

Midterm 2 Review

CS70 Summer 2016 - Lecture 6D

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UC Berkeley

Midterm 2: Format

8 questions, 190 points, 110 minutes (same as MT1).

Two pages (one double-sided sheet) of **handwritten** notes.

Coverage: we will assume knowledge of all the material from the beginning of the class to yesterday, but we will only explicitly test for material seen after MT1.

We will give you a formula sheet (see MT2 logistics post on Piazza to see it). On it: all the distributions we'll expect you to know (with expectation + variance), and Chernoff bounds.

Probability Basics

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Events can be combined using standard set operations.

Disjointness and Additivity

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Total probability: if A_1, \dots, A_n partition the entire sample space (disjoint, covers all of it), then $\Pr[B] = \Pr[B \cap A_1] + \dots + \Pr[B \cap A_n]$.

What are the probabilities?

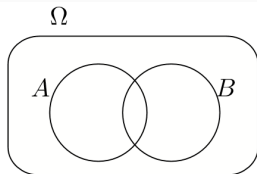


Figure : Two events

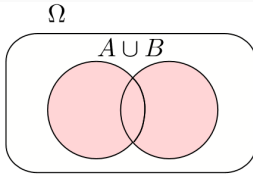


Figure : Union (or)

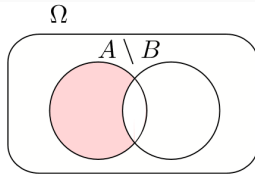


Figure : Difference (A , not B)

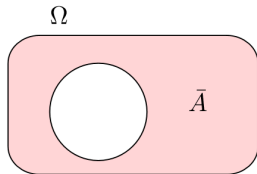


Figure : Complement (not)

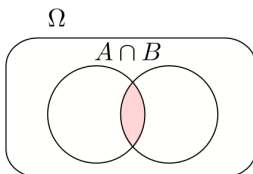


Figure : Intersection (and)

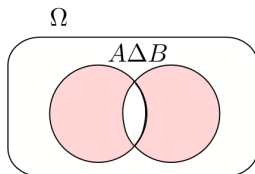


Figure : Symmetric difference (only one)

Conditional Probability and the Product Rule

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Definition:

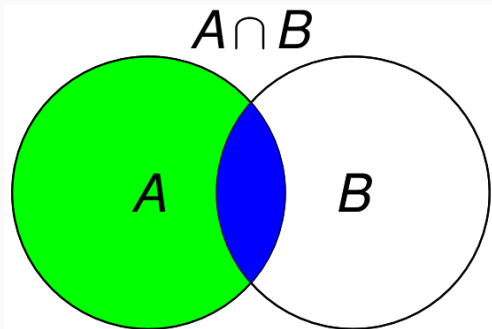
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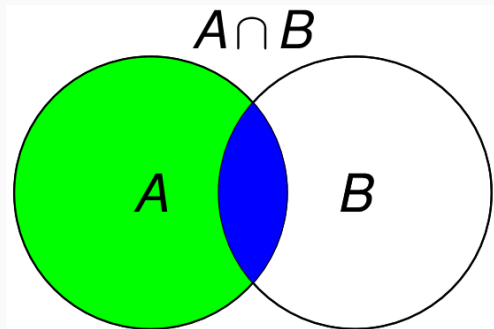


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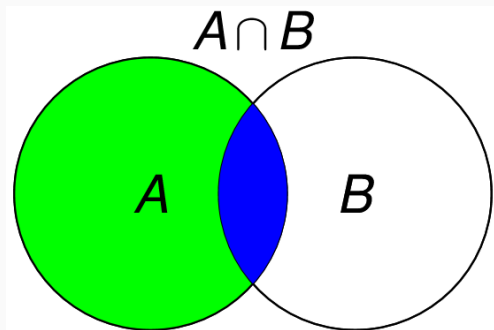
From definition: $\Pr[A \cap B] = \Pr[A] \Pr[B|A]$.

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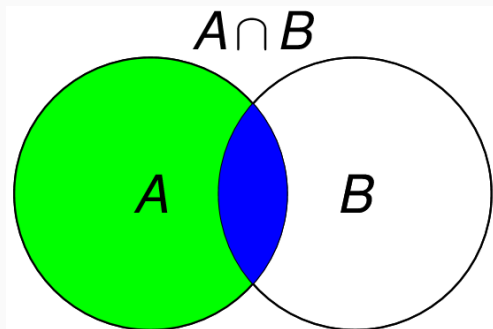
$$\Pr[A_1 \cap \dots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \dots \Pr[A_n|A_1 \cap \dots \cap A_{n-1}] .$$

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Correlation and Independence

“Knowing that A is true tells you nothing about B .” Independence:
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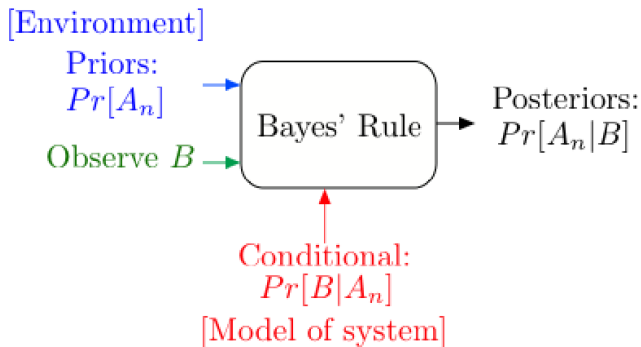
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Or maybe knowing that one is true tells you that the other is likely to be true, too.

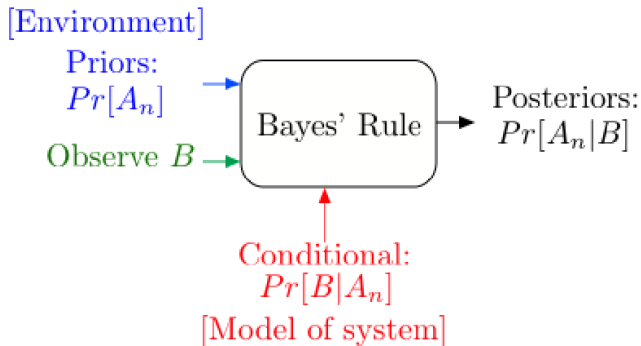
Positive Correlation: $\Pr[A \cap B] > \Pr[A] \Pr[B]$.

Negative Correlation: $\Pr[A \cap B] < \Pr[A] \Pr[B]$.

Bayes' Theorem



Bayes' Theorem



You know you will get a good grade in CS70 with some probability (prior). You take midterm 2 and get a good grade (observation). With this new information, figure out the probability that you get a good grade in CS70 (posterior).

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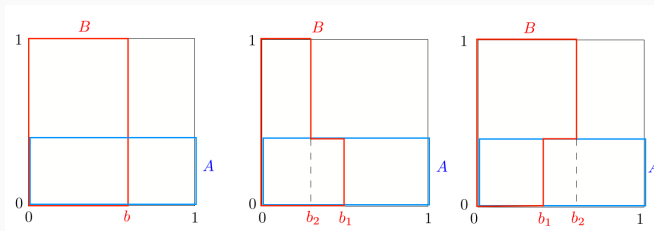
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Random Variables

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Random variable: function that assigns a real number $X(\omega)$ to each outcome ω in a probability space.

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Random variable: maps outcome to a number, e.g. {Psyduck, Typhlosion, Psyduck, Dratini, Typhlosion} \rightarrow 3.

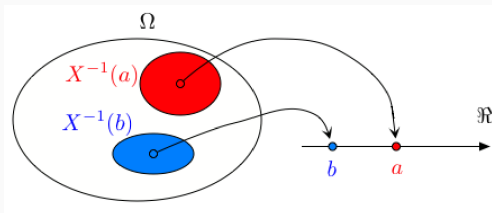
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Discrete distributions: when there are a finite number of values an R.V. can take: pairs of values and probabilities. Probability of R.V. taking on a value: probability that an event that maps onto that value occurs.



Continuous RVs

Probability space is infinite and maps onto a continuous set of reals.

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Distributions represented with a pdf

$$f_X(t) = \lim_{\delta \rightarrow 0} \frac{\Pr[X \in [t, t + \delta]]}{\delta}$$

...or, equivalently, a cdf:

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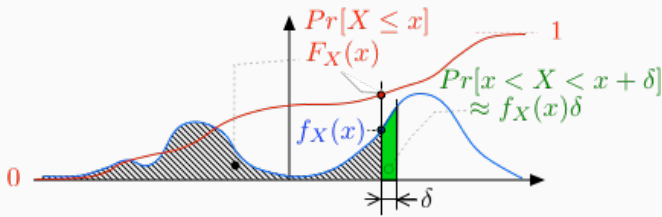
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If X, Y independent, then $f(X), g(Y)$ independent for all f, g .

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For independent X, Y : $E[XY] = E[X]E[Y]$.

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Standard deviation is defined as square root of variance.

If A is an event, let indicator r.v. be defined as 1 if our outcome is in the event and 0 otherwise.

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Expectation? Same as probability that event happened!

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Memorylessness.

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Tail Bounds and LLN

Confidence intervals: if X falls in $[a, b]$ with probability $1 - \alpha$, then we say that $[a, b]$ is an $1 - \alpha$ confidence interval for X .

For X non-negative, a positive:

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$

Not a very tight bound most of the time!

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Or: for monotone non-decreasing function f that takes non-negative values, and non-negative X :

$$\Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$$

for all a s.t. $f(a) > 0$.

$$\Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}$$

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How did we get this? Just use Markov and use $f(x) = x^2$ as our function.

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Chernoff

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General approach to derive these: note that

$$\Pr[X \geq t] = \Pr[e^{tX} \geq e^{ta}] .$$

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All the bounds you need are on the equation sheet on the exam.

If X_1, X_2, \dots are pairwise independent, and identically distributed with mean μ : $\Pr\left[\left|\frac{\sum_i X_i}{n} - \mu\right| \geq \epsilon\right] \rightarrow 0$ as $n \rightarrow \infty$.

LLN and CLT

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With many i.i.d. samples we converge not only to the mean, but also to a normal distribution with the same variance.

CLT: Suppose X_1, X_2, \dots are i.i.d. random variables with expectation μ and variance σ^2 . Let

$$S_n := \frac{A_n - n\mu}{\sigma\sqrt{n}} = \frac{(\sum_i X_i) - n\mu}{\sigma\sqrt{n}}$$

Then S_n tends towards $\mathcal{N}(0, 1)$ as $n \rightarrow \infty$.

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Or:

$$\Pr[S_n \leq a] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-x^2/2} dx$$

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This is an approximation, not a bound.

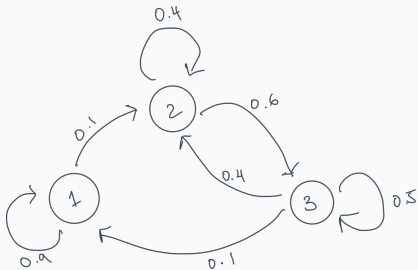
Markov Chains

Definitions

Set of states, transition probabilities, and initial distribution.

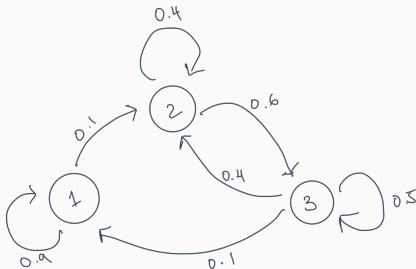
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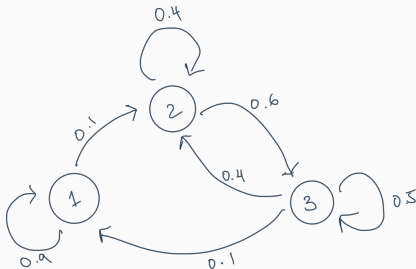


Also representable as a transition matrix.

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

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Distributions are row vectors. Timesteps correspond to matrix multiplication: $\pi \rightarrow \pi P$.

Hitting Time

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Set up system of linear equations and solve.

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Periodic Markov chain: any state is periodic.

Ergodic Markov chain: every state is ergodic. Any finite, irreducible, aperiodic Markov chain is ergodic.

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Distribution is unchanged by state. Intuitively: if I have a lot (approaching infinity) of people on the same MC: the number of people at each state is constant (even if the individual people may move around).

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- There is a unique stationary distribution π .
- For all j, i , the limit $\lim_{t \rightarrow \infty} P_{j,i}^t$ exists and is independent of j .
- $\pi_i = \lim_{t \rightarrow \infty} P_{j,i}^t = 1/h_{i,i}$

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Cover time (expected time that it takes to hit all the vertices, starting from the worst vertex possible): bounded above by $4|V||E|$.

Good luck on the midterm!