CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Psmoas, Dinh and Ye Discussion 6B Sol

1. Alice and Bob agree to try to meet for lunch between 12pm and 1pm at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch?

   Let the r.v. A be the time that Alice arrives and the r.v. B be the time when Bob arrives. Consider the following picture, plotting the space of all outcomes (a, b):
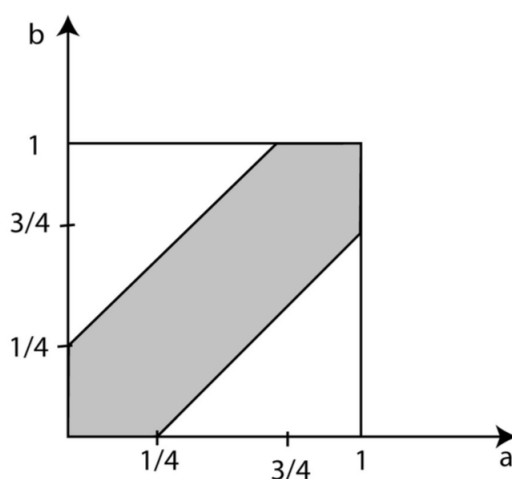
   The shaded region is the set of values (a, b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

   $$1 - 2 \times \left(\frac{1}{2} \times \left(\frac{3}{4} \right)^2 \right) = \frac{7}{16},$$

   since the area of the white triangle on the upper-left is $\frac{1}{2} \times (\frac{3}{4})^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $\frac{7}{16}$.

2. Let X be a continuous random variable whose pdf is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

   1. Find c

      Since our total probability must be equal to 1, $\int_0^1 cx^3 dx = 1 = [\frac{1}{4}cx^4]_{x=0}^1 = \frac{c}{4}$, so $c = 4$.

2. Find $\Pr\left[\frac{1}{3} \leq X \leq \frac{2}{3} \mid X \leq \frac{1}{2}\right]$.

$$
\begin{aligned}
\Pr\left[\frac{1}{3} \leq X \leq \frac{2}{3} \mid X \leq \frac{1}{2}\right] &= \Pr\left[\frac{1}{3} \leq X \leq \frac{2}{3} \cap X \leq \frac{1}{2}\right]/\Pr\left[X \leq \frac{1}{2}\right] \\
&= \Pr\left[\frac{1}{3} \leq X \leq \frac{1}{2}\right]/\Pr\left[X \leq \frac{1}{2}\right] \\
&= \left(\int_{\frac{1}{3}}^{\frac{1}{2}} 4x^3 dx\right)/\left(\int_0^{\frac{1}{2}} 4x^3 dx\right) \\
&= \left([x^4]_{x=\frac{1}{3}}^{\frac{1}{2}}\right)/\left([x^4]_{x=0}^{\frac{1}{2}}\right) \\
&= \left((\frac{1}{2})^4 - (\frac{1}{3})^4\right)/(\frac{1}{2})^4 \\
&= \frac{65}{81}
\end{aligned}
$$

3. Find $\mathrm{E}(X)$.

$$
\begin{aligned}
\mathrm{E}(X) &= \int_0^1 x \cdot 4x^3 dx \\
&= \int_0^1 4x^4 dx \\
&= [\frac{4}{5}x^5]_{x=0}^1 \\
&= \frac{4}{5}
\end{aligned}
$$

4. Find $\mathrm{Var}(X)$.

$$
\begin{aligned}
\mathrm{Var}(X) &= \int_0^1 x^2 \cdot 4x^3 dx - \mathrm{E}(X)^2 \\
&= \int_0^1 4x^5 dx - (\frac{4}{5})^2 \\
&= [\frac{2}{3}x^6]_{x=0}^1 - \frac{16}{25} \\
&= \frac{2}{75}
\end{aligned}
$$

3. You have two spinners, each having a circumference of 10, with values in the range $[0,10)$. If you spin both (independently) and let X be the position of the first spinner and Y be the position of the second spinner, what is the probability that $X \geq 5$, given that $Y \geq X$?

First we write down what we want and expand out the conditioning. $\Pr[X \geq 5 \mid Y \geq X] = \frac{\Pr[Y \geq X \cap X \geq 5]}{\Pr[Y \geq X]}$. $\Pr[Y \geq X] = \frac{1}{2}$ by symmetry (why should Y be any more likely to be bigger than X is?). To find $\Pr[Y \geq X \cap X \geq 5]$,

it helps a lot to just look at the picture of the probability space and use the continuous uniform law ($\Pr[A] = \frac{\text{area of } A}{\text{area of } \Omega}$). From the picture, one sees that $\Pr[Y \geq X \cap X \geq 5] = \frac{\frac{5 \cdot 5}{2}}{10 \cdot 10} = \frac{1}{8}$. So $\Pr[X \geq 5 \mid Y \geq X] = \frac{\frac{1}{8}}{\frac{1}{2}} = \boxed{\frac{1}{4}}$.
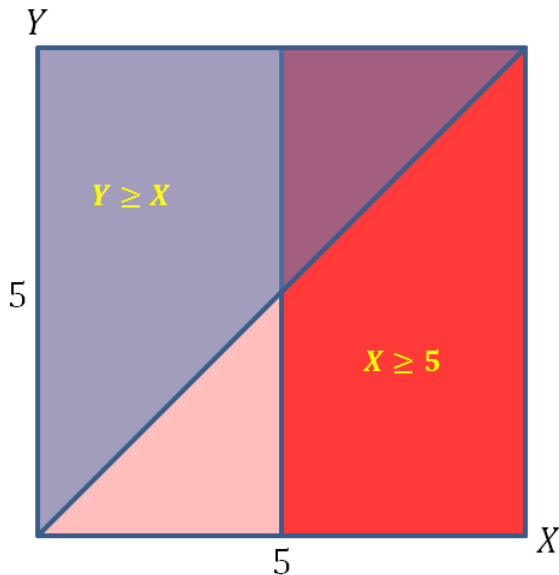
4. A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

   1. Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

      Let $X \sim Exp(\frac{1}{50})$ be the time until the bulb is broken. $\Pr[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot e^{-\frac{x}{50}}\right) dx = 1 - e^{-\frac{30}{50}} = \boxed{1 - e^{-\frac{3}{5}}} \approx .451$

   2. Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

      The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is $\Pr[Y > 30] = 1 - \Pr[Y < 30] = 1 - \left(1 - e^{-\frac{3}{5}}\right) = \boxed{e^{-\frac{3}{5}}} \approx .549$

   3. Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

      The bulb is memoryless, so $\Pr[X - 30 > 30 \mid X > 30] = \Pr[X > 30] = \boxed{e^{-\frac{3}{5}}} \approx .549$