# A Random Walk through CS70, Pt. II: Probability

CS70 Summer 2016 - Lecture 8C

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# Today

Same as yesterday (and tomorrow). Review, applications, gigs, cool examples, research questions...

Probability today!

# Fundamentals

Map of outcomes in a probability space $\Omega$ to values in $[0,1]$:
$\sum_{\omega \in \Omega} \Pr[\omega] = 1$

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Inclusion-Exclusion: $\Pr[A\cup B]=\Pr[A]+\Pr[B]-\Pr[A\cap B]$.

Union bound: $\Pr[A_1\cup A_2\cup\ldots\cup A_n]\leq\Pr[A_1]+\Pr[A_2]+\ldots\Pr[A_n]$.

Total probability: if $A_1,\ldots,A_n$ partition the entire sample space (disjoint, covers all of it), then $\Pr[B]=\Pr[B\cap A_1]+\ldots+\Pr[B\cap A_n]$.

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From definition: $\Pr[A \cap B] = \Pr[A] \Pr[B|A]$.

Or, generally: $\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \ldots \Pr[A_n|A_1 \cap \ldots \cap A_{n-1}]$.

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Useful theorem for inference (updating beliefs). Heavily used in AI. CS188.

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# Example: Random-SAT

Let's say I have some Boolean clause that looks like this ("3-CNF")

$$(a \vee b \vee \overline{c}) \wedge (\overline{b} \wedge d \wedge e) \wedge \ldots$$

$n$ clauses (three boolean variables, some may be negated). What is expected number of clauses that I satisfy with a random assignment?

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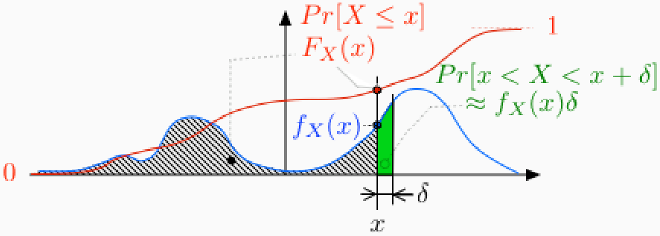
"Hardness of approximation". Ongoing topic of research.

# Random Variables: Continuous




Distributions represented with a pdf

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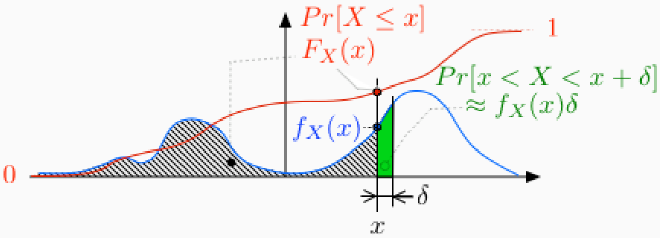


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$$\Pr[X \in [a,b]] = \int_a^b f_X(t)dt = F_X(b) - F_X(a)$$

# Expectation/Variance for Continuous

Sum $\rightarrow$ Integral. Most properties carry over.

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(t) f_X(t) dt$$

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# Application: Streaming Algorithm for Counting Uniques

Let's say that you're building a server that wants to count unique visitors. But you only have a very small amount of memory - enough to remember one number. How do you distinguish between a million unique visitors and a single IP address sending a million requests to your site?

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So PDF is $f(x) = n(1-x)^{n-1}$. Expectation: $\int_0^1 xn(1-x)^{n-1}dx = 1/(n+1)$. Just invert the minimum number to estimate number of unique visitors!

# Distributions

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Chernoff: Family of exponential bounds for sum of mutually independent 0-1 random variables. Derive by noting that $\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}]$, and then applying Markov to bound

$$\Pr[e^{tX} \geq e^{ta}] \leq \frac{E[e^{tX}]}{e^{ta}}$$

for a good value of $t$.

# Law of Large Numbers and CLT

If $X_1, X_2, \ldots$ are pairwise independent, and identically distributed with mean $\mu$: $\Pr\left[\left|\frac{\sum_i X_i}{n} - \mu\right| \geq \varepsilon\right] \to 0$ as $n \to \infty$.

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With many i.i.d. samples we converge not only to the mean, but also to a normal distribution with the same variance.

CLT: Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with expectation $\mu$ and variance $\sigma^2$. Let

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**This is an approximation, not a bound.**

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**Hitting time:** How long does it take us to get to some state $j$? Strategy: let $\beta(i)$ be the time it takes to get to $j$ from $i$, for each state $i$. $\beta(j) = 0$. Set up system of linear equations and solve.

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**Ergodic** state: aperiodic + recurrent.

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On the other hand, suppose $i$ is accessible from $j$. Suppose for contradiction that $i$ is recurrent. Then if we're at $j$, we *have* to hit $i$ again (because $i$ is recurrent, so we have to go back to $i$ if we go from $i$ to $j$).

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On the other hand, suppose $i$ is accessible from $j$. Suppose for contradiction that $i$ is recurrent. Then if we're at $j$, we *have* to hit $i$ again (because $i$ is recurrent, so we have to go back to $i$ if we go from $i$ to $j$). But when we're at $i$, we know that we're definitely going to hit $j$ sometime (because there's a nonzero chance of going to $j$ from $i$, and we'll be back at $i$ infinitely many times due to it being recurrent).

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# Markov Chain Classifications

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**Ergodic** Markov chain: every state is ergodic. Any finite, irreducible, aperiodic Markov chain is ergodic.

# Stationary Distributions

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Let $r_{i,j}^t$ be the probability that we first (if $i = j$, we don't count the zeroth timestep) hit $j$ exactly $t$ timesteps after we start at $i$. Then $h_{i,j} = \sum_{t \geq 1} t r_{i,j}^t$.

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Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

- There is a unqiue stationary distribution $\pi$.
- For all $j$, $i$, the limit $\lim_{t\to\infty} P_{j,i}^t$ exists and is independent of $j$.
- $\pi_i = \lim_{t\to\infty} P_{j,i}^t = 1/h_{i,i}$

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Cover time (expected time that it takes to hit all the vertices, starting from the worst vertex possible): bounded above by $4|V||E|$.

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So if I play machine 1 and machine 2 alternately, I should expect to end up broke too, right? Hmm...

# Parrondo's Paradox II

Let's say that the slot machines work as follows:

Machine 1: Put in some money. You gain a dollar w.p. 0.49 and lose a dollar w.p. 0.51. Pretty obvious that you lose money playing this game.

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Machine 2: You put in $d$ dollars.

- Case A: If $d$ is a multiple of 3 then you gain a dollar w.p. 0.09 and lose a dollar w.p. 0.91.
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What's the probability of winning a round? 1/3 probability of case A happening, so it would be

$$\frac{1}{3}(0.09) + \frac{2}{3}(0.74) = \frac{157}{300} > \frac{1}{2}$$

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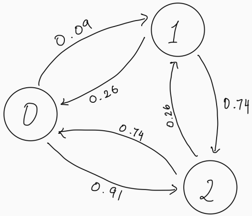
So how often do we end up with case $A$? Here's the approach: one state for each value of $d \pmod 3$.

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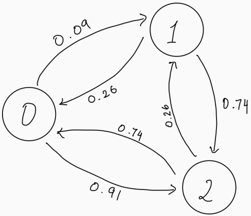


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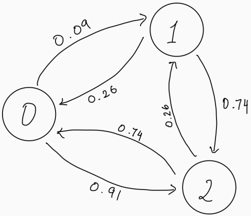


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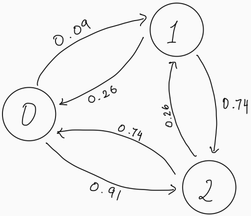


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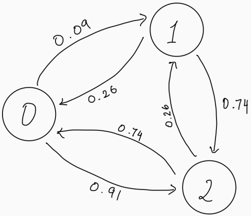


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Result: $\pi = [0.382604, 0.154728, 0.462668]$. Plug in:

$$0.3826(0.09) + (0.1547 + 0.4627)(0.74) = 0.4913 < \frac{1}{2}$$

So I lose money in the long run.

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I have $d$ dollars... if $d$ is a multiple of 3, probability of winning is:

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If $d$ isn't a multiple of 3, probability of winning is:

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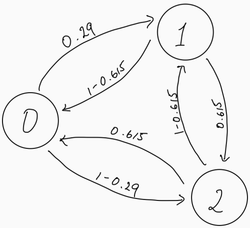

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Did we just break linearity of expectation? No! It doesn't make a whole lot of sense to talk about "expected winnings" for a state without taking into account the current state. Our distribution across states changes between the two games!

Questions?