Algebraic Structures and Polynomials

CS70 Summer 2016 - Lecture 7C

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UC Berkeley

Today

Review: Chinese Remainder Theorem and Blum Coin Flipping

Algebraic Structures: Groups, Rings, and Fields

Galois Fields

Polynomials

Applications: Secret Sharing and Erasure Codes

Motivation

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Can we express turn numbers and congruences in modular arithmetic into their own “number system”?

Define *algebraic structures* through axioms that define how they behave.

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Notice that there no commutativity requirement. "$\cdot$" may be non-commutative! If it is commutative, we refer to the group as *abelian*. Formally, Abelian groups must satisfy requires another axiom:

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Also, note that $+$ doesn't necessarily have to represent addition in the normal sense. Elements of G may not even be numbers!

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Examples: With addition and multiplication defined in the usual sense $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is a commutative ring but not a field.

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To be a multiplicative inverse: $\gcd(k,n) = 1$. How do we make sure that this holds for all $k \in \mathbb{Z}_n$?

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A polynomial is said to contain a point (x, y) if $p(x) = y$.

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$$
\begin{aligned}
y_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_d x_1^d \\
&\vdots \\
y_{d+1} &= a_0 + a_1 x_{d+1} + a_2 x_{d+1}^2 + \cdots + a_d x_{d+1}^d
\end{aligned}
$$

Or in matrix form:

$$
\begin{bmatrix}
1 & x_1 & x_1^2 & \ldots & x_1^d \\
1 & x_2 & x_2^2 & \ldots & x_2^d \\
1 & x_3 & x_3^2 & \ldots & x_3^d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{d+1} & x_{d+1}^2 & \ldots & x_{d+1}^d
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{d+1}
\end{bmatrix}
$$

(This matrix is called the *Vandermonde matrix*.)

Lagrange Interpolation (1/2)

How do we know the system of equations on the previous slide has a solution? Unfortunately, we don't. (If you know linear algebra you can prove directly through determinants or through linear independence that the Vandermonde matrix is nonsingular, but that's beyond the scope of this course.)

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$$y_1 \left(\frac{(x-x_2)(x-x_3)\dots(x-x_{d+1})}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_{d+1})} \right)$$

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$$\Delta_1(x) \coloneqq y_1 \frac{(x-x_2)(x-x_3)\dots(x-x_{d+1})}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_{d+1})}$$

Value at x_1?

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Value at x_1? y_1. Value at $x_2, \dots, x_{d+1}$? 0. General idea behind interpolation: make these polynomials for all i and add them together.

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must contain $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$.

$y_i \Delta_i(x_2)$ 1

for $i = 2$: term is y_2

for $i \neq 2$:

$$\frac{(x - x_1)(x - x_2) \cdots}{\text{const}}$$

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When does interpolation work? Notice that we need division also. **Polynomial must be over a field in order to guarantee that interpolation works.**

Uniqueness?

Now we have a polynomial passing through a collection of $d+1$ points. Is it the *only* polynomial passing through these points? Or: we know the system given by the Vandermonde matrix has a solution. Is it a unique solution or is the system underdetermined?

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We already know there is such a polynomial (we constructed one). Remains to show uniqueness.

Proof of Theorem 2

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Suppose that I have two polynomials $p(x)$, $q(x)$ with degree at most d that both contain $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$. Consider $r(x) = p(x) - q(x)$. It suffices to show that $r(x) = 0$ (i.e. $p(x) = q(x)$).

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$r(x_i) = \overbrace{p(x_i)}^{y_i} - \overbrace{q(x_i)}^{y_i}$

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Polynomial Division

Given a degree-d polynomial $f(x)$ and a polynomial $g(x)$ of degree at most d, we can use long division to write $f(x) = g(x)q(x) + r(x)$ for some polynomials $q(x), r(x)$ such that the degree of $r(x)$ is strictly smaller than the degree of $f(x)$. Method: same as elementary-school long division for numbers!

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Example: divide $x^3 - 2x^2 - 4$ by $x - 3$.

$$
\begin{array}{r|l}
 & x^2 + x + 3 \\
\hline
x - 3 & x^3 - 2x^2 + 0x - 4 \\
 & x^3 - 3x^2 \\
\hline
 & x^2 + 0x \\
 & x^2 - 3x \\
\hline
 & 3x - 4 \\
 & 3x - 9 \\
\hline
 & 5
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$$

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\underline{x^3 - 3x^2 \phantom{{}+0x-4}} \\
+x^2 + 0x \phantom{{}-4} \\
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+3x - 4 \\
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\begin{array}{rrrrr}
 & & x^2 & +\ x & +3 \\ \cline{2-5}
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 & x^3 & -\ 3x^2 & & \\ \cline{1-3}
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So $x^3 - 2x^2 - 4 = (x-3)(x^2 + x + 3) + 5$.

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Now suppose for induction that the lemma holds for some d. It suffices to show that we can express a degree-$d+1$ polynomial $p(x)$ with $d+1$ roots $a_1, \ldots, a_{d+1}$ as $p(x) = c(x-a_1)\ldots(x-a_{d+1})$.

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Up next...

Counting polynomials.

Applications: Shamir's secret sharing and error-correcting codes.

Polynomial identity testing and the Schwartz-Zippel lemma