Alex Psomas: Lecture 19.

1. Distributions
2. Tail bounds

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

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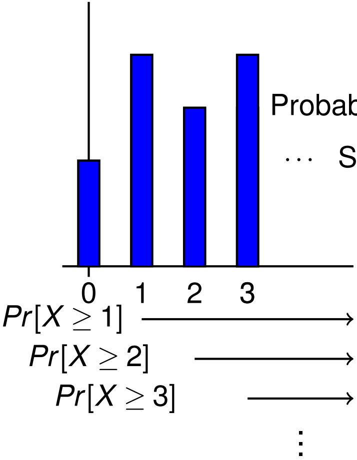

Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

A side step: Expected Value of Integer RV

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&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} (i-1) \times Pr[X \geq i] = \sum_{i=1}^{\infty} Pr[X \geq i].
\end{aligned}$$

$\square$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. **Theorem**

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

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$\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X]$ when p is small(ish).

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Random Variable: X - number of heads.

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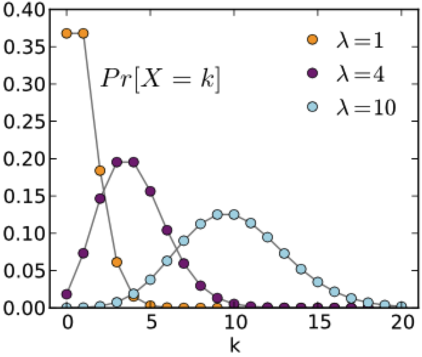

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$$Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}.$$

Poisson Distribution: Definition and Mean

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Fact: $E[X] = \lambda$.

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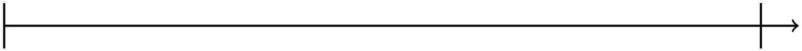

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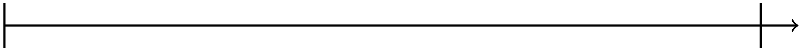

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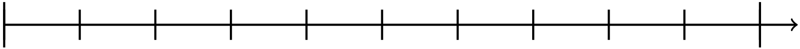

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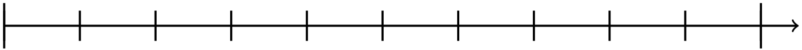

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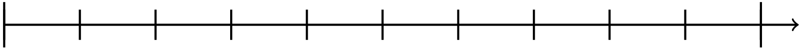

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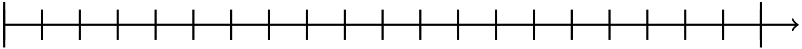

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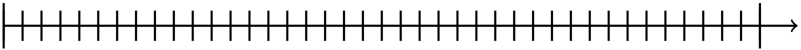

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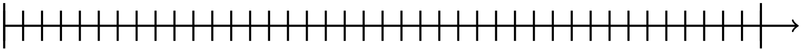

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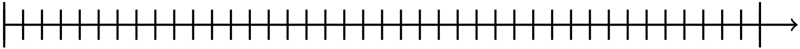

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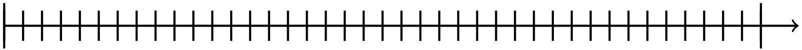

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...And we get the Poisson distribution!

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Examples:

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Simeon Poisson

The Poisson distribution is named after:

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Siméon Poisson

Siméon Denis Poisson (1781–1840)

“Life is good for only two things: doing mathematics and teaching it.”

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  $Var[X] = \frac{n^2-1}{12}$;
- $Geom(p) : Pr[X = n] =$

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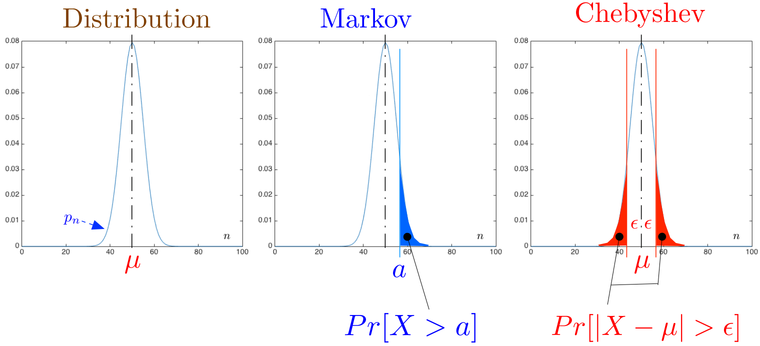
Inequalities: An Overview



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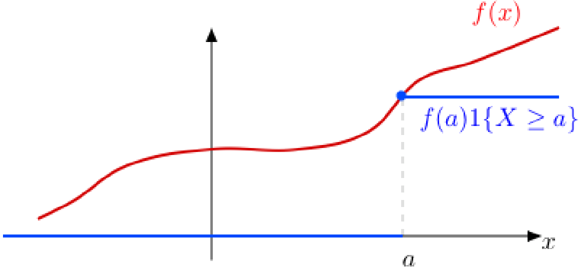
$$E[1\{X \geq a\}] \leq \frac{E[f(X)]}{f(a)}.$$

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

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A more common version of Markov is for $f(x) = x$:

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Theorem For a non-negative random variable X, and any $a > 0$,

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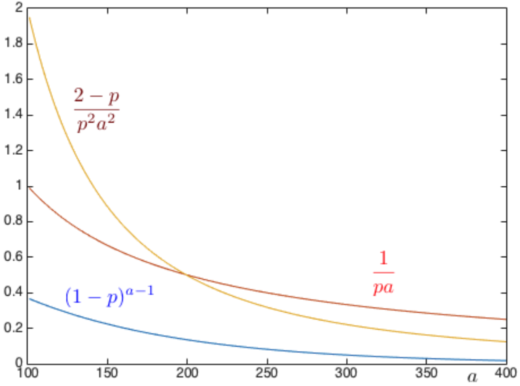

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This result confirms that the variance measures the "deviations from the mean."

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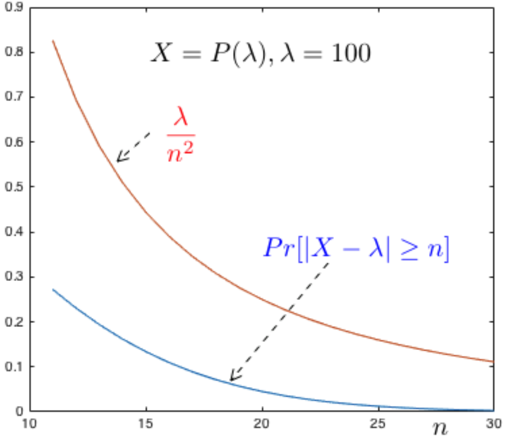

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We look at a general case next.

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For $\varepsilon = 0.01$ we get that $n \geq 50000$ coins are sufficient.

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Summary

- Variance of Geometric.
- Markov's Inequality
- Chebyshev's Inequality.