INTEGRO-DIFFERENTIAL ALLEN-CAHN EQUATIONS WITH GENERAL KERNELS: HALF-SPACE AND SADDLE-SHAPED SOLUTIONS

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Abstract. We study blablabla

Contents

Key words and phrases. Fractional Laplacian, extremal solution, Dirichlet problem, stable solutions.

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1. Introduction

ec:Introduction⟩? We study the equation

$$Lu = f(u) \quad \text{in } \mathbb{R}^{2m} \tag{1.1}$$ Eq:NonlocalAller

where f is of bistable type and L is an integro-differential operator of the form

$$Lu(x) = \mathrm{P.\,V.} \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x-y)\,\mathrm{d}y \tag{1.2}$$ Eq:DefOfLu

where $K \geq 0$ is a nonnegative kernel. We assume also

$$K(y) = K(-y) \quad \text{and} \quad \int_{\mathbb{R}^n} \min\left\{|y|^2, 1\right\} K(y)\,\mathrm{d}y < +\infty\,. \tag{1.3}$$ Eq:Symmetry&Inte

The most canonical example of such operators is the fractional Laplacian

$$(-\Delta)^\gamma u = \mathrm{P.\,V.}\, c_{n,\gamma} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma}}\,\mathrm{d}y\,,$$

Mirar a ver si esta es una buena referencia o se pone otra

where $c_{n,\gamma}$ is a normalizing constant (for its exact value see for instance [16]).

Throughout the paper, we assume that the operators of our study are uniformly elliptic in the sense of the following definition.

?⟨Def:L_0Class⟩? **Definition 1.1.** We say that an operator L of the form (1.2) belongs to the class $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ if its kernel $K \geq 0$ satisfies (1.3) and

$$c_{n,\gamma} \frac{\lambda}{|y|^{n+2\gamma}} \leq K(y) \leq c_{n,\gamma} \frac{\Lambda}{|y|^{n+2\gamma}}\,, \quad 0 < \lambda \leq \Lambda\,, \tag{1.4}$$ Eq:Ellipticity

where $c_{n,\gamma}$ is the constant appearing in the definition of the fractional Laplacian.

When there is no ambiguity, we shall write only $\mathcal{L}_0$ for simplicity.

[...]

Given f a C^1 nonlinearity, we define

$$G(u) = \int_u^1 f(t)\,\mathrm{d}t\,.$$

We have that G is a C^2 function satisfying $G' = -f$. In this paper, we assume some, or all, of the following conditions on f.

$$f \text{ is odd;} \tag{1.5}$$ Eq:HipothesisfOd

$$G \geq 0 \quad \text{in } \mathbb{R}, \quad G > 0 \ \text{ in } (-1,1), \quad \text{and} \quad G(\pm 1) = 0\,; \tag{1.6}$$ Eq:HipothesisGWe

$$f \text{ is concave in } (0,1). \tag{1.7}$$ Eq:HipothesisfCo

Note that (1.5) and (1.6) yield that $f(0) = f(\pm 1) = 0$.

Note that (1.5) is equivalent to say that G is even.

Note that (1.5), (1.6), and (1.7) and the fact that $f(1) = 0$ yield $f'(0) > 0$ and $f'(\pm 1) < 0$. As a consequence, $f > 0$ in $(0,1)$.

Comentario: $f'(\pm 1) < 0$ equivale a $G''(\pm 1) > 0$, que es la hipotesis junto con las otras para que exista el Layer

Note that, since f is concave in $(0,1)$ and $f(0) = 0$, then

$$f'(t)t \le f(t) \quad \text{for all } t \in (0,1)\,. \tag{1.8}$$ Eq:PropertyConca

The inequality is strict if we have strict concavity.

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[...]

The interest on this problem originates from the famous De Giorgi conjecture for the classical Allen-Cahn equation.

Conjecture 1.2 (De Giorgi, 1978)**.** *Let u be a bounded solution of the Allen-Cahn equation*

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n \tag{1.9}$$?Eq:LocalAllenCa

such that it is monotone in one direction, say $\partial_{x_n} u > 0$. Then, if $n \le 8$, u is one dimensional, i.e., u depends only on one Euclidean variable.

This conjecture was proved true in dimension $n = 2$ by Ghoussoub and Gui in [20], and in dimension $n = 3$ by Ambrosio and Cabré in [1]. For dimensions $4 \le n \le 8$, it was established by Savin in [29] but with the extra assumption of

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}\,. \tag{1.10}$$ Eq:SavinConditio

A counterexample to the conjecture was given by del Pino, Kowalczyk and Wei in [14].

One can also formulate the same conjecture in the nonlocal setting by changing the Laplacian by a nonlocal operator, the most canonical one being $(-\Delta)^\gamma$. In this framework, for the equation $(-\Delta)^\gamma u = f(u)$ in $\mathbb{R}^n$ and $\gamma \in (0,1)$, the conjecture has been proven to be true in dimension $n = 2$ by Cabré and Solà-Morales in [8] for $\gamma = 1/2$ and extended to every power $0 < \gamma < 1$ by Cabré and Sire in [7] and also by Sire and Valdinoci in [34]. In dimension 3 the conjecture has been proved by Cabré and Cinti for $1/2 \le \gamma < 1$ in [5, 6]. Recently, in [28] Savin has established the validity of the conjecture in dimensions $4 \le n \le 8$ and for $1/2 < \gamma < 1$, but assuming the condition (1.10). In that paper he has also announced that the same holds for $\gamma = 1/2$. The case $0 < \gamma < 1/2$ has been also treated by Dipierro, Serra and Valdinoci in [17] . The most recent result related to the conjecture is the one of Figalli and Serra in [19], where they have proven the conjecture in dimension $n = 4$ and $\gamma = 1/2$. Note that this is the only result that is available (by now) exclusively in the nonlocal setting and not for the Laplacian.

Revisar, dependeria de la clasificación de los conos

After all the years of study of the conjecture raised by De Giorgi, another question appeared naturally: do global minimizers of the energy associated to the equation

$-\Delta u = f(u)$ in $\mathbb{R}^n$ have one-dimensional symmetry? A deep result from Savin [29] is that in dimension $n \leq 7$ this is indeed true, and the conjecture is that for $n \geq 8$ is false. The answer to this question would provide some insights of the original conjecture of De Giorgi. This is due to a result by Jerison and Monneau in [22], where they show that a counterexample of the original conjecture in $\mathbb{R}^{n+1}$ can be constructed from a bounded, even with respect to each coordinate, global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$. Hence, finding a global minimizer that is not one-dimensional would give a natural counterexample to the original conjecture.

Saddle-shaped solutions are of special interest in the search for this counterexample. To define these solutions properly, we need to introduce some notation and definition.

First of all, recall that the Simons cone is defined in $\mathbb{R}^{2m}$ with $2m = n$ by

$$\mathscr{C} = \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\} . \tag{1.11}$$?Eq:SimonsCone?

The Simons cone is proven to be a (classical) stationary minimal surface. Moreover, if $n \geq 8$, is also minimizing (see 99 and the coments...). Through the paper we will also use the letters $\mathcal{O}$ and $\mathcal{I}$ to denote the outside and inside of the cone:

comentar algo despues con aquello del límite?

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \ \text{ and } \ \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\} . \tag{1.12}$$?Eq:DefOandI?

Let $SO(m)$ be the special orthogonal group of $\mathbb{R}^m$, that is, the group of rotations of $\mathbb{R}^m$. We will work with the group $SO(m)^2 = SO(m) \times SO(m)$. Note that $SO(m)^2 \subset SO(2m)$ and therefore, for any $R \in SO(m)^2$, $|Rx| = |x|$. Moreover, the sets $\mathcal{O}$ and $\mathcal{I}$ are invariant under the action of the group and belong to a more general class of domains defined next.

eRevolutionSet⟩? **Definition 1.3.** We say that a set $\Omega \subset \mathbb{R}^{2m}$ is of *double revolution* if it is invariant under $SO(m)^2$, i.e., if it is invariant under orthogonal transformations in the first m variables and also under orthogonal transformations in the last m variables.

More defs (introducir mejor)

ef:DoublyRadial⟩? **Definition 1.4.** We say that a function $w : \mathbb{R}^{2m} \to \mathbb{R}$ is *doubly radial* if it only depends on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, if $w(Rx) = w(x)$ for every $R \in SO(m)^2$.

Through the paper we will consider the following isometry:

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \, . \end{array} \tag{1.13}$$ Eq:DefStar

Note that this isometry satisfies

(1) $((\cdot)^\star)^{-1} = (\cdot)^\star$.
(2) $\mathcal{O}^\star = \mathcal{I}$ and $\mathcal{I}^\star = \mathcal{O}$.

ldwrtSimonsCone⟩? **Definition 1.5.** We say that a doubly radial function w is *odd with respect to the Simons cone* if $w(|x'|, |x''|) = -w(|x''|, |x'|)$ for every $x = (x', x'') \in \mathbb{R}^{2m}$, or equivalently, if $w(x) = -w(x^\star)$. Similarly, we say that a doubly radial function w is *even with* *respect to the Simons cone* if $w(|x'|, |x''|) = w(|x''|, |x'|)$ for every $x = (x', x'') \in \mathbb{R}^{2m}$, or equivalently, if $w(x) = w(x^\star)$.

With these definitions in hand, we can define now properly saddle-shaped solutions:

SaddleShapedSol⟩? **Definition 1.6.** We say that u is a *saddle-shaped solution* (or simply *saddle solution*) of (1.1) if

(1) u is doubly radial.
(2) u is odd with respect to the Simons cone.
(3) $u > 0$ in $\mathcal{O}$.

Note that these solutions are even with respect to the coordinate axis and that their zero level set is the Simons cone $\mathscr{C} = \{|x'| = |x''|\}$. Therefore, saddle-shaped solutions are candidates to build a counterexample of the De Giorgi conjecture in high dimensions, since if one could prove that they are global minimizers in $\mathbb{R}^8$, by the result in [22] one would have a counterexample of the De Giorgi conjecture in $\mathbb{R}^9$ (as an alternative to that of [14]).

Saddle-shaped solutions for the classical equation with the Laplacian were first studied by Dang, Fife, and Peletier in [13] in dimension $2m = 2$. They established the existence and uniqueness of this type of solutions, as well as some monotonicity properties and asymptotic behavior. In [31], Schatzman studied the instability property of saddle solutions in $\mathbb{R}^2$. In higher even dimensions, Cabré and Terra proved the existence of a saddle solution in every dimension $2m \geq 2$ and they established also some qualitative properties such as monotonicity properties, asymptotic behavior, as well as instability in dimensions $2m = 4$ and $2m = 6$ (see [9, 10]). The uniqueness in dimensions higher than 2 was established by Cabré in [4], where he also proved that saddle solutions are stable (see the definition below) in dimensions $2m \geq 14$. ver si lo quitamos o como lo ponemos

In the nonlocal framework, there are some works concerning saddle-shaped solutions to (1.1) with $L = (-\Delta)^\gamma$. In [11], Cinti established the existence of saddle-shaped solutions to $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^{2m}$, as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability in dimensions $2m = 4$ and $2m = 6$ (instability in dimension $2m = 2$ follows by a result of Cabré and Solà-Morales in [8]). More recently, she has extended the same results to all $\gamma \in (0, 1)$ in 99. Citar paper de Cinti

To the best of our knowledge, there are no more works studying the saddle-shaped solutions in the nonlocal setting. Moreover, the problem has not been studied for general operators $L \in \mathcal{L}_0$. Regarding the nonlocal Allen-Cahn equation with general kernels, we have to mention two works.

The first one is [12], where Cozzi and Passalacqua study layer solutions to the equation (1.1).

We should also mention [17] where they study nonlocal minimal surfaces with general kernels and blablabla.mal surfaces with general kernels and blablabla. CHECK

The second variation of the energy is

$$Q_u(\xi) := \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^{2m}} f'(u)\xi^2\,\mathrm{d}x\,. \tag{1.14}$$

?Eq:SecondVariat

⟨Def:Stability⟩ **Definition 1.7.** We say that a solution u of (1.1) is *stable* in a set $\Omega \subset \mathbb{R}^{2m}$ if

$$Q_u(\xi) = \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega} f'(u)\xi^2\,\mathrm{d}x \geq 0 \tag{1.15}$$

?Eq:StablityCond

for every $\xi \in C_c^\infty(\Omega)$.

lenceStability⟩? **Lemma 1.8** (see [21])**.** *A solution u is stable in Ω (in the sense of Definition 1.7) if and only if there exists a continuous function φ such that $\varphi > 0$ in Ω and $L\varphi \geq f'(u)\varphi$ in Ω.*

The proof of this result can be found in [21]. Nevertheless, one of the implications is quite simple and since is an argument that will be repeated in the paper, we show it here.

Assume that $\varphi > 0$ is a supersolution of the linearized operator in Ω. Then, let $\xi \in C_c^\infty(\Omega)$ and we compute

$$\begin{aligned}
\int_\Omega f'(u)\xi^2\,\mathrm{d}x &= \int_\Omega f'(u)\varphi\frac{\xi^2}{\varphi}\,\mathrm{d}x \\
&\leq \int_\Omega L\varphi\frac{\xi^2}{\varphi}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \big(\varphi(x)-\varphi(y)\big)\left(\frac{\xi^2(x)}{\varphi(x)}-\frac{\xi^2(y)}{\varphi(y)}\right)K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Here we have used that the kernel is positive and that

$$\big(\varphi(x)-\varphi(y)\big)\left(\frac{\xi^2(x)}{\varphi(x)}-\frac{\xi^2(y)}{\varphi(y)}\right) \leq |\xi(x)-\xi(y)|^2\,.$$

Indeed, developing the squares and the products, this last inequality is equivalent to

$$2\xi(x)\xi(y) \leq \frac{\varphi(x)}{\varphi(y)}\xi^2(y) + \frac{\varphi(y)}{\varphi(x)}\xi^2(x)\,,$$

which is equivalent to

$$\left(\xi(x)\sqrt{\frac{\varphi(y)}{\varphi(x)}} - \xi(y)\sqrt{\frac{\varphi(x)}{\varphi(y)}}\right)^2 \geq 0\,.$$

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⟨Th:Existence⟩ **Theorem 1.9** (Existence of the saddle-shaped solution)**.** *Let f satisfy .... Then, for every dimension $2m \geq 2$, there exists a saddle-shaped solution to* (1.1)*. In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$ and .*

Poner algo más?

⟨Th:Uniqueness⟩ **Theorem 1.10** (Uniqueness of the saddle-shaped solution)**.** *Let f satisfy .... Then, for every dimension $2m \geq 2$, there exists at most one saddle-shaped solution to* (1.1)*.*

juntar teoremas?

2. Preliminaries

2.1. **Regularity theory for nonlocal operators in the class $\mathcal{L}_0$.** This section is devoted to present the regularity results that will be used in the paper. For further reference, see [26, 32] and the references therein.

We start with a result on the interior regularity for linear equations.

Proposition 2.1. *Let $L \in \mathcal{L}_0$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to $Lw = h$ in B_1. Then,*

$$||w||_{C^{2\gamma}(B_{1/2})} \leq C \left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)} \right) . \tag{2.1}$$

Moreover, let $\alpha > 0$ and assume additionally that $w \in C^\alpha(\mathbb{R}^n)$. Then, if $\alpha + 2\gamma$ is not an integer,

$$||w||_{C^{\alpha+2\gamma}(B_{1/2})} \leq C \left(||h||_{C^\alpha(B_1)} + ||w||_{C^\alpha(\mathbb{R}^n)} \right) , \tag{2.2}$$

where C is a constant that depends only on n, γ, λ and Λ.

Throughout the paper we consider solutions u to (1.1) that satisfy $|u| \leq 1$ in $\mathbb{R}^n$. Hence, by applying (2.1) we find that for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} ||u||_{C^{2\gamma}(B_{1/2}(x_0))} &\leq C \left(||f(u)||_{L^\infty(B_1(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)} \right) \\ &\leq C \left(1 + ||f||_{L^\infty([-1,1])} \right) . \end{aligned}$$

Note, that the estimate is independent of the point x_0, and thus since the equation is satisfied in the whole $\mathbb{R}^n$,

$$||u||_{C^{2\gamma}(\mathbb{R}^n)} \leq C \left(1 + ||f||_{L^\infty([-1,1])} \right) .$$

Then, we use estimate (2.2) repeatedly and the same kind of arguments lead to the following conclusion.

Corollary 2.2. *Let $f \in C^1([-1,1])$, $L \in \mathcal{L}_0$ and let $-1 \leq u \leq 1$ be a bounded weak solution to (1.1). Then $u \in C^\alpha(\mathbb{R}^n)$ for some $\alpha > 1 + 2\gamma$. Moreover, the following estimate holds:*

$$||u||_{C^\alpha(\mathbb{R}^n)} \leq C \, , \tag{2.3}$$

for some constant C depending only on n, γ, λ, Λ, and $||f||_{C^1([-1,1])}$.

Sometimes we will need estimates in balls. With the same argument as in Corollaries 2.4 and 2.5 of [27], we deduce from Proposition 2.1 the following result.

Corollary 2.3. *Let $L \in \mathcal{L}_0$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to $Lw = h$ in B_1. Then,*

$$||w||_{C^{2\gamma}(B_{1/4})} \leq C \left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(B_1)} + \left|\left| \frac{w(x)}{(1+|x|)^{n+2\gamma}} \right|\right|_{L^1(\mathbb{R}^n)} \right) . \tag{2.4}$$

Moreover, let $\alpha > 0$ and assume additionally that $w \in C^{\alpha}(\mathbb{R}^n)$. Then, if $\alpha + 2\gamma$ is not an integer,

$$||w||_{C^{\alpha+2\gamma}(B_{1/4})} \leq C \left(||h||_{C^{\alpha}(\overline{B_1})} + ||w||_{C^{\alpha}(\overline{B_1})} + \left|\left| \frac{w(x)}{(1+|x|)^{n+2\gamma}} \right|\right|_{L^1(\mathbb{R}^n)} \right), \qquad (2.5)$$

Eq:Calpha->Calph

where C is a constant that depends only on n, γ, λ and Λ.

Therefore, assume now that u solves $Lu = f(u)$ in B_R and that $|u| \leq 1$ in $\mathbb{R}^n$ with $f \in C^{\alpha}([-1,1])$ for some $\alpha > 0$. Then, the combination of (2.1) and (2.5) yields

$$||u||_{C^{2s+\varepsilon}(B_{R/8})} \leq C \left(n,\ \gamma,\ \lambda,\ \Lambda,\ ||f||_{C^{\alpha}([-1,1])} \right). \qquad (2.6)$$

Eq:UniformC2alph

for some $\varepsilon > 0$.

Sometimes the previous regularity results will be used together with a compactness argument. Since it will be used repeatedly through the paper, we find it useful to state it here for easy reference. The result is an easy consequence of the Arzelà-Ascoli theorem and the compact embedding $C^{\alpha} \subset\subset C^{\beta}$ when $\beta < \alpha$.

CompactnessLemma⟩ **Lemma 2.4.** *Let $\Omega \subset \mathbb{R}^n$ a bounded domain, $L \in \mathcal{L}_0$ and let w_k be a sequence of functions satisfying*

- *$w_k \in C^{\alpha}(\overline{\Omega})$ with $\alpha > 2\gamma$ and*
$$||w_k||_{C^{\alpha}(\overline{\Omega})} \leq C$$
with a constant C independent of k.
- *$Lw_k = h_k$ with $h_k \in C^{\alpha'}(\overline{\Omega})$ for some $\alpha' > 0$ and such that $h_k \to h \in C^{\alpha'}(\overline{\Omega})$ uniformly.*

Then, for every $\beta \in (2\gamma, \alpha)$, a subsequence of w_k converges to some $w \in C^{\beta}(\overline{\Omega})$ with the $C^{\beta}(\overline{\Omega})$ norm and satisfies $Lw = h$ in Ω.

2.2. **The operator L acting on doubly radial functions.** The main purpose of this subsection is to deduce an alternative expression for the operator L acting on doubly radial functions. This expression is more suitable to work with and it will be used throughout the paper. Recall that we are always assuming that $L \in \mathcal{L}_0$ and that the kernel is radially symmetric, i.e., $K(z) = K(|z|)$.

Our first remark is that if w is invariant by $SO(m)^2$, so is Lw. Indeed, for every $R \in SO(m)^2$,

$$\begin{aligned} Lw(Rx) &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(y)\} K(|Rx - y|)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(R\tilde{y})\} K(|Rx - R\tilde{y}|)\,\mathrm{d}\tilde{y} \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(\tilde{y})\} K(|x - \tilde{y}|)\,\mathrm{d}\tilde{y} \\ &= Lw(x)\,. \end{aligned}$$

Here we have used that $w(R\cdot) = w(\cdot)$ for every $R \in SO(m)^2$ and the change $y = R\tilde{y}$.

Next, we present an alternative expression for the operator L acting on doubly radial functions.

eratorExpression⟩ **Lemma 2.5.** *Let w be a doubly radial function for which Lw is well-defined. Let K be the kernel associated to L and assume that it is radially symmetric and translation invariant, that is, $K(x,y) = K(|x-y|)$. Then, Lw can be expressed as*

$$Lw(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y$$

where $\overline{K}$ is symmetric, invariant by $SO(m)^2$ in both arguments and is defined by

$$\overline{K}(x,y) := \fint_{SO(m)^2} K(|Rx - y|)\,\mathrm{d}R\,. \tag{2.7}$$ Eq:KbarDef

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $SO(m)^2$.

Recall (see for instance 99) that the Haar measure on $SO(m)^2$ exists and it is unique up to a multiplicative constant. Let us state next the properties that will be used in the rest of the paper. In the following, the Haar measure is denoted by μ. First, since $SO(m)^2$ is a compact group, it is unimodular (see Chapter II, Proposition 13 of [23]). As a consequence, the measure μ is left and right invariant, that is, $\mu(R\Sigma) = \mu(\Sigma) = \mu(\Sigma R)$ for every subset $\Sigma \subseteq SO(m)^2$ and every $R \in SO(m)^2$. Moreover, it holds

$$\fint_{SO(m)^2} g(R^{-1})\,\mathrm{d}R = \fint_{SO(m)^2} g(R)\,\mathrm{d}R \tag{2.8}$$ Eq:Unimodular

for every $g \in L^1(SO(m)^2)$ —see [23] for the details.

Proof of Lemma 2.5. Since Lw is invariant by $SO(m)^2$ and $w(R\cdot) = w(\cdot)$ for every $R \in SO(m)^2$,

$$\begin{aligned} Lw(x) &= \fint_{SO(m)^2} Lw(Rx)\,\mathrm{d}R \\ &= \fint_{SO(m)^2} \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\}K(|Rx - y|)\,\mathrm{d}y\,\mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \fint_{SO(m)^2} K(|Rx - y|)\,\mathrm{d}R\,\mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y\,. \end{aligned}$$

Now, we show that $\overline{K}$ is symmetric:

$$\begin{aligned}\overline{K}(y,x) &= \fint_{SO(m)^2} K(|Ry-x|)\,\mathrm{d}R\\ &= \fint_{SO(m)^2} K(|R^{-1}(Ry-x)|)\,\mathrm{d}R\\ &= \fint_{SO(m)^2} K(|R^{-1}x-y)|)\,\mathrm{d}R\\ &= \overline{K}(x,y)\,.\end{aligned}$$

In this last step we have used property (2.8). It remains to show that $\overline{K}$ is invariant by $SO(m)^2$ in its two arguments. By the symmetry, it is enough to check it for the first one. Let $\tilde{R} \in SO^2(m)^2$. Then,

$$\overline{K}(\tilde{R}x,y) = \fint_{SO(m)^2} K(|R\tilde{R}x-y|)\,\mathrm{d}R = \fint_{SO(m)^2} K(|Rx-y|)\,\mathrm{d}R = \overline{K}(x,y)$$

where we have used the right invariance of the Haar measure. □

In the following lemma we present some properties of the involution $(\cdot)^\star$ defined by (1.13).

a:PropertiesStar⟩ **Lemma 2.6.** *Let $(\cdot)^\star$ be the isometry defined by $x^\star = (x',x'')^\star = (x'',x')$ (see (1.13)). Then,*

(1) For every $R \in SO(m)^2$, there exists $R_\star \in SO(m)^2$ defined by $R_\star := (R(\cdot)^\star)^\star$. Equivalently, if $R = (R_1, R_2)$ with R_1, $R_2 \in SO(m)$, then $R_\star = (R_2, R_1)$.

(2) The Haar integral on $SO(m)^2$ has the following invariance:

$$\int_{SO(m)^2} g(R_\star)\,\mathrm{d}R = \int_{SO(m)^2} g(R)\,\mathrm{d}R\,, \tag{2.9}$$ Eq:InvarianceByS

for every $g \in L^1(SO(m)^2)$.

(3) $\overline{K}(x^\star,y) = \overline{K}(x,y^\star)$.

(4) $\overline{K}(x^\star,y^\star) = \overline{K}(x,y)$.

(5) If w is a doubly radial function which is odd with respect to the Simons cone, then

$$Lw(x) = \int_{\mathcal{O}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x)+w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,.$$

Proof. The first statement is trivial. To check (2.9), we use Fubini:

$$\begin{aligned}
\int_{SO(m)^2} g(R_\star)\,\mathrm{d}R &= \int_{SO(m)} \mathrm{d}R_1 \int_{SO(m)} \mathrm{d}R_2 \quad g(R_2, R_1) \\
&= \int_{SO(m)} \mathrm{d}R_2 \int_{SO(m)} \mathrm{d}R_1 \quad g(R_2, R_1) \\
&= \int_{SO(m)} \mathrm{d}R_1 \int_{SO(m)} \mathrm{d}R_2 \quad g(R_1, R_2) \\
&= \int_{SO(m)^2} g(R)\,\mathrm{d}R\,.
\end{aligned}$$

To show the third statement, we use the definition of $R_\star$ and (2.9) to see that

$$\begin{aligned}
\overline{K}(x^\star, y) &= \fint_{SO(m)^2} K(|Rx^\star - y|)\,\mathrm{d}R \\
&= \fint_{SO(m)^2} K(|(Rx^\star - y)^\star|)\,\mathrm{d}R \\
&= \fint_{SO(m)^2} K(|(Rx^\star)^\star - y^\star|)\,\mathrm{d}R \\
&= \fint_{SO(m)^2} K(|R_\star x - y^\star|)\,\mathrm{d}R \\
&= \fint_{SO(m)^2} K(|Rx - y^\star|)\,\mathrm{d}R \\
&= \overline{K}(x, y^\star)\,.
\end{aligned}$$

As a consequence, we have that

$$\overline{K}(x^\star, y^\star) = \overline{K}(x, (y^\star)^\star) = \overline{K}(x, y)\,.$$

Finally, the last statement is just a computation with a change of variables. First, using the change $\bar{y} = y^\star$ and the odd symmetry of u, we see that

$$\begin{aligned}
\int_{\mathcal{I}} \{w(x) - w(y)\}\overline{K}(x, y)\,\mathrm{d}y &= \int_{\mathcal{O}^\star} \{w(x) - w(y)\}\overline{K}(x, y)\,\mathrm{d}y \\
&= \int_{\mathcal{O}} \{w(x) - w(y^\star)\}\overline{K}(x, y^\star)\,\mathrm{d}y \\
&= \int_{\mathcal{O}} \{w(x) + w(y)\}\overline{K}(x, y^\star)\,\mathrm{d}y\,.
\end{aligned}$$

Hence,

$$\begin{aligned} Lw(x) &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{I}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x) + w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,. \end{aligned}$$

□

To finish the subsection, we prove an important result that will be used repeatedly throughout the paper. It states a crucial inequality for the kernel $\overline{K}$ that is valid when the original kernel K satisfies that $K(\sqrt{|\cdot|})$ is convex.

qualityReflexion⟩ **Proposition 2.7.** *Assume that the kernel K is radially symmetric, satisfies hypothesis (1.3) and (1.4), and that $K(\sqrt{|\cdot|})$ is convex. Then,*

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \textit{for every } x,y \in \mathcal{O}\,, \tag{2.10}$$ Eq:KernelInequal

where $\overline{K}$ is defined by (2.7).

Proof. Let x, $y \in \mathcal{O}$ and let $x_0 = (|x'|e, |x''|e)$ and $y_0 = (|y'|e, |y''|e)$, for an arbitrary unitary vector $e \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. Hence, since $\overline{K}$ is invariant by $SO(m)^2$,

$$\overline{K}(x,y) = \overline{K}(x_0,y_0) \quad \text{and} \quad \overline{K}(x,y^\star) = \overline{K}(x_0,y_0^\star)\,.$$

To see this, let R_x, $R_y \in SO(m)^2$ satisfying $x = R_x x_0$ and $y = R_y y_0$. Then,

$$\overline{K}(x,y) = \overline{K}(R_x x_0, R_y y_0) = \overline{K}(x_0,y_0)$$

and, using that $(R_y y_0)^\star = (R_y)_\star y_0^\star$ (see Lemma 2.6),

$$\overline{K}(x,y^\star) = \overline{K}(R_x x_0, (R_y y_0)^\star) = \overline{K}(R_x x_0, (R_y)_\star y_0^\star) = \overline{K}(x_0,y_0^\star)\,.$$

Therefore, it is enough to show (2.10) for points x and y of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, with $e \in \mathbb{S}^{m-1}$ an arbitrary unitary vector.

Now, define

$$\begin{aligned} Q_1 &:= \big\{R = (R_1,R_2) \in SO(m)^2 \;:\; e \cdot R_1 e > |e \cdot R_2 e|\big\}\,, \\ Q_2 &:= \big\{R = (R_1,R_2) \in SO(m)^2 \;:\; e \cdot R_2 e > |e \cdot R_1 e|\big\}\,, \\ Q_3 &:= \big\{R = (R_1,R_2) \in SO(m)^2 \;:\; e \cdot R_1 e < -|e \cdot R_2 e|\big\}\,, \\ Q_4 &:= \big\{R = (R_1,R_2) \in SO(m)^2 \;:\; e \cdot R_2 e < -|e \cdot R_1 e|\big\}\,. \end{aligned}$$

Note that $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Moreover, the set

$$SO(m)^2 \setminus \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\} = \big\{R \in SO(m)^2 \;:\; |e \cdot R_1 e| = |e \cdot R_2 e|\big\}$$

has zero measure. Note also that

- If $R = (R_1,R_2) \in Q_2$, $R_\star = (R_2,R_1) \in Q_1$.
- If $R = (R_1,R_2) \in Q_3$, $-R = (-R_1,-R_2) \in Q_1$.

- If $R = (R_1, R_2) \in Q_4$, $-R_\star = (-R_2, -R_1) \in Q_1$.

Therefore,

$$\begin{aligned}\overline{K}(x,y) &= \fint_{SO(m)^2} K(|x-Ry|)\,\mathrm{d}R \\ &= \fint_{Q_1} K(|x-Ry|)\,\mathrm{d}R + \fint_{Q_2} K(|x-Ry|)\,\mathrm{d}R \\ &\quad + \fint_{Q_3} K(|x-Ry|)\,\mathrm{d}R + \fint_{Q_4} K(|x-Ry|)\,\mathrm{d}R \\ &= \fint_{Q_1} \{K(|x-Ry|) + K(|x+Ry|) \\ &\quad + K(|x-R_\star y|) + K(|x+R_\star y|)\}\,\mathrm{d}R\end{aligned}$$

and

$$\begin{aligned}\overline{K}(x,y^\star) &= \fint_{SO(m)^2} K(|x-Ry^\star|)\,\mathrm{d}R \\ &= \fint_{Q_1} \{K(|x-Ry^\star|) + K(|x+Ry^\star|) \\ &\quad + K(|x-R_\star y^\star|) + K(|x+R_\star y^\star|)\}\,\mathrm{d}R\end{aligned}$$

Thus, if we prove

$$\begin{aligned}&K(|x-Ry|) + K(|x+Ry|) + K(|x-R_\star y|) + K(|x+R_\star y|) \\ &\qquad \geq K(|x-Ry^\star|) + K(|x+Ry^\star|) + K(|x-R_\star y^\star|) + K(|x+R_\star y^\star|)\,,\end{aligned} \tag{2.11}$$

Eq:InequalityInt

Decir que todos los términos son finitos, los del rhs pq cada punto esta en un lado del cono y los otros pq sino implicaria que $e \cdot R_2 e = 1$ y esto no puede pasar en Q_1.

for every $R \in Q_1$, we immediately deduce (2.10) with a non strict inequality. To see that it is indeed a strict one, we must show that the inequality in (2.11) is strict for a.e. $R \in Q_1$.

For a short notation, we call

$$\alpha := e \cdot R_1 e \quad \text{and} \quad \beta := e \cdot R_2 e\,.$$

Note that

$$\begin{aligned}|x \pm Ry|^2 &= |x' \pm R_1 y'|^2 + |x'' \pm R_2 y''|^2 \\ &= |x'|^2 + |y'|^2 \pm 2x' \cdot R_1 y' + |x''|^2 + |y''|^2 \pm 2x'' \cdot R_2 y'' \\ &= \left(|x|^2 + |y|^2 \pm 2|x'||y'|\alpha \pm 2|x''||y''|\beta\right)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}|x \pm R_\star y| &= \left(|x|^2 + |y|^2 \pm 2|x'||y'|\beta \pm 2|x''||y''|\alpha\right)\,, \\ |x \pm Ry^\star| &= \left(|x|^2 + |y|^2 \pm 2|x'||y''|\alpha \pm 2|x''||y'|\beta\right)\,,\end{aligned}$$

and

$$|x \pm R_\star y^\star| = \left(|x|^2 + |y|^2 \pm 2|x'||y''|\beta \pm 2|x''||y'|\alpha\right)\,.$$

We define now

$$g(t) := K\left(\sqrt{|x|^2+|y|^2+2t}\right) + K\left(\sqrt{|x|^2+|y|^2-2t}\right)$$

we see that (2.11) is equivalent to

$$\begin{aligned} g\Big(|x'||y'|\alpha + |x''||y''|\beta\Big) + g\Big(|x'||y'|\beta + |x''||y''|\alpha\Big) \\ \geq g\Big(|x'||y''|\alpha + |x''||y'|\beta\Big) + g\Big(|x'||y''|\beta + |x''||y'|\alpha\Big)\,, \end{aligned} \tag{2.12}$$

Eq:InequalityInt

for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$. Let

$$\begin{aligned} A_{\alpha,\beta} &= |x'||y'|\alpha + |x''||y''|\beta\,, \quad & B_{\alpha,\beta} &= |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &= |x''||y'|\alpha + |x'||y''|\beta\,, \quad & D_{\alpha,\beta} &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

With this notation and taking into account that g is even, (2.12) is equivalent to

$$g(|A_{\alpha,\beta}|) + g(|D_{\alpha,\beta}|) \geq g(|C_{\alpha,\beta}|) + g(|B_{\alpha,\beta}|)\,, \tag{2.13}$$

Eq:InequalityInt

for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$. Note that g is defined in the open interval $I = (-(|x|^2+|y|^2)/2,\ (|x|^2+|y|^2)/2)$ and that $A_{\alpha,\beta}$, $B_{\alpha,\beta}$, $C_{\alpha,\beta}$, $D_{\alpha,\beta} \in I$ since $|\beta| < 1$.

To show (2.16), we use Proposition A.8 of the appendix. Hence, we should check that

$$\begin{cases} |A_{\alpha,\beta}| \geq |B_{\alpha,\beta}|,\ |A_{\alpha,\beta}| \geq |C_{\alpha,\beta}|,\ |A_{\alpha,\beta}| \geq |D_{\alpha,\beta}|\,, \\ |A_{\alpha,\beta}| + |D_{\alpha,\beta}| \geq |B_{\alpha,\beta}| + |C_{\alpha,\beta}|\,. \end{cases}$$

The verification of these inequalities is a simple but tedious computation and it is presented in the appendix (see point (1) of Lemma A.9). Once we have these inequalities, by Proposition A.8 we deduce (2.16).

To finish, we must see that the equality in (2.16) is never attained. By Proposition A.8, we know that a necessary condition for the equality to hold is that either $|A_{\alpha,\beta}| = |B_{\alpha,\beta}|$ and $|C_{\alpha,\beta}| = |D_{\alpha,\beta}|$, or $|A_{\alpha,\beta}| = |C_{\alpha,\beta}|$ and $|B_{\alpha,\beta}| = |D_{\alpha,\beta}|$. Nevertheless, by point (2) of Lemma (A.9), this yields $\alpha = \beta = 0$, but this cannot happen. □

Once we have presented a sufficient condition of the kernel to be positive, we can also obtain a necessary condition.

alityReflexion⟩? **Proposition 2.8.** *Assume that the kernel K is radially symmetric, nonincreasing and satisfies hypothesis (1.3) and (1.4). If*

$$\overline{K}(x,y) \geq \overline{K}(x,y^\star) \quad \text{for almost every } x,y \in \mathcal{O}\,, \tag{2.14}$$

Eq:KernelInequal

then $K(\sqrt{|\cdot|})$ cannot be strictly concave in any interval $I \subset [0,+\infty)$.

Proof. We prove it by contraposition. In fact, we will show that the existence of an interval where $K(\sqrt{|\cdot|})$ is strictly concave means the existence of a open set in $\mathcal{O} \times \mathcal{O}$ with positive measure where (2.14) is not satisfied.

Let $t_2 > t_1 > 0$ be such that $K(\sqrt{t})$ is strictly concave in (t_1, t_2) and define the set $\Omega_{t_1,t_2} \subset \mathbb{R}^{4m}$ as the points $(x, y) \in \mathcal{O} \times \mathcal{O}$ satisfying

$$\begin{cases} |x|^2 + |y|^2 & > & t_1, \\ |x|^2 + |y|^2 & < & t_2, \\ (|x'| - |x''|)^2 + (|y'| - |y''|)^2 & > & t_1, \\ (|x'| + |x''|)^2 + (|y'| + |y''|)^2 & < & t_2. \end{cases} \tag{2.15}$$

eq:OmegaSetDefin

First, it is easy to see that Ω_{t_1,t_2} is a nonempty open set. In fact, points of the form $(x', 0, y', 0) \in (\mathbb{R}^m)^4$ such that $t_1 \leq |x'|^2 + |y'|^2 < t_2$ belong to Ω_{t_1,t_2}. Then, if we prove that $\overline{K}(x, y) < \overline{K}(x, y^\star)$ in Ω_{t_1,t_2} we are done.

Let $x, y \in \Omega_{t_1,t_2}$, we are going to show that

$$\begin{aligned} K(|x - Ry|) &+ K(|x + Ry|) + K(|x - R_\star y|) + K(|x + R_\star y|) \\ &< K(|x - Ry^\star|) + K(|x + Ry^\star|) + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|)\,, \end{aligned} \tag{2.16}$$

Eq:InequalityInt

for any R in Q_1, where Q_1 is defined in the proof of Proposition 2.7. As before, we can assume that x and y are of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, with $e \in \mathbb{S}^{m-1}$ an arbitrary unitary vector. Then, defining α and β as in the previous proof we see that proving (2.16) is equivalent to prove that

$$g(A_{\alpha,\beta}) + g(D_{\alpha,\beta}) < g(B_{\alpha,\beta}) + g(C_{\alpha,\beta})\,, \tag{2.17}$$

Eq:InequalityInt

for every $\alpha, \beta \in [-1, 1]$ such that $\alpha > |\beta|$,

$$\begin{aligned} A_{\alpha,\beta} &= |x'||y'|\alpha + |x''||y''|\beta\,, \quad & B_{\alpha,\beta} &= |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &= |x''||y'|\alpha + |x'||y''|\beta\,, \quad & D_{\alpha,\beta} &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

and

$$g(t) = K\left(\sqrt{|x|^2 + |y|^2 + 2t}\right) + K\left(\sqrt{|x|^2 + |y|^2 - 2t}\right).$$

Now, from being $K(\sqrt{t})$ concave in (t_1, t_2) and $t_1 < |x|^2 + |y|^2 < t_2$, we obtain that g is concave in $\left(-\bar{t}, \bar{t}\right)$, with $\bar{t} = \min\left(\frac{t_2 - |x|^2 - |y|^2}{2}, \frac{|x|^2 + |y|^2 - t_1}{2}\right)$.

On the other hand it is easy to check that $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$ and $D_{\alpha,\beta}$ belong to the open interval $(-|x'||y'| - |x''||y''|, |x'||y'| + |x''||y''|)$ for every $\alpha, \beta \in [-1, 1]$ such that $\alpha > |\beta|$.

Therefore, since $x, y \in \Omega_{t_1,t_2}$, we obtain from the last inequalities in (2.15) that

$$\begin{cases} |x'||y'| + |x''||y''| & < & \dfrac{t_2 - |x|^2 - |y|^2}{2} \\ |x'||y'| + |x''||y''| & < & \dfrac{|x|^2 + |y|^2 - t_1}{2} \end{cases} \Longrightarrow |x'||y'| + |x''||y''| < \bar{t},$$

which means that $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$ and $D_{\alpha,\beta}$ belong to $(-\bar{t}, \bar{t})$ for every $\alpha, \beta \in [-1, 1]$ such that $\alpha > |\beta|$. Hence, by applying Lemma A.3 to the function $-g$ (taking into account Remark A.4) we obtain that inequality (2.17) is satisfied. Finally, from integrating (2.16) with respect to all the rotations $R \in Q_1$ we get

$$\overline{K}(x, y) < \overline{K}(x, y^\star),$$

for every $(x, y) \in \Omega_{t_1,t_2}$, contradicting (2.14). □

2.3. **Maximum principles for doubly radial odd functions.** In this subsection we prove a weak and a strong maximum principles for doubly radial functions that are odd with respect to the Simons cone. The key ingredient in these proofs is the kernel inequality of Proposition 2.7.

ctionsRotations⟩ **Proposition 2.9** (Weak maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $u \in C^\alpha(\mathbb{R}^{2m})$ with $\alpha > 2\gamma$ be a doubly radial function which is odd with respect to the Simons cone. Let $\Omega \subseteq \mathcal{O}$ and let $L \in \mathcal{L}_0$ be such that .... Assume that*

Hypothesis for the inequality of the kernels

$$\begin{cases} Lu \geq 0 & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \mathcal{O} \setminus \Omega\,, \end{cases}$$

and that either Ω is bounded or

Pensar si sabemos hacerlo para no acotados. En el paper no se usa pero quedaría mejor un resultado un poco más general

$$\liminf_{x \in \mathcal{O},\, |x| \to \infty} u(x) \geq 0\,.$$

Then, $u \geq 0$ in Ω.

Proof. By contradiction, assume that u takes negative values in Ω. Under the hypotheses we are assuming, a negative minimum must be achieved. Thus, there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{\Omega} u =: m < 0\,.$$

Then, using the expression of L for odd functions (see Lemma 2.6), we have

$$Lu(x_0) = \int_{\mathcal{O}} \{m - u(y)\}\overline{K}(x_0, y)\,\mathrm{d}y + \int_{\mathcal{O}} \{m + u(y)\}\overline{K}(x_0, y^\star)\,\mathrm{d}y\,.$$

Now, since $m - u(y) \leq 0$ in $\mathcal{O}$ and $\overline{K}(x_0, y) \geq \overline{K}(x_0, y^\star)$ (see Proposition 2.7), we have

$$\{m - u(y)\}\overline{K}(x_0, y) \leq \{m - u(y)\}\overline{K}(x_0, y^\star)$$

and therefore, since $m < 0$, we get

$$0 \leq Lu(x_0) \leq 2m \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y < 0\,,$$

a contradiction. □

The following is a strong maximum principle for odd functions.

ctionsRotations⟩ **Proposition 2.10** (Strong maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $u \in C^\alpha(\mathbb{R}^{2m})$ with $\alpha > 2\gamma$ be a doubly radial function which is odd with respect to the Simons cone. Let $\Omega \subseteq \mathcal{O}$ and assume that $Lu \geq 0$ in Ω, where $L \in \mathcal{L}_0$ such that .... Assume also that $u \geq 0$ in $\mathcal{O}$. Then, either $u \equiv 0$ or $u > 0$ in Ω.*

Hypothesis for the inequality of the kernels

Proof. Assume that $u \not\equiv 0$. We shall prove that $u > 0$ in Ω. By contradiction, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then, using the expression of L for odd functions given in Lemma 2.6, the kernel inequality of Proposition 2.7 and the fact that $u \geq 0$ in $\mathcal{O}$, we obtain

$$0 \leq Lu(x_0) = \int_{\mathcal{O}} u(y) \big\{ \overline{K}(x_0, y^\star) - \overline{K}(x_0, y) \big\} \, \mathrm{d}y < 0 \,,$$

a contradiction. □

3. Non-local Allen-Cahn Energy

lenCahn_Energy⟩? We follow the same notation as in [12].

ctionalSpaceHK⟩? **Definition 3.1.** Given a set $\Omega \subseteq \mathbb{R}^n$ and a kernel $K \in \mathcal{L}_0$, we define the space

$$\mathbb{H}^K(\Omega) := \left\{ w \in L^2(\Omega) \ : \ [w]^2_{\mathbb{H}^K(\Omega)} < +\infty \right\},$$

where

$$[w]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2} \int\int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} |w(x) - w(y)|^2 K(x-y) \, \mathrm{d}x \, \mathrm{d}y \, .$$

We also define

$$\begin{aligned} \mathbb{H}^K_0(\Omega) &:= \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\} \\ &= \left\{ w \in \mathbb{H}^K(\mathbb{R}^n) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\} . \end{aligned}$$

Assume that $\Omega \subseteq \mathbb{R}^{2m}$ is of double revolution. Then, we define

$$\widetilde{\mathbb{H}}^K(\Omega) := \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\} .$$

and

$$\widetilde{\mathbb{H}}^K_0(\Omega) := \left\{ w \in \mathbb{H}^K_0(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\} .$$

We will add the subscript 'odd' and 'even' to these spaces to consider only functions that are odd (respectively even) with respect to the Simons cone.

Note that when K satisfies (1.4), then $\mathbb{H}^K_0(\Omega) = \mathbb{H}^\gamma_0(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(y) = |y|^{-n-2\gamma}$. Furthermore, $\mathbb{H}^\gamma(\Omega) \subset H^\gamma(\Omega)$, the usual fractional Sobolev space (see [16]). For more comments on this, see [12].

?⟨Def:Energy⟩? **Definition 3.2.** Given a kernel $K \in \mathcal{L}_0$, a potential G and a function $w \in \mathbb{H}^K(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$, we define the energy

$$\mathcal{E}(w, \Omega) := \mathcal{E}_{\mathrm{K}}(w, \Omega) + \mathcal{E}_{\mathrm{P}}(w, \Omega) \, ,$$

where

$$\mathcal{E}_{\mathrm{K}}(w, \Omega) := \frac{1}{2} [w]^2_{\mathbb{H}^K(\Omega)} \quad \text{and} \quad \mathcal{E}_{\mathrm{P}}(w, \Omega) := \int_\Omega G(w) \, .$$

For short, we will denote $\mathcal{E}(w, \mathbb{R}^n) =: \mathcal{E}(w)$. Note that, for functions $w \in \mathbb{H}^K_0(\Omega)$, $\mathcal{E}_{\mathrm{K}}(w, \Omega) = \mathcal{E}_{\mathrm{K}}(w)$. Moreover, if $G \geq 0$, the energy satisfies

$$\mathcal{E}(w, \Omega) \leq \mathcal{E}(w, \Omega') \quad \text{whenever} \quad \Omega \subseteq \Omega' \, .$$

Let us introduce a notation that will make the expression of the kinetic energy more simple. For A, $B \subseteq \mathbb{R}^n$, we define formally

$$I_w(A, B) := \int_A \int_B |w(x) - w(y)|^2 K(|x-y|) \, \mathrm{d}x \, \mathrm{d}y \, .$$

Then, if $w \in \mathbb{H}^K(\Omega)$,

$$2\mathcal{E}_{\mathrm{K}}(w,\Omega) = \frac{1}{2} I_w(\Omega,\Omega) + I_w(\Omega, \mathbb{R}^n \setminus \Omega)\,. \tag{3.1}$$

Eq:EnergyWithInt

When working with spaces of even dimension, is it also convenient to consider the following interaction. If A, $B \subseteq \mathbb{R}^{2m}$ we denote

$$I_w^\star(A,B) := \int_A \int_B |w(x) - w(y^\star)|^2 K(|x-y^\star|)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Note that if $w(x^\star) = -w(x)$, then at least at a formal level we have

$$I_w^\star(A,B) = I_w(A,B^\star) = I_w(A^\star,B)\,.$$

From now on, we always assume that $n = 2m$. The first task is to write the energy using the kernel $\overline{K}$, and writing a different expression for the Energy of doubly radial and odd functions. This is done in the following lemma.

Mejorar

sWithOverlineK⟩? **Lemma 3.3.** *Let A, $B \subseteq \mathbb{R}^{2m}$ be two domains of double revolution and let w a doubly radial function for which $I_w(A,B)$ is well defined. Then, the following statements hold:*

(1) $I_w(A,B)$ can be written as

$$I_w(A,B) = \int_A \int_B |w(x) - w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

(2) If w is odd with respect to the isometry $(\cdot)^\star$, then

$$I_w^\star(A,B) = \int_A \int_B |w(x) + w(y)|^2 \overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.$$

(3) If Ω is a set of double revolution such that $\Omega^\star = \Omega$, then

$$\mathcal{E}_{\mathrm{K}}(w,\Omega) = \frac{1}{2}\big\{ I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + I_w^\star(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O})\big\} + I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + I_w^\star(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\,.$$

Proof. Given any $R \in SO(m)^2$ we have

$$\begin{aligned} I_w(A,B) &= \int_A \int_B |w(x) - w(y)|^2 K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_A \int_B |w(R\overline{x}) - w(y)|^2 K(|R\overline{x}-y|)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_A \int_B |w(x) - w(y)|^2 K(|R\overline{x}-y|)\,\mathrm{d}x\,\mathrm{d}y\,, \end{aligned}$$

where the first equality comes from the change of variables $x = R\tilde{x}$, which is an isometry, and the second one comes from the double radial symmetry of the function w. Now, if we integrate over all the rotations in $SO(m)^2$ we get the desired result.

That is,

$$\begin{aligned}
I_w(A,B) &= \fint_{SO(m)^2} I_w(A,B)\,\mathrm{d}R \\
&= \fint_{SO(m)^2} \int_A \int_B |w(x)-w(y)|^2 K(|R\overline{x}-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}R \\
&= \int_A \int_B |w(x)-w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

The second statement follows from the relation $I_w^\star(A,B) = I_w(A,B^\star)$ and the previous computation just using the change of variables $\overline{y} = y^\star$.

Finally, the last statement follows from the expression (3.1) and the relation $I_w^\star(A,B) = I_w(A,B^\star) = I_w(A^\star,B)$. We compute

$$\begin{aligned}
I_w(\Omega,\Omega) &= I_w(\Omega,\Omega\cap\mathcal{O}) + I_w(\Omega,\Omega\cap\mathcal{I}) \\
&= I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + I_w(\Omega\cap\mathcal{I},\Omega\cap\mathcal{O}) \\
&\qquad + I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{I}) + I_w(\Omega\cap\mathcal{I},\Omega\cap\mathcal{I}) \\
&= I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{I},\Omega\cap\mathcal{O}) + I_w(\Omega\cap\mathcal{I},\Omega\cap\mathcal{I}) \\
&= I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w((\Omega\cap\mathcal{O})^\star,\Omega\cap\mathcal{O}) + I_w((\Omega\cap\mathcal{O})^\star,(\Omega\cap\mathcal{O})^\star) \\
&= I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w^\star(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) \\
&= 2I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w^\star(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O})\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_w(\Omega,\mathbb{R}^n\setminus\Omega) &= I_w(\Omega,\mathcal{O}\setminus\Omega) + I_w(\Omega,\mathcal{I}\setminus\Omega) \\
&= I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + I_w(\Omega\cap\mathcal{I},\mathcal{O}\setminus\Omega) \\
&\qquad + I_w(\Omega\cap\mathcal{O},\mathcal{I}\setminus\Omega) + I_w(\Omega\cap\mathcal{I},\mathcal{I}\setminus\Omega) \\
&= I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + I_w((\Omega\cap\mathcal{O})^\star,\mathcal{O}\setminus\Omega) \\
&\qquad + I_w(\Omega\cap\mathcal{O},(\mathcal{O}\setminus\Omega)^\star) + I_w((\Omega\cap\mathcal{O})^\star,(\mathcal{O}\setminus\Omega)^\star) \\
&= I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + I_w^\star(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) \\
&\qquad + I_w^\star(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) \\
&= 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega) + 2I_w^\star(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\,.
\end{aligned}$$

□

As a consequence of this result, if $w \in \widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\mathbb{R}^{2m})$, then

$$\mathcal{E}_{\mathrm{K}}(w) = \frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} |w(x)-w(y)|^2\overline{K}(x,y) + |w(x)+w(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.2}$$

Eq:EnergyOddInOc

The following are two lemmas regarding the decrease of the energy under some operations. They are stated for $\mathcal{E}(w)$, but the proofs are exactly the same for $\mathcal{E}(w,\Omega)$ when $\Omega^\star = \Omega$.

s1DecreaseEnergy⟩ **Lemma 3.4.** *Given* $u \in \widetilde{\mathbb{H}}^K_{\rm odd}(\mathbb{R}^{2m})$, *we define*

$$v(x) = \begin{cases} |u(x)| & \text{if } \ x \in \mathcal{O}, \\ -|u(x)| & \text{if } \ x \in \mathcal{I}. \end{cases}$$

Then

$$\mathcal{E}(v) \leq \mathcal{E}(u)$$

Proof. First, it is clear by definition that $v \in \widetilde{\mathbb{H}}^K_{\rm odd}(\mathbb{R}^{2m})$. By defining

$$\mathcal{O}_0^+ = \{x \in \mathcal{O} \ : \ u(x) \geq 0\} \quad \text{ and } \quad \mathcal{O}_0^- = \{x \in \mathcal{O} \ : \ u(x) < 0\},$$

and using the expression (3.2), we get

$$\begin{aligned}
\mathcal{E}_{\rm K}(v) &= \frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} |v(x)-v(y)|^2\overline{K}(x,y) + |v(x)+v(y)|^2\overline{K}(x^\star,y)\,{\rm d}x\,{\rm d}y \\
&= \frac{1}{2}\int_{\mathcal{O}_0^+}\int_{\mathcal{O}_0^+} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x^\star,y)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_0^+}\int_{\mathcal{O}_0^-} |u(x)+u(y)|^2\overline{K}(x,y) + |u(x)-u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_0^-}\int_{\mathcal{O}_0^+} |-u(x)-u(y)|^2\overline{K}(x,y) + |-u(x)+u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_0^-}\int_{\mathcal{O}_0^-} |-u(x)+u(y)|^2\overline{K}(x,y) + |-u(x)-u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&= \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^+)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^-)} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^-)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^+)} |u(x)+u(y)|^2\overline{K}(x,y) + |u(x)-u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{\rm K}(u) &= \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^+)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^-)} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^-)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^+)} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{E}_{\rm K}(v) - \mathcal{E}_{\rm K}(u) &= \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^-)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^+)} \left\{|u(x)+u(y)|^2 - |u(x)-u(y)|^2\right\}\overline{K}(x,y)\,{\rm d}x\,{\rm d}y \\
&\quad + \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^-)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^+)} \left\{|u(x)-u(y)|^2 - |u(x)+u(y)|^2\right\}\overline{K}(x,y^\star)\,{\rm d}x\,{\rm d}y \\
&= \frac{1}{2}\iint_{(\mathcal{O}_0^+\times\mathcal{O}_0^-)\cup(\mathcal{O}_0^-\times\mathcal{O}_0^+)} 4u(x)u(y)\left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\}\,{\rm d}x\,{\rm d}y \leq 0
\end{aligned}$$

since $u(x)u(y) < 0$ in $(\mathcal{O}_0^+ \times \mathcal{O}_0^-) \cup (\mathcal{O}_0^- \times \mathcal{O}_0^+)$ and $\overline{K}(x,y) \geq \overline{K}(x,y^\star)$ in $\mathcal{O} \times \mathcal{O}$.

Concerning the potential energy, since G is an even function we have that $\mathcal{E}_{\mathrm{P}}(v) = \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result by adding the kinetic and potential energies. $\square$

s2DecreaseEnergy⟩ **Lemma 3.5.** *Given* $u \in \widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\mathbb{R}^{2m})$, *we define*

$$v(x) = \begin{cases} \min\{1, u(x)\} & \text{if } \; x \in \mathcal{O}, \\ \max\{-1, u(x)\} & \text{if } \; x \in \mathcal{I}. \end{cases}$$

Then,

$$\mathcal{E}(v) \leq \mathcal{E}(u)\,.$$

Proof. Note that without loss of generality, by Lemma 3.4 we can assume that u is nonnegative in $\mathcal{O}$. First, it is clear by definition that $v \in \widetilde{\mathbb{H}}^K_{\mathrm{odd}}(\mathbb{R}^{2m})$. By defining

$$\mathcal{O}_1^+ = \{x \in \mathcal{O} \ : \ u(x) \geq 1\} \quad \text{ and } \quad \mathcal{O}_1^- = \{x \in \mathcal{O} \ : \ u(x) < 1\}\ ,$$

we get

$$\begin{aligned}
\mathcal{E}_{\mathrm{K}}(v) &= \frac{1}{2}\int_{\mathcal{O}}\int_{\mathcal{O}} |v(x)-v(y)|^2\overline{K}(x,y) + |v(x)+v(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^+} |1-1|^2\overline{K}(x,y) + |1+1|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^-} |1-u(y)|^2\overline{K}(x,y) + |1+u(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+} |u(x)-1|^2\overline{K}(x,y) + |u(x)+1|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^-} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^+} 4\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+} |u(x)-1|^2\overline{K}(x,y) + |u(x)+1|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^-} |u(x)-u(y)|^2\overline{K}(x,y) + |u(x)+u(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

In the last equality we have used the symmetry of $\overline{K}$ and the property $\overline{K}(x,y^\star) = \overline{K}(x^\star,y)$.

Hence, by using the expression (3.2), we see that

$$
\begin{aligned}
\mathcal{E}_{\mathrm{K}}(v)-\mathcal{E}_{\mathrm{K}}(u) &= \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\left\{|u(x)-1|^2-|u(x)-u(y)|^2\right\}\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\left\{|u(x)+1|^2-|u(x)+u(y)|^2\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^+}\left\{4-|u(x)+u(y)|^2\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^+}|u(x)-u(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\{1+u(y)-2u(x)\}\{1-u(y)\}\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\{1+u(y)+2u(x)\}\{1-u(y)\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{\mathcal{O}_1^+}\int_{\mathcal{O}_1^+}\{2+u(x)+u(y)\}\{2-u(x)-u(y)\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

On the one hand, since $1-u(y)<0$ and $1+u(y)-2u(x)=2(1-u(x))+(u(y)-1)\geq 0$ in $\mathcal{O}_1^-\times\mathcal{O}_1^+$ and $\overline{K}(x,y)\geq\overline{K}(x,y^\star)$ in $\mathcal{O}\times\mathcal{O}$ we have

$$\{1+u(y)-2u(x)\}\{1-u(y)\}\overline{K}(x,y)\leq\{1+u(y)-2u(x)\}\{1-u(y)\}\overline{K}(x,y^\star)\ \text{ in }\ \mathcal{O}_1^-\times\mathcal{O}_1^+.$$

On the other hand, since $2+u(x)+u(y)\geq 0$ in $\mathcal{O}\times\mathcal{O}$, $2-u(x)-u(y)=(1-u(x))+(1-u(y))\leq 0$ in $\mathcal{O}_1^+\times\mathcal{O}_1^+$ and $\overline{K}\geq 0$ in $\mathbb{R}^{2m}\times\mathbb{R}^{2m}$ we have

$$\{2+u(x)+u(y)\}\{2-u(x)-u(y)\}\overline{K}(x,y^\star)\leq 0\ \text{ in }\ \mathcal{O}_1^+\times\mathcal{O}_1^+.$$

Thus,

$$
\begin{aligned}
\mathcal{E}_{\mathrm{K}}(v)-\mathcal{E}_{\mathrm{K}}(u) &\leq \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\{1+u(y)-2u(x)\}\{1-u(y)\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}\{1+u(y)+2u(x)\}\{1-u(y)\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathcal{O}_1^-}\int_{\mathcal{O}_1^+}2\{1+u(y)\}\{1-u(y)\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\leq 0.
\end{aligned}
$$

Concerning the potential energy, since G is such that $G(x)\geq G(1)=G(-1)=0$ if $|x|\geq 1$, then clearly $\mathcal{E}_{\mathrm{P}}(v)\leq\mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result by adding the kinetic and potential energies. □

We will also need the following decomposition lemma

DecompositionHK⟩ **Lemma 3.6.** *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a domain of double revolution such that $\Omega = \Omega^\star$. Then, the space $\widetilde{\mathbb{H}}^K_0(\Omega)$ can be decomposed as the following direct sum:*

$$\widetilde{\mathbb{H}}^K_0(\Omega) = \widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega) \oplus \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)\,.$$

Moreover,

$$\widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega) \perp \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)\,,$$

where we are considering the spaces equipped with the scalar product

$$\langle v, w\rangle_{\widetilde{\mathbb{H}}^K_0} := \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Proof. Obviously, every function in $w \in \widetilde{\mathbb{H}}^K_0(\Omega)$ can be written as

$$w(x) = \frac{w(x)+w(x^\star)}{2} + \frac{w(x)-w(x^\star)}{2}\,,$$

and such representation is unique if we show that

$$\widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega) \cap \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega) = 0\,.$$

This will follow from the orthogonality of $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega)$ and $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$. Therefore, it only remains to prove that if $v \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$ and $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{even}}(\Omega)$, then

$$\langle v, w\rangle_{\widetilde{\mathbb{H}}^K_0} = 0\,.$$

Let us show this by direct computation. Using the change given by $(\cdot)^\star$ in the integrals in $\mathcal{I}$, the symmetries of v and w, and the properties of Lemma 2.6, we see that

$$
\begin{aligned}
2\langle v,w\rangle_{\widetilde{\mathbb{H}}_0^K} &= \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}}\int_{\mathcal{I}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{I}}\int_{\mathcal{O}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{I}}\int_{\mathcal{I}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)-v(y^\star)\big)\big(w(x)-w(y^\star)\big)\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x^\star)-v(y)\big)\big(w(x^\star)-w(y)\big)\overline{K}(x^\star,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x^\star)-v(y^\star)\big)\big(w(x^\star)-w(y^\star)\big)\overline{K}(x^\star,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)+v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)+v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\mathcal{O}}\int_{\mathcal{O}} \big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= 0\,.
\end{aligned}
$$

□

The last result we will use is the following.

llTestFunctions⟩ **Proposition 3.7.** *Let $\Omega\subset\mathbb{R}^{2m}$ be a bounded set of double revolution. Let $u\in\widetilde{\mathbb{H}}_0^K(\Omega)$ such that*

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\xi(x)-\xi(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x)\,\mathrm{d}x$$

for every $\xi\in C_0^\infty(\Omega)$ that is doubly radial. Then, u is a weak solution of

$$\left\{\begin{array}{rcll} Lu &=& f(u) & \text{in } \Omega\,, \\ u &=& 0 & \text{in } \mathbb{R}^{2m}\setminus\Omega\,, \end{array}\right.$$

i.e.,

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}^{2m}}f(u(x))\eta(x)\,\mathrm{d}x$$

for every $\eta\in C_0^\infty(\Omega)$ (not necessarily symmetric).

Proof. Let $\eta\in C_0^\infty(\Omega)$. Then, given $R\in SO(m)^2$,

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}&\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\\&=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(R^{-1}x)-u(R^{-1}y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,,\end{aligned}$$

where we have used the change $x=R\tilde{x}$, $y=R\tilde{y}$. Integrating the previous expression with respect to R and taking the average, we get

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}&\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\\&=\fint_{SO(m)^2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}R\\&=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\left\{\fint_{SO(m)^2}\eta(Rx)\,\mathrm{d}R-\fint_{SO(m)^2}\eta(Ry)\,\mathrm{d}R\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Here we have used the notation

$$\overline{\eta}(x):=\fint_{SO(m)^2}\eta(Rx)\,\mathrm{d}R\,.$$

On the other hand, using the change $x=R\tilde{x}$, we have

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x=\int_\Omega f(u(R^{-1}x))\eta(x)\,\mathrm{d}x=\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,,$$

and integrating this expression with respect to R and taking the average, we get

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x=\fint_{SO(m)^2}\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,\mathrm{d}R=\int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\,.$$

Hence, since $\overline{\eta}\in C_0^\infty(\Omega)$ is double radially symmetric, we have

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}&\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\\&=0\,,\end{aligned}$$

and thus the result is proved. $\square$

Note that in the previous result we do not need to use the kernel $\overline{K}$.

3.1. **Energy estimate.** In this section we present a sharp estimate for the energy in B_S of minimizers in the space $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ of the energy in B_R.

In order to prove this result we need to define some auxiliary functions and sets. That is,

$$\Psi_S(x) := \max\left\{-1 + 2\min\{(|x| - S - 1)_+, 1\}, -\operatorname{dist}(x, \mathscr{C})\right\},$$

if $x \in \mathcal{O}$ and odd reflected in $\mathcal{I}$,

$$d_S(x) := \max\left\{1, \min\{S + 1 - |x|, \operatorname{dist}(x, \mathscr{C})\}\right\},$$

and

$$\Omega_S := \left(B_{S+2} \setminus \overline{B_s}\right) \cup \left(B_{S+2} \cap \{\operatorname{dist}(x, \mathscr{C}) < 1\}\right).$$

First note that both Ψ_S and d_s are Lipschitz functions, with Lipschitz norm independent of S. Moreover Ψ_S is odd and d_s even with respect to the Simon's cone. On the other hand, we can see $\overline{\Omega_S}$ as the preimage of 1 through d_S inside $\overline{B_{S+2}}$.

Now we show some auxiliary results concerning the previous definitions.

onWith_dFunction⟩ **Lemma 3.8.** *Given $S > 0$, if $(x, y) \in (\Omega_S \cap \mathcal{O}) \times \mathcal{I}$ or $(x, y) \in (B_{S+2} \cap \mathcal{O}) \times \mathcal{O}$, then*

$$|\Psi_S(x) - \Psi_S(y)| \leq C\frac{|x - y|}{d_S(x)} \qquad \textit{whenever} \ \ |x - y| \leq d_S(x),$$

with $C > 0$ independent of S.

Proof. On the one hand, if $x \in \Omega_S \cap \mathcal{O}$, then $d_S(x) = 1$ and the result is trivial by the Lipschitz continuity of Ψ_S.

Hence, it only rests to show the result for the case $x \in B_S \cap \{\operatorname{dist}(x, \mathscr{C}) \geq 1\} \cap \mathcal{O}$ and $y \in \mathcal{O}$. For this case we have $\Psi_S(x) = -1$. Moreover, since $x \in B_S$ and $\operatorname{dist}(x, \mathscr{C}) \geq 1$ we get

$$d_S(x) = \min\left\{S + 1 - |x|, \operatorname{dist}(x, \mathscr{C})\right\} \leq S + 1 - |x|,$$

and therefore

$$|y| \leq |x - y| + |x| \leq d_S(x) + |x| \leq S + 1.$$

In addition, if $y \in B_{S+1} \cap \{\operatorname{dist}(x, \mathscr{C}) \geq 1\} \cap \mathcal{O}$ the $\Psi_S(y) = -1$, and the result is trivial from being also $\Psi_S(x) = -1$.

Therefore, we have proven the result for all the cases with the exception of

$$\begin{cases} x \in B_S \cap \{\operatorname{dist}(\cdot, \mathscr{C}) \geq 1\} \cap \mathcal{O} & \Rightarrow \quad \Psi(x) = -1 \\ y \in B_{S+1} \cap \{\operatorname{dist}(\cdot, \mathscr{C}) \leq 1\} \cap \mathcal{O} & \Rightarrow \quad \Psi(y) = -\operatorname{dist}(y, \mathscr{C}). \end{cases}$$

Given $x, y \in \mathbb{R}^{2m}$ it is easy to prove by using the triangular inequality and the definition of distance to the cone that

$$\operatorname{dist}(x, \mathscr{C}) \leq |x - y| + \operatorname{dist}(y, \mathscr{C}). \tag{3.3}$$

eq: tirangularCo

Therefore we have

$$1-|x-y|-\operatorname{dist}(y,\mathscr{C}) \leq 1-\operatorname{dist}(x,\mathscr{C}) \leq 0 \tag{3.4}$$

eq: tirangularCo

Now, multiplying $|1-\operatorname{dist}(y,\mathscr{C})|$ by equation (3.3) and using (3.4) we obtain

$$\begin{aligned}
|1-\operatorname{dist}(y,\mathscr{C})|\ \operatorname{dist}(x,\mathscr{C}) &\leq |1-\operatorname{dist}(y,\mathscr{C})|\,(|x-y|+\operatorname{dist}(y,\mathscr{C}))\\
&= (1-\operatorname{dist}(y,\mathscr{C}))\,(|x-y|+\operatorname{dist}(y,\mathscr{C}))\\
&= |x-y|+\operatorname{dist}(y,\mathscr{C})\,\{-|x-y|+1-\operatorname{dist}(y,\mathscr{C})\}\\
&\leq |x-y|.
\end{aligned}$$

Hence,

$$|\Psi_S(x)-\Psi_S(y)| = |1-\operatorname{dist}(y,\mathscr{C})| \leq \frac{|x-y|}{\operatorname{dist}(x,\mathscr{C})} \leq \frac{|x-y|}{d_S(x)},$$

completing the proof. □

ility_dFunction⟩ **Lemma 3.9.** *Given $S>0$ we have*

$$\int_{B_{S+2}} d_S(x)^{-2\gamma}\,\mathrm{d}x \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if}\quad \gamma\in(0,1/2),\\ C\ \log(S)\,S^{2m-2\gamma} & \text{if}\quad \gamma=1/2,\\ C\ S^{2m-1} & \text{if}\quad \gamma\in(1/2,1),\end{cases}$$

with $C>0$ independent of S and only depending on m and γ.

Proof. In order to prove this result we first note that $d_S(x)=1$ in Ω_S. Thus, the contribution to the integral of this part is just its measure, which is well known to be of order $2m-1$ (see the proof of the energy estimate in [9]). That is,

$$\int_{\Omega_S} d_S(x)^{-2\gamma}\,\mathrm{d}x \leq C\,S^{2m-1}.$$

For the other part of the integral we can write

$$\begin{aligned}
\int_{B_{S+2}\setminus\Omega} d_S(x)^{-2\gamma}\,\mathrm{d}x &= \int_{B_S\cap\operatorname{dist}\{x,\mathscr{C}\}>1} d_S(x)^{-2\gamma}\,\mathrm{d}x\\
&\leq \int_{B_S\cap\operatorname{dist}\{x,\mathscr{C}\}>1} (S+1-|x|)^{-2\gamma}\,\mathrm{d}x + \int_{B_S\cap\operatorname{dist}\{x,\mathscr{C}\}>1} \operatorname{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

Note that from the computations of Savin and Valdinocci in [30], in order to complete the proof it only remains to estimate the second integral.

This integral can be estimated by writing it in (y,z) variables, since z is the distance to the cone. That is,

$$\begin{aligned}
\int_{B_S\cap \operatorname{dist}\{x,\mathscr{C}\}>1} \operatorname{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x &\le C\int\int_{B_S\cap\{y\ge|z|>1\}} |z|^{-2\gamma}\,(y^2-z^2)^{m-1}\,\mathrm{d}y\,\mathrm{d}z\\
&\le C\int\int_{B_S\cap\{y\ge|z|>1\}} |z|^{-2\gamma}\,y^{2m-2}\,\mathrm{d}y\,\mathrm{d}z\\
&\le C\int_1^S \mathrm{d}z\int_0^S \mathrm{d}y\;z^{-2\gamma}\,y^{2m-2}\\
&\le C\left(\int_1^S z^{-2\gamma}\,\mathrm{d}z\right)\left(\int_0^S \mathrm{d}y\,y^{2m-2}\right)\\
&\le \begin{cases} C\;S^{2m-2\gamma} & \text{if}\quad \gamma\in(0,1/2),\\ C\;\log(S)\,S^{2m-2\gamma} & \text{if}\quad \gamma=1/2,\\ C\;S^{2m-1} & \text{if}\quad \gamma\in(1/2,1),\end{cases}
\end{aligned}$$

□

Escribir bien las condiciones de las funciones

Lemma 3.10. *Let $A\subset\mathbb{R}^{2m}$ be a set of double revolution such that $A^\star=A$ and let be $\omega,\phi,\varphi\in$ such that*

$$\begin{cases}\omega=\phi\le\varphi & \text{in}\quad \mathcal{O}\setminus A\,,\\ \omega=\varphi\le\phi & \text{in}\quad \mathcal{O}\cap A\,.\end{cases}$$

Then,

$$\begin{aligned}
I_\omega(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I^\star_\omega(\mathcal{O}\cap A,\mathcal{O}\setminus A) \le{}& I_\phi(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I^\star_\phi(\mathcal{O}\cap A,\mathcal{O}\setminus A)\\
&+I_\varphi(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I^\star_\varphi(\mathcal{O}\cap A,\mathcal{O}\setminus A)
\end{aligned}$$

Proof. Let us begin by proving that given $x\in\mathcal{O}\cap A$ and $y\in\mathcal{O}\setminus A$ we have that

$$|\phi(x)-\phi(y)|^2+|\varphi(x)-\varphi(y)|^2\ge|\omega(x)-\omega(y)|^2.$$

That is,

$$\begin{aligned}
|\phi(x)-\phi(y)|^2&+|\varphi(x)-\varphi(y)|^2-|\omega(x)-\omega(y)|^2\\
&=|\phi(x)-\phi(y)|^2+|\varphi(x)-\varphi(y)|^2-|\varphi(x)-\phi(y)|^2\\
&=\phi^2(x)-2\phi(x)\phi(y)+\varphi^2(y)-2\varphi(x)\varphi(y)+2\varphi(x)\phi(y)\\
&=(\phi(x)-\varphi(y))^2+2\,(\phi(x)-\varphi(x))\,(\varphi(y)-\phi(y))\\
&\ge 0.
\end{aligned}$$

Therefore, by using this inequality and the kernel's reflexion property (Proposition 2.7) we obtain

Hay que alinear bien las ecuaciones

$$
\begin{aligned}
I_\phi(\mathcal{O}\cap A, \mathcal{O}\setminus A) &+ I_\phi^\star(\mathcal{O}\cap A, \mathcal{O}\setminus A) + I_\varphi(\mathcal{O}\cap A, \mathcal{O}\setminus A) + I_\varphi^\star(\mathcal{O}\cap A, \mathcal{O}\setminus A) \\
&\qquad - I_\omega(\mathcal{O}\cap A, \mathcal{O}\setminus A) - I_\omega^\star(\mathcal{O}\cap A, \mathcal{O}\setminus A) \\
&= \int_{\mathcal{O}\cap A} \mathrm{d}x \int_{\mathcal{O}\setminus A} \mathrm{d}y \, \Big[\big\{|\phi(x)-\phi(y)|^2 + |\varphi(x)-\varphi(y)|^2 - |\omega(x)-\omega(y)|^2\big\}\, \overline{K}(x,y) \\
&\qquad + \big\{|\phi(x)+\phi(y)|^2 + |\varphi(x)+\varphi(y)|^2 - |\omega(x)+\omega(y)|^2\big\}\, \overline{K}(x,y^\star)\Big] \\
&\geq \int_{\mathcal{O}\cap A} \mathrm{d}x \int_{\mathcal{O}\setminus A} \mathrm{d}y \, \big\{|\phi(x)-\phi(y)|^2 + |\varphi(x)-\varphi(y)|^2 - |\omega(x)-\omega(y)|^2 \\
&\qquad + |\phi(x)+\phi(y)|^2 + |\varphi(x)+\varphi(y)|^2 - |\omega(x)+\omega(y)|^2\big\}\, \overline{K}(x,y^\star) \\
&= \int_{\mathcal{O}\cap A} \mathrm{d}x \int_{\mathcal{O}\setminus A} \mathrm{d}y \, \big\{2\phi^2(x) + 2\varphi^2(y)\big\}\, \overline{K}(x,y^\star) \geq 0.
\end{aligned}
$$

Here we have used that $\omega(x) = \varphi(x)$ if $x \in \mathcal{O}\cap A$ and that $\omega(y) = \phi(y)$ if $y \in \mathcal{O}\setminus A$. □

n:EnergyEstimate⟩ **Theorem 3.11.** *Let u be a minimizer of the non-local Allen-Cahn energy in B_R, with $R > S + 2$, among functions that are doubly radial, odd with respect to the Simon's cone and zero outside B_R. Then*

$$
\mathcal{E}(u, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0, 1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2, 1), \end{cases}
$$

with C a positive constant depending only on m, γ, Λ and G.

Proof. Note that, by Lemmas 3.4 and 3.5 we can assume without loss of generality that $-1 \leq u \leq 1$ and that $u \geq 0$ in $\mathcal{O}$ and $u \leq 0$ in $\mathcal{I}$. In fact, it also true that $0 \leq u < 1$ in $\mathcal{O}$. In order to prove it we first need to show that u is a weak solution of

$$
\left\{ \begin{array}{rcll} Lu & = & f(u) & \text{in } B_R\,, \\ u & = & 0 & \text{in } \mathbb{R}^{2m} \setminus B_R. \end{array} \right. \tag{3.5}
$$

Eq:ProofExistenc

To see this, we consider on the one hand perturbations $u + \varepsilon\xi$, with $\xi \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ and such that ξ has compact support in B_R. Then,

$$
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u + \varepsilon\xi, B_R) = \langle u, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} - \langle f(u), \xi\rangle_{L^2(B_R)}\,.
$$

On the other hand, take $\xi \in \widetilde{\mathbb{H}}^K_{0,\mathrm{even}}(B_R)$. Since u is odd with respect to the Simons cone, so is $f(u)$. Then, by Lemma 3.6 and the same decomposition in $L^2(B_R)$, we find that

$$
\langle v_R, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} = 0 \quad \text{and} \quad \langle f(v_R), \xi\rangle_{L^2(B_R)} = 0\,.
$$

Therefore, we have that

$$
\langle u, \xi\rangle_{\widetilde{\mathbb{H}}^K_0(B_R)} = \langle f(u), \xi\rangle_{L^2(B_R)}
$$

for every $\xi \in \widetilde{\mathbb{H}}_0^K(B_R)$ with compact support in B_R. Therefore,

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\xi(x) - \xi(y)\} K(|x-y|) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x) \, \mathrm{d}x$$

for every $\xi \in C_0^\infty(\Omega)$ that is double radially symmetric.

By Proposition 3.7, u is a weak solution of (4.1), and by the regularity result of Corollary 2.2, since u is bounded, it is a classical solution.

From being u a classical solution it is easy to show that it cannot be 1 or -1 and therefore that it satisfies $0 \le u < 1$ in $\mathcal{O}$. That is, let us suppose that there exists $x_0 \in \mathbb{R}^{2m}$ such that $|u(x_0)| = 1$. It is clear that we can take $x_0 \in \mathcal{O} \cap B_R$. Then, from equation (4.1) and the fact of being x_0 an absolute maximum we can arrive at a contradiction:

$$\begin{aligned}
0 = f(1) = f(u(x_0)) = Lu(x_0) &= \int_{\mathcal{O}} (1 - u(y))\overline{K}(x,y) + (1 + u(y))\overline{K}(x,y^\star) \, \mathrm{d}y \\
&\ge \int_{\mathcal{O}} (1 - u(y))\overline{K}(x,y^\star) + (1 + u(y))\overline{K}(x,y^\star) \, \mathrm{d}y = 2 \int_{\mathcal{O}} \overline{K}(x,y^\star) \, \mathrm{d}y \\
&> 0.
\end{aligned}$$

Now we introduce the function

$$v(x) := \min\{u(x), \Psi_S(x)\},$$

if $x \in \mathcal{O}$ and odd reflected in $\mathcal{I}$. Since both u and v are equal outside $B_{S+2} \subset B_R$, v is going to be the competitor that will give us the energy estimate. Let us also define

$$A = \{v = \Psi_S\}.$$

Then it is easy to check that we have the inclusions

$$B_{S+1} \subseteq A \subseteq B_{S+2}.$$

Since A is symmetric with respect to the Simon's cone we only need to prove it inside $\mathcal{O}$. That is, on the one hand

$$x \in B_{S+1} \cap \mathcal{O} \Rightarrow \Psi_S(x) = \max\{-1, -\operatorname{dist}(x, \mathscr{C})\} \le 0 \le u(x) \Rightarrow v(x) = \Psi_S(x) \Rightarrow x \in A \cap \mathcal{O},$$

and on the other hand

$$x \in A \cap \mathcal{O} \Rightarrow \Psi_S(x) \le u(x) < 1 \Rightarrow x \in B_{S+2}.$$

Let us decompose the energy of u in B_R in terms of interactions between sets that involve A. That is,

$$\begin{aligned}\mathcal{E}(u,B_R) = &\frac{1}{2}I_u^\star(\mathcal{O}\cap A,\mathcal{O}\cap A)+\frac{1}{2}I_u(\mathcal{O}\cap A,\mathcal{O}\cap A)\\&+I_u^\star(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I_u(\mathcal{O}\cap A,\mathcal{O}\setminus A)\\&+\frac{1}{2}I_u^\star(\mathcal{O}\setminus A\cap B_R,\mathcal{O}\setminus A\cap B_R)+\frac{1}{2}I_u(\mathcal{O}\setminus A\cap B_R,\mathcal{O}\setminus A\cap B_R)\\&+I_u^\star(\mathcal{O}\setminus A\cap B_R,\mathcal{O}\setminus B_R)+I_u(\mathcal{O}\setminus A\cap B_R,\mathcal{O}\setminus B_R)\\&+\int_A G(u)+\int_{B_R\setminus A}G(u)\end{aligned}$$

Since u is a minimizer, $v=\Psi_S$ in A and $u=v$ out of A, we obtain from the previous expression

$$\begin{aligned}0&\le\mathcal{E}(v,B_R)-\mathcal{E}(u,B_R)\\&=\frac{1}{2}I^+_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\cap A)+\frac{1}{2}I^-_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\cap A)\\&\quad-\frac{1}{2}I^+_u(\mathcal{O}\cap A,\mathcal{O}\cap A)-\frac{1}{2}I^-_u(\mathcal{O}\cap A,\mathcal{O}\cap A)\\&\quad+I^+_v(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I^-_v(\mathcal{O}\cap A,\mathcal{O}\setminus A)\\&\quad-I^+_u(\mathcal{O}\cap A,\mathcal{O}\setminus A)-I^-_u(\mathcal{O}\cap A,\mathcal{O}\setminus A)\\&\quad+\int_A G(\Psi_S)-\int_A G(u)\end{aligned}$$

Since $v=\min\{u,\Psi_S\}$ in $\mathcal{O}$ we can apply Lemma 3.10 to obtain

$$\begin{aligned}&\frac{1}{2}I_u^\star(\mathcal{O}\cap A,\mathcal{O}\cap A)+\frac{1}{2}I_u(\mathcal{O}\cap A,\mathcal{O}\cap A)+\int_A G(u)\\&\qquad\le\frac{1}{2}I^\star_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\cap A)+\frac{1}{2}I_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\cap A)\\&\qquad\quad+I^\star_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\setminus A)+I_{\Psi_S}(\mathcal{O}\cap A,\mathcal{O}\setminus A)+\int_A G(\Psi_S)\\&\qquad=\mathcal{E}(\Psi_S,A)\le\mathcal{E}(\Psi_S,B_{S+2})\end{aligned}$$

Therefore we get an estimate of the energy of u in B_S. That is,

$$\begin{aligned}\mathcal{E}(u,B_S)&\le\frac{1}{2}I_u^\star(\mathcal{O}\cap A,\mathcal{O}\cap A)+\frac{1}{2}I_u(\mathcal{O}\cap A,\mathcal{O}\cap A)+\int_A G(u)\\&\quad+I_u^\star(\mathcal{O}\setminus B_{S+1},\mathcal{O}\cap B_S)+I_u(\mathcal{O}\setminus B_{S+1},\mathcal{O}\cap B_S)\\&\le I_u^\star(\mathcal{O}\setminus B_{S+1},\mathcal{O}\cap B_S)+I_u(\mathcal{O}\setminus B_{S+1},\mathcal{O}\cap B_S)\\&\quad+\mathcal{E}(\Psi_S,B_{S+2})\end{aligned}$$

Once we are at this point we only have to bound this three terms in order to obtain the desired energy estimate.

- Estimate for $\mathcal{E}(\Psi_S, B_{S+2})$
  In order to make this estimate we use the definition of the energy that involves the original kernel K and not the adapted one $\overline{K}$. That is,

$$
\begin{aligned}
\mathcal{E}(\Psi_S, B_{S+2}) &= \frac{1}{4}\int_{B_{S+2}}\int_{B_{S+2}} |\Psi_S(x)-\Psi(y)|^2 K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{B_{S+2}}\int_{\mathbb{R}^{2m}\setminus B_{S+2}} |\Psi_S(x)-\Psi(y)|^2 K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{S+2}} G(\Psi_S) \\
&\le \frac{1}{2}\int_{B_{S+2}}\int_{\mathbb{R}^{2m}} |\Psi_S(x)-\Psi(y)|^2 K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{S+2}} G(\Psi_S) \\
&= \int_{\mathcal{O}\cap B_{S+2}}\int_{\mathbb{R}^{2m}} |\Psi_S(x)-\Psi(y)|^2 K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{S+2}} G(\Psi_S),
\end{aligned}
$$

where last inequality comes from making the change of variables $x' = x^\star$ and $y' = y^\star$ and the fact that $|x-y| = |x^\star - y^\star|$. Now, by using the ellipticity condition:

$$
\begin{aligned}
\mathcal{E}(\Psi_S, B_{S+2}) &\le \Lambda \int_{\mathcal{O}\cap B_{S+2}}\int_{\mathbb{R}^{2m}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{S+2}} G(\Psi_S) \\
&= \Lambda \int_{\mathcal{O}\cap B_{S+2}}\int_{\mathcal{O}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \Lambda \int_{\Omega\cap\mathcal{O}}\int_{\mathcal{I}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \Lambda \int_{(B_{S+2}\setminus\Omega)\cap\mathcal{O}}\int_{\mathcal{I}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{S+2}} G(\Psi_S) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Let us compute this four integrals:

$$
\begin{aligned}
I_1 &= \Lambda \int_{\mathcal{O}\cap B_{S+2}} \int_{\mathcal{O}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \Lambda \int_{\mathcal{O}\cap B_{S+2}} \int_{\mathcal{O}\cap\{|x-y|\le d_S(x)\}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \Lambda \int_{\mathcal{O}\cap B_{S+2}} \int_{\mathcal{O}\cap\{|x-y|\ge d_S(x)\}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C \int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2} \int_{\mathcal{O}\cap\{|x-y|\le d_S(x)\}} |x-y|^{2-n-2\gamma}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + C \int_{\mathcal{O}\cap B_{S+2}} \int_{\mathcal{O}\cap\{|x-y|\ge d_S(x)\}} |x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C \int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2} \int_0^{d_S(x)} \rho^{1-2\gamma}\,\mathrm{d}\rho\,\mathrm{d}x + C \int_{\mathcal{O}\cap B_{S+2}} \mathrm{d}x \int_{d_S(x)}^{\infty} \rho^{-1-2\gamma}\,\mathrm{d}\rho \\
&\le C \int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2\gamma}\,\mathrm{d}x,
\end{aligned}
$$

where in the first inequality we have used Lemma 3.8 and the uniform bound of Ψ_S. The bound of I_2 is essentially the same. That is,

$$
\begin{aligned}
I_2 &= \Lambda \int_{\Omega\cap\mathcal{O}} \int_{\mathcal{I}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \Lambda \int_{\Omega\cap\mathcal{O}} \int_{\mathcal{I}\cap\{|x-y|\le d_S(x)\}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \Lambda \int_{\Omega\cap\mathcal{O}} \int_{\mathcal{I}\cap\{|x-y|\ge d_S(x)\}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le C \int_{\Omega\cap\mathcal{O}} d_S(x)^{-2} \int_0^{d_S(x)} \rho^{1-2\gamma}\,\mathrm{d}\rho\,\mathrm{d}x + C \int_{\Omega\cap\mathcal{O}} \mathrm{d}x \int_{d_S(x)}^{\infty} \rho^{-1-2\gamma}\,\mathrm{d}\rho \\
&\le C \int_{\Omega\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x \le C \int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}
$$

For the case of I_3 we use the fact that given $x \in (B_{S+2}\backslash\Omega)\cap\mathcal{O}$ then $\operatorname{dist}(x,\mathscr{C}) \geq d_S(x)$ and therefore $\mathcal{I} \subset \mathbb{R}^{2m} \setminus B_{d_S(x)}(x)$.

$$\begin{aligned}
I_3 &= \Lambda \int_{(B_{S+2}\backslash\Omega)\cap\mathcal{O}} \int_{\mathcal{I}} \frac{|\Psi_S(x)-\Psi(y)|^2}{|x-y|^{n+2\gamma}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq C \int_{(B_{S+2}\backslash\Omega)\cap\mathcal{O}} \int_{\mathbb{R}^{2m}\backslash B_{d_S(x)}(x)} |x-y|^{-n-2\gamma} \\
&\leq C \int_{\mathcal{O}\cap B_{S+2}} \int_{d_S(x)}^{\infty} \rho^{-1-2\gamma}\,\mathrm{d}\rho\,\mathrm{d}x \\
&\leq C \int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

Now, for the case of I_4, since $\Psi_S \equiv -1$ in Ω_S we have

$$I_4 = \int_{B_{S+2}} G(\Psi_S) = \int_{\Omega_S} G(\Psi_S) + \int_{B_{S+2}\backslash\Omega_S} G(\Psi_S) \leq C|B_{S+2}\setminus\Omega_S| \leq C\,S^{2m-1}$$

Then, we obtain

$$\begin{aligned}
\mathcal{E}(\Psi_S, B_{S+2}) &\leq C\left(\int_{\mathcal{O}\cap B_{S+2}} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1}\right) \\
&\leq C\left(\int_{\mathcal{O}\cap B_S} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1}\right)
\end{aligned}$$

- Estimate for $I_u(\mathcal{O}\setminus B_{S+1}, \mathcal{O}\cap B_S) + I_u^\star(\mathcal{O}\setminus B_{S+1}, \mathcal{O}\cap B_S)$ First we prove that if $x \in B_S\cap\mathcal{O}$ and $y \in \mathbb{R}^{2m}\setminus B_{S+1}$, then $|x-y| \geq d_S(x)$. It is clear that being $x \in B_S$ then $d_S(x) \leq S+1-|x|$ and therefore we have $|x-y| \geq |y|-|x| \geq |y| + d_S(x) - S - 1 \geq d_S(x)$. Thus we have

$$\begin{aligned}
&I_u(\mathcal{O}\setminus B_{S+1}, \mathcal{O}\cap B_S) + I_u^\star(\mathcal{O}\setminus B_{S+1}, \mathcal{O}\cap B_S) \\
&\qquad = \int_{\mathcal{O}\cap B_S} \mathrm{d}x \int_{\mathbb{R}^{2m}\backslash B_{S+1}} \mathrm{d}y\; |u(x)-u(y)|^2\,K(|x-y|) \\
&\qquad \leq \int_{\mathcal{O}\cap B_S} \mathrm{d}x \int_{\mathbb{R}^{2m}\backslash B_{S+1}} \mathrm{d}y\; \frac{|u(x)-u(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\qquad \leq \int_{\mathcal{O}\cap B_S} \mathrm{d}x \int_{|x-y|\geq d_S(x)} \mathrm{d}y\; |x-y|^{-2m-2\gamma} \\
&\qquad \leq C \int_{\mathcal{O}\cap B_S} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

Finally, by adding up this estimates and applying Lemma 3.9 we finally obtain the desired result. That is,

$$\begin{aligned}
\mathcal{E}(u, B_S) &\leq \mathcal{E}(\Psi_S, B_{S+2}) + I_u(\mathcal{O} \setminus B_{S+1}, \mathcal{O} \cap B_S) + I_u^\star(\mathcal{O} \setminus B_{S+1}, \mathcal{O} \cap B_S) \\
&\leq C \left(\int_{\mathcal{O} \cap B_S} d_S(x)^{-2\gamma} \, \mathrm{d}x + S^{2m-1} \right) \\
&\leq \begin{cases} C \; S^{2m-2\gamma} & \text{if} \quad \gamma \in (0, 1/2), \\ C \; \log(S) \, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C \; S^{2m-1} & \text{if} \quad \gamma \in (1/2, 1). \end{cases}
\end{aligned}$$

$\square$

4. Existence of the saddle-shaped solution

?⟨Sec:Existence⟩? We can now proceed with the proof of the existence of saddle-shaped solutions.

Proof of Theorem 1.9. Since $\mathcal{E}(w, B_R)$ is bounded below —by 0—, we can take a minimizing sequence $v_R^j \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Note that, by Lemmas 3.4 and 3.5 we can assume that $-1 \leq v_R^j \leq 1$ and that $v_R^j \geq 0$ in $\mathcal{O}$ and $v_R^j \leq 0$ in $\mathcal{I}$.

Now, using (1.4), $G \geq 0$ and the fact that v_R^j is a minimizing sequence, we deduce that

$$[v_R^j]_{H^s(B_R)} \leq \frac{c_{n,s}}{\lambda}[v_R^j]_{\mathbb{H}^K(B_R)} \leq \frac{2c_{n,s}}{\lambda}\mathcal{E}(v_R^j, B_R) \leq C$$

for a constant C that does not depend on j. Therefore, $\{v_R^j\}$ is bounded in $H^s(B_R)$ and then, by the compact embedding $H^s(B_R) \subset\subset L^2(B_R)$ (see Theorem 7.1 of [16]), there exists a subsequence, still denoted by v_R^j, that converges to some $v_R \in L^2(B_R)$, and thus, a.e. in B_R. By Fatou's lemma, we have

$$\mathcal{E}(v_R, B_R) \leq \liminf_{j\to\infty} \mathcal{E}(v_R^j, B_R) = \inf\left\{\mathcal{E}(w, B_R) \ : \ w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)\right\}.$$

Therefore, $v_R \in \widetilde{\mathbb{H}}^K_0(B_R)$ is a minimizer of $\mathcal{E}(\cdot, B_R)$ in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Moreover, it satisfies $-1 \leq v_R \leq 1$ in B_R, $v_R \geq 0$ in $\mathcal{O}$, $v_R(x) = -v_R(Sx)$ for every $x \in \mathbb{R}^{2m}$ and $v_R \equiv 0$ in $\mathbb{R}^{2m} \setminus B_R$.

Arguing exactly as in the proof of Theorem 3.11, we deduce that v_R is a classical solution of

$$\left\{\begin{array}{rcll} Lv_R & = & f(v_R) & \text{in } B_R\,, \\ v_R & = & 0 & \text{in } \mathbb{R}^{2m}\setminus B_R. \end{array}\right. \tag{4.1}$$ Eq:ProofExistenc

The next step is to make $R \to \infty$.

compactness argument

Therefore, we have u a solution of $Lu = f(u)$ in $\mathbb{R}^{2m}$ which is doubly radial. Furthermore, u is odd with respect to the Simons cone $\mathscr{C}$, i.e., $u(x) = -u(x^\star)$ for $x \in \mathbb{R}^{2m}$, and $0 \leq u \leq 1$ in $\mathcal{O}$.

Finally, we show that $0 < u < 1$ in $\mathcal{O}$. This will ensure that u is a saddle-shaped solution. First, note that $|u| < 1$ by the strong maximum principle (since u vanishes at $\mathscr{C}$, $u \not\equiv 1$ and $u \not\equiv -1$). Let us show that $u \not\equiv 0$. To do this, we use the energy estimate of Theorem 3.11. That is, if we consider u_R the minimizer of $\mathcal{E}(\cdot, B_R)$ with $R > 2$, we have

$$\mathcal{E}(u_R, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

for every $0 < S < R-2$ and with a constant C independent of R and S. By letting $R \to \infty$ in the estimate we get

$$\mathcal{E}(u, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

for every $S > 0$. and with a constant C independent of S. By contradiction, assume $u \equiv 0$. Then, the previous estimate leads to

$$c_m G(0) S^{2m} = \mathcal{E}(0, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ \log(S)\, S^{2m-2\gamma} & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

and this is a contradiction for S large enough. Therefore, since $u \not\equiv 0$, the strong maximum principle for odd functions (see Proposition B.3) yields that $u > 0$ in $\mathcal{O}$. □

4.1. **Alternative proof of Theorem 1.9.** First, we present a version of the monotone iteration procedure for doubly radial functions which are odd with respect to the Simons Cone $\mathscr{C}$.

oneIterationOdd⟩ **Proposition 4.1.** *Let $\underline{u} \leq \overline{u}$ be two bounded functions that are doubly radial and odd with respect to the Simons cone. Let L be an operator ... and assume that* Poner las condiciones mínimas

$$\begin{cases} L\overline{u} \geq f(\overline{u}) & \text{in } B_R \cap \mathcal{O}\,, \\ \overline{u} \geq \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases} \quad \text{and} \quad \begin{cases} L\underline{u} \leq f(\underline{u}) & \text{in } B_R \cap \mathcal{O}\,, \\ \underline{u} \leq \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases}$$

with f an odd C^2 function and φ a doubly radial function satisfying $\varphi(x) = -\varphi(Sx)$.

Then, there exists $u \in C^2(B_R)$ a solution of

$$\begin{cases} Lu = f(u) & \text{in } B_R\,, \\ u = \varphi & \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases}$$

such that u is doubly radial, odd with respect to the Simons cone and $\underline{u} \leq u \leq \overline{u}$ in $\mathcal{O}$.

Proof. The proof follows the classical monotone iteration method for elliptic equations (see for instance [18]). We just give here a sketch of the proof. First, let $M \geq 0$ be such that $-M \leq \underline{u} \leq \overline{u} \leq M$ and set

$$\Lambda := \max\left\{0, -\min_{[-M,M]} f'\right\} \geq 0\,.$$

Then one defines

$$\mathscr{L}w := Lw + \Lambda w$$

and

$$g(\theta) = f'(\theta) + \Lambda\theta\,.$$

Therefore, our problem is equivalent to find a solution to

$$\begin{cases} \mathscr{L}u = g(u) & \text{in } B_R\,, \\ u = \varphi & \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases}$$

such that u is doubly radial, odd with respect to the Simons cone and $\underline{u} \leq u \leq \overline{u}$ in $\mathcal{O}$. Here the main point is that g is also odd but satisfies $g'(\theta) \geq$ for $\theta \geq 0$. Moreover, since $\Lambda \geq 0$, then $\mathscr{L}$ satisfies the maximum principle for odd functions in $\mathcal{O}$ (see Proposition 2.9).

We define $u_0 = \underline{u}$ and, for $k \geq 1$, u_k as the solution to the linear problem

$$\begin{cases} \mathscr{L} u_k &= g(u_{k-1}) \quad \text{in } B_R\,, \\ \quad u_k &= \varphi \qquad\qquad \text{in } \mathbb{R}^{2m} \setminus B_R\,. \end{cases}$$

Them, using the maximum principle it is not difficult to show by induction that

$$\underline{u} = u_0 \leq u_1 \leq \ldots \leq u_k \leq u_{k+1} \leq \ldots \overline{u} \quad \text{ in } \mathcal{O}\,,$$

and that each function u_k is doubly radial and odd with respect to $\mathscr{C}$. Finally, by a compactness argument we see that, up to a subsequence, u_k converges in C^2 to the desired solution. □

We also need a characterization and some properties of the first odd eigenfunction and eigenvalue for the operator L.

ddEigenfunction⟩ **Lemma 4.2.** *Let Ω be a bounded subset of $\mathbb{R}^{2m}$ that is of double revolution and let $L \in \mathcal{L}_0$. Then, there exists an eigenvalue $\lambda_{1,\,\mathrm{odd}}(\Omega, L) > 0$ such that*

(1) The eigenvalue $\lambda_{1,\,\mathrm{odd}}(\Omega, L)$ is characterized by

$$\lambda_{1,\,\mathrm{odd}}(\Omega, L) = \min_{w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x) - w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{\Omega} w(x)^2\,\mathrm{d}x}\,. \tag{4.2}$$ Eq:DefLambda1

(2) There exists an eigenfunction $\phi_1 \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$, attaining the minimum in (4.2) and such that

$$\begin{cases} L\phi_1 &= \lambda_{1,\,\mathrm{odd}}(\Omega, L)\phi_1 \quad \text{in } \Omega\,, \\ \quad \phi_1 &= 0 \qquad\qquad\qquad\quad \text{in } \mathbb{R}^{2m} \setminus \Omega\,. \end{cases}$$

Moreover, $\phi \geq 0$ in $\mathcal{O}$ and $\phi > 0$ in $\Omega \cap \mathcal{O}$.

(3) There exists a constant C depending only on n, γ and Λ such that

$$\lambda_{1,\,\mathrm{odd}}(B_R, L) \leq C R^{-2\gamma}\,.$$

Proof. The first two statements are deduced exactly as in Proposition 9 of [33], with the help of Lemma 3.5 to guarantee that ϕ_1 is nonnegative in $\mathcal{O}$. The fact that $\phi > 0$ in $\Omega \cap \mathcal{O}$ follows from the strong maximum principle (see Proposition 2.10)

We show the third statement. Let $\widetilde{w}(x) := w(Rx)$ for every $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Then,

$$\begin{aligned}
&\min_{w\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} w(x)^2\,\mathrm{d}x} \\
&\qquad = \min_{\widetilde{w}\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x/R)-\widetilde{w}(y/R)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} \widetilde{w}(x/R)^2\,\mathrm{d}x} \\
&\qquad \le \min_{\widetilde{w}\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x/R)-\widetilde{w}(y/R)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} \widetilde{w}(x/R)^2\,\mathrm{d}x} \\
&\qquad = R^{-2\gamma}\min_{\widetilde{w}\in\widetilde{\mathbb{H}}^s_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x)-\widetilde{w}(y)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_1} \widetilde{w}(x)^2\,\mathrm{d}x} \\
&\qquad = \lambda_{1,\,\mathrm{odd}}(B_1,(-\Delta)^\gamma)\Lambda R^{-2\gamma}\,.
\end{aligned}$$

□

With these ingredients, we can proceed with the alternative argument to show Theorem 1.9.

Alternative proof of Theorem 1.9. The strategy is to build a suitable solution u_R of

$$\left\{\begin{array}{rcll} Lu_R &=& f(u_R) & \text{in } B_R\,, \\ u_R &=& 0 & \text{in } \mathbb{R}^{2m}\setminus B_R\,, \end{array}\right. \tag{4.3}$$

Eq:ProofExistenc

and then let $R\to\infty$ to get a saddle-shaped solution.

Let ϕ_1 be the first odd eigenfunction of L in $B_R\subset\mathbb{R}^{2m}$ given by Lemma 4.2 and $\lambda_1 := \lambda_{1,\,\mathrm{odd}}(B_R)$. Let $\underline{u}_R := \varepsilon\phi_1$. We claim that for R big enough and ε small enough, $\underline{u}_R$ is a subsolution of (4.3). To see this, first note that without loss of generality, we can assume that $||\phi_1||_{L^\infty(B_R)} = 1$. Then, since $\varepsilon\phi_1 > 0$ in B_R and using (1.8), we see that for every $x\in B_R$,

$$\frac{f(\varepsilon\phi_1(x))}{\varepsilon\phi_1(x)} > f'(\varepsilon\phi_1(x)) \ge f'(0)/2 > 0$$

if ε is small enough, independently of $x\in B_R$. Therefore, taking R big enough so that $\lambda_1 < f'(0)/2$ (see point (3) of Lemma 4.2), we have that for every $x\in B_R$, $f(\varepsilon\phi_1(x)) > \lambda_1\varepsilon\phi_1(x)$ and thus

$$L(\underline{u}_R) = \lambda_1\varepsilon\phi_1 < f(\varepsilon\phi_1) = f(\underline{u}_R) \quad \text{in } B_R\,.$$

Now, let $\overline{u}_R := \chi_{\mathcal{O}\cap B_R} - \chi_{\mathcal{I}\cap B_R}$, which is a supersolution of (4.1). Therefore, using the monotone iteration procedure (see Proposition 4.1), we obtain a solution u_R of (4.1) such that it is doubly radial, odd with respect to the Simons cone and $\underline{u}_R \leq u_R \leq \overline{u}_R$ in $\mathcal{O}$. Note that, since $\underline{u}_R > 0$ in $\mathcal{O} \cap B_R$, so is u_R.

The next step is to pass to the limit in R to obtain a solution in $\mathbb{R}^{2m}$. Let $S > 0$ and $T = 4\lceil 1/\gamma \rceil$ and consider the family $\{u_R\}$, for $R > S + T$, of solutions in B_{S+T}. By applying the estimate (2.6) in balls of radius 1 and centered at points in $\overline{B_S}$, we obtain a uniform $C^{2,\alpha}(\overline{B_S})$ bound for u_R. By the Arzela-Ascoli theorem, a subsequence of $\{u_R\}$ converges in $C^2(\overline{B_S})$ to a solution in B_S. Taking now $S = 1, 2, 3, \ldots$ and using a diagonal argument, we obtain a sequence u_{R_j} converging in $C^2_{\mathrm{loc}}(\mathbb{R}^{2m})$ to a solution $u \in C^2(\mathbb{R}^{2m})$ of (1.1).

Therefore, u is a solution of $Lu = f(u)$ in $\mathbb{R}^{2m}$ such that u is doubly radial, odd with respect to the Simons cone and $0 \leq u \leq 1$ in $\mathcal{O}$. Let us show that, indeed, $0 < u < 1$ in $\mathcal{O}$ and hence u is a saddle-shaped solution. The fact that $u > 0$ in $\mathcal{O}$ follows directly from the monotone iteration procedure, since $u_R > 0$ in $\mathcal{O} \cap B_R$ and $u_{2R} \geq u_R$ in $\mathcal{O}$ —that is, u_R is a subsolution for the problem in B_{2R}. On the other hand, $u < 1$ in $\mathcal{O}$ by the strong maximum principle (see Proposition 2.10).

□

5. Symmetry results

ymmetryResults⟩? This section is devoted to prove the following two symmetry results. The first one is a result for positive solutions in the whole space

metryWholeSpace⟩ **Theorem 5.1.** *Let L be an integro-differential operator with kernel K satisfying 99. Let u be a bounded solution to*

$$\begin{cases} Lu = f(u) & \text{in } \mathbb{R}^n\,, \\ u \geq 0 & \text{in } \mathbb{R}^n\,, \end{cases} \tag{5.1}$$

?Eq:PositiveWhol

with the nonlinearity $f \in C^1$ satisfying

- $f(0) = f(1) = 0$,
- $f'(0) > 0$,
- $f > 0$ *in* $(0,1)$, *and*
- $f < 0$ *in* $(1, +\infty)$.

Then, $u \equiv 0$ or $u \equiv 1$.

The second one is a symmetry result for equations in a half-space.

h:SymmHalfSpace⟩ **Theorem 5.2.** *Let L be an integro-differential operator with kernel K satisfying 99. Let u be a bounded solution to one of these two problems*

(P3) Eq:P3
$$\begin{cases} Lu = f(v) & \text{in } \mathbb{R}^n_+, \\ v > 0 & \text{in } \mathbb{R}^n_+, \\ v = 0 & \text{in } \overline{\mathbb{R}^n_-}, \end{cases}$$

(P4) Eq:P4
$$\begin{cases} Lv = f(v) & \text{in } \mathbb{R}^n_+, \\ v > 0 & \text{in } \mathbb{R}^n_+, \\ v(x', x_n) = -v(x', -x_n) & \text{in } \mathbb{R}^n. \end{cases}$$

Assume that the kernel K of the integral operator L satisfies

$$K(x-y) \geq K(x-y^*) \quad \text{for all} \quad x, y \in \mathbb{R}^n_+,$$

where y^* is the reflection of y with respect to $\{x_n = 0\}$. Suppose that the nonlinearity f is Lipsitchz and

- $f(0) = f(1) = 0$,
- $f'(0) > 0$, *and* $f'(t) \leq 0$ *for all* $t \in [1-\delta, 1]$ *for some* $\delta > 0$,
- $f > 0$ *in* $(0,1)$, *and*
- f *is odd in the case of* (P4).

Then, v depends only on x_n and it is increasing in that direction.

The result in the case of problem (P3) will not be used in this paper. However, since the proof is exactly the same as for (P4) we include it here for completeness and further reference.

5.1. **Preliminary: Parabolic Maximum principle.**

abolicmaxPrpBdd⟩ **Theorem 5.3.** *Assume that v is bounded and $\mathcal{C}^{2\gamma+\epsilon}(B_R \times (0,T])$ such that*

Acotada donde? Creo que no hace falta que sea en todo $\mathbb{R}^n$

$$\begin{cases} \partial_t v + Lv & \leq & 0 & \text{in } B_R \times (0,T]\,, \\ v_0 := v(x,0) & \leq & 0 & \text{in } B_R\,, \\ v & \leq & 0 & \text{in } (\mathbb{R}^n \setminus B_R) \times (0,T]\,. \end{cases}$$

Then,

$$v \leq 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T].$$

Proof. Let us proceed by contradiction. Assume v attains a positive maximum M. That is, $v(x_0,t_0) = M > 0$. For the exterior conditions x_0 must be in B_R. Now we distinguish two cases:

- If $t_0 \in (0,T)$, then (x_0,t_0) is an interior absolute maximum and it must satisfy $v_t(x_0,t_0) = 0$ and $Lv(x_0,t_0) > 0$, which is a contradiction with the equation.

- If $t_0 = T$, then it must satisfy $v_t(x_0,t_o) \geq 0$ and $Lv(x_0,t_0) > 0$, which is also a contradiction with the equation.

□

mma:NoBddSolL=1⟩ **Lemma 5.4.** *There is no bounded solution of*

$$Lv = 1 \quad \text{in} \quad \mathbb{R}^n.$$

Proof. Assume by contradiction that such solution exists. Then, by interior regularity (see Section 2) $v \in \mathcal{C}^1(\mathbb{R}^n)$ and $|\nabla v| \leq C$ in $\mathbb{R}^n$. By differentiating the equation with respect to x_i we obtain

$$\begin{cases} Lv_{x_i} & = & 0 & \text{in } \mathbb{R}^n\,, \\ |v_{x_i}| & \leq & C & \text{in } \mathbb{R}^n\,. \end{cases}$$

By Liouville Theorem, v_{x_i} is constant.

Quizás está bien escribir en algún sitio o como minimo referenciarlo

Hence, since this can be done for each partial derivative we obtain that ∇v is constant, and thus v is affine. But since u is bounded, v must be constant too, and we arrive to a contradiction with $Lv = 1$. □

⟨Lemma:SolBall⟩ **Lemma 5.5.** *Let L be an integral operator with kernel K satisfying 99-99 and let $R > 0$ be given. Then, there exists ϕ_R a solution of*

$$\begin{cases} L\phi_R & = & 1 & \text{in } B_R\,, \\ \phi_R & = & 0 & \text{in } \mathbb{R}^n \setminus B_R\,, \end{cases}$$

and satisfying

$$M_R := \sup_{B_R} \phi_R \to \infty \quad \text{as } R \to \infty\,.$$

Proof. The existence of a weak solution is given by Riesz representation theorem. Moreover, by regularity results (see section 2.1) it is in fact a classical solution and by the maximum principle, $\phi_R > 0$ in B_R. Now consider the new function

$$\varphi_R := \frac{\phi_R}{M_R},$$

which satisfies

$$\begin{cases} L\varphi_R &= 1/M_R \quad \text{in } B_R\,, \\ \varphi_R &= 0 \qquad\quad \text{in } \mathbb{R}^n \setminus B_R\,, \\ \|\varphi_R\|_{\mathrm{L}^\infty} &= 1. \end{cases} \tag{5.2}$$

Eq: varphi

Let us assume by contradiction that M_R does not tend to infinity. Then, since M_R is increasing (use the maximum principle to compare ϕ_R and $\phi_{R'}$ with $R > R'$), it must have a limit $M < +\infty$.

Therefore, applying Lemma 2.4 we deduce that φ_R converges (up to a subsequence) in $C^{2\gamma+\epsilon}$-norm to a function φ that is solution of

$$L\varphi = \frac{1}{M} \text{ in } \mathbb{R}^n\,,$$

and moreover $\|\varphi\|_{\mathrm{L}^\infty} = 1$ for being the uniform limit of functions with this norm.

But such φ cannot exists by Lemma 5.4. Thus, we arrive to a contradiction and M_R goes to infinity. ☐

⟨Lemma:SolBallToZero⟩ **Lemma 5.6.** *Let M_R be as in the previous Lemma. Then, there exists a function $\psi_R \geq 0$ solution of*

$$\begin{cases} L\psi_R &= -\frac{1}{M_R} \quad \text{in } B_R\,, \\ \psi_R &= 1 \qquad\quad \text{in } \mathbb{R}^n \setminus B_R\,, \end{cases}$$

such that

$$\psi_R \xrightarrow{as\ R\to\infty} 0.$$

Proof. Let us define

$$\psi_R := 1 - \frac{\phi_R}{M_R} = 1 - \varphi_R.$$

By Lemma 5.5, it is clear that ψ_R defined previously solves the problem and is nonnegative. Then we only need to show the limit condition. Note that this is equivalent to show that $\varphi_R \to 1$ as $R \to \infty$. Recall that φ_R solves problem (5.2). Therefore, letting R to infinity and knowing that $M_R \to \infty$ we can apply Lemma 2.4 to deduce that φ_R converges in $C^{2\gamma+\epsilon}$-norm (up to a subsequence) to a function $\varphi \geq 0$ that solves

$$L\varphi = 0 \quad \text{in} \quad \mathbb{R}^n.$$

By Liouville Theorem, φ must be constant, and since its L^∞-norm is one and it is nonnegative, then $\varphi \equiv 1$, and the result is proved. ☐

⟨Th:ParaMaxPrp⟩ **Theorem 5.7.** *Assume v is a bounded function such that*

$$\begin{cases} \partial_t v + Lv + c\,v &\leq 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty)\,, \\ v_0 := v(x,0) &\leq 0 \quad \text{in } \mathbb{R}^n\,, \end{cases}$$

with $c = c(x)$ a bounded function. Then,

$$v(x,t) \le 0 \quad \textit{in} \quad \mathbb{R}^n \times [0,+\infty).$$

Proof. First of all, note that with the change of function $\tilde{v}(x,t) = \mathrm{e}^{-\alpha\, t} v(x,t)$ we can reduce the initial problem to

$$\begin{cases} \partial_t \tilde{v} + L\tilde{v} & \le & 0 & \text{in } \Omega \subseteq \mathbb{R}^n \times (0,+\infty)\,, \\ \tilde{v} & \le & 0 & \text{in } (\mathbb{R}^n \times (0,+\infty)) \setminus \Omega\,, \\ \tilde{v}_0 & \le & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

if we take $\alpha > ||c||_{\mathrm{L}^\infty}$ and $\Omega = \{(x,t) \in \mathbb{R}^n \times (0,+\infty) \text{ such that } \; v(x,t) > 0\}$.

Now, consider the function

$$w_R(x,t) = ||v||_{\mathrm{L}^\infty} \left(\psi_R + \frac{t}{M_R} \right),$$

which satisfies

$$\begin{cases} \partial_t w_R + L w_R & = & 0 & \text{in } B_R \times (0,T]\,, \\ w_R(x,0) & \ge & 0 & \text{in } B_R\,, \\ w_R(x,t) & \ge & ||v||_{\mathrm{L}^\infty} & \text{in } (\mathbb{R}^n \setminus B_R) \times (0,T]\,. \end{cases}$$

By the maximum principle in $(B_R \times [0,T]) \cap \Omega$, Lemma 5.3, we conclude that $w_R \ge \tilde{v}$ in $B_R \times (0,T]$.

Now, given an arbitrary point (x_0,t_0), take $R_0 > 0$ and $T > 0$ such that $(x_0,t_0) \in B_{R_0} \times [0,T]$. Then

$$\tilde{v}(x_0,t_0) \le w_R(x_o,t_0) = ||v||_{\mathrm{L}^\infty} \left(\psi_R(x_0) + \frac{t_0}{M_R} \right), \quad \text{for any} \quad R \ge R_0.$$

Finally, letting $R \to \infty$ and using that $\psi_R(x_0) \to 0$ and $M_R \to \infty$ by Lemmas 5.5 and 5.6, we conclude

$$\tilde{v}(x_0,t_0) \le 0,$$

and therefore

$$v(x_0,t_0) = \mathrm{e}^{\alpha\, t_0}\, \tilde{v}(x_0,t_0) \le 0,$$

□

5.2. **A symmetry result for positive solutions in the whole space.**

Proof of Theorem 5.1. The proof follows the ideas of Berestycki, Hamel and Nadirashvili from Theorem 2.2. in [3] but adapted to the whole space and with a nonlocal operator.

Assume $u \not\equiv 0$. Then, by the strong maximum principle $u > 0$. Our goal is to show that $u \equiv 1$, and this will be accomplished in two steps.

Step 1: We show that $m := \inf_{\mathbb{R}^n} u > 0$.

By contradiction, we will assume $m = 0$. Then, there exists a sequence $\{x_k\}$ such that $u(x_k) \to 0$ as $k \to +\infty$. By the Harnack Inequality from Di Casto, Kuusi and Palatucci in [15], given any $R > 0$ we have

MIRAR LO DE MATTEO!!!!

$$\sup_{B_R(x_k)} u \le C_R \inf_{B_R(x_k)} u \le C\, u(x_k) \to 0 \text{ as } k \to +\infty. \tag{5.3}$$ Eq:Harnack

Since $f(0) = 0$ and $f'(0) > 0$, it is easy to show that $f(t) \ge f'(0)t/2$ if t is small enough. Indeed, since $f(0) = 0$,

$$f'(0) = \lim_{t\to 0} \frac{f(t)}{t}$$

and by the definition of limit we get that there exists $t_0 > 0$ such that

$$f(t) \ge \frac{f'(0)}{2} t \quad \text{for all} \quad 0 \le t \le t_0. \tag{5.4}$$ Eq:TaylorSimplif

Igual esto se puede quitar que es fácil

Therefore, with (5.3) and (5.4), we deduce that there exists $M(R) \in \mathbb{N}$ such that

$$Lu - \frac{f'(0)}{2} u \ge 0 \;\text{ in }\; B_R(x_{M(R)})\,. \tag{5.5}$$ Eq:WholeSpace2

On the other hand, let us define

$$\lambda_R^{x_0} = \inf_{\substack{\varphi \in \mathcal{C}_0^1(B_R(x_0))\\ \varphi \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y}{\displaystyle\int_{\mathbb{R}^n} \varphi^2\, \mathrm{d}x},$$

which decreases to zero uniformly in x_0 as R goes to infinity from being $L \in \mathcal{L}_0$ (see the proof of Lemma 4.2 and also Proposition 9 of [33]). Therefore, there exists $R_0 > 0$ such that

$$\lambda_R^x < \frac{f'(0)}{2}$$

for all $x \in \mathbb{R}^n$ and $R \ge R_0$. In particular, by choosing $x = x_{M(R_0)}$ there exists $w \in \mathcal{C}_0^1(B_{R_0}(x_{M(R_0)}))$ such that $w \not\equiv 0$ and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y < \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x. \tag{5.6}$$ Eq:Eigenfunction

If we multiply (5.5) by $w^2/u \ge 0$ and integrate

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^n} Lu \frac{w^2}{u}\, \mathrm{d}x - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \{u(x) - u(y)\} \left(\frac{w^2(x)}{u(x)} - \frac{w^2(y)}{u(y)} \right) K(x-y)\, \mathrm{d}x\, \mathrm{d}y - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \{w(x) - w(y)\}^2 K(x-y) - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x,
\end{aligned}$$

which contradicts (5.6). Then $\inf_{\mathbb{R}^n} u > 0$.

Step 2: We show that $u \equiv 1$.

Now, choose $0 < \xi_0 < \min\{1, m\}$, which is well define by Step 1, and let $\xi(t)$ be the solution of the ODE

$$\begin{cases} \dot{\xi}(t) &= f(\xi(t)) \quad \text{in } (0,\infty)\,, \\ \xi(0) &= \xi_0. \end{cases}$$

Since $f > 0$ in $(0,1)$ and $f(1) = 0$ we have that $\dot{\xi}(t) > 0$ for all $t \geq 0$ and $\lim_{t\to 0} \xi(t) = 1$.

Now, note that both $u(x)$ and $\xi(t)$ solve the parabolic equation

$$\partial_t w + Lw = f(w) \quad \text{in } \mathbb{R}^n \times (0,\infty)\,,$$

and satisfy

$$u(x) \geq m \geq \xi_0 = \xi(0).$$

Thus, by the parabolic maximum principle, Theorem 5.7, $u(x) \geq \xi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. By letting $t \to \infty$ we obtain

$$u(x) \geq 1 \quad \text{in } \mathbb{R}^n\,.$$

In a similar way, taking $\tilde{\xi}_0 > ||u||_{L^\infty} \geq 1$, using $f < 0$ in $(1,\infty)$, $f(1) = 0$ and the parabolic maximum principle, we obtain the upper bound $u \leq 1$.

□

5.3. **A one-dimensional symmetry result for positive solutions "in a half-space".** First we show that the solution is monotone. We do it using a moving planes argument, and for this reason we need the following maximum principle in "narrow" domains. Recall that for a domain $\Omega \subset \mathbb{R}^n$, we define the quantity $R(\Omega)$ as the smallest positive R for which

$$\frac{|B_R(x) \setminus \Omega|}{|B_R(x)|} \geq \frac{1}{2} \quad \text{for every } x \in \Omega.$$

If no such radius exists, we define $R(\Omega) = +\infty$. Thus, we say that a domain Ω is "narrow" ir $R(\Omega)$ is small.

Ellos lo demuestran dentro de la demo de los moving planes. En realidad sirve para los dos problemas. De todas formas, como el argumento es interesante y corto, lo ponemos aqui por completitud y para futura referencia. Hay que poner que es de ellos justo en el enunciado¿¿¿???

MaxPrpNarrowOdd⟩ **Proposition 5.8.** *Let H be a half-space in $\mathbb{R}^n$, and denote by x^* the reflection of any point x with respect to the hyperplane ∂H. Let L be an integro-differential operator with a positive kernel K satisfying*

$$K(x-y) \geq K(x-y^*), \quad \textit{for all } \; x,y \in H. \tag{5.7}$$

Eq:KernelSymmetr

Assume that v 99 satisfies

$$\left\{\begin{array}{rcll} Lv & \geq & c(x)\,v & \text{in } \Omega \subseteq H, \\ v & \geq & 0 & \text{in } H \setminus \Omega, \\ v(x) & = & -v(x^*) & \text{in } \mathbb{R}^n. \end{array}\right. \tag{5.8} \{?\}$$

Then, there exist a number $\overline{R}$ such that $v \geq 0$ whenever $R(\Omega) \leq \overline{R}$.

Proof. Let us begin by defining $\Omega_- = \{x \in \Omega \,:\, v < 0\}$. We shall prove that Ω_- is empty. Assume by contradiction that it is not empty. Then, we split

$$v = v_1 + v_2\,,$$

where

$$v_1(x) = \begin{cases} v(x) & \text{in } \Omega_-, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases} \qquad \text{and} \qquad v_2(x) = \begin{cases} 0 & \text{in } \Omega_-, \\ v(x) & \text{in } \mathbb{R}^n \setminus \Omega_-. \end{cases}$$

Let us first show that $Lv_2 \leq 0$ in Ω_-. To see this, let us take $x \in \Omega_-$ and thus

$$Lv_2(x) = \int_{\mathbb{R}^n \setminus \Omega_-} -v_2(y) K(x-y)\,\mathrm{d}y = -\int_{\mathbb{R}^n \setminus \Omega_-} v(y) K(x-y)\,\mathrm{d}y.$$

Now, we split $\mathbb{R}^n \setminus \Omega_-$ into

$$A_1 = \Omega_-^*, \qquad \text{and} \qquad A_2 = (H \setminus \Omega_-) \cup (H \setminus \Omega_-)^*\,,$$

and we compute

$$-\int_{A_1} v(y) K(x-y)\,\mathrm{d}y = -\int_{\Omega_-} v(y^*) K(x-y^*)\,\mathrm{d}y = \int_{\Omega_-} v(y) K(x-y^*)\,\mathrm{d}y \leq 0,$$

where the last inequality comes from being v negative in Ω_- and the kernel positive in all $\mathbb{R}^n$. On the other hand

$$\begin{aligned} -\int_{A_2} v(y) K(x-y)\,\mathrm{d}y &= -\int_{H \setminus \Omega_-} v(y) K(x-y)\,\mathrm{d}y - \int_{H \setminus \Omega_-} v(y^*) K(x-y^*)\,\mathrm{d}y \\ &= -\int_{H \setminus \Omega_-} v(y) \left\{ K(x-y) - K(x-y^*) \right\}\,\mathrm{d}y \leq 0, \end{aligned}$$

where we have use the kernel condition (5.7) and the odd symmetry of v. Thus, we get $Lv_2 \leq 0$ in Ω_-, which means

$$Lv_1 = Lv - Lv_2 \geq Lv \geq c(x)\,v = c(x)\,v_1 \quad \text{in} \;\; \Omega_-.$$

Therefore v_1 solves

$$\left\{\begin{array}{rcll} Lv_1 & \geq & c(x)\,v_1 & \text{in } \;\Omega_-, \\ v_1 & = & 0 & \text{in } \;\mathbb{R}^n \setminus \Omega_-, \end{array}\right.$$

and we can apply the usual maximum principle for narrow domains (see Theorem 2.4 in [25]) to v_1 in Ω_- in order to deduce that $v_1 \geq 0$ in all $\mathbb{R}^n$. But this is a contradiction with the definition of v_1 and the fact that the set Ω_- is not empty. □

This maximum principle in narrow domains for odd functions with respect to a hyperplane is the principal tool to use the moving plane argument. This allows us to show the following result.

notonyHalfSpace⟩ **Proposition 5.9.** *Let v be a bounded solution of one of the problems* (P3) *or* (P4)*, with L an integro-differential operator with a positive kernel K satisfying* (5.7) *and f a Lipschitz nonlinearity such that $f > 0$ in $(0, ||v||_{L^\infty(\mathbb{R}^n_+)})$. Then,*

$$\frac{\partial v}{\partial x_n} > 0 \quad \text{in } \mathbb{R}^n_+.$$

Proof. The proof is based on the moving planes method, and is exactly the same as the analogue proof of Theorem 3.1 in [25], where Quaas and Xia establish an equivalent result for the fractional Laplacian. For this reason, we give here just a sketch. As usual, for $\lambda > 0$ one defines $w_\lambda(x) = v(x', 2\lambda - x_n) - v(x', x_n)$ and since the nonlinearity is Lipschitz, w_λ solves, in any of the two cases —(P3) or (P4)—, the following problem:

$$\begin{cases} Lw_\lambda &= c_\lambda(x)\, w_\lambda & \text{in } \Sigma_\lambda \subseteq H_\lambda, \\ w_\lambda &\geq 0 & \text{in } H_\lambda \setminus \Sigma_\lambda, \\ w_\lambda(x) &= w_\lambda(x_\lambda) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Sigma_\lambda := \{x = (x', x_n) \ : \ 0 < x_n < \lambda\}$ and $H_\lambda := \{x = (x', x_n) \ : \ x_n < \lambda\}$ and c_λ is a bounded function. Note that w_λ is odd with respect to ∂H_λ. Then, using the maximum principle in narrow domains Proposition 5.8 one shows that, if λ is small enough, $w_\lambda > 0$ in Σ_λ. To conclude the proof, we define

$$\lambda^* := \sup\{\lambda \ : \ w_\eta > 0 \ \text{ in } \ \Sigma_\lambda \ \text{ for all } \ \eta < \lambda\}.$$

Note that λ^* is well defined by the previous argument. Then, to conclude the proof one has to show that $\lambda^* = \infty$. This is done by showing that, if λ^* is finite, then there exists a small $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ we have

$$w_{\lambda^*+\delta}(x) > 0 \quad \text{in } \Sigma_{\lambda^*-\varepsilon} \setminus \Sigma_\varepsilon$$

for some small ε. This is established using a compactness argument exactly as in Lemma 3.1 [25]. Finally, by the maximum principle in narrow domains we can deduce that $w_{\lambda^*+\delta}(x) > 0$ in $\Sigma_{\lambda^*+\delta}$, contradicting the definition of λ^*. □

alfSpaceLimUnif⟩ **Proposition 5.10.** *Let u be a bounded solution of one of the following problems*

(P1) Eq:P1
$$\begin{cases} Lu &= f(u) & \text{in } \mathbb{R}^n, \\ \lim\limits_{x_n \to \pm\infty} u(x', x_n) &= \pm 1 & \text{uniformly.} \end{cases}$$

(P2) Eq:P2
$$\begin{cases} Lu &= f(u) & \text{in } \mathbb{R}^n_+ = \{x_n > 0\}, \\ u &= 0 & \text{in } \overline{\mathbb{R}^n_-} = \{x_n \leq 0\}, \\ \lim\limits_{x_n \to +\infty} u(x', x_n) &= 1 & \text{uniformly.} \end{cases}$$

Assume that there exists $\delta > 0$ *such that*

$$f'(t) \leq 0 \quad in \quad [-1, -1+\delta] \cup [1-\delta, 1],$$

for problem (P1) *and*

$$f'(t) \leq 0 \quad in \quad [-1, -1+\delta]$$

for problem (P2).

Then, u *only depends on* x_n *and is increasing in that direction.*

Proof. The proof follows the ideas, sliding method, of Berestycki, Hamel and Monneau from Theorem 1 in [2] but adapted to a nonlocal operator.

Let us define $u^t(x) := u(x + \nu t)$ for any $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ and $\nu_n > 0$, and call $v^t(x) := u^t(x) - u(x)$. The aim is to show that $v^t(x) \geq 0$ for all $t \geq 0$.

Step 1: For $t > 0$ big enough $v^t \geq 0$.

Since the limits of u at infinity are uniform, let $A > 0$ be such that for all $|x_n| \geq A$ and $x' \in \mathbb{R}^{n-1}$

$$u(x', x_n) \notin (-1+\delta, 1-\delta). \tag{5.9}$$ Eq:DefinitionA

Let $H = \{v^t \geq 0\} = \{u^t(x) \geq u(x)\}$. Therefore, v^t solves

$$\begin{cases} Lv^t &= d^t(x)v^t & \text{in } \mathbb{R}^n \setminus H, \\ v^t &\geq 0 & \text{in } H, \end{cases}$$

with $d^t(x) \leq 0$. To show this, note that

$$Lv^t = Lu^t - Lu = f(u^t) - f(u) = \frac{f(u^t) - f(u)}{u^t - u} v^t = d^t(x)v^t. \tag{5.10}$$ Eq:Definition_dt

Then we have to show that $\frac{f(u^t)-f(u)}{u^t-u} \leq 0$ out of H for t large enough. In order to prove it we distinguish the two different problems, (P1) and (P2).

For (P1): Take $x \notin H$, that is $u^t(x) < u(x)$, and $t \geq 2A/\nu_n$, then

- If $x_n \geq -A$, then $1 - \delta \leq u^t(x) < u(x)$ and f is nonincreasing, which means
$$d^t(x) = \frac{f(u^t) - f(u)}{u^t - u} \leq 0.$$
- If $x_n \leq -A$, then $u^t(x) \leq u(x) \leq -1 + \delta$ and f is nonincreasing, which also means
$$d^t(x) = \frac{f(u^t) - f(u)}{u^t - u} \leq 0.$$

For (P2): Note that in this case $\mathbb{R}^n \setminus H$ is contained in $\mathbb{R}^n_+$. Take $x \notin H$, that is $u^t(x) < u(x)$, and $t \geq A/\nu_n$, then $1-\delta \leq u^t(x) \leq u(x)$ and f is nonincreasing, which means

$$d^t(x) = \frac{f(u^t) - f(u)}{u^t - u} \leq 0.$$

Therefore, by the maximum principle, $v^t \geq 0$ in $\mathbb{R}^n$ for t large enough. Note that we have quantify how large must be t in terms of the nonlinearity f and the vector ν.

Step 2: $v^t \geq 0$ for all $t \geq 0$.
For this, define

$$\tau = \inf\{t \ \text{ such that } \ u^t \geq u\},$$

and we claim that $\tau = 0$. Assume, in order to get a contradiction, that $\tau > 0$. Let us define

$$m = \inf_{\mathbb{R}^{n-1}\times[-A,A]} (u^\tau - u)$$

in the case of (P1) and

$$m = \inf_{\mathbb{R}^{n-1}\times[0,A]} (u^\tau - u)$$

in the case of (P2). Now we can distinguish two cases depending if $m > 0$ or $m = 0$.

Step 2.1: $v^t \geq 0$ for all $t \geq 0$ if $m > 0$.
By interior estimates for L, u is globally $\mathcal{C}^s$-Hölder continuous. Then, there exists η_0 small so that for any $t \in (\tau - \eta_0, \tau)$

$$u^t \geq u \ \text{ in } \ \mathbb{R}^{n-1} \times [-A, A] \ \big(\text{in } \ \mathbb{R}^{n-1} \times [0, A]\big) .$$

That is,

$$\begin{aligned} u^t(x) - u(x) &= u(x + t\nu) - u(x) - u(x + \tau\nu) + u(x + \tau\nu) \\ &\geq u^\tau(x) - u(x) - |u(x + t\nu) - u(x + \tau\nu)| \\ &\geq m - C|t - \tau|^s \geq 0, \end{aligned}$$

if $|t - \tau|^s \leq m/C$.

On the other hand, if we define $d^{\tau-\eta}(x)$ as in equation (5.10), we have

$$Lv^{\tau-\eta} = d^{\tau-\eta}(x)\, v^{\tau-\eta}.$$

Let us show now that $d^{\tau-\eta} \leq 0$ in the sets

$$\hat{H} = \{|x_n| \geq A\} \cap \big\{u^{\tau-\eta} < u\big\} \quad \text{for (P1)},$$

and

$$\hat{H} = \{x_n \geq A\} \cap \{u^{\tau-\eta} < u\} \ \text{ for (P2)}.$$

To see this fact, we have to proceed as before. Let $x \in \tilde{H}$. Then,

- If $x_n \leq -A$, then $u^{\tau-\eta}(x) < u(x) \leq -1 + \delta$ and f is nonincreasing, which means

$$d^{\tau-\eta}(x) = \frac{f(u^t) - f(u)}{u^t - u} \leq 0.$$

- If $x_n \geq A$, then $1 - \delta \leq u^{\tau-\eta}(x) < u(x)$ and f is nonincreasing, which also means

$$d^t(x) = \frac{f(u^t) - f(u)}{u^t - u} \leq 0.$$

Therefore, for $0 < \eta \le \eta_0$ we can apply the maximum principle in $\tilde{H}$ to obtain

$$u^{\tau-\eta} \ge u \quad \text{in} \quad \mathbb{R}^n,$$

which contradicts the minimality of τ. This means that $\tau = 0$.

Step 2.2: $v^t \ge 0$ for all $t \ge 0$ if $m = 0$.

By definition of m we can take a sequence $x_k = (x'_k, x^n_k) \in \mathbb{R}^{n-1} \times [-A, A]$ such that

$$u^\tau(x_k) - u(x_k) \to 0. \tag{5.11}$$ Eq:Limit

Since x^n_k is bounded, we can choose a subsequence that converges to $x^n_\infty \in [-A, A]$. Call $u_k(x) = u(x' + x'_k, x^n_k)$, which satisfies problem (P1) or (P2) for each k since the problems are invariant under translations. Therefore, by elliptic estimates, u_k converges to a function u_∞ which satisfies

$$Lu_\infty = f(u_\infty) \quad \text{in} \quad \mathbb{R}^n.$$

Note that since u satisfies (5.9) and the translations are only in the $n-1$ first variables, u_k and its limit u_∞ also satisfy this property.

Then, if we define $w = u^\tau_\infty - u_\infty$, it solves

$$\begin{cases} Lw &= d(x)w & \text{in } \mathbb{R}^n, \\ w &\ge 0 & \text{in } \mathbb{R}^n, \\ w(0, x^n_\infty) &= 0, & \end{cases}$$

with

$$d(x) = \begin{cases} \frac{f(u^\tau_\infty(x)) - f(u_\infty(x))}{u^\tau_\infty(x) - u_\infty(x)} & \text{if } u^\tau_\infty(x) \ne u_\infty(x) \\ 0 & \text{otherwise.} \end{cases}$$

That is, we have that $w \ge 0$ in $\mathbb{R}^n$ since $u^\tau \ge u$. And on the other hand, from (5.11) we get

$$\begin{aligned} w(0, x^n_\infty) &= \lim_{k\to\infty} u^\tau_k(0, x^n_k) - u_k(0, x^n_k) \\ &= \lim_{k\to\infty} u^\tau(x'_k, x^n_k) - u_k(x'_k, x^n_k) \\ &= \lim_{k\to\infty} u^\tau(x_k) - u_k(x_k) = 0 \end{aligned}$$

By applying the strong maximum principle we get $w \equiv 0$, which means that $u_\infty(x) = u_\infty(x + \tau\nu)$. But this is a contradiction with the fact that $u_\infty(x', x_n) \ge 1 - \delta$ for $x_n > A$, $u_\infty(x', x_n) \le -1 + \delta$ for $x_n < -A$ since $\delta << 1$ and $\tau_n > 0$. That is,

$$1 - \delta \le u_\infty(x', A) = u_\infty\left(x' - \left\lceil \frac{2A}{\tau\nu_n} \right\rceil \tau\nu', A - \left\lceil \frac{2A}{\tau\nu_n} \right\rceil \tau\nu_n\right) \le -1 + \delta.$$

Once we have shown that $v^t \ge 0$ for all $t > 0$, then it is clear that

$$\nabla u(x) \cdot \nu = \frac{\partial u}{\partial \nu}(x) = \lim_{t\to 0} \frac{u(x + t\nu) - u(x)}{t} \ge 0 \tag{5.12}$$ Eq:nuDerivative

for any $x \in \mathbb{R}^n$ and any $\nu \in \mathbb{R}^n$ such that $\nu_n > 0$. By making the limit ν_n to zero in the previous expression we get

$$\nabla u \cdot (\nu', 0) \geq 0 \text{ for all } \nu' \in \mathbb{R}^{n-1}.$$

And choosing now the opposite vector $-\nu'$ we finally get

$$\nabla u \cdot (\nu', 0) = 0 \text{ for all } \nu' \in \mathbb{R}^{n-1},$$

which is equivalent to say that u only depends on x_n. Moreover, if we choose $\nu = e_n$, then from (5.12) we obtain

$$\frac{\partial u}{\partial x_n} \geq 0.$$

□

With all these ingredients we can now proce Theorem 5.2

Proof of Theorem 5.2. Note that by Proposition 5.10 we only need to prove that

$$\lim_{x_n \to \infty} u(x', x_n) = 1$$

uniformly. Therefore we divide the proof in two steps, first we prove that the limit exists and is one and then we prove that it is uniform.

Step 1: Given $x' \in \mathbb{R}^{n-1}$, then $\lim_{x_n \to \infty} u(x', x_n) = 1$.

By Proposition 5.9 we know that u is strictly increasing in the direction x_n. Since u is also bounded by hypothesis, we know that that given $x' \in \mathbb{R}^{n-1}$, the one variable function $u(x', \cdot)$ has a limit, that is in fact positive for being $u(x', 0) = 0$ and $u_{x_n} > 0$. Let us define $\overline{u}(x')$ as this limit.

Let x_n^k be any increasing sequence tending to infinity. Then we define the following sequence of functions

$$u_k(x', x_n) = u(x', x_n + x_n^k),$$

which solves the problem

$$\begin{cases} Lu_k = f(u_k) & \text{in } \{x_n > -x_n^k\}, \\ u_k > 0 & \text{in } \{x_n > -x_n^k\}, \\ u = 0 \text{ or odd} & \text{in } \{x_n \leq -x_n^k\}, , \end{cases} \tag{5.13} \{?\}$$

and are uniformly bounded.

!!!!

By letting $k \to \infty$ and using compactness we have that u_k converges (up to a subsequence) to a function u_∞ that is solution of

$$\begin{cases} Lu_\infty = f(u_\infty) & \text{in } \mathbb{R}^n, \\ u_\infty \geq 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{5.14} \{?\}$$

By Theorem 5.1 it is clear that $u_\infty \equiv 0$ or $u_\infty \equiv 1$. But, by construction

$$u_\infty(x', 0) = \lim_{k \to \infty} u_k(x', 0) = \lim_{k \to \infty} u(x', x_n^k) = \overline{u}(x') > 0,$$

and therefore the only possibility is

$$\lim_{x_n\to\infty} u(x', x_n) = 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}.$$

Step 2: The limit is uniform in x'.

Let us proceed by contradiction. Suppose that the limit is not uniform. This means that given any $\epsilon > 0$ small enough, there exists a sequence of points (x'_k, x_n^k) with $x_n^k \to \infty$ such that $u(x'_k, x_n^k) = 1-\epsilon$. Then we define the following sequence of functions

$$u_k(x', x_n) = u(x' + x'_k, x_n + x_n^k),$$

which solves the problem

$$\begin{cases} Lu_k &= & f(u_k) & \text{in } \{x_n > -x_n^k\}, \\ u_k &> & 0 & \text{in } \{x_n > -x_n^k\}, \\ u &= & 0 \text{ or odd} & \text{in } \{x_n \le -x_n^k\},, \end{cases} \tag{5.15} \{?\}$$

and are uniformly bounded.

By letting $k \to \infty$ and using compactness we have that u_k converges (up to a subsequence) to a function u_∞ that is solution of

!!!!

$$\begin{cases} Lu_\infty &= & f(u_\infty) & \text{in } \mathbb{R}^n, \\ u_\infty &\ge & 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{5.16} \{?\}$$

By Theorem 5.1 it is clear that $u_\infty \equiv 0$ or $u_\infty \equiv 1$. But, by construction

$$u_\infty(0,0) = \lim_{k\to\infty} u_k(0,0) = \lim_{k\to\infty} u(x'_k, x_n^k) = 1 - \epsilon,$$

which is a contradiction for $\epsilon > 0$ small enough. Thus, the limit is uniform and applying Proposition 5.10 we get that u depends only on x_n and is increasing in that direction.

$\square$

6. Asymptotic results

⟨Sec:Asymptotic⟩?

Layer solution

ernelsDimension⟩?

Proposition 6.1. *Let L and $\tilde{L}$ be two symmetric and translation invariant integral operators with kernels K and $\tilde{K}$ in dimensions n and 1 respectively. Let also be $u: \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ such that $u(x) = v(x_n)$.*

If,

$$\tilde{K}(t) = |t|^{n-1} \int_{\mathbb{R}^n} K\left(t(\sigma, 1)\right) d\sigma,$$

then

(i) $Lu(x) = \tilde{L}v(x_n)$,

(ii) $L \in \mathcal{L}_0(n, s, \lambda, \Lambda) \Rightarrow \tilde{L} \in \mathcal{L}_0(n, s, \lambda, \Lambda)$.

Note that if L is the fractional Laplacian in dimension n, then $\tilde{L}$ is also the fractional Laplacian, but in dimension 1.

Proof. (i)

$$\begin{aligned} Lu(x) &= \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x-y) dy \\ &= \int_{\mathbb{R}^n} \{v(x_n) - v(y_n)\} K\left((x' - y', x_n - y_n\right) dy' dy_n. \end{aligned}$$

Now we make the change of variables $y' = x' - (x_n - y_n)\sigma$. That is,

$$\begin{aligned} Lu(x) &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} \int_{\mathbb{R}^{n-1}} K\left((\sigma(x_n - y_n), x_n - y_n)\right) |x_n - y_n|^{n-1} d\sigma dy_n \\ &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} |x_n - y_n|^{n-1} \int_{\mathbb{R}^{n-1}} K\left((x_n - y_n)(\sigma, 1)\right) d\sigma dy_n \\ &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} \tilde{K}(x_n - y_n dy_n = \tilde{L}v(x_n). \end{aligned}$$

(ii) We are only going to prove one of the bounds, since they are in fact the same. That is,

$$\begin{aligned} \tilde{K}(t) &= |t|^{n-1} \int_{\mathbb{R}^n} K\left(t(\sigma, 1)\right) d\sigma \geq |t|^{n-1} \int_{\mathbb{R}^n} c_{n,s} \frac{\lambda}{|t|^{n+2s}(\sigma^2 + 1)^{\frac{n+2s}{2}}} d\sigma \\ &= c_{n,s} \frac{\lambda}{|t|^{1+2s}} \int_{\mathbb{R}^n} \frac{d\sigma}{(\sigma^2 + 1)^{\frac{n+2s}{2}}} = c_{n,s} \frac{\lambda}{|t|^{1+2s}} \frac{c_{1,s}}{c_{n,s}} \\ &= c_{1,s} \frac{\lambda}{|t|^{1+2s}}, \end{aligned}$$

where the only inequality that appears comes from the lower bound of the kernel K for being in $\mathcal{L}_0$.

□

urSaddleSolution⟩ **Theorem 6.2.** *Let $f \in \mathcal{C}^{2,\alpha}(\mathbb{R})$ of the Allen-Cahn type. Let u be a saddle- shaped solution of*

$$Lu = f(u) \text{ in } \mathbb{R},$$

with kernel K satisfying . Then,

write conditions

$$||u - \mathcal{U}||_{\mathrm{L}^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla \mathcal{U}||_{\mathrm{L}^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left|D^2 u - D^2 \mathcal{U}\right|\right|_{\mathrm{L}^\infty(\mathbb{R}^n \setminus B_R)} \xrightarrow{as\ R\to\infty} 0,$$

where

$$\mathcal{U}(x) = u_0(z),$$

with z the distance to the cone and u_0 the layer 1D solution.

Proof. By contradiction, assume that the result does not hold. Then, there exists $\epsilon > 0$ and a sequence $\{x_k\}$, that we may assume without loss of generality in $\overline{\mathcal{O}}$ and by continuity out of $\mathscr{C}$, such that

$$|u(x_k) - \mathcal{U}(x_k)| + |\nabla u(x_k) - \nabla \mathcal{U}(x_k)| + |D^2 u(x_k) - D^2 \mathcal{U}(x_k)| > \epsilon. \tag{6.1}$$ Eq:Contradiction

We distinguish two cases:

CASE 1. $\{d_k := \operatorname{dist}(x_k, \mathscr{C})\}$ is an unbounded sequence. We may assume $d_k \geq 2k$. In such case, we define

$$w_k(x) = u(x + x_k) \text{ for } x \in B_{d_k}(0),$$

which satisfies $0 < w_k < 1$ and

$$Lw_k = f(w_k) \text{ in } B_k.$$

By letting k tend to infinity and using compactness we have that w_k converges to w, up to a subsequence, satisfying

Atencion

$$\begin{cases} Lw = f(w) & \text{in } \mathbb{R}^n, \\ w \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

By Theorem **??**, $w \equiv 0$ or $w \equiv 1$. First, w cannot be zero. Indeed, since w_k are stable in B_k, w is stable in $\mathbb{R}^n$, which means that the linear operator $L - f'(w)$ is a positive operator. Nevertheless, if $w \equiv 0$, the linear operator $L - f'(w) = L - f'(0)$ is negative for sufficiently large balls, since $f'(0) > 0$ and the first eigenvalue of L is of order R^{-2s} in balls of radius R. Therefore $w \equiv 1$.

On the other hand, since $d_k \to \infty$ and $\mathcal{U}(x_k) = u_0(z_k) = u_0(d_k)$ we get by the properties of the layer solution that $\mathcal{U}(x_k) \to 1$, $\nabla \mathcal{U}(x_k) \to 0$ and $D^2\mathcal{U}(x_k) \to 0$. From this and condition (**??**) we get

ver bien

$$|u(x_k) - 1| + |\nabla u(x_k)| + |D^2 u(x_k)| > \epsilon/2,$$

which means that $w \not\equiv 1$. Therefore we arrive to a contradiction with Theorem **??**.

CASE 2. $\{d_k := \operatorname{dist}(x_k, \mathscr{C})\}$ is a bounded sequence. Then, at least for a subsequence $d_k \to d$. Now, define for each x_k its projection on $\mathscr{C}$, x_k^0 and then we have that $\nu_k^0 := \frac{x_k - x_k^0}{d_k}$ is the unit normal to $\mathscr{C}$. Through a subsequence, $\nu_k^0 \to \nu$ with $|\nu| = 1$.

Define as before

$$w_k(x) = u(x + x_k^0),$$

which solves

$$Lw_k = f(w_k) \text{ in } \mathbb{R}^n.$$

Atencion

By letting k tend to infinity and using compactness we have that w_k converges to w, up to a subsequence, satisfying

$$\begin{cases} Lw = f(w) & \text{in } H := \{x \cdot \nu > 0\}, \\ w \geq 0 & \text{in } H, \\ w \text{ odd with respect to } H. \end{cases}$$

For the detail of the proof of how $\mathcal{O} \to H$ see [?].

As in the previous case, by stability w cannot be zero, and then, by Theorem **??**, w only depends on $x \cdot \nu$ and is increasing. Therefore, by the uniqueness of the layer solution, $w(x) = u_0(x \cdot \nu)$ and

$$\begin{aligned} u(x_k) &= w_k(x_k - x_k^0) = w(x_k - x_k^0) + o(1) \\ &= u_0((x_k - x_k^0) \cdot \nu) + o(1) \\ &= u_0((x_k - x_k^0) \cdot \nu_k^0) + o(1) \\ &= u_0(d_k |\nu_k^0|^2) + o(1) \\ &= u_0(d_k) + o(1) = \mathcal{U}(x_k) + o(1), \end{aligned}$$

contradicting (6.1). The same is done for ∇u and $D^2 u$. □

7. Maximum principles for the linearized operator and uniqueness of the saddle-shaped solution

aximumPrinciple⟩? In this section we show that the linearized operator $L - f'(u)$ satisfies the maximum principle in $\mathcal{O}$. This result combined with the asymptotic result of Theorem 6.2 gives the uniqueness of the saddle-shaped solution.

The maximum principle we establish is the following.

mumPrincipleInO⟩ **Proposition 7.1.** *Let $m \geq 1$, $\gamma \in (0,1)$, $\alpha > 2\gamma$ and let $v \in C^{\alpha}_{\rm loc}(\mathbb{R}^{2m}) \cap L^{\infty}(\mathbb{R}^{2m})$ be a doubly radial function. Let $\Omega \subseteq \mathcal{O}$ a domain (not necessarily bounded). Let $L \in \mathcal{L}_0$ such that .... Assume that v satisfies*

Write hypotheses

$$\begin{cases} Lv - f'(u)v - c(x)v & \leq & 0 & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathcal{O}\setminus\Omega\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} v(x) & \leq & 0\,, & \end{cases}$$

with $c \leq 0$ in Ω. Then, $v \leq 0$ in Ω.

In order to prove this result we need a maximum principle in narrow domains, stated next.

NarrowDomainsOdd⟩ **Proposition 7.2.** *Let $m \geq 1$, $\gamma \in (0,1)$, $\alpha > 2\gamma$ and let $v \in C^{\alpha}_{\rm loc}(\mathbb{R}^{2m}) \cap L^{\infty}(\mathbb{R}^{2m})$ be a doubly radial function. Let $\varepsilon > 0$ and $H \subseteq \{(x',x'') \in \mathbb{R}^m \times \mathbb{R}^m$ such that $|x''| < |x'| < |x''| + \varepsilon\} \subset \mathcal{O}$ a domain (not necessarily bounded). Let $L \in \mathcal{L}_0$ such that .... Assume that v satisfies*

Write hypotheses

$$\begin{cases} Lv + cv & \leq & 0 & \text{in } H\,, \\ v & \leq & 0 & \text{in } \mathcal{O}\setminus H\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in H,\ |x|\to\infty} v(x) & \leq & 0\,. & \end{cases} \tag{7.1}$$

Eq:AssumptionsMa

Under these assumptions there exists $\overline{\varepsilon} > 0$ such that, if $\varepsilon < \overline{\varepsilon}$, then $v \leq 0$ in H.

Proof. Assume, by contradiction, that

$$M := \sup_H > 0\,.$$

Under the assumptions (7.1), M must be attained at an interior point $x_0 \in H$, that we can assume without loss of generality that is of the form $x_0 = (|x_0'|e, |x_0''|e)$, with $e = (1,0,...,0) \in \mathbb{R}^m$. Then,

$$0 \geq Lv(x_0) + c(x_0)v(x_0) \geq Lv(x_0) - ||c_-||_{L^\infty(H)}\, M\,. \tag{7.2}$$

Eq:InequalitiesM

Now, we compute $Lv(x_0)$. Then, since v is doubly radial and odd with respect to the Simons cone, by Lemma 2.6 we can write

$$\begin{aligned} Lv(x_0) &= \int_{\mathcal{O}} \{v(x_0) - v(y)\}\overline{K}(x_0, y) + \{v(x_0) + v(y)\}\overline{K}(x_0, y^\star)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{M - v(y)\}\overline{K}(x_0, y) + \{M + v(y)\}\overline{K}(x_0, y^\star)\,\mathrm{d}y \\ &\geq \int_{\mathcal{O}} \{M - v(y)\}\overline{K}(x_0, y^\star) + \{M + v(y)\}\overline{K}(x_0, y^\star)\,\mathrm{d}y \\ &= 2M \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y, \end{aligned}$$

where the inequality comes from being M the supremum of v in $\mathcal{O}$ and the kernel inequality from Proposition 2.7.

Recalling (7.2), we have

$$\begin{aligned} 0 &\geq Lv(x_0) + c(x_0)v(x_0) \\ &\geq M\left\{2\int_{\mathcal{O}} \overline{K}(x_0, Sy)\,\mathrm{d}y - ||c_-||_{L^\infty(H)}\right\}. \end{aligned}$$

We claim that, for ε small enough,

$$2\int_{\mathcal{O}} \overline{K}(x_0, Sy)\,\mathrm{d}y > ||c_-||_{L^\infty(H)}\,. \tag{7.3}$$

Eq:ClaimMaxPNarr

If we assume this claim to be true, then we have the desired contradiction and we conclude that $v \leq 0$ in H.

Let us show (B.14) by proving that

$$\int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y = \int_{\mathcal{I}} \overline{K}(x_0, y)\,\mathrm{d}y = \int_{\mathcal{I}} K(|x_0 - y|\,\mathrm{d}y$$

can be as big as we want if ε is small enough.

By Lemma 4.2 in [9] and the narrow condition, there exists a point $\overline{x_0} = (\lambda e, \lambda e)$, which is a projection of x_0 in the Simons cone, with $\lambda > 0$ such that

$$|x_0 - \overline{x_0}| = \frac{|x_0'| - |x_0''|}{\sqrt{2}} \leq \frac{\sqrt{2}}{2}\varepsilon.$$

Then,

$$|x_0 - y| \leq |x_0 - \overline{x_0}| + |\overline{x_0} - y| \leq \frac{\sqrt{2}}{2}\varepsilon + |\overline{x_0} - y|$$

and since K is decreasing,

$$\int_{\mathcal{I}} K(|x_0 - y|)\,\mathrm{d}y \geq \int_{\mathcal{I}} K\left(\frac{\sqrt{2}}{2}\varepsilon + |\overline{x_0} - y|\right)\,\mathrm{d}y$$

The crucial point now is that, applying Monotone Convergence Theorem and the fact that K is continuous we get

$$\int_{\mathcal{I}} K\left(\frac{\sqrt{2}}{2}\varepsilon + |\overline{x_0} - y|\right)\,\mathrm{d}y \xrightarrow{\varepsilon\to 0} \int_{\mathcal{I}} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y.$$

Since $\overline{x_0} \in \mathscr{C}$, this last integral is infinite. That is,

$$\begin{aligned}\int_{\mathcal{I}} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y &\geq \int_{\mathcal{I}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathcal{I}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y + \frac{1}{2}\int_{\mathcal{O}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y^\star|\right)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathcal{I}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y + \frac{1}{2}\int_{\mathcal{O}\cap B_R(\overline{x_0})} K\left(|\overline{x_0}^\star - y|\right)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathcal{I}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y + \frac{1}{2}\int_{\mathcal{O}\cap B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{B_R(\overline{x_0})} K\left(|\overline{x_0} - y|\right)\,\mathrm{d}y = \frac{1}{2}\int_{B_R} K\left(|z|\right)\,\mathrm{d}z = +\infty.\end{aligned}$$

Note that the fact that $\overline{x_0} = (\lambda e, \lambda e)$ is fundamental in order to have $(\mathcal{I}\cap B_R(\overline{x_0}))^\star = \mathcal{O}\cap B_R(\overline{x_0})$. Therefore, for ε small enough,

$$\int_{\mathcal{I}} K(|x_0 - y|)\,\mathrm{d}y$$

is as big as we want, in particular bigger than $||c_-||_{L^\infty(H)}$. This shows (B.14) and concludes the proof. $\square$

Once this maximum principle in narrow domains is available, we can now proceed with the proof of Proposition 7.1.

Proof of Proposition 7.1. For the sake of simplicity, we will denote

$$\mathscr{L}w := Lw - f'(u)w - cw\,.$$

Let $\varepsilon > 0$ be such that the maximum principle of Proposition 7.2 is valid and define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{|x'| > |x''| + \varepsilon\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \Omega \cap \{|x''| < |x'| < |x''| + \varepsilon\}\,.$$

Recall that, by the asymptotic result, we have

$$u \geq \delta > 0 \quad \text{in } \overline{\Omega}_\varepsilon\,,$$

for some $\delta > 0$. Moreover, since f is strictly concave and $f(0) = 0$, , $f(\theta) > f'(\theta)\theta$ for $\theta \in (0,1)$. Therefore, CHECK

$$\mathscr{L}u = Lu - f'(u)u - cu \leq f(u) - f'(u)u > 0 \quad \text{in } \Omega \subseteq \mathcal{O}\,, \tag{7.4}$$

Eq:uSupersolLine

where we have used that $u > 0$ in $\mathcal{O}$ and that $c \leq 0$ in the first inequality.

By contradiction, assume that there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. Set $w := v - \tau u$. Since in $\overline{\Omega}_\varepsilon$ we have $u \geq \delta > 0$, we see that for τ big enough, $w < 0$ in $\overline{\Omega}_\varepsilon$. Moreover, since $v \leq 0$ in $\mathcal{O} \setminus \Omega$, we have

$$w \leq 0 \quad \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,.$$

Furthermore, we also have

$$\limsup_{x \in \mathcal{N}_\varepsilon,\ |x| \to \infty} w(x) \leq 0$$

and, by (B.15),

$$\mathscr{L} w = \mathscr{L} v - \tau \mathscr{L} u \leq 0 \text{ in } \mathcal{N}_\varepsilon\,.$$

Therefore, since w is odd with respect to $\mathscr{C}$, we can apply Proposition 7.2 with $H = \mathcal{N}_\varepsilon$ to deduce that

$$w \leq 0 \quad \text{in } \Omega\,,$$

if τ is big enough.

Now, define

$$\overline{\tau} := \inf\left\{\lambda > 0 \ : \ v - \tau u \leq 0 \ \text{ in } \Omega\right\}\,.$$

By the previous reasoning, $\overline{\tau}$ is well defined. Clearly, $v - \overline{\tau} u \leq 0$ in Ω. In addition, since $v(x_0) > 0$ and $u(x_0) > 0$, we have $-\overline{\tau} u(x_0) < v(x_0) - \overline{\tau} u(x_0) \leq 0$ and therefore $\overline{\tau} > 0$.

We claim that $v - \overline{\tau} u \not\equiv 0$. Indeed, if $v - \overline{\tau} u \equiv 0$ then $v = \overline{\tau} u$ and thus, since $\overline{\tau} > 0$, we get

$$0 \geq \mathscr{L} v(x_0) = \overline{\tau} \mathscr{L} u(x_0) > 0\,,$$

which is a contradiction.

Then, since $v - \overline{\tau} u \not\equiv 0$, the strong maximum principle (Proposition B.3) we have

$$v - \overline{\tau} u < 0 \quad \text{in } \Omega\,.$$

Therefore, by continuity and the assumption on v at infinity, there exists $0 < \eta < \overline{\tau}$ such that

$$\tilde{w} := v - (\overline{\tau} - \eta) u < 0 \quad \text{in } \overline{\Omega}_\varepsilon\,.$$

Using again the maximum principle in narrow domains with $\tilde{w}$ in $\mathcal{N}_\varepsilon$, we deduce that

$$v - (\overline{\tau} - \eta) u \leq 0 \quad \text{in } \Omega\,,$$

and this contradicts the definition of $\overline{\tau}$. Hence, $v \leq 0$ in Ω. □

With these ingredients available we can finally establish the uniqueness of the saddle-shaped solution.

Proof of Theorem 1.10. Let u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$ which is a doubly radial function that is odd with respect to $\mathscr{C}$. Then,

$$Lv = f(u_1) - f(u_2) \leq f'(u_2)(u_1 - u_2) = f'(u_2) v \quad \text{in } \mathcal{O}\,,$$

since f is concave in $(0,1)$. Moreover, by the asymptotic result (see Theorem 6.2), we have

$$\limsup_{x\in\mathcal{O},\ |x|\to\infty} v(x) = 0\,.$$

Then, by the maximum principle in $\mathcal{O}$ for the linearized operator $L - f'(u_2)$ (see Proposition 7.1), we are lead to $v \leq 0$ in $\mathcal{O}$, which means $u_1 \leq u_2$ in $\mathcal{O}$. Repeating the argument with $-v = u_2 - u_1$ we deduce $u_1 \geq u_2$ in $\mathcal{O}$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m}$. $\square$

Appendix A. Some auxiliary results on convex functions

uxiliaryResults⟩? In this appendix we present some auxiliary results that are needed in the paper. Such results concern convex functions. Recall that for measurable functions $f : \mathbb{R} \to \mathbb{R}$, convexity in an open interval I is equivalent to midpoint convexity, i.e.,

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) \qquad \text{for every } x,\, y \in I\,,$$

and the same is true for strict convexity with an strict inequality (see Chapter 1 of [24] and the references therein).

eflectionsConvex⟩ **Lemma A.1.** *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a measurable function. Then, $h(z)$ is convex in I if and only if $\widetilde{h}_c(z) := h(c+z) + h(c-z)$ is convex in $I_c := (-\operatorname{dist}\{c, \partial I\}, \operatorname{dist}\{c, \partial I\})$ for every $c \in I$. The statement remains true if we replace convexity by strict convexity.*

Proof. Assume first that h is convex in I. We call

$$x_+ = c + x\,, \quad x_- = c - x\,, \quad y_+ = c + y\,, \quad \text{and} \quad y_- = c - y\,.$$

Therefore, if $x,\, y \in I_c$, we have that $x_+,\, x_-,\, y_+,\, y_- \in I$. Hence, for all $t \in (0,1)$

$$\begin{aligned} t\widetilde{h}_c(x) + (1-t)\widetilde{h}_c(y) &= th(x_+) + (1-t)h(y_+) + th(x_-) + (1-t)h(y_-) \\ &\geq h(tx_+ + (1-t)y_+) + h(tx_- + (1-t)y_-) \\ &= h(c + tx + (1-t)y) + h(c - tx + (1-t)y) \\ &= \widetilde{h}_c(tx + (1-t)y)\,. \end{aligned}$$

Therefore, $\widetilde{h}_c(z)$ is convex in I_c for every $c \in I$.

Assume now that $\widetilde{h}_c(z)$ is convex in I_c for every $c \in I$. By contradiction, suppose that h is not convex in I. Then, there exist some $x,\, y \in I$ such that

$$\frac{h(x)+h(y)}{2} < h\left(\frac{x+y}{2}\right)\,. \tag{A.1}$$

Eq:Contradiction

Let $c = (x+y)/2$ and thus

$$\widetilde{h}_c(z) = h\left(\frac{x+y}{2} + z\right) + h\left(\frac{x+y}{2} - z\right)\,.$$

Define $x_0 := (x-y)/2$ and $y_0 := (y-x)/2$. It is clear that $x_0,\, y_0 \in I_c$. Therefore,

$$h(x) + h(y) = \frac{1}{2}\left(\widetilde{h}_c(x_0) + \widetilde{h}_c(y_0)\right) \geq \widetilde{h}_c\left(\frac{x_0+y_0}{2}\right) = 2h\left(\frac{x+y}{2}\right)\,,$$

and this contradicts (A.1). Hence, h is convex in I. □

<->K(sqrt)convex⟩ **Corollary A.2.** *Let $K : (0,+\infty) \to (0,+\infty)$ be a measurable function. Then, given any $c_1, c_2 > 0$, the function*

$$g(z) := K\left(c_1\sqrt{1+c_2 z}\right) + K\left(c_1\sqrt{1-c_2 z}\right)$$

is convex in $(-1/c_2, 1/c_2)$ if and only if $K(\sqrt{z})$ is convex in $(0,+\infty)$.

Proof. The function g is convex in $(-1/c_2, 1/c_2)$ if and only if

$$K\left(\sqrt{c_1^2 + c_1^2 c_2 z}\right) + K\left(\sqrt{c_1^2 - c_1^2 c_2 z}\right)$$

is convex in $(-1/c_2, 1/c_2)$, and this is equivalent to say that

$$K\left(\sqrt{c_1^2 + z}\right) + K\left(\sqrt{c_1^2 - z}\right)$$

is convex in $(-c_1^2, c_1^2)$. By Lemma A.1, this is equivalent to the convexity of $K(\sqrt{z})$ in $(0, +\infty)$. ☐

ConvexFunctions⟩ **Lemma A.3.** *Let $g\colon I \subset \mathbb{R} \to \mathbb{R}$ be a function defined in an open interval I such that g is measurable and nondecrasing in I. Then, we have the following equivalences:*

i) g is strictly convex in I.
ii) For any given real numbers A, B, C, $D \in I$ such that

$$\begin{cases} A = \max\{A, B, C, D\}\,, \\ A + D \geq B + C\,, \end{cases} \tag{A.2}$$

Eq:AssumptionsIn

it is satisfied that

$$g(A) + g(D) \geq g(B) + g(C)$$

and equality holds if and only if

$$A = B \qquad \textit{and} \qquad C = D,$$

or

$$A = C \qquad \textit{and} \qquad B = D.$$

Proof. i) $\Rightarrow$ *ii)* First note that from the strictly convexity condition of g we have that g is $C^0(I)$ and C^1 at all but at most countably many points and admits left and right derivatives (g'_- and g'_+ respectively), and these are monotonically increasing. Moreover, from being g strictly convex and nondecreasing, it is in fact increasing.

Without loss of generality, we can assume that $B \geq C$. Now, we distiguish two cases:

•$D \geq C$

This is the simpler case. Since g is nondecreasing and $A \geq B$, $D \geq C$, we have that $g(A) \geq g(B)$ and $g(D) \geq g(C)$. By adding up these two inequalities we get the desired result. Moreover, if $A > B$ or $D > C$ we obtain the strict inequality since g is in fact increasing.

•$D < C$

If $A = B$ we arrive at a contradiction from (A.3). Then, we have $A > B \geq C > D$.

Since g is piecewise C^1, let $\{z_i\}_{i=1}^n$ be the nondifferentiability points in the interval (B, A) with $z_0 = B$ and $z_{n+1} = A$, and let $\{\bar{z}_i\}_{i=1}^m$ be the nondifferentiability points in the interval (D, C) with $\bar{z}_0 = D$ and $\bar{z}_{m+1} = C$.

By applying Taylor's theorem in each interval of differentiability and adding the expressions we obtain

$$g(A) = g(B) + \sum_{i=0}^{n} (z_{i+1} - z_i) g'(\xi_i) \quad \text{ with } \xi_i \in (z_i, z_{i+1}) \,,$$

and

$$g(C) = g(D) + \sum_{i=0}^{m} (\bar{z}_{i+1} - \bar{z}_i) g'(\bar{\xi}_i) \quad \text{ with } \bar{\xi}_i \in (\bar{z}_i, \bar{z}_{i+1}) \,,$$

If we define

$$\xi = \xi_0 \qquad \text{and} \qquad \bar{\xi} = \bar{\xi}_m,$$

since g' is nondecreasing in the regular points we get

$$g(A) \geq g(B) + \sum_{i=0}^{n} (z_{i+1} - z_i) g'(\xi) = g(B) + (A - B) g'(\xi),$$

and

$$g(C) \leq g(D) + \sum_{i=0}^{m} (\bar{z}_{i+1} - \bar{z}_i) g'(\bar{\xi}) = g(D) + (C - D) g'(\bar{\xi}).$$

It is clear that $\xi > \bar{\xi}$ and then $g'(\xi) > g'(\bar{\xi})$. Therefore,

$$\begin{aligned} g(A) + g(D) - g(B) - g(C) &= (g(A) - g(B)) - (g(C) - g(D)) \\ &\geq (A - B) g'(\xi) - (C - D) g'(\bar{\xi}) \\ &= (A - B - C + D) g'(\xi) + (C - D)(g'(\xi) - g'(\bar{\xi})) \\ &> 0 \,, \end{aligned}$$

where we have used that $A - B - C + D \geq 0$, $C - D > 0$, $g' \geq 0$ in the regular points and $g'(\xi) > g'(\bar{\xi})$.

ii) $\Rightarrow$ *i)* Given $x \neq y$ in I, that we can suppose $x > y$ without loss of generality, we define $A = x$, $B = C = (x + y)/2$ and $D = y$. Then we get

$$g(x) + g(y) > 2g\left(\frac{x + y}{2}\right),$$

that from being g measurable means it is strictly convex. □

ConvexFunctions⟩ *Remark* A.4. Note that the condition of strict convexity is only needed in order to characterize when the equality is satisfied. That is, with only a convexity condition we also obtain the inequality although we are not able to determine when equality is satisfied.

reNondecreasing⟩ *Remark* A.5. The deduction of *i)* from *ii)* does not require g to be nondecreasing.

DecreasingConvex⟩ **Corollary A.6.** *Let $h\colon I \subset \mathbb{R} \to \mathbb{R}$ be a function defined in an open interval I such that h is measurable and nonincrasing in I. Then, we have the following equivalences:*

i) h is convex in I.

ii) For any given real numbers $a,\ b,\ c,\ d \in I$ such that

$$\begin{cases} a \geq b \geq c \geq d\,, \\ a + d \leq b + c\,. \end{cases} \tag{A.3}$$
Eq:AssumptionsIn

it is satisfied that

$$h(a) + h(d) \geq h(b) + h(c)\,.$$

Proof. Let us define $g(z) = h(a - z)$. It is clear that since h is measurable and nonincreasing, then g is measurable and nondecreasing. On the other hand, let $A = a - d$, $B = a - c$, $C = a - b$ and $D = 0$. Then, we have that condition $a \geq b \geq c \geq d$ is equivalent to $A \geq B \geq C \geq D$ and condition $a + d \leq b + c$ is equivalent to $A + D \geq B + C$. Therefore, we can apply Lemma A.3, taking into account Remark A.4, and obtain the desired equivalence. ☐

a:gNondecreasing⟩ **Lemma A.7.** *Let $K : (0, +\infty) \to (0, +\infty)$ be a measurable function such that*

$$\lim_{z \to \infty} K(z) = 0$$

and $K(\sqrt{z})$ is convex. Then, given any $c_1, c_2 > 0$ the function

$$g(z) := K(c_1\sqrt{1 + c_2 z}) + K(c_1\sqrt{1 - c_2 z})$$

is nondecreasing $(0, 1/c_2)$.

Proof. First of all, we can note that from the hypothesis of K we can deduce that $K(\sqrt{z})$, and also $K(c_1\sqrt{z})$, is convex and nonincreasing.

Now, given $x \geq y$, we have

$$\begin{aligned} g(x) - g(y) = {}& K(c_1\sqrt{1 + c_2 x}) + K(c_1\sqrt{1 - c_2 x}) \\ & - K(c_1\sqrt{1 + c_2 y}) - K(c_1\sqrt{1 - c_2 y}) \geq 0, \end{aligned}$$

where the inequality follows from applying Corollary A.6 with $h(z) = K(c_1\sqrt{z})$, $a = 1 + c_2 x$, $b = 1 + c_2 y$, $c = 1 - c_2 y$ and $d = 1 - c_2 x$. ☐

ex<->Inequality⟩ **Proposition A.8.** *Let $K : (0, +\infty) \to (0, +\infty)$ be a continuous and nonincreasing function such that*

$$\lim_{z \to +\infty} K(z) = 0\,.$$

Then, the following statements are equivalent:

i) $K(\sqrt{z})$ is strictly convex in $(0, +\infty)$.
ii) For every $c_1,\ c_2,\ A,\ B,\ C$ and D satisfying
  (a) $c_1 > 0,\ c_2 > 0$.
  (b) $A,\ B,\ C,\ D \in (0, 1/c_2)$.
  (c) $A = \max\{A,\ B,\ C,\ D\}$.
  (d) $A + D \geq B + C$.

it holds

$$g(A) + g(D) \geq g(B) + g(C)\,,$$

where

$$g(z) := K(c_1\sqrt{1+c_2 z}) + K(c_1\sqrt{1-c_2 z})\,.$$

Moreover, equality holds if and only if

$$A = B \qquad \text{and} \qquad C = D,$$

or

$$A = C \qquad \text{and} \qquad B = D.$$

Proof. $i) \Rightarrow ii)$ By Lemma A.7 and Corollary A.2, g is nondecreasing and strictly convex. Therefore, point $ii)$ follows from Lemma A.3.

$ii) \Rightarrow i)$ By Lemma A.3 and in view of Remark A.5, g is strictly convex. Hence, using Lemma A.2 we deduce that $K(\sqrt{z})$ is strictly convex in $(0, +\infty)$. □

ComputationABCD⟩ **Lemma A.9.** *Let α, β two real numbers satisfying $\alpha \geq |\beta|$. Let $x = (x', x'')$, $y = (y', y'') \in \mathcal{O} \subset \mathbb{R}^{2m}$. Define*

$$\begin{array}{ll} A = |x'||y'|\alpha + |x''||y''|\beta\,, & B = |x'||y''|\alpha + |x''||y'|\beta\,, \\ C = |x''||y'|\alpha + |x'||y''|\beta\,, & D = |x''||y''|\alpha + |x'||y'|\beta\,. \end{array}$$

Then,

(1) It holds

$$\begin{cases} |A| \geq |B|, \ |A| \geq |C|, \ |A| \geq |D|\,, \\ |A| + |D| \geq |B| + |C|\,. \end{cases}$$

(2) If either

$$|A| = |B| \qquad \text{and} \qquad |C| = |D|,$$

or

$$|A| = |C| \qquad \text{and} \qquad |B| = |D|,$$

then necessarily $\alpha = \beta = 0$.

Proof. Since $\alpha \geq |\beta|$,

$$\alpha \geq 0 \quad \text{and} \quad -\alpha \leq \beta \leq \alpha\,.$$

Moreover, since $x, y \in \mathcal{O}$ it holds

$$|x'| > |x''| \quad \text{and} \quad |y'| > |y''|\,.$$

These inequalities will be used in all the proof.

We start with the first point. First, we show that $A \geq 0$ and that

$$A \geq |B|, \ A \geq C, \ |A| \geq D\,.$$

- $A \geq 0$:

$$A = |x'||y'|\alpha + |x''||y''|\beta \geq (|x'||y'| - |x''||y''|)\alpha \geq 0\,.$$

- $A \geq |B|$:
$$A \pm B = (|x'|\alpha - |x''|\beta)(|y'| \pm |y''|) \geq 0\,.$$
- $A \geq |C|$:
$$A \pm C = (|y'|\alpha - |y''|\beta)(|x'| \pm |x''|) \geq 0\,.$$
- $A \geq |D|$:
$$A \pm D = (|x'||y'| \pm |x''||y''|)(\alpha \pm \beta) \geq 0\,.$$

It remains to show
$$A + |D| \geq |B| + |C|\,.$$
The proof of it is just a computation considering all the eight possible configurations of the signs of B, C and D. Since the roles of B and C are completely interchangeable, we may assume that $B \geq C$ and we only need to check six cases. To do it, note first that
$$A + D - B - C = (|x'| - |x''|)(|y'| - |y''|)(\alpha + \beta) \geq 0\,, \tag{A.4}$$ Eq:LemmaABCDProc
$$A - D - B + C = (|x'| + |x''|)(|y'| - |y''|)(\alpha - \beta) \geq 0\,, \tag{A.5}$$ Eq:LemmaABCDProc
and
$$A + D + B + C = (|x'| + |x''|)(|y'| + |y''|)(\alpha + \beta) \geq 0\,, \tag{A.6}$$ Eq:LemmaABCDProc
With these three relations at hand we check the six cases.

- If $B \geq 0$, $C \geq 0$ and $D \geq 0$, then by (A.4) we have
$$A + |D| - |B| - |C| = A + D - B - C = \geq 0\,.$$
- If $B \geq 0$, $C \geq 0$ and $D \leq 0$, we use that $D \leq 0$ and (A.4) to see that
$$A + |D| - |B| - |C| = A - D - B - C = A + D - B - C - 2D \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$ and $D \geq 0$, we use that $D \geq 0$ and (A.5) to see that
$$A + |D| - |B| - |C| = A + D - B + C = A - D - B + C + 2D \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$ and $D \leq 0$, then by (A.5) we have
$$A + |D| - |B| - |C| = A - D - B + C = \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$ and $D \geq 0$, then by (A.6) we have
$$A + |D| - |B| - |C| = A + D + B + C = \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$ and $D \leq 0$, we use that $D \leq 0$ and (A.6) to see that
$$A + |D| - |B| - |C| = A - D + B + C = A + D + B + C - 2D \geq 0\,.$$

This concludes the proof of the first statement.

We prove now the second point of the lemma. Since the roles of B and C are completely interchangeable, we only need to show the result in the case $|A| = |B|$ and $|C| = |D|$.

Recall that $A \geq 0$. Hence, since $A = |B|$ and $|C| = |D|$, a simple computation shows that
$$\alpha = \operatorname{sign}(B)\frac{|x''|}{|x'|}\beta \quad \text{ and } \quad \beta = \operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|}{|x'|}\alpha\,.$$

Hence,

$$\alpha = \operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|^2}{|x'|^2}\alpha$$

and if we assume $\alpha \neq 0$, then necessarily $|x'| = |x''|$, but this is a contradiction with $x \in \mathcal{O}$. Therefore, $\alpha = 0$ and thus $\beta = 0$. Therefore, $\alpha = 0$ and thus $\beta = 0$. □

Appendix B. Computations in s and t coordinates

stcomputations⟩?

B.1. **The nonlocal Allen-Cahn equation in the (s,t) variables.** The goal of this subsection is to write equation (1.1) in the (s,t) variables. For this, we must find a new kernel in these variables.

orInSTVariables⟩

Lemma B.1. *Let $m \geq 1$, $\gamma \in (0,1)$ and let $u \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$ with $\alpha > 2\gamma$ be a function depending only on the variables s and t. Let L be an operator of the form (1.2) with a kernel $K \geq 0$ satisfying $K(y) = K(|y|)$. Then, for any $x = (sx_s, tx_t)$ with x_s, x_t $\in \mathbb{S}^{m-1}$ (x_s, $x_t = \pm 1$ in the case $m = 1$), $Lu(x)$ can be written in the following way:*

Is singular in $s = 0$ or $t = 0$??

$$Lu(x) = \widetilde{L}u(s,t) := \int_0^{+\infty}\int_0^{+\infty} \sigma^{m-1}\tau^{m-1}\big(u(s,t) - u(\sigma,\tau)\big)J(s,t,\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \tag{B.1}$$

Eq:OperatorInSTV

where:

(1) If $m = 1$,

$$J(s,t,\sigma,\tau) := \sum_{i=0}^{1}\sum_{j=0}^{1} K\Big(\sqrt{s^2 + t^2 + \sigma^2 + \tau^2 - 2s\sigma(-1)^i - 2t\tau(-1)^j}\Big)\,. \tag{B.2}$$

Eq:KernelInSTVar

(2) If $m \geq 2$,

$$\begin{aligned} J(s,t,\sigma,\tau) &:= \int_{\mathbb{S}^{m-1}}\int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + t^2 + \sigma^2 + \tau^2 - 2s\sigma\omega_1 - 2t\tau\tilde{\omega}_1}\Big)\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega} && \text{(B.3)}\\ &= c_m \int_{-1}^{1}\int_{-1}^{1}(1-\theta^2)^{\frac{m-3}{2}}(1-\overline{\theta}^2)^{\frac{m-3}{2}} \\ &\qquad K\Big(\sqrt{s^2 + t^2 + \sigma^2 + \tau^2 - 2s\sigma\theta - 2t\tau\overline{\theta}}\Big)\,\mathrm{d}\theta\,\mathrm{d}\overline{\theta}\,, && \text{(B.4)}\end{aligned}$$

Eq:KernelSTVaria

Eq:KernelSTVaria

with

$$c_m = \left(\frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}\right)^2.$$

Proof. We start with the case $m = 1$. In this case, we will use explicitly that since u is a function of s and t, then u is even with respect to the coordinate axis. Using this symmetry and the change $y = -\tilde{y}$, we have

$$Lu(x) = \int_{\{y_2 > -y_1\}} \big(u(x) - u(y)\big)\big\{K(|x-y|) + K(|x+y|)\big\}\,\mathrm{d}y\,.$$

If we call

$$I(\Omega,x) := \int_\Omega \big(u(x)-u(y)\big)\{K(|x-y|)+K(|x+y|)\}\,\mathrm{d}y\,,$$

then

$$Lu(x) = I(\{y_2>|y_1|\},x) + I(\{y_1>|y_2|\},x)\,.$$

We will check that $I(\{y_2>|y_1|\},x)$ can be written in the form (B.1) (integrated in the set $\tau>\sigma$). The computations for $I(\{y_1>|y_2|\},x)$ are completely analogous.

First, note that the set $\{y_2>|y_1|\}$ can be written as

$$\{y_2>y_1>0\}\cup\phi(\{y_2>y_1>0\})\cup\{y_2>0,y_1=0\}$$

where ϕ is the reflection with respect to the y_2-axis. Therefore,

$$\begin{aligned}I(\{y_2>|y_1|\},x) = &\int_{\{y_2>y_1>0\}} \big(u(x)-u(y)\big)\{K(|x-y|)+K(|x+y|)\}\,\mathrm{d}y\\ &+\int_{\phi(\{y_2>y_1>0\})} \big(u(x)-u(y)\big)\{K(|x-y|)+K(|x+y|)\}\,\mathrm{d}y\,.\end{aligned}$$

By performing the change $\phi=\phi^{-1}$ in the second integral and using the symmetry of u, we end up with

$$\begin{aligned}I(\{y_2>|y_1|\},x) = \int_{\{y_2>y_1>0\}} &\big(u(x)-u(y)\big)\cdot\\ &\{K(|x-y|)+K(|x+y|)+K(|x-\phi_1(y)|)+K(|x+\phi_1(y)|)\}\,\mathrm{d}y\,.\end{aligned}$$

Then, if in the previous expression we write

$$x=(s\,\mathrm{sign}(x_1),t\,\mathrm{sign}(x_2))\quad\text{ and }\quad =(\sigma\,\mathrm{sign}(y_1),\tau\,\mathrm{sign}(y_2))\,,$$

we find that

$$I(\{y_2>|y_1|\},x) = \int_0^\infty \mathrm{d}\sigma\int_\sigma^\infty \mathrm{d}\tau\;\;\big(u(s,t)-u(\sigma,\tau)\big)J(s,t,\sigma,\tau)\,,$$

with J as in (B.2). Indeed, in $\{y_2>y_1>0\}$ we have that $y=(\sigma,\tau)$, and it is not difficult to check that the expression

$$K(|x-y|)+K(|x+y|)+K(|x-\phi_1(y)|)+K(|x+\phi_1(y)|)$$

does not depend on $\mathrm{sign}(x_1)$ nor $\mathrm{sign}(x_1)$, so we can assume that $x=(s,t)$ and then

$$\begin{aligned}&K(|x-y|)+K(|x+y|)+K(|x-\phi_1(y)|)+K(|x+\phi_1(y)|)=\\ &K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2+2s\sigma+2t\tau}\Big)+K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2+2s\sigma-2t\tau}\Big)\\ &+K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma+2t\tau}\Big)+K\Big(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma-2t\tau}\Big)\,.\end{aligned}$$

In a completely analogous way, we find that

$$I(\{y_1>|y_2|\},x) = \int_0^\infty \mathrm{d}\sigma\int_0^\sigma \mathrm{d}\tau\;\;\big(u(s,t)-u(\sigma,\tau)\big)J(s,t,\sigma,\tau)\,,$$

and hence

$$Lu(x) = \int_0^\infty \mathrm{d}\sigma \int_0^\infty \mathrm{d}\tau \quad \big(u(s,t) - u(\sigma,\tau)\big) J(s,t,\sigma,\tau) =: \widetilde{L}u(s,t)\,.$$

This concludes the proof when $m = 1$.

We deal now with the case $m = 2$. Let $x = (sx_s, tx_t)$ with $x_s,\ x_t \in \mathbb{S}^{m-1}$ and $y = (\sigma y_\sigma, \tau y_\tau)$ with $y_\sigma,\ y_\tau \in \mathbb{S}^{m-1}$. Then,

$$\begin{aligned} Lu(x) &= \int_{\mathbb{R}^{2m}} \big(u(x) - u(y)\big) K(|x-y|)\,\mathrm{d}y \\ &= \int_0^{+\infty} \int_0^{+\infty} \sigma^{m-1}\tau^{m-1}\big(u(s,t) - u(\sigma,\tau)\big) \\ &\qquad \left(\int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma y_\sigma|^2 + |tx_t - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau \right) \mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

Now, we make some manipulations to the term

$$J(x_s, x_t, s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma y_\sigma|^2 + |tx_t - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau\,. \tag{B.5}$$ Eq:KernelSTVaria

First of all, it is easy to see that (B.5) does not depend on x_s nor x_t. Indeed, consider a different point (z_s, z_t) and let M_s and M_t be two orthogonal transformations such that $M_s(x_s) = z_s$ and $M_t(x_t) = z_t$. Then, making the change of variables $y_\sigma = M_s(\tilde{y}_\sigma)$ and $y_\tau = M_t(\tilde{y}_\tau)$, and using that $M_s(\mathbb{S}^{m-1}) = M_t(\mathbb{S}^{m-1}) = \mathbb{S}^{m-1}$, we find out that

$$\begin{aligned} &J(z_s, z_t, s, t, \sigma, \tau) = \\ &\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma y_\sigma|^2 + |tM_t(x_t) - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau \\ &\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma M_s(\tilde{y}_\sigma)|^2 + |tM_t(x_t) - \tau M_t(\tilde{y}_\tau)|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\ &\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|M_s(sx_s - \sigma \tilde{y}_\sigma)|^2 + |M_t(tx_t - \tau \tilde{y}_\tau)|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\ &\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma \tilde{y}_\sigma|^2 + |tx_t - \tau \tilde{y}_\tau|^2}\Big)\,\mathrm{d}\tilde{y}_\sigma\,\mathrm{d}\tilde{y}_\tau \\ &\quad = J(x_s, x_t, s, t, \sigma, \tau)\,. \end{aligned}$$

Therefore, we can replace x_s and x_t in (B.5) by $e = (1, 0, \dots, 0) \in \mathbb{S}^{m-1}$. Thus, we have

$$J(s,t,\sigma,\tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2}\Big)\,\mathrm{d}y_\sigma\,\mathrm{d}y_\tau\,. \tag{B.6}$$?Eq:KernelSTVari

For an easier notation, we rename $\omega = y_\sigma$ and $\tilde\omega = y_\tau$, and thus we have

$$\begin{aligned}|se-\sigma y_\sigma|^2 + |te-\tau y_\tau|^2 &= |se-\sigma\omega|^2 + |te-\tau\tilde\omega|^2\\ &= s^2+\sigma^2-2s\sigma e\cdot\omega + t^2+\tau^2-2t\tau e\cdot\tilde\omega\\ &= s^2+\sigma^2-2s\sigma\omega_1+t^2+\tau^2-2t\tau\tilde\omega_1\,.\end{aligned}$$

This gives (B.3). Finally, we use polar coordinates in $\mathbb{S}^{m-1}$ to obtain (B.4). ☐

B.2. **Weak and strong maximum principle.**

Falta poner el weak max principle en estas variables

Finally, in this subsection we show a strong maximum principle for functions that are odd with respect to the Simons cone $\mathscr{C}$. To do this, we need to show first the following inequality for the kernel in the (s,t) variables.

elInequalityCone⟩ **Lemma B.2.** *Let $m \geq 2$ and let J the kernel defined in (B.4). Then, if $s > t > 0$ and $\sigma > \tau > 0$, we have*

$$J(s,t,\sigma,\tau) \geq J(s,t,\tau,\sigma)\,. \tag{B.7}$$

Eq:KernelInequal

Check if we can put $t = 0$ or $\tau = 0$

Proof. We will just consider some simplifications of (B.7). Eventually, we will use Lemma A.3 to deduce the desired inequality.

First of all, note that it is enough to show that

Kernel strictly decreasing???

$$\int_{-1}^{1}\int_{-1}^{1}(1-\alpha^2)^{\frac{m-3}{2}}(1-\beta^2)^{\frac{m-3}{2}}\left\{\tilde K\left(\sqrt{1-2s\sigma\alpha-2t\tau\beta}\right)-\tilde K\left(\sqrt{1-2s\tau\alpha-2t\sigma\beta}\right)\right\}\,d\alpha\,d\beta \geq 0, \tag{B.8}$$

Eq:KernelInequal

Where $\tilde K(z) = K((s^2+t^2+\sigma^2+\tau^2)z)$. To see that this is equivalent to (B.7), we just normalize the variables in the following way:

$$\tilde s = \frac{s}{\sqrt{s^2+t^2+\sigma^2+\tau^2}}\,,\quad \tilde t = \frac{t}{\sqrt{s^2+t^2+\sigma^2+\tau^2}}\,,$$

$$\tilde\sigma = \frac{\sigma}{\sqrt{s^2+t^2+\sigma^2+\tau^2}}\quad\text{and}\quad \tilde\tau = \frac{\tau}{\sqrt{s^2+t^2+\sigma^2+\tau^2}}\,.$$

The second simplification is the following. Since $s > t > 0$ and $\sigma > \tau > 0$, we may write

$$s = (1+\varepsilon)t \quad\text{and}\quad \sigma = (1+\delta)\tau$$

with ε, $\delta > 0$. Then, (B.8) is equivalent to

$$\int_{-1}^{1}\int_{-1}^{1}(1-\alpha^2)^{\frac{m-3}{2}}(1-\beta^2)^{\frac{m-3}{2}}\left\{\tilde K\left(\sqrt{1-2t\tau\{(1+\varepsilon)(1+\delta)\alpha+\beta\}}\right)-\tilde K\left(\sqrt{1-2t\tau\{(1+\varepsilon)\alpha+(1+\delta)\right.\right. \tag{B.9}$$

Eq:KernelInequal

Now, we make some changes of variables to reduce the domain of integration. First, we divide $(-1,1)^2 \setminus \{|\alpha| = |\beta|\}$ into four sectors:

$$Q_1 = \{\alpha > |\beta|\}\,,\quad Q_2 = \{\beta > |\alpha|\}\,,\quad Q_3 = \{\alpha < -|\beta|\}\,,\quad\text{and}\quad Q_4 = \{\beta < -|\alpha|\}\,.$$

Consider the changes

$$\begin{aligned}\psi_2\colon Q_2 &\to Q_1\\ (\alpha,\beta) &\mapsto (\beta,\alpha)\end{aligned}$$

$$\begin{aligned}\psi_3\colon Q_3 &\to Q_1\\ (\alpha,\beta) &\mapsto (-\alpha,-\beta)\end{aligned}$$

$$\begin{aligned}\psi_4\colon Q_4 &\to Q_1\\ (\alpha,\beta) &\mapsto (-\beta,-\alpha)\end{aligned}$$

Then, (B.9) is now equivalent to show

$$\begin{aligned}\int\int_{Q_1} \Big\{ &\tilde K\Big(\sqrt{1-2t\tau\{(1+\varepsilon)(1+\delta)\alpha+\beta\}}\Big) + \tilde K\Big(\sqrt{1+2t\tau\{(1+\varepsilon)(1+\delta)\alpha+\beta\}}\Big)\\ &+\tilde K\Big(\sqrt{1-2t\tau\{(1+\varepsilon)(1+\delta)\beta+\alpha\}}\Big) + \tilde K\Big(\sqrt{1+2t\tau\{(1+\varepsilon)(1+\delta)\beta+\alpha\}}\Big)\\ &-\tilde K\Big(\sqrt{1-2t\tau\{(1+\varepsilon)\alpha+(1+\delta)\beta\}}\Big) - \tilde K\Big(\sqrt{1+2t\tau\{(1+\varepsilon)\alpha+(1+\delta)\beta\}}\Big)\\ &-\tilde K\Big(\sqrt{1-2t\tau\{(1+\varepsilon)\beta+(1+\delta)\alpha\}}\Big) - \tilde K\Big(\sqrt{1+2t\tau\{(1+\varepsilon)\beta+(1+\delta)\alpha\}}\Big)\Big\}\cdot\\ &\cdot(1-\alpha^2)^{\frac{m-3}{2}}(1-\beta^2)^{\frac{m-3}{2}}\,\mathrm{d}\alpha\,\mathrm{d}\beta \geq 0\,.\end{aligned}$$

Note that we are not considering anymore the set $\{|\alpha|=|\beta|\}$, that has measure zero.

Now, if we call

$$g(z) := \tilde K\Big(\sqrt{1-2t\tau z}\Big) + \tilde K\Big(\sqrt{1+2t\tau z}\Big)\,,$$

the previous inequality reads

$$\begin{aligned}\int\int_{Q_1} \Big\{ &g\Big((1+\varepsilon)(1+\delta)\alpha+\beta\Big) + g\Big((1+\varepsilon)(1+\delta)\beta+\alpha\Big)\\ &-g\Big((1+\varepsilon)\alpha+(1+\delta)\beta\Big) - g\Big((1+\varepsilon)\beta+(1+\delta)\alpha\Big)\Big\}\cdot\\ &\cdot(1-\alpha^2)^{\frac{m-3}{2}}(1-\beta^2)^{\frac{m-3}{2}}\,\mathrm{d}\alpha\,\mathrm{d}\beta \geq 0\,.\end{aligned} \tag{B.10}$$

?Eq:KernelInequa

We claim that

$$g\Big((1+\varepsilon)(1+\delta)\alpha+\beta\Big)+g\Big((1+\varepsilon)(1+\delta)\beta+\alpha\Big) \geq g\Big((1+\varepsilon)\alpha+(1+\delta)\beta\Big)+g\Big((1+\varepsilon)\beta+(1+\delta)\alpha\Big)\,. \tag{B.11}$$

Eq:KernelInequal

This will conclude the proof.

To prove the claim, we want to use Lemma A.3 with

$$\begin{aligned} A &= (1+\varepsilon)(1+\delta)\alpha+\beta\,, \quad B = (1+\varepsilon)\alpha+(1+\delta)\beta\,,\\ C &= (1+\delta)\alpha+(1+\varepsilon)\beta\,, \quad D = \alpha+(1+\varepsilon)(1+\delta)\beta\,.\end{aligned}$$

Note that, since $\tilde K$ and $\sqrt{1-z}$ are nonincreasing, g is nondecreasing $[0,1)$. Moreover, since $\tilde K$ is convex and nonincreasing and $\sqrt{1-z}$ is concave, g is convex in $[0,1)$. But since A, B, C and $D \in (-1,1)$ —by the normalizations we have made— and g is even, we cannot apply directly Lemma A.3 (g is nonincreasing in $(-1,0]$). Instead, if we do it for $|A|$, $|B|$, $|C|$ and $|D|$, then we can use the lemma in $[0,1)$. Hence, we should check that

$$\begin{cases} |A| \geq |B|, \ |C|, \ |D| \,, \\ |A| + |D| \geq |B| + |C| \,. \end{cases}$$

The verification of these inequalities is a simple but tedious computation and it is presented in the appendix (see Lemma A.9). Once we have these inequalities, we use Lemma A.3 to deduce

$$g(|A|) + g(|D|) \geq g(|B|) + g(|C|) \,,$$

which is equivalent to (B.11) since g is even. This concludes the proof of (B.7).

Finally, to justify that the inequality in (B.7) is strict up to a set of measure zero, we consider the points where $J(s,t,\sigma,\tau) = J(s,t,\tau,\sigma)$. Following the previous arguments, this is equivalent to say that we have an equality in (B.9), and since the integrand of that equality is nonnegative, it is equivalent to say that we have an equality in (B.11) up to a set of measure zero. Then, we take into account the following: since the function $\sqrt{1-z}$ is increasing and strictly convex and the kernel K is decreasing, then $g'' > 0$ in $(0,1)$. Therefore, in view of Remark (**??**), the equality $g(A) + g(D) = g(B) + g(C)$ is only possible in the case $A = B = C = D$, and it is easy to verify that this cannot happen unless $A = B = C = D = 0$. This is equivalent to $s = t$ and $\sigma = \tau$, something that is impossible in $\mathcal{O}$. Check again

□

eForOddFunctions⟩ **Proposition B.3** (Strong maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $u \in C^{2s}(\mathbb{R}^{2m})$ and assume that $u = u(s,t)$ and that u is odd with respect to the Simons cone $\mathscr{C}$, i.e., $u(s,t) = -u(t,s)$. Assume that $Lu \geq 0$ in $\mathcal{O}$, where $L \in \mathcal{L}_0$ such that .... Then, either $u \equiv 0$ or $u > 0$ in $\mathcal{O}$.* write well the space

Proof. Assume that $u \not\equiv 0$. We shall prove that $u > 0$ in $\mathcal{O}$. By contradiction, assume that there exists a point $x_0 \in \mathcal{O}$ with $|x_0|^2 = s_0^2 + t_0^2$ such that $u(s_0,t_0) = 0$. Then, we have Hypothesis for the inequality of the kernels

$$\begin{aligned} 0 \leq Lu(x_0) &= \widetilde{L}u(s_0,t_0) \\ &= -\int_0^{+\infty} \mathrm{d}\sigma \int_0^{+\infty} \mathrm{d}\tau \ \ \sigma^{m-1}\tau^{m-1} u(\sigma,\tau) J(s_0,t_0,\sigma,\tau) \\ &= -\int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \ \ \sigma^{m-1}\tau^{m-1} u(\sigma,\tau) J(s_0,t_0,\sigma,\tau) \\ &\quad - \int_0^{+\infty} \mathrm{d}\sigma \int_\sigma^{+\infty} \mathrm{d}\tau \ \ \sigma^{m-1}\tau^{m-1} u(\sigma,\tau) J(s_0,t_0,\sigma,\tau) \,. \end{aligned}$$

Now, we compute the second term. By changing the order of integration and performing the change $(\sigma,\tau)=(\tilde\tau,\tilde\sigma)$, we get

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}\sigma\int_\sigma^{+\infty}\mathrm{d}\tau\ \ \sigma^{m-1}\tau^{m-1}u(\sigma,\tau)J(s_0,t_0,\sigma,\tau) &= \int_0^{+\infty}\mathrm{d}\tau\int_0^{\tau}\mathrm{d}\sigma\ \ \sigma^{m-1}\tau^{m-1}u(\sigma,\tau)J(s_0,t_0,\sigma,\tau)\\ &= \int_0^{+\infty}\mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\tau\ \ \sigma^{m-1}\tau^{m-1}u(\tau,\sigma)J(s_0,t_0,\tau,\sigma)\\ &= -\int_0^{+\infty}\mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\tau\ \ \sigma^{m-1}\tau^{m-1}u(\sigma,\tau)J(s_0,t_0,\tau,\sigma)\,.\end{aligned}$$

Now, since $u\geq 0$ in $\mathcal{O}$ and $u\not\equiv 0$, by Lemma B.2 we get

$$0\leq Lu(x_0)=\int_0^{+\infty}\mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\tau\ \ \sigma^{m-1}\tau^{m-1}u(\sigma,\tau)\big(J(s_0,t_0,\tau,\sigma)-J(s_0,t_0,\sigma,\tau)\big)<0\,,$$

a contradiction. □

B.3. In (s,t) variables.

eInOVariablesst⟩? **Proposition B.4.** *Let $m\geq 1$, $s\in(0,1)$, $\alpha>2s$ and let $v\in C^\alpha_{\mathrm{loc}}(\mathbb{R}^{2m})\cap L^\infty(\mathbb{R}^{2m})$ be a function depending only on s and t. Let $\Omega\subseteq\mathcal{O}$ a domain (not necessarily bounded). Let $L\in\mathcal{L}_0$ such that .... Assume that v satisfies*

Write hypotheses

$$\begin{cases} Lv-f'(u)v-c(x)v &\leq& 0 & \text{in } \Omega\,,\\ v &\leq& 0 & \text{in } \mathcal{O}\setminus\Omega\,,\\ v(s,t) &=& -v(t,s) & \text{for all } s,t\geq 0\,,\\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} v(x) &\leq& 0\,, \end{cases}$$

with $c\leq 0$ in Ω. Then, $v\leq 0$ in Ω.

sOddVariablesst⟩? **Proposition B.5.** *Let $m\geq 1$, $s\in(0,1)$, $\alpha>2s$ and let $v\in C^\alpha_{\mathrm{loc}}(\mathbb{R}^{2m})\cap L^\infty(\mathbb{R}^{2m})$ be a function depending only on s and t. Let $\varepsilon>0$ and $H\subseteq\{t<s<t+\varepsilon\}\subset\mathcal{O}$ a domain (not necessarily bounded). Let $L\in\mathcal{L}_0$ such that .... Assume that v satisfies*

Write hypotheses

$$\begin{cases} Lv+cv &\leq& 0 & \text{in } H\,,\\ v &\leq& 0 & \text{in } \mathcal{O}\setminus H\,,\\ v(s,t) &=& -v(t,s) & \text{for all } s,t\geq 0\,,\\ \limsup\limits_{x\in H,\ |x|\to\infty} v(x) &\leq& 0\,. \end{cases} \tag{B.12}$$

Eq:AssumptionsMa

Under these assumptions there exists $\varepsilon_\star>0$ such that, if $\varepsilon<\varepsilon_\star$, then $v\leq 0$ in Ω.

Proof. Assume, by contradiction, that

$$M:=\sup_H>0\,.$$

Under the assumptions (B.12), S must be attained at an interior point $x_0\in H$. Then,

$$0\geq Lv(x_0)+c(x_0)v(x_0)\geq Lv(x_0)-||c_-||_{L^\infty(H)}\,M\,. \tag{B.13}$$

?Eq:Inequalities

Now, we compute $Lv(x_0)$. Assume that $x_0 = (s_0 x_s, t_0 x_t)$ with x_s, $x_t \in \mathbb{S}^{m-1}$. Then, by Lemma B.1 we can write

$$\begin{aligned} Lv(x_0) &= \widetilde{L}v(s_0,t_0) \\ &= \int_0^{+\infty} \mathrm{d}\sigma \int_0^{+\infty} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(v(s_0,t_0) - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau) \\ &= \int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau) \\ &\quad + \int_0^{+\infty} \mathrm{d}\sigma \int_\sigma^{+\infty} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau)\,. \end{aligned}$$

We treat each term separately. For the first term, since $M \geq v$ in $\mathcal{O}$, by Lemma B.2 we have

$$\begin{aligned} &\int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau) \\ &\geq \int_0^{+\infty} \mathrm{d}\sigma \int_\sigma^{+\infty} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\tau,\sigma)\,. \end{aligned}$$

We compute now the second term. By changing the order of integration and performing the change $(\sigma,\tau) = (\tilde{\tau},\tilde{\sigma})$, we get

$$\begin{aligned} &\int_0^{+\infty} \mathrm{d}\sigma \int_\sigma^{+\infty} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau) \\ &= \int_0^{+\infty} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\sigma \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\sigma,\tau)\big) J(s_0,t_0,\sigma,\tau) \\ &= \int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M - v(\tau,\sigma)\big) J(s_0,t_0,\tau,\sigma) \\ &= \int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1}\big(M + v(\sigma,\tau)\big) J(s_0,t_0,\tau,\sigma)\,. \end{aligned}$$

Hence, we deduce that

$$\widetilde{L}v(s_0,t_0) \geq 2M \int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1} J(s_0,t_0,\tau,\sigma)\,.$$

Recalling (7.2), we have

$$\begin{aligned} 0 &\geq Lv(x_0) + c(x_0)v(x_0) \\ &\geq M\left\{ 2\int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \;\; \sigma^{m-1}\tau^{m-1} J(s_0,t_0,\tau,\sigma) - \|c_-\|_{L^\infty(H)} \right\}. \end{aligned}$$

We claim that, for ε small enough,

$$2\int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \quad \sigma^{m-1}\tau^{m-1}J(s_0,t_0,\tau,\sigma) > ||c_-||_{L^\infty(H)}\,. \tag{B.14}$$ Eq:ClaimMaxPNarr

If we assume this claim to be true, then we have the desired contradiction and we conclude that $v \le 0$ in H.

Let us show (B.14). Note first that $J(s_0,t_0,\tau,\sigma) = J(t_0,s_0,\sigma,\tau)$. We will show that

$$\int_0^{+\infty} \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\tau \quad \sigma^{m-1}\tau^{m-1}J(t_0,s_0,\sigma,\tau)$$

can be as big as we want if ε is small enough. To see this, we recall that $x_0 = (s_0x_s, t_0x_t)$ and define

$$z_0 := (t_0x_s, s_0x_t)\,.$$

In view of Lemma B.1, what we want to show is that

$$\int_{\mathcal{O}} K(|z_0-y|)\,\mathrm{d}y$$

is as big as desired if ε is small enough.

Justificar que es única

Let $z_\star$ be the projection of z_0 into the Simons cone $\mathscr{C}$. Since $x_0 \in \{t < s < t+\varepsilon\}$, then $|z_0 - z_\star| \le \sqrt{2}\varepsilon$. Then,

$$|z_0 - y| \le \sqrt{2}\varepsilon + |z_\star - y|$$

justify better

and since K is decreasing,

$$\int_{\mathcal{O}} K(|z_0-y|)\,\mathrm{d}y \ge \int_{\mathcal{O}} K(\sqrt{2}\varepsilon + |z_\star-y|)\,\mathrm{d}y$$

The crucial point now is that, as $\varepsilon \to 0$, the integral of the right-hand side converges to

$$\int_{\mathcal{O}} K(|z_\star-y|)\,\mathrm{d}y\,,$$

escribir en el enunciado o al ppio del paper

and since $z_\star \in \mathscr{C}$, this last integral is infinite — see (??). Therefore, for ε small enough,

$$\int_{\mathcal{O}} K(|z_0-y|)\,\mathrm{d}y$$

is as big as we want, in particular bigger than $||c_-||_{L^\infty(H)}$. This shows (B.14) and concludes the proof. ☐

Once this maximum principle in narrow domains is available, we can now proceed with the proof of Proposition 7.1.

Proof of Proposition 7.1. For the sake of simplicity, we will denote

$$\mathscr{L}w := Lw - f'(u)w - cw\,.$$

Let $\varepsilon > 0$ such that the maximum principle of Proposition 7.2 is valid and define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{s > t + \varepsilon\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \Omega \cap \{t < s < t + \varepsilon\}\,.$$

Recall that, by (??), we have

Poner esto bien

$$u \geq \delta > 0 \quad \text{in } \overline{\Omega}_\varepsilon\,,$$

for some $\delta > 0$. Moreover, since $f(\theta) > f'(\theta)\theta$ for $\theta > 0$, we see that

$$\mathscr{L}u = Lu - f'(u)u - cu = f(u) - f'(u)u - cu > 0 \quad \text{in } \Omega \subseteq \mathcal{O}\,, \tag{B.15}$$

Eq:uSupersolLine

where we have used that $u > 0$ in $\mathcal{O}$.

Assume, in order to get a contradiction, that there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. Let $w = v - \lambda u$. Since in $\overline{\Omega}_\varepsilon$ we have $u \geq \delta > 0$, we see that for λ big enough, $w < 0$ in $\overline{\Omega}_\varepsilon$. Moreover, since $v \leq 0$ in $\mathcal{O} \setminus \Omega$, we have

$$w \leq 0 \quad \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,.$$

Furthermore, we also have

$$\limsup_{x \in \mathcal{N}_\varepsilon,\ |x| \to \infty} w(x) \leq 0$$

and, by (B.15),

$$\mathscr{L}w = \mathscr{L}v - \lambda \mathscr{L}u \leq 0 \text{ in } \mathcal{N}_\varepsilon\,.$$

Therefore, since w is odd with respect to $\mathscr{C}$, we can apply Proposition 7.2 with $H = \mathcal{N}_\varepsilon$ to deduce that

$$w \leq 0 \quad \text{in } \Omega\,,$$

if λ is big enough.

Now, define

$$\lambda_\star := \inf\{\lambda > 0 \ :\ v - \lambda u \leq 0 \ \text{ in } \Omega\}\,.$$

By the previous reasoning, $\lambda_\star$ is well defined. Clearly, $v - \lambda_\star u \leq 0$ in Ω. In addition, since $v(x_0) > 0$ and $u(x_0) > 0$, we have $-\lambda_\star u(x_0) < v(x_0) - \lambda_\star u(x_0) \leq 0$ and therefore $\lambda_\star > 0$.

We claim that $v - \lambda_\star u \not\equiv 0$. Indeed, if $v - \lambda_\star u \equiv 0$ then $v = \lambda_\star u$ and thus, since $\lambda_\star > 0$, we get

$$0 \geq \mathscr{L}v(x_0) = \lambda_\star \mathscr{L}u(x_0) > 0\,,$$

a contradiction.

Then, by the strong maximum principle (Proposition B.3), since $v - \lambda_\star u \not\equiv 0$ we have

$$v - \lambda_\star u < 0 \quad \text{in } \Omega\,.$$

Therefore, by continuity there exists $0 < \eta < \lambda_\star$ such that

$$\tilde{w} := v - (\lambda_\star - \eta)u < 0 \quad \text{in } \overline{\Omega}_\varepsilon\,.$$

Using again the maximum principle in narrow domains with $\tilde{w}$ in $\mathcal{N}_\varepsilon$, we deduce that

$$v - (\lambda_\star - \eta)u \leq 0 \quad \text{in } \Omega\,,$$

and this contradicts the definition of $\lambda_\star$. Hence, $v \le 0$ in Ω. $\square$

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