Alex Psomas: Lecture 17.

Random Variables: Expectation, Variance

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1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance

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$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

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- $(X - a)^2$
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- $(X - Y)^2$
- $X \cos(2\pi Y + Z)$.

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$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

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$$\sum_{\omega} X(\omega) Pr[\omega] = 3\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 0\frac{1}{8}.$$

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Also,

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The fact that this average converges to $E[X]$ is a theorem:

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An Illustration: Rolling Dice

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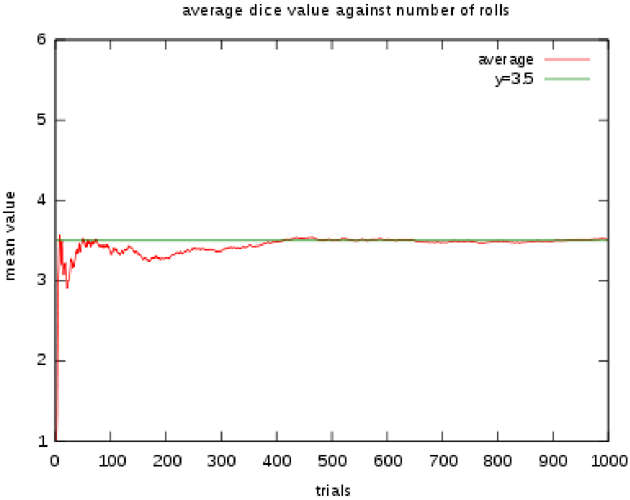

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Thus, we will write $X = 1_A$.

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Theorem:

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&\quad = a_1 \sum_{\omega} X_1(\omega) Pr[\omega] + \cdots + a_n \sum_{\omega} X_n(\omega) Pr[\omega] \\
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\end{aligned}$$

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Note: If we had defined $Y=a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y]=\sum_y yPr[Y=y]$, we would have been in trouble!

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Flip n coins with heads probability p. X - number of heads

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For a group of 28 it's about 1. For 100 it's 13.5. For 280 it's 107.

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Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

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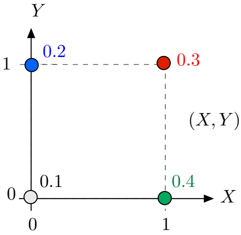

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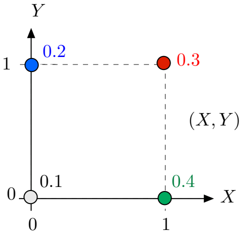

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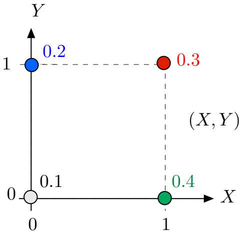

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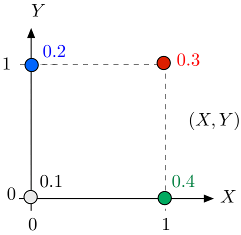

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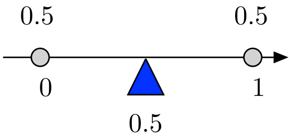
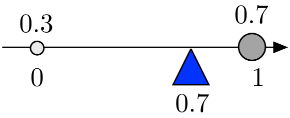
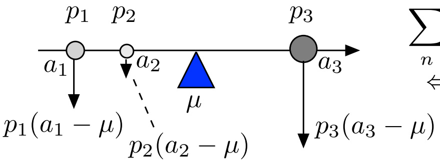
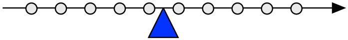


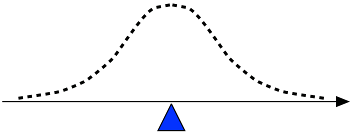




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$





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Unfortunately, we won't talk about this in this class...

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X and Y are independent if and only if

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Let X, Y be independent RV. Then

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Examples

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Mutually Independent Random Variables

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The events $A,B,C,\dots$ are pairwise (resp. mutually) independent iff the random variables $1_A,1_B,1_C,\dots$ are pairwise (resp. mutually) independent.

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$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

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$$\begin{aligned} E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\ &= E[Y] E[X_{n+1}], \\ &\quad \text{because } Y, X_{n+1} \text{ are independent} \\ &= E[X_1] \cdots E[X_n] E[X_{n+1}]. \end{aligned}$$

$\square$

Variance

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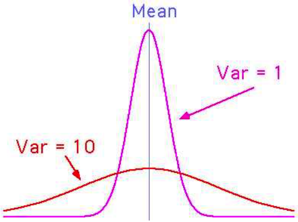

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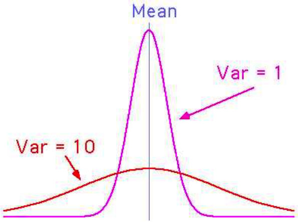

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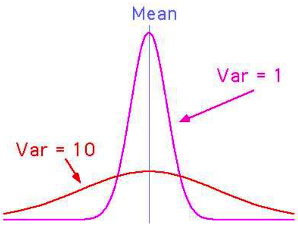

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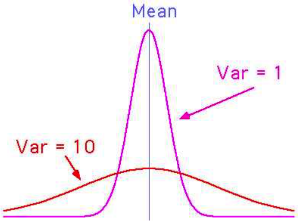

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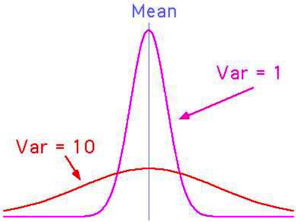

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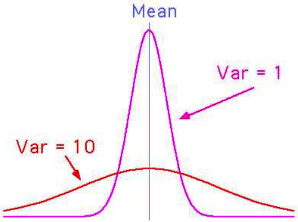

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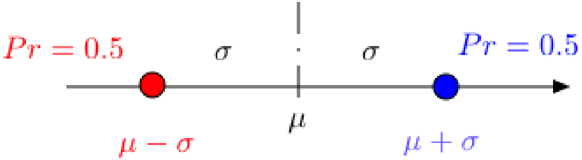

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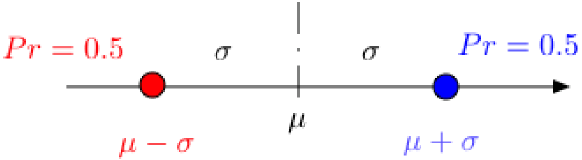

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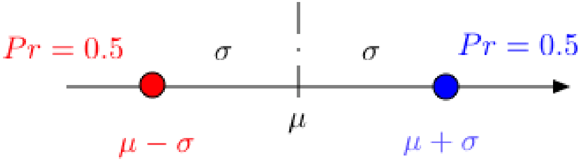

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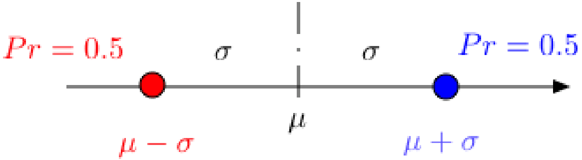

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

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How come this show is still around?

Today's gig: Lies!

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2. Monty Hall.
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Wait... Wrong Simpson.

The paradox

Summary

Random Variables

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- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $g(X, Y, Z)$ assigns the value .... .
- $E[X] := \sum_a a Pr[X = a]$.
- Expectation is Linear.
- Independent Random Variables.
- Variance.