SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS, I: ODD SOLUTIONS WITH RESPECT TO THE SIMONS CONE

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Abstract. This is the first of two papers concerning saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator and f is of Allen-Cahn type.

Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only on this set. By the odd symmetry, L_K coincides with a new operator $L_K^{\mathcal{O}}$ which acts on functions defined only on one side of the Simons cone, $\{|x'| > |x''|\}$, and that vanish on it. This operator $L_K^{\mathcal{O}}$, which corresponds to reflect a function oddly and then apply L_K, has a kernel on $\{|x'| > |x''|\}$ which is different from K.

In this first paper, we characterize the kernels K for which the new kernel is positive and therefore one can develop a theory on the saddle-shaped solution. The necessary and sufficient condition for this turns out to be that K is radially symmetric and $\tau \mapsto K(\sqrt{\tau})$ is a strictly convex function.

Assuming this, we prove an energy estimate for doubly radial odd minimizers and the existence of saddle-shaped solution. In part II further properties of saddle-shaped solutions will be established, such as their asymptotic behavior, a maximum principle for the linearized operator, and their uniqueness.

Contents

Key words and phrases. Integro-differential semilinear equation, odd symmetry, Simons cone, maximum principle for odd functions, energy estimate, saddle-shaped solution.

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1. Introduction

In this paper we study solutions to the semilinear integro-differential equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m} \tag{1.1}$$

which are odd with respect to the Simons cone —see (1.6). The interest on these solutions is motivated by the nonlocal version of a conjecture by De Giorgi (see the details below) with the aim of finding a counterexample in high dimensions through the so-called saddle-shaped solutions (see the definition at the end of this introduction). Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces.

Equation (1.1) is driven by an integro-differential operator L_K of the form

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x-y)\, \mathrm{d}y, \tag{1.2}$$

where the kernel K satisfies

$$K \geq 0\,, \quad K(z) = K(-z) \quad \text{and} \quad \int_{\mathbb{R}^n} \min\left\{|z|^2, 1\right\} K(z)\, \mathrm{d}z < +\infty\,. \tag{1.3}$$

The integral in (1.2) has to be understood in the principal value sense, as well as all the integrals involving nonlocal operators in the rest of the paper. The most canonical example of such operators is the fractional Laplacian, defined for $\gamma \in (0,1)$ as

$$(-\Delta)^\gamma u = c_{n,\gamma} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma}}\, \mathrm{d}y\,,$$

where $c_{n,\gamma}$ is a normalizing constant (for its exact value see for instance [17]).

Recall that the fractional Laplacian has an associated extension problem (see [9]) that allows the use of local arguments to deal with equations such as (1.1). This is not the case for general operators such as L_K, and therefore some purely nonlocal techniques are developed along this work.

Throughout the paper, we assume that the operators of our study are uniformly elliptic. That is, their kernels are bounded from above and below by a positive multiple of the one of the fractional Laplacian:

$$\lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}} \leq K(z) \leq \Lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}}\,, \quad \text{with } 0 < \lambda \leq \Lambda\,, \tag{1.4}$$

where $c_{n,\gamma}$ is the constant appearing in the definition of the fractional Laplacian. This condition is one of the most frequently adopted when dealing with nonlocal operators of the form (1.2), since it is known to yield Hölder regularity of solutions (see [26, 31]). The family of linear operators satisfying conditions (1.3) and (1.4) is the so-called $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$ and we will make explicit the parameters only when needed. Note that, under the assumption (1.4), $L_K u$ is well-defined if $u \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^{2m}) \cap L^\infty(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$.

Moreover, for some purposes we will need the operators to be invariant under rotations. This is equivalent to saying that the kernel is radially symmetric, $K(z) = K(|z|)$.

Concerning the nonlinearity of the equation, given f a C^1 function, we define the potential

$$G(u) = \int_u^1 f(\tau)\,\mathrm{d}\tau\,.$$

Then, we have that G is a C^2 function satisfying $G' = -f$. In this paper, we assume the following conditions on G:

$$G \text{ is even and } G \geq G(\pm 1) = 0 \text{ in } \mathbb{R}\,. \tag{1.5}$$

Note that the previous conditions on G yield that f is an odd function with $f(0) = f(\pm 1) = 0$. In some cases, as in Theorem 1.4 below, we will further assume that $G(0) > 0$. In such situation, equation (1.1) can be seen as a model for phase transitions. The Allen-Cahn nonlinearity, $f(u) = u - u^3$, is a model example.

The Simons cone will be a central object along this paper. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \;:\; |x'| = |x''| \right\}. \tag{1.6}$$

This cone is of special importance in the theory of local and nonlocal minimal surfaces, and its variational properties are related to the conjecture of De Giorgi (see the end of this introduction for more details). Through the whole article we will use $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \;:\; |x'| > |x''| \right\} \text{ and } \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \;:\; |x'| < |x''| \right\}.$$

Both $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. These are sets that are invariant under orthogonal transformations in the first m variables and also under orthogonal transformations in the last m variables. That is, $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution if $R\Omega = \Omega$ for every given transformation $R \in O(m)^2 = O(m) \times O(m)$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

In this paper we deal with functions that are *doubly radial*. These are functions $w : \mathbb{R}^{2m} \to \mathbb{R}$ that only depend on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$.

In order to define certain symmetries of functions with respect to the Simons cone, we consider the following isometry, that will play a significant role in this article:

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x')\,. \end{array} \tag{1.7}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$. Similarly, we say that a doubly radial function w is *even with respect to the Simons cone* if $w(x) = w(x^\star)$.

Regarding the doubly radial symmetry we define the following variables

$$s := |x'| \quad \text{and} \quad t := |x''|\,.$$

These variables are specially useful when dealing with the Laplacian, since the operator can be written very easily in these variables and the resulting PDE in $(0,+\infty)\times(0,+\infty)$ is suitable to work with. Recall that the Laplacian of a doubly radial function w can be written as

$$\Delta w = w_{ss} + w_{tt} + \frac{m-1}{s} w_s + \frac{m-1}{t} w_t \,. \tag{1.8}$$

The same happens in the case of the fractional Laplacian thanks to the local extension problem. Having a PDE in the two variables $(s,t)\in\mathbb{R}^2$ is useful to perform certain computations (see [7, 8, 3, 2] for the local case and [11, 12, 20] for the fractional framework).

If we try to follow the same strategy by writing a rotation invariant operator L_K in (s,t) variables, the expression of the new operator is quite complex. Indeed, if $w:\mathbb{R}^{2m}\to\mathbb{R}$ is doubly radial and we define $\widetilde{w}(s,t) := w(s,0,...,0,t,0,...,0)$, it holds

$$L_K w(x) = \widetilde{L}_K \widetilde{w}(|x'|,|x''|),$$

with

$$\widetilde{L}_K \widetilde{w}(s,t) := \int_0^{+\infty}\int_0^{+\infty} \sigma^{m-1}\tau^{m-1}\big(\widetilde{w}(s,t)-\widetilde{w}(\sigma,\tau)\big) J(s,t,\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \tag{1.9}$$

where

$$J(s,t,\sigma,\tau) := \int_{\mathbb{S}^{m-1}}\int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2+\sigma^2-2s\sigma\omega_1+t^2+\tau^2-2t\tau\tilde{\omega}_1}\Big)\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega}\,.$$

Note that $\widetilde{L}_K$ is an integro-differential operator in $(0,+\infty)\times(0,+\infty)$, but the expression of its kernel is quite involved. Indeed, such an expression does not become simpler even when L_K is the fractional Laplacian. In this case, the kernel J involves hypergeometric functions of two variables, the so-called Appell functions (see Appendix C for more details on it), but this does not simplify any computation.

Instead of working with the (s,t) variables, we follow another approach that conceptually is the same, but that we find more clear and concise. It consists on rewriting the operator L_K with a different kernel $\overline{K}:\mathbb{R}^{2m}\times\mathbb{R}^{2m}\to\mathbb{R}$ that is doubly radial, but in such a way that it still acts on functions defined in $\mathbb{R}^{2m}$ —and not in $(0,+\infty)^2$. As it is explained with more details in Section 2, if K is a radially symmetric kernel, then we can write L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y\,, \tag{1.10}$$

where $\overline{K}:\mathbb{R}^{2m}\times\mathbb{R}^{2m}\to\mathbb{R}$ is defined by

$$\overline{K}(x,y) := \fint_{O(m)^2} K(|Rx-y|)\,\mathrm{d}R\,. \tag{1.11}$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$ (see Section 2 for the details).

This new expression (1.10) has some advantages compared with (1.9). First, the computations with this new expression are shorter and more clear than the analogous using (s,t) variables. This also makes the notation more concise. Furthermore we avoid some issues of the (s,t) variables such as the special treatment of the set $\{st = 0\}$. Although in this paper we do not work in (s,t) variables, we include an appendix at the end of the article with some computations in these variables (see Appendix C). We think that this could be useful in future works.

The leitmotiv of this paper is that the odd symmetry of functions can be used to work with the operator L_K in a smart way. Let us be more precise. If we consider doubly radial functions that are odd with respect to the Simons cone, we can rewrite L_K acting on such functions as a new operator $L_K^{\mathcal{O}}$ that only takes into account the values of the functions in $\mathcal{O}$ (since $L_K^{\mathcal{O}}$ itself incorporates the odd symmetry). The new operator is the following:

$$L_K^{\mathcal{O}} w(x) := \int_{\mathcal{O}} \{w(x) - w(y)\}\{\overline{K}(x,y) - \overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,, \tag{1.12}$$

where $(\cdot)^\star$ is defined in (1.7).

Note that this operator, in contrast with L_K, acts on functions that only need to be defined in $\mathcal{O}$. Moreover, as we show in Section 2, $L_K^{\mathcal{O}}$ acting on a doubly radial function $w : \mathcal{O} \to \mathbb{R}$ corresponds to consider the odd extension of w with respect to the Simons cone and apply the operator L_K to this extended function. This can be easily seen by using the change of variables given by $(\cdot)^\star$.

Our first main result concerns necessary and sufficient conditions on the original kernel K in order to this operator have a positive kernel. As we will stress through this paper, and also in the forthcoming work [19], the positivity of the kernel in (1.12) is crucial in order to develop a theory on the saddle-shaped solution. In particular, under this assumption a maximum principle for doubly radial odd functions holds (see Proposition 1.2 below).

Theorem 1.1. *Let $K : (0,+\infty) \to (0,+\infty)$ and consider the radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (1.11).*

If

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau\,, \tag{1.13}$$

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \text{for every } x, y \in \mathcal{O}\,. \tag{1.14}$$

In addition, if $K \in C^2((0,+\infty))$, then (1.13) is not only a sufficient condition for (1.14) to hold, but also a necessary one.

This theorem is proved in Section 2 (see Propositions 2.4 and 2.5). It is based on a suitable division of the group $O(m)^2$ and a result on convex functions proved in Appendix A (Proposition A.1).

In [21], Jarohs and Weth study solutions to general integro-differential equations which are odd with respect to a hyperplane. The sufficient condition that they ask on the kernel K (not necessarily radially symmetric) is to be decreasing in the orthogonal direction to the hyperplane. This assumption follows readily after making a change of variables given by the symmetry with respect to such hyperplane. In our case, since we deal with a more complex symmetry, the proof of Theorem 1.1 is quite involved and requires a finer argument. Indeed, if we simply make the change $y \mapsto y^\star$ in (1.2), following [21], we should prove that $K(|x-y|) > K(|x-y^\star|)$ for every x and y in $\mathcal{O}$, but this is false even in the easiest case $L_K = (-\Delta)^\gamma$ and $m = 1$. Instead, if we write L_K in the form (1.10) with the kernel $\overline{K}$, the analogous positivity condition, (1.14), holds. We should remark that the use of (s,t) variables does not simplify the proof of Theorem 1.1. As mentioned in Appendix C, an analogous result can be established for the kernel J in the (s,t) variables, but the proof presents exactly the same difficulties as the one for $\overline{K}$.

Regarding the sufficient conditions on the kernel K, since we are dealing with a more complex symmetry than the one in [21], the kernel is required to satisfy further assumptions than just monotonicity. First, we assume that K is radially symmetric to be able to write L_K in the form (1.10). Moreover, we require the convexity condition (1.13), which is stronger than the decreasing assumption in [21] (note that if K satisfies (1.3), an assumption that we make through the paper, then (1.13) yields that K is radially decreasing).

The first direct consequence of the positivity condition (1.14) is the following maximum principle.

Proposition 1.2 (Maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $\Omega \subset \mathcal{O}$ an open set and let L_K be an integro-differential operator with a radially symmetric kernel K satisfying the positivity condition (1.14). Let $u \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone.*

(i) (Weak maximum principle) Assume that

$$\begin{cases} L_K u + c(x) u \geq 0 & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \mathcal{O}\setminus\Omega\,, \end{cases}$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x\in\mathcal{O},\,|x|\to+\infty} u(x) \geq 0\,.$$

Then, $u \geq 0$ in Ω.

(ii) (Strong maximum principle) Assume that $L_K u + c(x)u \geq 0$ in Ω, with $c(x)$ any function, and that $u \geq 0$ in $\mathcal{O}$. Then, either $u \equiv 0$ in $\mathcal{O}$ or $u > 0$ in Ω.

This statement differs from the usual maximum principle for L_K in the fact that we only assume that w is nonpositive in $\mathcal{O}\setminus\Omega$, instead of in $\mathbb{R}^{2m}\setminus\Omega$ (an assumption that makes no sense for odd functions). This form of maximum principle is analogous to the ones in [10, 21], where similar statements are considered for functions that are odd with respect to a hyperplane.

Since in this paper we will always consider doubly radial functions w which are odd with respect to the Simons cone, $L_K w = L_K^{\mathcal{O}} w$ in $\mathcal{O}$. Thus, to simplify the notation we will always write L_K, and we will say that L_K has a maximum principle in $\mathcal{O}$ when acting on doubly radial odd functions to mean that Proposition 1.2 holds.

Let us now turn to the variational problem from which equation (1.1) comes. As it is well known, (1.1) is the Euler-Lagrange equation associated to the energy functional

$$\mathcal{E}(w,\Omega) := \frac{1}{4}\int\int_{(\mathbb{R}^n)^2\setminus(\mathbb{R}^n\setminus\Omega)^2} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega G(w)\,\mathrm{d}x\,. \tag{1.15}$$

Using the same type of arguments as for the operator L_K, we can rewrite the energy of doubly radial and odd functions with a suitable new expression that involves the kernel $\overline{K}$ and that only takes into account the values of the functions at $\mathcal{O}$. To do it, we introduce the following notation. For A, $B \subset \mathcal{O}$ two sets of double revolution, we define

$$\begin{aligned} I_w(A,B) := 2\int_A\int_B & |w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ & + 4\int_A\int_B \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned}$$

Then, as proved in Section 3 (see Lemma 3.2), we can rewrite the energy of a doubly radial odd function w as

$$\mathcal{E}(w,\Omega) = \frac{1}{4}\big\{I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\big\} + 2\int_{\Omega\cap\mathcal{O}} G(w)\,\mathrm{d}x\,. \tag{1.16}$$

Note that this expression of the energy has only integrals computed in $\mathcal{O}$, and this is useful in some arguments (see Sections 3 and 4).

Thanks to this new expression for the energy, we are able to establish the second main result of this paper. It is the following energy estimate for doubly radial and odd minimizers of $\mathcal{E}$. In the next statement, $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ denotes the space of doubly radial and odd functions that vanish outside B_R and for which the energy $\mathcal{E}$ is well defined (see Section 3 for the precise definition).

Theorem 1.3. *Let K be a radially symmetric kernel such that $L_K \in \mathcal{L}_0(2m,\gamma,\lambda,\Lambda)$ and satisfying the positivity condition (1.14). Assume that G is a potential satisfying (1.5). Let $S>2$ and let u be a minimizer of the energy $\mathcal{E}$ in B_R, with $R > S+5$, among functions that are in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Then*

$$\mathcal{E}(u,B_S) \le \begin{cases} C\;S^{2m-2\gamma} & \text{if}\quad \gamma\in(0,1/2),\\ C\;\log(S)\,S^{2m-1} & \text{if}\quad \gamma = 1/2,\\ C\;S^{2m-1} & \text{if}\quad \gamma\in(1/2,1),\end{cases}$$

with C a positive constant depending only on m, γ, λ, Λ, and G.

This result has been proved in the case of the half-Laplacian by Cabré and Cinti [4], and extended to all the powers $0<\gamma<1$ by Cinti [12], both using the local extension problem (see [5] for non-doubly radial minimizers). In our case, since this technique is not available, we follow the arguments of Savin and Valdinoci in [29], where they prove a similar energy estimate for minimizers without any symmetry. The strategy they follow to establish the result is to compare the energy of u with the one of a suitable competitor which is constructed by taking the minimum between u and a radially symmetric auxiliary function —see (4.1). Such competitor is not permitted in our case, since it is not odd with respect to the Simons cone. Nevertheless, we show in Section 3 how to adapt the auxiliary functions of [29] to our setting in order to establish Theorem 1.3. In the arguments, the assumption (1.14) is crucial.

As an application of the previous results, we prove, by using standard variational methods, the existence of saddle-shaped solutions to (1.1) when f is of Allen-Cahn type. We say that a bounded solution u to (1.1) is a *saddle-shaped* solution if u is doubly radial, odd with respect to the Simons cone, and positive in $\mathcal{O}$.

Theorem 1.4 (Existence of saddle-shaped solutions)**.** *Let $f = -G'$ satisfy (1.5), and such that $G(0) > 0$. Let K be a radially symmetric kernel such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$ and satisfying the positivity condition (1.14). Then, for every even dimension $2m \geq 2$, there exists a saddle-shaped solution to (1.1). In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

We are interested in the study of this type of solutions since they are relevant in connection with a famous conjecture for the (classical) Allen-Cahn equation raised by De Giorgi, that reads as follows. Let u be a bounded monotone (in some direction) solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$, then, if $n \leq 8$, u depends only on one Euclidean variable, that is, all its level sets are hyperplanes. This conjecture is not completely closed (see [18] and references therein) but a counterexample in dimension $n = 9$ was build in [16] by using the so-called gluing method. Saddle-shaped solutions are natural objects to build a counterexample in a simpler way, as explained next. On the one hand, Jerison and Monneau [22] showed that a counterexample to the conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate, but is not one-dimensional. On the other hand, by the Γ-converge results from Modica and Mortola (see [23, 24]) and the fact that the Simons cone is the simplest nonplanar minimizing minimal surface, saddle-shaped solutions are expected to be global minimizers of the Allen-Cahn equation in dimensions $2m \geq 8$ (this is still an open problem).

Similar facts happen in the nonlocal setting (see the introduction of [20] for further details). For this reason, saddle-shaped solutions are of great interest in the study of the nonlocal version of the conjecture of De Giorgi for equation (1.1).

Saddle-shaped solutions to the classical Allen-Cahn equation involving the Laplacian were studied in [14, 30, 7, 8, 3]. In these works, it is established the existence, uniqueness, and some qualitative properties of this type of solutions, such as instability when $2m \leq 6$ and stability if $2m \geq 14$. Stability in dimensions $8, 10$, and 12 is still an open problem, as well as minimality in dimensions $2m \geq 8$.

In the fractional framework, there are only three works concerning saddle-shaped solutions to the equation $(-\Delta)^\gamma u = f(u)$. In [11, 12], Cinti proved the existence of a saddle-shaped solution as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability in low dimensions. In a previous paper by the authors [20], further properties of these solutions have been established, the main ones being uniqueness and, when $2m \geq 14$, stability. To our knowledge, the present paper together with its second part [19] are the first ones studying saddle-shaped solutions for general integro-differential equations of the form (1.1). In the three previous papers [11, 12, 20], the main tool used is the extension problem for the fractional Laplacian (see [9]). As mentioned, this technique cannot be carried out for general integro-differential operators different from the fractional Laplacian. Therefore, some purely nonlocal techniques are developed through this paper.

In the forthcoming paper [19], we study saddle-shaped solutions to (1.1) in more detail taking advantage of the setting for odd functions established in the present article. We present an alternative proof for the existence of a saddle-shaped solution by using monotone iteration and maximum principle techniques. As in the proof of Theorem 1.4, the assumtion (1.14) is crucial. In [19], we also show the uniqueness of the saddle-shaped solution through its asymptotic behavior and a maximum principle for the linearized operator, which we also establish in that article.

Let us make some final remarks on the minimality and stability properties of the Simons cone. Recall that, in the classical theory of minimal surfaces it is well known that the Simons cone has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in all even dimensions, and it is a minimizer of the perimeter functional when $2m \geq 8$. Concerning the nonlocal setting, $\mathscr{C}$ has also zero nonlocal mean curvature in all even dimensions, although it is not known if it is a minimizer of the nonlocal perimeter in dimensions $2m \geq 4$ (in [28] it is proven that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat). In higher dimensions, the only available results appear in [15, 20] and concern stability, a weaker property than minimality. In [15], Dávila, del Pino, and Wei characterize the stability of Lawson cones —a more general class of cones that includes $\mathscr{C}$— through an inequality involving only two hypergeometric constants which depend only on γ and the dimension n. This inequality is checked numerically in [15], finding that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. Numerics also shows that all Lawson cones in dimension 7 are stable if γ is close to zero. These results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem. In [20], the authors prove, by using the saddle-shaped solution to the fractional Allen-Cahn equation and a Γ-convergence result of [6], that the Simons cone is a stable (2γ)-minimal cone in dimensions $2m \geq 14$. To the best of our knowledge, this is the first analytical proof of a stability result for the Simons cone in dimensions $2m \geq 4$.

This paper is organized as follows. Section 2 is devoted to study the operator L_K acting on doubly radial odd functions. We deduce the expression of the kernel $\overline{K}$ and rewrite the operator acting on doubly radial odd functions, finding the expression (1.12). We also show Theorem 1.1 and Proposition 1.2. In Section 3 we study the energy functional associated to (1.1) and in Section 4 we establish the energy estimate stated in Theorem 1.3. Finally, in Section 5 we prove the existence of a saddle-shaped solution to the integro-differential Allen-Cahn equation. At the end of the paper there are three appendices. Appendix A is devoted to some results on convex functions, and Appendix B contains some auxiliary computations. Both are used in the proof of Theorem 1.1. In Appendix C we include some results and expressions in (s, t) variables for future reference.

2. Rotation invariant operators acting on doubly radial odd functions

This section is devoted to study rotation invariant operators of the class $\mathcal{L}_0$ when they act on doubly radial odd functions. First, we deduce an alternative expression for the operator in terms of a doubly radial kernel $\overline{K}$. Then, we present necessary and sufficient conditions on the kernel K in order to (1.14) hold (we establish Theorem 1.1). Finally, we show two maximum principles for doubly radial odd functions (Proposition 1.2).

2.1. Alternative expressions for the operator L_K.

The main purpose of this subsection is to deduce an alternative expression for a rotation invariant operator $L_K \in \mathcal{L}_0$ acting on doubly radial functions. This expression is more suitable to work with and it will be used throughout the paper. Our first remark is that if w is invariant by $O(m)^2$, the same holds for $L_K w$. Indeed, for every $R \in O(m)^2$,

$$\begin{aligned} L_K w(Rx) &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(y)\} K(|Rx - y|)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(R\tilde{y})\} K(|Rx - R\tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(\tilde{y})\} K(|x - \tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= L_K w(x)\,. \end{aligned}$$

Here we have used the change $y = R\tilde{y}$ and the fact that $w(R\cdot) = w(\cdot)$ for every $R \in O(m)^2$.

Next, we present an alternative expression for the operator L_K acting on doubly radial functions. This expression involves the new kernel $\overline{K}$, which is also doubly radial.

Lemma 2.1. *Let $L_K \in \mathcal{L}_0(2m, \gamma)$ have a radially symmetric kernel K, and let w be a doubly radial function such that $L_K w$ is well-defined. Then, $L_K w$ can be expressed as*

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\, \mathrm{d}y$$

where $\overline{K}$ is symmetric, invariant by $O(m)^2$ in both arguments, and it is defined by

$$\overline{K}(x, y) := \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,.$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$.

Recall (see for instance [25]) that the Haar measure on $O(m)^2$ exists and it is unique up to a multiplicative constant. Let us state next the properties of this measure that will be used in the rest of the paper. In the following, the Haar measure is denoted by μ. First, since $O(m)^2$ is a compact group, it is unimodular (see Chapter II, Proposition 13 of [25]). As a consequence, the measure μ is left and right invariant, that is, $\mu(R\Sigma) = \mu(\Sigma) = \mu(\Sigma R)$ for every subset $\Sigma \subset O(m)^2$ and every $R \in O(m)^2$. Moreover, it holds

$$\fint_{O(m)^2} g(R^{-1})\,\mathrm{d}R = \fint_{O(m)^2} g(R)\,\mathrm{d}R \tag{2.1}$$

for every $g \in L^1(O(m)^2)$ —see [25] for the details.

Proof of Lemma 2.1. Since $L_K w(x) = L_K w(Rx)$ for every $R \in O(m)^2$, by taking the mean over all the transformations in $O(m)^2$, we get

$$\begin{aligned} L_K w(x) &= \fint_{O(m)^2} L_K w(Rx)\,\mathrm{d}R = \fint_{O(m)^2} \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} K(|Rx - y|)\,\mathrm{d}y\,\mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \fint_{O(m)^2} K(|Rx - y|)\,\mathrm{d}R\,\mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\,\mathrm{d}y\,. \end{aligned}$$

Now, we show that $\overline{K}$ is symmetric. Using property (2.1), we get

$$\begin{aligned} \overline{K}(y, x) &= \fint_{O(m)^2} K(|Ry - x|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R^{-1}(Ry - x)|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|R^{-1}x - y)|)\,\mathrm{d}R = \overline{K}(x, y)\,. \end{aligned}$$

It remains to show that $\overline{K}$ is invariant by $O(m)^2$ in its two arguments. By the symmetry, it is enough to check it for the first one. Let $\tilde{R} \in O(m)^2$. Then,

$$\overline{K}(\tilde{R}x, y) = \fint_{O(m)^2} K(|R\tilde{R}x - y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|Rx - y|)\,\mathrm{d}R = \overline{K}(x, y)\,,$$

where we have used the right invariance of the Haar measure. □

In the following lemma we present some properties of the involution $(\cdot)^\star$ defined by (1.7) and its relation with the doubly radial kernel $\overline{K}$ and the transformations of $O(m)^2$. In particular, in the proof of Theorem 1.1 it will be useful to consider the following transformation. For every $R \in O(m)^2$, we define $R_\star \in O(m)^2$ by

$$R_\star := (R(\cdot)^\star)^\star\,. \tag{2.2}$$

Equivalently, if $R = (R_1, R_2)$ with R_1, $R_2 \in O(m)$, then $R_\star = (R_2, R_1)$.

Lemma 2.2. *Let $(\cdot)^\star : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be the involution defined by $x^\star = (x', x'')^\star = (x'', x')$ —see (1.7). Then,*

(1) The Haar integral on $O(m)^2$ has the following invariance:

$$\int_{O(m)^2} g(R_\star)\,\mathrm{d}R = \int_{O(m)^2} g(R)\,\mathrm{d}R\,, \tag{2.3}$$

for every $g \in L^1(O(m)^2)$.

(2) $\overline{K}(x^\star, y) = \overline{K}(x, y^\star)$*. As a consequence,* $\overline{K}(x^\star, y^\star) = \overline{K}(x, y)$*.*

Proof. The first statement is easy to check by using Fubini:

$$\begin{aligned}\int_{O(m)^2} g(R_\star)\,\mathrm{d}R &= \int_{O(m)} \mathrm{d}R_1 \int_{O(m)} \mathrm{d}R_2 \;\; g(R_2, R_1) = \int_{O(m)} \mathrm{d}R_2 \int_{O(m)} \mathrm{d}R_1 \;\; g(R_2, R_1) \\ &= \int_{O(m)} \mathrm{d}R_1 \int_{O(m)} \mathrm{d}R_2 \;\; g(R_1, R_2) = \int_{O(m)^2} g(R)\,\mathrm{d}R\,.\end{aligned}$$

To show the second statement, we use the definition of $R_\star$ and (2.3) to see that

$$\begin{aligned}\overline{K}(x^\star, y) &= \fint_{O(m)^2} K(|Rx^\star - y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|(Rx^\star - y)^\star|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|(Rx^\star)^\star - y^\star|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R_\star x - y^\star|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|Rx - y^\star|)\,\mathrm{d}R = \overline{K}(x, y^\star)\,.\end{aligned}$$

As a consequence, we have that $\overline{K}(x^\star, y^\star) = \overline{K}(x, (y^\star)^\star) = \overline{K}(x, y)$. $\square$

To conclude this subsection, we present two alternative expressions for the operator L_K when it acts on doubly radial odd functions. These expressions are suitable in the rest of the paper and also in the forthcoming one [19], since the integrals appearing in the expression are computed only in $\mathcal{O}$, and this is important to prove maximum principle and other properties.

Lemma 2.3. *Let w be a doubly radial function which is odd with respect to the Simons cone. Let $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$ be a rotation invariant operator and let $L_K^{\mathcal{O}}$ be defined by (1.12).*

Then, for every $x \in \mathcal{O}$,

$$L_K w(x) = L_K^{\mathcal{O}} w(x).$$

Indeed,

$$\begin{aligned}L_K w(x) &= \int_{\mathcal{O}} \{w(x) - w(y)\}\overline{K}(x, y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x) + w(y)\}\overline{K}(x, y^\star)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\}\{\overline{K}(x, y) - \overline{K}(x, y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x, y^\star)\,\mathrm{d}y\,.\end{aligned}$$

Moreover,

$$\frac{1}{C}\,\mathrm{dist}(x, \mathscr{C})^{-2\gamma} \le \int_{\mathcal{O}} \overline{K}(x, y^\star)\,\mathrm{d}y \le C\,\mathrm{dist}(x, \mathscr{C})^{-2\gamma}, \tag{2.4}$$

with $C > 0$ depending only on $m, \gamma, \lambda,$ and Λ.

Proof. The first statement is just a computation. Indeed, using the change of variables $\bar{y} = y^\star$ and the odd symmetry of w, we see that

$$\begin{aligned}\int_{\mathcal{I}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y &= \int_{\mathcal{O}} \{w(x) - w(y^\star)\}\overline{K}(x,y^\star)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) + w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,.\end{aligned}$$

Hence,

$$\begin{aligned}L_K w(x) &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{I}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x) + w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,.\end{aligned}$$

By adding and subtracting $w(x)\overline{K}(x,y^\star)$ in the last integrand, we immediately deduce

$$L_K w(x) = \int_{\mathcal{O}} \{w(x) - w(y)\}\{\overline{K}(x,y) - \overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,.$$

Note that we can add and subtract the term $w(x)\overline{K}(x,y^\star)$ since it is integrable with respect to y in $\mathcal{O}$. This is a consequence of (2.4).

Let us show now (2.4). In the following arguments we will use the letters C and c to denote positive constants, depending only on m, γ, λ, and Λ, that may change its value in each inequality.

The upper bound in (2.4) is the simplest one since we only need to use the ellipticity of the kernel and the inclusion $\mathcal{I} \subset \{y \in \mathbb{R}^{2m} : |x-y| \geq \operatorname{dist}(x,\mathscr{C})\}$ for every $x \in \mathcal{O}$. Indeed,

$$\begin{aligned}\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{O}} K(|x-y^\star|)\,\mathrm{d}y = \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \leq \int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} K(|x-y|)\,\mathrm{d}y \\ &\leq C\int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} |x-y|^{-2m-2\gamma}\,\mathrm{d}y = C\int_{\operatorname{dist}(x,\mathscr{C})}^{\infty} \rho^{-1-2s}\,\mathrm{d}\rho \\ &= C\operatorname{dist}(x,\mathscr{C})^{-2s}\,.\end{aligned}$$

In order to prove the lower bound in (2.4), let us define $x_0 = x/\operatorname{dist}(x,\mathscr{C})$. Note that $\operatorname{dist}(x_0,\mathscr{C}) = 1$. Then, we have

$$\begin{aligned}\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \\ &\geq c\int_{\mathcal{I}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y = c\int_{\mathcal{I}} |x_0\operatorname{dist}(x,\mathscr{C}) - y|^{-2m-2\gamma}\,\mathrm{d}y \\ &= c\operatorname{dist}(x,\mathscr{C})^{-2s}\int_{\mathcal{I}} |x_0 - \tilde{y}|^{-2m-2\gamma}\,\mathrm{d}\tilde{y}\,.\end{aligned}$$

To conclude the proof, we claim that

$$\int_{\mathcal{I}} |x_0 - y|^{-2m-2\gamma}\, \mathrm{d}y \geq c > 0 \quad \text{for every } x_0 \text{ such that } \operatorname{dist}(x_0, \mathscr{C}) = 1,$$

with a constant c independent of x_0. To establish this claim we will use three facts. The first one is that, since $\operatorname{dist}(x_0, \mathscr{C}) = 1$, there exists a point $\overline{x_0} \in \mathscr{C}$ realizing this distance. Thus, we can easily deduce that $B_{3k+3}(\overline{x_0}) \setminus \overline{B_{3k+2}(\overline{x_0})} \subset B_{3k+4}(x_0) \setminus \overline{B_{3k+1}(x_0)}$ for every $k \geq 0$. The second fact is the identity $\mathcal{I} = \mathcal{I} \cap B_1^c(x_0)$, that also follows from $\operatorname{dist}(x_0, \mathscr{C}) = 1$. Finally, the last fact is a property of the Simons cone: $|B_R(z) \cap \mathcal{I}| = 1/2|B_R|$ for every $z \in \mathscr{C}$ (see Lemma 2.5 in [20] for the proof). Combining these three facts, we get

$$\begin{aligned}
\int_{\mathcal{I}} |x_0 - y|^{-2m-2\gamma}\, \mathrm{d}y &= \sum_{k=0}^{\infty} \int_{I \cap (B_{3k+4}(x_0) \setminus B_{3k+1}(x_0))} |x_0 - y|^{-2m-2\gamma}\, \mathrm{d}y \\
&\geq \sum_{k=0}^{\infty} \int_{I \cap (B_{3k+4}(x_0) \setminus B_{3k+1}(x_0))} (3k+4)^{-2m-2\gamma}\, \mathrm{d}y \\
&= \sum_{k=0}^{\infty} (3k+4)^{-2m-2\gamma} |I \cap (B_{3k+4}(x_0) \setminus B_{3k+1}(x_0))\,| \\
&\geq \sum_{k=0}^{\infty} (3k+4)^{-2m-2\gamma} |I \cap (B_{3k+3}(\overline{x_0}) \setminus B_{3k+2}(\overline{x_0}))\,| \\
&= c \sum_{k=0}^{\infty} (3k+4)^{-2m-2\gamma} \left\{ (3k+3)^{2m} - (3k+2)^{2m} \right\} \\
&\geq c \sum_{k=0}^{\infty} (3k+4)^{-1-2\gamma} = c.
\end{aligned}$$

□

2.2. **Necessary and sufficient conditions for ellipticity.** In this subsection, we establish Theorem 1.1. As we have mentioned in the introduction, the kernel inequality (1.14) is crucial in the rest of the results of this paper, as well as in the ones in [19]. We will see in the next subsection that this inequality guarantees that the operator L_K has a maximum principle for odd functions (see Proposition 1.2).

First, we give a sufficient condition on a radially symmetric kernel K so that $\overline{K}$ satisfies (1.14). It is the following result.

Proposition 2.4. *Let $K : (0, +\infty) \to \mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (1.11). Assume that $K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$. Then, the associated kernel $\overline{K}$ satisfies*

$$\overline{K}(x, y) > \overline{K}(x, y^{\star}) \quad \textit{for every } x, y \in \mathcal{O}\,. \tag{2.5}$$

Proof. Since $\overline{K}$ is invariant by $O(m)^2$, it is enough to show (2.5) for points $x, y \in \mathcal{O}$ of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, with $e \in \mathbb{S}^{m-1}$ an arbitrary unitary vector.

Now, define

$$\begin{aligned} Q_1 &:= \left\{ R = (R_1, R_2) \in O(m)^2 \ : \ e \cdot R_1 e > |e \cdot R_2 e| \right\}, \\ Q_2 &:= \left\{ R = (R_1, R_2) \in O(m)^2 \ : \ e \cdot R_2 e > |e \cdot R_1 e| \right\} = (Q_1)_\star, \\ Q_3 &:= \left\{ R = (R_1, R_2) \in O(m)^2 \ : \ e \cdot R_1 e < -|e \cdot R_2 e| \right\} = -Q_1, \\ Q_4 &:= \left\{ R = (R_1, R_2) \in O(m)^2 \ : \ e \cdot R_2 e < -|e \cdot R_1 e| \right\} = -(Q_1)_\star. \end{aligned} \tag{2.6}$$

Recall that given $R = (R_1, R_2) \in O(m)^2$, then $R_\star = (R_2, R_1) \in O(m)^2$ —see (2.2). Moreover, note that the sets Q_i are disjoint, have the same measure and cover all $O(m)^2$ up to a set of measure zero.

Therefore,

$$\begin{aligned} 4\overline{K}(x, y) &= 4 \fint_{O(m)^2} K(|x - Ry|)\, \mathrm{d}R \\ &= \fint_{Q_1} K(|x - Ry|)\, \mathrm{d}R + \fint_{Q_2} K(|x - Ry|)\, \mathrm{d}R \\ &\quad + \fint_{Q_3} K(|x - Ry|)\, \mathrm{d}R + \fint_{Q_4} K(|x - Ry|)\, \mathrm{d}R \\ &= \fint_{Q_1} \{K(|x - Ry|) + K(|x + Ry|) \\ &\quad + K(|x - R_\star y|) + K(|x + R_\star y|)\}\, \mathrm{d}R \end{aligned}$$

and

$$\begin{aligned} 4\overline{K}(x, y^\star) &= 4 \fint_{O(m)^2} K(|x - Ry^\star|)\, \mathrm{d}R \\ &= \fint_{Q_1} \{K(|x - Ry^\star|) + K(|x + Ry^\star|) \\ &\quad + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|)\}\, \mathrm{d}R. \end{aligned}$$

Thus, if we prove

$$\begin{aligned} &K(|x - Ry|) + K(|x + Ry|) + K(|x - R_\star y|) + K(|x + R_\star y|) \\ &\qquad \geq K(|x - Ry^\star|) + K(|x + Ry^\star|) + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|), \end{aligned} \tag{2.7}$$

for every $R \in Q_1$, we immediately deduce (2.5) with a non strict inequality. To see that it is indeed a strict one, we will show that the inequality in (2.7) is strict for every $R \in Q_1$.

For a short notation, we call

$$\alpha := e \cdot R_1 e \quad \text{and} \quad \beta := e \cdot R_2 e\,. \tag{2.8}$$

Now, note that since $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, we have

$$\begin{aligned}|x \pm Ry|^2 &= |x' \pm R_1 y'|^2 + |x'' \pm R_2 y''|^2 \\ &= |x'|^2 + |y'|^2 \pm 2x' \cdot R_1 y' + |x''|^2 + |y''|^2 \pm 2x'' \cdot R_2 y'' \\ &= |x|^2 + |y|^2 \pm 2|x'||y'|\alpha \pm 2|x''||y''|\beta.\end{aligned}$$

Similarly,

$$\begin{aligned}|x \pm R_\star y|^2 &= |x|^2 + |y|^2 \pm 2|x'||y'|\beta \pm 2|x''||y''|\alpha, \\ |x \pm R y^\star|^2 &= |x|^2 + |y|^2 \pm 2|x'||y''|\alpha \pm 2|x''||y'|\beta,\end{aligned}$$

and

$$|x \pm R_\star y^\star|^2 = |x|^2 + |y|^2 \pm 2|x'||y''|\beta \pm 2|x''||y'|\alpha.$$

We define now

$$g(\tau) := K\left(\sqrt{|x|^2 + |y|^2 + 2\tau}\right) + K\left(\sqrt{|x|^2 + |y|^2 - 2\tau}\right).$$

Thus, proving (2.7) is equivalent to show that, for every α, $\beta \in [-1, 1]$ such that $\alpha > |\beta|$, it holds

$$\begin{aligned}&g\Big(|x'||y'|\alpha + |x''||y''|\beta\Big) + g\Big(|x'||y'|\beta + |x''||y''|\alpha\Big) \\ &\qquad \geq g\Big(|x'||y''|\alpha + |x''||y'|\beta\Big) + g\Big(|x'||y''|\beta + |x''||y'|\alpha\Big).\end{aligned} \tag{2.9}$$

Let

$$\begin{aligned}A_{\alpha,\beta} &:= |x'||y'|\alpha + |x''||y''|\beta\,, \quad B_{\alpha,\beta} := |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &:= |x''||y'|\alpha + |x'||y''|\beta\,, \quad D_{\alpha,\beta} := |x''||y''|\alpha + |x'||y'|\beta\,.\end{aligned}$$

With this notation and taking into account that g is even, (2.9) is equivalent to

$$g(|A_{\alpha,\beta}|) + g(|D_{\alpha,\beta}|) \geq g(|C_{\alpha,\beta}|) + g(|B_{\alpha,\beta}|)\,, \tag{2.10}$$

for every α, $\beta \in [-1, 1]$ such that $\alpha > |\beta|$. Note that g is defined in the open interval $I = (-(|x|^2 + |y|^2)/2,\ (|x|^2 + |y|^2)/2)$ and that $A_{\alpha,\beta}$, $B_{\alpha,\beta}$, $C_{\alpha,\beta}$, $D_{\alpha,\beta} \in I$.

To show (2.10), we use Proposition A.1 of the Appendix A. There, it is stated that in order to establish (2.10) it is enough to check that

$$\begin{cases}|A_{\alpha,\beta}| \geq |B_{\alpha,\beta}|, \quad |A_{\alpha,\beta}| \geq |C_{\alpha,\beta}|, \quad |A_{\alpha,\beta}| \geq |D_{\alpha,\beta}|\,, \\ |A_{\alpha,\beta}| + |D_{\alpha,\beta}| \geq |B_{\alpha,\beta}| + |C_{\alpha,\beta}|\,.\end{cases}$$

The verification of these inequalities is a simple but tedious computation and it is presented in Appendix B —see point (1) of Lemma B.1. Once this is proved, we deduce (2.10) by Proposition A.1.

To finish, we see that the inequality in (2.10) is always strict for every α, $\beta \in [-1, 1]$ such that $\alpha > |\beta|$ (that corresponds to Q_1). By contradiction, assume that equality holds in (2.10). Thus, by Proposition A.1, if follows that the sets $\{|A_{\alpha,\beta}|, |D_{\alpha,\beta}|\}$ and $\{|B_{\alpha,\beta}|, |C_{\alpha,\beta}|\}$ coincide. This fact and point (2) of Lemma B.1 yield $\alpha = \beta = 0$, a contradiction. Thus, the inequality in (2.10) is strict, as well as the inequality in (2.7). This leads to (2.5). □

Now, we give a necessary condition on the kernel K so that inequality (1.14) holds.

Proposition 2.5. *Let $K:(0,+\infty)\to\mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K}:\mathbb{R}^{2m}\times\mathbb{R}^{2m}\to\mathbb{R}$ by (1.11).*
If

$$\overline{K}(x,y)>\overline{K}(x,y^\star)\qquad \textit{for almost every } x,y\in\mathcal{O}\,, \tag{2.11}$$

then $K(\sqrt{\cdot})$ cannot be concave in any interval $I\subset[0,+\infty)$.

Proof. We prove it by contraposition. In fact, we will show that if there exists an interval where $K(\sqrt{\cdot})$ is concave, then we can find an open set in $\mathcal{O}\times\mathcal{O}$ with positive measure where (2.11) is not satisfied.

Let $\ell_2>\ell_1>0$ be such that $K(\sqrt{\cdot})$ is concave in (ℓ_1,ℓ_2) and define the set $\Omega_{\ell_1,\ell_2}\subset\mathbb{R}^{4m}$ as the points $(x,y)\in\mathcal{O}\times\mathcal{O}$ satisfying

$$\left\{\begin{array}{lcl}(|x'|-|y'|)^2+(|x''|-|y''|)^2 & > & \ell_1,\\ (|x'|+|y'|)^2+(|x''|+|y''|)^2 & < & \ell_2.\end{array}\right. \tag{2.12}$$

First, it is easy to see that Ω_{ℓ_1,ℓ_2} is a nonempty open set. In fact, points of the form $(x',0,y',0)\in(\mathbb{R}^m)^4$ such that $(|x'|-|y'|)^2>\ell_1$ and $(|x'|+|y'|)^2<\ell_2$ belong to Ω_{ℓ_1,ℓ_2}. Then, if we prove that $\overline{K}(x,y)\le\overline{K}(x,y^\star)$ in Ω_{ℓ_1,ℓ_2} we are done.

Given $x,y\in\Omega_{\ell_1,\ell_2}$, we are going to show, as in the previous proof, that

$$\begin{aligned}&K(|x-Ry|)+K(|x+Ry|)+K(|x-R_\star y|)+K(|x+R_\star y|)\\&\qquad\le K(|x-Ry^\star|)+K(|x+Ry^\star|)+K(|x-R_\star y^\star|)+K(|x+R_\star y^\star|)\,,\end{aligned} \tag{2.13}$$

for any $R\in Q_1$, where Q_1 is defined in (2.6) (see the proof of Proposition 2.4). As before, we can assume that x and y are of the form $x=(|x'|e,|x''|e)$ and $y=(|y'|e,|y''|e)$, with $e\in\mathbb{S}^{m-1}$ an arbitrary unitary vector. Then, by defining α and β as in (2.8), we see that proving (2.13) is equivalent to establish that

$$g(A_{\alpha,\beta})+g(D_{\alpha,\beta})\le g(B_{\alpha,\beta})+g(C_{\alpha,\beta})\,, \tag{2.14}$$

for every $\alpha,\beta\in[-1,1]$ such that $\alpha>|\beta|$, where

$$\begin{aligned}A_{\alpha,\beta}&=|x'||y'|\alpha+|x''||y''|\beta\,, & B_{\alpha,\beta}&=|x'||y''|\alpha+|x''||y'|\beta\,,\\ C_{\alpha,\beta}&=|x''||y'|\alpha+|x'||y''|\beta\,, & D_{\alpha,\beta}&=|x''||y''|\alpha+|x'||y'|\beta\,.\end{aligned}$$

and

$$g(\tau)=K\left(\sqrt{|x|^2+|y|^2+2\tau}\right)+K\left(\sqrt{|x|^2+|y|^2-2\tau}\right).$$

Now, by (2.12), we have $\ell_1<|x|^2+|y|^2<\ell_2$. As a consequence of this and the concavity of $K(\sqrt{\cdot})$ in (ℓ_1,ℓ_2), it is easy to see (by using Lemma A.2 in the Appendix A) that g is concave in $\left(-\overline{\ell},\overline{\ell}\right)$, and decreasing in $(0,\overline{\ell})$, where

$$\overline{\ell}:=\min\left\{\frac{\ell_2-|x|^2-|y|^2}{2},\frac{|x|^2+|y|^2-\ell_1}{2}\right\}.$$

Note that, since $\ell_1<|x|^2+|y|^2<\ell_2$, we have $\overline{\ell}>0$.

We claim that $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to $(-\overline{\ell},\overline{\ell})$ for every $\alpha,\beta\in[-1,1]$ such that $\alpha>|\beta|$. Indeed, it is easy to check that for every $\alpha,\beta\in[-1,1]$ such that $\alpha>|\beta|$, the numbers $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to the open interval $(-|x'||y'|-|x''||y''|, |x'||y'|+|x''||y''|)$. Furthermore, since $x,y\in\Omega_{\ell_1,\ell_2}$, we obtain from (2.12) that

$$\begin{cases} |x'||y'|+|x''||y''| & < & \dfrac{\ell_2-|x|^2-|y|^2}{2} \\ |x'||y'|+|x''||y''| & < & \dfrac{|x|^2+|y|^2-\ell_1}{2} \end{cases}$$

and thus $|x'||y'|+|x''||y''|<\overline{\ell}$ and the claim is proved.

Finally, by applying Lemma A.2 to the function $-g$ in $(0,\overline{\ell})$ (using again point (1) of Lemma B.1), we obtain that inequality (2.14) is satisfied, which yields (2.13). Finally, by integrating (2.13) with respect to all the rotations $R\in Q_1$ we get

$$\overline{K}(x,y)\le\overline{K}(x,y^\star),$$

for every $(x,y)\in\Omega_{\ell_1,\ell_2}$, contradicting (2.11). □

From the two previous results, Theorem 1.1 follows immediately.

Proof of Theorem 1.1. The first statement is exactly the same as Proposition 2.4. Assume now that K is a C^2 function and that (1.14) holds. Then, by Proposition 2.5, $K(\sqrt{\cdot})$ is not concave in any interval of $[0,+\infty)$. By the regularity of K, this automatically yields that $K(\sqrt{\cdot})$ is strictly convex. □

Remark 2.6. Note that a priori we cannot relax the C^2 assumption in the necessary condition of Theorem 1.1, since there are C^1 functions that are neither convex nor concave in any interval (they can be constructed as a primitive of a Weierstrass function, whose graph is a non rectifiable curve with fractal dimension). Besides these "exotic" examples, there are also simple radially symmetric kernels K that are not C^1 for which we do not know if the kernel inequality (1.14) holds. For instance, given $0<\gamma<1$, if we consider the kernel

$$K(\tau)=\frac{1}{\tau^{2m+2\gamma}}\chi_{(0,1)}(\tau)+\frac{1}{10\tau^{2m+2\gamma}-9}\chi_{[1,+\infty)}(\tau),$$

it is easy to check that K is continuous and decreasing but $K(\sqrt{\tau})$ is not convex in $(0,+\infty)$ even though it does not have any interval of concavity (see Figure 1).

2.3. **Maximum principles for doubly radial odd functions.** In this subsection we prove Proposition 1.2, a weak and a strong maximum principles for doubly radial functions that are odd with respect to the Simons cone. The formulation of these maximum principles is very suitable since all the hypotheses refer to the set $\mathcal{O}$ and not $\mathbb{R}^{2m}$. The key ingredient in the proofs is the kernel inequality (1.14).

Proof of Proposition 1.2. (i) By contradiction, suppose that u takes negative values in Ω. Under the hypotheses we are assuming, a negative minimum must be achieved.

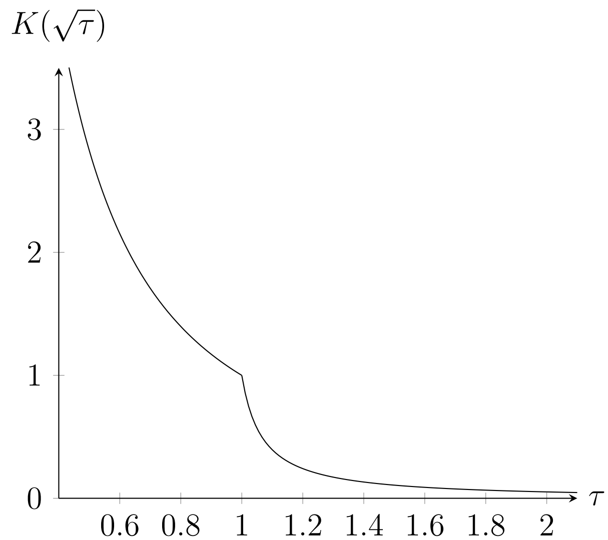

FIGURE 1. An example of kernel $K(\sqrt{\tau})$ ($m = 1$ and $\gamma = 1/2$) which is not strictly convex in $(0, +\infty)$ but does not have any interval of concavity.

Thus, there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{\Omega} u =: m < 0\,.$$

Then, using the expression of L_K for odd functions (see Lemma 2.3), we have

$$L_K u(x_0) = \int_{\mathcal{O}} \{m - u(y)\}\{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\}\,\mathrm{d}y + 2m \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y\,.$$

Now, since $m - u(y) \le 0$ in $\mathcal{O}$, $m < 0$, $c \ge 0$, and $\overline{K}(x_0, y) \ge \overline{K}(x_0, y^\star) > 0$ —by (1.14)—, we get

$$0 \le L_K u(x_0) + c(x_0)u(x_0) \le m\left(2\int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y + c(x_0)\right) < 0\,,$$

a contradiction.

(ii) Assume that $u \not\equiv 0$ in $\mathcal{O}$. We shall prove that $u > 0$ in Ω. By contradiction, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then, using the expression of L_K for odd functions given in Lemma 2.3, the kernel inequality (1.14), and the fact that $u \ge 0$ in $\mathcal{O}$, we obtain

$$0 \le L_K u(x_0) + c(x_0)u(x_0) = -\int_{\mathcal{O}} u(y)\big\{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\big\}\,\mathrm{d}y < 0\,,$$

a contradiction. □

Remark 2.7. Note that since the operator L_K includes itself a positive zero order term in addition to the integro-differential part, the condition $c \ge 0$ in point (i) of the previous proposition can be lightly relaxed. Indeed, if we follow the proof of the result, we can deduce that the hypothesis on c that we can assume is

$$c(x) > -2 \int_{\mathcal{O}} \overline{K}(x, y^\star)\, \mathrm{d}y.$$

This hypothesis seems hard to be checked for applications apart from the case $c \geq 0$. Nevertheless, recall that by Lemma 2.3 we have an explicit lower bound for the quantity $\int_{\mathcal{O}} \overline{K}(x, y^\star)\, \mathrm{d}y$ in terms of the function $\operatorname{dist}(x, \mathscr{C})$ that could be used to check the previous condition.

3. The energy functional for doubly radial odd functions

This section is devoted to the energy functional associated to the semilinear equation (1.1). We first define appropriately the functional spaces where we are going to apply classic techniques of calculus of variations. Next we rewrite the energy in terms of the new kernel $\overline{K}$ and we give an alternative expression for the energy of doubly radial odd functions. Finally, we establish some results that are useful when using variational techniques, and that will be exploited in the next section.

Let us start by defining the functional spaces that we are going to consider in the rest of the paper. Given a set $\Omega \subset \mathbb{R}^n$ and a translation invariant and positive kernel K satisfying (1.3), we define the space

$$\mathbb{H}^K(\Omega) := \left\{ w \in L^2(\Omega) \ : \ [w]^2_{\mathbb{H}^K(\Omega)} < +\infty \right\},$$

where

$$[w]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2} \int\int_{(\mathbb{R}^n)^2 \setminus (\mathbb{R}^n \setminus \Omega)^2} |w(x) - w(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y \,.$$

We also define

$$\begin{aligned}\mathbb{H}^K_0(\Omega) &:= \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\} \\ &= \left\{ w \in \mathbb{H}^K(\mathbb{R}^n) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\}.\end{aligned}$$

Assume that $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution. Then, we define

$$\widetilde{\mathbb{H}}^K(\Omega) := \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

and

$$\widetilde{\mathbb{H}}^K_0(\Omega) := \left\{ w \in \mathbb{H}^K_0(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

We will add the subscript 'odd' and 'even' to these spaces to consider only functions that are odd (respectively even) with respect to the Simons cone.

Remark 3.1. If $\widetilde{\mathbb{H}}^K_0(\Omega)$ is equipped with the scalar product

$$\langle v, w \rangle_{\widetilde{\mathbb{H}}^K_0(\Omega)} := \frac{1}{2} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \big(v(x) - v(y)\big)\big(w(x) - w(y)\big) K(x-y)\, \mathrm{d}x\, \mathrm{d}y \,,$$

then, it is easy to check that $\widetilde{\mathbb{H}}^K_0(\Omega)$ can be decomposed as the orthogonal direct sum of $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega)$ and $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$.

Note that when K satisfies (1.4), then $\mathbb{H}^K_0(\Omega) = \mathbb{H}^\gamma_0(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(z) = c_{n,\gamma} |z|^{-n-2\gamma}$. Furthermore, $\mathbb{H}^\gamma(\Omega) \subset H^\gamma(\Omega)$, the usual fractional Sobolev space (see [17]). For more comments on this, see [13], and the references therein.

Once presented the functional setting of our problem, we proceed with the study of the energy functional associated to equation (1.1).

Given a kernel K satisfying (1.3) and a potential G, and given a function $w \in \mathbb{H}^K(\Omega)$, with $\Omega \subset \mathbb{R}^n$, we write the energy defined in (1.15) as

$$\mathcal{E}(w,\Omega) = \mathcal{E}_{\mathrm{K}}(w,\Omega) + \mathcal{E}_{\mathrm{P}}(w,\Omega)\,,$$

where

$$\mathcal{E}_{\mathrm{K}}(w,\Omega) := \frac{1}{2}[w]^2_{\mathbb{H}^K(\Omega)} \quad \text{and} \quad \mathcal{E}_{\mathrm{P}}(w,\Omega) := \int_\Omega G(w)\,\mathrm{d}x\,.$$

We will call $\mathcal{E}_{\mathrm{K}}$ and $\mathcal{E}_{\mathrm{P}}$ the *kinetic* and *potential* energy respectively.

Note that, for functions $w \in \mathbb{H}^K_0(\Omega)$, it holds $\mathcal{E}_{\mathrm{K}}(w,\Omega) = \mathcal{E}_{\mathrm{K}}(w,\mathbb{R}^n)$. Moreover, if $G \geq 0$, the energy satisfies $\mathcal{E}(w,\Omega) \leq \mathcal{E}(w,\Omega')$ whenever $\Omega \subset \Omega'$.

Sometimes it is useful to rewrite the kinetic energy as

$$\begin{aligned}\mathcal{E}_{\mathrm{K}}(w,\Omega) := \frac{1}{4}\bigg\{&\int_\Omega\int_\Omega |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\bigg\}\,.\end{aligned} \tag{3.1}$$

Roughly speaking, we have split the kinetic energy into two parts: "interactions inside-inside" and "interactions inside-outside". Our goal is to rewrite the kinetic energy in terms of the doubly radial kernel $\overline{K}$ and with integrals computed only in $\mathcal{O}$, in the same spirit as in the previous section for the operator L_K. In particular, we are interested in finding an expression similar to (3.1) for the kinetic energy. To do this, we introduce the following notation for the interaction. Given A, $B \subset \mathcal{O}$ sets of double revolution, we define

$$\begin{aligned}I_w(A,B) := 2\int_A\int_B &|w(x)-w(y)|^2\left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ &+ 4\int_A\int_B \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned} \tag{3.2}$$

Thanks to this notation, we rewrite the kinetic energy as follows.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^{2m}$ be a set of double revolution that is symmetric with respect to the Simons cone, i.e., $\Omega^\star = \Omega$, and let $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$. Let K be a radially symmetric kernel satisfying* (1.3)*. Then,*

$$\mathcal{E}_{\mathrm{K}}(w,\Omega) = \frac{1}{4}\big\{I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\big\}, \tag{3.3}$$

where $I_w(\cdot,\cdot)$ is the interaction defined in (3.2)*.*

Proof. First, in (3.1) we consider the change $y = R\tilde{y}$. Since w is doubly radial and Ω is of double revolution, by taking the average among all $R \in O(m)^2$ as in Lemma 2.1, we obtain

$$\mathcal{E}_{\mathrm{K}}(w,\Omega) = \frac{1}{4}\int_\Omega\int_\Omega |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Now we split Ω into $\Omega\cap\mathcal{O}$ and $\Omega\setminus\mathcal{O}$. By using the change of variables given by $(\cdot)^\star$ and the symmetries of Ω and w, we get

$$\begin{aligned}
4\mathcal{E}_{\mathrm{K}}(w,\Omega) &= 2\int_{\Omega\cap\mathcal{O}}\int_{\Omega\cap\mathcal{O}} |w(x)-w(y)|^2\overline{K}(x,y) + |w(x)+w(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\int_{\Omega\cap\mathcal{O}}\int_{\mathcal{O}\setminus\Omega} |w(x)-w(y)|^2\overline{K}(x,y) + |w(x)+w(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_{\Omega\cap\mathcal{O}}\int_{\Omega\cap\mathcal{O}} |w(x)-w(y)|^2\left\{\overline{K}(x,y)-\overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\int_{\Omega\cap\mathcal{O}}\int_{\Omega\cap\mathcal{O}} \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\int_{\Omega\cap\mathcal{O}}\int_{\mathcal{O}\setminus\Omega} |w(x)-w(y)|^2\left\{\overline{K}(x,y)-\overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 8\int_{\Omega\cap\mathcal{O}}\int_{\mathcal{O}\setminus\Omega} \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega).
\end{aligned}$$

□

Using the previous expression for the energy, we can establish now the following lemma regarding the decrease of the energy under some operations.

Lemma 3.3. *Let $\Omega\subset\mathbb{R}^{2m}$ be a set of double revolution that is symmetric with respect to the Simons cone, and let K be a radially symmetric kernel satisfying the positivity condition (1.14). Given $u\in\widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\Omega)$, we define*

$$v(x)=\begin{cases} |u(x)| & \text{if } x\in\mathcal{O},\\ -|u(x)| & \text{if } x\in\mathcal{I},\end{cases} \qquad \text{and} \qquad w(x)=\begin{cases} \min\{1,u(x)\} & \text{if } x\in\mathcal{O},\\ \max\{-1,u(x)\} & \text{if } x\in\mathcal{I}.\end{cases}$$

If G satisfies (1.5), then

$$\mathcal{E}(v,\Omega)\le\mathcal{E}(u,\Omega) \qquad \text{and} \qquad \mathcal{E}(w,\Omega)\le\mathcal{E}(u,\Omega)\,.$$

Proof. We first establish the result for v. Let us show that $\mathcal{E}_{\mathrm{K}}(v)\le\mathcal{E}_{\mathrm{K}}(u)$. Note that $v\in\widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\Omega)$. Thus, by using the expression of the kinetic energy given in (3.3) and the fact that $\overline{K}(x,y)>\overline{K}(x,y^\star)>0$ if $x,y\in\mathcal{O}$ —see (1.14)—, we only need to check that $|v(x)-v(y)|^2\le|u(x)-u(y)|^2$ and $v^2(x)\le u^2(x)$ whenever $x,y\in\mathcal{O}$. The first condition follows from the equivalence

$$\big||u(x)|-|u(y)|\big|^2\le|u(x)-u(y)|^2 \Longleftrightarrow u(x)u(y)\le|u(x)u(y)|,$$

while the second one is trivial and it is in fact an equality. Concerning the potential energy, since G is an even function we have that $\mathcal{E}_{\mathrm{P}}(v)=\mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result for v by adding the kinetic and potential energies.

We show now the result for w. Let us show that $\mathcal{E}_{\mathrm{K}}(w)\le\mathcal{E}_{\mathrm{K}}(u)$. As before, $w\in\widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\Omega)$ and thus, in view of (3.3) and the kernel inequality (1.14), we only need to check that $|w(x)-w(y)|^2 \le |u(x)-u(y)|^2$ and $w^2(x) \le u^2(x)$ whenever $x, y \in \mathcal{O}$. The first inequality is trivial whenever $u(x) \le 1$ and $u(y) \le 1$, or $u(x) \ge 1$ and $u(y) \ge 1$. If $u(x) \ge 1$ and $u(y) \le 1$, then $|u(x)-u(y)|^2 - |w(x)-w(y)|^2 = |u(x)-u(y)|^2 - |1-u(y)|^2 = (u(x)-1))^2 + 2(u(x)-1)(1-u(y)) \ge 0$. The second inequality follows from the fact that $w^2(x) = u^2(x)$ when $u(x) \le 1$, while $w^2(x) = 1 \le u^2(x)$ if $u(x) \ge 1$. Concerning the potential energy, since G is such that $G(x) \ge G(1) = G(-1) = 0$ if $|x| \le 1$, then clearly $\mathcal{E}_{\mathrm{P}}(w) \le \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result by adding the kinetic and potential energies. □

Next we present a result that will be used later, and concerns weak solutions to semilinear Dirichlet problems. Its main consequence is that a function $u \in \widetilde{\mathbb{H}}_0^K(\Omega)$ that minimizes the energy $\mathcal{E}$, but only among doubly radial functions, is actually a weak solution to a semilinear Dirichlet problem in Ω. We remark that to show the following result we do not need to use the kernel $\overline{K}$.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set of double revolution and let $L_K \in \mathcal{L}_0$ with kernel K radially symmetric. Let $u \in \widetilde{\mathbb{H}}_0^K(\Omega)$ such that*

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\xi(x) - \xi(y)\} K(|x-y|) \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x) \,\mathrm{d}x$$

for every $\xi \in C_c^\infty(\Omega)$ that is doubly radial. Then, u is a weak solution to

$$\begin{cases} L_K u = f(u) & \text{in } \Omega\,, \\ u = 0 & \text{in } \mathbb{R}^{2m} \setminus \Omega\,, \end{cases}$$

i.e.,

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\eta(x) - \eta(y)\} K(|x-y|) \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\eta(x) \,\mathrm{d}x$$

for every $\eta \in C_c^\infty(\Omega)$ (not necessarily doubly radial).

Proof. Let $\eta \in C_c^\infty(\Omega)$. We define its associated doubly radial function as

$$\overline{\eta}(x) := \fint_{O(m)^2} \eta(Rx) \,\mathrm{d}R\,.$$

Now, on the one hand, given $R \in O(m)^2$ and using the change $x = R\tilde{x}$, $y = R\tilde{y}$ and the fact that u is doubly radial, we get

$$\begin{aligned} &\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\eta(x) - \eta(y)\} K(|x-y|) \,\mathrm{d}x \,\mathrm{d}y = \\ &\qquad = \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\eta(Rx) - \eta(Ry)\} K(|x-y|) \,\mathrm{d}x \,\mathrm{d}y\,. \end{aligned}$$

Taking the average in the previous equality among all $R \in O(m)^2$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} &\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \fint_{O(m)^2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

On the other hand, using also the change $x = R\tilde{x}$, we have

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x = \int_\Omega f(u(R^{-1}x))\eta(x)\,\mathrm{d}x = \int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,.$$

Similarly as before, taking the average among all $R \in O(m)^2$, we get

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x = \fint_{O(m)^2}\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,\mathrm{d}R = \int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\,.$$

Hence, since $\overline{\eta} \in C_c^\infty(\Omega)$ is doubly radial, we have

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} &\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f(u(x))\eta(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x \\ &= 0\,,\end{aligned}$$

and thus the result is proved. $\square$

Remark 3.5. This proposition combined with some regularity results for operators in the class $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ yield that bounded minimizers among doubly radial functions of the energy $\mathcal{E}(\cdot,\Omega)$ are classical solutions to $L_K u = f(u)$ in Ω. Indeed, if $w \in L^\infty(\mathbb{R}^n)$ is a weak solution to $L_K w = h$ in $B_1 \subset \mathbb{R}^n$, then

$$||w||_{C^{2\gamma}(\overline{B_{1/2}})} \le C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)}\right). \tag{3.4}$$

If, in addition, $w \in C^\alpha(\mathbb{R}^n)$ with $\alpha + 2\gamma$ not an integer, then

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/2}})} \le C\left(||h||_{C^\alpha(\overline{B_1})} + ||w||_{C^\alpha(\mathbb{R}^n)}\right), \tag{3.5}$$

where the previous two constants C depend only on n, γ, λ, and Λ (see [26, 31] and the references therein).

From the previous estimates and using the same argument as in Corollaries 2.4 and 2.5 of [27], (3.4) and (3.5) yield, respectively, the estimates

$$||w||_{C^{2\gamma}(\overline{B_{1/4}})} \le C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(B_1)} + \left|\left|\frac{w(x)}{(1+|x|)^{n+2\gamma}}\right|\right|_{L^1(\mathbb{R}^n)}\right), \tag{3.6}$$

and

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/4}})} \leq C \left(||h||_{C^{\alpha}(\overline{B_1})} + ||w||_{C^{\alpha}(\overline{B_1})} + \left|\left| \frac{w(x)}{(1+|x|)^{n+2\gamma}} \right|\right|_{L^1(\mathbb{R}^n)} \right). \tag{3.7}$$

Therefore, by applying these estimates (maybe a translated and rescaled version of them) to a weak solution $u \in L^\infty(\mathbb{R}^{2m})$ of $L_K u = f(u)$ in Ω, with f a C^1 nonlinearity, we easily conclude that u is a classical solution, that is, the equation makes sense pointwise.

4. An energy estimate for doubly radial odd minimizers

In this section we present an estimate for the energy in the ball B_S of minimizers in the space $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ with $R > S + 5$. That is, we prove Theorem 1.3. In order to establish this result, we follow the ideas of Savin and Valdinoci in [29], where they show the same type of estimate but for minimizers without any symmetry. The strategy used in [29] is to compare the energy of u, the minimizer, with the energy of a suitable competitor which is constructed taking the minimum between u and the radially symmetric auxiliary function

$$\phi_S(x) := -1 + 2\min\{(|x| - S - 1)_+, 1\}\,. \tag{4.1}$$

In our case, such construction would not give us a doubly radial function which is odd with respect to the Simons cone $\mathscr{C}$, and therefore it would not be an admissible competitor —recall that we consider u to be a minimizer among functions in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Thus, we need to adapt the auxiliary function ϕ_S to obtain the desired competitor (that must be doubly radial and odd with respect to the Simons cone).

The auxiliary function needed to build the competitor is defined as follows. For points $x \in \mathcal{O}$ and μ a positive real number, we set

$$\Psi_S(x) := \begin{cases} \phi_S(x)\min\{1, \mu\,\mathrm{dist}(x, \mathscr{C})\} & x \in B_{S+4} \cap \mathcal{O}\,, \\ 1 & x \in \mathcal{O} \setminus B_{S+4}\,, \end{cases} \tag{4.2}$$

and we extend it to the whole $\mathbb{R}^{2m}$ by setting $\Psi_S(x) = 0$ for every $x \in \mathscr{C}$ and then considering its odd reflection with respect to the cone, that is $\Psi_S(x) = -\Psi_S(x^\star)$ for $x \in \mathcal{I}$. It is clear that Ψ_S is a bounded function with $||\Psi_S||_{L^\infty(\mathbb{R}^{2m})} = 1$. Moreover, it is uniformly Lipschitz in $\overline{B_{S+3}}$ with Lipschitz constant depending only on m and μ, but independent of S. For an schematic description of Ψ_S see Figure 2 (a).

In our arguments we will also use the following function and set:

$$d_S(x) := \max\{1, \min\{S + 1 - |x|, \mu\,\mathrm{dist}(x, \mathscr{C})\}\}\,,$$

and

$$\Omega_S := \left(\overline{B_{S+2}} \setminus B_S\right) \cup \left(\overline{B_{S+2}} \cap \{\mu\,\mathrm{dist}(x, \mathscr{C}) \leq 1\}\right). \tag{4.3}$$

The set Ω_S is the preimage of 1 through d_S in $\overline{B_{S+2}}$ —see Figure 2 (b). Furthermore, it is easy to see that its measure is of order $2m - 1$. That is,

$$|\Omega_S| \leq C\, S^{2m-1}, \tag{4.4}$$

with a constant C depending only on m and μ. To see this, follow the computations in the proof of the energy estimate for the local equation in Theorem 1.3 of [7].

Now we show some auxiliary results concerning the previous definitions, needed in the proof of Theorem 1.3.

Lemma 4.1. *Given $S > 0$, if either $(x, y) \in (\Omega_S \cap \mathcal{O}) \times \mathcal{I}$ or $(x, y) \in (B_{S+2} \cap \mathcal{O}) \times \mathcal{O}$, then*

$$|\Psi_S(x) - \Psi_S(y)| \leq C\frac{|x - y|}{d_S(x)} \qquad \textit{whenever} \;\; |x - y| \leq d_S(x),$$

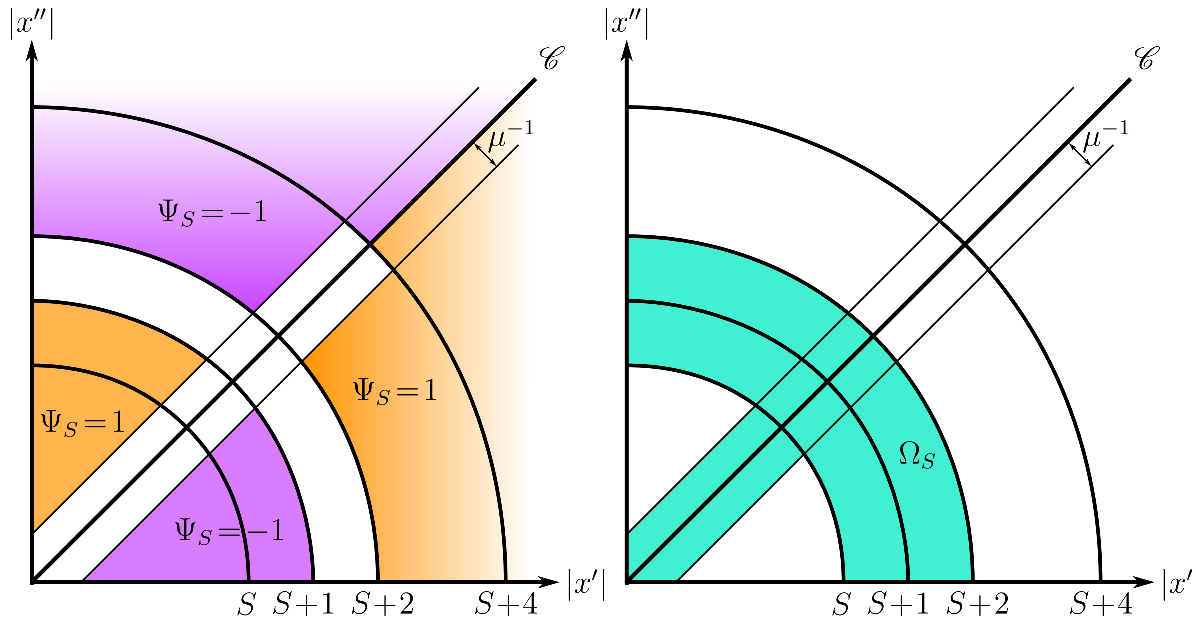

FIGURE 2. (a) The 1 and -1 level sets of Ψ_S. (b) The set Ω_S.

with $C > 0$ depending only on m and μ, and thus independent of S.

Proof. First, note that if $x \in \Omega_S$, then $d_S(x) = 1$ and $x, y \in B_{S+3}$ whenever $|x - y| \leq d_S(x) = 1$. Therefore, the result follows from the Lipschitz continuity of Ψ_S in B_{S+3}. Hence, we only need to establish the result for the case $x \in B_S \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) > 1\} \cap \mathcal{O}$ and $y \in \mathcal{O}$. Under these assumptions, we have that $\Psi_S(x) = -1$ (see Figure 2) and $d_S(x) = \min\{S + 1 - |x|, \mu \operatorname{dist}(x, \mathscr{C})\} \leq S + 1 - |x|$. Thus, if $|x - y| \leq d_S(x)$, it holds

$$|y| \leq |x - y| + |x| \leq d_S(x) + |x| \leq S + 1.$$

Therefore, we have to show

$$|1 + \Psi_S(y)| \leq C \frac{|x - y|}{\min\{S + 1 - |x|, \mu \operatorname{dist}(x, \mathscr{C})\}} \tag{4.5}$$

for $x \in B_S \cap \{\mu \operatorname{dist}(x, \mathscr{C}) > 1\} \cap \mathcal{O}$ and $y \in B_{S+1} \cap \mathcal{O}$.

Now we distinguish two cases: either $\mu \operatorname{dist}(y, \mathscr{C}) \geq 1$ or $\mu \operatorname{dist}(y, \mathscr{C}) < 1$. Assume first that $y \in B_{S+1} \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) \geq 1\} \cap \mathcal{O}$. Then, $\Psi_S(y) = -1$ and (4.5) is trivial. Hence, it only remains to show (4.5) in the case $y \in B_{S+1} \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) < 1\} \cap \mathcal{O}$. Note that under these assumptions, $\Psi_S(x) = -1$ and $\Psi_S(y) = -\mu \operatorname{dist}(y, \mathscr{C})$.

We claim that for $x \in \{\mu \operatorname{dist}(\cdot, \mathscr{C}) \geq 1\}$ and $y \in \{\mu \operatorname{dist}(\cdot, \mathscr{C}) \leq 1\}$ it holds

$$|1 - \mu \operatorname{dist}(y, \mathscr{C})| \leq \frac{|x - y|}{\operatorname{dist}(x, \mathscr{C})}. \tag{4.6}$$

Assuming this claim to be true, we have that if $x \in B_S \cap \{\mu\, \mathrm{dist}(x,\mathscr{C}) \geq 1\} \cap \mathcal{O}$ and $y \in B_{S+1} \cap \{\mu\, \mathrm{dist}(\cdot,\mathscr{C}) \leq 1\} \cap \mathcal{O}$, then

$$|\Psi_S(x) - \Psi_S(y)| = |1 - \mu\, \mathrm{dist}(y,\mathscr{C})| \leq \frac{|x-y|}{\mathrm{dist}(x,\mathscr{C})} \leq \mu \frac{|x-y|}{d_S(x)},$$

completing the proof.

Let us show now (4.6). Given $x, y \in \mathbb{R}^{2m}$ it is easy to prove by using the triangular inequality and the definition of distance to the cone that

$$\mathrm{dist}(x,\mathscr{C}) \leq |x-y| + \mathrm{dist}(y,\mathscr{C}). \tag{4.7}$$

Therefore, since $\mu\, \mathrm{dist}(x,\mathscr{C}) \geq 1$, we have

$$1 - \mu|x-y| - \mu\, \mathrm{dist}(y,\mathscr{C}) \leq 1 - \mu\, \mathrm{dist}(x,\mathscr{C}) \leq 0 \tag{4.8}$$

Now, multiplying (4.7) by $|1 - \mu\, \mathrm{dist}(y,\mathscr{C})|$, and using that $\mu\, \mathrm{dist}(y,\mathscr{C}) \leq 1$ and (4.8), we obtain

$$\begin{aligned} |1 - \mu\, \mathrm{dist}(y,\mathscr{C})|\ \mathrm{dist}(x,\mathscr{C}) &\leq (1 - \mu\, \mathrm{dist}(y,\mathscr{C}))\,(|x-y| + \mathrm{dist}(y,\mathscr{C})) \\ &= |x-y| + \mathrm{dist}(y,\mathscr{C})\,\{1 - \mu|x-y| - \mu\, \mathrm{dist}(y,\mathscr{C})\} \\ &\leq |x-y|. \end{aligned}$$

□

Another auxiliary result that we will need in the proof of Theorem 1.3 is the following estimate for the function d_S.

Lemma 4.2. *Given $\gamma \in (0,1)$ and $B_{S+2} \subset \mathbb{R}^{2m}$ with $S > 2$, we have*

$$\int_{B_{S+2}} d_S(x)^{-2\gamma}\, \mathrm{d}x \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-1} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

with $C > 0$ independent of S and depending only on m, γ, and μ.

Proof. In order to prove this result we first note that $d_S(x) = 1$ in Ω_S. Thus, the contribution to the integral of this part is just the measure of the set Ω_S —see (4.4). That is,

$$\int_{\Omega_S} d_S(x)^{-2\gamma}\, \mathrm{d}x = |\Omega_S| \leq C\, S^{2m-1}.$$

Since $S > 2$, the desired estimate for this integral follows straightforward.

For the other part of the integral we can write

$$\begin{aligned}\int_{B_{S+2}\setminus\Omega_S} d_S(x)^{-2\gamma}\,\mathrm{d}x &= \int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} d_S(x)^{-2\gamma}\,\mathrm{d}x \\ &\le \int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} (S+1-|x|)^{-2\gamma}\,\mathrm{d}x \\ &\quad + \int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x.\end{aligned}$$

The desired estimate for the first integral can be found in [29]. Therefore, in order to complete the proof, it only remains to estimate the second integral. This can be done by writing it in (y,z) variables, where

$$y := \frac{|x'|+|x''|}{\sqrt{2}} \qquad \text{and } z := \frac{|x'|-|x''|}{\sqrt{2}}\,.$$

Note that z is the signed distance to the cone (see Lemma 4.2 in [7]). Thus,

$$\begin{aligned}\int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x &\le C\int\int_{B_S\cap\{y\ge|z|>1/\mu\}} |z|^{-2\gamma}\,(y^2-z^2)^{m-1}\,\mathrm{d}y\,\mathrm{d}z \\ &\le C\int\int_{B_S\cap\{y\ge|z|>1/\mu\}} |z|^{-2\gamma}\,y^{2m-2}\,\mathrm{d}y\,\mathrm{d}z \\ &\le C\int_{1/\mu}^{S}\mathrm{d}z\;z^{-2\gamma}\int_0^S\mathrm{d}y\;y^{2m-2} \\ &\le \begin{cases} C\;S^{2m-2\gamma} & \text{if} \quad \gamma\in(0,1/2), \\ C\;\log(S)\,S^{2m-1} & \text{if} \quad \gamma=1/2, \\ C\;S^{2m-1} & \text{if} \quad \gamma\in(1/2,1). \end{cases}\end{aligned}$$

□

Note that the (y,z) variables used in the previous computation are usually very helpful when dealing with doubly radial odd functions —see [7, 8, 3, 11, 12, 20].

The last auxiliary result we need in order to establish the energy estimate is the following inequality.

Lemma 4.3. *Let $\rho>0$ and let $A\subset B_\rho\subset\mathbb{R}^{2m}$ be a set of double revolution that is symmetric with respect to the Simons cone, i.e., $A^\star=A$. Let $\omega,\phi,\varphi\in\widetilde{\mathbb{H}}^K(B_\rho)$, with K radially symmetric, be such that*

$$\begin{cases}\omega=\phi\le\varphi & \text{in} \quad \mathcal{O}\setminus A\,,\\ \omega=\varphi\le\phi & \text{in} \quad A\cap\mathcal{O}\,.\end{cases} \tag{4.9}$$

Then, if K satisfies (1.14), it holds

$$I_\omega(A\cap\mathcal{O},\mathcal{O}\setminus A)\le I_\phi(A\cap\mathcal{O},\mathcal{O}\setminus A)+I_\varphi(A\cap\mathcal{O},\mathcal{O}\setminus A)\,,$$

where $I_w(\cdot,\cdot)$ is the interaction defined in (3.2).

Proof. A simple computation shows that if $x \in A \cap \mathcal{O}$ and $y \in \mathcal{O} \setminus A$ we have that

$$|\phi(x) - \phi(y)|^2 + |\varphi(x) - \varphi(y)|^2 \geq |\omega(x) - \omega(y)|^2.$$

Indeed, using (4.9) we see that

$$\begin{aligned}
|\phi(x) - \phi(y)|^2 &+ |\varphi(x) - \varphi(y)|^2 - |\omega(x) - \omega(y)|^2 \\
&= |\phi(x) - \phi(y)|^2 + |\varphi(x) - \varphi(y)|^2 - |\varphi(x) - \phi(y)|^2 \\
&= \phi^2(x) - 2\phi(x)\phi(y) + \varphi^2(y) - 2\varphi(x)\varphi(y) + 2\varphi(x)\phi(y) \\
&= (\phi(x) - \varphi(y))^2 + 2\,(\phi(x) - \varphi(x))\,(\varphi(y) - \phi(y)) \\
&\geq 0.
\end{aligned}$$

Therefore, by using this inequality and the reflexion property of the kernel, (1.14), we obtain

$$\begin{aligned}
&I_\phi(A \cap \mathcal{O}, \mathcal{O} \setminus A) + I_\varphi(A \cap \mathcal{O}, \mathcal{O} \setminus A) - I_\omega(A \cap \mathcal{O}, \mathcal{O} \setminus A) = \\
&\quad = \int_{A\cap\mathcal{O}} \mathrm{d}x \int_{\mathcal{O}\setminus A} \mathrm{d}y \Big(4\left\{\phi^2(x) + \phi^2(y) + \varphi^2(x) + \varphi^2(y) - \omega^2(x) - \omega^2(y)\right\} \overline{K}(x, y^\star) \\
&\qquad + 2\left\{|\phi(x) - \phi(y)|^2 + |\varphi(x) - \varphi(y)|^2 - |\omega(x) - \omega(y)|^2\right\}\left\{\overline{K}(x, y) - \overline{K}(x, y^\star)\right\}\Big) \\
&\quad \geq 4 \int_{A\cap\mathcal{O}} \mathrm{d}x \int_{\mathcal{O}\setminus A} \mathrm{d}y \; \left\{\phi^2(x) + \varphi^2(y)\right\} \overline{K}(x, y^\star) \geq 0.
\end{aligned}$$

□

With all these ingredients we can establish now the desired energy estimate.

Proof of Theorem 1.3. Note that since u is a minimizer of $\mathcal{E}$ in B_R, by Lemma 3.3 we can assume without loss of generality that $-1 \leq u \leq 1$, $u \geq 0$ in $\mathcal{O}$, and $u \leq 0$ in $\mathcal{I}$.

Step 1. We show that $0 \leq u < 1$ in $\mathcal{O}$.

In order to prove it, we first show that u is a classical solution to

$$\begin{cases} L_K u = f(u) & \text{in } B_R\,, \\ \quad u = 0 & \text{in } \mathbb{R}^{2m} \setminus B_R. \end{cases} \tag{4.10}$$

To see this, we consider perturbations $u + \varepsilon\xi$ with $\varepsilon > 0$. On the one hand, take $\xi \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Then, since u is a minimizer among functions in $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u + \varepsilon\xi, B_R) = \langle u, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} - \langle f(u), \xi\rangle_{L^2(B_R)}\,.$$

On the other hand, take $\xi \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(B_R)$. Since u is odd with respect to the Simons cone, the same holds for $f(u)$ —recall that f is odd. Then, by Remark 3.1 we find that

$$\langle u, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} = 0 \quad \text{ and } \quad \langle f(u), \xi\rangle_{L^2(B_R)} = 0\,.$$

Therefore,

$$\langle u, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} = \langle f(u), \xi\rangle_{L^2(B_R)}$$

for every $\xi \in \widetilde{\mathbb{H}}_0^K(B_R)$ with compact support in B_R. By approximation, ξ can be taken to be $C_c^\infty(B_R)$. Thus,

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\xi(x) - \xi(y)\} K(|x-y|) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x) \, \mathrm{d}x$$

for every $\xi \in C_c^\infty(\Omega)$ that is doubly radial. Finally, by Proposition 3.4, u is a weak solution to (4.10), and in view of the regularity theory for operators in $\mathcal{L}_0$ (see Remark 3.5), since u is bounded, it is a classical solution.

Since u is a classical solution that is odd and continuous, it follows that $u \not\equiv 1$ in $\mathcal{O}$. Therefore, by the usual strong maximum principle (recall that $u \leq 1$), we get $0 \leq u < 1$ in $\mathcal{O}$.

Step 2. We build a suitable competitor for u and compare their energies. For $x \in \overline{\mathcal{O}}$ we define

$$v(x) := \min\{u(x), \Psi_S(x)\},$$

and we define it in $\mathcal{I}$ by considering its odd reflection with respect to the Simons cone. Recall that Ψ_S is defined in (4.2) and it is zero on the cone $\mathscr{C}$. Let us also define

$$A := (\{\Psi_S < u\} \cap \mathcal{O}) \cup (\{\Psi_S < u\} \cap \mathcal{O})^\star.$$

Thus, $v = \Psi_S < u$ in $A \cap \mathcal{O}$ and $v = u \leq \Psi_S$ in $\mathcal{O} \setminus A$.

We claim that if we take

$$\mu = ||u||_{\mathrm{Lip}(\overline{B_{S+4}})}\,, \tag{4.11}$$

we have the inclusions

$$B_{S+1} \setminus \mathscr{C} \subset A \subset B_{S+2}\,. \tag{4.12}$$

Let us show this. Note first that by symmetry and the fact that $\Psi_S = u = 0$ in $\mathscr{C}$, we only need to prove that

$$B_{S+1} \cap \mathcal{O} \subset A \cap \mathcal{O} \subset B_{S+2} \cap \mathcal{O}\,.$$

On the one hand, if $x \in B_{S+1} \cap \mathcal{O}$, then $\Psi_S(x) = -\min\{1, \mu \operatorname{dist}(x, \mathscr{C})\} < 0 \leq u(x)$. Thus, $x \in A \cap \mathcal{O}$. On the other hand, if $x \in A \cap \mathcal{O}$ then $\Psi_S(x) < u(x) < 1$ and this can only happen if $x \in B_{S+2} \cap \mathcal{O}$. Let us show this. First, note that $\Psi_S \equiv 1$ in $\mathcal{O} \setminus B_{S+4}$ and in $(B_{S+4} \setminus B_{S+2}) \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) \geq 1\} \cap \mathcal{O}$ —see Figure 2 (a). Therefore, we only need to see that

$$u(x) \leq \Psi(x) = \mu \operatorname{dist}(x, \mathscr{C}) \quad \text{for every } x \in (B_{S+4} \setminus B_{S+2}) \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) < 1\} \cap \mathcal{O}.$$

By the choice of μ in (4.11), we know that

$$|u(x) - u(y)| \leq \mu |x - y| \quad \text{for every } x,\ y \in \overline{B_{S+4}}.$$

Thus, by taking $x \in (B_{S+4} \setminus B_{S+2}) \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) < 1\} \cap \mathcal{O}$ and $y \in \mathscr{C}$ to be a point realizing $\operatorname{dist}(x, \mathscr{C})$, we obtain the desired inequality, concluding the proof of (4.12). Note that if $x \in \overline{B_{S+4}}$, it is easy to check that $y \in \overline{B_{S+4}}$.

By (4.12), both u and v are equal outside $B_{S+2} \subset B_R$, and therefore v is an admissible competitor. Comparing the energies of u and v we will obtain the desired estimate.

Using the expression (3.3), let us decompose the energy of u in B_R in the following way:

$$\begin{aligned}\mathcal{E}(u,B_R) &= \frac{1}{4}I_u(A\cap\mathcal{O},A\cap\mathcal{O}) + \frac{1}{2}I_u(A\cap\mathcal{O},\mathcal{O}\setminus A)\\ &\quad + \frac{1}{4}I_u\big((\mathcal{O}\setminus A)\cap B_R,(\mathcal{O}\setminus A)\cap B_R\big) + \frac{1}{2}I_u\big((\mathcal{O}\setminus A)\cap B_R,\mathcal{O}\setminus B_R\big)\\ &\quad + \int_A G(u)\,\mathrm{d}x + \int_{B_R\setminus A} G(u)\,\mathrm{d}x\end{aligned}$$

Since u is a minimizer, $v=\Psi_S$ in $A\cap\mathcal{O}$, and $u=v$ in $\mathcal{O}\setminus A$, from the previous expression we obtain

$$\begin{aligned}0\le \mathcal{E}(v,B_R)-\mathcal{E}(u,B_R) &= \frac{1}{4}I_{\Psi_S}(A\cap\mathcal{O},A\cap\mathcal{O}) - \frac{1}{4}I_u(A\cap\mathcal{O},A\cap\mathcal{O})\\ &\quad + \frac{1}{2}I_v(A\cap\mathcal{O},\mathcal{O}\setminus A) - \frac{1}{2}I_u(A\cap\mathcal{O},\mathcal{O}\setminus A) + \int_A G(\Psi_S)\,\mathrm{d}x - \int_A G(u)\,\mathrm{d}x.\end{aligned}$$

Now, recalling that $v=\min\{u,\Psi_S\}$ in $\mathcal{O}$, we can apply Lemma 4.3 with $\omega=v$, $\Psi_S=\varphi$, and $u=\phi$, to get $I_v(A\cap\mathcal{O},\mathcal{O}\setminus A)\le I_u(A\cap\mathcal{O},\mathcal{O}\setminus A)+I_{\Psi_S}(A\cap\mathcal{O},\mathcal{O}\setminus A)$. Therefore,

$$\begin{aligned}\frac{1}{4}I_u(A\cap\mathcal{O},A\cap\mathcal{O}) + \int_A G(u)\,\mathrm{d}x &\le \frac{1}{4}I_{\Psi_S}(A\cap\mathcal{O},A\cap\mathcal{O}) + \frac{1}{2}I_{\Psi_S}(A\cap\mathcal{O},\mathcal{O}\setminus A)\\ &\quad + \int_A G(\Psi_S)\,\mathrm{d}x\\ &= \mathcal{E}(\Psi_S,A)\le \mathcal{E}(\Psi_S,B_{S+2}).\end{aligned}$$

From this and using the inclusions of (4.12), we deduce an estimate for the energy of u in B_S as follows.

$$\begin{aligned}\mathcal{E}(u,B_S) &\le \frac{1}{4}I_u(A\cap\mathcal{O},A\cap\mathcal{O}) + \int_A G(u)\,\mathrm{d}x + \frac{1}{2}I_u(B_S\cap\mathcal{O},\mathcal{O}\setminus B_{S+1})\\ &\le \mathcal{E}(\Psi_S,B_{S+2}) + \frac{1}{2}I_u(B_S\cap\mathcal{O},\mathcal{O}\setminus B_{S+1}).\end{aligned}$$

Thus, to obtain the desired energy estimate we only have to bound the right-hand side of the last inequality.

Step 3. We estimate the remaining terms.

In the following arguments, we use the definition of the energy that involves the original kernel K and not $\overline{K}$. Moreover, the letter C will denote different constants depending on m, γ, μ, G, and Λ.

3.1. Estimate for $\mathcal{E}(\Psi_S, B_{S+2})$. First, by using the change of variables given by $(\cdot)^\star$ and the ellipticity of K —that is, condition (1.4)—, we obtain

$$\begin{aligned}
\mathcal{E}(\Psi_S, B_{S+2}) &= \frac{1}{4}\int_{B_{S+2}} \mathrm{d}x \int_{B_{S+2}} \mathrm{d}y\ |\Psi_S(x)-\Psi_S(y)|^2 K(|x-y|) \\
&\quad + \frac{1}{2}\int_{B_{S+2}} \mathrm{d}x \int_{\mathbb{R}^{2m}\setminus B_{S+2}} \mathrm{d}y\ |\Psi_S(x)-\Psi_S(y)|^2 K(|x-y|) \\
&\quad + \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x \\
&\le \frac{1}{2}\int_{B_{S+2}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y\ |\Psi_S(x)-\Psi_S(y)|^2 K(|x-y|) + \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x \\
&= \int_{B_{S+2}\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y\ |\Psi_S(x)-\Psi_S(y)|^2 K(|x-y|) + \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x \\
&\le \Lambda\, c_{n,\gamma} \int_{B_{S+2}\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y\ \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x.
\end{aligned}$$

Now, we split the kinetic energy into three terms. We get

$$\begin{aligned}
\mathcal{E}(\Psi_S, B_{S+2}) &\le \Lambda\, c_{n,\gamma} \int_{B_{S+2}\cap\mathcal{O}} \mathrm{d}x \int_{\mathcal{O}} \mathrm{d}y \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \Lambda\, c_{n,\gamma} \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_{\mathcal{I}} \mathrm{d}y \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \Lambda\, c_{n,\gamma} \int_{(B_{S+2}\setminus\Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{\mathcal{I}} \mathrm{d}y \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x \\
&=: \Lambda\, c_{n,\gamma}(I_1+I_2+I_3) + I_G,
\end{aligned}$$

where Ω_S is defined in (4.3).

Let us estimate these four integrals. To control I_1, we use Lemma 4.1 and the fact that Ψ_S is bounded by 1.

$$\begin{aligned}
I_1 &= \int_{B_{S+2}\cap\mathcal{O}} \int_{\mathcal{O}} \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{B_{S+2}\cap\mathcal{O}} \int_{\mathcal{O}\cap\{|x-y|\le d_S(x)\}} \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \int_{B_{S+2}\cap\mathcal{O}} \int_{\mathcal{O}\cap\{|x-y|\ge d_S(x)\}} \frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}}\,\mathrm{d}y\,\mathrm{d}x \\
&\le C\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2}\left(\int_{\mathcal{O}\cap\{|x-y|\le d_S(x)\}} |x-y|^{2-2m-2\gamma}\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad + C\int_{B_{S+2}\cap\mathcal{O}} \left(\int_{\mathcal{O}\cap\{|x-y|\ge d_S(x)\}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y\right)\mathrm{d}x \\
&\le C\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2}\left(\int_0^{d_S(x)} \rho^{1-2\gamma}\,\mathrm{d}\rho\right)\mathrm{d}x + C\int_{B_{S+2}\cap\mathcal{O}}\left(\int_{d_S(x)}^{\infty}\rho^{-1-2\gamma}\,\mathrm{d}\rho\right)\mathrm{d}x \\
&\le C\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

The bound of I_2 is essentially the same using also Lemma 4.1 and the inclusion $\Omega_S\subset B_{S+2}$. That is,

$$\begin{aligned}
I_2 &\le C\int_{\Omega_S\cap\mathcal{O}} d_S(x)^{-2}\left(\int_0^{d_S(x)}\rho^{1-2\gamma}\,\mathrm{d}\rho\right)\mathrm{d}x + C\int_{\Omega_S\cap\mathcal{O}}\left(\int_{d_S(x)}^{\infty}\rho^{-1-2\gamma}\,\mathrm{d}\rho\right)\mathrm{d}x \\
&\le C\int_{\Omega_S\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x \le C\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

In the case of I_3, we use the fact that given $x\in(B_{S+2}\setminus\Omega_S)\cap\mathcal{O}$, then $\mathrm{dist}(x,\mathscr{C})\ge d_S(x)/\mu$ and therefore, for such x, it holds $\mathcal{I}\subset\mathbb{R}^{2m}\setminus B_{d_S(x)/\mu}(x)$. We obtain

$$\begin{aligned}
I_3 &= \int_{(B_{S+2}\setminus\Omega_S)\cap\mathcal{O}}\mathrm{d}x\int_{\mathcal{I}}\mathrm{d}y\;\frac{|\Psi_S(x)-\Psi_S(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le C\int_{(B_{S+2}\setminus\Omega_S)\cap\mathcal{O}}\mathrm{d}x\int_{\mathbb{R}^{2m}\setminus B_{d_S(x)/\mu}(x)}\mathrm{d}y\;|x-y|^{-2m-2\gamma} \\
&\le C\int_{B_{S+2}\cap\mathcal{O}}\left(\int_{d_S(x)/\mu}^{\infty}\rho^{-1-2\gamma}\,\mathrm{d}\rho\right)\mathrm{d}x \le C\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

Finally, we estimate I_G. Since Ψ_S is either 1 or -1 in $B_{S+2}\setminus\Omega_S$, and $G(-1)=G(1)=0$, we have

$$I_G = \int_{B_{S+2}} G(\Psi_S)\,\mathrm{d}x = \int_{\Omega_S} G(\Psi_S)\,\mathrm{d}x \le C|\Omega_S| \le C\,S^{2m-1},$$

where we have used (4.4). Therefore, we obtain

$$\mathcal{E}(\Psi_S, B_{S+2}) \le C \left(\int_{B_{S+2}\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1} \right) \le C \left(\int_{B_S\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1} \right).$$

3.2. Estimate for $I_u(B_S \cap \mathcal{O}, \mathcal{O} \setminus B_{S+1})$. First, we claim that $|x-y| \ge d_S(x)$ whenever $x \in B_S$ and $y \in \mathbb{R}^{2m} \setminus B_{S+1}$. Indeed, if $x \in B_S$, then it is easy to see that $d_S(x) \le S+1-|x|$ and therefore we have $|x-y| \ge |y|-|x| \ge |y|+d_S(x)-S-1 \ge d_S(x)$, since $|y| \ge S+1$. Thus, using this inequality and the ellipticity of K, we get

$$\begin{aligned} I_u(B_S \cap \mathcal{O}, \mathcal{O} \setminus B_{S+1}) &\le C \int_{B_S\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}\setminus B_{S+1}} \mathrm{d}y\, \frac{|u(x)-u(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\le C \int_{B_S\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\ge d_S(x)} \mathrm{d}y\, |x-y|^{-2m-2\gamma} \\ &\le C \int_{B_S\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x. \end{aligned}$$

Step 4. Conclusion.

Finally, by adding up the estimates of Step 3 and applying Lemma 4.2, we obtain the desired result. That is,

$$\begin{aligned} \mathcal{E}(u, B_S) &\le \mathcal{E}(\Psi_S, B_{S+2}) + \frac{1}{2} I_u(B_S \cap \mathcal{O}, \mathcal{O} \setminus B_{S+1}) \le C \left(\int_{B_S\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1} \right) \\ &\le \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-1} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases} \end{aligned}$$

with a constant C depending only on m, γ, μ, G, and Λ.

To conclude, it remains to show that μ can be controlled by a constant that depends only on m, γ, λ, Λ, and f (and thus, independent of R and S). Indeed, by Step 1, we know that u solves $L_K u = f(u)$ in B_R with $R > S+5$. Therefore, by applying repeatedly the estimates (3.6) and (3.7) in balls centered at points of B_{S+4}, it is easy to see that

$$\mu = ||u||_{\mathrm{Lip}(\overline{B_{S+4}})} \le C,$$

with a constant C depending only on m, γ, λ, Λ, and $||f||_{C^1([-1,1])}$. $\square$

5. Existence of saddle-shaped solution: variational method

In this section we establish the existence of saddle-shaped solutions to the integro-differential Allen-Cahn equation. The proof is based on the direct method of the calculus of variations, and it uses most of the results appearing in the previous sections.

Proof of Theorem 1.4. Since $\mathcal{E}(w, B_R)$ is bounded from below —by 0—, we can take a minimizing sequence $u_R^j \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Note that, by Lemma 3.3 we can assume that $-1 \le u_R^j \le 1$ and that $u_R^j \ge 0$ in $\mathcal{O}$ and $u_R^j \le 0$ in $\mathcal{I}$.

Now, using (1.4), $G \ge 0$, and the fact that u_R^j is a minimizing sequence, we deduce that

$$[u_R^j]^2_{H^\gamma(B_R)} \le \frac{c_{n,\gamma}}{\lambda}[u_R^j]^2_{\mathbb{H}^K(B_R)} \le \frac{2c_{n,\gamma}}{\lambda}\mathcal{E}(u_R^j, B_R) \le C$$

for a constant C that does not depend on j. Therefore, by combining this with the fractional Poincaré inequality (recall that $u_R^j \equiv 0$ in $\mathbb{R}^{2m} \setminus B_R$) we get that the sequence $\{u_R^j\}$ is bounded in $H^\gamma(B_R)$. Hence, by the compact embedding $H^\gamma(B_R) \subset\subset L^2(B_R)$ (see Theorem 7.1 of [17]), there exists a subsequence, still denoted by u_R^j, that converges to some doubly radial $u_R \in L^2(B_R)$, and thus, a.e. in B_R. By Fatou's lemma, we have

$$\mathcal{E}(u_R, B_R) \le \liminf_{j\to\infty} \mathcal{E}(u_R^j, B_R) = \inf\left\{\mathcal{E}(w, B_R) \ : \ w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)\right\}.$$

Therefore, $u_R \in \widetilde{\mathbb{H}}^K(B_R)$. In addition, $u_R(x) = -u_R(x^\star)$ for every $x \in \mathbb{R}^{2m}$, and $u_R \equiv 0$ in $\mathbb{R}^{2m} \setminus B_R$. Thus, u_R is a minimizer of $\mathcal{E}(\cdot, B_R)$ in $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Moreover, it satisfies $-1 \le u_R \le 1$ in B_R and $u_R \ge 0$ in $\mathcal{O}$.

Arguing exactly as in the proof of Theorem 1.3 (using Proposition 3.4), we deduce that u_R is a classical solution to

$$\begin{cases} L_K u_R = f(u_R) & \text{in } B_R\,, \\ \quad u_R = 0 & \text{in } \mathbb{R}^{2m} \setminus B_R. \end{cases}$$

The next step is to pass to the limit in R to obtain a solution in $\mathbb{R}^{2m}$. This is done using a compactness argument. Let $S > 0$ and consider the family $\{u_R\}$, for $R > S+1$, of solutions to $L_K u_R = f(u_R)$ in B_S. Note first that, if w solves $L_K w = f(w)$ in B_ρ and $|w| \le 1$ in $\mathbb{R}^{2m}$ with $f \in C^\alpha([-1,1])$ for some $\alpha > 0$, the combination of (3.4) and (3.7) yields

$$||w||_{C^{2\gamma+\varepsilon}(B_{\rho/8})} \le C\left(n,\ \gamma,\ \lambda,\ \Lambda,\ ||f||_{C^\alpha([-1,1])}\right). \tag{5.1}$$

for some $\varepsilon > 0$. By applying the estimate (5.1) to u_R in balls of radius $\rho = 1$ and centered at points in $\overline{B_S}$, we obtain a uniform $C^{2\gamma+\varepsilon}(\overline{B_S})$ bound for u_R. By the Arzelà-Ascoli theorem, as $R \to +\infty$, a subsequence of $\{u_R\}$ converges in $C^{2\gamma+\varepsilon/2}(\overline{B_S})$ to a (pointwise) solution in B_S. Taking now $S = 1, 2, 3, \ldots$ and using a diagonal argument, we obtain a sequence u_{R_j} converging uniformly on compacts in the $C^{2\gamma+\varepsilon/2}$ norm to a solution $u \in C^{2\gamma+\varepsilon/2}(\mathbb{R}^{2m})$ of (1.1).

Therefore, we have obtained a solution u to $L_K u = f(u)$ in $\mathbb{R}^{2m}$ which is doubly radial. Furthermore, u is odd with respect to the Simons cone $\mathscr{C}$, i.e., $u(x) = -u(x^\star)$ for $x \in \mathbb{R}^{2m}$, and $0 \le u \le 1$ in $\mathcal{O}$.

Finally, we show that $0 < u < 1$ in $\mathcal{O}$. This will ensure that u is a saddle-shaped solution. First, note that $|u| < 1$ by the usual strong maximum principle (since u vanishes at $\mathscr{C}$ and is continuous, we have $u \not\equiv 1$ and $u \not\equiv -1$ in $\mathbb{R}^{2m}$). Let us show that $u \not\equiv 0$. To do this, we use the energy estimate of Theorem 1.3. That is, if we consider u_R the minimizer of $\mathcal{E}(\cdot, B_R)$ with $R > 8$, we have

$$\mathcal{E}(u_R, B_S) \le \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

for every $2 < S < R - 5$ and with a constant C independent of R and S. By letting $R \to \infty$ we obtain the same estimate for u. By contradiction, assume $u \equiv 0$. Then, the previous estimate leads to

$$c_m G(0) S^{2m} = \mathcal{E}(0, B_S) \le \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

and, since $G(0) > 0$, this is a contradiction for S large enough. Therefore, $u \not\equiv 0$ and the strong maximum principle for odd functions (see Proposition 1.2) yields that $u > 0$ in $\mathcal{O}$. $\square$

Appendix A. Some auxiliary results on convex functions

In this appendix we present some auxiliary results concerning convex functions. The main result, used in the proof of Theorem 1.1, is the following.

Proposition A.1. *Let $K : (0, +\infty) \to (0, +\infty)$ be a measurable function. Then, the following statements are equivalent:*

i) $K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$.

ii) For every positive constants c_1 and c_2, the function $g : (0, 1/c_2) \to \mathbb{R}$ defined by

$$g(z) := K(c_1\sqrt{1 + c_2 z}) + K(c_1\sqrt{1 - c_2 z}) \tag{A.1}$$

satisfies

$$g(A) + g(D) \geq g(B) + g(C) \tag{A.2}$$

whenever A, B, C, and D belong to $(0, 1/c_2)$ and satisfy

$$A = \max\{A,\, B,\, C,\, D\} \quad \text{and} \quad A + D \geq B + C.$$

In addition, still assuming $A = \max\{A, B, C, D\}$ and $A + D \geq B + C$, equality holds in (A.2) if and only if the sets $\{A, D\}$ and $\{C, B\}$ coincide.

To prove this proposition, we need a lemma on convex functions.

Lemma A.2. *Let $0 < M \leq +\infty$ and let $h : (0, M) \to \mathbb{R}$ be a measurable nondecreasing function. Then, the following statements are equivalent.*

(a) h is convex in $(0, M)$.

(b) For every $0 \leq L \leq 2M$, the function $h_L(x) := h(x) + h(L - x)$ is convex in $(\max\{L - M, 0\}, \min\{L, M\})$.

(c) For every A, B, C, D in the interval $(0, M)$ such that

$$A = \max\{A,\, B,\, C,\, D\} \quad \text{and} \quad A + D \geq B + C\,,$$

it holds

$$h(A) + h(D) \geq h(B) + h(C)\,. \tag{A.3}$$

Proof. $(a) \Rightarrow (c)$. Since B and C are interchangeable and h is nondecreasing, we may assume that $A \geq B \geq C \geq D$. Now, let M_C be the maximum slope of the supporting lines of h at C, and let m_B be the minimum slope of the supporting lines of h at B. By the convexity and monotonicity of h, it holds $m_B \geq M_C \geq 0$ and also

$$h(x) \geq h(B) + m_B(x - B) \quad \text{and} \quad h(x) \geq h(C) + M_C(x - C)$$

for every $x \in (0, M)$.

Hence, since $A - B \geq C - D \geq 0$, we have

$$h(A) - h(B) \geq m_B(A - B) \geq M_C(C - D) \geq h(C) - h(D)\,.$$

$(c) \Rightarrow (b)$. Let x, $y \in (\max\{L - M, 0\}, \min\{L, M\})$ and assume that $x > y$. By taking $A = x$, $B = C = (x + y)/2$, and $D = y$ in (A.3), we get

$$\frac{h(x) + h(y)}{2} \geq h\left(\frac{x + y}{2}\right).$$

Similarly, by taking $A = L - y$, $B = C = L - (x + y)/2$, and $D = L - x$ in (A.2), we get

$$\frac{h(L-x) + h(L-y)}{2} \geq h\left(L - \frac{x+y}{2}\right).$$

By adding up the previous two inequalities we obtain

$$\frac{h_L(x) + h_L(y)}{2} \geq h_L\left(\frac{x+y}{2}\right).$$

$(b) \Rightarrow (a)$. Let x_0, $y_0 \in (0, M)$ and choose $L = x_0 + y_0 \leq 2M$. By (b) we have

$$\frac{h(x) + h(x_0+y_0-x) + h(y) + h(x_0+y_0-y)}{2} \geq h\left(\frac{x+y}{2}\right) + h\left(x_0 + y_0 - \frac{x+y}{2}\right),$$

for every x and y in the interval $(\max\{L - M, 0\}, \min\{L, M\})$. By choosing $x = x_0$ and $y = y_0$ we obtain

$$h(x_0) + h(y_0) \geq 2h\left(\frac{x_0+y_0}{2}\right).$$

□

Remark A.3. We can replace convexity by strict convexity in (a) and (b), and then the inequality in (A.3) is strict unless the sets $\{A, D\}$ and $\{C, B\}$ coincide.

Remark A.4. Note that the function h_L is even with respect to $L/2$. Thus, if it is convex, it is nondecreasing in $(L/2, \min\{L, M\})$.

Remark A.5. The assumption of h being nondecreasing is only used to deduce (c) from (a). It is not required to show the equivalence between (a) and (b), neither to deduce (a) from (c).

With this result available we can show now Proposition A.1

Proof. $i) \Rightarrow ii)$ We take $M = +\infty$ and $h(\cdot) = K(\sqrt{\cdot})$ in Lemma A.2. Since h is strictly convex, the function h_L is strictly convex in $(0, L)$ for every $L > 0$ (recall that we do not need to assume that h is monotone to deduce this, see Remark A.5). Moreover, by Remark A.4, h_L is nondecreasing in $(L/2, L)$. Thus, the function $\phi(\cdot) = h_L(\cdot + L/2)$ is strictly convex in $(-L/2, L/2)$ and nondecreasing in $(0, L/2)$. If we choose $L = 2c_1^2$, we have that $\phi((L/2)c_2\cdot) = g(\cdot)$, where g is defined by (A.1). Therefore, g is strictly convex in $(-1/c_2, 1/c_2)$ and nondecreasing in $(0, 1/c_2)$. Thus, the result follows by applying Lemma A.2 to g in $(0, 1/c_2)$ (taking into account Remark A.3).

$ii) \Rightarrow i)$ By Lemma A.2 applied to g we deduce that g is strictly convex and nondecreasing in $(0, 1/c_2)$ —take $C = D$ to see that g is monotone. Thus, since g is even and nondecreasing, g is strictly convex in $(-1/c_2, 1/c_2)$ and $\varphi(\cdot) = g(\cdot/(c_1^2 c_2))$ is strictly convex in $(-c_1^2, c_1^2)$. Hence, if we call $h(\cdot) := K(\sqrt{\cdot})$ and $L := 2c_1^2$, we have that $\varphi(\cdot - c_1^2) = h(\cdot) + h(L - \cdot) =: h_L(\cdot)$, and thus h_L is strictly convex in $(0, L)$. Note that since $c_1 > 0$ is arbitrary, h_L is strictly convex in $(0, L)$ for all $L > 0$. Therefore, by Lemma A.2, with $M = +\infty$, we conclude that $h(\cdot) = K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$. □

Appendix B. An auxiliary computation

In this appendix we present an auxiliary computation that is needed in Section 2 in order to complete the proof of Proposition 2.4.

Lemma B.1. *Let α, β be two real numbers satisfying $\alpha \geq |\beta|$. Let $x = (x', x'')$, $y = (y', y'') \in \mathcal{O} \subset \mathbb{R}^{2m}$. Define*

$$\begin{aligned} A &= |x'||y'|\alpha + |x''||y''|\beta\,, \qquad & B &= |x'||y''|\alpha + |x''||y'|\beta\,,\\ C &= |x''||y'|\alpha + |x'||y''|\beta\,, \qquad & D &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

Then,

(1) It holds

$$\begin{cases} |A| \geq |B|, \ |A| \geq |C|, \ |A| \geq |D|\,,\\ |A| + |D| \geq |B| + |C|\,. \end{cases}$$

(2) If the sets $\{|A|, |D|\}$ and $\{|C|, |B|\}$ coincide, then necessarily $\alpha = \beta = 0$.

Proof. The proof is elementary but requires to check some cases. In all of them we will use the following inequalities. Since $\alpha \geq |\beta|$,

$$\alpha \geq 0 \quad \text{ and } \quad -\alpha \leq \beta \leq \alpha\,.$$

Moreover, since $x, y \in \mathcal{O}$, it holds

$$|x'| > |x''| \quad \text{ and } \quad |y'| > |y''|\,.$$

We start establishing the first statement. We show next that $A \geq 0$ and that

$$A \geq |B|, \ A \geq |C|, \ A \geq |D|\,.$$

- $A \geq 0$:
$$A = |x'||y'|\alpha + |x''||y''|\beta \geq (|x'||y'| - |x''||y''|)\alpha \geq 0\,.$$
- $A \geq |B|$:
$$A \pm B = (|x'|\alpha - |x''|\beta)(|y'| \pm |y''|) \geq 0\,.$$
- $A \geq |C|$:
$$A \pm C = (|y'|\alpha - |y''|\beta)(|x'| \pm |x''|) \geq 0\,.$$
- $A \geq |D|$:
$$A \pm D = (|x'||y'| \pm |x''||y''|)(\alpha \pm \beta) \geq 0\,.$$

It remains to show

$$A + |D| \geq |B| + |C|\,.$$

The proof of this fact is just a computation considering all the eight possible configurations of the signs of B, C, and D. Since the roles of B and C are completely interchangeable, we may assume that $B \geq C$ and we only need to check six cases. To do it, note first that

$$A + D - B - C = (|x'| - |x''|)(|y'| - |y''|)(\alpha + \beta) \geq 0\,, \tag{B.1}$$

$$A - D - B + C = (|x'| + |x''|)(|y'| - |y''|)(\alpha - \beta) \geq 0\,, \tag{B.2}$$

and

$$A + D + B + C = (|x'| + |x''|)(|y'| + |y''|)(\alpha + \beta) \geq 0\,, \tag{B.3}$$

With these three relations at hand we check the six cases.

- If $B \geq 0$, $C \geq 0$, and $D \geq 0$, then by (B.1) we have
$$A + |D| - |B| - |C| = A + D - B - C \geq 0\,.$$
- If $B \geq 0$, $C \geq 0$, and $D \leq 0$, we use the sign of D and (B.1) to see that
$$A + |D| - |B| - |C| = A - D - B - C = (A + D - B - C) + (-2D) \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$, and $D \geq 0$, we use the sign of D and (B.2) to see that
$$A + |D| - |B| - |C| = A + D - B + C = (A - D - B + C) + 2D \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$, and $D \leq 0$, then by (B.2) we have
$$A + |D| - |B| - |C| = A - D - B + C \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$, and $D \geq 0$, then by (B.3) we have
$$A + |D| - |B| - |C| = A + D + B + C \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$, and $D \leq 0$, we use the sign of D and (B.3) to see that
$$A + |D| - |B| - |C| = A - D + B + C = (A + D + B + C) + (-2D) \geq 0\,.$$

This concludes the proof of the first statement.

We prove now the second point of the lemma. Since the roles of B and C are completely interchangeable, we only need to show the result in the case $|A| = |B|$ and $|C| = |D|$.

Recall that $A \geq 0$. Hence, since $A = |B|$ and $|C| = |D|$, a simple computation shows that

$$\alpha = \operatorname{sign}(B)\frac{|x''|}{|x'|}\beta \quad \text{ and } \quad \beta = \operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|}{|x'|}\alpha\,.$$

Hence, combining both equalities we obtain

$$\alpha = \operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|^2}{|x'|^2}\alpha.$$

Finally, if we assume $\alpha \neq 0$, then necessarily $\operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D) = 1$ and $|x'| = |x''|$, but this is a contradiction with $x \in \mathcal{O}$. Therefore, $\alpha = 0$ and thus $\beta = 0$. □

Appendix C. The integro-differential operator L_K in the (s,t) variables

The goal of this appendix is to take advantage of the doubly radial symmetry of the functions we are dealing with to write equation (1.1) in (s,t) variables, passing from an equation in $\mathbb{R}^{2m}$ to an equation in $(0,+\infty) \times (0,+\infty) \subset \mathbb{R}^2$. Although we do not use these computations in this paper, we include them here for future reference, since we think they could be useful.

Lemma C.1. *Let $m \geq 1$, $\gamma \in (0,1)$, and let $w \in C^\alpha(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function, i.e., depending only on the variables s and t. Let L_K be a rotation invariant operator, that is, $K(z) = K(|z|)$, of the form (1.2). Then, if we define $\tilde{w} : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ by $\tilde{w}(s,t) = w(s,0,...,0,t,0,...,0)$, it holds*

$$L_K w(x) = \tilde{L}_K \tilde{w}(|x'|,|x''|),$$

with

$$\widetilde{L}_K \tilde{w}(s,t) := \int_0^{+\infty}\int_0^{+\infty} \sigma^{m-1}\tau^{m-1}\big(\tilde{w}(s,t) - \tilde{w}(\sigma,\tau)\big) J(s,t,\sigma,\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau\,,$$

where:

(1) If $m = 1$,

$$J(s,t,\sigma,\tau) := \sum_{i=0}^{1}\sum_{j=0}^{1} K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma(-1)^i-2t\tau(-1)^j}\Big)\,. \tag{C.1}$$

(2) If $m \geq 2$,

$$\begin{aligned} J(s,t,\sigma,\tau) := c_m^2 \int_{-1}^{1}\int_{-1}^{1} &(1-\theta^2)^{\frac{m-2}{2}}(1-\overline{\theta}^2)^{\frac{m-2}{2}} \\ &K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma\theta-2t\tau\overline{\theta}}\Big)\, \mathrm{d}\theta\, \mathrm{d}\overline{\theta}\,, \end{aligned} \tag{C.2}$$

with

$$c_m = \frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}.$$

Proof. Let $x = (sx_s, tx_t)$ with $x_s,\ x_t \in \mathbb{S}^{m-1}$ and $y = (\sigma y_\sigma, \tau y_\tau)$ with $y_\sigma,\ y_\tau \in \mathbb{S}^{m-1}$. Then, decomposing $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ and using spherical coordinates in each $\mathbb{R}^m$ we obtain

$$\begin{aligned} L_K u(x) &= \int_{\mathbb{R}^{2m}} \big(u(x)-u(y)\big) K(|x-y|)\, \mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{+\infty} \sigma^{m-1}\tau^{m-1}\big(u(s,t)-u(\sigma,\tau)\big) \\ &\qquad \left(\int_{\mathbb{S}^{m-1}}\int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s-\sigma y_\sigma|^2+|tx_t-\tau y_\tau|^2}\Big)\, \mathrm{d}y_\sigma\, \mathrm{d}y_\tau\right) \mathrm{d}\sigma\, \mathrm{d}\tau \end{aligned}$$

Now, we define the kernel

$$J(x_s,x_t,s,t,\sigma,\tau) := \int_{\mathbb{S}^{m-1}}\int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s-\sigma y_\sigma|^2+|tx_t-\tau y_\tau|^2}\Big)\, \mathrm{d}y_\sigma\, \mathrm{d}y_\tau\,. \tag{C.3}$$

First of all, it is easy to see that J does not depend on x_s nor x_t. Indeed, consider a different point $(z_s,z_t) \in \mathbb{S}^{m-1}\times\mathbb{S}^{m-1}$ and let M_s and M_t be two orthogonal transformations such that $M_s(x_s) = z_s$ and $M_t(x_t) = z_t$. Then, making the change of variables

$y_\sigma = M_s(\tilde{y}_\sigma)$ and $y_\tau = M_t(\tilde{y}_\tau)$, and using that $M_s(\mathbb{S}^{m-1}) = M_t(\mathbb{S}^{m-1}) = \mathbb{S}^{m-1}$, we find out that

$$\begin{aligned}
&J(z_s, z_t, s, t, \sigma, \tau) = \\
&\quad= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma y_\sigma|^2 + |tM_t(x_t) - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau \\
&\quad= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma M_s(\tilde{y}_\sigma)|^2 + |tM_t(x_t) - \tau M_t(\tilde{y}_\tau)|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\
&\quad= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|M_s(sx_s - \sigma \tilde{y}_\sigma)|^2 + |M_t(tx_t - \tau \tilde{y}_\tau)|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\
&\quad= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma \tilde{y}_\sigma|^2 + |tx_t - \tau \tilde{y}_\tau|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\
&\quad= J(x_s, x_t, s, t, \sigma, \tau)\,.
\end{aligned}$$

Therefore, we can replace x_s and x_t in (C.3) by $e = (1, 0, \dots, 0) \in \mathbb{S}^{m-1}$. Thus, we have

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau\,.$$

For an easier notation, we rename $\omega = y_\sigma$ and $\tilde{\omega} = y_\tau$, and thus we have

$$\begin{aligned}
|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2 &= |se - \sigma\omega|^2 + |te - \tau\tilde{\omega}|^2 \\
&= s^2 + \sigma^2 - 2s\sigma e \cdot \omega + t^2 + \tau^2 - 2t\tau e \cdot \tilde{\omega} \\
&= s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1\,.
\end{aligned}$$

Then, we can rewrite J as

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega}\,.$$

At this point we have to distinguish the cases $m = 1$ and $m \geq 2$. For the fist one, since $\mathbb{S}^0 = \{-1, 1\}$ we directly obtain (C.1). For the second one, since the integrand only depends on ω_1 and $\tilde{\omega}_1$, we proceed as follows

$$\begin{aligned}
J(s, t, \sigma, \tau) &= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega} \\
&= \int_{-1}^{1} \mathrm{d}\omega_1 \int_{\partial B_{\rho(\omega_1)}} \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_m \int_{-1}^{1} \mathrm{d}\tilde{\omega}_1 \int_{\partial B_{\rho(\tilde{\omega}_1)}} \mathrm{d}\tilde{\omega}_2 \cdots \mathrm{d}\tilde{\omega}_m \\
&\qquad\qquad K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big) \\
&= \int_{-1}^{1} \int_{-1}^{1} |\partial B_{\rho(\omega_1)}||\partial B_{\rho(\tilde{\omega}_1)}| \\
&\qquad\qquad K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\,\mathrm{d}\omega_1\,\mathrm{d}\tilde{\omega}_1.
\end{aligned}$$

where $\rho(r) = \sqrt{1-r^2}$. Finally, we obtain (C.2) once we replace $|\partial B_r| = c_m\, r^{m-2}$, where c_m is the measure of the boundary of the ball of radius one in $\mathbb{R}^{m-1}$. $\square$

If the operator L_K is the fractional Laplacian, the previous expression of the kernel J can be rewritten in terms of a hypergeometric function of two variables, the so-called Appell function F_2 (see [1] for its definition). This expression does not simplify any of the arguments of this paper. Nevertheless, we think that it is worthy to point out the relation between J and F_2, since the known properties of the last one could provide some information about the kernel J.

Lemma C.2. *Let F_2 be the Appell hypergeometric function defined in [1]. If $L_K = (-\Delta)^\gamma$ and $m \geq 2$, then*

$$J(s,t,\sigma,\tau) = \frac{c_{2m,\gamma}\pi^m \Gamma\left(\frac{m}{2}\right)^2}{\Gamma\left(\frac{m-1}{2}\right)^2 \Gamma\left(\frac{m+1}{2}\right)^2} \frac{F_2\left(m+\gamma; \frac{m}{2}, m; \frac{m}{2}, m; \frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}, \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right)}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}. \tag{C.4}$$

Proof. If we take $K(z) = c_{2m,\gamma}|z|^{-2m-2\gamma}$ in (C.2) we get

$$J(s,t,\sigma,\tau) = c_{2m,\gamma}c_m^2 \int_{-1}^1 \int_{-1}^1 \frac{(1-\theta^2)^{\frac{m-2}{2}}(1-\overline{\theta}^2)^{\frac{m-2}{2}}}{(s^2+t^2+\sigma^2+\tau^2-2s\sigma\theta-2t\tau\overline{\theta})^{m+\gamma}}\, \mathrm{d}\theta\, \mathrm{d}\overline{\theta}\,.$$

Then, if we make the change of variables $\theta = 2\varpi_1 - 1$ and $\overline{\theta} = 2\varpi_2 - 1$ we arrive at

$$\begin{aligned} J(s,t,\sigma,\tau) &= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}} \cdot \\ &\qquad \int_0^1\int_0^1 \frac{\varpi_1^{\frac{m-2}{2}}(1-\varpi_1)^{\frac{m-2}{2}}\varpi_2^{\frac{m-2}{2}}(1-\varpi_2)^{\frac{m-2}{2}}}{\left(1-\frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}\,\varpi_1 - \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\,\varpi_2\right)^{m+\gamma}}\,\mathrm{d}\varpi_1\,\mathrm{d}\varpi_2 \\ &= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}} \frac{\Gamma\left(\frac{m}{2}\right)^4}{\Gamma(m)^2} \cdot \\ &\qquad F_2\left(m+\gamma; \frac{m}{2}, m; \frac{m}{2}, m; \frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}, \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right). \end{aligned}$$

We finally obtain (C.4) by using the duplication formula for the Γ-function. $\square$

To conclude the appendix, we rewrite the kernel inequality (1.14) in (s,t) variables and in terms of the kernel J. We do not present a proof of this result since it is identical to the one of Proposition 2.4 but changing the notation.

Lemma C.3. *Let $m \geq 1$ and let J the kernel defined in (C.1)-(C.2) with $K(\sqrt{\cdot})$ strictly convex. Then, if $s > t$ and $\sigma > \tau$, we have*

$$J(s,t,\sigma,\tau) > J(s,t,\tau,\sigma)\,.$$

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