# A Random Walk through CS70

CS70 Summer 2016 - Lecture 8B

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UC Berkeley

# Today (and tomorrow, and Wednesday)

Review: what have we done in class?

Future classes: where do you go next?

Applications: how is the stuff you learned in 70 useful in the real world?

Research frontiers: what are people in academia working on (related to 70) right now?

Gigs: interesting stuff with material for fun and practice!

**Announcement:** No scantron HKN surveys now (or ever again!). Everything's online now. You should have received an email about this sometime in the previous week or two.

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Sanity check: Why is "$\forall q \in \mathbb{R} : |q| \geq q$" a statement but "$\forall x \in \mathbb{R} : xy = 0$" not a statement?

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How big do circuits need to be in order to compute some function that you're interested in? *Circuit lower bounds.* Hard problem. Lots of research going on about this. Absolute lower bounds that don't depend on unproven assumptions are pretty primitive (think lower bounds for computing whether you have an even number of 1s as input).

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What's the probability that some formula has a satisfying assignment? Counting/probabilistic arguments. Interesting results. Phase transitions.

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**These are all techniques you can compose!**

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Two quantities have to be the same. So we have proved the claim.

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Also $a^p \equiv a \pmod{p}$ by inductive hypothesis.

So
$$(a+1)^p - (a+1) = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k - (a+1)$$

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Also $a^p \equiv a \pmod{p}$ by inductive hypothesis.

So
$$\begin{aligned}(a+1)^p - (a+1) &= 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k - (a+1) \\ &\equiv 1 + a + 0 - (a+1) \pmod{p}\end{aligned}$$

# Example: FLT

**Fermat's little theorem:** For all $p$ prime, $p|(a^p - a)$.

**Proof:** by induction on $a$. Base case: obviously $p$ must divide $0 = (0^p - 0) = (1^p - 1)$.

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as desired. $\square$

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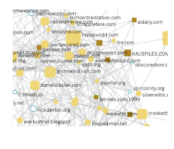
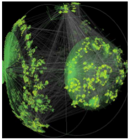
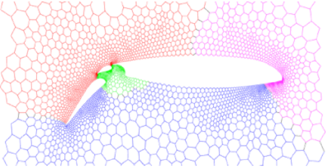
(remember that a Markov chain represented by a strongly connected graph is irreducible).

# Aside: Interesting Applications of Graphs

Web hyperlinks and social networks. Meshes in simulations and scientific computing.[1]



Maps and grids. Games.

Finding paths in graphs is really useful. How does Google maps find a route to your destination? Finding paths in graphs! CS170, CS188.

[1] Images from Aydin Buluc's CS267 slides,
`https://people.eecs.berkeley.edu/~demmel/cs267_Spr16/Lectures/CS267_March17_Buluc_2016_4pp.pdf`

# Walks and Tours

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What about if we say that we want to touch every vertex? Interesting question... and hard. CS170.

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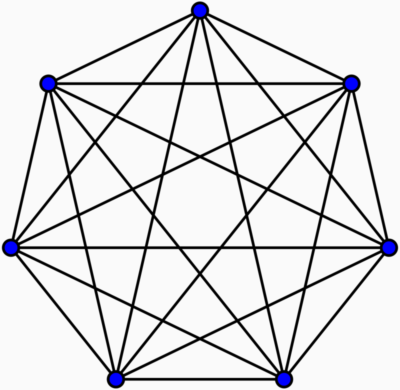

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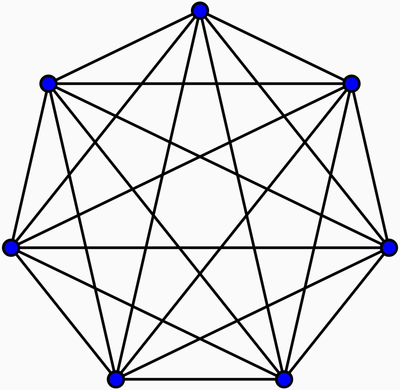

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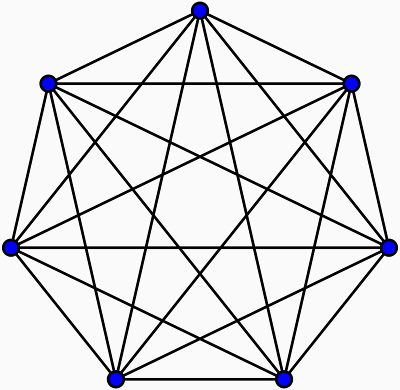

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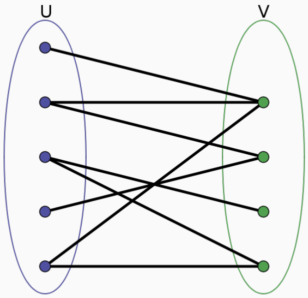
Bipartite graphs. Vertices can be partitioned into two sets such that there are no edges between edges in the same set.



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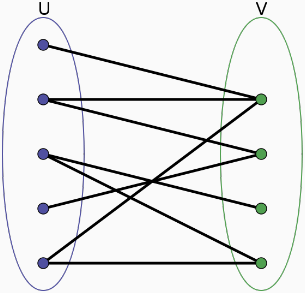


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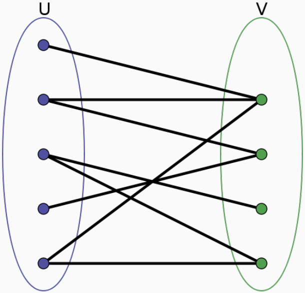


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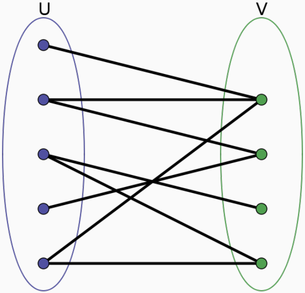


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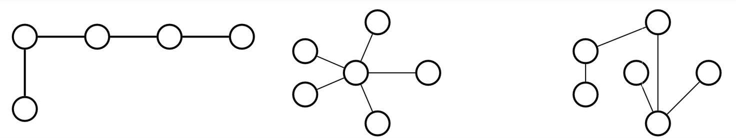
No odd length cycles. Random walk on a bipartite graph is periodic.

# Trees



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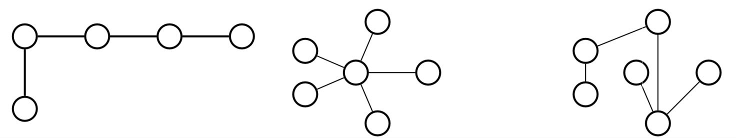

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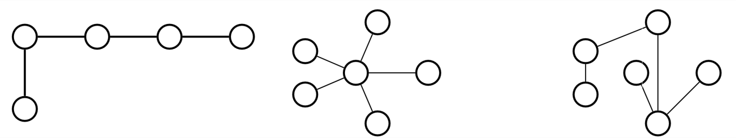

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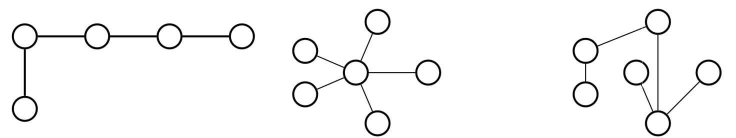

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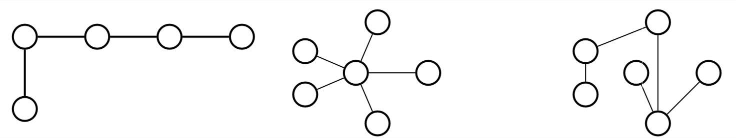

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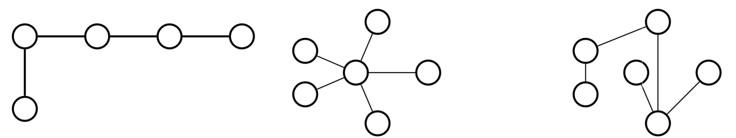

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Which definition is correct? All of them are equivalent. Good practice exercise: prove it!

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Decision trees. Every vertex represents a decision you can make. CS188

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**Practice problems:** try doing these proofs yourself (without looking at the old slides).

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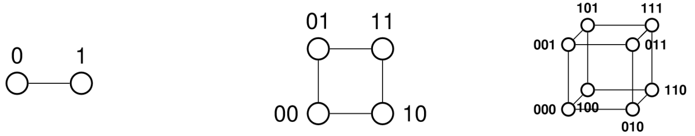
Register optimization! Touched on in CS164.

# Hypercubes

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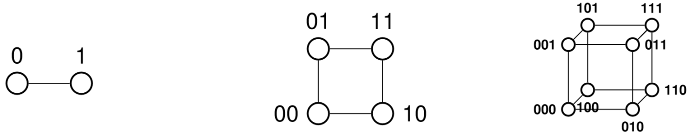


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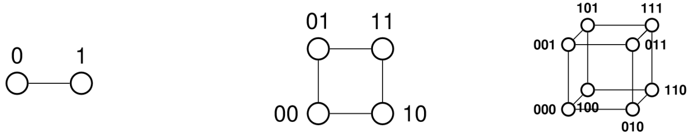


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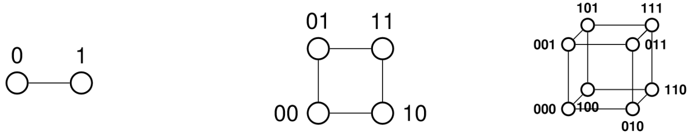


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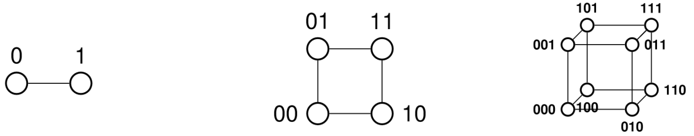


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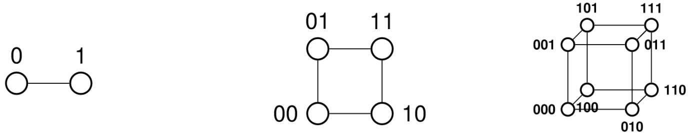


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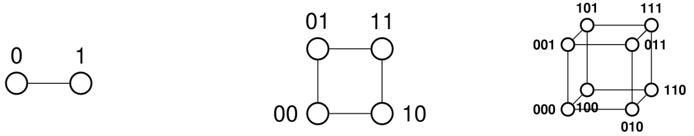


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Ease of routing and difficulty to cut make hypercubes really useful for distributed systems. Hypercube topology to be very common in supercomputers: Intel iPSC, nCube.

Now being used for routing messages in the Ethereum network.[2]

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More review on this tomorrow.

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TMA produces *male optimal* stable matching! Male optimal = best pairing for men among all possible matchings.

**Theorem:** male optimal = female pessimal.

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TMA produces *male optimal* stable matching! Male optimal = best pairing for men among all possible matchings.

**Theorem:** male optimal = female pessimal.

What about stable roommates? Nope. No stable pairing necessarily exists.

# Stable Marriage and Matchings

Want to match men and women in a stable manner. No rogue couples (people who would both rather be with each other than with their current partners).

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Used in hospital residency matching systems. Matching in general is a well studied problem. Used in programs like kidney exchanges.

# Counting

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| Order doesn't matter | $\binom{n+k-1}{n-1}$ | $\binom{n}{k}$ |

Selecting $k$ TAs out of $n$ to run a review session?

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If you like counting, take Math 172.

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And so on!

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So halting is undecidable!

Questions?