Non-linear dynamics numerical assignment

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March 2022

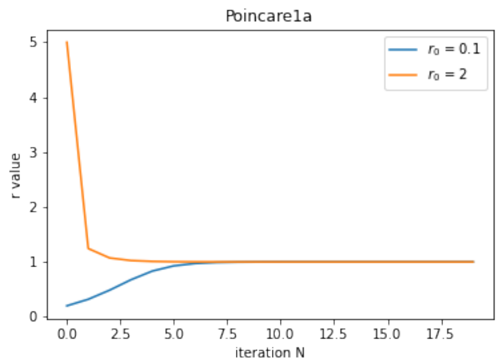
1 Task 1

1.1 1a)

From example 8.7.1 in Strogatz' book we change T from 2π to $\frac{2\pi}{\theta'} = \frac{2\pi}{4\pi} = \frac{1}{2}$ which changes the resulting equation for the Poincaré map to

$$P(r) = (1 + e^{-2*T} * (r^{-2} - 1))^{-\frac{1}{2}} = (1 + e^{-1} * (r^{-2} - 1))^{-\frac{1}{2}} \tag{1}$$

From ICs $= \{r_1 = 0.1, r_2 = 2\}$ we get the following plot:



1.2 1b)

We have the equation Plan is to integrate LHS of the equation

$$\int_{r_n}^{r_{n+1}} \frac{dr}{A(r-\pi)e^{-wt_n}} = \int_0^{2\pi} dt = T \tag{2}$$

we solve for r_1. By substituting $u = A(r-\pi)e^{-wt_n}$ and $\frac{du}{dr} = A$ we obtain

$$\frac{\ln A(r_{n+1}-\pi)e^{-wt_1} - \ln A(r_n - \pi)e^{-wt_0}}{A} \tag{3}$$

$$= \frac{\ln \frac{A(r_{n+1}-\pi)e^{-wt_1}}{A(r_n-\pi)e^{-wt_0}}}{A} \tag{4}$$

$$= \frac{\ln \frac{(r_{n+1}-\pi)}{(r_n-\pi)} e^{-wT}}{A} = LHS = T \tag{5}$$

Multiplying by A and taking the exponent, and then solving for r_{n+1} we get

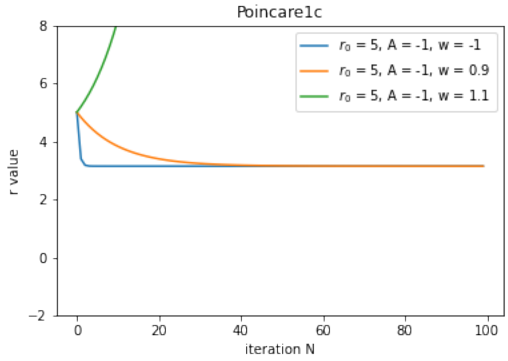
$$e^{AT} = \frac{(r_{n+1}-\pi)}{(r_n-\pi)} e^{-wT} \tag{6}$$

$$e^{AT}(r_n - \pi)e^{wT} = e^{T(A+w)}(r_n - \pi) = r_{n+1} = P(r_n) \tag{7}$$

From this expression we see that if $A + w > 0$ then $r_n > \pi$ will make $P(r_n) > r_n$, and it will grow out of bounds. For $A + w < 0$ this will not necessary be the case, and if a limit cycle exist it will be for parameters $A + w < 0$.

1.3 1c)

We plot the cobweb diagram and get

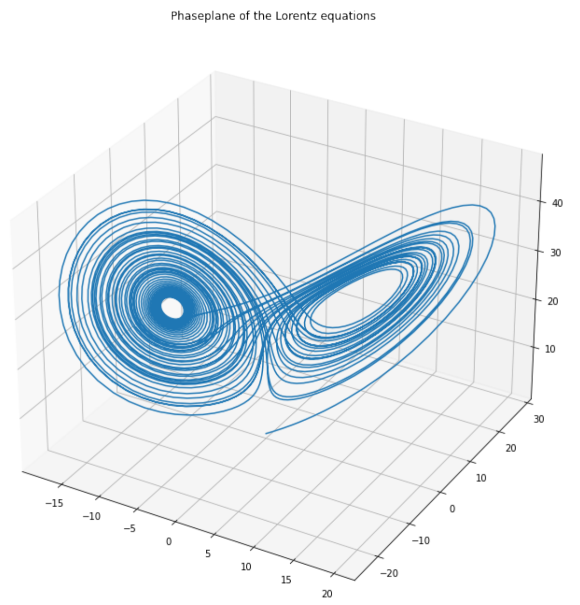


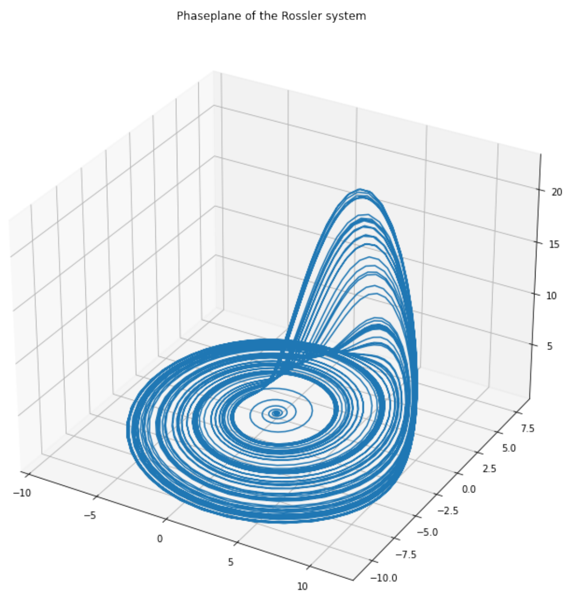
How did this compare to the hypothesis in 2b? Very good, actually. As $A + w < 0$ the plots converge to a limit cycle, but as soon as $A + w > 0$ the behaviour changes to divergence.

2 Task 2 & 3

Generall remarks on solver in solving the Lorentz and Rossler system: We solve the systems of equation given by Lorentz and Rossler by the scypy.integrate.solve_ivp schemes for arbitrary systems of ODEs. Here a stepsize varying RK4-5 method is used which for every iteration use timestep just small enough to reach a given tolerance. By requiring the relative tolerance of 10^{-9} we will get good solutions computed fast.

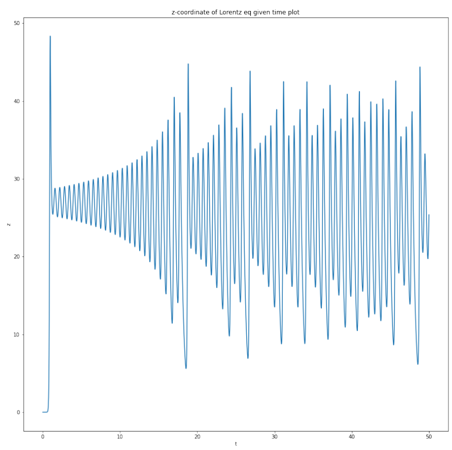
2.1 Phasespaces for the systems in task 2 and 3

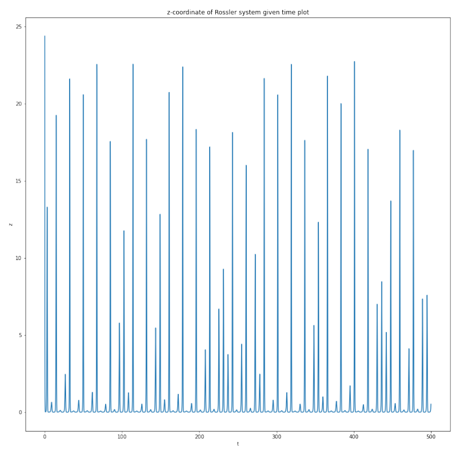




2.2 a)

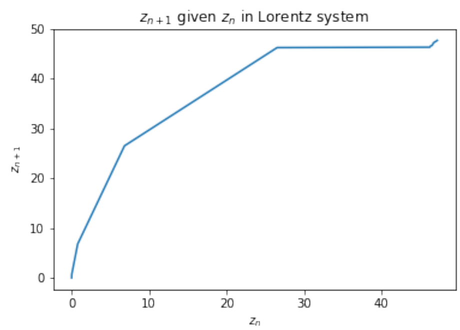
From solving up to time 50 and 500 respectably for Loretnz and Rossler, we plot the xz plane. As seen the plots are irregular and non deterministic, but always withing the bounds of the stability of the corresponding attractors.

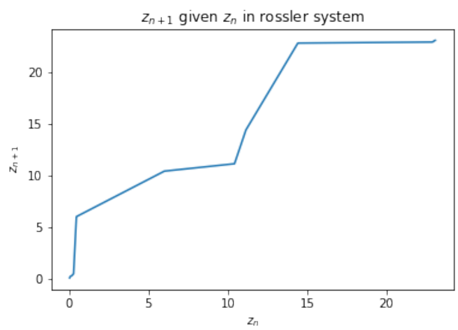




2.3 b)

To acquire some indication of wether a limit cycle in the systems exist, the plot of z_{n+1} z_n for the systems are then plotted by setting z_n as the n-th maximum value in the z coordinate. If a limit cycle exist, the plot should be looking like the cobweb plots from the first task, a steady increasing function converging to some value which is the maximum value along a limit cycle.

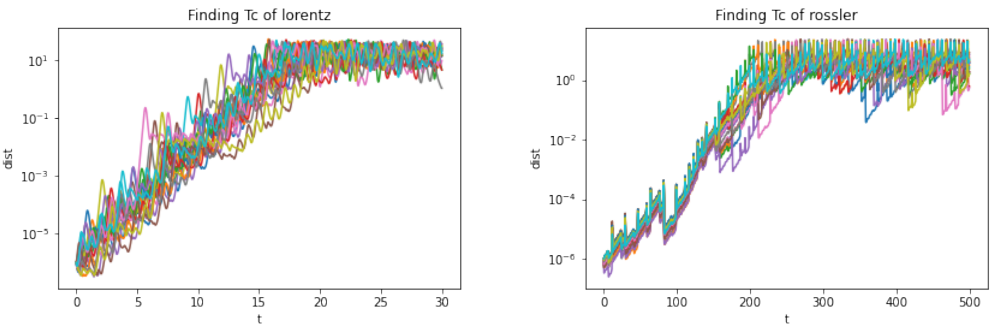




From the plots we see no such pattern, which is proof for there being no limit cycle in the phase spaces of the correspond systems.

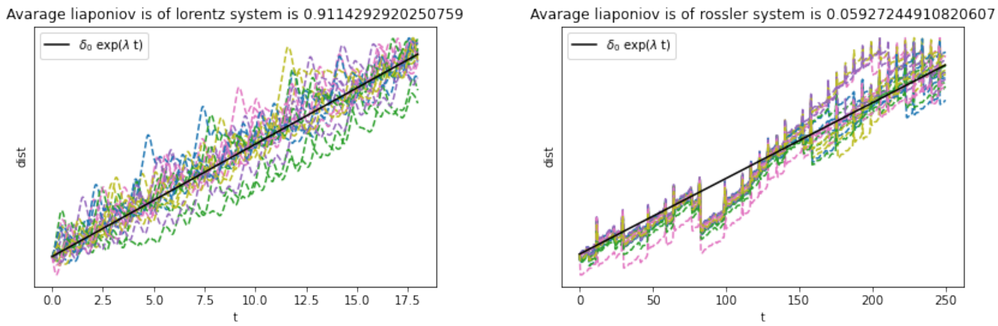
2.4 b i-vi)

The task is not to study the chaotic systems and to look for some order in them. This we do by trying to approximate the value for the Liapunov coefficient of the system, i. e the rate of which the distance between to neighbouring point are expected to grow with time until they are indistinguishable from from two random points in the attractor. For this we initialise a random point near the origin, integrate it until $T_end = 50$, and then initialise $x_0 = x_sol$. This we do 20 times to get some different starting values x_0. For all of these we integrate x_0 and a randomly generated x_1 within the distance of 10^{-6} until the time of $T_end = 30$ and $T_end = 500$ for the two systems, and below we plot the 20 different plots of the distance between the two initial conditions trajectories. What we are looking for is a point where the average distance stops growing.



Getting the exact cutoff time T_c across the 20 trajectories is not easy to do, so a measure made with eyesight on the figures above yields $T_c = 18$ and $T_c = 250$.

Now, we want to generate 20 new initial conditions the same way as previously. But now we want to stop integrating at the cutoff point T_c. Then an average is taken over all the log-transformed distance-trajectories from the result, and the average slope is calculated between the last and the first slope. Plotting this approximation of the slope of the linear together with all the trajectories gives the following plot.



2.5 Comparing the results

From 2.2 it looks like Lorentz system the value of z oscillates in x with a slowly growing amplitude with increasing x with exceptions from abrupt discontinues falls. This is in contrast to the Rossler system, where no oscillation occurs with x, and the two coordinates seems uncorrelated.

Figure 2.4 and 2.4 then indicates that the Lorentz system uses much less time to mix than the Rossler system. This way one can say that the Lorentz system is more sensitive to initial conditions that then its counterpart. With the same uncertainty in measures of initial conditions, the Rossler should therefore be able to be predicted better for a longer time than in the case of systems with Lorentz equations as the underlying dynamics.

From 2.3, and several more runs of it, a trend is that the graph in the Lorentz system has a log shape, while the Rossler system is way more irregular. A smooth looking curve like Lorentz have indicates that each maximum found differs from the last with an decreasing amount, which corresponds to a slowly growing amplitude of an oscillation and "smooth" behaviour in the z-coordinate. For Rosslers system the behaviour of the projection into the z-axis is highly irregular and non smooth.

Generally, it seems like the Lorentz behaves more smoothly for most times, but with huge altering behaviour happening at some instances. This contrast to Rosslers system which always seems to be irregular in its z-coordinate for the $x > 0, y < 0$ quadrant in its xy plane by its phase portrait drawing. So "most" of the time, Lorentz system behaves smoothly, while Rosslers never does.