Alex Psomas: Lecture 17.

Random Variables: Expectation, Variance

1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance

Random Variables: Definitions

Definition
A random variable, X, for a random experiment with sample space Ω is a variable that takes as value one of the random samples.
NO!

Random Variables: Definitions

Definition
A random variable, X, for a random experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions
(a) For $a \in \Re$, one defines the **event**

$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}.$$

(b) For $A \subset \Re$, one defines the **event**

$$X^{-1}(A) := \{\omega \in \Omega \mid X(\omega) \in A\}.$$

(c) The probability that $X = a$ is defined as

$$Pr[X = a] = Pr[X^{-1}(a)].$$

(d) The probability that $X \in A$ is defined as

$$Pr[X \in A] = Pr[X^{-1}(A)].$$

(e) The distribution of a random variable X, is

$$\{(a, Pr[X = a]) : a \in \mathscr{A}\},$$

where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X =$ number of H's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$.

- Range of X? $\{0, 1, 2, 3\}$. All the values X can take.
- $X^{-1}(2)$? $X^{-1}(2) = \{HHT, HTH, THH\}$. All the **outcomes** ω such that $X(\omega) = 2$.
- Is $X^{-1}(1)$ an event? **YES**. It's a subset of the outcomes.
- $Pr[X]$? This doesn't make any sense bro....
- $Pr[X = 2]$?

$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

Random Variables: Definitions

Definition
Let X, Y, Z be random variables on Ω and $g : \Re^3 \to \Re$ a function. Then $g(X, Y, Z)$ is the random variable that assigns the value $g(X(\omega), Y(\omega), Z(\omega))$ to ω.

Thus, if $V = g(X, Y, Z)$, then $V(\omega) := g(X(\omega), Y(\omega), Z(\omega))$.

Examples:

- X^k
- $(X - a)^2$
- $a + bX + cX^2 + (Y - Z)^2$
- $(X - Y)^2$
- $X \cos(2\pi Y + Z)$.

Expectation - Definition

Definition: The **expected value** (or mean, or expectation) of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Theorem:

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

An Example

Flip a fair coin three times.
$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. $X =$ number of H's: $\{3, 2, 2, 2, 1, 1, 1, 0\}$. Thus,

$$\sum_{\omega} X(\omega)Pr[\omega] = 3\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 0\frac{1}{8}.$$

Also,

$$\sum_{a} a \times Pr[X = a] = 3\frac{1}{8} + 2\frac{3}{8} + 1\frac{3}{8} + 0\frac{1}{8}.$$

Win or Lose.

Expected winnings for heads/tails games, with 3 flips?
Recall the definition of the random variable X:
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \to \{3, 1, 1, -1, 1, -1, -1, -3\}$.

$$E[X] = 3\frac{1}{8} + 1\frac{3}{8} - 1\frac{3}{8} - 3\frac{1}{8} = 0.$$

Can you ever win 0?

Apparently: Expected value is not a common value. It doesn't have to be in the range of X.

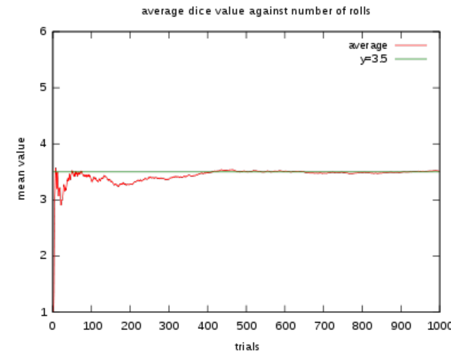
The expected value of X is not the value that you expect!
It is the average value per experiment, if you perform the experiment many times. Let X_1 be your winnings the first time you play the game, X_2 are your winnings the second time you play the game, and so on. (Notice that X_i's have the same distribution!) When $n \gg 1$:

$$\frac{X_1 + \cdots + X_n}{n} \to 0$$

The fact that this average converges to $E[X]$ is a theorem: the Law of Large Numbers. (See later.)

Law of Large Numbers

An Illustration: Rolling Dice



Indicators

Definition
Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1X_1 + \cdots + a_nX_n] = a_1E[X_1] + \cdots + a_nE[X_n].$$

Proof:

$$\begin{aligned} & E[a_1X_1 + \cdots + a_nX_n] \\ &= \sum_{\omega}(a_1X_1 + \cdots + a_nX_n)(\omega)Pr[\omega] \\ &= \sum_{\omega}(a_1X_1(\omega) + \cdots + a_nX_n(\omega))Pr[\omega] \\ &= a_1\sum_{\omega}X_1(\omega)Pr[\omega] + \cdots + a_n\sum_{\omega}X_n(\omega)Pr[\omega] \\ &= a_1E[X_1] + \cdots + a_nE[X_n]. \end{aligned}$$

□

Note: If we had defined $Y = a_1X_1 + \cdots + a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y = y]$, we would have been in trouble!

Using Linearity - 1: Dots on dice

Roll a die n times.

X_m = number of dots on roll m.

$X = X_1 + \cdots + X_n$ = total number of dots in n rolls.

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because the } X_m \text{ have the same distribution} \end{aligned}$$

Now,

$$E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

Using Linearity - 2: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X = i]$, for each i.

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X = i] = \sum_i i \times \binom{n}{i} p^i (1-p)^{n-i}.$$

Better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{"heads"}] + 0 \times Pr[\text{"tails"}] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Using Linearity - 3: The birthday paradox

Let X be the random variable indicating the number of pairs of people, in a group of k people, sharing the same birthday. What's $E(X)$?

Let $X_{i,j}$ be the indicator random variable for the event that two people i and j have the same birthday. $X = \sum_{i,j} X_{i,j}$.

$$E[X] = E[\sum_{i,j} X_{i,j}]$$
$$= \sum_{i,j} E[X_{i,j}]$$
$$= \sum_{i,j} Pr[X_{i,j}]$$
$$= \sum_{i,j} \frac{1}{365} = \binom{k}{2} \frac{1}{365} = \frac{k(k-1)}{2} \frac{1}{365}$$

For a group of 28 it's about 1. For 100 it's 13.5. For 280 it's 107.

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

$$E[g(X)] = \sum_{x \in \mathscr{A}(X)} g(x) Pr[X = x].$$

Proof:

$$\begin{aligned} E[g(X)] &= \sum_\omega g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\ &= \sum_x g(x) Pr[X = x]. \end{aligned}$$

□

An Example

Let X be uniform in $\{-2,-1,0,1,2,3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\ &= \{4+1+0+1+4+9\} \frac{1}{6} = \frac{19}{6}. \end{aligned}$$

Method 1 - We find the distribution of $Y = X^2$:

$$Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}$$

Thus,

$$E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.$$

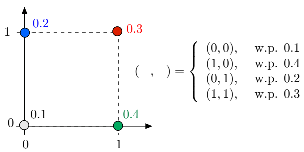
Calculating $E[g(X,Y,Z)]$

We have seen that $E[g(X)] = \sum_x g(x) Pr[X = x]$.

Using a similar derivation, one can show that

$$E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) Pr[X = x, Y = y, Z = z].$$

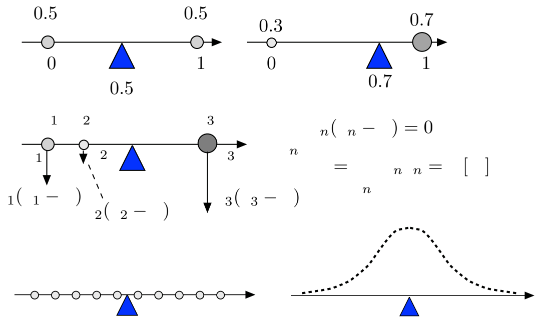
An Example. Let X, Y be as shown below:



$$\begin{aligned} E[\cos(2\pi X + \pi Y)] &= 0.1\cos(0) + 0.4\cos(2\pi) + 0.2\cos(\pi) + 0.3\cos(3\pi) \\ &= 0.1 \times 1 + 0.4 \times 1 + 0.2 \times (-1) + 0.3 \times (-1) = 0. \end{aligned}$$

Center of Mass

The expected value has a *center of mass* interpretation:



Best Guess: Least Squares

If you only know the distribution of X, it seems that $E[X]$ is a 'good guess' for X.

The following result makes that idea precise.

Theorem
The value of a that minimizes $E[(X-a)^2]$ is $a = E[X]$.

Unfortunately, we won't talk about this in this class...

Independent Random Variables.

Definition: Independence

The random variables X and Y are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$

Fact:

X, Y are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b], \text{ for all } a \text{ and } b.$$

Obvious.

Independence: Examples

Example 1
Roll two die. X = number of dots on the first one, Y = number of dots on the other one. X, Y are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. X = total number of dots, Y = number of dots on die 1 minus number on die 2. X and Y are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$.

Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

Mean of product of independent RV

Theorem
Let X, Y be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

Proof:
Recall that $E[g(X,Y)] = \sum_{x,y} g(x,y)Pr[X = x, Y = y]$. Hence,

$$\begin{aligned} E[XY] &= \sum_{x,y} xyPr[X = x, Y = y] = \sum_{x,y} xyPr[X = x]Pr[Y = y], \text{ by ind.} \\ &= \sum_x [\sum_y xyPr[X = x]Pr[Y = y]] = \sum_x [xPr[X = x](\sum_y yPr[Y = y])] \\ &= \sum_x [xPr[X = x]E[Y]] = E[X]E[Y]. \end{aligned}$$

$\square$

Examples

(1) Assume that X, Y, Z are (pairwise) independent, with $E[X] = E[Y] = E[Z] = 0$ and $E[X^2] = E[Y^2] = E[Z^2] = 1$.

Wait. Isn't X independent with itself? No. If I tell you the value of X, then you know the value of X.

Then

$$\begin{aligned} &E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ] \\ &= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0 \\ &= 14. \end{aligned}$$

(2) Let X, Y be independent and take values from $\{1, 2, \ldots n\}$ uniformly at random. Then

$$\begin{aligned} E[(X - Y)^2] &= E[X^2 + Y^2 - 2XY] = 2E[X^2] - 2E[X]^2 \\ &= \frac{1 + 3n + 2n^2}{3} - \frac{(n+1)^2}{2}. \end{aligned}$$

Mutually Independent Random Variables

Definition

X, Y, Z are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$

Theorem
The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.
Proof:

$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

□

Functions of pairwise independent RVs

If X, Y, Z are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

Example: Flip two fair coins,
$X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, X, Y, Z are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of X.

Functions of mutually independent RVs

One has the following result:
Theorem
Functions of disjoint collections of mutually independent random variables are mutually independent.
Example:
Let $\{X_n, n \geq 1\}$ be mutually independent. Then,
$Y_1 := X_1 X_2 (X_3 + X_4)^2, Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\}, Y_3 := X_9 \cos(X_{10} + X_{11})$
are mutually independent.
Proof:
Let $B_1 := \{(x_1, x_2, x_3, x_4) \mid x_1 x_2 (x_3 + x_4)^2 \in A_1\}$. Similarly for B_2, B_3.
Then

$$\begin{aligned} & Pr[Y_1 \in A_1, Y_2 \in A_2, Y_3 \in A_3] \\ &= Pr[(X_1, \ldots, X_4) \in B_1, (X_5, \ldots, X_8) \in B_2, (X_9, \ldots, X_{11}) \in B_3] \\ &= Pr[(X_1, \ldots, X_4) \in B_1] Pr[(X_5, \ldots, X_8) \in B_2] Pr[(X_9, \ldots, X_{11}) \in B_3] \\ &= Pr[Y_1 \in A_1] Pr[Y_2 \in A_2] Pr[Y_3 \in A_3] \end{aligned}$$

□

Operations on Mutually Independent Events

Theorem

Operations on disjoint collections of mutually independent events produce mutually independent events.

For instance, if A, B, C, D, E are mutually independent, then $A \Delta B, C \setminus D, \bar{E}$ are mutually independent.

Product of mutually independent RVs

Theorem
Let $X_1, \ldots, X_n$ be mutually independent RVs. Then,

$$E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n].$$

Proof:

Assume that the result is true for n. (It is true for $n = 2$.)

Then, with $Y = X_1 \cdots X_n$, one has

$$\begin{aligned} E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\ &= E[Y]E[X_{n+1}], \\ & \quad \text{because } Y, X_{n+1} \text{ are independent} \\ &= E[X_1] \cdots E[X_n]E[X_{n+1}]. \end{aligned}$$

□

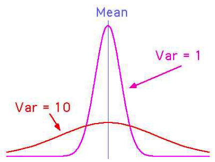
Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.
Let X be the RV indicating how much money you make.
$E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars.
Let Y be the RV indicating how much money you make.
$E(Y) = 0$.

Any other measures??? What else that's informative can we say?

Variance



The variance measures the deviation from the mean value.

Definition: The variance of X is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

Variance and Standard Deviation

Fact:

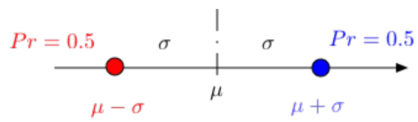
$$var[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2 \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \\ &= E[X^2] - 2E[X]E[X] + E[X]^2, \\ &= E[X^2] - E[X]^2. \end{aligned}$$

A simple example

This example illustrates the term 'standard deviation.'



Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then, $E[X] = \mu$ and $(X - E[X])^2 = \sigma^2$. Hence,

$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

Example

Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= 1 \times 0.99 + (99)^2 \times 0.01 \approx 100. \\ Var(X) &\approx 100 \implies \sigma(X) \approx 10. \end{aligned}$$

Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
   Scales by c^2.
2. $Var(X + c) = Var(X)$, where c is a constant.
   Shifts center.

Proof:

$$\begin{aligned} Var(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2E(X^2) - c^2(E(X))^2 = c^2(E(X^2) - E(X)^2) \\ &= c^2 Var(X) \\ Var(X + c) &= E((X + c - E(X + c))^2) \\ &= E((X + c - E(X) - c)^2) \\ &= E((X - E(X))^2) = Var(X) \end{aligned}$$

$\square$

Variance of sum of two independent random variables

Theorem:
If X and Y are independent, then

$$Var(X + Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\begin{aligned} var(X + Y) &= E((X + Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \\ &= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 = var(X) + var(Y). \end{aligned}$$

Variance of sum of independent random variables

Theorem:
If $X, Y, Z, \dots$ are pairwise independent, then

$$var(X+Y+Z+\cdots) = var(X)+var(Y)+var(Z)+\cdots.$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$

Hence,

$$\begin{aligned} var(X+Y+Z+\cdots) &= E((X+Y+Z+\cdots)^2) \\ &= E(X^2+Y^2+Z^2+\cdots+2XY+2XZ+2YZ+\cdots) \\ &= E(X^2)+E(Y^2)+E(Z^2)+\cdots+0+\cdots+0 \\ &= var(X)+var(Y)+var(Z)+\cdots. \end{aligned}$$

□

Today's gig: Lies!

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
3. Birthday Paradox.
4. St. Petersburg paradox

Today: Simpson's paradox.
How come this show is still around?

Wait... Wrong Simpson.

The paradox

In 1314 English women were surveyed in 1972-1974 and again after 20 years about smoking:

Smoker	Dead	Alive	Total	% Dead
Yes	139	443	582	24
No	230	502	732	31
Total	369	945	1314	28

Not smoking kills!

The paradox

A closer look:

Age group	18–24		25–34		35–44		45–54		55–54	
Smoker	Y	N	Y	N	Y	N	Y	N	Y	N
Dead	2	1	3	5	11	7	27	12	51	40
Alive	53	61	121	152	95	114	103	66	64	81
Ratio	2.3		0.75		2.4		1.44		1.61	

In each separate category, the percentage of fatalities among smokers is higher, and yet the overall percentage of fatalities among smokers is lower!

Summary

Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X=a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega)=a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X=a]), a \in \mathscr{A}\}$.
- $g(X,Y,Z)$ assigns the value .... .
- $E[X] := \sum_a aPr[X=a]$.
- Expectation is Linear.
- Independent Random Variables.
- Variance.