
CS 70 Discrete Mathematics and Probability Theory

Summer 2016 Dinh, Psomas, and Ye Discussion 4D Sol

1. Boy or Girl?

The following are variants of the famous “boy or girl paradox”.

Note: For both parts, assume that the probability of a boy or girl being born is the same, a child is equally likely to be born on any day of the week, and the genders of all children are independent of each other and independent of the day of the week.

- a) Mr. and Mrs. Smith have two children, at least one of whom is a girl. What is the probability that both children are girls? Let G_1 be the event that the first child is a girl, and G_2 be the event that the second child is a girl. We are asked to find $\Pr[(G_1 \cap G_2)|(G_1 \cup G_2)]$:

$$\begin{aligned}\Pr[(G_1 \cap G_2)|(G_1 \cup G_2)] &= \frac{\Pr[(G_1 \cap G_2) \cap (G_1 \cup G_2)]}{\Pr[G_1 \cup G_2]} \\ &= \frac{\Pr[G_1 \cap G_2]}{\Pr[G_1] + \Pr[G_2] - \Pr[G_1 \cap G_2]} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{2} - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3}.\end{aligned}$$

Note: you can check “Boy or Girl paradox” from https://en.wikipedia.org/wiki/Boy_or_Girl_paradox. In Wikipedia, it is mentioned that there are two cases:

- Case 1: From all families with two children, at least one of whom is a boy, a family is chosen at random. This would yield the answer of $1/3$.
- Case 2: From all families with two children, one child is selected at random, and the sex of that child is specified to be a boy. This would yield an answer of $1/2$.

We follow Case 1 in this question.

- b) Mr. and Mrs. Brown have two children, one of whom is a boy born on a Tuesday. What is the probability that both children are boys? Consider a 14 by 14 table listing out all possible combinations of each child’s gender and birth day-of-week. We know that at least one boy is born on a Tuesday, so we are only concerned with one row and one column of the table (the boy-born-on-Tuesday row and column). There are $14 + 14 - 1 = 27$ cells in this part of the table. In 13 of these 27 cells, there are two boys. Therefore, Mr. and Mrs. Brown have a $\frac{13}{27}$ probability of having two boys, given that one of his two children is a boy born on Tuesday.

Alternatively, consider working with probabilities. We define the following events:

- A = 1st child is a boy;
- B = 1st child was born on Tuesday;
- C = 2nd child is a boy;
- D = 2nd child was born on Tuesday;
- E = at least one child is a boy born on Tuesday.

From this, we can see that we are interested in

$$\begin{aligned}\Pr[(A \cap C)|E] &= \frac{\Pr[(A \cap C) \cap E]}{\Pr[E]} \\ &= \frac{\Pr[E|(A \cap C)] \Pr[A \cap C]}{\Pr[E]}.\end{aligned}$$

To calculate $\Pr[E]$:

$$\begin{aligned}\Pr[E] &= \Pr[(A \cap B) \cup (C \cap D)] \\ &= \Pr[A \cap B] + \Pr[C \cap D] - \Pr[(A \cap B) \cap (C \cap D)] \\ &= \Pr[A] \Pr[B] + \Pr[C] \Pr[D] - \Pr[A \cap B] \Pr[C \cap D] \\ &= \frac{1}{2} \cdot \frac{1}{7} + \frac{1}{2} \cdot \frac{1}{7} - \left(\frac{1}{2} \cdot \frac{1}{7}\right) \left(\frac{1}{2} \cdot \frac{1}{7}\right) \\ &= \frac{27}{196}.\end{aligned}$$

Plugging into our expression from before,

$$\Pr[(A \cap C)|E] = \frac{\Pr[(A \cap C) \cap E]}{\Pr[E]} = \frac{(\frac{1}{7} + \frac{1}{7} - \frac{1}{49}) \frac{1}{2} \cdot \frac{1}{2}}{\frac{27}{196}} = \frac{\frac{13}{196}}{\frac{27}{196}} = \frac{13}{27}.$$

2. Wedding in the Desert

Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year, so the prior probability of rain is just $\frac{5}{365}$. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 5% of the time. What is the probability that it will rain on the day of Marie's wedding?

Let R be the event that it rains, and W be the event that the weatherman predicts rain.

$$\begin{aligned}\Pr[\text{rain tomorrow}] &= \Pr[R|W] \\ &= \frac{\Pr[W|R] \Pr[R]}{\Pr[W]} \\ &= \frac{\Pr[W|R] \Pr[R]}{\Pr[W|R] \Pr[R] + \Pr[W|\bar{R}] \Pr[\bar{R}]} \\ &= \frac{0.9 \cdot 5/365}{0.9 \cdot 5/365 + 0.05 \cdot 360/365} \\ &= 0.2,\end{aligned}$$

so maybe Marie doesn't have to cancel those wedding plans after all!

3. Three Diseases

A doctor assumes that a patient has exactly one of three diseases d_1 , d_2 , or d_3 . Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability 0.8 if the patient has d_1 , 0.6 if the patient has disease d_2 , and 0.4 if the patient has disease d_3 . Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases? Let D_i be the event that the patient has disease d_i , and T be the event that the test comes back positive.

$$\Pr[D_1] = \Pr[D_2] = \Pr[D_3] = \frac{1}{3};$$

$$\Pr[T|D_1] = 0.8; \Pr[T|D_2] = 0.6; \Pr[T|D_3] = 0.4;$$

$$\begin{aligned}\Pr[T] &= \Pr[T|D_1]\Pr[D_1] + \Pr[T|D_2]\Pr[D_2] + \Pr[T|D_3]\Pr[D_3] = 0.8 \cdot \frac{1}{3} + 0.6 \cdot \frac{1}{3} + 0.4 \cdot \frac{1}{3} \\ &= 0.6; \\ \Pr[D_1|T] &= \frac{\Pr[T|D_1] \Pr[D_1]}{\Pr[T]} = \frac{0.8 \cdot 1/3}{0.6} = \frac{4}{9}; \\ \Pr[D_2|T] &= \frac{\Pr[T|D_2] \Pr[D_2]}{\Pr[T]} = \frac{0.6 \cdot 1/3}{0.6} = \frac{1}{3}; \\ \Pr[D_3|T] &= \frac{\Pr[T|D_3] \Pr[D_3]}{\Pr[T]} = \frac{0.4 \cdot 1/3}{0.6} = \frac{2}{9},\end{aligned}$$

so our final answer is $\Pr[D_1|T] = \frac{4}{9}$, $\Pr[D_2|T] = \frac{1}{3}$, and $\Pr[D_3|T] = \frac{2}{9}$. Note that we're not much more confident than before that the patient has any one particular disease!