Alex Psomas: Lecture 18.

Random Variables: Variance

1. Variance
2. Distributions

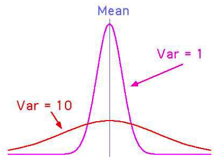
Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) = 0$.

Any other measures???
What else that's informative can we say?

Variance



The variance measures the deviation from the mean value.

Definition: The variance of X is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \\ &= E[X^2] - 2E[X]E[X] + E[X]^2, \\ &= E[X^2] - E[X]^2. \end{aligned}$$

Example

Consider X with

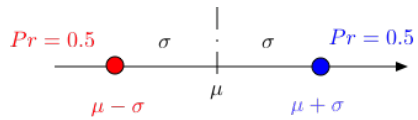
$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= (-1)^2 \times 0.99 + (99)^2 \times 0.01 \approx 100. \\ Var(X) &\approx 100 \implies \sigma(X) \approx 10. \end{aligned}$$

A simple example

This example illustrates the term 'standard deviation.'



Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then, $E[X] = \mu$ and $E[(X - E[X])^2] = \sigma^2$. Hence,

$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X+c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2E(X^2) - c^2(E(X))^2 = c^2(E(X^2) - E(X)^2) \\ &= c^2 Var(X) \\ Var(X+c) &= E((X+c-E(X+c))^2) \\ &= E((X+c-E(X)-c)^2) \\ &= E((X-E(X))^2) = Var(X) \end{aligned}$$

□

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2+2XY+Y^2) \\ &= E(X^2)+2E(XY)+E(Y^2) = E(X^2)+E(Y^2) \\ &= var(X)+var(Y). \end{aligned}$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X)+var(Y)+var(Z)+\cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2+Y^2+Z^2+\cdots+2XY+2XZ+2YZ+\cdots) \\ &= E(X^2)+E(Y^2)+E(Z^2)+\cdots+0+\cdots+0 \\ &= var(X)+var(Y)+var(Z)+\cdots. \end{aligned}$$

□

Distributions

- Bernoulli
- Binomial
- Uniform
- Geometric
- Poisson

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

$p = 0 \implies Var(X) = 0$
$p = 1 \implies Var(X) = 0$

Jacob Bernoulli

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.
So, we get $Pr[\omega] = p^i(1-p)^{n-i}$.
Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{"heads"}] + 0 \times Pr[\text{"tails"}] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p).$

$X = X_1 + X_2 + \ldots X_n.$

X_i and X_j are independent: $Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1]$.

$$Var(X) = Var(X_1 + \cdots X_n) = np(1-p).$$

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$.
In that case,

$$E[X] = \sum_{m=1}^{n} mPr[X = m] = \sum_{m=1}^{n} m \times \frac{1}{n} = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n}\sum_{i=1}^{n} i^2 \\ &= \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$\omega_1 = H$, or
$\omega_2 = T\ H$, or
$\omega_3 = T\ T\ H$, or
$\omega_n = T\ T\ T\ T \cdots T\ H.$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

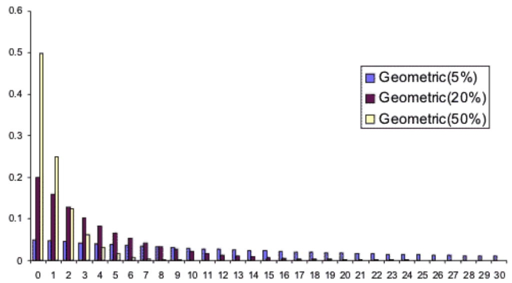
Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] = (1-p)^{n-1}p, \ n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned} E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\ (1-p)E[X] &= \quad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\ pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\ & \quad \text{by subtracting the previous two identities} \\ &= \sum_{n=1}^{\infty}(1-p)^{n-1}p = \sum_{n=1}^{\infty} Pr[X = n] = 1. \end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145\ldots,56765\ldots\}$
Random Variable: X - length of outcome.

Before: $Pr[X \geq n\ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$$Pr[\text{"getting } i\text{th distinct coupon}|\text{"got } i-1 \text{ distinct coupons"}]$$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1,2,\ldots,n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x}dx = \ln(n).$$

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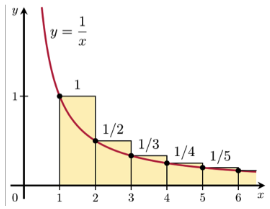
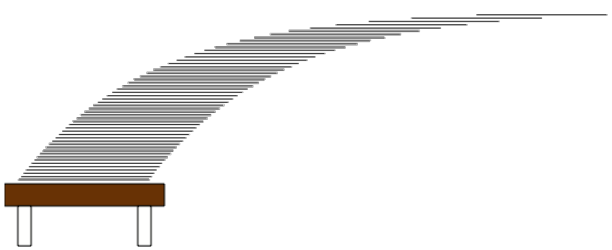


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

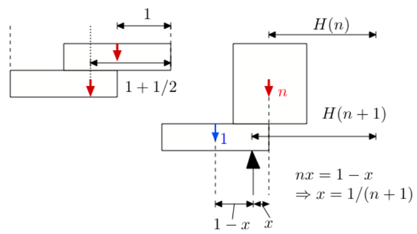
Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Stacking



The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

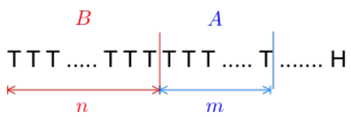
$$Pr[X > n+m|X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned} Pr[X > n+m|X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\ &= \frac{Pr[X > n+m]}{Pr[X > n]} \\ &= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m \\ &= Pr[X > m]. \end{aligned}$$

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m|X > n] = Pr[X > m], m, n \geq 0.$$



The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = Geom(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

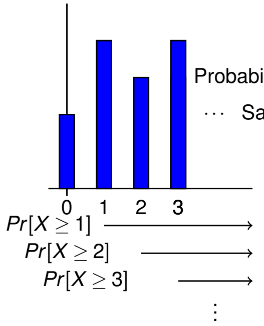
Proof: One has

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\ &= \sum_{i=1}^{\infty} i(Pr[X \geq i] - Pr[X \geq i+1]) \\ &= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1]) \\ &= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} i \times Pr[X \geq i+1] \\ &= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} (i-1) \times Pr[X \geq i] = \sum_{i=1}^{\infty} Pr[X \geq i]. \end{aligned}$$

□

Theorem: For a r.v. X that takes values in $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$



Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + \ldots] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + \ldots \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
&\quad -(p + p(1-p) + p(1-p)^2 + \ldots) \quad 1. \\
pE[X^2] &= 2E[X] - 1 \\
&= 2(\frac{1}{p}) - 1 = \frac{2-p}{p}
\end{aligned}$$

$\Longrightarrow E[X^2] = (2-p)/p^2$ and
$var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.
$\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X]$ when p is small(ish).

Review: Distributions

- $Bern(p) : Pr[X = 1] = p$;
  $E[X] = p$;
  $Var[X] = p(1-p)$;
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0,\ldots,n$;
  $E[X] = np$;
  $Var[X] = np(1-p)$;
- $U[1,\ldots,n] : Pr[X = m] = \frac{1}{n}, m = 1,\ldots,n$;
  $E[X] = \frac{n+1}{2}$;
  $Var[X] = \frac{n^2-1}{12}$;
- $Geom(p) : Pr[X = n] = (1-p)^{n-1}p, n = 1,2,\ldots$;
  $E[X] = \frac{1}{p}$;
  $Var[X] = \frac{1-p}{p^2}$;

Today's gig: Two envelopes problem.

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
3. Birthday Paradox.
4. St. Petersburg paradox.
5. Simpson's paradox.

Today: Two envelopes problem.

Summary

Random Variables

- Variance.
- Distributions.