Algebraic Structures and Polynomials

CS70 Summer 2016 - Lecture 7C

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UC Berkeley

Today

Review: Chinese Remainder Theorem and Blum Coin Flipping

Algebraic Structures: Groups, Rings, and Fields

Galois Fields

Polynomials

Applications: Secret Sharing and Erasure Codes

Motivation

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Can we express turn numbers and congruences in modular arithmetic into their own “number system”?

Define *algebraic structures* through axioms that define how they behave.

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- **Associativity:** For all $a, b, c \in G$: $a + (b + c) = (a + b) + c$.
- **Existence of Identity:** There exists some element $e \in G$ such that for all $a \in G$, $e + a = a$.
- **Existence of inverse:** For all $a \in G$, exists $0 \in G$ such that $a + 0 = 0 + a = e$.

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Notice that there no commutativity requirement. "$\cdot$" may be non-commutative! If it is commutative, we refer to the group as *abelian*. Formally, Abelian groups must satisfy requires another axiom:

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Also, note that $+$ doesn't necessarily have to represent addition in the normal sense. Elements of G may not even be numbers!

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Examples: With addition and multiplication defined in the usual sense $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is a commutative ring but not a field.

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Definition: For prime p, the field $(\mathbb{Z}_p, +, \cdot)$, with $+$ and $\cdot$ defined as modular arithmetic $\pmod{p}$, is known as the prime field[1] of order p, denoted $GF(p)$.

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A polynomial is said to contain a point (x, y) if $p(x) = y$.

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One way to do it: try plugging in the points and solving for the coefficients. Say I give you $(x_1, y_1), (x_2, y_,), \ldots, (x_{d+1}, y_{d+1})$.

$$\begin{aligned} y_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_d x_1^d \\ &\vdots \\ y_{d+1} &= a_0 + a_1 x_{d+1} + a_2 x_{d+1}^2 + \cdots + a_d x_{d+1}^d \end{aligned}$$

Or in matrix form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^d \\ 1 & x_2 & x_2^2 & \ldots & x_2^d \\ 1 & x_3 & x_3^2 & \ldots & x_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{d+1} & x_{d+1}^2 & \ldots & x_{d+1}^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{d+1} \end{bmatrix}$$

(This matrix is called the *Vandermonde matrix*.)

Lagrange Interpolation (1/2)

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$$\Delta_1(x) := y_1 \frac{(x-x_2)(x-x_3)\dots(x-x_{d+1})}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_{d+1})}$$

Value at x_1?

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Value at x_1? y_1. Value at $x_2, \dots, x_{d+1}$? 0. General idea behind interpolation: make these polynomials for all i and add them together.

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Now we have a polynomial passing through a collection of $d+1$ points. Is it the *only* polynomial passing through these points? Or: we know the system given by the Vandermonde matrix has a solution. Is it a unique solution or is the system underdetermined?

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We already know there is such a polynomial (we constructed one). Remains to show uniqueness.

Proof of Theorem 2

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Polynomial Division

Given a degree-d polynomial $f(x)$ and a polynomial $g(x)$ of degree at most d, we can use long division to write $f(x) = g(x)q(x) + r(x)$ for some polynomials $q(x), r(x)$ such that the degree of $r(x)$ is strictly smaller than the degree of $f(x)$. Method: same as elementary-school long division for numbers!

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$$
\begin{array}{rrrrr}
 & & x^2 & +\; x & +3 \\
\cline{2-5}
x-3 \,\big) & x^3 & -2x^2 & +0x & -4 \\
 & x^3 & -\;3x^2 & & \\
\cline{1-3}
 & & +x^2 & +0x & \\
 & & +x^2 & -3x & \\
\cline{3-4}
 & & & +3x & -4 \\
 & & & +3x & -9 \\
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\begin{array}{r}
x^2 + x + 3 \\
x - 3 \overline{\smash{)} x^3 - 2x^2 + 0x - 4} \\
\underline{x^3 - 3x^2} \\
+x^2 + 0x \\
\underline{+x^2 - 3x} \\
+3x - 4 \\
\underline{+3x - 9} \\
+5
\end{array}
$$

So $x^3 - 2x^2 - 4 = (x-3)(x^2 + x + 3) + 5$.

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Lemma 2: If a degree-d polynomial $p(x)$ has d distinct roots $a_1, \ldots, a_d$, then it can be written as $p(x) = c(x - a_1) \ldots (x - a_d)$ for some constant c.

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Up next...

Counting polynomials.

Applications: Shamir's secret sharing and error-correcting codes.

Polynomial identity testing and the Schwartz-Zippel lemma