

# Alex Psomas: Lecture 18.

## Random Variables: Variance

1. Variance
2. Distributions

# Variance

Flip a coin: If H you make a dollar. If T you lose a dollar.

Let  $X$  be the RV indicating how much money you make.

$$E(X) = 0.$$

Flip a coin: If H you make a million dollars. If T you lose a million dollars.

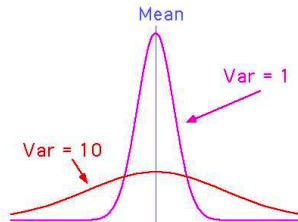
Let  $Y$  be the RV indicating how much money you make.

$$E(Y) = 0.$$

Any other measures???

What else that's informative can we say?

# Variance



The variance measures the deviation from the mean value.

**Definition:** The **variance** of  $X$  is

$$\sigma^2(X) := \text{var}[X] = E[(X - E[X])^2].$$

$\sigma(X)$  is called the **standard deviation** of  $X$ .

# Variance and Standard Deviation

**Fact:**

$$\text{var}[X] = E[X^2] - E[X]^2.$$

Indeed:

$$\begin{aligned}\text{var}(X) &= E[(X - E[X])^2] \\&= E[X^2 - 2XE[X] + E[X]^2] \\&= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity} \\&= E[X^2] - 2E[X]E[X] + E[X]^2, \\&= E[X^2] - E[X]^2.\end{aligned}$$

## Example

Consider  $X$  with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

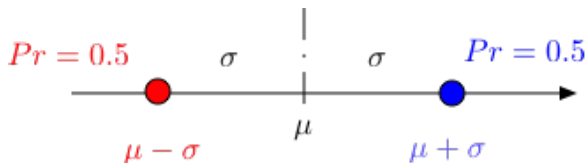
$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$

$$E[X^2] = (-1)^2 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$

$$\text{Var}(X) \approx 100 \implies \sigma(X) \approx 10.$$

## A simple example

This example illustrates the term ‘standard deviation.’



Consider the random variable  $X$  such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

Then,  $E[X] = \mu$  and  $E[(X - E[X])^2] = \sigma^2$ . Hence,

$$\text{var}(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

# Properties of variance.

1.  $\text{Var}(cX) = c^2 \text{Var}(X)$ , where  $c$  is a constant.  
Scales by  $c^2$ .
2.  $\text{Var}(X + c) = \text{Var}(X)$ , where  $c$  is a constant.  
Shifts center.

## Proof:

$$\begin{aligned}\text{Var}(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\ &= c^2 \text{Var}(X)\end{aligned}$$

$$\begin{aligned}\text{Var}(X + c) &= E((X + c - E(X + c))^2) \\ &= E((X + c - E(X) - c)^2) \\ &= E((X - E(X))^2) = \text{Var}(X)\end{aligned}$$



# Variance of sum of two independent random variables

## Theorem:

If  $X$  and  $Y$  are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

## Proof:

Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that  $E(X) = 0$  and  $E(Y) = 0$ .

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

$$\text{Var}(X) = E(X^2), \text{Var}(Y) = E(Y^2).$$

Hence,

$$\begin{aligned} \text{var}(X + Y) &= E((X + Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \\ &= \text{var}(X) + \text{var}(Y). \end{aligned}$$



# Variance of sum of independent random variables

## Theorem:

If  $X, Y, Z, \dots$  are pairwise independent, then

$$\text{var}(X + Y + Z + \dots) = \text{var}(X) + \text{var}(Y) + \text{var}(Z) + \dots.$$

## Proof:

Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that  $E[X] = E[Y] = \dots = 0$ .

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \dots = 0.$$

Hence,

$$\begin{aligned} \text{var}(X + Y + Z + \dots) &= E((X + Y + Z + \dots)^2) \\ &= E(X^2 + Y^2 + Z^2 + \dots + 2XY + 2XZ + 2YZ + \dots) \\ &= E(X^2) + E(Y^2) + E(Z^2) + \dots + 0 + \dots + 0 \\ &= \text{var}(X) + \text{var}(Y) + \text{var}(Z) + \dots. \end{aligned}$$



# Distributions

- ▶ Bernoulli
- ▶ Binomial
- ▶ Uniform
- ▶ Geometric

# Bernoulli

Flip a coin, with heads probability  $p$ .

Random variable  $X$ : 1 is heads, 0 if not heads.

$X$  has the Bernoulli distribution.

Distribution:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1 - p \end{cases}$$

$$E[X] = p$$

$$E[X^2] = 1^2 \times p + 0^2 \times (1 - p) = p$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$$

Notice that:

$$p = 0 \implies \text{Var}(X) = 0$$

$$p = 1 \implies \text{Var}(X) = 0$$

# Jacob Bernoulli



# Binomial

Flip  $n$  coins with heads probability  $p$ .

Random variable: number of heads.

Binomial Distribution:  $Pr[X = i]$ , for each  $i$ .

How many sample points in event " $X = i$ "?

$i$  heads out of  $n$  coin flips  $\implies \binom{n}{i}$

Sample space:  $\Omega = \{HHH\dots HH, HHH\dots HT, \dots\}$

What is the probability of  $\omega$  if  $\omega$  has  $i$  heads?

Probability of heads in any position is  $p$ .

Probability of tails in any position is  $(1 - p)$ .

So, we get  $Pr[\omega] = p^i(1 - p)^{n-i}$ .

Probability of " $X = i$ " is sum of  $Pr[\omega]$ ,  $\omega \in "X = i"$ .

$$Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}, i = 0, 1, \dots, n : B(n, p) \text{ distribution}$$

# Expectation of Binomial Distribution

Indicator for the  $i$ -th coin:

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = 1 \times \text{Pr}[\text{"heads"}] + 0 \times \text{Pr}[\text{"tails"}] = p.$$

Moreover  $X = X_1 + \cdots + X_n$  and

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = n \times E[X_i] = np.$$

## Variance of Binomial Distribution.

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$

$$\text{Var}(X_i) = p - (E(X_i))^2 = p - p^2 = p(1 - p).$$

$$X = X_1 + X_2 + \dots X_n.$$

$$X_i \text{ and } X_j \text{ are independent: } Pr[X_i = 1 | X_j = 1] = Pr[X_i = 1].$$

$$\text{Var}(X) = \text{Var}(X_1 + \dots X_n) = np(1 - p).$$

# Uniform Distribution

Roll a six-sided balanced die. Let  $X$  be the number of pips (dots). Then  $X$  is equally likely to take any of the values  $\{1, 2, \dots, 6\}$ . We say that  $X$  is *uniformly distributed* in  $\{1, 2, \dots, 6\}$ .

More generally, we say that  $X$  is uniformly distributed in  $\{1, 2, \dots, n\}$  if  $\Pr[X = m] = 1/n$  for  $m = 1, 2, \dots, n$ . In that case,

$$E[X] = \sum_{m=1}^n m \Pr[X = m] = \sum_{m=1}^n m \times \frac{1}{n} = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$



## Variance of Uniform

$$E[X] = \frac{n+1}{2}.$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^n i^2 \Pr[X=i] = \frac{1}{n} \sum_{i=1}^n i^2 \\ &= \frac{1+3n+2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

This gives

$$\text{var}(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

# Geometric Distribution

Let's flip a coin with  $Pr[H] = p$  until we get  $H$ .



For instance:

$$\omega_1 = H, \text{ or}$$

$$\omega_2 = T H, \text{ or}$$

$$\omega_3 = T T H, \text{ or}$$

$$\omega_n = T T T T \dots T H.$$

Note that  $\Omega = \{\omega_n, n = 1, 2, \dots\}$ .

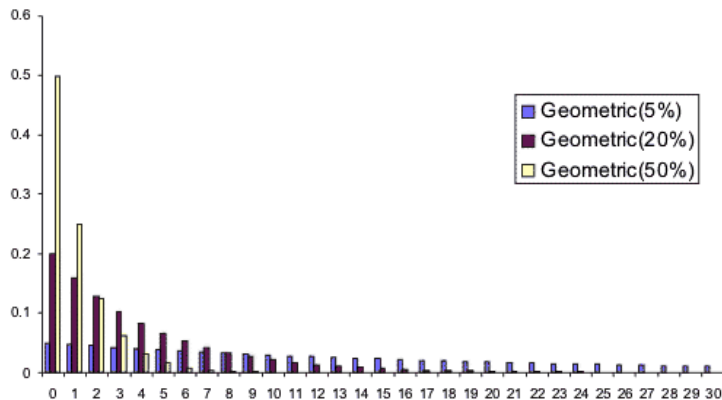
Let  $X$  be the number of flips until the first  $H$ . Then,  $X(\omega_n) = n$ .

Also,

$$Pr[X = n] = (1 - p)^{n-1} p, \quad n \geq 1.$$

# Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$



# Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X = n] = \sum_{n=1}^{\infty} (1 - p)^{n-1} p = p \sum_{n=1}^{\infty} (1 - p)^{n-1} = p \sum_{n=0}^{\infty} (1 - p)^n.$$

We want to analyze  $S := \sum_{n=0}^{\infty} a^n$  for  $|a| < 1$ .  $S = \frac{1}{1-a}$ . Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \dots \\ aS &= a + a^2 + a^3 + a^4 + \dots \\ (1 - a)S &= 1 + a - a + a^2 - a^2 + \dots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X = n] = p \frac{1}{1 - (1 - p)} = 1.$$

## Geometric Distribution: Expectation

$$X \sim \text{Geom}(p), \text{ i.e., } \Pr[X = n] = (1 - p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} n \Pr[X = n] = \sum_{n=1}^{\infty} n(1 - p)^{n-1}p.$$

Thus,

$$\begin{aligned} E[X] &= p + 2(1 - p)p + 3(1 - p)^2p + 4(1 - p)^3p + \dots \\ (1 - p)E[X] &= (1 - p)p + 2(1 - p)^2p + 3(1 - p)^3p + \dots \\ pE[X] &= p + (1 - p)p + (1 - p)^2p + (1 - p)^3p + \dots \end{aligned}$$

by subtracting the previous two identities

$$= \sum_{n=1}^{\infty} (1 - p)^{n-1}p = \sum_{n=1}^{\infty} \Pr[X = n] = 1.$$

Hence,

$$E[X] = \frac{1}{p}.$$

# Coupon Collectors Problem.

**Experiment:** Get coupons at random from  $n$  until collect all  $n$  coupons.

**Outcomes:**  $\{123145\dots, 56765\dots\}$

**Random Variable:**  $X$  - length of outcome.

Before:  $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$ .

Today:  $E[X]$ ?

## Time to collect coupons

$X$ -time to get  $n$  coupons.

$X_1$  - time to get first coupon. Note:  $X_1 = 1$ .  $E(X_1) = 1$ .

$X_2$  - time to get second (distinct) coupon after getting first.

$$\Pr[\text{"get second distinct coupon"} | \text{"got first coupon"}] = \frac{n-1}{n}$$

$$E[X_2]? \text{ Geometric ! ! } \implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}.$$

$$\Pr[\text{"getting } i\text{th distinct coupon"} | \text{"got } i-1 \text{ distinct coupons"}]$$

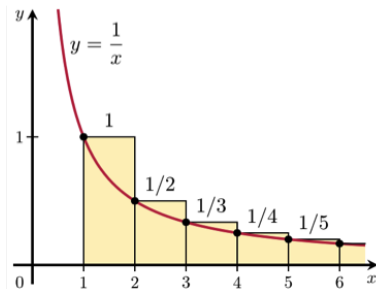
$$= \frac{n - (i-1)}{n} = \frac{n-i+1}{n}$$

$$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \dots, n.$$

$$\begin{aligned} E[X] &= E[X_1] + \dots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{1} \\ &= n\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

## Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$



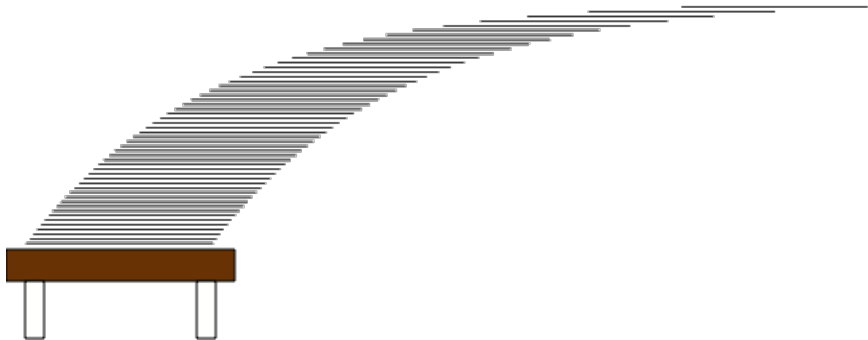
A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$



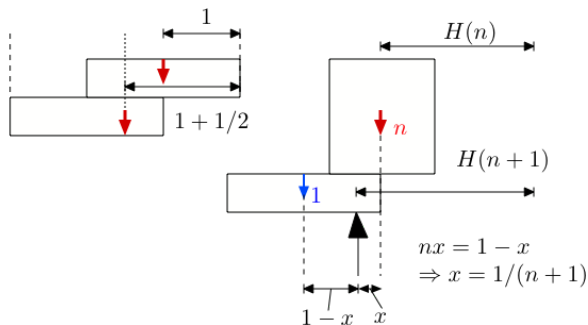
## Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend  $H(n)$  to the right of the table. As  $n$  increases, you can go as far as you want!

# Stacking



The cards have width 2. Induction shows that the center of gravity after  $n$  cards is  $H(n)$  away from the right-most edge.

## Geometric Distribution: Memoryless

Let  $X$  be  $\text{Geom}(p)$ . Then, for  $n \geq 0$ ,

$$\Pr[X > n] = \Pr[\text{first } n \text{ flips are } T] = (1 - p)^n.$$

### Theorem

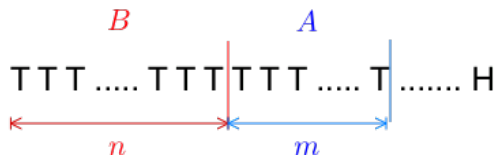
$$\Pr[X > n + m | X > n] = \Pr[X > m], m, n \geq 0.$$

### Proof:

$$\begin{aligned}\Pr[X > n + m | X > n] &= \frac{\Pr[X > n + m \text{ and } X > n]}{\Pr[X > n]} \\&= \frac{\Pr[X > n + m]}{\Pr[X > n]} \\&= \frac{(1 - p)^{n+m}}{(1 - p)^n} = (1 - p)^m \\&= \Pr[X > m].\end{aligned}$$

# Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$



The coin is memoryless, therefore, so is  $X$ .

## Geometric Distribution: Yet another look

**Theorem:** For a r.v.  $X$  that takes the values  $\{0, 1, 2, \dots\}$ , one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If  $X = \text{Geom}(p)$ , then  $Pr[X \geq i] = Pr[X > i - 1] = (1 - p)^{i-1}$ .

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1 - p)^{i-1} = \sum_{i=0}^{\infty} (1 - p)^i = \frac{1}{1 - (1 - p)} = \frac{1}{p}.$$

## A side step: Expected Value of Integer RV

**Theorem:** For a r.v.  $X$  that takes values in  $\{0, 1, 2, \dots\}$ , one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

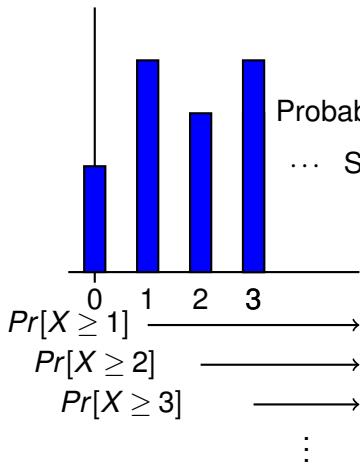
**Proof:** One has

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\ &= \sum_{i=1}^{\infty} i (Pr[X \geq i] - Pr[X \geq i+1]) \\ &= \sum_{i=1}^{\infty} (i \times Pr[X \geq i] - i \times Pr[X \geq i+1]) \\ &= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} i \times Pr[X \geq i+1] \\ &= \sum_{i=1}^{\infty} i \times Pr[X \geq i] - \sum_{i=1}^{\infty} (i-1) \times Pr[X \geq i] = \sum_{i=1}^{\infty} Pr[X \geq i]. \end{aligned}$$



**Theorem:** For a r.v.  $X$  that takes values in  $\{0, 1, 2, \dots\}$ , one has

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$



Probability mass at  $i$ , counted  $i$  times.

... Same as  $\sum_{i=1}^{\infty} i \times \Pr[X = i]$ .

## Variance of geometric distribution.

$X$  is a geometrically distributed RV with parameter  $p$ .

Thus,  $\Pr[X = n] = (1 - p)^{n-1} p$  for  $n \geq 1$ . Recall  $E[X] = 1/p$ .

$$\begin{aligned} E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \dots \\ -(1-p)E[X^2] &= -[p(1-p) + 4p(1-p)^2 + \dots] \\ pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + \dots \\ &= 2(p + 2p(1-p) + 3p(1-p)^2 + \dots) \quad E[X]! \\ &\quad - (p + p(1-p) + p(1-p)^2 + \dots) \quad 1. \\ pE[X^2] &= 2E[X] - 1 \\ &= 2\left(\frac{1}{p}\right) - 1 = \frac{2-p}{p} \end{aligned}$$

$$\begin{aligned} \implies E[X^2] &= (2-p)/p^2 \text{ and} \\ \text{var}[X] &= E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}. \\ \sigma(X) &= \frac{\sqrt{1-p}}{p} \approx E[X] \text{ when } p \text{ is small(ish).} \end{aligned}$$



# Review: Distributions

- ▶  $Bern(p) : Pr[X = 1] = p;$   
 $E[X] = p;$   
 $Var[X] = p(1 - p);$
- ▶  $Bin(n, p) : Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, m = 0, \dots, n;$   
 $E[X] = np;$   
 $Var[X] = np(1 - p);$
- ▶  $U[1, \dots, n] : Pr[X = m] = \frac{1}{n}, m = 1, \dots, n;$   
 $E[X] = \frac{n+1}{2};$   
 $Var[X] = \frac{n^2-1}{12};$
- ▶  $Geom(p) : Pr[X = n] = (1 - p)^{n-1} p, n = 1, 2, \dots;$   
 $E[X] = \frac{1}{p};$   
 $Var[X] = \frac{1-p}{p^2};$

# Today's gig: Two envelopes problem.

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
3. Birthday Paradox.
4. St. Petersburg paradox.
5. Simpson's paradox.

Today: Two envelopes problem.

## Two envelopes

I put  $x$  dollars in an envelope, and  $2x$  dollars in another envelope, and seal both envelopes.

You pick one at random (you don't know which is which).

Before you open it you think: What will happen if I switch?

Well, if I picked the one I picked has  $y$  dollars, then the other either  $2y$  or  $\frac{y}{2}$ .

In the first case, I win  $y$ . In the second case, I lose  $\frac{y}{2}$ .

Therefore, in expectation, my net gain is:  $\frac{1}{2}y - \frac{1}{2}\frac{y}{2} = \frac{y}{4}$ .

Therefore, I should switch.

Before you open the new envelope you think: What will happen if I switch?

# Summary

## Random Variables

- ▶ Variance.
- ▶ Distributions.