# RSA, the Chinese Remainder Theorem, and Remote Coin Flipping

## CS70 Summer 2016 - Lecture 7B

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UC Berkeley

# Agenda

RSA

The Chinese remainder theorem

Euler's Criterion

Blum's coin-flipping scheme *Slides marked with an asterisk* are considered enrichment material and will not be tested on the exam. Think of them as gigs.*

# Encryption

# Motivation

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Goal: transmit my credit card number to Amazon without any eavesdroppers knowing what they are.

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Example: let's say my credit card has a bit representation of 01101. Pick key 11001. Ciphertext is 10100. Easy to verify that bitwise xor of 10100 and 11001 is 01101.

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- Needs a key to be shared before the transmission is done. If I need to walk into Amazon HQ to give them a secret key before sending them my CC number, why not just go to a store?

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Big idea: Amazon gives everyone a mathematical safe that they can put stuff into, but can’t unlock.

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**Decrypt:** Amazon computes $D(c) = mod(c^d, N)$. We'll show (next slide) this actually gives the plaintext $x$ back.

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- Case 1: $p$ divides $x$. Then obviously it also divides $x(x^{k(p-1)(q-1)} - 1)$, as desired.
- Case 2: $p$ doesn't divide $x$. Then $x^{k(p-1)(q-1)} = (x^{p-1})^{k(q-1)}$. Applying Fermat's little theorem, $x^{p-1} \equiv 1 \pmod{p}$. So $x^{k(p-1)(q-1)} - 1 \equiv 1^{k(q-1)} - 1 \equiv 0 \pmod{p}$, so $x(x^{k(p-1)(q-1)} - 1)$ must be a multiple of $p$.

Argument for $q$ is exactly the same. Therefore $p|(x^{ed} - x)$. $\square$

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The security RSA, like all almost all encryption schemes, relies on *hardness assumptions*. We need to assume something is hard in order to show that decrypting something, or even getting some information about the plaintext, *even with full information*, is hard.

# Message Indistinguishability*

How do we formalize this notion of ”hard to get information about the plaintext”?

Quasi-formally: under some hardness assumptions, this must hold for *all* pairs of strings $m^{(1)}$, $m^{(0)}$: for any *probabilistically polynomial time* (”PPT”) algorithm $A$ that knows the length of the strings and the public key, the probability that $A$ returns 1 given the public key and the encryption of $m^{(1)}$ must be ”extremely close” to the probability that it returns 1 on $m^{(0)}$.

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$$\left|\Pr[A^{E(1^k,PK)}(1^k, PK, E(1^k, PK, m_k^{(1)}) = 1] - \Pr[A^{E(1^k,PK)}(1^k, PK, E(1^k, PK, m_k^{(0)}) = 1]\right|$$

is “negligible” in $k$.

Intuitively? There is no algorithm (even if we allow the algorithm access to the public key) that runs in a reasonable amount of time that can distinguish between the ciphertexts for two different plaintexts.

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**Security of breaking RSA requires on hardness of factoring large integers.**

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Main takeaway: primes aren't too uncommon. If we select a few hundred 512-bit numbers, there will probably be a prime among them.

Problem: how do we figure out if something's a prime?

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Suppose $k$ is composite. Call $a$ such that $a^{k-1} \not\equiv 1 \pmod{k}$ "Fermat witnesses" and $a$ such that $a^{k-1} \equiv 1 \pmod{k}$ "Fermat liars".

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Let's say $a$ is a Fermat witness and $b_1, \ldots, b_l$ are a Fermat liar. Then

$$(ab_i)^{k-1} \equiv a^{k-1} b_i^{k-1} \equiv a^{k-1} 1 \not\equiv 1 \pmod{k} \ .$$

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AKS primality test [Agrawal–Kayal–Saxena '02]: you can find primes "efficiently" (roughly # of bits to the sixth power) without using randomness.

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AKS primality test [Agrawal–Kayal–Saxena '02]: you can find primes "efficiently" (roughly # of bits to the sixth power) without using randomness.

Fundamental question in computer science: how much additional computational power does randomness give you? Can you do things with randomness efficiently that you can't without randomness?

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Major open problem! There are problems that we know how to solve with randomness, but we don't know how to solve deterministically.

# The Chinese Remainder Theorem, Euler's Criterion, and an Application to Flipping Coins

# Simultaneous Congruences

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So $x = a + m(c + nk) = (a + mc) + mnk$, i.e. $x \equiv m(c + nk) \pmod{mn}$; this is a unique solution to the equations mod $mn$. $\square$

# Chinese Remainder Theorem

We can generalize this to multiple primes!

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Add the last equation back. Since $m_{k+1}$ is relatively prime to each of $m_1, ..., m_k$, it is relatively prime to $m_1 m_2 ... m_k$. So by the previous theorem, there is a unique solution mod $(m_1 m_2 ... m_k) m_{k+1}$. □

# Euler’s Criterion

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Number theory to the rescue!

# Square Roots in Modular Arithmetic

**Theorem (Euler's Criterion):** Suppose $p$ is an odd prime and $a$ is some integer relatively prime to $p$. Then $a^{(p-1)/2}$ is 1 if and only if there exists some integer $x$ such that $a \equiv x^2 \pmod{p}$ and $-1$ otherwise.

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**Proof:** If direction:

$$a^{(p-1)/2} = (x^2)^{(p-1)/2} = x^{p-1} \equiv 1 \pmod{p}$$

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Notice that if $a \equiv 3 \pmod{4}$, then we can find square roots easily. In fact, if the solutions to $x^2 \equiv a \pmod{p}$ are given by $x \equiv \pm a^{(p+1)/4} \pmod{p}$. Why?

$$(\pm a^{(p+1)/4})^2 \equiv a^{(p+1)/2} \equiv a^{(p-1)/2} a \equiv 1a \equiv a \pmod{p}$$

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Four square roots mod $pq$!

Combine sqare root formula on previous slide for single prime congruent to $3 \pmod{4}$ with trick here gives us an easy way to compute square roots of numbers mod $pq$ where $p, q$ are congruent to $3 \pmod{4}$.

Products of distinct primes both congruent to $3 \pmod{4}$ are called "Blum integers".

# Blum's Coin-Flipping Scheme

Here's how to flip a coin over the telephone [Blum-'82]:

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3. Alex, armed with knowledge of $p$, $q$, computes the square roots $\pm x, \pm y$ of $a$, mod $n$, and sends one to David.
4. If David got $\pm x$, then he says Bob guessed correctly. Otherwise, if he gets $\pm y$, he can factor $n$ and use that to prove that he won.

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If David got $\pm y$: he now has two different square roots of $a \pmod{n}$. Now he can use this to factor $n$: since $x^2 \equiv a \equiv y^2 \pmod{n}$ (with $x, y$ distinct), $pq|(x+y)(x-y)$, so each prime divides either $(x+y)$ or $(x-y)$ but not both. So $\gcd(x+y, n)$ and $\gcd(x-y, n)$ provide the two prime factors. All David has to do is compute $x^2 - y^2 = (x+y)(x-y)$ and run EGCD twice!

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After the game is over each side can verify the other's honesty: David asks Alex for the factors $p, q$ to make sure they're Blum integers and check that they're primes.

Questions?