Markov Chains

CS70 Summer 2016 - Lecture 6B/C

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25 July 2016

UC Berkeley

Agenda

What are Markov Chains? State machine and matrix representations.

Hitting Time

Convergence, limiting and stationary distributions

А. А. Марков (1886).

Motivation

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Solution: Markov chains!

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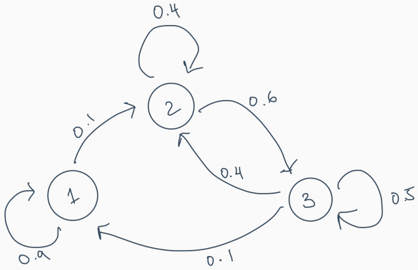

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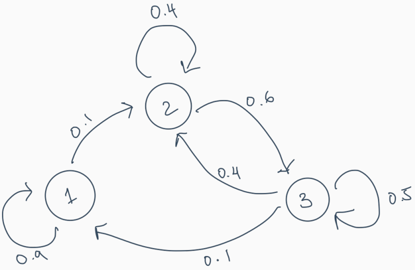

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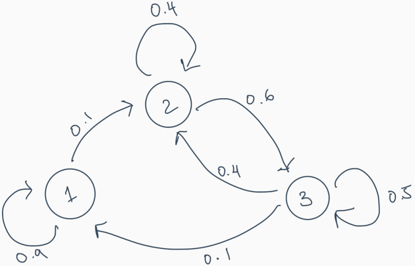

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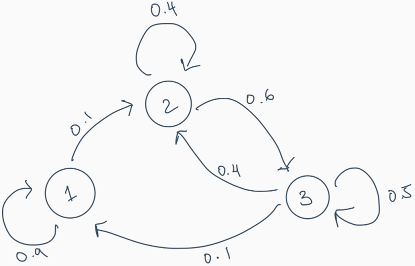

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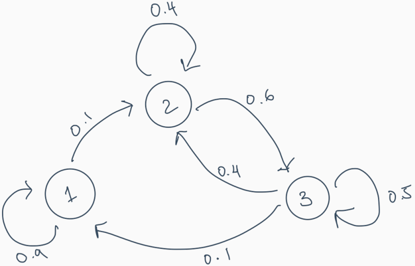

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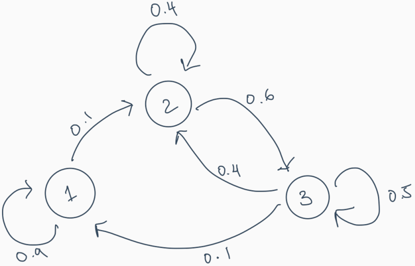

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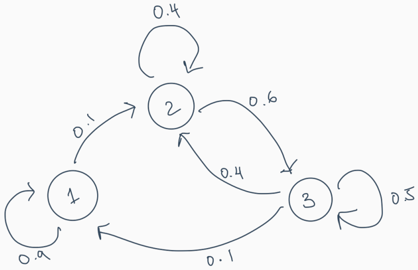
Markov chains are **memoryless** - they don't remember anything other than what state they are.

Formally Speaking...



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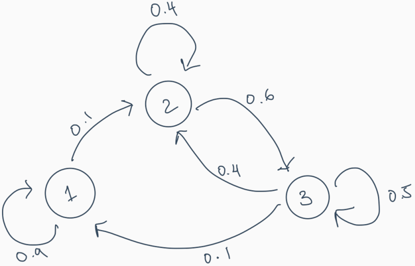

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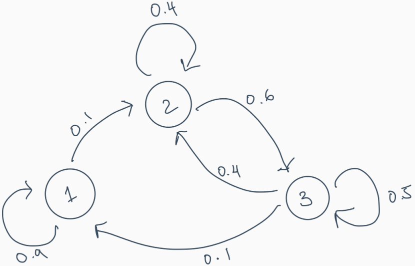

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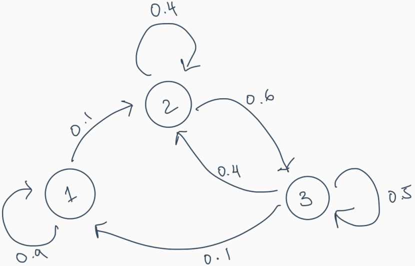

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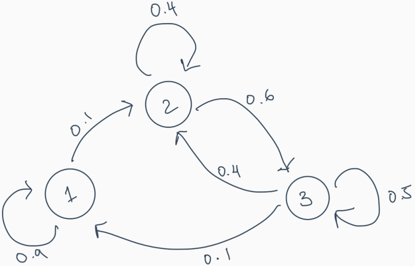

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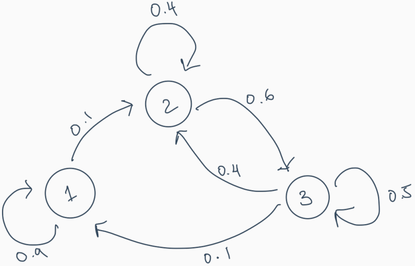

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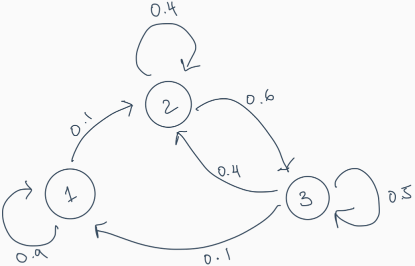

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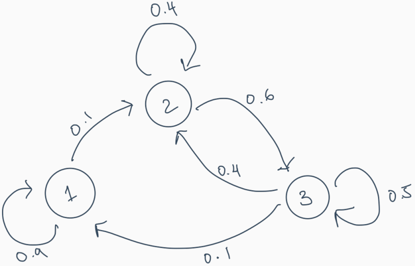

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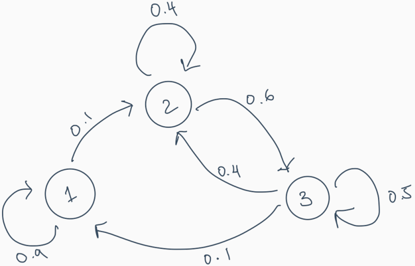

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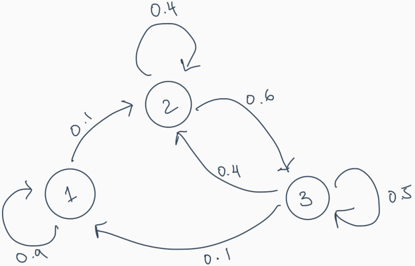

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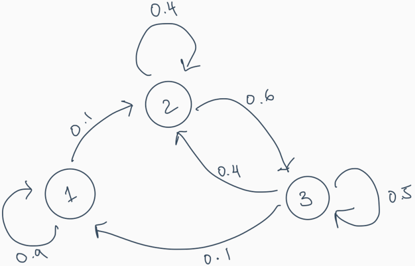

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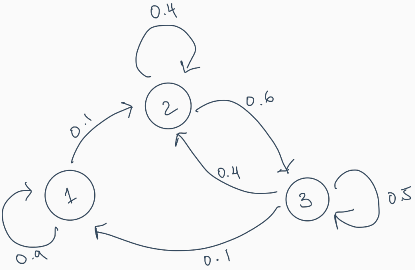
- $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i, j), i, j \in \mathcal{X}$.

One Small (Time)step for a State



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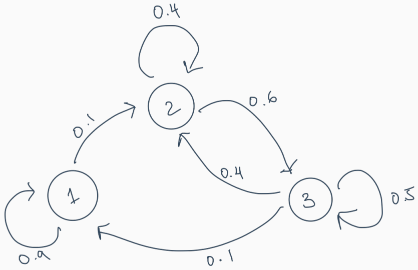

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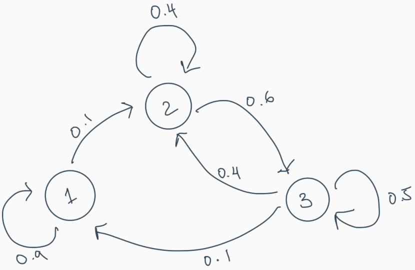

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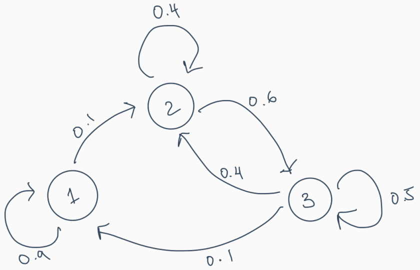

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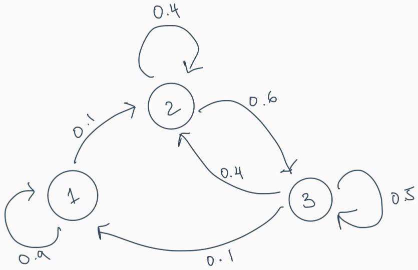
Probability depends on the previous state, but is independent of how it got to the previous state. (It's not independent of states before the previous state - but any dependence is captured in the previous state.)



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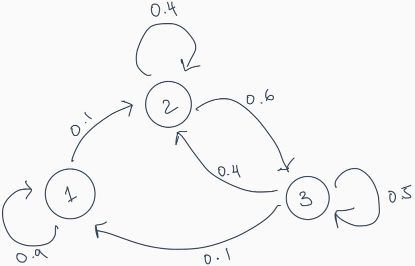
One Giant Leap with Conditional Probability



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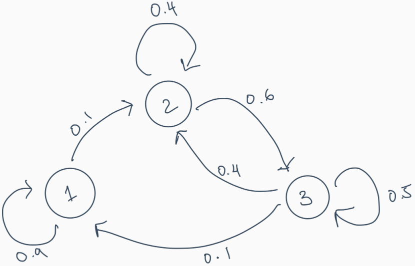

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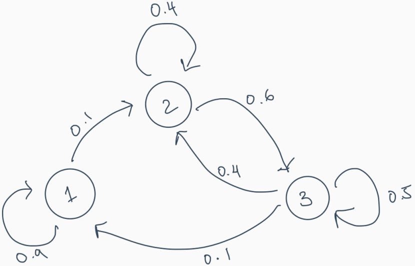

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$$\Pr[X_{t+1} = 1] \quad =$$

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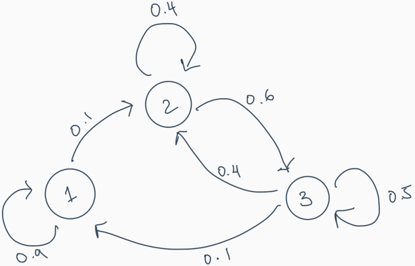

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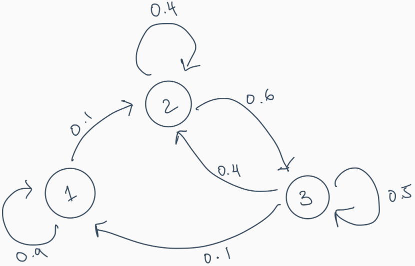

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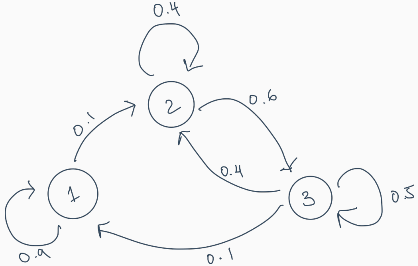

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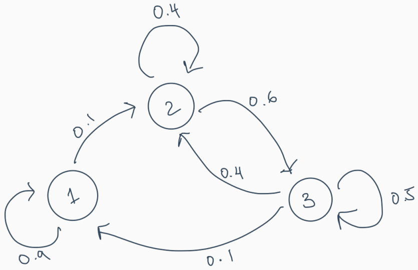

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Rest of distribution for X_{t+1} can be found similarly.

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Very quick linear algebra intro:

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Matrix Multplication

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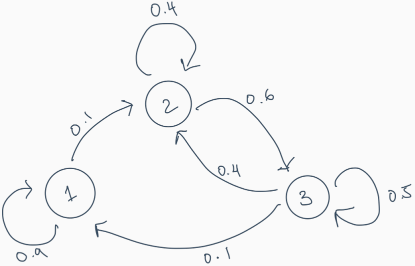

Markov chains have a very nice translation to matrices! Transition probabilities form an *transition matrix* P whose i, jth entry is $P_{i,j}$.

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Probabilities from a state sum to 1...

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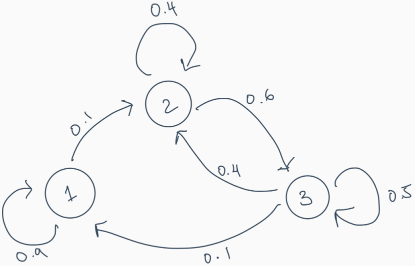

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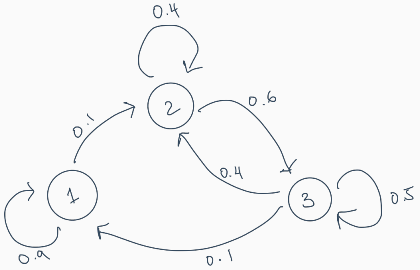

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$$\begin{bmatrix} 0.2 * 0.9 + 0.3 * 0 + 0.5 * 0.1 \\ 0.2 * 0.1 + 0.3 * 0.4 + 0.5 * 0.4 \\ 0.2 * 0 + 0.3 * 0.6 + 0.5 * 0.5 \end{bmatrix}^T$$

Stepping with Multiplication

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

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This is the distribution of X_{t+1}.

An Example

California driving test: you get 3 retakes before you have to start the application process all over again. Suppose someone passes a driving test w.p. 0.6, unless it's their final retake, in which case they're more careful and pass w.p. 0.8.

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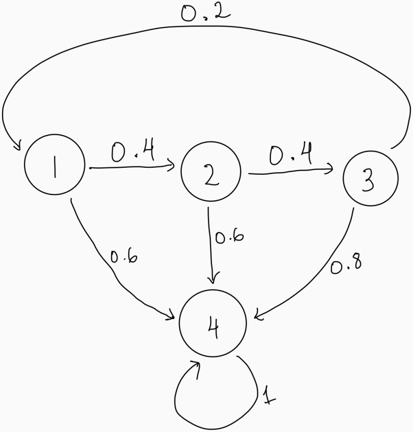

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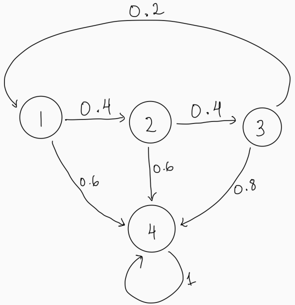
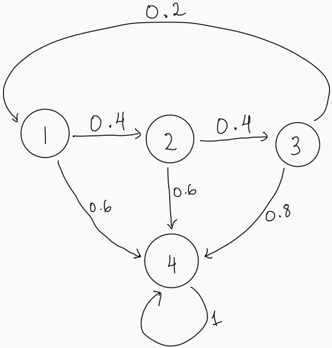


Initial distribution? $\pi_0 = [1\ 0\ 0\ 0]$
Transition matrix?

$$T = \begin{bmatrix} 0 & .4 & 0 & .6 \\ 0 & 0 & .4 & .6 \\ .2 & 0 & 0 & .8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hitting Time

Motivation



How long does it take to get a driver's license, in expectation?

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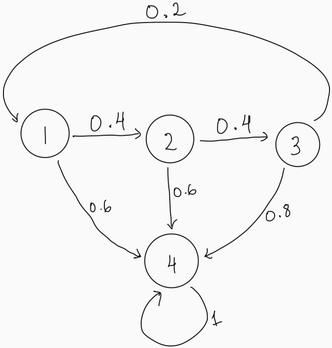
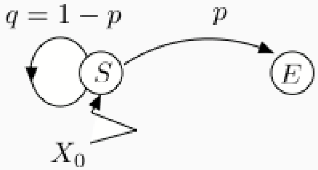


How long does it take to get a driver’s license, in expectation?

Generally: given a Markov chain and an initial distribution, how many timesteps do we expect to take before reaching a particular state?

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



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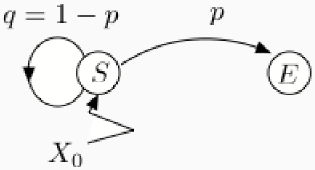

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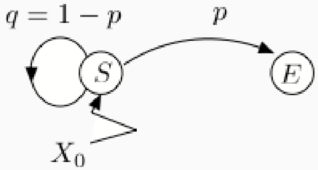

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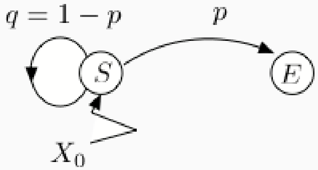

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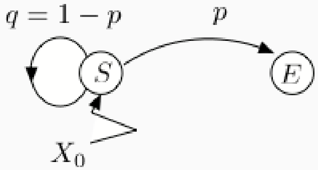

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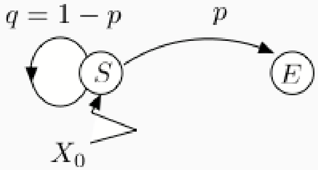

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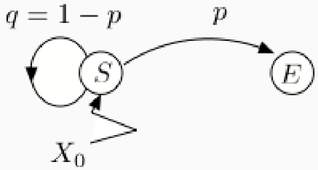

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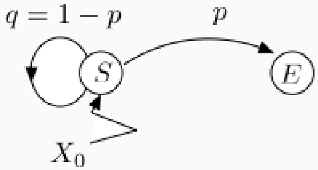

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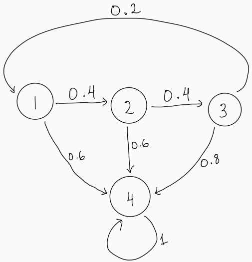
Note: Time until E is $G(p)$. We have rediscovered that the mean of $G(p)$ is $1/p$.

How Long to Get a Driver's License?



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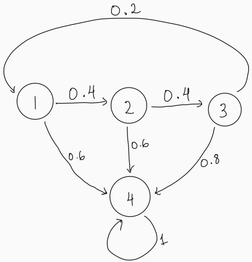

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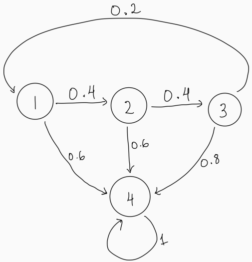

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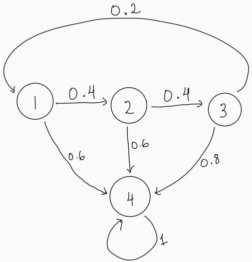

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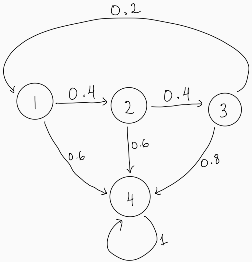

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Drunkard’s Climb

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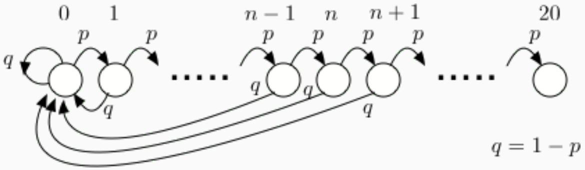

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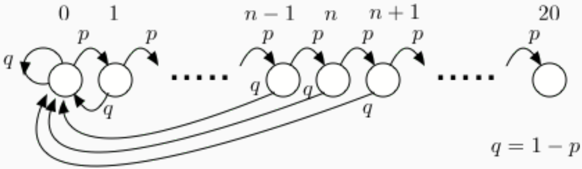

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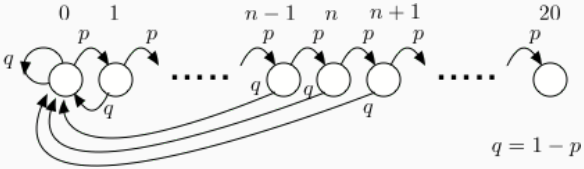

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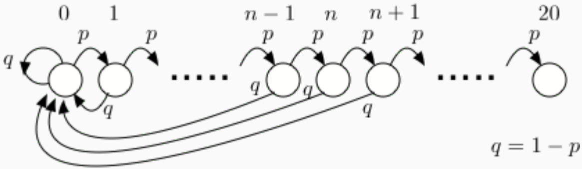

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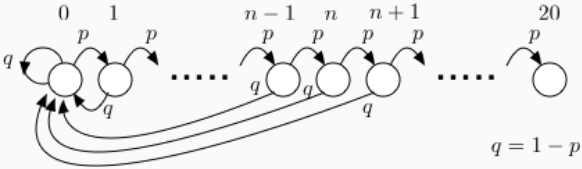

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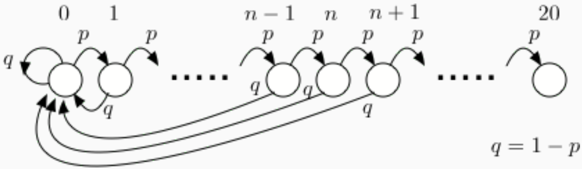

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See Lecture Note 24 for algebra.

Questions?