Continuous Probability

CS70 Summer 2016 - Lecture 6A

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25 July 2016

UC Berkeley

Logistics

Tutoring Sections - M/W 5-8PM in 540 Cory.

- Conceptual discussions of material
- No homework discussion (take that to OH/HW party, please)

Midterm is this Friday - 11:30-1:30, same rooms as last time.

- Covers material from MT1 to this Wednesday...
- ...but we will expect you to know everything we've covered from the start of class.
- One **double**-sided sheet of notes allowed (our advice: reuse sheet from MT1 and add MT2 topics to the other side).
- Students with time conflicts and DSP students should have been contacted by us - if you are one and you haven't heard from us, get in touch ASAP.

Today

- What is continuous probability?
- Expectation and variance in the continuous setting.
- Some common distributions.

Continuous Probability

Motivation I

Sometimes you can't model things discretely.

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What is an event in continuous probability?

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Chance of getting one event in an infinite sized uniform sample space?

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Not so simple to define events in continuous probability!

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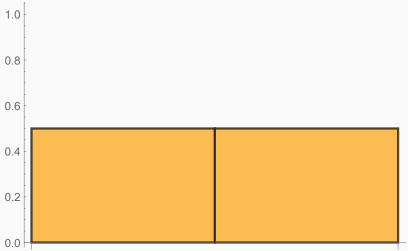

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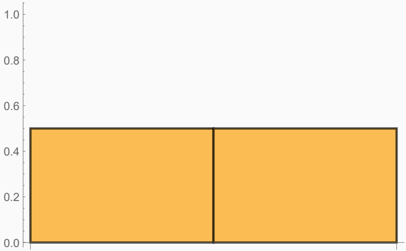

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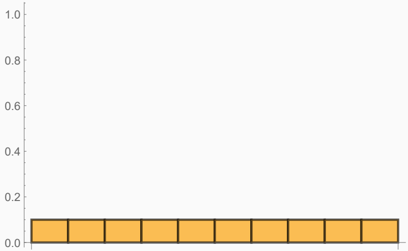

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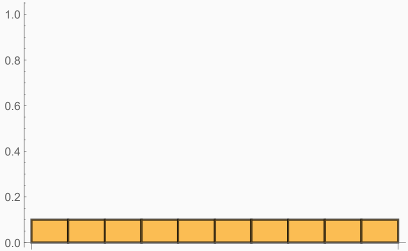

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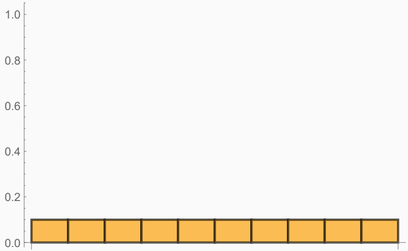

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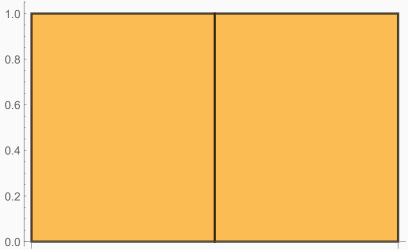

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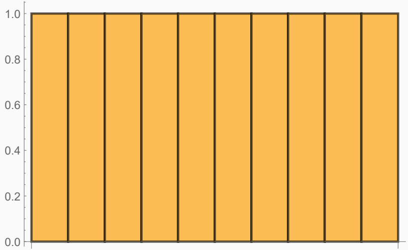

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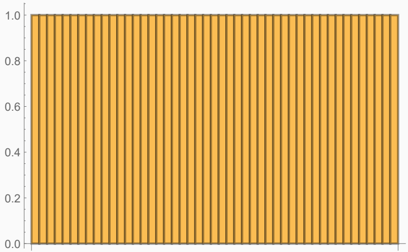

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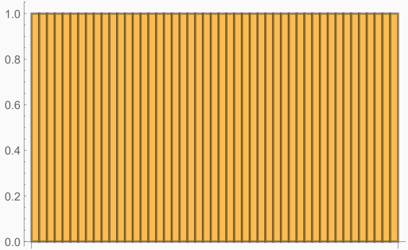

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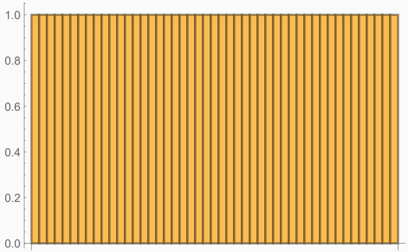

The resulting curve as $k \to \infty$ is the **probability density function (PDF).**

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Total probability is 1: $\int_{-\infty}^{\infty} f_X(t)dt = 1$

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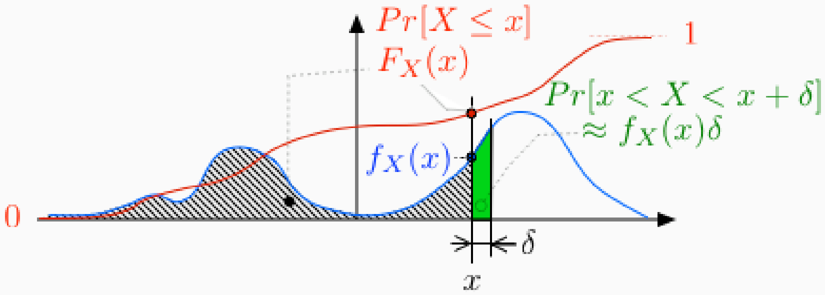
In Pictures



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Exercise: try proving these yourself.

Variance

Variance is defined exactly like it is for the discrete case.

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The standard properties for variance hold in the continuous case as well.

$$Var(aX) = a^2 Var(X)$$

For independent r.v. X, Y:

$$Var(X + Y) = Var(X) + Var(Y)$$

.

Target shooting

Suppose an archer always hits a circular target with 1 meter radius, and the exact point that he hits is distributed uniformly across the target. What is the distribution the distance between his arrow and the center (call this r.v. X)?

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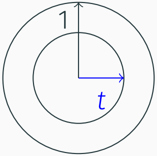

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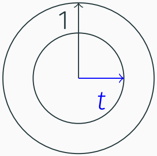

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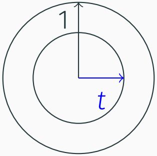

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$$Pr[X \leq t] \quad = \quad \frac{\text{area of small circle}}{\text{area of dartboard}}$$

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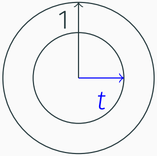

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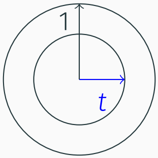

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Target shooting II

CDF:

$$F_Y(t) = Pr[Y \leq t] = \begin{cases} 0 & \text{for } t < 0 \\ t^2 & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t > 1 \end{cases}$$

PDF?

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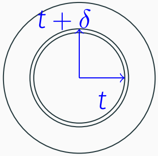
$$f_Y(t) = F_Y(t)' = \begin{cases} 2t & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Target shooting III

Another way of attacking the same problem: what's the probability of hitting some ring with inner radius t and outer radius $t + \delta$ for small δ?



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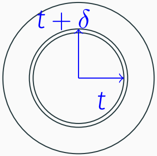

Area of circle: π

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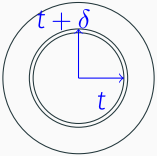

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Area of ring:

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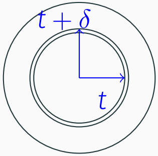

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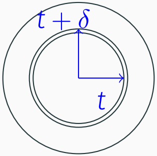

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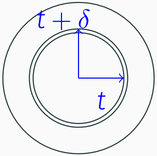

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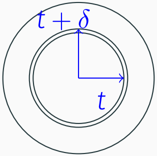

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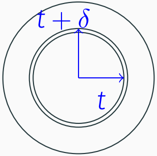

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PDF for $t \leq 1$: $2t$

Shifting & Scaling

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Continuous Distributions

Uniform Distribution: CDF and PDF

PDF is constant over some interval $[a, b]$, zero outside the interval.

What's the value of the constant in the interval?

$$\int_{-\infty}^{\infty} k dt = \int_{a}^{b} k dt = b - a = 1$$

so PDF is $1/(b-a)$ in $[a, b]$ and 0 otherwise.

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CDF?

$$\int_{-\infty}^{t} 1/(b-a) dz$$

0 for $t < a$, $(t-a)/(b-a)$ for $a < t < b$, and 1 for $t > b$.

Uniform Distribution: CDF and PDF, Graphically

$$f_X(t) = \begin{cases} 1/(b-a) & a < t < b \\ 0 & \text{otherwise} \end{cases}$$

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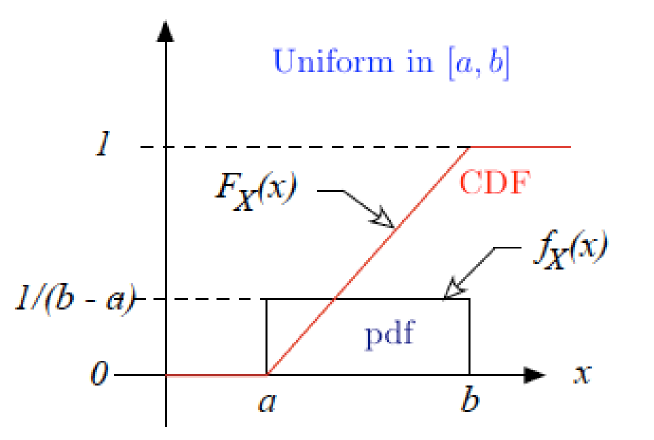

Uniform Distribution: Expectation and Variance

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&= \tfrac{(a-b)^2}{12}
\end{aligned}$$

Exponential Distribution: Motivation

Continuous-time analogue of the geometric distribution.

How long until a server fails?

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How long until a server fails? How long does it take you to run into pokemon?

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Can't "continuously flip a coin". What do we do?

Look at geometric distributions representing processes with higher and higher granularity.

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Probability that server fails on the same day as time t:

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This is the PDF of the **exponential distribution**!

Exponential Distribution: PDF and CDF

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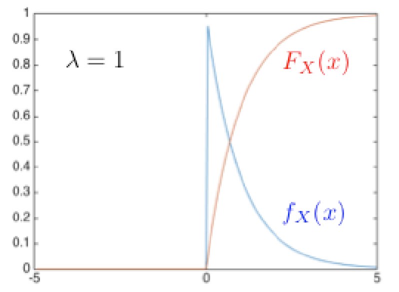
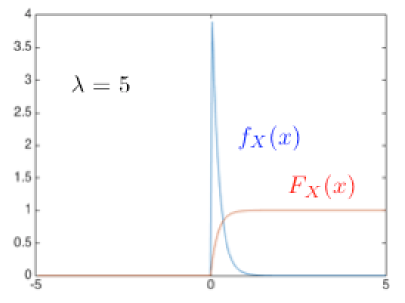
$$f_X(t) = \begin{cases} 0, & \text{if } t < 0 \\ \lambda e^{-\lambda t}, & \text{if } t \geq 0. \end{cases} \qquad F_X(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - e^{-\lambda t}, & \text{if } t \geq 0. \end{cases}$$





Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Expectation & Variance of the Exponential Distribution

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Variance: $1/\lambda^2$

Properties of the Exponential Distribution: Memorylessness

Similar to memorylessness for geometric distributions.

"If your server doesn't fail today, it's in the same state as it was before today."

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Thus, $a \times Expo(\lambda) = Expo(\lambda/a)$.

Properties of the Exponential Distribution: Scaling

Let $X = Expo(\lambda)$ and $Y = aX$ for some $a > 0$. Then

$$\begin{aligned} Pr[Y > t] &= Pr[aX > t] = Pr[X > t/a] \\ &= e^{-\lambda(t/a)} = e^{-(\lambda/a)t} = Pr[Z > t] \text{ for } Z = Expo(\lambda/a). \end{aligned}$$

Thus, $a \times Expo(\lambda) = Expo(\lambda/a)$. Also, $Expo(\lambda) = \frac{1}{\lambda} Expo(1)$.

Normal Distribution

Continuous counterpart to Binomial dist. (more on this later)

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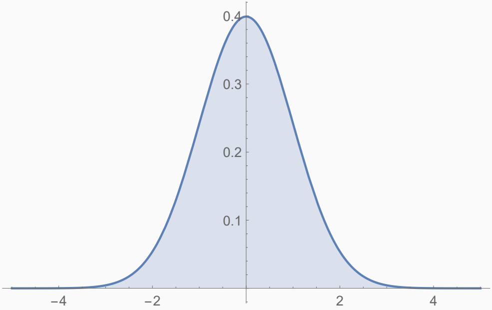

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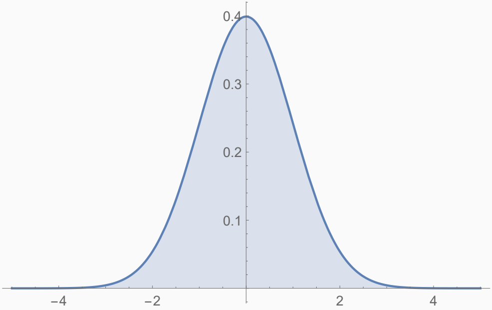

Sometimes called a ”bell curve”. Above: $\mathcal{N}(0, 1)$, the ”standard normal”.

Normal Distribution: Properties

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“n-sigma events” - sometimes used as a shorthand to describe the probability of the event as being the same probability of something falling over n standard deviations away from the mean in a normal distribution.

How Many Sigmas, Exactly?

Range	Expected Fraction of Population Inside Range	Approximate Expected Frequency Outside Range	Approximate Frequency for Daily Event
μ ± 0.5σ	0.382 924 922 548 026	2 in 3	Four times a week
μ ± σ	0.682 689 492 137 086	1 in 3	Twice a week
μ ± 1.5σ	0.866 385 597 462 284	1 in 7	Weekly
μ ± 2σ	0.954 499 736 103 642	1 in 22	Every three weeks
μ ± 2.5σ	0.987 580 669 348 448	1 in 81	Quarterly
μ ± 3σ	0.997 300 203 936 740	1 in 370	Yearly
μ ± 3.5σ	0.999 534 741 841 929	1 in 2149	Every six years
μ ± 4σ	0.999 936 657 516 334	1 in 15 787	Every 43 years (twice in a lifetime)
μ ± 4.5σ	0.999 993 204 653 751	1 in 147 160	Every 403 years (once in the modern era)
μ ± 5σ	0.999 999 426 696 856	1 in 1 744 278	Every 4776 years (once in recorded history)
μ ± 5.5σ	0.999 999 962 020 875	1 in 26 330 254	Every 72 090 years (thrice in history of modern humankind)
μ ± 6σ	0.999 999 998 026 825	1 in 506 797 346	Every 1.38 million years (twice in history of humankind)
μ ± 6.5σ	0.999 999 999 919 680	1 in 12 450 197 393	Every 34 million years (twice since the extinction of dinosaurs)
μ ± 7σ	0.999 999 999 997 440	1 in 390 682 215 445	Every 1.07 billion years (a quarter of Earth's history)
μ ± xσ	$\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$	1 in $\frac{1}{1-\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)}$	Every $\frac{1}{1-\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)}$ days

Central Limit Theorem

Basically: if you take a lot of i.i.d random variables from any* distribution with zero mean and the same variance and sum them up, the sum will converge to a random Gaussian with the same mean and variance.

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Suppose $X_1, X_2, \dots$ are i.i.d. random variables with expectation μ and variance σ^2. Let

$$S_n := \frac{A_n - n\mu}{\sigma\sqrt{n}} = \frac{(\sum_i X_i) - n\mu}{\sigma\sqrt{n}}$$

Then S_n tends towards $\mathcal{N}(0,1)$ as $n \to \infty$.

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Sum of Bernoullis (binomial) tends towards normal!

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Central limit theorem: everything converges to normal if we take enough samples

Today’s Gig: Cauchy Distribution

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$$\tan\theta = t$$

$$\theta = \tan^{-1} t$$
$$d\theta = \frac{1}{1+t^2} dt$$
$$\frac{d\theta}{\pi} = \frac{1}{1+t^2}\frac{dt}{\pi}$$

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Main takeaway: there are some really badly-behaved distributions out there.

Questions?