1. **(Sanity Check!)** Derive Chebyshev's inequality using Markov's inequality for random variable X.

   Answer: We're interested in the probability $\Pr(|X - \mathbf{E}[X]| \geq k) = \Pr((X - \mathbf{E}[X])^2 \geq k^2)$. We simply apply Markov's inequality and we find that $\Pr((X - \mathbf{E}[X])^2 \geq k^2) \leq \mathrm{Var[X]}/\mathrm{k}^2$.

2. **(Coin Flips)**

   (a) Suppose we flip a fair coin n times and we wish to understand the probability that we get at least $3n/4$ heads. Use Markov's inequality to come up with an upper bound for this probability.

   Answer: Let X be a random variable for the number of heads. Let X_i be an indicator for the event that the i-th flip is heads. Since X_i is an indicator, $E[X_i] = \Pr(X_i = 1) = 1/2$. Since $X = \sum_{i=1}^n X_i$,

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbf{E}[X_i] = \frac{n}{2}$$

   We want to bound the probability that $X \geq 3n/4$. Since X is nonnegative, we can use Markov's inequality to get

$$\Pr\left(X \geq \frac{3n}{4}\right) \leq \frac{\mathbf{E}[X]}{\frac{3n}{4}} = \frac{\frac{n}{2}}{\frac{3n}{4}} = \frac{2}{3}$$

   (b) Use Markov's inequality to come up with a similar upper bound on the probability that the number of heads is at least n.

   Answer: This time, we want to bound the probability that $X \geq n$. By Markov's inequality,

$$\Pr(X \geq n) \leq \frac{\mathbf{E}[X]}{n} = \frac{\frac{n}{2}}{n} = \frac{1}{2}$$

   (c) Find the true probability that there are at least n heads in a sequence of n fair coin flips. Is the bound you derived in the previous part tight?

   Answer: Since X can't be greater than n,

$$\Pr(X \geq n) = \Pr(X = n) = \left(\frac{1}{2}\right)^n$$

   So we can see that Markov's inequality gives a very loose bound; it bounds $\Pr(X \geq n)$ by a constant, whereas in reality this probability decreases exponentially as n increases.

3. **Working with the Law of Large Numbers**

(a) A fair coin is tossed and you win a prize if there are more than 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

Solution: 10 tosses.

(b) A fair coin is tossed and you win a prize if there are more than 40% heads. Which is better: 10 tosses or 100 tosses? Explain.

Solution: 100 tosses. Based on the first part, consider the inverse of the event "more than 60% heads" and the symmetry of heads and tails.

(c) A coin is tossed and you win a prize if there are between 40% and 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

Solution: 100 tosses. Based on the first part, consider the union of the events "more than 60% heads" and "more than 60% tails" ("less than 40% heads").

(d) A coin is tossed and you win a prize if there are exactly 50% heads. Which is better: 10 tosses or 100 tosses? Explain.

Solution: 10 tosses. Compare the probability of getting equal number of heads and tails between $2n$ and $2n+2$ tosses.

$$\begin{aligned}
\Pr[n \text{ heads in } 2n \text{ tosses}] &= \binom{2n}{n}/2^{2n} \\
\Pr[n+1 \text{ heads in } 2n+2 \text{ tosses}] &= \binom{2n+2}{n+1}/2^{2n+2} \\
&= \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{1}{2^{2n+2}} \\
&= \frac{(2n+2)(2n+1)2n!}{(n+1)(n+1)n!n!} \cdot \frac{1}{2^{2n+2}} \\
&= \frac{2n+2}{n+1} \cdot \frac{2n+1}{n+1} \binom{2n}{n} \cdot \frac{1}{2^{2n+2}} \\
&< \left(\frac{2n+2}{n+1}\right)^2 \binom{2n}{n} \cdot \frac{1}{2^{2n+2}} \\
&= 4\binom{2n}{n} \cdot \frac{1}{2^{2n+2}} = \binom{2n}{n}/2^{2n} = \Pr[n \text{ heads in } 2n \text{ tosses}]
\end{aligned}$$

The larger n is, the less probability we'll get 50% heads. □

4. **Vegas**

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Given the results of your experiment, how should you estimate p?

(b) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

Solution:

(a) You count the fraction of heads that you see. If q is this fraction then $q \simeq 1/2(1-p)+p = 1/2+p/2$. So you declare p to be $2q-1$.

(b) We want $2q-1$ to be within 0.05 of its mean. This means that q should be within 0.025 of its mean, or the sum should be within $0.025n$ of its mean. The variance of each coin flip is $q(1-q)$, therefore Chebyshev tells us that

$$\Pr[|\sum_{i=1}^{n} X_i - qn| \geq 0.025n] \leq nq(1\text{-}q)/(0.025n)^2$$

We have $q(1-q) < 1/4$. So for the probability to be bounded by 5% we can have $n =$ 8000