

Alex Psomas: Lecture 16.

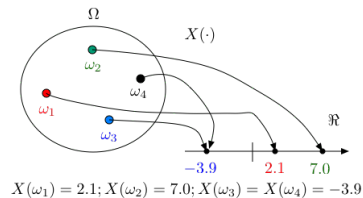
Random Variables

- ▶ Regrade requests open.
- ▶ Quiz due tomorrow.
- ▶ Quiz coming out today.
- ▶ Non-technical office hours tomorrow 1-3pm.
- ▶ Anonymous questionnaire tonight or tomorrow.

Random Variables.

A **random variable**, X , for an experiment with sample space Ω is a **function** $X: \Omega \rightarrow \mathbb{R}$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.



The function $X(\cdot)$ is defined on the outcomes Ω .

A random variable X is **not random, not a variable!**

What varies at random (from experiment to experiment)? The outcome!

Random Variables

1. Random Variables.
2. Distributions.
3. Combining random variables.
4. Expectation

Example 1 of Random Variable

Experiment: roll two dice.

Sample Space: $\{(1,1), (1,2), \dots, (6,6)\} = \{1, \dots, 6\}^2$

Random Variable X : number of pips.

$X(1,1) = 2$

$X(1,2) = 3,$

\vdots

$X(6,6) = 12,$

$X(a,b) = a + b, (a,b) \in \Omega.$

Questions about outcomes ...

Experiment: roll two dice.

Sample Space: $\{(1,1), (1,2), \dots, (6,6)\} = \{1, \dots, 6\}^2$

How many dots?

Experiment: flip 100 coins.

Sample Space: $\{HHH \dots H, THH \dots H, \dots, TTT \dots T\}$

How many heads in 100 coin tosses?

Experiment: choose a random student in cs70.

Sample Space: $\{Peter, Phoebe, \dots\}$

What midterm score?

Experiment: hand back assignments to 3 students at random.

Sample Space: $\{123, 132, 213, 231, 312, 321\}$

How many students get back their own assignment?

In each scenario, each outcome gives a number.

The number is a (known) function of the outcome.

Example 2 of Random Variable

Experiment: flip three coins

Sample Space: $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$

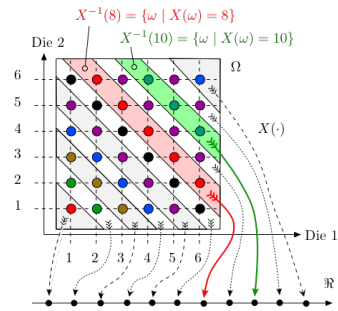
Winnings: if win 1 on heads, lose 1 on tails: X

$X(HHH) = 3$ $X(THH) = 1$ $X(HTH) = 1$ $X(TTH) = -1$

$X(HHT) = 1$ $X(THT) = -1$ $X(HTT) = -1$ $X(TTT) = -3$

Number of dots in two dice.

"What is the likelihood of seeing n dots?"



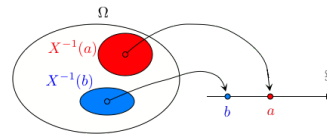
$$Pr[X = 10] = 3/36 = Pr[X^{-1}(10)] = \sum_{\omega \in X^{-1}(10)} Pr[\omega]$$

$$Pr[X = 8] = 5/36 = Pr[X^{-1}(8)].$$

Distribution

The probability of X taking on a value a .

Definition: The **distribution** of a random variable X , is $\{(a, Pr[X = a]) : a \in \mathcal{A}\}$, where \mathcal{A} is the range of X .



$$Pr[X = a] := Pr[X^{-1}(a)] \text{ where } X^{-1}(a) := \{\omega \mid X(\omega) = a\}.$$

Handing back assignments

Experiment: hand back assignments to 3 students at random.

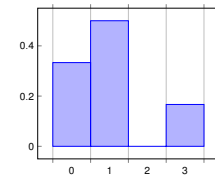
Sample Space: $\Omega = \{123, 132, 213, 231, 312, 321\}$

How many students get back their own assignment?

Random Variable: values of $X(\omega) : \{3, 1, 1, 0, 0, 1\}$

Distribution:

$$X = \begin{cases} 0, & \text{w.p. } 1/3 \\ 1, & \text{w.p. } 1/2 \\ 3, & \text{w.p. } 1/6 \end{cases}$$



Flip three coins

Experiment: flip three coins

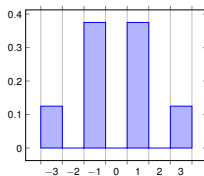
Sample Space: $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$

Winnings: if win 1 on heads, lose 1 on tails. X

Random Variable: $\{3, 1, 1, -1, 1, -1, -1, -3\}$

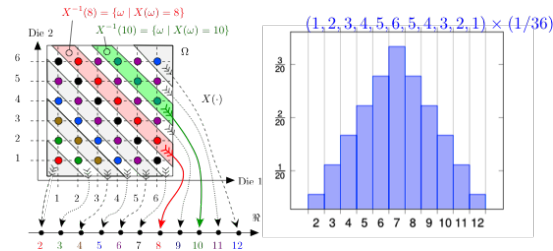
Distribution:

$$X = \begin{cases} -3, & \text{w.p. } 1/8 \\ -1, & \text{w.p. } 3/8 \\ 1, & \text{w.p. } 3/8 \\ 3, & \text{w.p. } 1/8 \end{cases}$$



Number of dots.

Experiment: roll two dice.



The Bernoulli distribution

Flip a coin, with heads probability p .

Random variable X : 1 is heads, 0 if not heads.

X has the Bernoulli distribution.

We will also call this an **indicator random variable**. It indicates whether the event happened.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

The binomial distribution.

Flip n coins with heads probability p .

Random variable: number of heads.

Binomial Distribution: $Pr[X = i]$, for each i .

How many sample points in event " $X = i$ "?

i heads out of n coin flips $\Rightarrow \binom{n}{i}$

Sample space: $\Omega = \{HHH...HH, HHH...HT, \dots\}$

What is the probability of ω if ω has i heads?

Probability of heads in any position is p .

Probability of tails in any position is $(1-p)$.

So, we get $Pr[\omega] = p^i(1-p)^{n-i}$.

Probability of " $X = i$ " is sum of $Pr[\omega]$, $\omega \in "X = i"$.

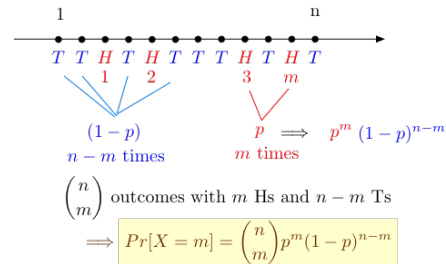
$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \dots, n: B(n, p) \text{ distribution}$$

Combining Random Variables

Other random variables:

- ▶ $X^k : \Omega \rightarrow \Re$ is defined by $X^k(\omega) = [X(\omega)]^k$.
In the dice example, $X^3(a, b) = a^3$.
- ▶ $(X-2)^2 + 4XY$ assigns the value $(X(\omega)-2)^2 + 4X(\omega)Y(\omega)$ to ω .
- ▶ $g(X, Y, Z)$ assigned the value $g(X(\omega), Y(\omega), Z(\omega))$ to ω .

The binomial distribution.



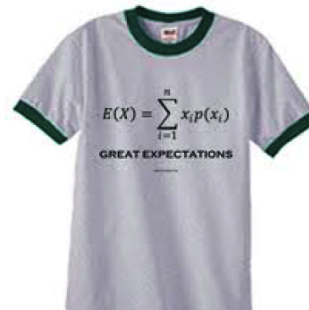
Expectation.

How did people do on the midterm?

Distribution.

Summary of distribution?

Average!



Combining Random Variables.

Let X and Y be two RV on the same probability space.

That is, $X : \Omega \rightarrow \Re$ assigns the value $X(\omega)$ to ω . Also,

$Y : \Omega \rightarrow \Re$ assigns the value $Y(\omega)$ to ω .

Then $Z = X + Y$ is a random variable: It assigns the value

$$Z(\omega) = X(\omega) + Y(\omega)$$

to outcome ω .

Experiment: Roll two dice. X = outcome of first die, Y = outcome of second die.

$$X(a, b) = a \text{ and } Y(a, b) = b \text{ for } (a, b) \in \Omega = \{1, \dots, 6\}^2.$$

Then $Z = X + Y$ = sum of two dice is defined by

$$Z(a, b) = X(a, b) + Y(a, b) = a + b.$$

Expectation - Intuition

Flip a loaded coin with $Pr[H] = p$ a large number N of times.

We expect heads to come up a fraction p of the times and tails a fraction $1-p$.

Say that you get 5 for every H and 3 for every T .

If there are N_H outcomes equal to H and N_T outcomes equal to T , you collect

$$5 \times N_H + 3 \times N_T.$$

Your average gain per experiment is

$$\frac{5N_H + 3N_T}{N}.$$

Since $\frac{N_H}{N} \approx p = Pr[X = 5]$ and $\frac{N_T}{N} \approx 1-p = Pr[X = 3]$, we find that the average gain per outcome is approximately equal to

$$5Pr[X = 5] + 3Pr[X = 3].$$

We use this frequentist interpretation as a definition.

Expectation - Definition

Definition: The **expected value** of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

a in the range of X .

The expected value is also called the mean.

According to our intuition, we expect that if we repeat an experiment a large number N of times and if X_1, \dots, X_N are the successive values of the random variable, then

$$\frac{X_1 + \dots + X_N}{N} \approx E[X].$$

That is indeed the case, in the same way that the fraction of times that $X = x$ approaches $Pr[X = x]$.

This (nontrivial) result is called the **Law of Large Numbers**.

Expectation and Average.

There are n students in the class;

$X(m)$ = score of student m , for $m = 1, 2, \dots, n$.

"Average score" of the n students: add scores and divide by n :

$$\text{Average} = \frac{X(1) + X(2) + \dots + X(n)}{n}.$$

Experiment: choose a student uniformly at random.

Uniform sample space: $\Omega = \{1, 2, \dots, n\}$, $Pr[\omega] = 1/n$, for all ω .

Random Variable: midterm score: $X(\omega)$.

Expectation:

$$E(X) = \sum_{\omega} X(\omega) Pr[\omega] = \sum_{\omega} X(\omega) \frac{1}{n}.$$

Hence,

$$\text{Average} = E(X).$$

Our intuition matches the math.

Expectation: A Useful Fact

Theorem:

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \times Pr[\omega].$$

Proof:

$$\begin{aligned} E[X] &= \sum_a a \times Pr[X = a] \\ &= \sum_a a \times \sum_{\omega: X(\omega)=a} Pr[\omega] \\ &= \sum_a \sum_{\omega: X(\omega)=a} a \times Pr[\omega] \\ &= \sum_a \sum_{\omega: X(\omega)=a} X(\omega) Pr[\omega] \\ &= \sum_{\omega} X(\omega) Pr[\omega] \end{aligned}$$

Handing back assignments

We give back assignments randomly to three students.

What is the expected number of students that get their own assignment back?

The expected number of **fixed points** in a random permutation.

Expected value of a random variable:

$$E[X] = \sum_a a \times Pr[X = a].$$

For 3 students (permutations of 3 elements):

$$Pr[X = 3] = 1/6, Pr[X = 1] = 3/6, Pr[X = 0] = 2/6.$$

$$E[X] = 3 \times \frac{1}{6} + 1 \times \frac{3}{6} + 0 \times \frac{2}{6} = 1.$$

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

X = number of H 's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$.

Thus,

$$\sum_{\omega} X(\omega) Pr[\omega] = \{3 + 2 + 2 + 2 + 1 + 1 + 1 + 0\} \times \frac{1}{8}.$$

Also,

$$\sum_a a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

Win or Lose.

Expected winnings for heads/tails games, with 3 flips?

Every time it's H , I get 1. Every time it's T , I lose 1.

$$E[X] = 3 \times \frac{1}{8} + 1 \times \frac{3}{8} - 1 \times \frac{3}{8} - 3 \times \frac{1}{8} = 0.$$

Can you ever win 0?

Apparently: expected value is not a common value, by any means.

Expectation

Recall: $X : \Omega \rightarrow \mathfrak{R}; Pr[X = a] = Pr[X^{-1}(a)]$;

Definition: The **expectation** of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Indicator:

Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the **indicator** of the event A .

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

The random variable X is sometimes written as

$$1_{\{\omega \in A\}} \text{ or } 1_A(\omega).$$

Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

X = number of students that get their own assignment back.

$X = X_1 + \dots + X_n$ where

$X_m = 1_{\{\text{student } m \text{ gets his/her own assignment back}\}}$.

One has

$$\begin{aligned} E[X] &= E[X_1 + \dots + X_n] \\ &= E[X_1] + \dots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because all the } X_m \text{ have the same distribution} \\ &= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\ &= n(1/n), \text{ because student 1 is equally likely} \\ &\quad \text{to get any one of the } n \text{ assignments} \\ &= 1. \end{aligned}$$

Note that linearity holds even though the X_m are not independent (whatever that means).

Linearity of Expectation

Theorem:

$$E[X] = \sum_{\omega} X(\omega) \times Pr[\omega].$$

Theorem: Expectation is linear

$$E[a_1 X_1 + \dots + a_n X_n] = a_1 E[X_1] + \dots + a_n E[X_n].$$

Proof:

$$\begin{aligned} E[a_1 X_1 + \dots + a_n X_n] &= \sum_{\omega} (a_1 X_1 + \dots + a_n X_n)(\omega) Pr[\omega] \\ &= \sum_{\omega} (a_1 X_1(\omega) + \dots + a_n X_n(\omega)) Pr[\omega] \\ &= a_1 \sum_{\omega} X_1(\omega) Pr[\omega] + \dots + a_n \sum_{\omega} X_n(\omega) Pr[\omega] \\ &= a_1 E[X_1] + \dots + a_n E[X_n]. \end{aligned}$$

□

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p . X - number of heads

Binomial Distribution: $Pr[X = i]$, for each i .

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X = i] = \sum_i i \times \binom{n}{i} p^i (1-p)^{n-i}.$$

No no no no no. **NO** ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = 1 \times Pr[\text{"heads"}] + 0 \times Pr[\text{"tails"}] = p.$$

Moreover $X = X_1 + \dots + X_n$ and

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n] = n \times E[X_i] = np.$$

Using Linearity - 1: Dots on dice

Roll a die n times.

X_m = number of dots on roll m .

$X = X_1 + \dots + X_n$ = total number of dots in n rolls.

$$\begin{aligned} E[X] &= E[X_1 + \dots + X_n] \\ &= E[X_1] + \dots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because the } X_m \text{ have the same distribution} \end{aligned}$$

Now,

$$E[X_1] = 1 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Today's gig: St. Petersburg paradox

I offer the following game:

We start with a pot of 2 dollars.

Flip a fair coin. If it's tails, you take the pot. If it's heads, I double the pot.

So, if the sequence is *HHT*, you make 8 dollars.

How much would you be willing to pay?

Today's gig: St. Petersburg paradox

Well, how much money should you expect to make?

Let X be the random variable indicating how much money you make for each outcome:

$X = 2$ with probability $\frac{1}{2}$

$X = 4$ with probability $\frac{1}{4}$

$X = 8$ with probability $\frac{1}{8}$

$$E[X] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \dots$$

$$= 1 + 1 + 1 + \dots = \infty$$

So, if you were rational you would be willing to pay anything!

Is there a trick here?

Today's gig: St. Petersburg paradox

What if I didn't have infinite money?

Banker	Bankroll	Expected value of lottery
Friendly game	\$100	\$7.56
Millionaire	\$1,000,000	\$20.91
Billionaire	\$1,000,000,000	\$30.86
Bill Gates (2015)	\$79,200,000,000 ^[5]	\$37.15
U.S. GDP (2007)	\$13.8 trillion ^[6]	\$44.57
World GDP (2007)	\$54.3 trillion ^[6]	\$46.54
Googolaire	10^{100}	\$333.14

Summary

Random Variables

- ▶ A random variable X is a function $X : \Omega \rightarrow \Re$.
- ▶ $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- ▶ $Pr[X \in A] := Pr[X^{-1}(A)]$.
- ▶ The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathcal{A}\}$.
- ▶ $g(X, Y, Z)$ assigns the value
- ▶ $E[X] := \sum_a a Pr[X = a]$.
- ▶ Expectation is Linear.
- ▶ $B(n, p)$.