CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Psmoas, Dinh and Ye Discussion 7B

1. Baby Fermat

Assume that a does have a multiplicative inverse $\pmod{m}$. Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

- Consider the sequence $a, a^2, a^3, \ldots \pmod{m}$. Prove that this sequence has repetitions.

  Solution: There are only m possible values $\pmod{m}$, and so after the m-th term we should see repetitions.

- Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

  Solution: If we multiply both sides by $(a^*)^j$, where a^* is the multiplicative inverse, we get $a^{i-j} \equiv 1 \pmod{m}$.

- Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j?

  Solution: We can rewrite $a^{i-j} \equiv 1 \pmod{m}$ as $a^{i-j-1}a \equiv 1 \pmod{m}$. Therefore a^{i-j-1} is the multiplicative inverse of $a \pmod{m}$.

2. Product of Two

Suppose that $p > 2$ is a prime number and S is a set of numbers between 1 and $p-1$ such that $|S| > \frac{p}{2}$. Prove that any number $1 \leq x \leq p-1$ can be written as the product of two (not necessarily distinct) numbers in S, mod p.

Solution: Given x, consider the set T defined as $\{xy^{-1} \pmod{p} : y \in S\}$. Note that the set T has the same cardinality as S, because for $y_1 \neq y_2 \pmod{p}$, we have $xy_1^{-1} \neq xy_2^{-1} \pmod{p}$ (if not, we can multiply both sides by x^{-1}, and take the inverse to get a contradiction).

Therefore the set S and T must have a nonempty intersection. So there must be $y_1, y_2 \in S$ such that $xy_1^{-1} = y_2 \pmod{p}$. But this means that $x = y_1 y_2 \pmod{p}$.

3. RSA

In this problem you play the role of Amazon, who wants to use RSA to be able to receive messages securely.

1. Amazon first generates two large primes p and q. She picks $p = 13$ and $q = 19$ (in reality these should be 512-bit numbers). She then computes $N = pq$. Amazon chooses e from $e = 37, 38, 39$. Only one of those values is legitimate, which one? (N, e) is then the public key.

   Solution: Since 38 and 39 are not relatively prime to $p-1 = 12$ and $q-1 = 18$, they cannot be inverted mod $(p-1)\cdot(q-1) = 216$, so a decryption key cannot be obtained for them. Thus, only $e = 37$ works. The public key then is $(N, e) = (247, 37)$.

2. Amazon generates her private key d. She keeps d as a secret. Find d. Explain your calculation.

   Solution: We compute $d \equiv e^{-1} \equiv 37^{-1} \pmod{216}$.

```
e-gcd(216,37)
  e-gcd(37,31)
    e-gcd(31,6)
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```
      e-gcd(6,1)
        e-gcd(1, 0)
        return (1,1,0)
      return (1,0,1)
    return (1,1,-5)
  return (1,-5,6)
return (1,6,-35)
```

Thus $d \equiv -35 \equiv 181 \pmod{216}$.

3. Bob wants to send Amazon the message $x = 102$. How does he encrypt his message using the public key, and what is the result?

   Note: For this problem you may find the following trick of fast exponentiation useful. To compute x^k, first write k in base 2 then use repeated squaring to compute each power of 2. For example, $x^7 = x^{4+2+1} = x^4 \cdot x^2 \cdot x^1$.

   Solution: The encrypted message is $y \equiv x^e \equiv 102^{37} \pmod{247}$. Using fast exponentiation, we compute:

$$
\begin{aligned}
102^2 &\equiv 30 \pmod{247} \\
102^4 &\equiv 30^2 \equiv 159 \pmod{247} \\
102^8 &\equiv 159^2 \equiv 87 \pmod{247} \\
102^{16} &\equiv 87^2 \equiv 159 \pmod{247} \\
102^{32} &\equiv 159^2 \equiv 87 \pmod{247}
\end{aligned}
$$

   Then, $y \equiv 102^{37} \equiv 102^{32} \cdot 102^4 \cdot 102 \equiv 102 \pmod{247}$. Notice that the encrypted message is the same as the original!

4. Amazon receives an encrypted message $y = 141$ from Charlie. What is the unencrypted message that Charlie sent her?

   Solution: We decrypt the message by raising to the dth power: $x \equiv y^d \equiv 141^{181} \pmod{247}$. We compute the powers:

$$
\begin{aligned}
141^2 &\equiv 121 \pmod{247} \\
141^4 &\equiv 121^2 \equiv 68 \pmod{247} \\
141^8 &\equiv 68^2 \equiv 178 \pmod{247} \\
141^{16} &\equiv 178^2 \equiv 68 \pmod{247} \\
141^{32} &\equiv 68^2 \equiv 178 \pmod{247} \\
141^{64} &\equiv 178^2 \equiv 68 \pmod{247} \\
141^{128} &\equiv 68^2 \equiv 178 \pmod{247}
\end{aligned}
$$

   Then $x \equiv 141^{181} \equiv 141^{128} \cdot 141^{32} \cdot 141^{16} \cdot 141^4 \cdot 141 \equiv 141 \pmod{247}$.

   By now, you may have guessed that $\forall x \in \{0, \dots, 246\}, x^{37} \equiv x \pmod{247}$. We can prove this by noting that $e = 37 \equiv 1 \pmod{p-1}$ and $e = 37 \equiv 1 \pmod{q-1}$. Thus, $e = 1 + j(p-1) = 1 + k(q-1)$ for some j and k. By Fermat's little theorem, $x^{e-1} = x^{j(p-1)} \equiv 1 \pmod{p}$ and $x^{e-1} = x^{k(q-1)} \equiv 1 \pmod{q}$ where x is coprime with p and q. Then by the Chinese remainder theorem, $x^{e-1} \equiv 1 \pmod{pq}$, so $x^e \equiv x \pmod{pq}$. Though we omit it here, we can also show that $x^e \equiv x \pmod{pq}$ when x is not coprime with p and q. See the very similar RSA proof for details.

Moral of the story: stick with $e = 3$!