Alex Psomas: Lecture 17.

Random Variables: Expectation, Variance

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1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance

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$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

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- $(X - Y)^2$
- $X \cos(2\pi Y + Z)$.

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$$E[X] = \sum_{\omega} X(\omega) \times Pr[\omega].$$

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$$\sum_{\omega} X(\omega) Pr[\omega] = 3\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 0\frac{1}{8}.$$

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An Illustration: Rolling Dice

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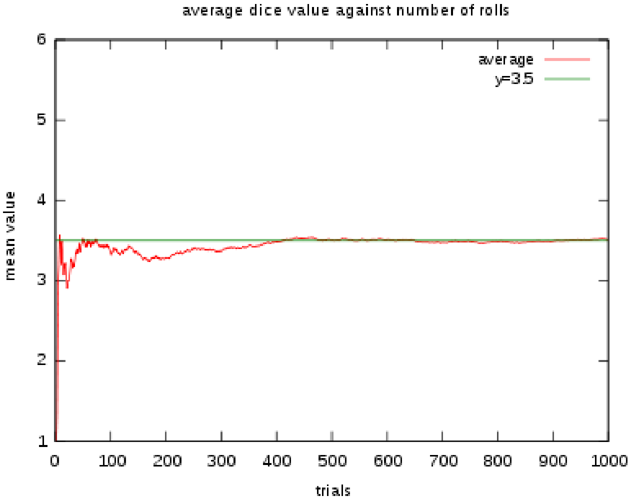

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Thus, we will write $X = 1_A$.

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Note: If we had defined $Y = a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y=y]$, we would have been in trouble!

Using Linearity - 1: Dots on dice

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Flip n coins with heads probability p.

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Flip n coins with heads probability p. X - number of heads

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Using Linearity - 3

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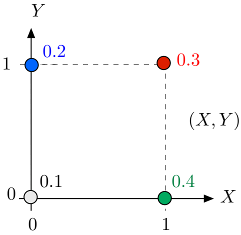

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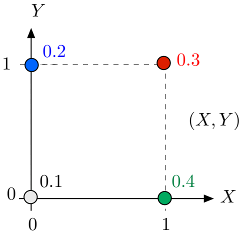

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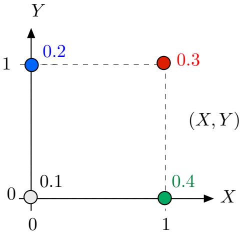

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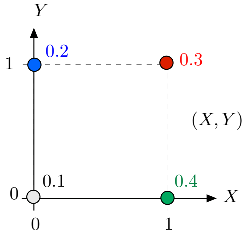

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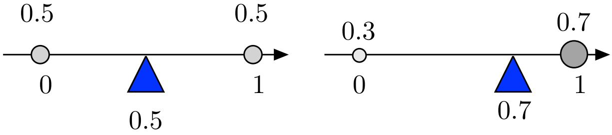
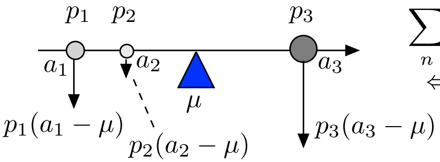
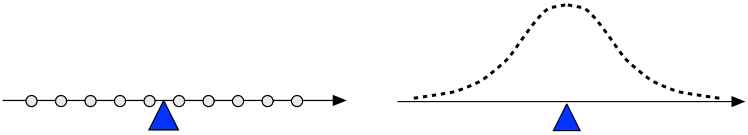




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$



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Unfortunately, we won't talk about this in this class...

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$\square$

Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

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Let X, Y be independent RVs. Then

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Examples

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Mutually Independent Random Variables

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$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$

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$$E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n].$$

Proof:

Assume that the result is true for n. (It is true for $n = 2$.)

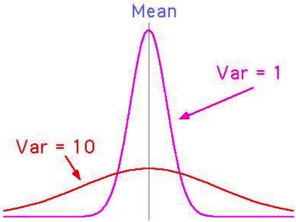
Then, with $Y = X_1 \cdots X_n$, one has

$$\begin{aligned}
E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\
&= E[Y]E[X_{n+1}], \\
&\quad \text{because } Y, X_{n+1} \text{ are independent} \\
&= E[X_1] \cdots E[X_n]E[X_{n+1}].
\end{aligned}$$

$\square$

Variance

Variance



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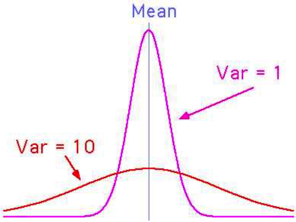

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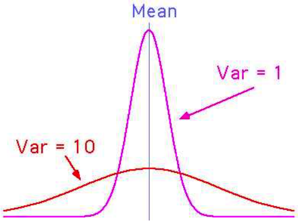

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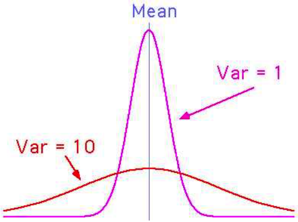

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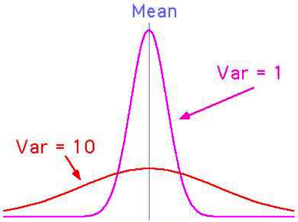

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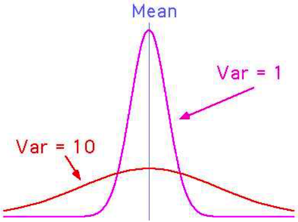

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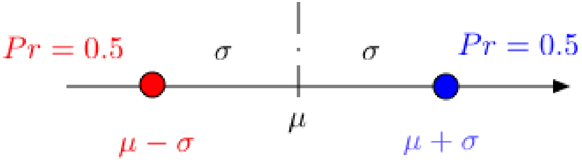

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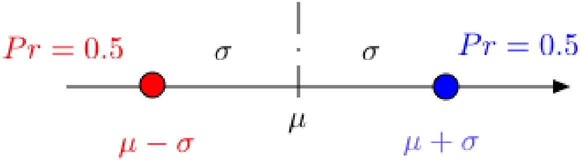

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$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

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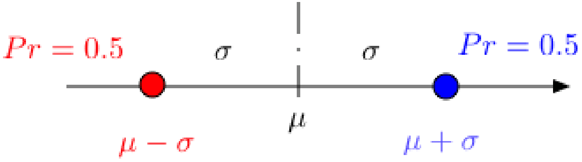

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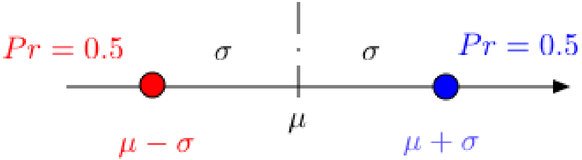

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Consider X with

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

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1. How to tell random from human.

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Wait... Wrong Simpson.

The paradox

Summary

$\boxed{\text{Random Variables}}$

Summary

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $g(X, Y, Z)$ assigns the value .... .
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.
- Independent Random Variables.
- Variance.