SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS, II: ONE-DIMENSIONAL AND SADDLE-SHAPED SOLUTIONS TO THE ALLEN-CAHN EQUATION

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ABSTRACT. This paper, which is the follow-up to part I, concerns saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator with a radially symmetric kernel and f is of Allen-Cahn type. Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only in this set.

Following the setting previously established in part I for doubly radial odd functions, we show existence, asymptotic behavior, and uniqueness of the saddle-shaped solution. For this, we prove, among others, some symmetry results and maximum principles in "narrow" sets. We also need to study one-dimensional solutions to $L_K u = f(u)$ in $\mathbb{R}^n$, that we do by relating this equation with an analogous one in $\mathbb{R}$, for a related operator.

CONTENTS

Key words and phrases. Integro-differential semilinear equation, saddle-shaped solution, odd symmetry, Simons cone, symmetry results.

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1. Introduction

ec:Introduction⟩? In this paper, which is the second part of [38], we study saddle-shaped solutions to the semilinear equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{1.1}$$

Eq:NonlocalAllen

where L_K is a linear integro-differential operator of the form (1.2) and f is of Allen-Cahn type —see (1.6). These solutions (see Definition 1.1 below) are particularly interesting in relation to the nonlocal version of a conjecture by De Giorgi, with the aim of finding a counterexample in high dimensions (see the details below). Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces.

In the previous paper [38] we established an appropriate setting to study solutions to (1.1) which are doubly radial and odd with respect to the Simons cone (this terminology is described more precisely below), a property satisfied by saddle-shaped solutions. We deduced an alternative expression for the operator L_K when acting on doubly radial odd functions —see (1.11). This expression was used to deduce some maximum principles for odd functions under certain assumptions on the kernel K of the operator L_K. Moreover, we proved an energy estimate for doubly radial and odd minimizers of the energy associated to the equation, as well as the existence of saddle-shaped solutions to (1.1).

Using the results obtained in [38], in the present paper we further study saddle-shaped solutions to (1.1). First, we prove existence of this type of solutions, Theorem 1.4, by using the monotone iteration method (as an alternative to the proof in [38]). After this, we establish the asymptotic behavior of saddle-shaped solutions, Theorem 1.5. To do it, we use two symmetry results for semilinear equations similar to (1.1): Theorems 4.1 and 4.2, proved in Section 4. In the study of the asymptotic behavior of saddle-shaped solutions we establish further properties of the so-called *layer solution* u_0 (see Section 5). Finally, we show the uniqueness of the saddle-shaped solution, Theorem 1.6, by using a maximum principle for the linearized operator $L_K - f'(u)$ (Proposition 6.1).

As in [38], equation (1.1) is driven by a linear integro-differential operator L_K of the form

$$L_K w(x) = \int_{\mathbb{R}^n} \{w(x) - w(y)\} K(x-y)\, \mathrm{d}y, \tag{1.2}$$

Eq:DefOfLu

where the kernel K satisfies

$$K \geq 0\,, \quad K(y) = K(-y) \quad \text{and} \quad \int_{\mathbb{R}^n} \min\left\{|y|^2, 1\right\} K(y)\, \mathrm{d}y < +\infty\,. \tag{1.3}$$

Eq:Symmetry&Inte

The integral in (1.2) has to be understood in the principal value sense, as well as all the integrals involving nonlocal operators in the rest of the paper. The most canonical example of such operators is the fractional Laplacian

$$(-\Delta)^\gamma w = c_{n,\gamma} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2\gamma}}\, \mathrm{d}y\,,$$

where $\gamma \in (0,1)$ and $c_{n,\gamma}$ is a normalizing constant given by

$$c_{n,\gamma} = \gamma \frac{2^{2\gamma}\Gamma(\frac{n}{2}+\gamma)}{\pi^{n/2}\Gamma(1-\gamma)}, \tag{1.4}$$

Eq:ConstantFracL

see for instance [9].

Throughout the paper, we assume that the operator L_K is uniformly elliptic. That is, its kernel is bounded from above and below by a positive multiple of the one of the fractional Laplacian:

$$\lambda \frac{c_{n,\gamma}}{|y|^{n+2\gamma}} \leq K(y) \leq \Lambda \frac{c_{n,\gamma}}{|y|^{n+2\gamma}}\,, \qquad \text{with } 0 < \lambda \leq \Lambda\,, \tag{1.5}$$

Eq:Ellipticity

where $c_{n,\gamma}$ is given in (1.4). This condition is very frequently adopted when dealing with nonlocal operators of the form (1.2) since it is known to yield Hölder regularity of solutions (see [50, 57] and Section 2). The family of linear operators satisfying conditions (1.3) and (1.5) is the so-called $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$, and we will make explicit the parameters only when needed.

Following [38], in this paper we will sometimes assume that the operator L_K is rotational invariant, that is, that its kernel is radially symmetric. This extra assumption allows us to rewrite the operator in the setting of doubly radial odd functions, as explained in [38].

Note that, for operators in the class $\mathcal{L}_0$, the minimal assumption on w so that $L_K w$ is well defined in an open set Ω is that $w \in C^\alpha(\Omega) \cap L^1_\gamma(\mathbb{R}^n)$ for some $\alpha > 2\gamma$, where $w \in L^1_\gamma(\mathbb{R}^n)$ means that

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{1+|x|^{n+2\gamma}}\,\mathrm{d}x < +\infty\,.$$

Concerning the nonlinearity in (1.1), we will assume that f is a C^1 function satisfying

$$f \text{ is odd}, \quad f(\pm 1) = 0, \quad \text{and} \quad f \text{ is strictly concave in } (0,1). \tag{1.6}$$

Eq:Hypothesesf

Note that this yields $f > 0$ in $(0,1)$, $f'(0) > 0$ and $f'(\pm 1) < 0$. A property of f that will be used through the paper is that, since f is strictly concave in $(0,1)$ and $f(0) = 0$, then

$$f'(\tau)\tau < f(\tau) \quad \text{for all } \tau \in (0,1)\,. \tag{1.7}$$

Eq:PropertyConca

Before introducing saddle-shaped solutions, we need to recall some definitions. First, we present the Simons cone, which is a central object along this paper. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\}\,.$$

This cone is of importance in the theory of minimal surfaces. It has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in all even dimensions, and it is a minimizer of the perimeter functional when $2m \geq 8$. Concerning the nonlocal setting, $\mathscr{C}$ has also zero nonlocal mean curvature in all even dimensions, although it is not known if it is a minimizer of the nonlocal perimeter (see the introduction of [39] and the references therein for more details).

As in [38], we will use the letters $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \text{ and } \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\}.$$

Both sets $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. They are sets that are invariant under orthogonal transformations in the first m variables and also under orthogonal transformations in the last m variables. That is, $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution if $R\Omega = \Omega$ for any given transformation $R \in O(m)^2 = O(m) \times O(m)$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

In this paper we deal with functions that are *doubly radial*. These are functions $w : \mathbb{R}^{2m} \to \mathbb{R}$ that depend only on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$.

In order to define certain symmetries of a function with respect to the Simons cone, we consider the following isometry, that will play a significant role in this article:

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \,. \end{array} \tag{1.8}$$

Eq:DefStar

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$ for every $x \in \mathbb{R}^{2m}$. Similarly, we say that a doubly radial function w is *even with respect to the Simons cone* if $w(x) = w(x^\star)$ for every $x \in \mathbb{R}^{2m}$.

With these definitions at hand we can now introduce saddle-shaped solutions.

SaddleShapedSol⟩ **Definition 1.1.** We say that a bounded solution u to (1.1) is a *saddle-shaped solution* (or simply *saddle solution*) if

(1) u is doubly radial.
(2) u is odd with respect to the Simons cone.
(3) $u > 0$ in $\mathcal{O} = \{|x'| > |x''|\}$.

Note that these solutions are even with respect to the coordinate axis and that their zero level set is the Simons cone $\mathscr{C} = \{|x'| = |x''|\}$.

Saddle-shaped solutions to the classical Allen-Cahn equation involving the Laplacian were studied in [30, 56, 19, 20, 12]. In these works, it was established the existence, uniqueness and some qualitative properties of this type of solutions, such as instability when $2m \leq 6$ and stability if $2m \geq 14$. The possible stability in dimensions 8, 10, and 12 is still an open problem, as well as the possible minimality of this solution in dimensions $2m \geq 8$.

In the fractional framework, there are only three works concerning saddle-shaped solutions to the equation $(-\Delta)^\gamma u = f(u)$. In [25, 26], Cinti proved the existence of a saddle-shaped solution as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability in low dimensions. In a previous paper by the authors [39], further properties of these solutions were proved, the main ones being uniqueness and, when $2m \geq 14$, stability. Recall that we say that a bounded solution w to $L_K w = f(w)$ in $\Omega \subset \mathbb{R}^n$ is *stable* in Ω if the second variation of the energy at w is nonnegative. That is, if

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega} f'(w)\xi^2\,\mathrm{d}x \geq 0 \tag{1.9}$$

Eq:StablityCondi

for every $\xi \in C_0^\infty(\Omega)$.

To our knowledge, the present paper together with its first part [38] are the first ones studying saddle-shaped solutions for general integro-differential equations of the form (1.1). In the three previous papers [25, 26, 39], the main tool used is the extension problem for the fractional Laplacian (see [21]). Nevertheless, this technique has the limitation that it cannot be carried out for general integro-differential operators different from the fractional Laplacian. Therefore, some purely nonlocal techniques were developed in [38] to study saddle-shaped solutions, and we exploit them in the present paper.

Let us collect now the main results of the previous paper [38] that will be used in the present one. Recall that, if K is a radially symmetric kernel, then we can rewrite the operator L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y\,,$$

where $\overline{K}$ is doubly radial in both variables and is defined by

$$\overline{K}(x,y) := \fint_{O(m)^2} K(|Rx-y|)\,\mathrm{d}R\,. \tag{1.10}$$

Eq:KbarDef

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$ (see Section 2 of [38] for the details).

Moreover, if we consider doubly radial functions that are odd with respect to the Simons cone, we can use the involution $(\cdot)^\star$ —defined in (1.8)—, to find that

$$L_K w(x) = \int_{\mathcal{O}} \{w(x)-w(y)\}\{\overline{K}(x,y)-\overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,. \tag{1.11}$$

Eq:OperatorOddF

Furthermore,

$$\frac{1}{C}\operatorname{dist}(x,\mathscr{C})^{-2\gamma} \leq \int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y \leq C\operatorname{dist}(x,\mathscr{C})^{-2\gamma}, \tag{1.12}$$

Eq:ZeroOrderTerm

with $C>0$ depending only on m, γ, λ and Λ (see the details in [38]).

Note that the expression (1.11) has an integro-differential part plus a zero order term with a positive coefficient. Thus, the natural assumption to make for that operator to have a maximum principle is that its "kernel" is positive. That is, $\overline{K}(x,y)-\overline{K}(x,y^\star) > 0$.

One of the main results in [38] established necessary and sufficient conditions on the original kernel K in order to L_K have a maximum principle in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone. This is given in the next result.

erizationLstar⟩? **Theorem 1.2** ([38]). *Let $K:(0,+\infty)\to\mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K}:\mathbb{R}^{2m}\times\mathbb{R}^{2m}\to\mathbb{R}$ by* (1.10).

If

$$K(\sqrt{\tau}) \textit{ is a strictly convex function of } \tau\,, \tag{1.13}$$ Eq:SqrtConvex

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \textit{for every } x,y\in\mathcal{O}\,. \tag{1.14}$$ Eq:KernelInequal

In addition, if $K\in C^2((0,+\infty))$, (1.13) *is not only a sufficient condition for* (1.14) *to hold, but also a necessary one.*

This lead us in [38] to introduce the following definition.

Definition 1.3 (Ellipticity class $\mathcal{L}_\star$). Let $L_K\in\mathcal{L}_0(2m,\gamma,\lambda,\Lambda)$ with kernel K radially symmetric. We say that $L_K\in\mathcal{L}_\star(2m,\gamma,\lambda,\Lambda)$ whenever the associated kernel $\overline{K}$ satisfies (1.14).

The importance of $\mathcal{L}_\star$ is that the operators in this class have a maximum principle for doubly radial odd functions (see Section 2). This is a key ingredient to prove the first main result of this paper: the existence of saddle-shaped solutions.

⟨Th:Existence⟩ **Theorem 1.4** (Existence of saddle-shaped solutions). *Let f satisfy* (1.6) *and let $L_K\in\mathcal{L}_\star$. Then, for every even dimension $2m\geq 2$, there exists a saddle-shaped solution to* (1.1). *In addition, u satisfies $|u|<1$ in $\mathbb{R}^{2m}$.*

The previous theorem was already proved in [38] using variational techniques. Here, instead, we use the monotone iteration method and the existence of a positive supersolution to a semilinear Dirichlet problem in a ball (see Section 3 for the details). Let us remark that in both proofs the crucial ingredient is inequality (1.14).

The second main result of this paper is Theorem 1.5 below, on the asymptotic behavior of saddle-shaped solutions at infinity. Before state it, let us introduce an important type of solutions in the study of the integro-differential Allen-Cahn equation: the layer solutions.

We say that a solution v to $L_K v = f(v)$ in $\mathbb{R}^n$ is a *layer solution* if v is increasing in one direction, say $e\in\mathbb{S}^{n-1}$ and $v(x)\to\pm1$ as $x\cdot e\to\pm\infty$ (not necessarily uniformly). By a result of Cozzi and Passalacqua in [29], under the assumptions (1.6) on f, for every kernel K_1 such that $L_{K_1}\in\mathcal{L}_0(1,\gamma,\Lambda,\lambda)$ there exist a layer solution to $L_{K_1}w=f(w)$ in $\mathbb{R}$ which is unique up to translations and is odd with respect to some point (in the case of the fractional Laplacian this result was proved in [18, 17] by using the extension problem).

In $\mathbb{R}^n$, a special case of layer solutions are the one-dimensional ones. Actually, in relation with the available results concerning a conjecture by De Giorgi, in low dimensions all layer solutions are one-dimensional (see the comments at the end of this introduction). One-dimensional layer solutions in $\mathbb{R}^n$ are in correspondence with the ones in $\mathbb{R}$ as explained next —see also [29]. Let v be a layer solution to $L_K v = f(v)$ in $\mathbb{R}^n$ depending only on one direction, say $v(x) = w(x_n)$, and assume that $L_K \in \mathcal{L}_0(n, \gamma, \Lambda, \lambda)$. Then w is a layer solution to $L_{K_1} w = f(w)$ in $\mathbb{R}$ with K_1 given by

$$K_1(t) := \int_{\mathbb{R}^{n-1}} K(\theta, t)\, \mathrm{d}\theta = |t|^{n-1} \int_{\mathbb{R}^{n-1}} K(t\sigma, t)\, \mathrm{d}\sigma.$$

Moreover $L_{K_1} \in \mathcal{L}_0(1, \gamma, \Lambda, \lambda)$. For more details see Proposition 5.1 in Section 5 and [29].

A particular layer solution, denoted by u_0, plays an important role in this paper. It is defined to be the unique solution of the following problem.

$$\begin{cases} L_{K_1} u_0 &= f(u_0) \quad \text{in } \mathbb{R}, \\ \dot{u}_0 &> 0 \quad \text{in } \mathbb{R}, \\ u_0(x) &= -u_0(-x) \quad \text{in } \mathbb{R}, \\ \lim_{x \to \pm\infty} u_0(x) &= \pm 1. \end{cases} \tag{1.15}$$ Eq:LayerSolution

Note that, by the previous comments, $v(x) = u_0(x_n)$ is a one-dimensional layer solution to $L_K v = f(v)$ in $\mathbb{R}^n$. Moreover, the same holds for $u_0(x \cdot e)$ for every $e \in \mathbb{S}^{n-1}$ whenever the kernel is radially symmetric.

The importance of the layer solution u_0 in relation with saddle-shaped solutions is that the associated function

$$U(x) := u_0 \left(\frac{|x'| - |x''|}{\sqrt{2}} \right) \tag{1.16}$$ Eq:DefOfU

describes the asymptotic behavior of saddle solutions at infinity. Note that $(|x'| - |x''|)/\sqrt{2}$ is the signed distance to the Simons cone (see Lemma 4.2 in [20]). Therefore, we can understand the function U as the layer solution u_0 centered at each point of the Simons cone and oriented in the normal direction to the cone.

The precise statement on the asymptotic behavior of saddle-shaped solutions at infinity is the following.

orSaddleSolution⟩ **Theorem 1.5.** *Let $f \in C^2(\mathbb{R})$ satisfy (1.6). Let u be a saddle-shaped solution to (1.1) with $L_K \in \mathcal{L}_\star$. Let U be the function defined by (1.16).*

Then,

$$||u - U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left|D^2 u - D^2 U\right|\right|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0$$

as $R \to +\infty$.

To establish the asymptotic behavior of saddle-shaped solutions we use a compactness argument as in [20, 25, 26] and we need two important symmetry results established in Section 4. The first one, Theorem 4.1, is a symmetry result for nonnegative solutions to a semilinear equation in the whole space. The second one, Theorem 4.2, is a symmetry result for solutions to a semilinear equation which are odd with respect to a hyperplane and nonnegative at one of its sides. Both results are well known for the classical Laplacian, and are also available for $(-\Delta)^\gamma$. Here we present the proof for a general integro-differential operator L_K, following some arguments that are used for the analogous results for $-\Delta$ and $(-\Delta)^\gamma$.

The last main result of this paper is the uniqueness of the saddle-shaped solution, stated next.

⟨Th:Uniqueness⟩ **Theorem 1.6** (Uniqueness of the saddle-shaped solution)**.** *Let f satisfy* (1.6) *and let $L \in \mathcal{L}_\star$. Then, for every dimension $2m \geq 2$, there exists a unique saddle-shaped solution to* (1.1)*.*

To prove this result we need two ingredients. The first one is the asymptotic behavior of saddle solutions given in Theorem 1.5. The second one is a maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$, which is given in Proposition 6.1. To establish it, we will need to use a maximum principle in "narrow" sets that we also prove in Section 6.

To conclude this introduction, let us make some comments on the importance of problem (1.1) and its relation with the theory of (classical and nonlocal) minimal surfaces and a famous conjecture raised by De Giorgi.

A main open problem (even in the local case) is to determine whether the saddle-shaped solution is a minimizer of the energy functional associated to the equation, depending on the dimension $2m$. This question is deeply related to the regularity theory of local and nonlocal minimal surfaces, as explained next.

In the seventies, Modica and Mortola (see [47, 48]) proved that, considering an appropriately rescaled version of the (local) Allen-Cahn equation, the corresponding energy functionals Γ-converge to the perimeter functional. Thus, the minimizers of the equation converge to the characteristic function of a set of minimal perimeter. This same fact holds for the equation with the fractional Laplacian, though we have two different scenarios depending on the parameter $\gamma \in (0,1)$. If $\gamma \geq 1/2$, the rescaled energy functionals associated to the equation Γ-converge to the classical perimeter (see [1, 43]), while in the case $\gamma \in (0, 1/2)$ they Γ-converge to the fractional perimeter (see [55]). As a consequence, if the saddle-shaped solution was proved to be a minimizer in a certain dimension for some $\gamma \in (0, 1/2)$, it would follow that the Simons cone $\mathscr{C}$ would be a minimizing nonlocal (2γ)-minimal surface in such dimensions. This last statement on the saddle-shaped solution is an open problem in any dimension (although it is known that the Simons cone is not a minimizer in dimension $2m = 2$). The only available result related to this question is the recent result in [39] that states that the saddle-shaped solution to the fractional Allen-Cahn equation is stable in dimensions $2m \geq 14$. As a consequence of this and a result in [15], the Simons cone is a stable nonlocal (2γ)-minimal surface in dimensions $2m \geq 14$ (see the details in [39]).

Moreover, as explained below, saddle-shaped solutions are natural objects to build a counterexample to a famous conjecture raised by De Giorgi, that reads as follows. Let u be a bounded solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$ which is monotone in one direction, say $\partial_{x_n} u > 0$. Then, if $n \leq 8$, u is one dimensional, i.e., u depends only on one Euclidean variable. This conjecture was proved to be true in dimensions $n = 2$ and $n = 3$ (see

[42, 2]), and in dimensions $4 \leq n \leq 8$ with the extra assumption of

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{ for all } x' \in \mathbb{R}^{n-1}\,, \tag{1.17}$$

Eq:SavinConditic

(see [52]). A counterexample to the conjecture in dimensions $n \geq 9$ was given in [31] by using the gluing method.

An alternative approach to the one of [31] to construct a counterexample to the conjecture was given by Jerison and Monneau in [45]. They showed that a counterexample in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate but is not one-dimensional. The saddle-shaped solution is of special interest in the search for this counterexample, since it is even with respect to all the coordinate axis and it is canonically associated to the Simons cone, which in turn is the simplest nonplanar minimizing minimal surface. Therefore, by proving that the saddle solution to the classical Allen-Cahn equation is a minimizer in some dimension $2m$, one would obtain automatically a counterexample to the conjecture in $\mathbb{R}^{2m+1}$.

The corresponding conjecture in the fractional setting, where one replaces the operator $-\Delta$ by $(-\Delta)^\gamma$, has been widely studied in the last years. In this framework, the conjecture has been proven to be true for all $\gamma \in (0,1)$ in dimensions $n = 2$ (see [18, 16, 59]) and $n = 3$ (see [13, 14, 33]). The conjecture is also true in dimension $n = 4$ in the case of $\gamma = 1/2$ (see [41]) and if $\gamma \in (0, 1/2)$ is close to $1/2$ (see [15]). Assuming the additional hypothesis (1.17), the conjecture is true in dimensions $4 \leq n \leq 8$ for $1/2 \leq \gamma < 1$ (see [53, 54]), and also for $\gamma \in (0, 1/2)$ if γ is close to $1/2$ (see [34]). A counterexample to the De Giorgi conjecture for the fractional Allen-Cahn equation in dimensions $n \geq 9$ for $\gamma \in (1/2, 1)$ has been very recently announced in [22].

Concerning the conjecture with more general operators like L_K, fewer results are known. In dimension $n = 2$ the conjecture is proved in [44, 8, 37], under different assumptions on the kernel K and even for more general nonlinear operators. Note also that the results of [34] also hold for a particular class of kernels in $\mathcal{L}_0$.

The paper is organized as follows. In Section 2 we present some preliminary results that will be used in the rest of the article. Section 3 contains the proof of Theorem 1.4 on the existence of saddle-shaped solutions. In Section 4 we establish two symmetry results, Theorems 4.1 and 4.2. Section 5 is devoted to the layer solution u_0 of problem (1.1) and the proof of the asymptotic behavior of saddle-shaped solutions, Theorem 1.5. Finally, Section 6 concerns the proof of a maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$ (Proposition 6.1), as well as the proof of Theorem 1.6, establishing the uniqueness of the saddle-shaped solution.

2. Preliminaries

In this section we collect some preliminary results that will be used in the rest of this paper. First, we state the regularity results needed in the forthcoming sections. Then, we present the functional setting which we are going to work with and finally we recall the two basic maximum principles for doubly radial odd functions proved in [38].

2.1. Regularity theory for nonlocal operators in the class $\mathcal{L}_0$.

In this subsection we present the regularity results that will be used in the paper. For further details, see [50, 57] and the references therein.

We start with a result on the interior regularity for linear equations.

Proposition 2.1 ([50, 57]). *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to $L_K w = h$ in B_1. Then,*

$$||w||_{C^{2\gamma}(B_{1/2})} \le C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)}\right). \tag{2.1}$$

Moreover, let $\alpha > 0$ and assume additionally that $w \in C^\alpha(\mathbb{R}^n)$. Then, if $\alpha + 2\gamma$ is not an integer,

$$||w||_{C^{\alpha+2\gamma}(B_{1/2})} \le C\left(||h||_{C^\alpha(B_1)} + ||w||_{C^\alpha(\mathbb{R}^n)}\right), \tag{2.2}$$

where C is a constant that depends only on n, γ, λ and Λ.

Throughout the paper we consider saddle solutions u to (1.1) that satisfy $|u| \le 1$ in $\mathbb{R}^n$. Hence, by applying (2.1) we find that for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} ||u||_{C^{2\gamma}(B_{1/2}(x_0))} &\le C\left(||f(u)||_{L^\infty(B_1(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)}\right) \\ &\le C\left(1 + ||f||_{L^\infty([-1,1])}\right). \end{aligned}$$

Note that the estimate is independent of the point x_0, and thus since the equation is satisfied in the whole $\mathbb{R}^n$,

$$||u||_{C^{2\gamma}(\mathbb{R}^n)} \le C\left(1 + ||f||_{L^\infty([-1,1])}\right).$$

Then, we use estimate (2.2) repeatedly and the same kind of arguments wield that, if $f \in C^k([-1,1])$, then $u \in C^\alpha(\mathbb{R}^n)$ for some $\alpha > k + 2\gamma$. Moreover, the following estimate holds:

$$||u||_{C^\alpha(\mathbb{R}^n)} \le C,$$

for some constant C depending only on n, γ, λ, Λ, k and $||f||_{C^k([-1,1])}$.

ctional setting⟩? 2.2. **Functional setting.** In this section we define the functional spaces that we are going to consider in some parts of this paper. These were also the spaces used in the previous article [38], and we refer the reader to that work for more details.

Given a set $\Omega \subset \mathbb{R}^n$ and a translation invariant and positive kernel K satisfying (1.3), we define the space

$$\mathbb{H}^K(\Omega) := \left\{ w \in L^2(\Omega) \ : \ [w]^2_{\mathbb{H}^K(\Omega)} < +\infty \right\},$$

where

$$[w]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2} \int\int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} |w(x) - w(y)|^2 K(x-y) \, \mathrm{d}x \, \mathrm{d}y \,.$$

We also define

$$\begin{aligned} \mathbb{H}^K_0(\Omega) &:= \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\} \\ &= \left\{ w \in \mathbb{H}^K(\mathbb{R}^n) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\}. \end{aligned}$$

Assume that $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution. Then, we define

$$\widetilde{\mathbb{H}}^K(\Omega) := \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

and

$$\widetilde{\mathbb{H}}^K_0(\Omega) := \left\{ w \in \mathbb{H}^K_0(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

We will add the subscript 'odd' and 'even' to these spaces to consider only functions that are odd (respectively even) with respect to the Simons cone.

Recall that when K satisfies (1.5), then $\mathbb{H}^K_0(\Omega) = \mathbb{H}^\gamma_0(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(y) = c_{n,\gamma}|y|^{-n-2\gamma}$. Furthermore, $\mathbb{H}^\gamma(\Omega) \subset H^\gamma(\Omega)$, the usual fractional Sobolev space (see [32, 29]).

:MaxPrinciples⟩? 2.3. **Maximum principles for doubly radial odd functions.** In this last subsection, we state two basic maximum principles for doubly radial odd functions. Note that in both results we only need assumptions on the functions at one side of the Simons cone thanks to their symmetry. Both results where proved in [38] using the key inequality (1.14) for the kernel $\overline{K}$.

The first result is a weak maximum principle for odd functions with respect to $\mathscr{C}$.

eForOddFunctions⟩ **Proposition 2.2** ([38]). *Let $\Omega \subset \mathcal{O}$ an open set and let $L_K \in \mathcal{L}_\star(2m, \gamma)$. Let $w \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone. Assume that*

$$\left\{ \begin{array}{rcll} L_K w + c(x) w & \geq & 0 & \text{in } \Omega \,, \\ w & \geq & 0 & \text{in } \mathcal{O} \setminus \Omega \,, \end{array} \right.$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x \in \mathcal{O},\, |x| \to +\infty} w(x) \geq 0 \,.$$

Then, $w \geq 0$ in Ω.

The second result is the strong maximum principle for odd functions with respect to $\mathscr{C}$.

eForOddFunctions⟩ **Proposition 2.3** ([38]). *Let $\Omega \subset \mathcal{O}$ an open set and let $L_K \in \mathcal{L}_\star(2m, \gamma)$. Let $w \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone. Assume that $L_K w + c(x) w \geq 0$ in Ω, with $c(x)$ any function, and that $w \geq 0$ in $\mathcal{O}$. Then, either $w \equiv 0$ or $w > 0$ in Ω.*

cipleSingularity⟩ *Remark* 2.4. The regularity assumptions on w in the previous results can be weakened allowing $L_K w$ to take the value $+\infty$ at the points of Ω where w is not regular enough for $L_K w$ to be finite. This will be used in the proof of Theorem 1.4 in order to apply this maximum principle with a function that is no more regular than C^γ.

3. Existence of saddle-shaped solution: monotone iteration method

⟨Sec:Existence⟩ In this section we give a proof of Theorem 1.4 based on the maximum principle and the existence of a positive subsolution. To do this, we need a version of the monotone iteration procedure for doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. Along this section we will call odd sub/supersolutions to problem (3.2) the functions that are doubly radial, odd with respect to the Simons cone and satisfy the corresponding problem in (3.1). In view of Remark 2.4, we do not need the operator to be finite in the whole set when applied to a subsolution (respectively supersolution), it can be $-\infty$ (respectively $+\infty$) at some points.

oneIterationOdd⟩ **Proposition 3.1.** *Let $L_K \in \mathcal{L}_\star(2m,\gamma)$ for some $\gamma\in(0,1)$. Assume that $\underline{v}\leq\overline{v}$ are two bounded functions which are doubly radial and odd with respect to the Simons cone. Furthermore, assume that $\underline{v}\in C^\gamma(\mathbb{R}^{2m})$ and that $\underline{v}$ and $\overline{v}$ satisfy respectively*

$$\left\{\begin{array}{rcll} L_K\underline{v} & \leq & f(\underline{v}) & \text{in } B_R\cap\mathcal{O}\,,\\ \underline{v} & \leq & \varphi & \text{in } \mathcal{O}\setminus B_R\,,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} L_K\overline{v} & \geq & f(\overline{v}) & \text{in } B_R\cap\mathcal{O}\,,\\ \overline{v} & \geq & \varphi & \text{in } \mathcal{O}\setminus B_R\,,\end{array}\right. \tag{3.1}$$ Eq:SemilinearSub

with f a C^1 odd function and φ a doubly radial odd function.

Then, there exists $v\in C^{2\gamma+\varepsilon}(B_R)$ for some $\varepsilon>0$, a solution to

$$\left\{\begin{array}{rcll} L_K v & = & f(v) & \text{in } B_R\,,\\ v & = & \varphi & \text{in } \mathbb{R}^{2m}\setminus B_R\,,\end{array}\right. \tag{3.2}$$ Eq:SemilinearSol

such that v is doubly radial, odd with respect to the Simons cone and $\underline{v}\leq v\leq\overline{v}$ in $\mathcal{O}$.

Proof. The proof follows the classical monotone iteration method for elliptic equations (see for instance [35]). We just give here a sketch of the proof. First, let $M\geq 0$ be such that $-M\leq\underline{v}\leq\overline{v}\leq M$ and set

$$b := \max\left\{0, -\min_{[-M,M]} f'\right\}\geq 0\,.$$

Then one defines

$$\tilde{L}_K w := L_K w + bw \quad \text{ and } \quad g(\tau) := f(\tau) + b\tau\,.$$

Therefore, our problem is equivalent to find a solution to

$$\left\{\begin{array}{rcll} \tilde{L}_K v & = & g(v) & \text{in } B_R\,,\\ v & = & \varphi & \text{in } \mathbb{R}^{2m}\setminus B_R\,,\end{array}\right.$$

such that v is doubly radial, odd with respect to the Simons cone and $\underline{v}\leq v\leq\overline{v}$ in $\mathcal{O}$. Here the main point is that g is also odd but satisfies $g'(\tau)\geq 0$ for $\tau\in[-M,M]$. Moreover, since $b\geq 0$, $\tilde{L}_K$ satisfies the maximum principle for odd functions in $\mathcal{O}$ (as in Proposition 2.2).

We define $v_0=\underline{v}$ and, for $k\geq 1$, let v_k be the solution to the linear problem

$$\left\{\begin{array}{rcll} \tilde{L}_K v_k & = & g(v_{k-1}) & \text{in } B_R\,,\\ v_k & = & \varphi & \text{in } \mathbb{R}^{2m}\setminus B_R\,.\end{array}\right.$$

It is easy to see by induction and the regularity results from Proposition 2.1 that $v_k \in L^\infty(B_R)\cap C^{2\gamma+2\varepsilon}(B_R)$ for some $\varepsilon>0$. Moreover, given $\Omega\subset B_R$ a compact set, then $||v_k||_{C^{2\gamma+2\varepsilon}(\Omega)}$ is uniformly bounded in k.

Then, using the maximum principle it is not difficult to show by induction that

$$\underline{v} = v_0 \le v_1 \le \ldots \le v_k \le v_{k+1} \le \ldots \overline{v} \quad \text{ in } \mathcal{O}\,,$$

and that each function v_k is doubly radial and odd with respect to $\mathscr{C}$. Finally, by Arzelà-Ascoli theorem and the compact embedding of Hölder spaces we see that, up to a subsequence, v_k converges uniformly on compacts in $C^{2\gamma+\varepsilon}$ norm to the desired solution. □

In order to construct a positive subsolution, we also need a characterization and some properties of the first odd eigenfunction and eigenvalue for the operator L_K, which are presented next.

OddEigenfunction⟩ **Lemma 3.2.** *Let $\Omega\subset\mathbb{R}^{2m}$ be a bounded set of double revolution and let $L_K\in\mathcal{L}_\star(2m,\gamma,\lambda,\Lambda)$. Let us define*

$$\lambda_{1,\,\mathrm{odd}}(\Omega,L_K) := \inf_{w\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)} \frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_\Omega w(x)^2\,\mathrm{d}x}\,. \tag{3.3}$$

?Eq:DefLambda1?

Then, such infimum is attained at a function $\phi_1\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)\cap L^\infty(\Omega)$ which solves

$$\begin{cases} L_K\phi_1 &= \lambda_{1,\,\mathrm{odd}}(\Omega,L_K)\phi_1 & \text{in } \Omega\,,\\ \phi_1 &= 0 & \text{in } \mathbb{R}^{2m}\setminus\Omega\,,\end{cases}$$

and satisfies that $\phi_1>0$ in $\Omega\cap\mathcal{O}$. We call such function the first odd eigenfunction *of L_K* in Ω *and $\lambda_{1,\,\mathrm{odd}}(\Omega,L_K)$ the* first odd eigenvalue.

Moreover, in the case $\Omega=B_R$, there exists a constant C depending only on n, γ and Λ such that

$$\lambda_{1,\,\mathrm{odd}}(B_R,L_K)\le CR^{-2\gamma}\,.$$

Proof. The first two statements are deduced exactly as in Proposition 9 of [58], using the same arguments as in Lemma 3.4. of [38] to guarantee that ϕ_1 is nonnegative in $\mathcal{O}$. The fact that $\phi_1>0$ in $\Omega\cap\mathcal{O}$ follows from the strong maximum principle (see Proposition 2.3).

We show the third statement. Let $\widetilde{w}(x) := w(Rx)$ for every $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Then,

$$
\begin{aligned}
\min_{w\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)} & \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} w(x)^2\,\mathrm{d}x} \\
&\leq \min_{\widetilde{w}\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x/R)-\widetilde{w}(y/R)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} \widetilde{w}(x/R)^2\,\mathrm{d}x} \\
&= R^{-2\gamma} \min_{\widetilde{w}\in\widetilde{\mathbb{H}}^s_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x)-\widetilde{w}(y)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_1} \widetilde{w}(x)^2\,\mathrm{d}x} \\
&= \lambda_{1,\,\mathrm{odd}}(B_1,(-\Delta)^\gamma)\Lambda R^{-2\gamma}\,.
\end{aligned}
$$

□

rstEigenfunction⟩ *Remark* 3.3. Note that, by standard regularity results for L_K, we have that $\phi_1 \in C^\gamma(\overline{\Omega}) \cap C^\infty(\Omega)$, and the regularity up to the boundary is optimal (see [50] and the references therein for the details). Due to this and the fact that $\phi_1 > 0$ in $\Omega \cap \mathcal{O}$ while $\phi_1 = 0$ in $\mathbb{R}^{2m} \setminus \Omega$, it is easy to check by using (1.11) that $-\infty < L_K\phi_1 < 0$ in $\mathcal{O} \setminus \overline{\Omega}$ and $L_K\phi_1 = -\infty$ in $\partial\Omega \cap \mathcal{O}$.

With these ingredients, we can proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. The strategy is to build a suitable solution u_R of

$$
\left\{ \begin{array}{rcll} L_K u_R & = & f(u_R) & \text{in } B_R\,, \\ u_R & = & 0 & \text{in } \mathbb{R}^{2m}\setminus B_R\,, \end{array} \right. \tag{3.4}
$$
Eq:ProofExistenc

and then let $R \to +\infty$ to get a saddle-shaped solution.

Let $\phi_1^{R_0}$ be the first odd eigenfunction of L_K in $B_{R_0} \subset \mathbb{R}^{2m}$, given by Lemma 3.2, and let $\lambda_1^{R_0} := \lambda_{1,\,\mathrm{odd}}(B_{R_0}, L_K)$. Then, we claim that for R_0 big enough and ε small enough, $\underline{u}_R = \varepsilon\phi_1^{R_0}$ is an odd subsolution of (3.4) for every $R \geq R_0$. To see this, first note that, without loss of generality, we can assume that $||\phi_1^{R_0}||_{L^\infty(B_R)} = 1$. Then, since $\varepsilon\phi_1^{R_0} > 0$ in $B_{R_0} \cap \mathcal{O}$ and using (1.7), we see that for every $x \in B_{R_0} \cap \mathcal{O}$,

$$
\frac{f(\varepsilon\phi_1^{R_0}(x))}{\varepsilon\phi_1^{R_0}(x)} > f'(\varepsilon\phi_1^{R_0}(x)) \geq f'(0)/2
$$

if ε is small enough, independently of x. Therefore, since $f'(0) > 0$, taking R_0 big enough so that $\lambda_1^{R_0} < f'(0)/2$ (see the last statement of Lemma 3.2), we have that for every $x \in B_{R_0} \cap \mathcal{O}$, $f(\varepsilon\phi_1^{R_0}(x)) > \lambda_1\varepsilon\phi_1^R(x)$. Thus,

$$
L_K\underline{u}_R = \lambda_1^{R_0}\varepsilon\phi_1^{R_0} < f(\varepsilon\phi_1^{R_0}) = f(\underline{u}_R) \quad \text{in } B_{R_0} \cap \mathcal{O}\,.
$$

In addition, if $x \in (B_R \setminus B_{R_0}) \cap \mathcal{O}$, by Remark 3.3 we have that

$$L_K \underline{u}_R < 0 = f(0) = f(\underline{u}_R) \quad \text{ in } (B_R \setminus B_{R_0}) \cap \mathcal{O}\,.$$

Hence, the claim is proved.

Now, if we define $\overline{u}_R := \chi_{\mathcal{O}\cap B_R} - \chi_{\mathcal{I}\cap B_R}$, a simple computation shows that it is an odd supersolution to (3.4). Therefore, using the monotone iteration procedure given in Proposition 3.1 (taking into account Remarks 2.4 and 3.3 when using the maximum principle), we obtain a solution u_R to (3.4) such that it is doubly radial, odd with respect to the Simons cone and $\varepsilon\phi_1^{R_0} = \underline{u}_R \leq u_R \leq \overline{u}_R$ in $\mathcal{O}$. Note that, since $\underline{u}_R > 0$ in $\mathcal{O} \cap B_{R_0}$, the same holds for u_R.

Using a standard compactness argument as in [38], we let $R \to +\infty$ to obtain a sequence u_{R_j} converging on compacts in $C^{2\gamma+\eta}(\mathbb{R}^{2m})$ norm, for some $\eta > 0$, to a solution $u \in C^{2\gamma+\eta}(\mathbb{R}^{2m})$ of $L_K u = f(u)$ in $\mathbb{R}^{2m}$. Note that u is doubly radial, odd with respect to the Simons cone and $0 \leq u \leq 1$ in $\mathcal{O}$. Let us show that $0 < u < 1$ in $\mathcal{O}$ and hence u is a saddle-shaped solution. Indeed, the usual strong maximum principle yields $u < 1$ in $\mathcal{O}$. Moreover, since $u_R \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$ for $R > R_0$, also the limit $u \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$. Therefore, by applying the strong maximum principle for odd functions (see Proposition 2.3) we obtain that $0 < u < 1$ in $\mathcal{O}$. □

The fact of being u positive in $\mathcal{O}$ yields that u is stable in this set, as explained in the following remark.

⟨Remark:Stability⟩ *Remark* 3.4. Note that if w is a bounded positive solution to $L_K w = f(w)$ in a set $\Omega \subset \mathbb{R}^n$, then w is stable, that is, (1.9) holds. The proof of this is rather simple and we present it next. It is a consequence of the fact that, under these assumptions, w is a positive supersolution of the linearized operator $L_K - f'(w)$ (a more detailed discussion can be found in [44]).

On the one hand, note that by (1.7), we have that $f'(w)w < f(w)$ in Ω, since w is positive in Ω. On the other hand, the following holds for all functions φ and ξ, with $\varphi > 0$:

$$\big(\varphi(x) - \varphi(y)\big)\left(\frac{\xi^2(x)}{\varphi(x)} - \frac{\xi^2(y)}{\varphi(y)}\right) \leq |\xi(x) - \xi(y)|^2\,. \tag{3.5}$$ Eq:IdentityStabi

Indeed, developing the squares and the products, this last inequality is equivalent to

$$2\xi(x)\xi(y) \leq \frac{\varphi(x)}{\varphi(y)}\xi^2(y) + \frac{\varphi(y)}{\varphi(x)}\xi^2(x)\,,$$

which in turn is equivalent to

$$\left(\xi(x)\sqrt{\frac{\varphi(y)}{\varphi(x)}} - \xi(y)\sqrt{\frac{\varphi(x)}{\varphi(y)}}\right)^2 \geq 0\,.$$

Using these two facts, for every $\xi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega f'(w)\xi^2\,\mathrm{d}x &\le \int_\Omega \frac{\xi^2}{w} f(w)\,\mathrm{d}x = \int_\Omega \frac{\xi^2}{w} L_K w\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \big(w(x)-w(y)\big)\left(\frac{\xi^2(x)}{w(x)} - \frac{\xi^2(y)}{w(y)}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Thus, w is stable in Ω.

4. Symmetry results

SymmetryResults⟩ This section is devoted to prove the following two symmetry results. Both results will be needed in the following section to establish the asymptotic behavior of the saddle-shaped solution. The first one is a result for positive solutions in the whole space.

metryWholeSpace⟩ **Theorem 4.1.** *Let $L_K \in \mathcal{L}_0(n,\gamma)$ and let v be a bounded solution to*

$$\left\{\begin{array}{rcll} L_K v &=& f(v) & \text{in } \mathbb{R}^n\,, \\ v &\geq& 0 & \text{in } \mathbb{R}^n\,, \end{array}\right. \tag{4.1}$$

Eq:PositiveWhole

with the nonlinearity $f \in C^1$ satisfying

- $f(0) = f(1) = 0$,
- $f'(0) > 0$,
- $f > 0$ *in* $(0,1)$, *and*
- $f < 0$ *in* $(1,+\infty)$.

Then, $v \equiv 0$ or $v \equiv 1$.

Similar classification results have been proved for the fractional Laplacian in [24, 46] (either using the extension problem or not) with the method of moving spheres, which uses crucially the scale invariance of the operator $(-\Delta)^\gamma$. To the best of our knowledge, there is no similar result available in the literature for general kernels in the ellipticity class $\mathcal{L}_0$ (which are not necessarily scale invariant). Thus, we present here a proof based on the techniques used in [7] for a local equation with the classical Laplacian. It relies on the maximum principle and the translation invariance of the operator (combined with a Harnack inequality and a stability argument), and thus can be carried out for all operators in $\mathcal{L}_0$.

The second one is a symmetry result for equations in a half-space. Here we use the notation $\mathbb{R}^n_+ = \{x_n > 0\}$.

Notacion primas

h:SymmHalfSpace⟩ **Theorem 4.2.** *Let $L_K \in \mathcal{L}_0(n,\gamma)$ and let v be a bounded solution to one of these two problems:*

(P1) Eq:P1
$$\left\{\begin{array}{rcll} L_K v &=& f(v) & \text{in } \mathbb{R}^n_+, \\ v &>& 0 & \text{in } \mathbb{R}^n_+, \\ v(x',x_n) &=& -v(x',-x_n) & \text{in } \mathbb{R}^n. \end{array}\right.$$

(P2) Eq:P2
$$\left\{\begin{array}{rcll} L_K v &=& f(v) & \text{in } \mathbb{R}^n_+, \\ v &>& 0 & \text{in } \mathbb{R}^n_+, \\ v &=& 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{array}\right.$$

Assume that, in $\mathbb{R}^n_+$, the kernel K of the operator L_K is decreasing in the direction of x_n, that is, it satisfies

$$K(x-y) \geq K(x-y^*) \quad \text{for all } \; x,y \in \mathbb{R}^n_+,$$

where y^* is the reflection of y with respect to $\{x_n = 0\}$. Suppose that the nonlinearity $f \in C^1$ and

- $f(0) = f(1) = 0$,
- $f'(0) > 0$, *and* $f'(t) \le 0$ *for all* $t \in [1-\delta, 1]$ *for some* $\delta > 0$,
- $f > 0$ *in* $(0,1)$, *and*
- f *is odd in the case of* (P1).

Then, v depends only on x_n and it is increasing in that direction.

The result for (P2) has been proved for the fractional Laplacian under some assumptions on f (weaker than the ones in Theorem 4.2) in [49, 3, 4, 36]. Instead, to the best of our knowledge (P1) has not been treated even for the fractional Laplacian. In our case, the fact that f is of Allen-Cahn type allows us to use rather simple arguments that work for both problems (P1) and (P2) —moving planes and sliding methods. Moreover, the fact that we replace the kernel of the operator by a general K satisfying (1.5) do not affect significantly the proof. Although (P2) will not be used in this paper, we state it and prove it here for future reference, since the proof is analogous to the one of (P1).

4.1. **A symmetry result for positive solutions in the whole space.** In the proof of Theorem 4.1 we will need two main ingredients, that we present next. The first one is a Harnack inequality for a solution to the semilinear problem (4.1). This inequality follows readily from the results of Cozzi in [27], although the precise result that we need is not stated there. For the reader's convenience and for future reference, we present the result here and indicate how to deduce it from the results in [27].

arnackSemilinear⟩ **Proposition 4.3.** *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let w be a solution to (4.1) with f a Lipschitz nonlinearity such that $f(0) = 0$. Then, for every $x_0 \in \mathbb{R}^n$ and every $R > 0$, it holds*

$$\sup_{B_R(x_0)} w \le C \inf_{B_R(x_0)} w,$$

with $C > 0$ depending only on $n, \gamma, \lambda, \Lambda$ and R.

Proof. Following the notation of [27], since f is Lipschitz and $f(0) = 0$, we have

$$|f(u)| \le d_1 + d_2 |u|^{q-1} \quad \text{in } \mathbb{R}^n,$$

with $d_1 = 0$, $d_2 = ||f||_{\mathrm{Lip}}$ and $q = 2$. With this choice of the parameters, we only need to repeat the proof of Proposition 8.5 from [27] (with $p = 2$ and $\Omega = \mathbb{R}^n$) in order to obtain that u belongs to the fractional De Giorgi class $\mathrm{DG}^{\gamma,2}(\mathbb{R}^n, 0, H, -\infty, 2\gamma/n, 2\gamma, +\infty)$ for some constant $H > 0$ (see [27] for the precise definition of these classes). Therefore, the Harnack inequality follows from Theorem 6.9 in [27]. $\square$

The second ingredient that we need in the proof of Theorem 4.1 is the following parabolic maximum principle in the unbounded set $\mathbb{R}^n \times (0, +\infty)$.

⟨Prop:ParaMaxPrp⟩ **Proposition 4.4.** *Let $L_K \in \mathcal{L}_0(n,\gamma)$ and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, such that*

$$\begin{cases} \partial_t v + L_K v + c(x)\, v & \leq & 0 & \text{in } \mathbb{R}^n \times (0,+\infty)\,, \\ v_0 := v(x,0) & \leq & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

with $c(x)$ a bounded function. Then,

$$v(x,t) \leq 0 \quad \text{in} \quad \mathbb{R}^n \times [0,+\infty).$$

This result can be deduced from the usual parabolic maximum principle in a bounded (in space and time) set with a rather simple argument. Since we have not found a specific reference where such result is stated, let us present its proof with full detail for the sake of clarity. First of all, we present the usual parabolic maximum principle in a bounded set. The proof for cylindrical sets can be found for instance in [5]. Although the argument is essentially the same, since it is short we include here the complete proof for the sake of completeness.

abolicmaxPrpBdd⟩ **Lemma 4.5.** *Let L_K be an integro-differential operator of the form (1.2) with kernel satisfying (1.3), and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, satisfying*

$$\begin{cases} \partial_t v + L_K v & \leq & 0 & \text{in } \Omega \subset B_R \times (0,T)\,, \\ v_0 := v(x,0) & \leq & 0 & \text{in } \overline{\Omega} \cap \{t=0\} \subset B_R\,, \\ v & \leq & 0 & \text{in } (\mathbb{R}^n \times (0,T)) \setminus \Omega\,. \end{cases}$$

Then, $v \leq 0$ in $\mathbb{R}^n \times [0,T]$.

Proof. By contradiction, for every $\varepsilon > 0$ assume that

$$M := \sup_{\mathbb{R}^n \times (0,T-\varepsilon)} v > 0$$

By the initial conditions and since $v \leq 0$ in $(\mathbb{R}^n \times (0,T)) \setminus \Omega$, v attains this positive value M at a point $(x_0,t_0) \in \Omega$ with $t_0 \leq T-\varepsilon$. If $t_0 \in (0,T-\varepsilon)$, then (x_0,t_0) is an interior global maximum (in $\mathbb{R}^n \times (0,T-\varepsilon)$) and it must satisfy $v_t(x_0,t_0) = 0$ and $L_K v(x_0,t_0) > 0$, which contradicts the equation. If $t_0 = T-\varepsilon$, then $v_t(x_0,t_0) \geq 0$ and $L_K v(x_0,t_0) > 0$, which is also a contradiction with the equation. Thus, $v \leq 0$ in $\mathbb{R}^n \times [0,T-\varepsilon)$ and since this holds for $\varepsilon > 0$ arbitrarily small, we deduce $v \leq 0$ in $\mathbb{R}^n \times [0,T)$, and by continuity in $\mathbb{R}^n \times [0,T]$. □

To establish Proposition 4.4 from Lemma 4.5, we need to introduce an auxiliary function enjoying certain properties (see Lemma 4.7 below). Before presenting such function, we need the following result.

mma:NoBddSolL=1⟩ **Lemma 4.6.** *There is no bounded solution to $L_K v = 1$ in $\mathbb{R}^n$ for any $L_K \in \mathcal{L}_0(n,\gamma)$.*

Proof. Assume by contradiction that such solution exists. Then, by interior regularity (see Section 2) $v \in C^1(\mathbb{R}^n)$ and $|\nabla v| \le C$ in $\mathbb{R}^n$. For every $i = 1, \dots, n$, we differentiate the equation with respect to x_i to obtain

$$\left\{ \begin{array}{rcll} L_K v_{x_i} & = & 0 & \text{in } \mathbb{R}^n \,, \\ |v_{x_i}| & \le & C & \text{in } \mathbb{R}^n \,. \end{array} \right.$$

By the Liouville Theorem for the operator L_K (it is proved exactly as in [51], see also [57]), v_{x_i} is constant. Hence, ∇v is constant, and thus v is affine. But since u is bounded, v must be constant too, and we arrive to a contradiction with $L_K v = 1$. □

With this result we can introduce the auxiliary function that we will use to prove the parabolic maximum principle of Proposition 4.4.

ma:SolBallToZero⟩ **Lemma 4.7.** *Let $L_K \in \mathcal{L}_0(n, \gamma)$. Then, for every $R > 0$ there exists a constant $M_R > 0$ and a function $\psi_R \ge 0$ solution to*

$$\left\{ \begin{array}{rcll} L_K \psi_R & = & -1/M_R & \textit{in } B_R \,, \\ \psi_R & = & 1 & \textit{in } \mathbb{R}^n \setminus B_R \,, \end{array} \right. \tag{4.2}$$

Eq:psiRProblem

satisfying

$$\psi_R \to 0 \quad \textit{and} \quad M_R \to +\infty \quad \textit{as } R \to +\infty \,.$$

Proof. First, consider ϕ_R the solution to

$$\left\{ \begin{array}{rcll} L_K \phi_R & = & 1 & \text{in } B_R \,, \\ \phi_R & = & 0 & \text{in } \mathbb{R}^n \setminus B_R \,. \end{array} \right.$$

Note that the existence of a weak solution to the previous problem is given by the Riesz representation theorem. Moreover, by standard regularity results (see Section 2.1), ϕ_R is in fact a classical solution and by the maximum principle, $\phi_R > 0$ in B_R.

Define $M_R := \sup_{B_R} \phi_R$. Since M_R is increasing (use the maximum principle to compare ϕ_R and $\phi_{R'}$ with $R > R'$), it must have a limit M. Assume by contradiction that $M < +\infty$. Now, consider the new function $\varphi_R := \phi_R / M_R$, which satisfies

$$\left\{ \begin{array}{rcll} L_K \varphi_R & = & 1/M_R & \text{in } B_R \,, \\ \varphi_R & = & 0 & \text{in } \mathbb{R}^n \setminus B_R \,, \\ \varphi_R & \le & 1 \,. & \end{array} \right. \tag{4.3}$$

Eq:varphiRProble

Therefore, by a standard compactness argument, we deduce that φ_R converges (up to a subsequence) as $R \to +\infty$ to a function φ that is solution to $L_K \varphi = 1/M$ in $\mathbb{R}^n$ and satisfies $|\varphi| \le 1$. This contradicts Lemma 4.6 and therefore, $M_R \to +\infty$ as $R \to +\infty$.

Define now $\psi_R := 1 - \phi_R / M_R = 1 - \varphi_R$, which solves trivially (4.2). Thus, we only need to show that $\phi \to 0$ as $R \to +\infty$. We will see that $\varphi_R \to 1$ as $R \to +\infty$. Recall that φ_R solves problem (4.3), and by the previous arguments, letting $R \to +\infty$ we deduce that φ_R converges to a bounded function $\varphi \ge 0$ that solves $L_K \varphi = 0$ in $\mathbb{R}^n$. By the Liouville Theorem, φ must be constant, and since its L^∞ norm is 1 and $\varphi \ge 0$, we conclude $\varphi \equiv 1$. □

With these ingredients, we establish now the parabolic maximum principle in $\mathbb{R}^n \times (0,+\infty)$.

Proof of Proposition 4.4. First of all, note that with the change of function $\tilde{v}(x,t) = \mathrm{e}^{-\alpha t} v(x,t)$ we can reduce the initial problem to

$$\begin{cases} \partial_t \tilde{v} + L_K \tilde{v} & \leq & 0 & \text{in } \Omega \subset \mathbb{R}^n \times (0,+\infty)\,, \\ \tilde{v} & \leq & 0 & \text{in } (\mathbb{R}^n \times (0,+\infty)) \setminus \Omega\,, \\ \tilde{v}_0 & \leq & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

if we take $\alpha > ||c||_{L^\infty}$ and $\Omega = \{(x,t) \in \mathbb{R}^n \times (0,+\infty) \ : \ v(x,t) > 0\}$.

Now, consider the function

$$w_R(x,t) := ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} \left(\psi_R + \frac{t}{M_R} \right),$$

where ψ_R and M_R are defined in Lemma 4.7. Then, it is easy to check that w_R satisfies

$$\begin{cases} \partial_t w_R + L_K w_R & = & 0 & \text{in } B_R \times (0,T)\,, \\ w_R(x,0) & \geq & 0 & \text{in } B_R\,, \\ w_R(x,t) & \geq & ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} & \text{in } (\mathbb{R}^n \setminus B_R) \times (0,T)\,, \end{cases}$$

for every $T > 0$ and $R > 0$. Since $w_R \geq 0$, by the maximum principle in $(B_R \times (0,T)) \cap \Omega$ (see Lemma 4.5) we can easily deduce that $w_R \geq \tilde{v}$ in $B_R \times (0,T)$.

Finally, given an arbitrary point $(x_0,t_0) \in \Omega$, take $R_0 > 0$ and $T > 0$ such that $(x_0,t_0) \in B_{R_0} \times (0,T)$. Thus,

$$\tilde{v}(x_0,t_0) \leq w_R(x_0,t_0) = ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} \left(\psi_R(x_0) + \frac{t_0}{M_R} \right), \quad \text{for every} \quad R \geq R_0.$$

Letting $R \to +\infty$ and using that $\psi_R(x_0) \to 0$ and $M_R \to +\infty$ (see Lemma 4.7), we conclude $\tilde{v}(x_0,t_0) \leq 0$, and therefore $v(x_0,t_0) = \mathrm{e}^{\alpha t_0} \tilde{v}(x_0,t_0) \leq 0$. □

By using the Harnack inequality and the parabolic maximum principle we can now establish Theorem 4.1. The proof follows the ideas of Berestycki, Hamel, and Nadirashvili from Theorem 2.2. in [7] but adapted to the whole space and with an integro-differential operator.

Proof of Theorem 4.1. Assume $v \not\equiv 0$. Then, by the strong maximum principle $v > 0$. Our goal is to show that $v \equiv 1$, and this will be accomplished in two steps.

Step 1: We show that $m := \inf_{\mathbb{R}^n} v > 0$.

By contradiction, we will assume $m = 0$. Then, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $v(x_k) \to 0$ as $k \to +\infty$. By the Harnack Inequality from Proposition 4.3, given any $R > 0$ we have

$$\sup_{B_R(x_k)} v \leq C_R \inf_{B_R(x_k)} v \leq C_R\, v(x_k) \to 0 \text{ as } k \to +\infty. \tag{4.4}$$

Eq:Harnack

Since $f(0) = 0$ and $f'(0) > 0$, it is easy to show that $f(t) \geq f'(0)t/2$ if t is small enough. Therefore, from this and (4.4) we deduce that there exists $M(R) \in \mathbb{N}$ such that

$$L_K v - \frac{f'(0)}{2} v \geq 0 \quad \text{in} \quad B_R(x_{M(R)})\,. \tag{4.5}$$ Eq:WholeSpace2

On the other hand, let us define

$$\lambda_R^{x_0} = \inf_{\substack{\varphi \in C_0^1(B_R(x_0)) \\ \varphi \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^2 \, K(x-y)\, \mathrm{d}x \, \mathrm{d}y}{\displaystyle\int_{\mathbb{R}^n} \varphi(x)^2 \, \mathrm{d}x},$$

which decreases to zero uniformly in x_0 as $R \to +\infty$ from being $L_K \in \mathcal{L}_0$ (see the proof of Lemma 3.2 and also Proposition 9 of [58]). Therefore, there exists $R_0 > 0$ such that $\lambda_R^x < f'(0)/2$ for all $x \in \mathbb{R}^n$ and $R \geq R_0$. In particular, by choosing $x = x_{M(R_0)}$ there exists $w \in C_0^1(B_{R_0}(x_{M(R_0)}))$ such that $w \not\equiv 0$ and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(x) - w(y)|^2 \, K(x-y)\, \mathrm{d}x \, \mathrm{d}y < \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2 \, \mathrm{d}x. \tag{4.6}$$ Eq:Eigenfunction

Now, multiply (4.5) by $w^2/v \geq 0$ and integrate in $\mathbb{R}^n$, we get

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^n} \frac{w^2}{v} L_K v \, \mathrm{d}x - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \{v(x) - v(y)\} \left(\frac{w^2(x)}{v(x)} - \frac{w^2(y)}{v(y)} \right) K(x-y) \, \mathrm{d}x \, \mathrm{d}y - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2 \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(x) - w(y)|^2 K(x-y) - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2 \, \mathrm{d}x,
\end{aligned}$$

which contradicts (4.6). Here we have used that the kernel is positive and the inequality (3.5). Therefore, $\inf_{\mathbb{R}^n} v > 0$.

Step 2: We show that $v \equiv 1$.

Now, choose $0 < \xi_0 < \min\{1, m\}$, which is well defined by Step 1, and let $\xi(t)$ be the solution of the ODE

$$\begin{cases} \dot{\xi}(t) &= f(\xi(t)) \quad \text{in } (0, \infty)\,, \\ \xi(0) &= \xi_0\,. \end{cases}$$

Since $f > 0$ in $(0,1)$ and $f(1) = 0$ we have that $\dot{\xi}(t) > 0$ for all $t \geq 0$ and $\lim_{t \to 0} \xi(t) = 1$.

Now, note that both $v(x)$ and $\xi(t)$ solve the parabolic equation

$$\partial_t w + L_K w = f(w) \quad \text{in } \mathbb{R}^n \times (0, \infty)\,,$$

and satisfy

$$v(x) \geq m \geq \xi_0 = \xi(0).$$

Thus, by the parabolic maximum principle (Proposition 4.4) applied to $v - \xi$ and taking $c(x) = -\{f(v) - f(\xi)\}/(v - \xi)$, we deduce that $v(x) \geq \xi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. By letting $t \to +\infty$ we obtain

$$v(x) \geq 1 \quad \text{in } \mathbb{R}^n\,.$$

In a similar way, taking $\tilde{\xi}_0 > ||v||_{L^\infty} \geq 1$, using $f < 0$ in $(1, +\infty)$, $f(1) = 0$ and the parabolic maximum principle, we obtain the upper bound $v \leq 1$.

□

4.2. **A one-dimensional symmetry result for positive solutions in a half-space.** In this subsection we establish Theorem 4.2. To do it, we proceed in three steps. First, we show that the solution is monotone in the x_n direction by using a moving planes argument (see Proposition 4.8 below). Once this is shown, we can deduce that the solution v has uniform limits as $x_n \pm \to \infty$. Finally, by using the sliding method (see Proposition 4.14 below), we deduce the one-dimensional symmetry of the solution.

We proceed now with the details of the arguments. As we have said, the first step is to show that the solution is monotone. We establish the following result.

onotonyHalfSpace⟩ **Proposition 4.8.** *Let v be a bounded solution to one of the problems (P1) or (P2), with $L_K \in \mathcal{L}_0(n, \gamma)$ such that the kernel K satisfies (4.8). Let f be a Lipschitz nonlinearity such that $f > 0$ in $(0, ||v||_{L^\infty(\mathbb{R}^n_+)})$. Then,*

$$\frac{\partial v}{\partial x_n} > 0 \quad \textit{in } \mathbb{R}^n_+.$$

To prove it, we use a moving planes argument, and for this reason we need a maximum principle in "narrow" sets for odd functions with respect to a hyperplane. Recall that for a set $\Omega \subset \mathbb{R}^n$, we define the quantity $R(\Omega)$ as the smallest positive R for which

$$\frac{|B_R(x) \setminus \Omega|}{|B_R(x)|} \geq \frac{1}{2} \quad \text{for every } x \in \Omega. \tag{4.7}$$ Eq:DefNarrow

If no such radius exists, we define $R(\Omega) = +\infty$. Thus, we say that a set Ω is "narrow" ir $R(\Omega)$ is small depending on certain quantities.

An important result that we need is the following ABP-type estimate. It is proved in [49] for the fractional Laplacian, following the arguments in [10] (see also [11]). The proof for a general operator L_K do not differ significantly from the one for the fractional Laplacian. Nevertheless, we include it here for the sake of completeness.

⟨Th:ABPEstimate⟩ **Theorem 4.9.** *Let $\Omega \subset \mathbb{R}^n$ with $R(\Omega) < +\infty$. Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $\sup_\Omega v < +\infty$ and satisfying*

$$\begin{cases} L_K v - c(x) v & \leq & h & \textit{in } \Omega\,, \\ v & \leq & 0 & \textit{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $c(x) \leq 0$ in Ω and $h \in L^\infty(\Omega)$.

Then,

$$\sup_\Omega v \leq C R(\Omega)^{2\gamma} \, ||h||_{L^\infty(\Omega)} \,,$$

where C is a constant depending on n, γ and Λ.

The only ingredient needed to show Theorem 4.9 is the following weak Harnack inequality proved in [28].

Prop:WeakHarnack⟩ **Proposition 4.10** (see Corollary 4.4 of [28]). *Let $\Omega \subset \mathbb{R}^n$ and $L_K \in (n, \gamma, \lambda, \Lambda)$. Let $w \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $w \geq 0$ in $\mathbb{R}^n$. Assume that w satisfies weakly $L_K w \geq h$ in Ω, for some $h \in L^\infty(\Omega)$. Then, there exists an exponent $\varepsilon > 0$ and a constant $C > 1$, both depending on n, γ and Λ, such that*

$$\left(\fint_{B_{R/2}(x_0)} w^\varepsilon \right)^{1/\varepsilon} \leq C \left(\inf_{B_R(x_0)} w + R^{2\gamma} \, ||h||_{L^\infty(\Omega)} \right)$$

for every $x_0 \in \Omega$ and $0 < R < \operatorname{dist}(x_0, \partial\Omega)$.

With the previous weak Harnack inequality we can now establish the ABP estimate.

Proof of Theorem 4.9. First, note that it is enough to show it for $v > 0$ in Ω satisfying

$$\begin{cases} L_K v & \leq & h & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Indeed, if we consider $\Omega_0 = \{x \in \Omega \ : \ v > 0\}$, then since $c \leq 0$ we have $L_K v \leq L_K v - c(x) v \leq h$ in Ω_0.

Assume first that Ω is bounded. Then the supremum of v must be achieved at an interior point $x_0 \in \Omega$. Define $M := v(x_0) = \sup_\Omega v$ and consider the function $w := M - v^+$. Note that $0 \leq w \leq M$, $w(x_0) = 0$ and $w \equiv M$ in $\mathbb{R}^n \setminus \Omega$. If we extend h to be 0 outside Ω, we can easily see that $L_K w \geq -h$ in $B_R(x_0)$.

Now, by choosing $R = 2R(\Omega)$, and using the weak Harnack inequality of Proposition 4.10, we get

$$\begin{aligned} M \frac{1}{2^\varepsilon} &\leq \left(M^\varepsilon \frac{|B_{R/2}(x_0) \setminus \Omega|}{|B_{R/2}(x_0)|} \right)^{1/\varepsilon} = \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0) \setminus \Omega} w^\varepsilon \right)^{1/\varepsilon} \\ &\leq \left(\fint_{B_{R/2}(x_0)} w^\varepsilon \right)^{1/\varepsilon} \leq C \left(\inf_{B_R(x_0)} w + R^{2\gamma} \, ||h||_{L^\infty(\Omega)} \right). \end{aligned}$$

The conclusion follows from the fact that $w(x_0) = \inf_{B_R(x_0)} w = 0$.

In the case of Ω being unbounded, the proof is the same with minor changes. We define M as before and we consider, for every $\delta > 0$, a point x_0 such that $M - \delta \leq v(x_0)$. We proceed as before and the desired estimate follows by letting $\delta \to 0$. □

As a consequence of this result, one can deduce easily a general maximum principle in "narrow" sets.

leNarrowDomains⟩ **Corollary 4.11.** *Let $\Omega \subset \mathbb{R}^n$ with $R(\Omega) < +\infty$. Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $\sup_\Omega v < +\infty$ and satisfying*

$$\begin{cases} L_K v + c(x) v & \leq & 0 & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $c(x)$ bounded below.

Then, there exists a number $\bar{R} > 0$ such that $v \leq 0$ in Ω whenever $R(\Omega) < \bar{R}$.

Proof. We write $c = c^+ - c^-$, and therefore $L_K v - (-c^+)v \leq c^- v^+$. By Theorem 4.9 we get

$$\sup_\Omega v \leq C R(\Omega)^{2\gamma} \left|\left|c^- v^+\right|\right|_{L^\infty(\Omega)} \leq C R(\Omega)^{2\gamma} \left|\left|c^-\right|\right|_{L^\infty(\Omega)} \sup_\Omega v\,.$$

Hence, if $CR(\Omega)^{2\gamma} \left|\left|c^-\right|\right|_{L^\infty(\Omega)} < 1$, we deduce that $v \leq 0$ in Ω. □

The previous maximum principle in "narrow" sets is not powerful enough to apply the moving planes method. The reason for this is that in the hypotheses of Corollary 4.11 there is a prescribed constant sign of a function outside the set. Nevertheless, in the application of the moving planes argument, since our functions are odd with respect to a hyperplane, they cannot have a constant sign in the exterior of a "narrow" band. Thus, we need another version of a maximum principle in "narrow" sets that applies to odd functions and only requires a constant sign of the function at one side of a hyperplane.

MaxPrpNarrowOdd⟩ **Proposition 4.12.** *Let H be a half-space in $\mathbb{R}^n$, and denote by x^* the reflection of any point x with respect to the hyperplane ∂H. Let $L_K \in \mathcal{L}_0(n,\gamma)$ with a positive kernel K satisfying*

$$K(x-y) \geq K(x-y^*), \quad \text{for all } \; x, y \in H. \tag{4.8}$$

Eq:KernelSymmetr

Assume that $v \in C^\beta(\Omega)$, with $\beta > 2\gamma$, satisfies

$$\begin{cases} L_K v & \geq & c(x)\, v & \text{in } \Omega \subset H, \\ v & \geq & 0 & \text{in } H \setminus \Omega, \\ v(x) & = & -v(x^*) & \text{in } \mathbb{R}^n. \end{cases}$$

Then, there exist a number $\overline{R}$ such that $v \geq 0$ whenever $R(\Omega) \leq \overline{R}$.

Proof. Let us begin by defining $\Omega_- = \{x \in \Omega \;:\; v < 0\}$. We shall prove that Ω_- is empty. Assume by contradiction that it is not empty. Then, we split $v = v_1 + v_2$, where

$$v_1(x) = \begin{cases} v(x) & \text{in } \Omega_-, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases} \qquad \text{and} \qquad v_2(x) = \begin{cases} 0 & \text{in } \Omega_-, \\ v(x) & \text{in } \mathbb{R}^n \setminus \Omega_-. \end{cases}$$

Let us first show that $Lv_2 \leq 0$ in Ω_-. To see this, let us take $x \in \Omega_-$ and thus

$$L_K v_2(x) = \int_{\mathbb{R}^n \setminus \Omega_-} -v_2(y) K(x-y)\, \mathrm{d}y = -\int_{\mathbb{R}^n \setminus \Omega_-} v(y) K(x-y)\, \mathrm{d}y.$$

Now, we split $\mathbb{R}^n \setminus \Omega_-$ into

$$A_1 = \Omega_-^*, \qquad \text{and} \qquad A_2 = (H \setminus \Omega_-) \cup (H \setminus \Omega_-)^*\,,$$

and we compute the previous integral in these two sets separately. On the one hand,

$$-\int_{A_1} v(y) K(x-y)\, \mathrm{d}y = -\int_{\Omega_-} v(y^*) K(x-y^*)\, \mathrm{d}y = \int_{\Omega_-} v(y) K(x-y^*)\, \mathrm{d}y \leq 0,$$

where the last inequality follows from being v negative in Ω_- and the kernel positive in all $\mathbb{R}^n$. On the other hand,

$$-\int_{A_2} v(y)K(x-y)\,\mathrm{d}y = -\int_{H\setminus\Omega_-} v(y)K(x-y)\,\mathrm{d}y - \int_{H\setminus\Omega_-} v(y^*)K(x-y^*)\,\mathrm{d}y$$
$$= -\int_{H\setminus\Omega_-} v(y)\left\{K(x-y)-K(x-y^*)\right\}\,\mathrm{d}y \le 0,$$

where we have use the kernel condition (4.8) and the odd symmetry of v. Thus, we get $L_K v_2 \le 0$ in Ω_-, which means

$$L_K v_1 = L_K v - L_K v_2 \ge L_K v \ge c(x)\,v = c(x)\,v_1 \quad \text{in } \ \Omega_-.$$

Therefore v_1 solves

$$\begin{cases} L_K v_1 & \ge & c(x)\,v_1 & \text{in } \ \Omega_-, \\ v_1 & = & 0 & \text{in } \ \mathbb{R}^n \setminus \Omega_-, \end{cases}$$

and we can apply the usual maximum principle for "narrow" sets (Corollary 4.11) to v_1 in Ω_- in order to deduce that $v_1 \ge 0$ in all $\mathbb{R}^n$. But this is a contradiction with the definition of v_1 and the fact that the set Ω_- is not empty. □

Remark 4.13. A maximum principle such as Proposition 4.12 was already proved for the fractional Laplacian in [23], but with the additional hypothesis that either Ω is bounded or $\liminf_{x\in\Omega,\ |x|\to\infty} v(x) \ge 0$. In the proof of Theorem 3.1 in [49], Quaas and Xia use a suitable argument (the truncation used in the previous proof, previously used by Felmer and Wang in [40]) to avoid the requirement of such additional hypotheses on Ω or v.

With the maximum principle in "narrow" sets for odd functions with respect to a hyperplane we can use the moving plane argument. Now we can show Proposition 4.8.

Proof of Proposition 4.8. The proof is based on the moving planes method, and is exactly the same as the analogue proof of Theorem 3.1 in [49], where Quaas and Xia establish an equivalent result for the fractional Laplacian. For this reason, we give here just a sketch. As usual, for $\lambda > 0$ we defines $w_\lambda(x) = v(x', 2\lambda - x_n) - v(x', x_n)$ and since the nonlinearity is Lipschitz, w_λ solves, in both cases —(P1) or (P2)—, the following problem:

$$\begin{cases} L_K w_\lambda & = & c_\lambda(x)\,w_\lambda & \text{in } \Sigma_\lambda \subset H_\lambda, \\ w_\lambda & \ge & 0 & \text{in } H_\lambda \setminus \Sigma_\lambda, \\ w_\lambda(x', 2\lambda - x_n) & = & -w_\lambda(x', x_n) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Sigma_\lambda := \{x = (x', x_n) \ : \ 0 < x_n < \lambda\}$ and $H_\lambda := \{x = (x', x_n) \ : \ x_n < \lambda\}$ and c_λ is a bounded function. Note that w_λ is odd with respect to ∂H_λ. Then, using the maximum principle in "narrow" sets Proposition 4.12 one shows that, if λ is small enough, $w_\lambda > 0$ in Σ_λ. To conclude the proof, we define

$$\lambda^* := \sup\{\lambda \ : \ w_\eta > 0 \ \text{ in } \ \Sigma_\lambda \ \text{ for all } \ \eta < \lambda\}.$$

Note that λ^* is well defined (but may be infinite) by the previous argument. To conclude the proof, one has to show that $\lambda^* = \infty$. This is done by proving that, if λ^* is finite, then there exists a small $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ we have

$$w_{\lambda^*+\delta}(x) > 0 \quad \text{in } \Sigma_{\lambda^*-\varepsilon} \setminus \Sigma_\varepsilon$$

for some small ε. This is established using a compactness argument exactly as in Lemma 3.1 of [49] (in the argument a Harnack inequality is needed, one can use for instance Proposition 4.3). Finally, by the maximum principle in "narrow" sets we can deduce that $w_{\lambda^*+\delta}(x) > 0$ in $\Sigma_{\lambda^*+\delta}$ if δ is small enough, contradicting the definition of λ^*. □

Now, we present the other important ingredient needed in the proof of Theorem 4.2. It is the following symmetry result.

HalfSpaceLimUnif⟩ **Proposition 4.14.** *Let $L_K \in \mathcal{L}_0(n, \gamma)$ and let v be a bounded solution to one of the following problems:*

(P3) Eq:P3
$$\begin{cases} L_K v &= f(v) \quad \text{in } \mathbb{R}^n\,, \\ \lim_{x_n \to \pm\infty} v(x', x_n) &= \pm 1 \quad \text{uniformly.} \end{cases}$$

(P4) Eq:P4
$$\begin{cases} L_K v &= f(v) \quad \text{in } \mathbb{R}^n_+ = \{x_n > 0\}\,, \\ v &= 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+ = \{x_n \le 0\}\,, \\ \lim_{x_n \to +\infty} v(x', x_n) &= 1 \quad \text{uniformly.} \end{cases}$$

Assume that there exists a $\delta > 0$ such that

$$f'(t) \le 0 \quad \text{in} \quad [-1, -1+\delta] \cup [1-\delta, 1],$$

for problem (P3) and

$$f'(t) \le 0 \quad \text{in} \quad [1-\delta, 1]$$

for problem (P4).

Then, v depends only on x_n and is increasing in that direction.

Proof. It is based on the sliding method, exactly as in the proof of Theorem 1 in [6]. The idea is, as usual, to define $v^t(x) := v(x + \nu t)$ for every $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ and $\nu_n > 0$, and the aim is to show that $v^t(x) - v(x) \ge 0$ for all $t \ge 0$. Despite the fact that L_K is a nonlocal operator, the proof is exactly the same as the one in [6] —it only relies on the maximum principle, the translation invariance of the operator and the symmetry result of Theorem 4.1. Therefore, we do not include here the details. □

Finally, we can proceed with the proof of Theorem 4.2.

Proof of Theorem 4.2. Note that by Proposition 4.14 we only need to prove that

$$\lim_{x_n \to +\infty} v(x', x_n) = 1$$

uniformly. Therefore we divide the proof in two steps: first, we prove that the limit exists and is 1, and then we prove that it is uniform.

Step 1: Given $x' \in \mathbb{R}^{n-1}$, then $\lim_{x_n \to +\infty} v(x', x_n) = 1$.

By Proposition 4.8 we know that v is strictly increasing in the direction x_n. Since v is also bounded by hypothesis, we know that, given $x' \in \mathbb{R}^{n-1}$, the one variable function $v(x', \cdot)$ has a limit, that we call $\overline{v}(x')$. Note that, since $v(x', 0) = 0$ and $v_{x_n} > 0$, we deduce that $\overline{v}(x') > 0$.

Let x_n^k be any increasing sequence tending to infinity. Define $v_k(x', x_n) := v(x', x_n + x_n^k)$. By the regularity theory of the operator L_K (see Section 2) and a standard compactness argument, we see that, up to a subsequence, v_k converge uniformly on compact sets to a function v_∞ that is a classical solution to

$$\left\{ \begin{array}{rcll} L_K v_\infty & = & f(v_\infty) & \text{in } \mathbb{R}^n, \\ v_\infty & \geq & 0 & \text{in } \mathbb{R}^n. \end{array} \right. \tag{4.9}$$

Eq:ProofSymmHalf

By Theorem 4.1, either $v_\infty \equiv 0$ or $v_\infty \equiv 1$. But, by construction,

$$v_\infty(x', 0) = \lim_{k \to +\infty} v_k(x', 0) = \lim_{k \to +\infty} v(x', x_n^k) = \overline{v}(x') > 0,$$

and therefore the only possibility is

$$\lim_{x_n \to \infty} v(x', x_n) = 1 \quad \text{ for all } \ x' \in \mathbb{R}^{n-1}.$$

Step 2: The limit is uniform in x'.

Let us proceed by contradiction. Suppose that the limit is not uniform. This means that given any $\varepsilon > 0$ small enough, there exists a sequence of points (x'_k, x_n^k) with $x_n^k \to +\infty$ such that $v(x'_k, x_n^k) = 1 - \varepsilon$. Similarly as before, the sequence of functions $\tilde{v}_k(x', x_n) = v(x' + x'_k, x_n + x_n^k)$ converge uniformly on compact sets to a function $\tilde{v}_\infty$ that solves also (4.9). By Theorem 4.1, either $\tilde{v}_\infty \equiv 0$ or $\tilde{v}_\infty \equiv 1$. But, by construction

$$\tilde{v}_\infty(0, 0) = \lim_{k \to +\infty} \tilde{v}_k(0, 0) = \lim_{k \to +\infty} v(x'_k, x_n^k) = 1 - \varepsilon,$$

which is a contradiction for $\varepsilon > 0$ small enough. Thus, the limit is uniform.

Finally, by applying Proposition 4.14, we get that v depends only on x_n and is increasing in that direction. $\square$

5. Asymptotic behavior of a saddle-shaped solution

⟨Sec:Asymptotic⟩ In this section, we establish Theorem 1.5, concerning the asymptotic behavior of the saddle-shaped solution.

In order to study this behavior, it is important to relate the Allen-Cahn equation in $\mathbb{R}^{2m}$ with the same equation in $\mathbb{R}$. In the local case, this is very easy, since if v is a solution to $-\ddot{v} = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves $-\Delta w = f(w)$ in $\mathbb{R}^n$ for every unitary vector $e \in \mathbb{R}^n$. The same fact also happens for the fractional Laplacian, that is, if v is a solution to $(-\Delta)^\gamma v = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves the same equation in $\mathbb{R}^n$. We can easily see this relation via the local extension problem.

Nevertheless, for a general operator L_K this is not true anymore and we need a way to relate a solution to a one-dimensional problem with a one-dimensional solution to a n-dimensional problem. This is given in the next result. Some of its points appear in [29] with a different notation but we state and prove them here for completeness.

KernelsDimension⟩ **Proposition 5.1.** *Let L_K be a symmetric and translation invariant integro-differential operator of the form (1.2) with kernel $K : \mathbb{R}^n \to \mathbb{R}$ satisfying (1.3). Define the one dimensional kernel K_1 by*

$$K_1(t) := \int_{\mathbb{R}^{n-1}} K(\theta, t)\, \mathrm{d}\theta = |t|^{n-1} \int_{\mathbb{R}^{n-1}} K(t\sigma, t)\, \mathrm{d}\sigma. \tag{5.1}$$

Eq:OneDimKernel

(i) Let $v : \mathbb{R} \to \mathbb{R}$ and consider $w : \mathbb{R}^n \to \mathbb{R}$ defined by $w(x) = v(x_n)$. Then, $L_K w(x) = L_{K_1} v(x_n)$. If we assume moreover that K is radially symmetric, then the same happens with $w(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$. That is, $L_K w(x) = L_{K_1} v(x \cdot e)$.

(ii) If K is nonincreasing/decreasing in the x_n-direction in $\{x_n > 0\}$, then $K_1(t)$ is nonincreasing/decreasing in $(0, +\infty)$.

(iii) If $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$, then $L_{K_1} \in \mathcal{L}_0(1, \gamma, \lambda, \Lambda)$. In particular, if L_K is the fractional Laplacian in dimension n, then L_{K_1} is the fractional Laplacian in dimension 1.

Proof. We start proving point (i). We write $y = (y', y_n)$, with $y' \in \mathbb{R}^{n-1}$.

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}^n} \{w(x) - w(y)\} K(x - y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \{v(x_n) - v(y_n)\} K(x' - y', x_n - y_n)\, \mathrm{d}y'\, \mathrm{d}y_n. \end{aligned}$$

Now we make the change of variables $\theta = x' - y'$. That is,

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} \int_{\mathbb{R}^{n-1}} K(\theta, x_n - y_n)\, \mathrm{d}\theta\, \mathrm{d}y_n \\ &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} K_1(x_n - y_n)\, \mathrm{d}y_n = L_{K_1} v(x_n). \end{aligned}$$

This shows the first equality in (5.1). The alternative expression of the kernel K_1, that is useful in some cases, can be obtained from the change of variable $\theta = t\sigma$. Furthermore, in the case of K radially symmetric, the result is valid for $u(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$ after a change of variables using the previous computations.

The proof of point (ii) follows directly from the first expression of the unidimensional kernel K_1. That is,

$$K_1(t_2) - K_1(t_1) = \int_{\mathbb{R}^{n-1}} \{K(\theta, t_2) - K(\theta, t_1)\} \,\mathrm{d}\theta \geq 0 \quad \text{for any} \quad t_2 > t_1 > 0.$$

We establish now point (iii). To do it, we bound the kernel K_1 using the ellipticity condition on K:

$$\begin{aligned}
K_1(t) = |t|^{n-1} \int_{\mathbb{R}^{n-1}} K\left(t(\sigma, 1)\right) \mathrm{d}\sigma &\geq |t|^{n-1} \int_{\mathbb{R}^{n}} c_{n,\gamma} \frac{\lambda}{|t|^{n+2\gamma}(|\sigma|^2+1)^{\frac{n+2s}{2}}} \,\mathrm{d}\sigma \\
&= c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\sigma}{(|\sigma|^2+1)^{\frac{n+2\gamma}{2}}} = c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty \frac{r^{n-2}}{(r^2+1)^{\frac{n+2\gamma}{2}}} \,\mathrm{d}r \\
&= c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{n}{2}+\gamma)} = c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{c_{1,\gamma}}{c_{n,\gamma}} = c_{1,\gamma} \frac{\lambda}{|t|^{1+2\gamma}},
\end{aligned}$$

where we have used (1.4) and the definition of the Beta and Gamma functions. The upper bound for K_1 is obtained in the same way. Note that the previous computation is an equality with $\lambda = 1$ in the case of the fractional Laplacian. □

In the proof of Theorem 1.5 we will use some properties of the layer solution, which are presented next. First, in [29] it is proved that there exists a constant C such that

$$|u_0(x) - \operatorname{sign}(x)| \leq C|x|^{-2\gamma} \quad \text{and} \quad |\dot{u}_0(x)| \leq C|x|^{-1-2\gamma} \quad \text{for large } |x|. \tag{5.2}$$ Eq:PropertiesLay

In our arguments we need also to show that the second derivative of the layer goes to zero at infinity. This is the first statement of the following lemma.

dDerivativeLayer⟩ **Lemma 5.2.** *Let $K_1 : \mathbb{R} \setminus \{0\} \to (0, +\infty)$ be a kernel satisfying (1.3) and assume that it is decreasing in $(0, +\infty)$. Let u_0 be the layer solution associated to the kernel K_1, that is, u_0 solving (1.15). Then,*

(i) $\ddot{u}_0(x) \to 0$ as $x \to \pm\infty$.
(ii) $\ddot{u}_0(x) < 0$ in $(0, +\infty)$.

We prove here the first statement of this lemma, and we postpone the proof of the second one until the next section, since we need to use a maximum principle for the linearized operator $L_{K_1} - f'(u_0)$.

Proof of point (i) of Lemma 5.2. By contradiction, suppose that there exists an unbounded sequence $\{x_j\}$ satisfying $|\ddot{u}_0(x_j)| > \varepsilon$ for some $\varepsilon > 0$. Note that by the symmetry of u_0 we may assume that $x_j \to +\infty$. Now define $w_j(x) := \ddot{u}_0(x + x_j)$. By differentiating twice the equation of the layer solution, we see that $\ddot{u}_0$ solves

$$L_{K_1} \ddot{u}_0 = f''(u_0)\dot{u}_0^2 + f'(u_0)\ddot{u}_0 \text{ in } \mathbb{R}.$$

Hence, as $x_j \to +\infty$ a standard compactness argument combined with (5.2) yield that w_j converges on compact sets to a function w that solves

$$L_{K_1} w = f'(1) w \quad \text{in } \mathbb{R}.$$

In addition, since $|\ddot{u}_0(x_j)| > \varepsilon$ we have $|w(0)| \geq \varepsilon$.

At this point we use Lemma 4.3 of [29] to deduce that, since $f'(1) < 1$, then $w \to 0$ as $|x| \to +\infty$. Therefore, if w is not identically zero, it has either a positive maximum or a negative minimum, but this contradicts the maximum principle (recall that $f'(1) < 1$). We conclude that $w \equiv 0$ in $\mathbb{R}$, but this is a contradiction with $|w(0)| \geq \varepsilon$. □

Now we have all the ingredients to establish the asymptotic behavior of the saddle-solution.

Proof of Theorem 1.5. By contradiction, assume that the result does not hold. Then, there exists an $\varepsilon > 0$ and an unbounded sequence $\{x_k\}$, such that

$$|u(x_k) - U(x_k)| + |\nabla u(x_k) - \nabla U(x_k)| + |D^2 u(x_k) - D^2 U(x_k)| > \varepsilon. \tag{5.3}$$

Eq:Contradiction

By the symmetry of u, we may assume without loss of generality that $x_k \in \overline{\mathcal{O}}$, and by continuity we can further assume $x_k \notin \mathscr{C}$.

Let $d_k := \operatorname{dist}(x_k, \mathscr{C})$. We distinguish two cases:

Case 1: $\{d_k\}$ is an unbounded sequence. In this situation, we may assume that $d_k \geq 2k$. Define

$$w_k(x) := u(x + x_k),$$

which satisfies $0 < w_k < 1$ in $\overline{B_k}$ and

$$L_K w_k = f(w_k) \ \text{ in } B_k.$$

By letting $k \to +\infty$, by the uniform estimates for the operators of the class $\mathcal{L}_0$ and the Arzelà-Ascoli theorem, we have that, up to a subsequence, w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w &= f(w) & \text{in } \mathbb{R}^n, \\ w &\geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Then, by Theorem 4.1, either $w \equiv 0$ or $w \equiv 1$. First, note that w cannot be zero. Indeed, since w_k are stable with respect to perturbations supported in B_k (see Remark 3.4), w is stable in $\mathbb{R}^n$, which means that the linearized operator $L_K - f'(w)$ is a positive operator. Nevertheless, if $w \equiv 0$, then the linearized operator $L_K - f'(w) = L_K - f'(0)$ is negative for sufficiently large balls, since $f'(0) > 0$ and the first eigenvalue of L_K is of order $R^{-2\gamma}$ in balls of radius R (as in Lemma 3.2). Therefore $w \equiv 1$.

On the other hand, since $d_k \to +\infty$ and $U(x_k) = u_0(d_k)$, we get by the properties of the layer solution that $U(x_k) \to 1$, $\nabla U(x_k) \to 0$ and $D^2 U(x_k) \to 0$ —see (5.2) and Lemma 5.2. From this and condition (5.3) we get

$$|u(x_k) - 1| + |\nabla u(x_k)| + |D^2 u(x_k)| > \varepsilon/2,$$

for k big enough. This yields that

$$|w_k(0) - 1| + |\nabla w_k(0)| + |D^2 w_k(0)| > \varepsilon/2,$$

and this contradicts $w \equiv 1$.

Case 2: $\{d_k\}$ is a bounded sequence. In this situation, at least for a subsequence, we have that $d_k \to d$. Now, for each x_k we define x_k^0 as its projection on $\mathscr{C}$. Therefore, we have that $\nu_k^0 := (x_k - x_k^0)/d_k$ is the unit normal to $\mathscr{C}$. Through a subsequence, $\nu_k^0 \to \nu$ with $|\nu| = 1$.

We define

$$w_k(x) := u(x + x_k^0),$$

which solves

$$L_K w_k = f(w_k) \text{ in } \mathbb{R}^n.$$

Similarly as before, by letting $k \to +\infty$, up to a subsequence w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w &= f(w) \quad \text{in } H := \{x \cdot \nu > 0\}\,, \\ \quad w &\geq 0 \qquad \text{in } H\,, \\ \quad w \text{ odd with respect to } H. \end{cases}$$

For the details about the fact that $\mathcal{O} \to H$, see [19].

As in the previous case, by stability w cannot be zero, and thus $w > 0$ in H (by the strong maximum principle for odd functions with respect to a hyperplane, see [23]). Hence, by Theorem 4.2, w only depends on $x \cdot \nu$ and is increasing. Finally, by the uniqueness of the layer solution, $w(x) = u_0(x \cdot \nu)$ and

$$\begin{aligned} u(x_k) &= w_k(x_k - x_k^0) = w(x_k - x_k^0) + \mathrm{o}(1) \\ &= u_0((x_k - x_k^0)\cdot\nu) + \mathrm{o}(1) = u_0((x_k - x_k^0)\cdot\nu_k^0) + \mathrm{o}(1) \\ &= u_0(d_k|\nu_k^0|^2) + \mathrm{o}(1) = u_0(d_k) + \mathrm{o}(1) = U(x_k) + \mathrm{o}(1), \end{aligned}$$

contradicting (5.3). The same is done for ∇u and $D^2 u$. □

⟨Remark:u>delta⟩ *Remark* 5.3. The previous result yields that, for $\varepsilon > 0$ the saddle-shaped solution satisfies $u \geq \delta$ in the set $\mathcal{O}_\varepsilon := \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| + \varepsilon < |x'|\}$, for some positive constant δ. That is, thanks to the asymptotic result, and since $U(x) \geq u_0(\varepsilon/\sqrt{2})$ for $x \in \mathcal{O}_\varepsilon$, there exists a radius $R > 0$ such that $u(x) \geq U(x)/2 \geq u_0(\varepsilon/\sqrt{2})/2$ if $x \in \mathcal{O}_\varepsilon \setminus B_R$. Moreover, since u is positive in the compact set $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R}$ it has a positive minimum in this set, say $m > 0$. Therefore, if we choose $\delta = \min\{m, u_0(\varepsilon/\sqrt{2})/2\}$ we obtain the desired result.

6. Maximum principles for the linearized operator and uniqueness of the saddle-shaped solution

laximumPrinciple⟩ In this section we show that the linearized operator $L_K - f'(u)$ satisfies the maximum principle in $\mathcal{O}$. This result combined with the asymptotic result of Theorem 1.5 yields the uniqueness of the saddle-shaped solution.

The maximum principle we establish is the following.

ncipleLinearized⟩ **Proposition 6.1.** *Let $\Omega \subset \mathcal{O}$ be an open set (not necessarily bounded) and let $L_K \in \mathcal{L}_\star(2m,\gamma)$. Let $v \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$, be a doubly radial function satisfying*

$$\begin{cases} L_K v - f'(u)v - c(x)v & \leq & 0 & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathcal{O}\setminus\Omega\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} v(x) & \leq & 0\,, & \end{cases}$$

with $c \leq 0$ in Ω. Then, $v \leq 0$ in Ω.

In order to prove this result we need a maximum principle in "narrow" sets, stated next.

larrowDomainsOdd⟩ **Proposition 6.2.** *Let $\varepsilon > 0$ and let*

$$H \subset \{(x',x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| < |x'| < |x''| + \varepsilon\} \subset \mathcal{O}$$

be an open set (not necessarily bounded). Let $L_K \in \mathcal{L}_\star(2m,\gamma)$ and let $v \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$, be a doubly radial function satisfying

$$\begin{cases} L_K v + c(x)v & \leq & 0 & \text{in } H\,, \\ v & \leq & 0 & \text{in } \mathcal{O}\setminus H\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in H,\ |x|\to\infty} v(x) & \leq & 0\,. & \end{cases} \tag{6.1}$$
Eq:AssumptionsMa

Under these assumptions there exists $\overline{\varepsilon} > 0$ depending only on λ, m, γ and $||c_-||_{L^\infty}$ such that, if $\varepsilon < \overline{\varepsilon}$, then $v \leq 0$ in H.

Proof. Assume, by contradiction, that

$$M := \sup_H v > 0\,.$$

Under the assumptions (6.1), M must be attained at an interior point $x_0 \in H$. Then,

$$0 \geq L_K v(x_0) + c(x_0)v(x_0) \geq L_K v(x_0) - ||c_-||_{L^\infty(H)}\,M\,. \tag{6.2}$$
Eq:InequalitiesM

Now, we compute $L_K v(x_0)$. Since v is doubly radial and odd with respect to the Simons cone, we can use the expression (1.11) to write

$$\begin{aligned} L_K v(x_0) &= \int_{\mathcal{O}} \big(M - v(y)\big)\big(\overline{K}(x_0,y) - \overline{K}(x_0,y^\star)\big)\,\mathrm{d}y + 2M\int_{\mathcal{O}} \overline{K}(x_0,y^\star)\,\mathrm{d}y \\ &\geq 2M\int_{\mathcal{O}} \overline{K}(x_0,y^\star)\,\mathrm{d}y, \end{aligned}$$

where the inequality follows from being M the supremum of v in $\mathcal{O}$ and the kernel inequality (1.14). Combining this last inequality with (6.2), we obtain

$$0 \geq L_K v(x_0) + c(x_0)v(x_0) \geq M\left\{2\int_{\mathcal{O}} \overline{K}(x_0,y^\star)\,\mathrm{d}y - ||c_-||_{L^\infty(H)}\right\}.$$

Finally, if we use the lower bound of the integral term from (1.12) and the fact that $\operatorname{dist}(x_0,\mathscr{C}) \leq \varepsilon/\sqrt{2}$, we get

$$\begin{aligned} 0 &\geq M\left\{2\int_{\mathcal{O}} \overline{K}(x_0,y^\star)\,\mathrm{d}y - ||c_-||_{L^\infty(H)}\right\} \geq M\left(\frac{1}{C}\operatorname{dist}(x_0,\mathscr{C})^{-2\gamma} - ||c_-||_{L^\infty(H)}\right) \\ &\geq M\left(\frac{1}{C}\varepsilon^{-2\gamma} - ||c_-||_{L^\infty(H)}\right). \end{aligned}$$

Therefore, for ε small enough, we arrive at a contradiction that follows from assuming that the supremum is positive. □

Remark 6.3. Proposition 6.2 can be extended to general doubly radial "narrow" sets — in the sense of (4.7)— and without requiring any assumption at infinity, just repeating the exact same arguments as in the proof of Proposition 4.12. Indeed, we only need to replace symmetry with respect to a hyperplane by symmetry with respect to the Simons cone and use the kernel inequality (1.14). Nevertheless, we preferred to present the result for sets that are contained in an ε-neighborhood of the Simons cone, since we are only going to use the maximum principle in such sets. In addition, the crucial fact that the sets are contained in $\{(x',x'') \in \mathbb{R}^m \times \mathbb{R}^m \;:\; |x''| < |x'| < |x''| + \varepsilon\}$ makes the argument rather simple.

Once this maximum principle in "narrow" sets is available, we can now proceed with the proof of Proposition 6.1.

Proof of Proposition 6.1. For the sake of simplicity, we will denote

$$\mathscr{L}w := L_K w - f'(u)w - cw\,.$$

The crucial point in this proof is the fact that u is a positive supersolution of the operator $\mathscr{L}$. Indeed, by (1.7) we get

$$\mathscr{L}u = L_K u - f'(u)u - cu \geq f(u) - f'(u)u > 0 \quad \text{in } \Omega \subset \mathcal{O}\,, \tag{6.3}$$ Eq:uSupersolLine

where in the first inequality we have used that $u > 0$ in $\mathcal{O}$ and that $c \leq 0$.

Let $\varepsilon > 0$ be such that the maximum principle of Proposition 6.2 is valid and define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{|x'| > |x''| + \varepsilon\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \Omega \cap \{|x''| < |x'| < |x''| + \varepsilon\}\,.$$

By contradiction, assume that there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. Set $w := v - \tau u$. By the asymptotic result, we have

$$u \geq \delta > 0 \quad \text{in } \overline{\Omega}_\varepsilon\,, \tag{6.4}$$

Eq:u>delta

for some $\delta > 0$ (see Remark 5.3). Therefore, $w < 0$ in $\overline{\Omega}_\varepsilon$ if τ is big enough. Moreover, since $v \leq 0$ in $\mathcal{O} \setminus \Omega$, we have

$$w \leq 0 \quad \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,.$$

Furthermore, we also have

$$\limsup_{x \in \mathcal{N}_\varepsilon,\ |x| \to \infty} w(x) \leq 0$$

and, by (6.3),

$$\mathscr{L} w = \mathscr{L} v - \tau \mathscr{L} u \leq 0 \text{ in } \mathcal{N}_\varepsilon\,.$$

Thus, since w is odd with respect to $\mathscr{C}$, we can apply Proposition 6.2 with $H = \mathcal{N}_\varepsilon$ to deduce that

$$w \leq 0 \quad \text{in } \Omega\,,$$

if τ is big enough.

Now, define

$$\tau_0 := \inf\{\tau > 0 \ : \ v - \tau u \leq 0 \ \text{ in } \Omega\}\,.$$

By the previous reasoning, τ_0 is well defined. Clearly, $v - \tau_0 u \leq 0$ in Ω. In addition, since $v(x_0) > 0$, we have $-\tau_0 u(x_0) < v(x_0) - \tau_0 u(x_0) \leq 0$ and therefore, since $u(x_0) > 0$, we deduce that $\tau_0 > 0$.

We claim that $v - \tau_0 u \not\equiv 0$. Indeed, if $v - \tau_0 u \equiv 0$ then $v = \tau_0 u$ and thus, by using (6.3), the equation for v and the fact that $\tau_0 > 0$, we get

$$0 \geq \mathscr{L} v(x_0) = \tau_0 \mathscr{L} u(x_0) > 0\,,$$

which is a contradiction.

Then, since $v - \tau_0 u \not\equiv 0$, the strong maximum principle (Proposition 2.3) yields

$$v - \tau_0 u < 0 \quad \text{in } \Omega\,.$$

Therefore, by continuity, the assumption on v at infinity and (6.4), there exists $0 < \eta < \tau_0$ such that

$$\tilde{w} := v - (\tau_0 - \eta) u < 0 \quad \text{in } \overline{\Omega}_\varepsilon\,.$$

Using again the maximum principle in "narrow" sets with $\tilde{w}$ in $\mathcal{N}_\varepsilon$, we deduce that

$$v - (\tau_0 - \eta) u \leq 0 \quad \text{in } \Omega\,,$$

and this contradicts the definition of τ_0. Hence, $v \leq 0$ in Ω. $\square$

The same argument used in the previous proof can be used to establish the remaining statement of Lemma 5.2.

Proof of point (ii) of Lemma 5.2. Let $v = \ddot{u}_0$. First we show that $v \le 0$ in $(0,+\infty)$. To see this, note that since f is concave and by point (i) of Lemma 5.2, it follows that

$$\begin{cases} L_{K_1} v - f'(u_0)v & \le & 0 & \text{in } (0,+\infty)\,. \\ v(x) & = & -v(-x) & \text{for every } x \in \mathbb{R}\,, \\ \limsup\limits_{x\to+\infty} v(x) & = & 0\,. & \end{cases}$$

Now, we follow the proof of Proposition 6.1 but with the previous problem, replacing u by u_0 and using that

$$L_{K_1} u_0 - f'(u_0)u_0 > 0 \quad \text{in } (0,+\infty)\,.$$

All the arguments are the same, using the maximum principle of Proposition 4.12 in the set $(0,\varepsilon)$, and yield that $v \le 0$ in $(0,+\infty)$.

The fact that the inequality is strict follows from the strong maximum principle for odd functions in $\mathbb{R}$, as follows. Suppose by contradiction that there exists a point $x_0 \in (0,+\infty)$ such that $v(x_0) = 0$. Then,

$$\begin{aligned} 0 \ge L_{K_1} v(x_0) &= -\int_{-\infty}^{+\infty} v(y) K_1(x_0 - y)\,\mathrm{d}y \\ &= -\int_{-\infty}^{+\infty} v(y)\{K_1(x_0 - y) - K_1(x_0 + y)\}\,\mathrm{d}y > 0\,, \end{aligned}$$

arriving at a contradiction. Here we have used that $v \not\equiv 0$ and the fact that K_1 is decreasing in $(0,+\infty)$, which yields $K_1(x-y) \ge K_1(x+y)$ for every $x > 0$ and $y > 0$. □

With these ingredients available, we can finally establish the uniqueness of the saddle-shaped solution.

Proof of Theorem 1.6. Let u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$ which is a doubly radial function that is odd with respect to $\mathscr{C}$. Then,

$$L_K v = f(u_1) - f(u_2) \le f'(u_2)(u_1 - u_2) = f'(u_2)v \quad \text{in } \mathcal{O}\,,$$

since f is concave in $(0,1)$. Moreover, by the asymptotic result (see Theorem 1.5), we have

$$\limsup_{x\in\mathcal{O},\ |x|\to\infty} v(x) = 0\,.$$

Then, by the maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u_2)$ (see Proposition 6.1), we are lead to $v \le 0$ in $\mathcal{O}$, which means $u_1 \le u_2$ in $\mathcal{O}$. Repeating the argument with $-v = u_2 - u_1$ we deduce $u_1 \ge u_2$ in $\mathcal{O}$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m}$. □

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References

uchitteSeppecher [1] G. Alberti, G. Bouchitté, and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. **144** (1998), 1–46.

AmbrosioCabre [2] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), 725–739.

tAl-Monotonicity [3] B. Barrios, L. Del Pezzo, J. García-Melián, and A. Quaas, *Monotonicity of solutions for some nonlocal elliptic problems in half-spaces*, Calc. Var. Partial Differential Equations **56** (2017), Art. 39, 16.

iosEtAl-Symmetry [4] ———, *Symmetry results in the half space for a semi-linear fractional Laplace equation through a one-dimensional analysis*, preprint (2017).

alSoriaValdinoci [5] B. Barrios, I. Peral, F. Soria, and E. Valdinoci, *A Widder's type theorem for the heat equation with nonlocal diffusion*, Arch. Ration. Mech. Anal. **213** (2014), 629–650.

yckiHamelMonneau [6] H. Berestycki, F. Hamel, and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (2000), 375–396.

restyckiHamelNadi [7] H. Berestycki, F. Hamel, and N. Nadirashvili, *The speed of propagation for KPP type problems II: General domains*, J. Amer. Math. Soc. **23** (2010), 1–34.

Bucur [8] C. Bucur, *A symmetry result in $\mathbb{R}^2$ for global minimizers of a general type of nonlocal energy*, preprint (2017).

BucurValdinoci [9] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, Springer International Publishing, 2016.

Cabre-ABP [10] X. Cabré, *On the Alexandroff–Bakel'man–Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations.*, Comm. Pure Appl. Math. **48** (1995), 539–570.

Cabre-Topics [11] ———, *Topics in regularity and qualitative properties of solutions of nonlinear elliptic equations*, Discrete Contin. Dyn. Syst. **8** (2002), 331–359.

Cabre-Saddle [12] ———, *Uniqueness and stability of saddle-shaped solutions to the Allen–Cahn equation*, J. Math. Pures Appl. **98** (2012), 239–256.

inti-EnergyHalfL [13] X. Cabré and E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. **28** (2010), 1179–1206.

inti-SharpEnergy [14] ———, *Sharp energy estimates for nonlinear fractional diffusion equations*, Calc. Var. Partial Differential Equations **49** (2014), 233–269.

intiSerra-Stable [15] X. Cabré, E. Cinti, and J. Serra, *Stable solutions to the fractional Allen-Cahn equation*, preprint (2018).

CabreSireI [16] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 23–53.

CabreSireII [17] ———, *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. **367** (2015), 911–941.

CabreSolaMorales [18] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58** (2005), 1678–1732.

CabreTerraI [19] X. Cabré and J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of $\mathbb{R}^{2m}$*, J. Eur. Math. Soc. **11** (2009), 819–843.

CabreTerraII [20] ———, *Qualitative properties of saddle-shaped solutions to bistable diffusion equations*, Comm. Partial Differential Equations **35** (2010), 1923–1957.

farelliSilvestre [21] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

ChanLiuWei [22] H. Chan, Y. Liu, and J. Wei, *A gluing construction for fractional elliptic equations. Part I: a model problem on the catenoid*, preprint (2017).

ChenLiLi [23] W. Chen, C. Li, and Y. Li, *A direct method of moving planes for the fractional Laplacian*, Adv. Math. **308** (2017), 404–437.

ChenLiZhang [24] W. Chen, Y. Li, and R. Zhang, *A direct method of moving spheres on fractional order equations*, J. Funct. Anal. **272** (2017), 4131–4157.

Cinti-Saddle [25] E. Cinti, *Saddle-shaped solutions of bistable elliptic equations involving the half-Laplacian*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), 623–664.

Cinti-Saddle2 [26] ———, *Saddle-shaped solutions for the fractional Allen-Cahn equation*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), 441–463.

iorgiClassesLong [27] M. Cozzi, *Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: A unified approach via fractional De Giorgi classes*, J. Funct. Anal. (2017), no. 11, 4762–4837.

orgiClassesShort [28] ———, *Fractional De Giorgi classes and applications to nonlocal regularity theory*, preprint (2018).

CozziPassalacqua [29] M. Cozzi and T. Passalacqua, *One-dimensional solutions of non-local Allen–Cahn-type equations with rough kernels*, J. Differential Equations **260** (2016), 6638–6696.

DangFifePeletier [30] H. Dang, P.C. Fife, and L.A. Peletier, *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. **43** (1992), 984–998.

PinoKowalczykWei [31] M. del Pino, M. Kowalczyk, and J. Wei, *On De Giorgi's conjecture in dimension $N \geq 9$*, Ann. of Math. **174** (2011), 1485–1569.

HitchhikerGuide [32] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

oFarinaValdinoci [33] S. Dipierro, A. Farina, and E. Valdinoci, *A three-dimensional symmetry result for a phase transition equation in the genuinely nonlocal regime*, Calc. Var. Partial Differential Equations **57** (2018).

roSerraValdinoci [34] S. Dipierro, J. Serra, and E. Valdinoci, *Improvement of flatness for nonlocal phase transitions*, preprint (2017).

Evans [35] L. C. Evans, *Partial Differential Equations: Second Edition*, 2nd ed., Graduate Studies in Mathematics, AMS, 2010.

WethMonotonicity [36] M. Fall and T. Weth, *Monotonicity and nonexistence results for some fractional elliptic problems in the half-space*, Commun. Contemp. Math. **18** (2016), 1550012, 25.

FazlySire [37] M. Fazly and Y. Sire, *Symmetry properties for solutions of nonlocal equations involving nonlinear operators*, preprint (2018).

groDifferentialI [38] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone*, preprint (2018).

SaddleFractional [39] ———, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, preprint (2018).

FelmerWang [40] P. Felmer and Y. Wang, *Radial symmetry of positive solutions to equations involving the fractional Laplacian*, Commun. Contemp. Math. **16** (2014), 1350023, 24.

FigalliSerra [41] A. Figalli and J. Serra, *On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4+1*, preprint (2017).

GhoussoubGui [42] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), 481–491.

Gonzalez [43] M.d.M González, *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Partial Differential Equations **36** (2009), 173–210.

tonSireValdinoci [44] F. Hamel, X. Ros-Oton, Y. Sire, and E. Valdinoci, *A one-dimensional symmetry result for a class of nonlocal semilinear equations in the plane*, Ann. Institut H. Poincaré **34** (2017), 469–482.

JerisonMonneau [45] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Ann. Mat. Pura Appl. (4) **183** (2004), 439–467.

LiZhang [46] Y. Li and L. Zhang, *Liouville-type theorems and harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math. **90** (2003), 27–87.

Modica [47] L. Modica, Γ-*convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, pp. 223–244.

ModicaMortola [48] L. Modica and S. Mortola, *Un esempio di* Γ^{-}-*convergenza*, Boll. Un. Mat. Ital. B (5) **14** (1977), 285–299.

QuaasXia [49] A. Quaas and A. Xia, *Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space*, Calc. Var. Partial Differential Equations **52** (2015), 641–659.

RosOton-Survey [50] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*, Publ. Mat. **60** (2016), 3–26.

OtonSerra-Stable [51] X. Ros-Oton and J. Serra, *Regularity theory for general stable operators*, J. Differential Equations **260** (2016), 8675–8715.

Savin-DeGiorgi [52] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. **169** (2009), 41–78.

Savin-Fractional [53] ———, *Rigidity of minimizers in nonlocal phase transitions*, Anal. PDE **11** (2018), 1881–1900.

avin-Fractional2 [54] ———, *Rigidity of minimizers in nonlocal phase transitions II*, preprint (2018).

GammaConvergence [55] O. Savin and E. Valdinoci, Γ-*convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 479–500.

Schatzman [56] M. Schatzman, *On the stability of the saddle solution of Allen–Cahn's equation*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1241–1275.

+alphaRegularity [57] J. Serra, $C^{\sigma+\alpha}$ *regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels*, Calc. Var. Partial Differential Equations **54** (2015), 3571–3601.

ervadeiValdinoci [58] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105–2137.

SireValdinoci [59] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), 1842–1864.

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