Markov Chains

CS70 Summer 2016 - Lecture 6B

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Agenda

Quiz is out! Due: Friday at noon.

What are Markov Chains? State machine and matrix representations.

Hitting Time

А. А. Марков (1886).

Motivation

Suppose we flip a coin until we get a three heads in a row. How many coin flips should we expect to do?

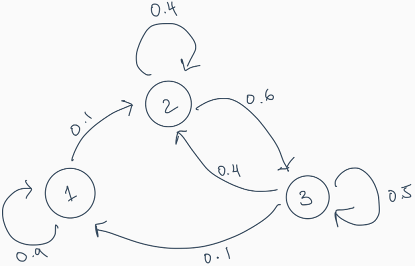
Drunkard on an arbitrary graph (remember HW?). When does the drunkard come home?

Try solving directly? Problem: conditioning gets really messy.

Need some way to express **state**.

Solution: Markov chains!

Intuition



A finite Markov chain consists of *states*, *transition probabilities* between states, and an *initial distribution*.

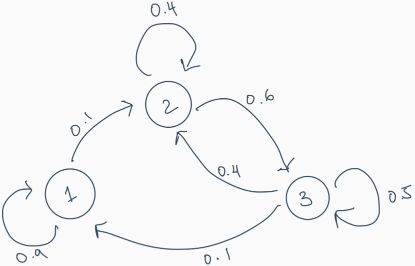
State: where are you now?

Transition probability: From where you are, where do you go next?

Initial distribution: how do you start?

Markov chains are **memoryless** - they don't remember anything other than what state they are.

Formally Speaking...



A finite set of states: $\mathcal{X} = \{1, 2, \ldots, K\}$

A initial probability distribution π_0 on $\mathcal{X} : \pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$

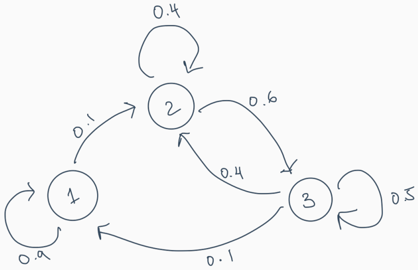
Transition probabilities: $P(i, j)$ for $i, j \in \mathcal{X}$

- $P(i, j) \geq 0, \forall i, j$; $\sum_j P(i, j) = 1, \forall i$

$\{X_n, n \geq 0\}$ is defined so that:

- $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$ (initial distribution)
- $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i, j), i, j \in \mathcal{X}$.

One Small (Time)step for a State



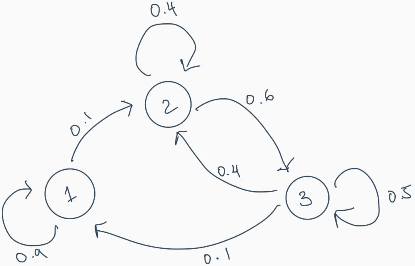
At each timestep t we are in some **state** $X_t \in \mathcal{X}$. (random variable.)

Where do we go next?

$$\Pr[X_{t+1} = j | X_t = i] = P_{i,j}$$

Probability depends on the previous state, but is independent of how it got to the previous state. (It's not independent of states before the previous state - but any dependence is captured in the previous state.)

One Giant Leap with Conditional Probability



At some point we might have a distribution for X_t - say, it's 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Distribution for X_{t+1}? Probability that it goes to 1?

$$\begin{aligned}\Pr[X_{t+1} = 1] &= \sum_i \Pr[X_{t+1} = 1 | X_t = i] \Pr[X_t = i] = \sum_i P_{i,1} \Pr[X_t = i] \\ &= 0.9 * 0.2 + 0 * 0.3 + 0.1 * 0.5 = 0.23\end{aligned}$$

Rest of distribution for X_{t+1} can be found similarly.

Linear Algebra Intro

Very quick linear algebra intro:

Matrices: two-dimensional collection of numbers. $n \times m$ matrix has n rows, m columns. Element at ith row, jth column denoted A_{ij}.

$$\begin{bmatrix} 1 & 6 & 7 & 2 \\ 6 & 5 & 6 & 3 \\ 8 & 6 & 2 & 2 \\ 2 & 5 & 3 & 8 \end{bmatrix}$$

Vector: one-dimensional collection of numbers. We deal with **row vectors** - $n \times 1$ matrices.

$$\begin{bmatrix} 5 & 9 & 3 & 0 \end{bmatrix}$$

Matrix Multplication

For $n \times m$ matrix A and $m \times p$ matrix B:

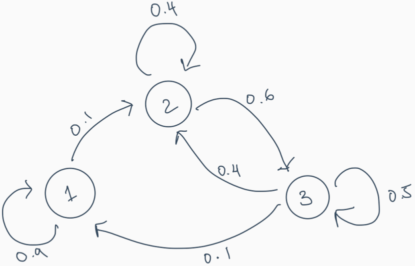
$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Or for vector x:

$$(xA)_i = \sum_k x_k A_{ki}$$

$$\begin{bmatrix} 5 & 9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 6 & 7 & 2 \\ 6 & 5 & 6 & 3 \\ 8 & 6 & 2 & 2 \\ 2 & 5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1*5+6*9+8*3+2*0 \\ 6*5+5*9+6*3+5*0 \\ 7*5+6*9+2*3+3*0 \\ 2*5+3*9+2*3+8*0 \end{bmatrix}^T$$

Matrix Markov



Markov chains have a very nice translation to matrices! Transition probabilities form an *transition matrix* P whose i, jth entry is $P_{i,j}$.

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Probabilities from a state sum to 1...rows sum to 1... **(right) stochastic matrix**.

Stepping with Multiplication

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Distributions are vectors. Suppose that X_t is distributed 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Write distribution as vector!

$$\pi_t = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix}$$

What's the product of π_t and P?

$$\begin{bmatrix} 0.2 * 0.9 + 0.3 * 0 + 0.5 * 0.1 \\ 0.2 * 0.1 + 0.3 * 0.4 + 0.5 * 0.4 \\ 0.2 * 0 + 0.3 * 0.6 + 0.5 * 0.5 \end{bmatrix}^T = \begin{bmatrix} 0.23 & 0.34 & 0.43 \end{bmatrix}$$

This is the distribution of X_{t+1}.

Multiple Steps with Matrix Powers

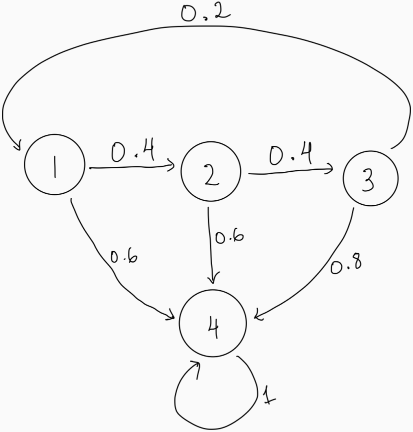
One step: $\pi_t \to \pi_t P$

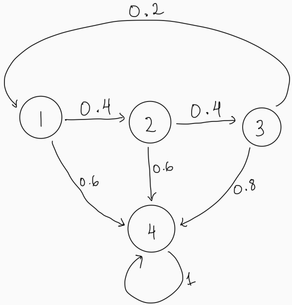
What if we take two steps? What's the distribution?
$\pi_t \to (\pi_t P)P = \pi_t P^2$

n steps? $\pi_t P^n$.

This will be very useful when we start talking about limiting distributions (next lecture).

California driving test: you get 3 retakes before you have to start the application process all over again. Suppose someone passes a driving test w.p. 0.6, unless it's their final retake, in which case they're more careful and pass w.p. 0.8.



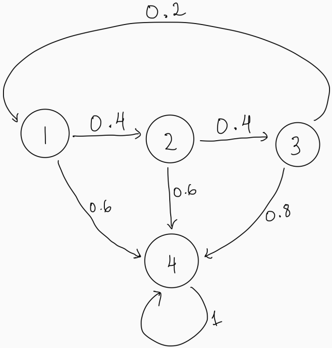


Initial distribution? $\pi_0 = [1\ 0\ 0\ 0]$
Transition matrix?

$$T = \begin{bmatrix} 0 & .4 & 0 & .6 \\ 0 & 0 & .4 & .6 \\ .2 & 0 & 0 & .8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hitting Time

Motivation

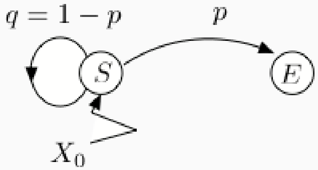


How long does it take to get a driver's license, in expectation?

Generally: given a Markov chain and an initial distribution, how many timesteps do we expect to take before reaching a particular state?

A Simple Example

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



Let $\beta(S)$ be the average time until E, starting from S. Then,

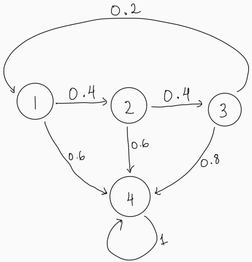
$$\beta(S) = 1 + q\beta(S) + p0.$$

Hence,

$$p\beta(S) = 1, \text{ so that } \beta(S) = 1/p.$$

Note: Time until E is $G(p)$. We have rediscovered that the mean of $G(p)$ is $1/p$.

How Long to Get a Driver's License?



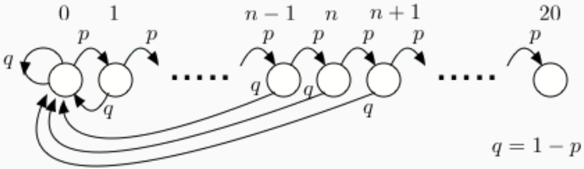
Let $\beta(S)$ denote expected time to get a driver's license from S.

$$
\begin{aligned}
\beta(1) &= 1 + 0.6 * 0 + 0.4 * \beta(2) \\
\beta(2) &= 1 + 0.6 * 0 + 0.4 * \beta(3) \\
\beta(3) &= 1 + 0.8 * 0 + 0.2 * \beta(1)
\end{aligned}
$$

Solves to $\beta(1) \approx 1.61$.

Driving test

A driving test consists of 20 maneuvers that must be done properly. The examinee succeeds w.p. $p = 0.9$ for each maneuver. Otherwise, he fails the driving test and has to start all over again. How many maneuvers does it take to pass the test?



$$\beta(n) = 1 + p\beta(n+1) + q\beta(0), 0 \le n < 19$$
$$\beta(19) = 1 + p0 + q\beta(0)$$

$$\Rightarrow \beta(0) = \frac{p^{-20} - 1}{1 - p} \approx 72.$$

See Lecture Note 24 for algebra.

Gig: Random names, random headlines