

Alex Psomas: Lecture 17.

Random Variables: Expectation, Variance

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1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance

Random Variables: Definitions

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$$Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]$$

$$= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}$$

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- ▶ $(X - Y)^2$
- ▶ $X \cos(2\pi Y + Z)$.

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$$\sum_{\omega} X(\omega)Pr[\omega] = 3\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 2\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 1\frac{1}{8} + 0\frac{1}{8}.$$

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Also,

$$\sum_a a \times Pr[X = a] = 3\frac{1}{8} + 2\frac{3}{8} + 1\frac{3}{8} + 0\frac{1}{8}.$$

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The fact that this average converges to $E[X]$ is a theorem:

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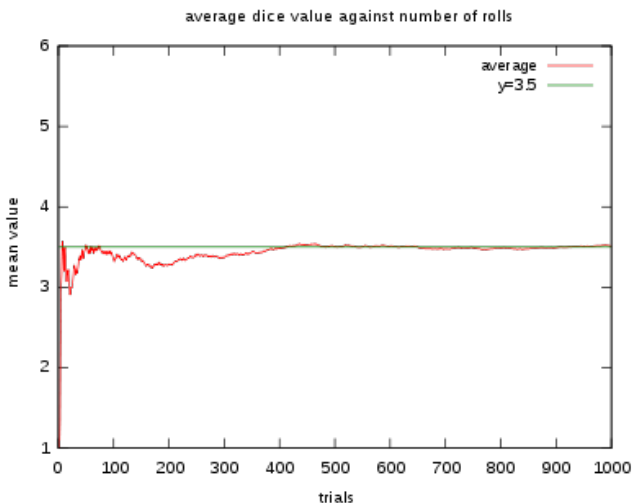
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An Illustration: Rolling Dice

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Thus, we will write $X = 1_A$.

Linearity of Expectation

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Note: If we had defined $Y = a_1 X_1 + \cdots + a_n X_n$ and had tried to compute $E[Y] = \sum_y y Pr[Y = y]$, we would have been in trouble!

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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For a group of 28 it's about 1. For 100 it's 13.5. For 280 it's 107.

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$$\begin{aligned} E[g(X)] &= \sum_{\omega} g(X(\omega)) Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \end{aligned}$$

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X .

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y :

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \mathfrak{X} : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:

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$$\begin{aligned} E[g(X)] &= \sum_{x=-2}^3 x^2 \frac{1}{6} \\ &= \{4 + 1 + 0 + 1 + 4 + 9\} \frac{1}{6} = \frac{19}{6}. \end{aligned}$$

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Thus,

$$E[Y] = 4 \frac{2}{6} + 1 \frac{2}{6} + 0 \frac{1}{6} + 9 \frac{1}{6} = \frac{19}{6}.$$

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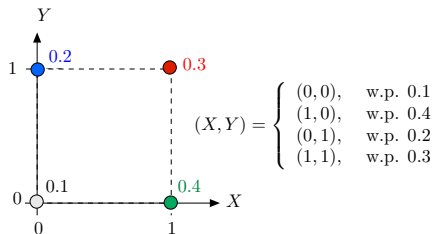
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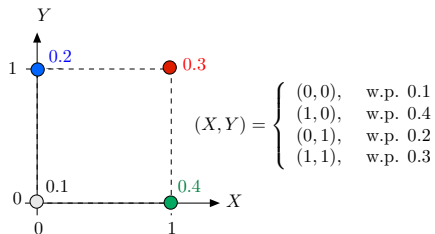
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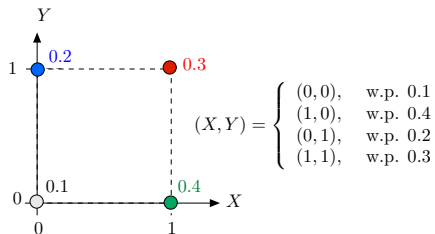
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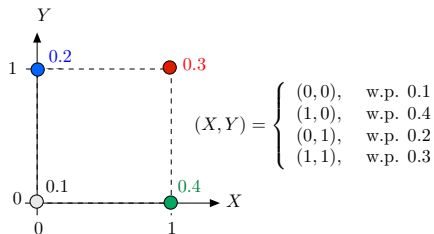
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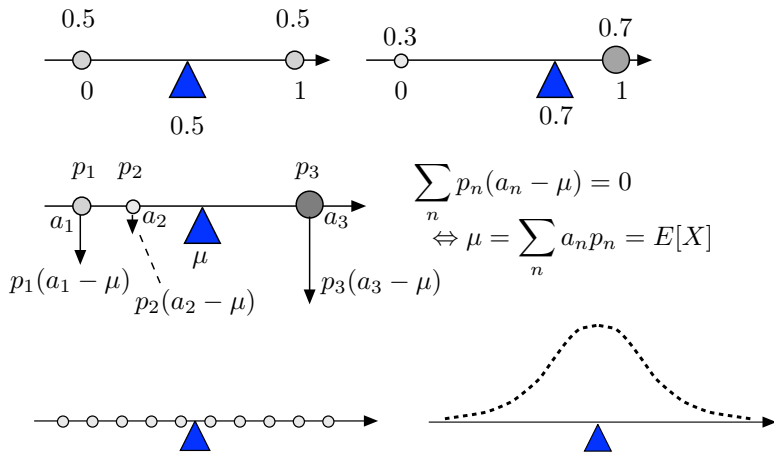
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Unfortunately, we won't talk about this in this class...

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Roll two die. X = number of dots on the first one, Y = number of dots on the other one. X, Y are independent.

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Functions of Independent random Variables

Theorem Functions of independent RVs are independent
Let X, Y be independent RV. Then

$f(X)$ and $g(Y)$ are independent, for all $f(\cdot), g(\cdot)$.

Mean of product of independent RV

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Mutually Independent Random Variables

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$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \dots$$



Functions of pairwise independent RVs

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$$\begin{aligned} E[X_1 \cdots X_n X_{n+1}] &= E[Y X_{n+1}], \\ &= E[Y] E[X_{n+1}], \\ &\quad \text{because } Y, X_{n+1} \text{ are independent} \end{aligned}$$

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Variance

Flip a coin:

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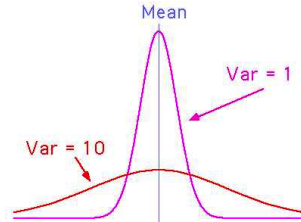
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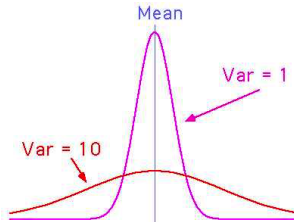
Any other measures??? What else that's informative can we say?

Variance

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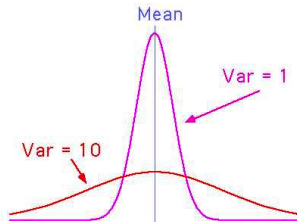


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The variance measures the deviation from the mean value.

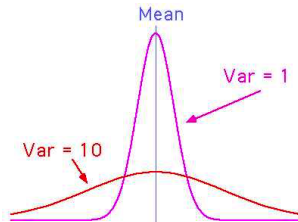
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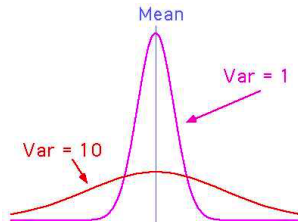


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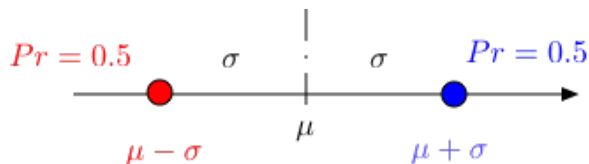
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A simple example

This example illustrates the term 'standard deviation.'

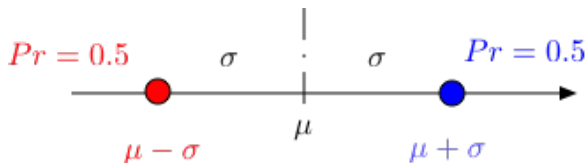
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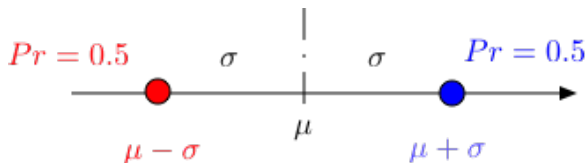


Consider the random variable X such that

$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

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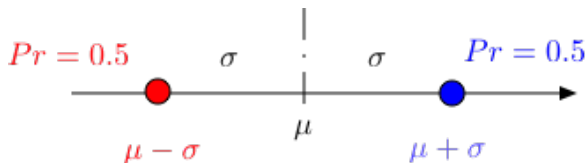
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$$\begin{aligned}\text{Var}(cX) &= E((cX)^2) - (E(cX))^2 \\ &= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\ &= c^2 \text{Var}(X)\end{aligned}$$

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Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where c is a constant.
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Wait... Wrong Simpson.

The paradox

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Smoker	Dead	Alive	Total	% Dead
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Age group	18–24		25–34		35–44		45–54		55–54	
Smoker	Y	N	Y	N	Y	N	Y	N	Y	N
Dead	2	1	3	5	11	7	27	12	51	40
Alive	53	61	121	152	95	114	103	66	64	81
Ratio	2.3		0.75		2.4		1.44		1.61	

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A closer look:

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In each separate category, the percentage of fatalities among smokers is higher, and yet the overall percentage of fatalities among smokers is lower!

Summary

Random Variables

- ▶ A random variable X is a function $X : \Omega \rightarrow \mathfrak{R}$.
- ▶ $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- ▶ $Pr[X \in A] := Pr[X^{-1}(A)]$.
- ▶ The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathcal{A}\}$.
- ▶ $g(X, Y, Z)$ assigns the value
- ▶ $E[X] := \sum_a a Pr[X = a]$.
- ▶ Expectation is Linear.
- ▶ Independent Random Variables.
- ▶ Variance.