Alex Psomas: Lecture 17.

Random Variables: Variance

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1. Variance
2. Distributions

Variance

Flip a coin:

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Any other measures???

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Any other measures??? What else that's informative can we say?

Variance

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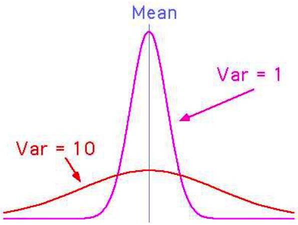

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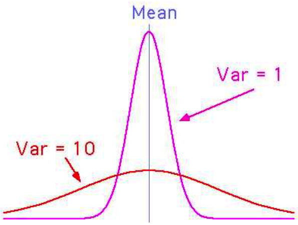

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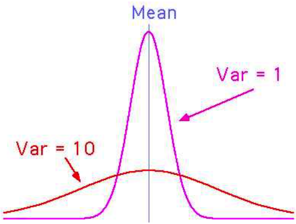

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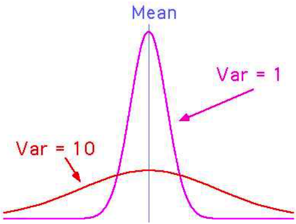

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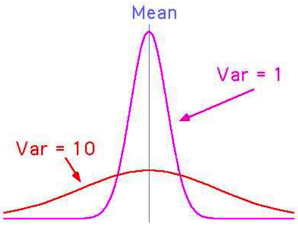

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$\sigma(X)$ is called the standard deviation of X.

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Fact:

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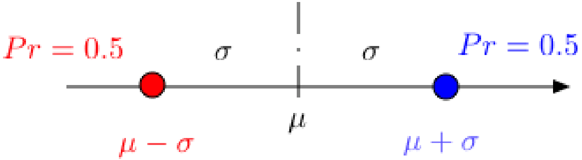

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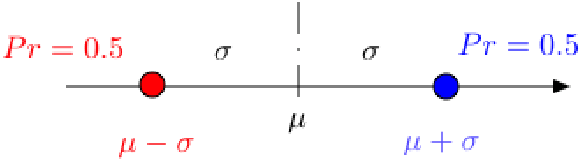

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$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

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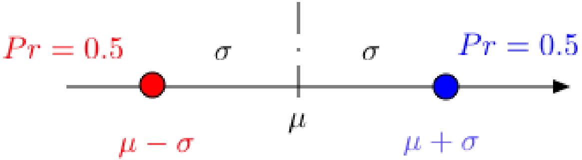

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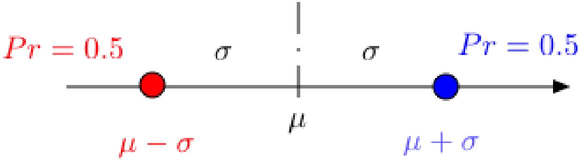

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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$\square$

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$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2+Y^2+Z^2+\cdots+2XY+2XZ+2YZ+\cdots) \\ &= E(X^2)+E(Y^2)+E(Z^2)+\cdots+0+\cdots+0 \\ &= var(X)+var(Y)+var(Z)+\cdots. \end{aligned}$$

□

Distributions

- Bernoulli
- Binomial
- Uniform
- Geometric
- Poisson

Bernoulli

Flip a coin, with heads probability p.

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$$Pr[X = i] = \binom{n}{i} p^i(1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

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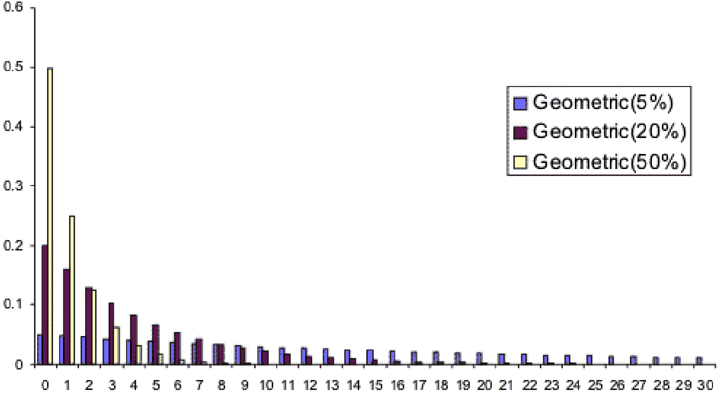

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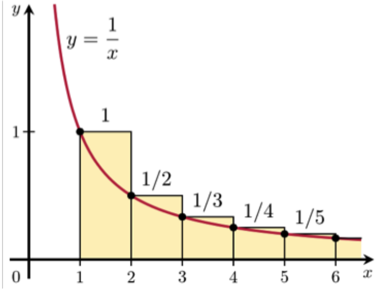

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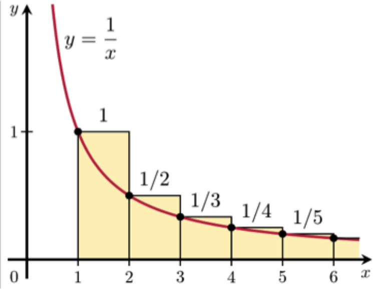

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

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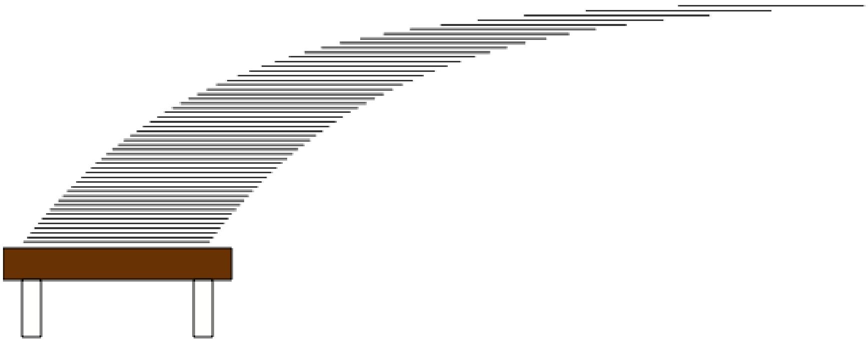

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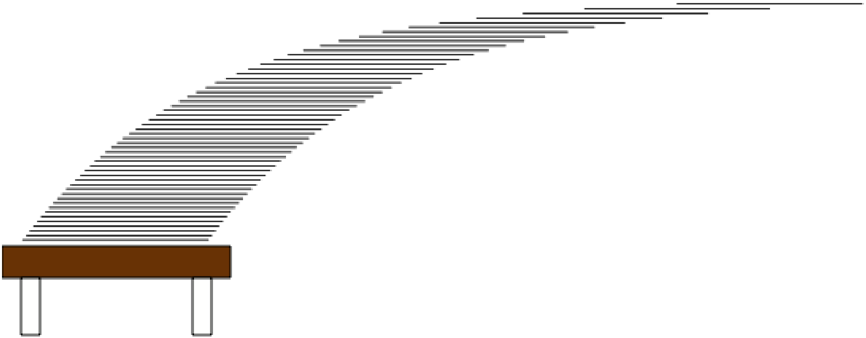

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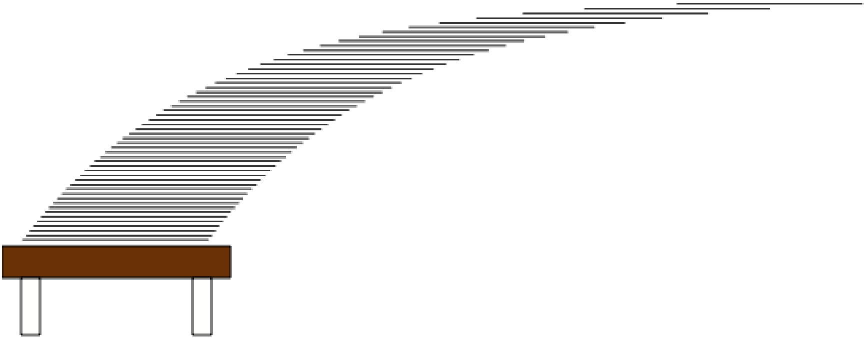

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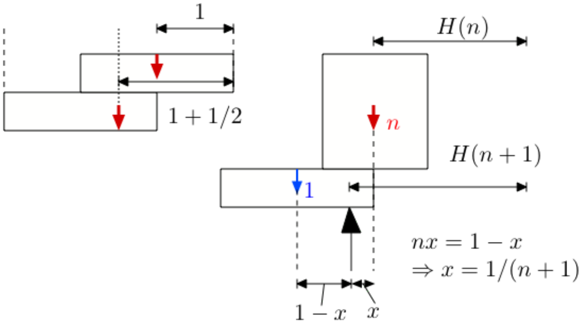
Stacking



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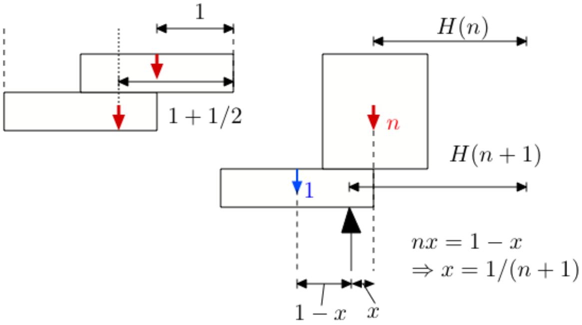

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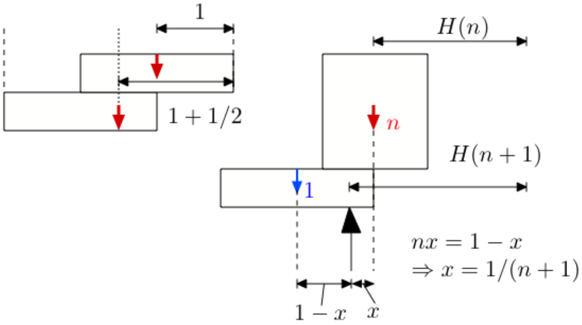

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

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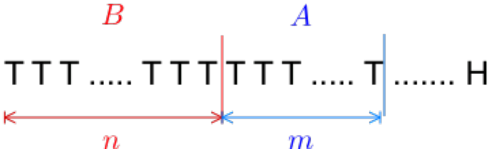

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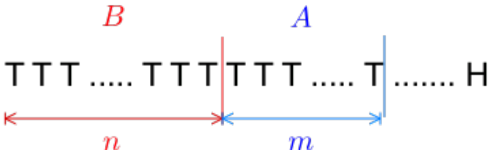

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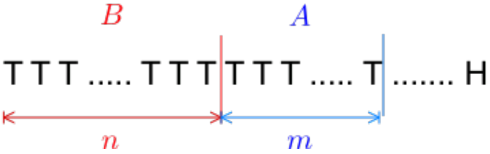

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The coin is memoryless, therefore, so is X.

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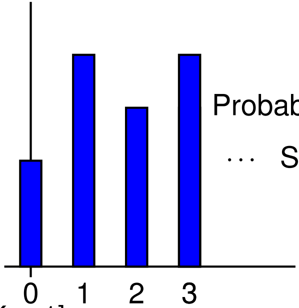

Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

$Pr[X \geq 1] \longrightarrow$

$Pr[X \geq 2] \longrightarrow$

$Pr[X \geq 3] \longrightarrow$

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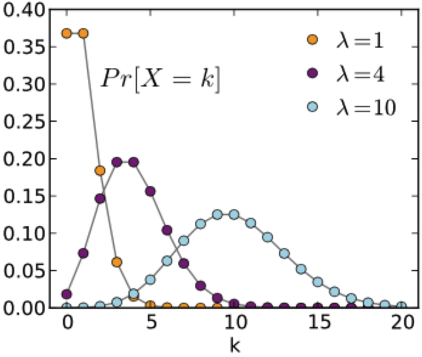

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