Alex Psomas: Lecture 18.

$\boxed{\text{Random Variables: Variance}}$

Alex Psomas: Lecture 18.

Random Variables: Variance

1. Variance
2. Distributions

Variance

Flip a coin:

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) =$

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin:

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) =$

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) = 0$.

Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) = 0$.

Any other measures???

Variance

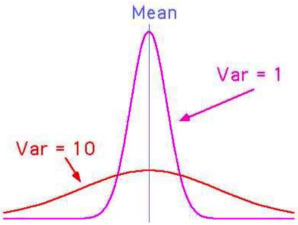
Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) = 0$.

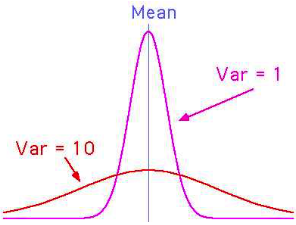
Any other measures???
What else that's informative can we say?

Variance

Variance

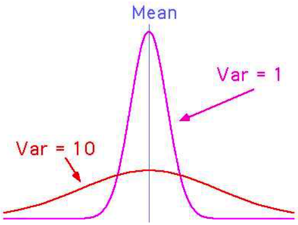


Variance



The variance measures the deviation from the mean value.

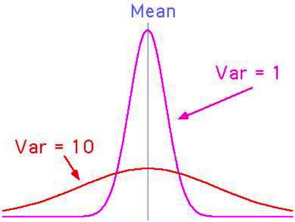
Variance



The variance measures the deviation from the mean value.

Definition: The variance of X is

Variance

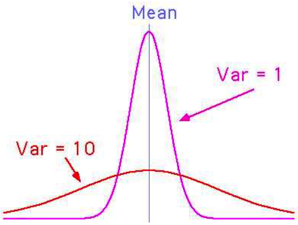


The variance measures the deviation from the mean value.

Definition: The variance of X is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

Variance



The variance measures the deviation from the mean value.

Definition: The variance of X is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$var(X) = E[(X - E[X])^2]$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \end{aligned}$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \end{aligned}$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \end{aligned}$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned}
var(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \\
&= E[X^2] - 2E[X]E[X] + E[X]^2,
\end{aligned}$$

Variance and Standard Deviation

Fact:

$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned}
var(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \\
&= E[X^2] - 2E[X]E[X] + E[X]^2, \\
&= E[X^2] - E[X]^2.
\end{aligned}$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$E[X] \quad = \quad -1 \times 0.99 + 99 \times 0.01 = 0.$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= (-1)^2 \times 0.99 + (99)^2 \times 0.01 \approx 100. \end{aligned}$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

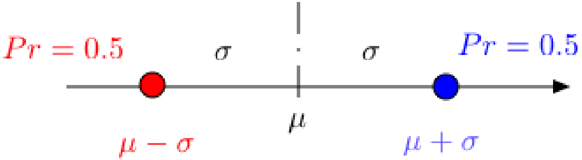
$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= (-1)^2 \times 0.99 + (99)^2 \times 0.01 \approx 100. \\ Var(X) &\approx 100 \implies \sigma(X) \approx 10. \end{aligned}$$

A simple example

This example illustrates the term ‘standard deviation.’

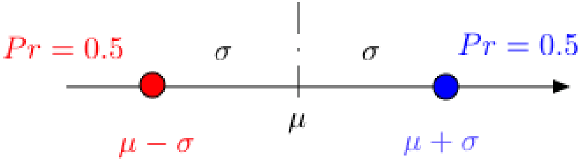
A simple example

This example illustrates the term ‘standard deviation.’



A simple example

This example illustrates the term 'standard deviation.'

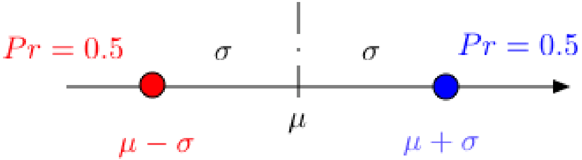


Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

A simple example

This example illustrates the term ‘standard deviation.’



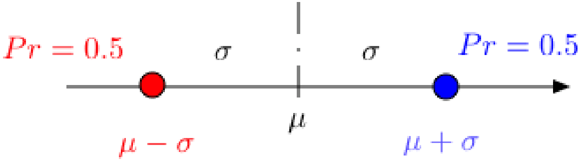
Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then, $E[X] = \mu$ and $E[(X - E[X])^2] = \sigma^2$.

A simple example

This example illustrates the term ‘standard deviation.’



Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then, $E[X] = \mu$ and $E[(X - E[X])^2] = \sigma^2$. Hence,

$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$Var(cX) \quad = \quad E((cX)^2) - (E(cX))^2$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2 E(X^2) - c^2 (E(X))^2 \end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X+c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\ &= c^2 Var(X) \end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned}
Var(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2(E(X^2) - E(X)^2) \\
&= c^2 Var(X) \\
Var(X + c) &= E((X + c - E(X + c))^2)
\end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned}
Var(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
&= c^2 Var(X) \\
Var(X + c) &= E((X + c - E(X + c))^2) \\
&= E((X + c - E(X) - c)^2)
\end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned}
Var(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
&= c^2 Var(X) \\
Var(X + c) &= E((X + c - E(X + c))^2) \\
&= E((X + c - E(X) - c)^2) \\
&= E((X - E(X))^2)
\end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X+c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned}
Var(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
&= c^2 Var(X) \\
Var(X+c) &= E((X+c-E(X+c))^2) \\
&= E((X+c-E(X)-c)^2) \\
&= E((X-E(X))^2) = Var(X)
\end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X+c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned}
Var(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
&= c^2 Var(X) \\
Var(X+c) &= E((X+c-E(X+c))^2) \\
&= E((X+c-E(X)-c)^2) \\
&= E((X-E(X))^2) = Var(X)
\end{aligned}$$

$\square$

Variance of sum of two independent random variables

Variance of sum of two independent random variables

Theorem:

If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$var(X+Y) \quad = \quad E((X+Y)^2)$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$var(X+Y) \quad = \quad E((X+Y)^2) = E(X^2 + 2XY + Y^2)$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) \end{aligned}$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \end{aligned}$$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$Var(X) = E(X^2), Var(Y) = E(Y^2).$$

Hence,

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \\ &= var(X) + var(Y). \end{aligned}$$

Variance of sum of independent random variables

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots .$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X)+var(Y)+var(Z)+\cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots .$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0.$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots .$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$var(X + Y + Z + \cdots) \quad = \quad E((X + Y + Z + \cdots)^2)$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X) + var(Y) + var(Z) + \cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2+Y^2+Z^2+\cdots+2XY+2XZ+2YZ+\cdots) \end{aligned}$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X) + var(Y) + var(Z) + \cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2+Y^2+Z^2+\cdots+2XY+2XZ+2YZ+\cdots) \\ &= E(X^2)+E(Y^2)+E(Z^2)+\cdots+0+\cdots+0 \end{aligned}$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \ldots$ are pairwise independent, then

$$var(X + Y + Z + \cdots) = var(X) + var(Y) + var(Z) + \cdots .$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned}
var(X + Y + Z + \cdots) &= E((X + Y + Z + \cdots)^2) \\
&= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \\
&= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0 \\
&= var(X) + var(Y) + var(Z) + \cdots .
\end{aligned}$$

$\square$

Distributions

- Bernoulli
- Binomial
- Uniform
- Geometric
- Poisson

Bernoulli

Flip a coin, with heads probability p.

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1 - p \end{cases}$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] =$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p)$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1 - p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1 - p) = p$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] =$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

$p = 0$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

$p = 0 \implies Var(X) = 0$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

$p = 0 \implies Var(X) = 0$
$p = 1$

Bernoulli

Flip a coin, with heads probability p.

Random variable X: 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1-p) = p$$

$$Var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Notice that:

$$p = 0 \implies Var(X) = 0$$
$$p = 1 \implies Var(X) = 0$$

Jacob Bernoulli

Binomial

Flip n coins with heads probability p.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \dots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1 - p)$.
So, we get $Pr[\omega] = p^i$

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \dots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.
So, we get $Pr[\omega] = p^i(1-p)^{n-i}$.

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.
So, we get $Pr[\omega] = p^i(1-p)^{n-i}$.
Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

Binomial

Flip n coins with heads probability p.

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i.

How many sample points in event "$X = i$"?
i heads out of n coin flips $\implies \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \ldots\}$
What is the probability of ω if ω has i heads?
Probability of heads in any position is p.
Probability of tails in any position is $(1-p)$.
So, we get $Pr[\omega] = p^i(1-p)^{n-i}$.
Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = 1 \times Pr[\text{"}heads\text{"}] + 0 \times Pr[\text{"}tails\text{"}]$$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[``heads"] + 0 \times Pr[``tails"] = p.$

Expectation of Binomial Distribution

Indicator for the *i*-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{``}heads\text{''}] + 0 \times Pr[\text{``}tails\text{''}] = p.$

Moreover $X = X_1 + \cdots X_n$ and

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{``}heads''] + 0 \times Pr[\text{``}tails''] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n]$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[``heads''] + 0 \times Pr[``tails''] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i]$

Expectation of Binomial Distribution

Indicator for the i-th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[``heads''] + 0 \times Pr[``tails''] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2)$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p)$$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$
$Var(X_i) = p - (E(X_i))^2$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$
$Var(X_i) = p - (E(X_i))^2 = p - p^2$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \dots X_n$.

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \ldots X_n$.

X_i and X_j are independent:

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \ldots X_n$.

X_i and X_j are independent: $Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1]$.

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \ldots X_n$.

X_i and X_j are independent: $Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1]$.

$$Var(X) = Var(X_1 + \cdots X_n)$$

Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p$.
$Var(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \ldots X_n$.

X_i and X_j are independent: $Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1]$.

$$Var(X) = Var(X_1 + \cdots X_n) = np(1-p).$$

Uniform Distribution

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots).

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$.
In that case,

$$E[X] = \sum_{m=1}^{n} mPr[X = m]$$

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1,2,\ldots,6\}$. We say that X is *uniformly distributed* in $\{1,2,\ldots,6\}$.

More generally, we say that X is uniformly distributed in $\{1,2,\ldots,n\}$ if $Pr[X=m]=1/n$ for $m=1,2,\ldots,n$.
In that case,

$$E[X]=\sum_{m=1}^{n} mPr[X=m]=\sum_{m=1}^{n} m\times\frac{1}{n}$$

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1, 2, \ldots, 6\}$. We say that X is *uniformly distributed* in $\{1, 2, \ldots, 6\}$.

More generally, we say that X is uniformly distributed in $\{1, 2, \ldots, n\}$ if $Pr[X = m] = 1/n$ for $m = 1, 2, \ldots, n$.
In that case,

$$E[X] = \sum_{m=1}^{n} m Pr[X = m] = \sum_{m=1}^{n} m \times \frac{1}{n} = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$E[X^2] \quad = \quad \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n}\sum_{i=1}^{n} i^2 \\ &= \frac{1+3n+2n^2}{6}, \end{aligned}$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X=i] = \frac{1}{n}\sum_{i=1}^{n} i^2 \\ &= \frac{1+3n+2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 \\ &= \frac{1+3n+2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

Geometric Distribution

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\omega_1 = H, \text{ or}$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\omega_1 = H, \text{ or}$$

$$\omega_2 = T\ H, \text{ or}$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\omega_1 = H, \text{ or}$$
$$\omega_2 = T\ H, \text{ or}$$
$$\omega_3 = T\ T\ H, \text{ or}$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\omega_1 = H, \text{ or}$$
$$\omega_2 = T\ H, \text{ or}$$
$$\omega_3 = T\ T\ H, \text{ or}$$
$$\omega_n = T\ T\ T\ T\ \cdots\ T\ H.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Let X be the number of flips until the first H. Then, $X(\omega_n) =$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] =$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}
\omega_1 &= H, \text{ or} \\
\omega_2 &= T\ H, \text{ or} \\
\omega_3 &= T\ T\ H, \text{ or} \\
\omega_n &= T\ T\ T\ T\ \cdots\ T\ H.
\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

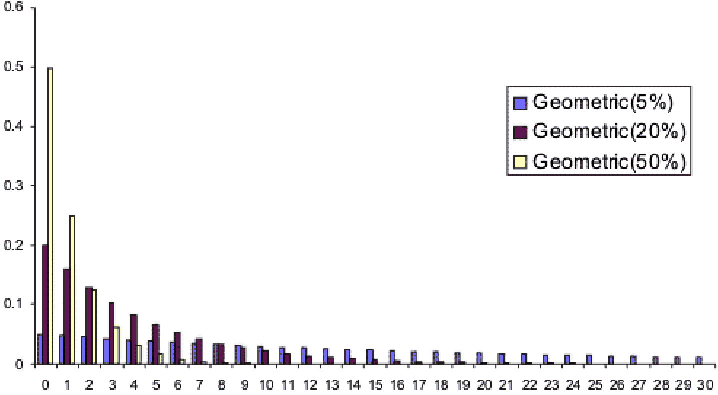
$$Pr[X = n] = (1-p)^{n-1}p, \ n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] =$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p =$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1}$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$.

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty}(1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^{n}.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$.

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$S = 1 + a + a^2 + a^3 + \cdots$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1} p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1} p = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \sum_{n=0}^{\infty} (1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \phantom{1 +{}} a + a^2 + a^3 + a^4 + \cdots \end{aligned}$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty}(1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad\;\; a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] =$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad\; a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} =$$

Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty}(1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

We want to analyze $S := \sum_{n=0}^{\infty} a^n$ for $|a| < 1$. $S = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad\ a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$E[X] \quad = \quad p$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$E[X] \quad = \quad p + 2(1-p)p$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$E[X] \quad = \quad p + 2(1-p)p + 3(1-p)^2 p$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$E[X] \quad = \quad p + 2(1-p)p + 3(1-p)^2 p + 4(1-p)^3 p + \cdots$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned} E[X] &= p + 2(1-p)p + 3(1-p)^2 p + 4(1-p)^3 p + \cdots \\ (1-p)E[X] &= \qquad\quad (1-p)p + 2(1-p)^2 p + 3(1-p)^3 p + \cdots \end{aligned}$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \phantom{p+{}} (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots
\end{aligned}$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X=n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X=n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \qquad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots
\end{aligned}$$

by subtracting the previous two identities

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \qquad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\
& \qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1}p =
\end{aligned}$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \qquad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\
&\qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1}p = \sum_{n=1}^{\infty} Pr[X = n] =
\end{aligned}$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \qquad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\
&\qquad \qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1}p = \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Geometric Distribution: Expectation

$$X \sim Geom(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= \phantom{p+{}} (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + \ (1-p)p + \ (1-p)^2p + \ (1-p)^3p + \cdots \\
&\qquad\qquad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1}p = \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145...,56765...\}$

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145...,56765...\}$
Random Variable: X - length of outcome.

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145..., 56765...\}$
Random Variable: X - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145...,56765...\}$
Random Variable: X - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?

Time to collect coupons

X-time to get n coupons.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}]$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[$"get second distinct coupon"$|$"got first coupon$''] = \frac{n-1}{n}$

$E[X_2]$?

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric !

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! !

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon}''] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! !

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} =$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}}$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Pr[“get second distinct coupon”|“got first coupon$''$] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith distinct coupon|“got $i-1$ distinct coupons”]

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[$"get second distinct coupon"$|$"got first coupon$'']= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[$"getting ith distinct coupon$|$"got $i-1$ distinct coupons"$]$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{“getting } i\text{th distinct coupon}|\text{“got } i-1 \text{ distinct coupons”}]$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i]$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Pr[“get second distinct coupon”|“got first coupon$''$] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith distinct coupon|“got $i-1$ distinct coupons”]

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p}$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{"getting } i\text{th distinct coupon}|\text{"got } i-1 \text{ distinct coupons"}]$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}$,

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon"}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{"getting } i\text{th distinct coupon}|\text{"got } i-1 \text{ distinct coupons"}]$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Pr[“get second distinct coupon”|“got first coupon$''$] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith distinct coupon|“got $i-1$ distinct coupons”]

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$$E[X] = E[X_1] + \cdots + E[X_n] =$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{"get second distinct coupon"}|\text{"got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{"getting } i\text{th distinct coupon}|\text{"got } i-1 \text{ distinct coupons"}]$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1,2,\ldots,n.$

$$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

$Pr[\text{“get second distinct coupon”}|\text{“got first coupon''}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$$Pr[\text{“getting } i\text{th distinct coupon}|\text{“got } i-1 \text{ distinct coupons”}]$$

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1,2,\ldots,n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \end{aligned}$$

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Pr[“get second distinct coupon”|“got first coupon$''$] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith distinct coupon|“got $i-1$ distinct coupons”]

$$= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1,2,\ldots,n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

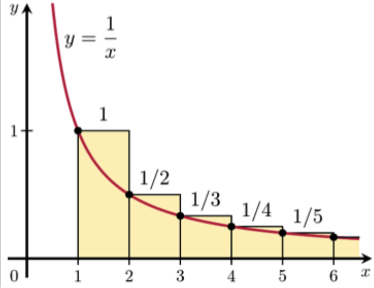
$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

-

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

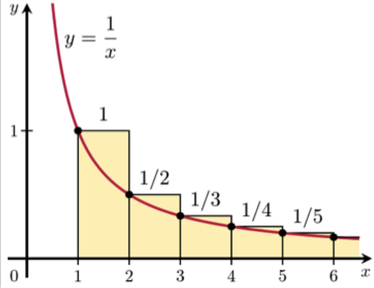
.



Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

.



A good approximation is

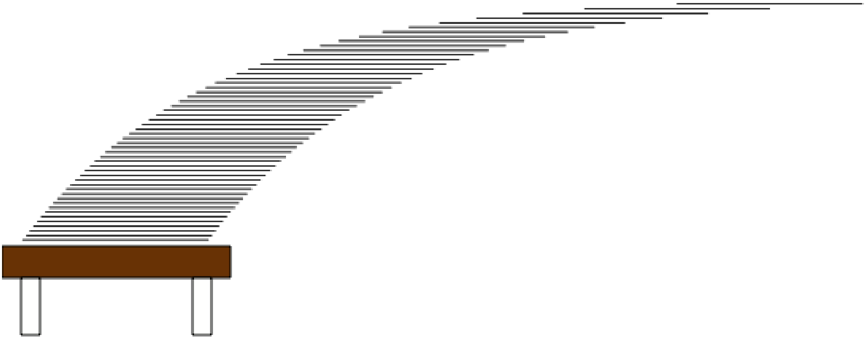
$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

Consider this stack of cards (no glue!):

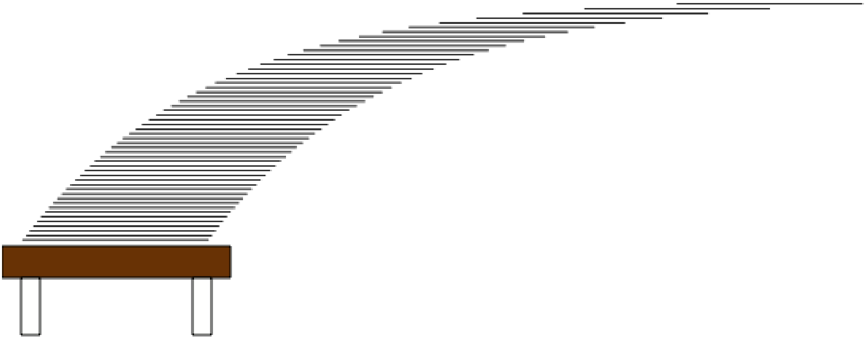
Harmonic sum: Paradox

Consider this stack of cards (no glue!):



Harmonic sum: Paradox

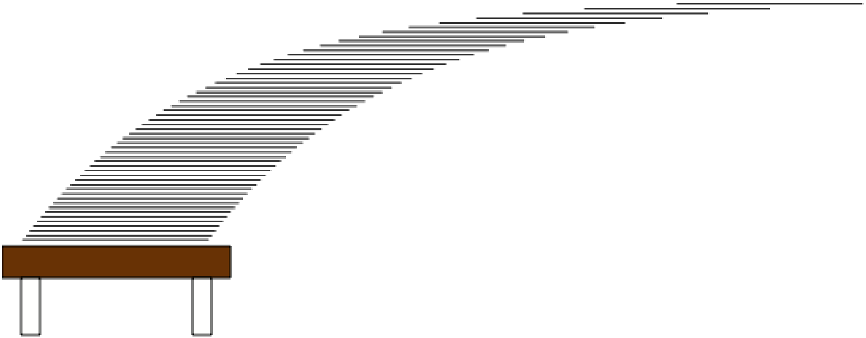
Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table.

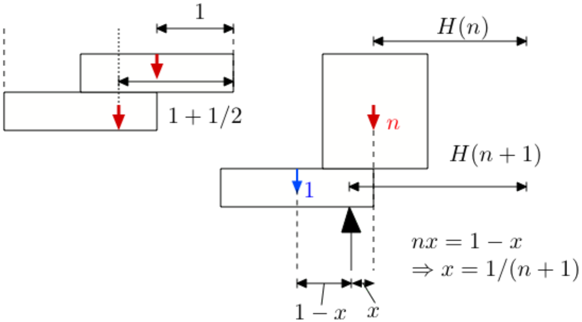
Harmonic sum: Paradox

Consider this stack of cards (no glue!):

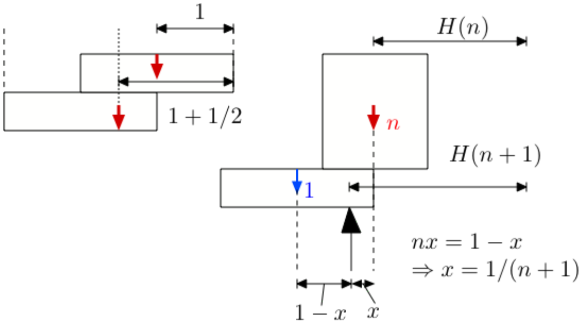


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Stacking

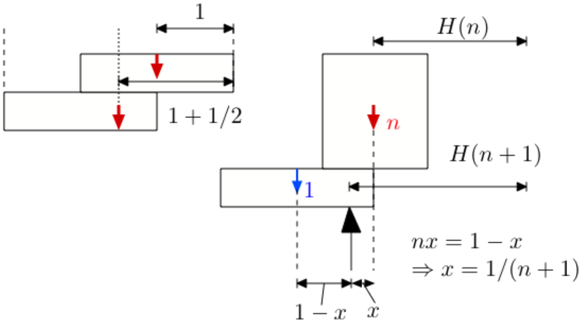


Stacking



The cards have width 2.

Stacking



The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] =$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] =$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$Pr[X > n+m | X > n] \quad =$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$Pr[X > n+m | X > n] \quad = \quad \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]}$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned} Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\ &= \frac{Pr[X > n+m]}{Pr[X > n]} \end{aligned}$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned}
Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\
&= \frac{Pr[X > n+m]}{Pr[X > n]} \\
&= \frac{(1-p)^{n+m}}{(1-p)^n} =
\end{aligned}$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned}
Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\
&= \frac{Pr[X > n+m]}{Pr[X > n]} \\
&= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m
\end{aligned}$$

Geometric Distribution: Memoryless

Let X be $Geom(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

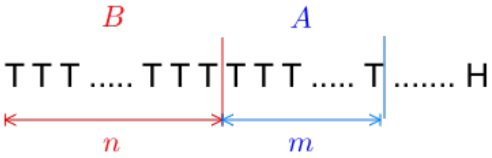
$$\begin{aligned} Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\ &= \frac{Pr[X > n+m]}{Pr[X > n]} \\ &= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m \\ &= Pr[X > m]. \end{aligned}$$

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

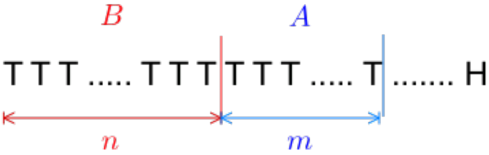
Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$



Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$



The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = Geom(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = Geom(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i}$$

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = Geom(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)} =$$

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = Geom(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\ &= \sum_{i=1}^{\infty} i\,(Pr[X \geq i] - Pr[X \geq i+1]) \end{aligned}$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i(Pr[X \geq i] - Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1])
\end{aligned}$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i\,(Pr[X \geq i] - Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} i \times Pr[X \geq i+1]
\end{aligned}$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i (Pr[X \geq i] - Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} i \times Pr[X \geq i+1] \\
&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} (i-1) \times Pr[X \geq i]
\end{aligned}$$

A side step: Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

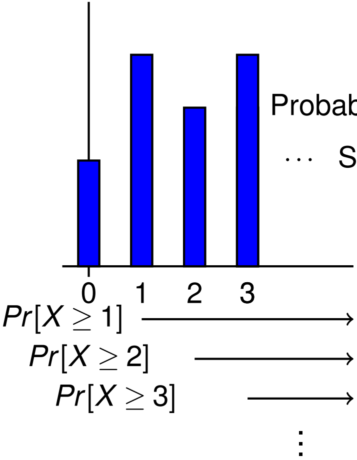
Proof: One has

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i (Pr[X \geq i] - Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1]) \\
&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} i \times Pr[X \geq i+1] \\
&= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} (i-1) \times Pr[X \geq i] = \sum_{i=1}^{\infty} Pr[X \geq i].
\end{aligned}$$

$\square$

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$



Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$.

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$E[X^2] \quad = \quad p+4p(1-p)+9p(1-p)^2+...$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + \ldots]
\end{aligned}$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + ... \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + ...] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + ...
\end{aligned}$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + ... \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + ...] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + ... \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
&\quad -(p + p(1-p) + p(1-p)^2 + ...) \quad 1.
\end{aligned}$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$
\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+... \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+...] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+... \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& \quad -(p+p(1-p)+p(1-p)^2+...) \quad 1.
\end{aligned}
$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$
\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+... \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+...] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+... \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& \quad -(p+p(1-p)+p(1-p)^2+...) \quad 1.
\end{aligned}
$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + ... \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + ...] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + ... \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
& \quad -(p + p(1-p) + p(1-p)^2 + ...) \quad 1. \\
pE[X^2] &= 2E[X] - 1
\end{aligned}$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$
\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + ... \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + ...] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + ... \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
&\quad -(p + p(1-p) + p(1-p)^2 + ...) \quad 1. \\
pE[X^2] &= 2E[X] - 1 \\
&= 2(\frac{1}{p}) - 1
\end{aligned}
$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$
\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + \ldots] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + \ldots \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
&\quad -(p + p(1-p) + p(1-p)^2 + \ldots) \quad 1. \\
pE[X^2] &= 2E[X] - 1 \\
&= 2(\frac{1}{p}) - 1 = \frac{2-p}{p}
\end{aligned}
$$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n\geq 1$. Recall $E[X]=1/p$.

$$
\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+\ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+\ldots] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+\ldots \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
&\quad -(p+p(1-p)+p(1-p)^2+\ldots) \quad 1. \\
pE[X^2] &= 2E[X]-1 \\
&= 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{aligned}
$$

$\Longrightarrow E[X^2]=(2-p)/p^2$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$\begin{array}{rcl}
E[X^2] & = & p+4p(1-p)+9p(1-p)^2+\ldots \\
-(1-p)E[X^2] & = & \quad -[p(1-p)+4p(1-p)^2+\ldots] \\
pE[X^2] & = & p+3p(1-p)+5p(1-p)^2+\ldots \\
 & = & 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
 & & -(p+p(1-p)+p(1-p)^2+\ldots) \quad 1. \\
pE[X^2] & = & 2E[X]-1 \\
 & = & 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{array}$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$
\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+... \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+...] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+... \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& \quad -(p+p(1-p)+p(1-p)^2+...) \quad 1. \\
pE[X^2] &= 2E[X]-1 \\
&= 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{aligned}
$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+\ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+\ldots] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+\ldots \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& \quad -(p+p(1-p)+p(1-p)^2+\ldots) \quad 1. \\
pE[X^2] &= 2E[X]-1 \\
&= 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{aligned}$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$.
$\sigma(X)=\frac{\sqrt{1-p}}{p}$

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$
\begin{aligned}
E[X^2] &= p+4p(1-p)+9p(1-p)^2+... \\
-(1-p)E[X^2] &= \quad -[p(1-p)+4p(1-p)^2+...] \\
pE[X^2] &= p+3p(1-p)+5p(1-p)^2+... \\
&= 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& \quad -(p+p(1-p)+p(1-p)^2+...) \quad 1. \\
pE[X^2] &= 2E[X]-1 \\
&= 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{aligned}
$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$.
$\sigma(X)=\frac{\sqrt{1-p}}{p}\approx E[X]$ when p is small(ish).

Variance of geometric distribution.

X is a geometrically distributed RV with parameter p.
Thus, $Pr[X=n]=(1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X]=1/p$.

$$\begin{array}{rcl}
E[X^2] & = & p+4p(1-p)+9p(1-p)^2+... \\
-(1-p)E[X^2] & = & \quad -[p(1-p)+4p(1-p)^2+...] \\
pE[X^2] & = & p+3p(1-p)+5p(1-p)^2+... \\
& = & 2(p+2p(1-p)+3p(1-p)^2+..) \quad E[X]! \\
& & -(p+p(1-p)+p(1-p)^2+...) \quad 1. \\
pE[X^2] & = & 2E[X]-1 \\
& = & 2(\frac{1}{p})-1=\frac{2-p}{p}
\end{array}$$

$\implies E[X^2]=(2-p)/p^2$ and
$var[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}$.
$\sigma(X)=\frac{\sqrt{1-p}}{p} \approx E[X]$ when p is small(ish).

Review: Distributions

Review: Distributions

- $Bern(p) : Pr[X = 1] = p$;

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \ldots, n;$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1 - p);$
- $Bin(n, p) : Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, m = 0, \ldots, n;$
  $E[X] = np;$
  $Var[X] = np(1 - p);$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p$;
  $E[X] = p$;
  $Var[X] = p(1-p)$;
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \ldots, n$;
  $E[X] = np$;
  $Var[X] = np(1-p)$;
- $U[1, \ldots, n] : Pr[X = m] = \frac{1}{n}, m = 1, \ldots, n$;

Review: Distributions

- $Bern(p) : Pr[X = 1] = p$;
  $E[X] = p$;
  $Var[X] = p(1 - p)$;
- $Bin(n, p) : Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, m = 0, \ldots, n$;
  $E[X] = np$;
  $Var[X] = np(1 - p)$;
- $U[1, \ldots, n] : Pr[X = m] = \frac{1}{n}, m = 1, \ldots, n$;
  $E[X] = \frac{n+1}{2}$;
  $Var[X] = \frac{n^2 - 1}{12}$;

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \ldots, n;$
  $E[X] = np;$
  $Var[X] = np(1-p);$
- $U[1,\ldots,n] : Pr[X = m] = \frac{1}{n}, m = 1, \ldots, n;$
  $E[X] = \frac{n+1}{2};$
  $Var[X] = \frac{n^2-1}{12};$
- $Geom(p) : Pr[X = n] =$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \ldots, n;$
  $E[X] = np;$
  $Var[X] = np(1-p);$
- $U[1, \ldots, n] : Pr[X = m] = \frac{1}{n}, m = 1, \ldots, n;$
  $E[X] = \frac{n+1}{2};$
  $Var[X] = \frac{n^2-1}{12};$
- $Geom(p) : Pr[X = n] = (1-p)^{n-1} p, n = 1, 2, \ldots;$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \ldots, n;$
  $E[X] = np;$
  $Var[X] = np(1-p);$
- $U[1,\ldots,n] : Pr[X = m] = \frac{1}{n}, m = 1, \ldots, n;$
  $E[X] = \frac{n+1}{2};$
  $Var[X] = \frac{n^2-1}{12};$
- $Geom(p) : Pr[X = n] = (1-p)^{n-1} p, n = 1, 2, \ldots;$
  $E[X] = \frac{1}{p};$
  $Var[X] = \frac{1-p}{p^2};$

Review: Distributions

- $Bern(p) : Pr[X = 1] = p;$
  $E[X] = p;$
  $Var[X] = p(1-p);$
- $Bin(n,p) : Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0,\ldots,n;$
  $E[X] = np;$
  $Var[X] = np(1-p);$
- $U[1,\ldots,n] : Pr[X = m] = \frac{1}{n}, m = 1,\ldots,n;$
  $E[X] = \frac{n+1}{2};$
  $Var[X] = \frac{n^2-1}{12};$
- $Geom(p) : Pr[X = n] = (1-p)^{n-1} p, n = 1,2,\ldots;$
  $E[X] = \frac{1}{p};$
  $Var[X] = \frac{1-p}{p^2};$

Today’s gig: Two envelopes problem.

Today's gig: Two envelopes problem.

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
3. Birthday Paradox.
4. St. Petersburg paradox.
5. Simpson's paradox.

Today:

Today's gig: Two envelopes problem.

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
3. Birthday Paradox.
4. St. Petersburg paradox.
5. Simpson's paradox.

Today: Two envelopes problem.

Summary

$\boxed{\text{Random Variables}}$

Summary

$\boxed{\text{Random Variables}}$

- Variance.
- Distributions.