(l,t) = 0$

iii. $u(x,0) = x$ between $x = 0$ and $x = l$

Where l being the length of bar.

Heat equation (CO3)

Sol: 1-d Heat equation is given by

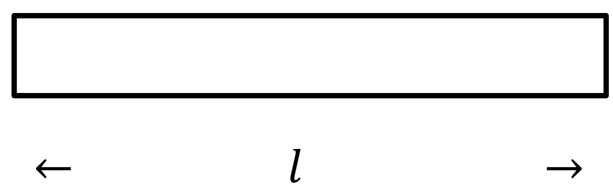
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

s.t.

i. $u(0,t) = 0$

ii. $u(l,t) = 0$

iii. $u(x,0) = x$ between $x = 0$ and $x = l$



Heat equation (CO3)

Solution of (1) by method of separation variable

$\Rightarrow u(x,t) = \ (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t} \ \cdots (2)$

$\because u(0,t) = 0$

Put $x = 0$ in (2)

$\Rightarrow u(0,t) = \ (c_1 \cos 0 + c_2 \sin 0)\, c_3\, e^{-c^2p^2t}$

$\Rightarrow 0 = c_1 c_3 e^{-c^2p^2t}$

$\Rightarrow c_1 = 0$

from here $c_3 \neq 0$ & $e^{-c^2p^2t} \neq 0$

if $c_3 = 0$ then $u = 0$.

Now solution becomes if we put $c_1 = 0$

Heat equation (CO3)

$\Rightarrow u(x,t) = \; c_2\, c_3 \sin px\, e^{-c^2p^2t}$

Again $\because u(l,t) = 0$

Put $x = l$ Then

$\Rightarrow u(l,t) = \; c_2\, c_3 \sin lp\, e^{-c^2p^2t}$

$\Rightarrow c_2\, c_3 \sin lp\, e^{-c^2p^2t} \;= 0$

$\Rightarrow \sin lp = 0$ from here $c_2, c_3 \neq 0$ & $e^{-c^2p^2t} \neq 0$

if $c_3 = 0$ or $c_2 = 0$ then $u = 0$.

$\Rightarrow \sin lp = \sin n\pi$

$\Rightarrow p = \frac{n\pi}{l}$

Now solution becomes

Heat equation (CO3)

$$\Rightarrow u(x,t) = \ c_2\, c_3 \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)}$$

$$\Rightarrow u(x,t) = \ b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)} \text{where } b_n = c_2c_3$$

Complete solution is given by-

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \ b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)}$$

$\because u(x,0) = x$

Put $t = 0$ we get

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} \ b_n \sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

This is half range Fourier Sine series, so b_n is given by

$$b_n = \frac{2}{l}\int_0^l F(x)\, sin\left(\frac{n\pi x}{l}\right) dx$$

$\Rightarrow b_n = \frac{2}{l}\int_0^l x\, sin\left(\frac{n\pi x}{l}\right) dx$

$\Rightarrow b_n = -\frac{2l}{\pi}\frac{cosn\pi}{n}$

Now solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)}$$

Heat equation (CO3)

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{2l}{\pi}\frac{cosn\pi}{n}\sin\left(\frac{n\pi x}{l}\right)e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)}$$

$$u(x,t) = -\frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{cosn\pi}{n}\sin\left(\frac{n\pi x}{l}\right)e^{-\left(\frac{n^2\pi^2c^2t}{l^2}\right)}$$

Example-4: Solve the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ under boundary condition

i. $u \neq 0\ if\ t \to \infty$

ii. $\frac{\partial u}{\partial x} = 0\ for\ x = 0\ and\ x = l$

iii. $u = lx - x^2\ for\ t = 0$ between $x = 0\ and\ x = l$

Heat equation (CO3)

Sol: 1-d Heat equation is given by

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \cdots\cdots (1)$$

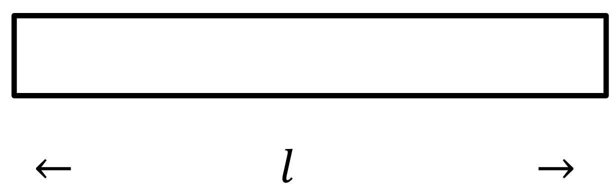
Solution of (1) by method of separation variable

$$\Rightarrow u(x,t) = \ (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-p^2 kt} \ \cdots (2)$$

Equation (2) satisfies the condition $u \neq 0 \; if \; t \to \infty$

Now by using condition $\frac{\partial u}{\partial x} = 0 \; for \; x = 0 \; and \; x = l$ in equation (2)

$$c_2 = 0 \; and \; p = \frac{n\pi}{l} \; , n \in I$$



Now (2) solution becomes

$$\Rightarrow u(x,t) = c_1 c_3 \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}}$$

Heat equation (CO3)

$$\Rightarrow u(x,t) = a_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}} \cdots (3)$$

Again second possible solution is

$$\Rightarrow u(x,t) = \ (c_1 x + c_2)\, c_3 \cdots (4)$$

Now by using condition $\frac{\partial u}{\partial x} = 0\ for\ x = 0\ and\ x = l$ in equation (4)

$c_1 = 0$

$$\text{u} = c_2 c_3 = \frac{a_0}{2} \ \ldots\ldots (5)$$

The general solution is the sum of solutions (3) and (5)

$$\Rightarrow u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}} \cdots (6)$$

Now by using condition $u = lx - x^2\ for\ t = 0$ between $x = 0\ and\ x = l$ in equation (6)

Heat equation (CO3)

$$\Rightarrow lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Here $a_0 = \frac{2}{l}\int_0^l (lx - x^2)dx = \frac{l^2}{3}$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \begin{cases} -\dfrac{4l^2}{n^2\pi^2}, & when\ n\ is\ even \\ 0, & when\ n\ is\ odd \end{cases}$$

Hence the solution is

Heat equation (CO3)

$$u(x,t) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,6,..}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2 kt}{l^2}}$$

Put $\mathrm{n} = 2m$

$$\Rightarrow u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-\frac{4m^2\pi^2 kt}{l^2}}$$

Example-5: The temperature in a bar of length π which is perfectly insulated at ends $x = 0$ & $x = \pi$ is governed by PDE $u_t = u_{xx}$. Assuming initial temperature as $u(x,0) = cos2x$. Find the temperature distribution at any instant of time.

Heat equation (CO3)

Sol: 1-d Heat equation is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \cdots\cdots (1) \quad \text{here } c^2 = 1$$

s.t.

$$\frac{\partial u}{\partial x}(0, t) = 0$$

$$\frac{\partial u}{\partial x}(\pi, t) = 0$$

$$u(x, 0) = \cos 2x$$

Solution of (1) by method of separation variable

$$\Rightarrow u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-p^2 t} \cdots (2)$$

Heat equation (CO3)

$\because \frac{\partial u}{\partial x}(0,t) = 0$

Now $\frac{\partial u}{\partial x} = \ (-pc_1 \sin px + pc_2 \cos px)\, c_3\, e^{-p^2 t}$

Put $x = 0$

$\Rightarrow \frac{\partial u}{\partial x}(0,t) = \ (-pc_1 \sin 0 + pc_2 \cos 0)\, c_3\, e^{-p^2 t}$

$\Rightarrow 0 = pc_2 c_3 e^{-p^2 t}$

$\Rightarrow c_2 = 0$ from here $c_3 \neq 0$ & $e^{-p^2 t} \neq 0$

if $c_3 = 0$ then $u = 0$.

Now solution becomes if we put $c_2 = 0$

$\Rightarrow u(x,t) = \ c_1\, c_3 \cos px\, e^{-p^2 t}$

Heat equation (CO3)

Again $\frac{\partial u}{\partial x} = -c_1\, c_3 p \sin px\, e^{-p^2 t}$

$\because \frac{\partial u}{\partial x}(\pi, t) = 0$

Put $x = \pi$

$\Rightarrow c_1\, c_3 \sin p\pi\, e^{-p^2 t} = 0$

$\Rightarrow \sin p\pi = 0$ from here $c_1, c_3 \neq 0$ & $e^{-p^2 t} \neq 0$

if $c_3 = 0$ or $c_1 = 0$ then $u = 0$.

$\Rightarrow \sin p\pi = \sin n\pi$

$\Rightarrow p = n$

Heat equation (CO3)

Now solution becomes

$u(x,t) = \ c_1 \, c_3 \cos nx \, e^{-n^2 t}$

$\Rightarrow u(x,t) = \ b_n \cos nx \, e^{-n^2 t}$ where $b_n = c_2 c_3$

Complete solution is given by-

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \ b_n \cos nx \, e^{-n^2 t}$

$\because u(x,0) = \cos 2x$

Put $t = 0$ we get

$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} \ b_n \cos nx$

$\Rightarrow \cos 2x = b_1 \cos x + \ b_2 \cos 2x + b_3 \cos 3x + \cdots$

Comparing coefficient

$$b_1 = 0, b_2 = 1, b_3 = 0 \ldots \ldots \ldots$$

Heat equation (CO3)

So solution is

$$u(x,t) = \cos 2x \, e^{-4t}$$

Daily Quiz (CO3)

Q1. A rod of length l with insulated sides is initially at a uniform temperature u_0. Its end are suddenly cooled to $0°C$ and are kept at that temperature. Find the temperature function $u(x, t)$.

Q2. The heat flow in a bar of length 10 cm of homogeneous material is governed by PDE $u_t = c^2 u_{xx}$. The ends of the bar are kept at temp. $0°C$ and initial temp. is $f(x) = x(10 - x)$. Find the temperature distribution in the bar at any instant of time.

Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details (CO3)

- Introduction to Partial differential Equation
  https://youtu.be/FkhIRX2bN9k
- Homogeneous linear PDE with constant Coefficients
  https://youtu.be/4cvFNmtytFw
- Homogeneous linear PDE with constant Coefficients
  https://youtu.be/5WHOL66MOh0
- Non Homogeneous linear PDE with constant Coefficients
  https://youtu.be/uo5i3W7ErRs

Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details (CO3)

- Classification of second order Partial differential equation
  https://youtu.be/L1GWz5POILg
- Method of separation of variables
  https://youtu.be/m0brqZCPg
- Solution of One Dimensional Heat Equation
  https://youtu.be/0H9DUqKtWPI
- Solution of one dimensional Wave Equation
  https://youtu.be/KK8PCjvbbZY
- Solution of two dimensional heat equation in steady state
  (Laplace Equation) https://youtu.be/9VOC5DodMek

MCQ's (CO3)

1. Solution of the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x+y)$ is

a) $u = f_1(y+ix) + f_2(y-ix) + (x+y)^2$

b) $u = f_1(y+x) + f_2(y-x) + (x+y)^2$

c) $u = f_1(y+ix) + f_2(y-ix) + (x-y)^2$

d) None

2. Solution of the equation $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$ is

a) $z = f_1(y-x) + f_2(y-2x) + \frac{(x-y)^3}{36}$

b) $z = f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}$

c) $z = f_1(y+x) + f_2(y+2x) + \frac{(x-y)^3}{36}$

d) $z = f_1(y+x) + f_2(y+2x) + \frac{(x-y)^3}{36}$

MCQ's(CO3)

3.Solution of equation $D(D-2D'-3)z=e^{x+2y}$ is

$a)\ z=f_1(y)+e^{3x}f_2(y+2x)-\frac{1}{6}e^{x+2y}$

$b)\ z=f_1(x)+e^{3x}f_2(y+2x)-\frac{1}{6}e^{x+2y}$

$c)\ z=f_1(y)+e^{3x}f_2(y+2x)+\frac{1}{6}e^{x+2y}$

d) None

4.Solution of PDE $(D^2-DD'+D'-1)z=Cos(x+2y)$

$a)\ z=e^{-x}f_1(y)+e^{x}f_2(y+x)+\frac{1}{2}Sin(x+2y)$

$b)\ z=e^{x}f_1(y)+e^{-x}f_2(y+x)+\frac{1}{2}Sin(x+2y)$

$c)\ z=e^{-x}f_1(y)+e^{x}f_2(y+x)-\frac{1}{2}Sin(x+2y)$

$d)\ z=e^{x}f_1(y)+e^{-x}f_2(y+x)+\frac{1}{2}Sin(x-2y)$

MCQ (CO3)

5. Classify the PDE $4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

a) Parabolic

b) Elliptic

c) Hyperbolic

d) None

6. Solve by method of Separation of Variables, The Solution of $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ is

a) $u(x.y) = c_1 c_2 e^{k(x-y)}$

b) $u(x.y) = ce^{k(x+y)}$

c) $u(x.y) = c_1 c_2 e^{kxy}$

d) $u(x.y) = c_1 c_2 e^{k(x+y)}$

MCQ (CO3)

7. Which of the following is a two-dimensional heat equation?

a) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

b) $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

c) $u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

d) None

MCQ (CO2)

8. Solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with conditions $u(x, 0) = 3 Sinn\pi x,\ u(0, t) = 0, u(l, t) = 0$, where $0 < x < l$ is

a) $u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\ Sinn\pi x$

b) $u(x, t) = 3 \sum_{n=1}^{\infty} e^{n^2\pi^2 t}\ Sinn\pi x$

c) $u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 x}\ Sinn\pi t$

d) *None*

Glossary Questions

1. Pick out the correction option from Glossary-

I. Heat equation

II. Wave equation

III. Laplace equation

IV. Steady State

A. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

B. $\frac{\partial u}{\partial t} = c^2 (\frac{\partial^2 u}{\partial x^2})$

C. $\frac{\partial u}{\partial t} = 0$

D. $\frac{\partial u}{\partial t} = c^2 (\frac{\partial^2 u}{\partial x^2})$

Glossary Questions

2. Pick out the correction option from Glossary-

I. $(D^2 + 4DD' + D'^2)z = x + y$

II. $(D^2 - D')z = 0$

III. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x$

IV. $\frac{\partial u}{\partial t} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$

A. Non homogeneous pde
B. 1st order linear pde
C. Homogeneous pde
D. Two dimensional heat equation

First Sessional Paper

Printed page:2 **Subject Code: AAS0301A**

Roll No: | | | | | | | | | | | | | | |

NOIDA INSTITUTE OF ENGINEERING AND TECHNOLOGY, GREATER NOIDA

(An Autonomous Institute Affiliated to AKTU, Lucknow)

B.Tech (CSE/CS/IT)

(SEM. III SESSIONAL EXAMINATION –I) (2021-2022)

Subject Name: Eng. Mathematics III

Time: 1.15 Hours **Max. Marks:30**

General Instructions:

- All questions are compulsory. Answers should be brief and to the point.
- This Question paper consists of 2 pages & 5 questions.
- It comprises of three Sections, A, B, and C. You are to attempt all the sections.
- Section A -Question No- 1 is objective type questions carrying 1 mark each, Question No- 2 is very short answer type carrying 2 mark each. You are expected to answer them as directed.
- Section B - Question No-3 is short answer type questions carrying 5 marks each. You need to attempt any two out of three questions given.
- Section C -Question No. 4 & 5 Long answer type (within unit choice) questions carrying 6 marks each. You need to attempt any one-part *a* or *b*.
- Students are instructed to cross the blank sheets before handing over the answer sheet to the invigilator.

First Sessional Paper

- No sheet should be left blank. Any written material after a blank sheet will not be evaluated/checked.
- **Blooms Level:** *K1: Remember, K2: Understand, K3: Apply, K4: Analyze, K5: Evaluate, K6: Create*

		SECTION – A	**[8]**	**CO**	**Blooms level**
1.	**Attempt all parts**		**(4×1=4)**	**CO**	
	a.	$\lim_{z \to 0} \frac{z}{\bar{z}}$ (i) Limit exists (ii) Limit does not exist (iii) Limit exists and equal to 1 (iv) None of these	**(1)**	**1**	**K5**
	b.	If $f(z) = \frac{z}{z^2+9}$ then (i) $f(z)$ is continuous (ii) $f(z)$ is discontinuous at $z = \pm 3i$ (iii) $\lim_{z \to i} \frac{z}{z^2+9} = -\frac{i}{8}$ (iv) Both B & C	**(1)**	**1**	**K2**
	c.	Function $f(z) = z\|z\|$ is (i) Analytic anywhere (ii) Not analytic anywhere (ii) Harmonic (iv) None of these	**(1)**	**1**	**K3**

First Sessional Paper

	d.	There exists no analytic function $f(z)$ if (i) $real\ f(z) = y - 2x$ (ii) $real\ f(z) = y^2 - 2x$ (ii) $real\ f(z) = y^2 - x^2$ (iv) $real\ f(z) = y - x$	(1)	1	K2
2.	**Attempt all parts**		**(2×2=4)**	**CO**	
	a.	Show that if $f(z)$ is analytic and $Imf(z) = constant$ then $f(z)$ is constant.	(2)	1	K3
	b.	Find the bilinear transformation which maps the points $z = 0,1,\infty$ into the points $w = i, -1, -i$ respectively.	(2)	1	K5
SECTION – B					
3.	**Answer any two of the following-**		**[2×5=10]**	**CO**	
	a.	Examine the nature of the function $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$ $, z \neq 0, f(0) = 0$, prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner and also that $f(z)$ is not analytic at $z = 0$.	(5)	1	K4
	b.	Find the image of $\lvert z-1 \rvert = 1$ under the transformation $w = \frac{1}{z}$.	(5)	1	K5
	c.	Show that $f(z) = \cos z$ is analytic in entire complex plane.	(5)	1	K3

First Sessional Paper

		SECTION – C			
4	**Answer any one of the following-**		**[2×6=12]**	**CO**	
	a.	Determine an analytic function $f(z)$ in terms of z whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.	**(6)**	**1**	**K5**
	b.	If $w = \varphi + i\psi$ represent the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$. Determine the function φ.	**(6)**	**1**	**K5**
5.	**Answer any one of the following-**				
	a.	Determine an analytic function $f(z)$ in terms of z if $3u + v = 3\sin x \cos hy + \cos x . \sin hy$.	**(6)**	**1**	**K5**
	b.	Find an analytic function $f(z)$ in terms of z if $Re[f'(z)] = 3x^2 - 4y - 3y^2$ and $f(1+i) = 0$ & $f'(0) = 0$.	**(6)**	**1**	**K5**

Second Sessional Paper

Printed page:2 **Subject Code: AAS0301A**

Roll No: | | | | | | | | | | | | | | |

NOIDA INSTITUTE OF ENGINEERING AND TECHNOLOGY, GREATER NOIDA

(An Autonomous Institute)

Affiliated to Dr. A.P. J. Abdul Kalam Technical University, Uttar Pradesh, Lucknow

Course: B.Tech **Branch: CSE/IT/CS**

Semester: III **Sessional Examination: II** **Year: (2020-2021)**

Subject Name: Eng. Maths III

Time: 1.15Hours **[SET-1]** **Max. Marks:30**

General Instructions:

- This Question paper consists of 2 pages & 5 questions. It comprises of three Sections, A, B, and C.
- **Section A** -Question No- 1 is objective type questions carrying 1 mark each, Question No- 2 is very short answer type carrying 2 mark each. You are expected to answer them as directed.
- **Section B** - Question No-3 is short answer type questions carrying 5 marks each. You need to attempt any two out of three questions given.
- **Section C** -Question No. 4 & 5 Long answer type (within unit choice) questions carrying 6 marks each. You need to attempt any one-part *a* or *b*.

Second Sessional Paper

1	Attempt all parts		(4×1=4)	
	a.	$\int_0^{2+i}(x^2+iy)dz$ along the path $y = x$ is equal to $(i)\ \left(\frac{2}{3}+\frac{14}{3}i\right)$ $(ii)\ \left(\frac{3}{2}+\frac{3}{14}i\right)$ $(iii)\ \left(\frac{2}{3}-\frac{14}{3}i\right)$ (iv) None of these	(1)	CO2
	b.	Residue of $z\cos(1/z)$ at $z = 0$ is (i) 0 (ii) 1 (iii) $-1/2$ (iv) $1/2$	(1)	CO2
	c.	The region of validity for Taylor's series about $z = 0$ of the function e^z is (i) $\|z\| = 0$ (ii) $\|z\| < 1$ (iii) $\|z\| > 1$ (iv) $\|z\| < \infty$	(1)	CO2
	d.	If $f(z) = \frac{sinz}{z^4}$, then $z = 0$ is (i) Removable singularity (ii) Pole of order 4 (iii) Pole of order 3 (iv) None of these	(1)	CO2

Second Sessional Paper

2.	**Attempt all parts**		**(2×2=4)**	
	a.	State Cauchy Integral formula.	**(2)**	**CO2**
	b.	Evaluate the integral $\int_C \lvert z\rvert\, dz$ where C is the left half of the unit circle $\lvert z\rvert = 1$ from $z = -i$ to $z = i$.	**(2)**	**CO2**
SECTION – B			**[10 Marks]**	
3.	**Answer any two of the following-**		**(2×5=10)**	
	a.	Verify Cauchy integral theorem for $f(z) = z^2$ taken over the boundary of square with vertices $1 \pm i, -1 \pm i$.	**(5)**	**CO2**
	b.	Using Cauchy integral formula, evaluate $\int_C \frac{z^2+1}{z^2-1} dz$ where C is circle (i) $\lvert z\rvert = 3/2$ (ii) $\lvert z-1\rvert = 1$	**(5)**	**CO2**
	c.	Evaluate $\int_C \frac{1}{z^2(z^2-4)e^z} dz$ where C is $\lvert z\rvert = 1$.	**(5)**	**CO2**

Second Sessional Paper

SECTION – C			**[12 Marks]**	
4	**Answer any one of the following-**		**(1×6=6)**	
	a.	Expand $f(z) = \frac{1}{(z+1)(z+3)}$ $(i)\ \ \|z\| < 1 \qquad (ii)\ \ 1 < \|z\| < 3$	**(6)**	**CO2**
	b.	State & Prove Cauchy Residue Theorem.	**(6)**	**CO2**
5.	**Answer any one of the following-**		**(1×6=6)**	
	a.	Evaluate $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,d\theta$ using contour integration.	**(6)**	**CO2**
	b.	Prove that $\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ using contour integration.	**(6)**	**CO2**

Third Sessional Paper

Printed page:2 **Subject Code: AAS0301A**

Roll No: | | | | | | | | | | | | | | |

<u>NOIDA INSTITUTE OF ENGINEERING AND TECHNOLOGY, GREATER NOIDA</u>

(An Autonomous Institute)

Affiliated to Dr. A.P. J. Abdul Kalam Technical University, Uttar Pradesh, Lucknow

Course: B.Tech Branch: CSE/IT/CS

Semester: III **Sessional Examination: III** **Year: (2021-2022)**

Subject Name: Eng. Maths III

Time: 1.15 Hours **[SET-2]** **Max. Marks:30**

<u>General Instructions:</u>

- This Question paper consists of 2 pages & 5 questions. It comprises of three Sections, A, B, and C.
- **<u>Section A</u>** -Question No- 1 is objective type questions carrying 1 mark each, Question No- 2 is very short answer type carrying 2 mark each. You are expected to answer them as directed.
- **<u>Section B</u>** - Question No-3 is short answer type questions carrying 5 marks each. You need to attempt any two out of three questions given.
- **<u>Section C</u>** -Question No. 4 & 5 Long answer type (within unit choice) questions carrying 6 marks each. You need to attempt any one-part *a* or *b*.

Blooms Level: K1: Remember, K2: Understand, K3: Apply, K4: Analyze, K5: Evaluate, K6: Create

Third Sessional Paper

		SECTION – A	[8 Marks]	CO	Blooms level
1	Attempt all parts		(4×1=4)		
	a.	The solution of PDE $(D + 4D' + 5)^2 z = 0$ is (i) $z = e^{-5x} f_1(y - 4x) + xe^{-5x} f_2(y - 4x)$ (ii) $z = e^{-5x} f_1(y + 4x) + xe^{-5x} f_2(y + 4x)$ (iii) $z = e^{5x} f_1(y + 4x) + xe^{5x} f_2(y + 4x)$ (iv) None of these	(1)	CO3	K5
	b.	PDE: $Bu_{xx} + Au_{xy} + Cu_{yy} + f(x, y, u, u_x, u_y) = 0$ is elliptic if ________	(1)	CO3	K4
	c.	While solving a PDE using a Variable Separable method, we equate the ratio to a Constant which? (i) Can be Positive or Negative Integer or Zero (ii) Can be Positive or Negative rational number or Zero (iii) Must be a Positive Integer (iv) Must be a Negative Integer	(1)	CO3	K1
	d.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is two-dimensional heat equation in ________state.	(1)	CO3	K1

Third Sessional Paper

2.	**Attempt all parts**		**(2×2=4)**		
	a.	Find the P.I. of $(D^2 - 2DD')z = \sin x.\cos 2y$	**(2)**	**CO3**	**K5**
	b.	Classify the PDE: $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$ about the line $y = x$.	**(2)**	**CO3**	**K4**
<u>SECTION – B</u>			**[10 Marks]**		
3.	**Answer any <u>two</u> of the following-**		**[2×5=10]**		
	a.	Solve the PDE $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ subject to the condition $u(0,y) = 4e^{-y} - e^{-5y}$ by method of separation of variables.	**(5)**	**CO3**	**K5**
	b.	Solve the PDE: $(D^2 + DD' - 6D'^2)z = y\, sin\, x$	**(5)**	**CO3**	**K5**
	c.	Solve the PDE: $\left(D^2 - D'^2 - 3D + 3D'\right)z = xy + e^{x+2y}$	**(5)**	**CO3**	**K5**

Third Sessional Paper

		SECTION – C	**[12 Marks]**		
4	**Answer any one of the following-**		**[2×6=12]**		
	a.	A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position is given by $y=y_0\, sin^3\frac{\pi x}{l}$. If it released from rest from this position, find the displacement $y(x,t)$.	**(6)**	**CO3**	**K5**
	b.	Solve the PDE $\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}=0$ subject to the condition: $u(x,0)=0, u(x,\pi)=0, u(0,y)=4\sin 3y$ by method of separation of variables.	**(6)**	**CO3**	**K5**
5.	**Answer any one of the following-**				
	a.	Find the temperature of the bar of length 2 whose ends are kept at zero and internal surface insulated by if the initial temperature is $sin\frac{\pi x}{2}+3\sin\frac{5\pi x}{2}$.	**(6)**	**CO3**	**K5**
	b.	Find the solution of Laplace equation subject to the condition: $u(0,y)=u(1,y)=u(x,0)=0, u(x,1)=100\sin\pi x$	**(6)**	**CO3**	**K5**

Expected Questions for University Exam (CO3)

1. Solve: $t = \sin xy$.
2. Solve: $(y^2 + z^2)p - xyq = -zx$.
3. Solve: $2zx - px^2 - 2qxy + pq = 0$.
4. Use Cauchy's method of characteristics to solve the following 1st order PDE : $u_x - u_y = 2$; $u(0, y) = -y$.
5. Solve: $(D^2 + 3DD' - 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x + 2y)$.
6. Solve: $(D^2 - D'^2)z = \sin x \cos y$.
7. Solve: $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$.
8. Solve: $(D^2 + DD' - 6D'^2)z = y\, cos\, x$.
9. Solve: $\left(D^2 - D'^2 - 3D + 3D'\right)z = xy + e^{x+2y}$.
10. Solve: $\left(3D^2 - 2D'^2 + D - 1\right)z = 4e^{x+y}\, cos\,(x + y)$.
11. Solve: $x^2 \frac{\partial^2 z}{\partial x^2} - 4y^2 \frac{\partial^2 z}{\partial y^2} - 4y \frac{\partial z}{\partial y} - z = x^2y^2 \log x$

Expected Questions for University Exam(CO3)

12. Classify the equation:

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0.$$

13. Solve the equation $2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} + 5u = 0; u(0, y) = 2e^{-y}$, by method of separation of variables.

14. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u = 0; u(x, 0) = 10e^{-x} - 6e^{-4x}$, by method of separation of variables.

15. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$, by method of separation of variables.

16. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A\, sin\, \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by $y(x, t) = A\, sin\, \frac{\pi x}{l} cos\, \frac{\pi ct}{l}$.

Expected Questions for University Exam(CO3)

17. A tightly stretched string with fix end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 sin^3\left(\frac{\pi x}{l}\right)$.If it is released from rest from this position, find displacement $y(x, t)$.

18. A tightly stretched flexible string has its end fixed at $x = 0$ and $x = l$. At time $t = 0$ the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.

19. A rod of length l with insulated sides is initially at a uniform temperature u_0. Its end are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x, t)$.

Summary(CO3)

We discussed following points in this unit.

- ✓ Order and degree of partial differential equation.
- ✓ Homogenous Partial Differential equation
- ✓ C.F. for Homogenous Partial Differential equation.
- ✓ P.I. for Homogenous Partial Differential equation.
- ✓ Non-Homogenous Partial Differential equation
- ✓ C.F. for Non-Homogenous Partial Differential equation.
- ✓ P.I. for Non-Homogenous Partial Differential equation.
- ✓ Classification of second order partial differential equations
- ✓ Method of separation of variables for solving partial differential equations
- ✓ Solution of one dimensional wave and heat conduction equations.

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Thank You