Alex Psomas: Lecture 15.

$\boxed{\text{Bayes' Rule, Mutual Independence, Collisions and Collecting}}$

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Bayes' Rule, Mutual Independence, Collisions and Collecting

1. Conditional Probability
2. Independence
3. Bayes' Rule
4. Balls and Bins
5. Coupons

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Monty Hall

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Question: What should you do in order to maximize the probability of winning?

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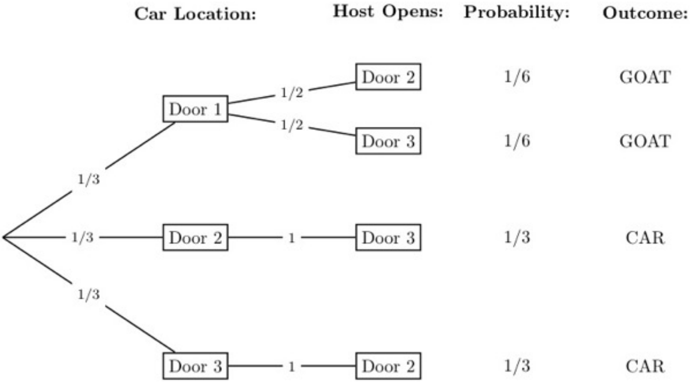
The door I opened wasn't random! I knew it didn't have a prize!! Therefore, switching, is like getting to pick two doors at the beginning!


Car Location:
Host Opens:
Probability:
Outcome:
Door 1
Door 2
Door 3
1/3
1/3
1/3
1/2
1/2
1
1
Door 2
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1/6
1/6
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GOAT
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CAR
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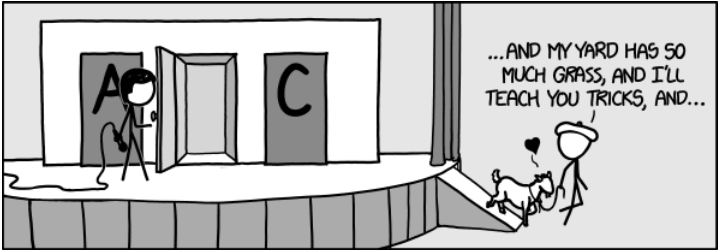
A
C
...AND MY YARD HAS SO MUCH GRASS, AND I'LL TEACH YOU TRICKS, AND...

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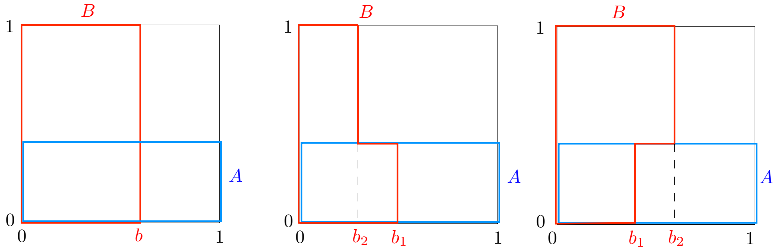
Illustrations: Pick a point uniformly in the unit square



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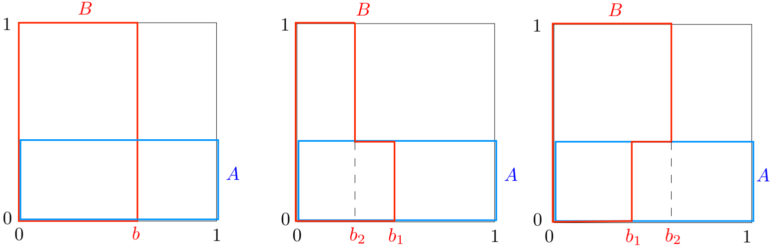

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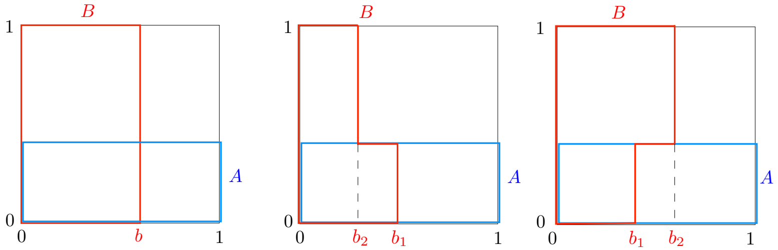

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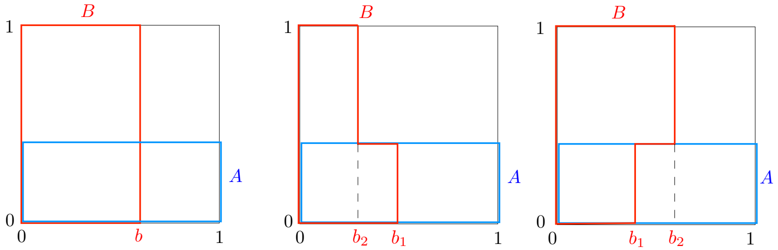

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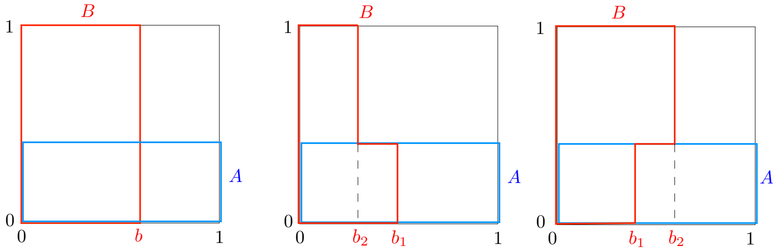

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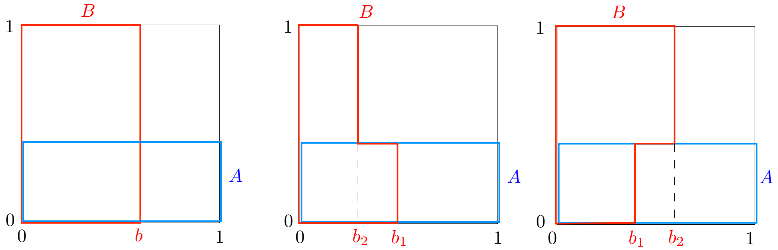

- Left: A and B are independent. $Pr[B] = b; Pr[B|A] =$

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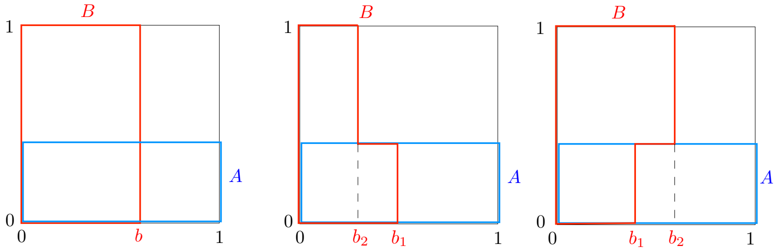

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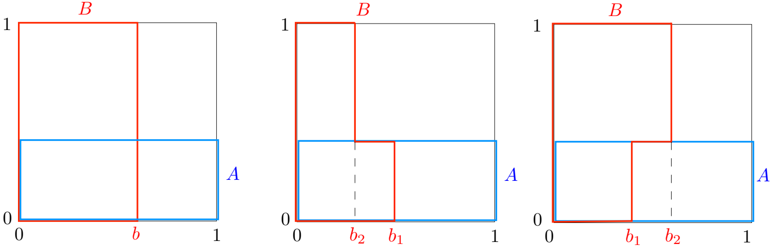

- Left: *A* and *B* are independent. $Pr[B] = b; Pr[B|A] = b$.
- Middle: *A* and *B* are

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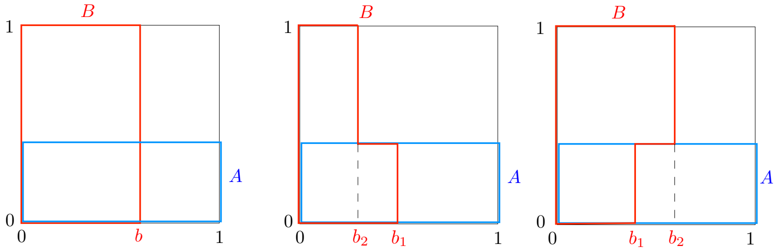

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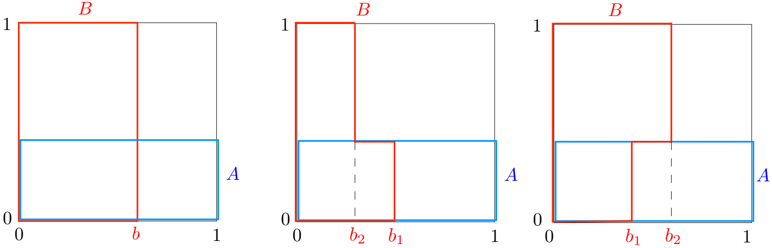

- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
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  $Pr[B|A] =$

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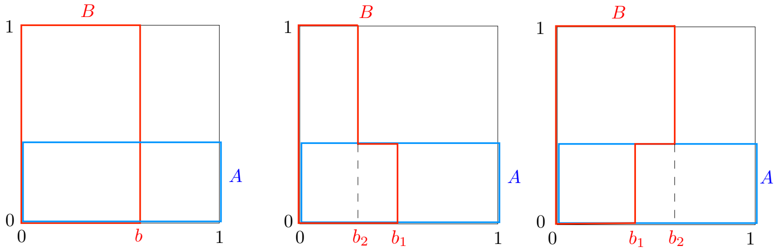

- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
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  $Pr[B|A] = b_1 > Pr[B|\bar{A}] =$

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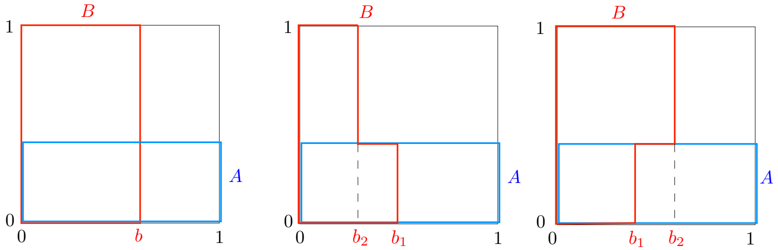

- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- Middle: A and B are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$.

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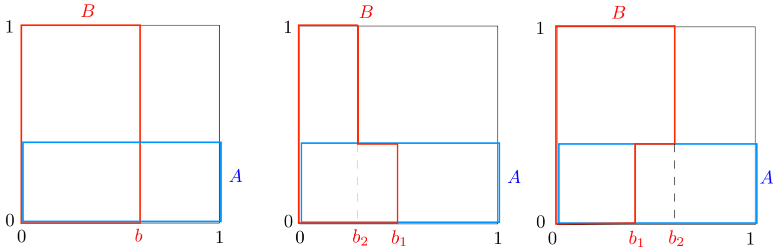

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- Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

Conditional Probability: Pictures

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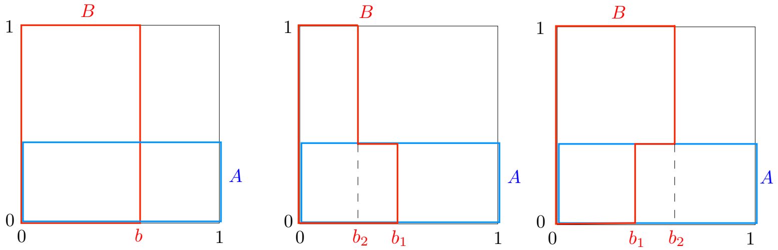

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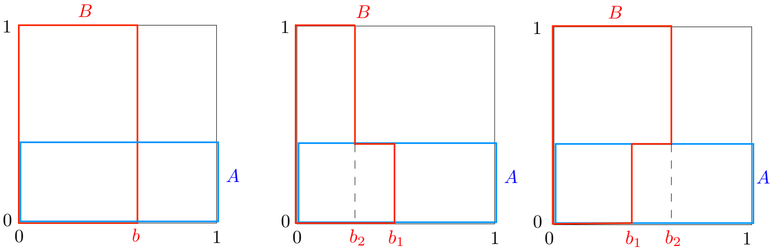

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Conditional Probability: Pictures

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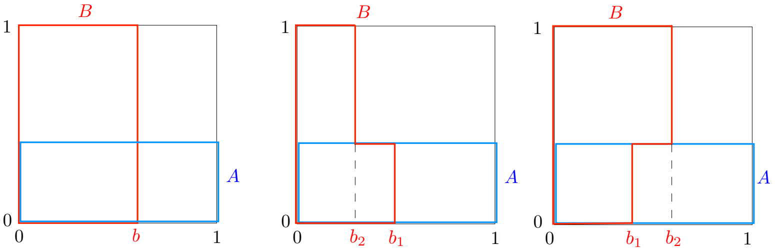
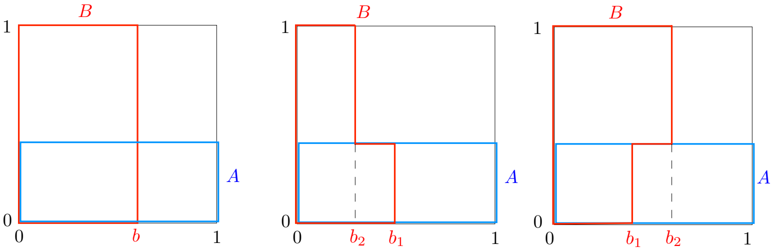


- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
- Right: A and B are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square



- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
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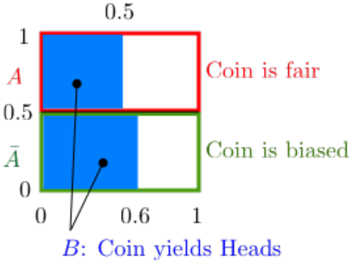
Bayes and Biased Coin

Bayes and Biased Coin



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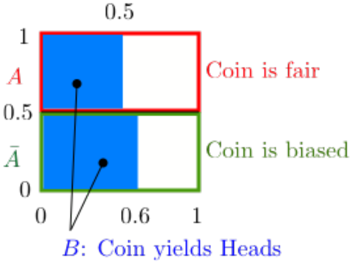

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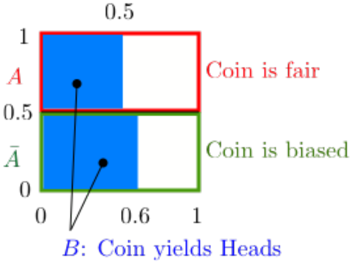
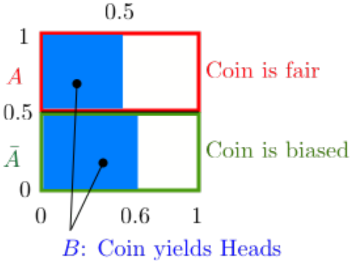
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] =$$

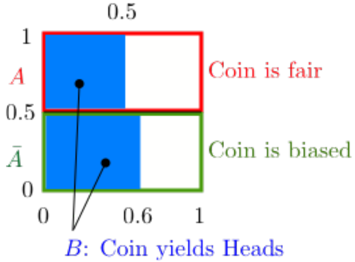
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5;$

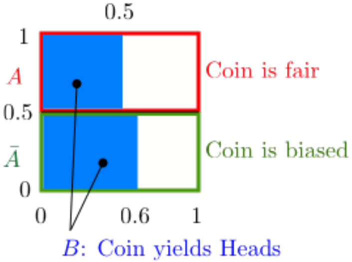
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] =$$

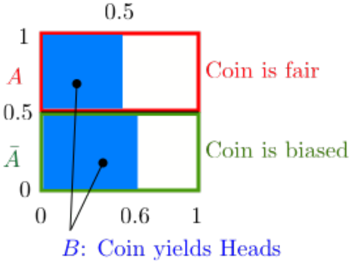
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$

Bayes and Biased Coin

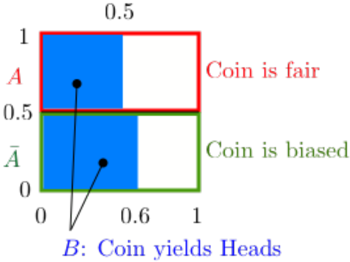


Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

$Pr[B|A] =$

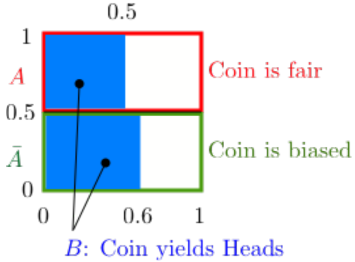
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5;$$

Bayes and Biased Coin



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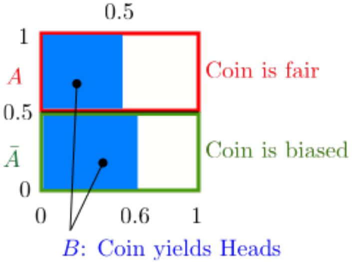
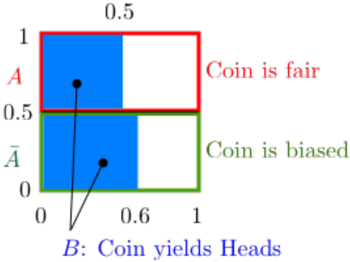
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6;$$

Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
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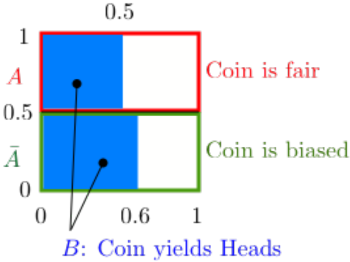
Bayes and Biased Coin



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$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
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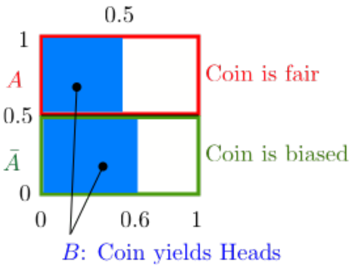
Bayes and Biased Coin



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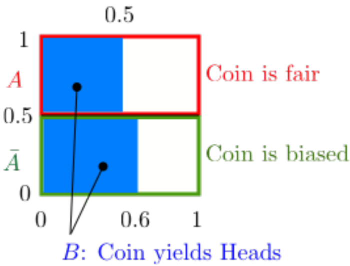
$$Pr[B] =$$

Bayes and Biased Coin



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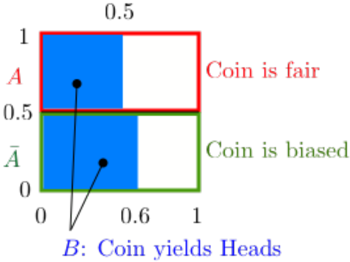
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6$$

Bayes and Biased Coin



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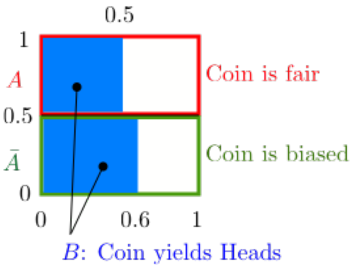
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$

Bayes and Biased Coin



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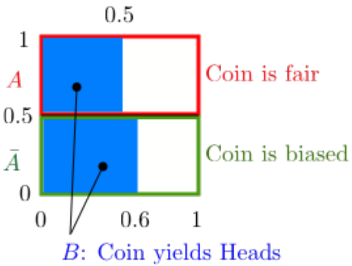
$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6}$$

Bayes and Biased Coin



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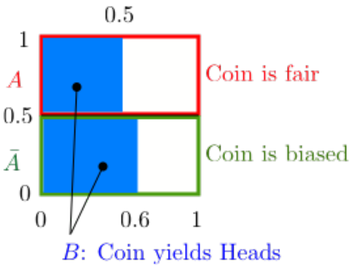
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$

Bayes and Biased Coin



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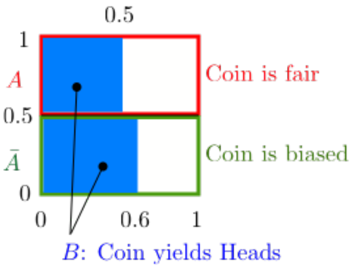
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46$$

Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$
\begin{aligned}
Pr[A] &= 0.5; Pr[\bar{A}] = 0.5 \\
Pr[B|A] &= 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5 \\
Pr[B] &= 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \\
Pr[A|B] &= \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]} \\
&\approx 0.46 = \text{fraction of } B \text{ that is inside } A
\end{aligned}
$$

Bayes: General Case

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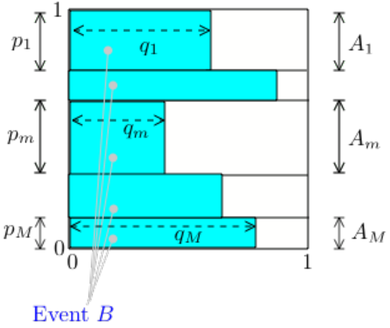
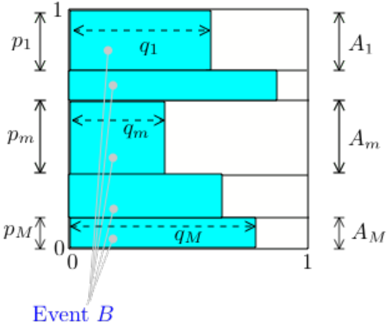


Bayes: General Case



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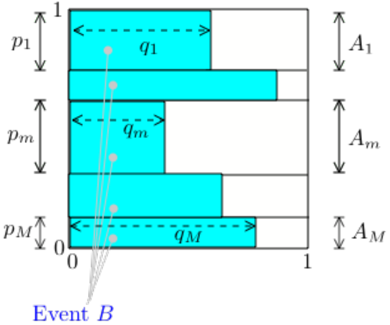
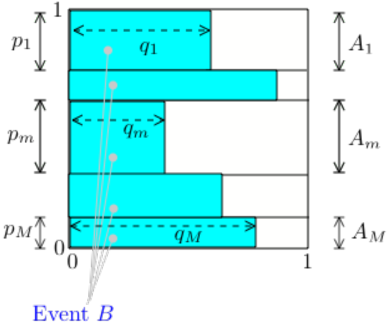
Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$

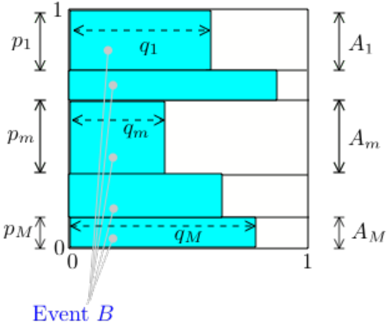
Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M;$$

Bayes: General Case



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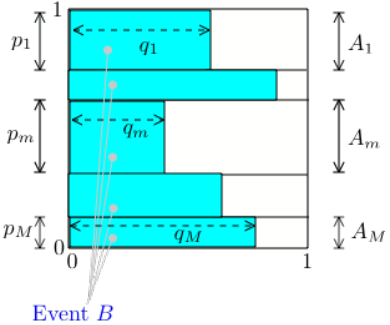
$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] =$$

Bayes: General Case



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$$Pr[A_m] = p_m, m = 1, \ldots, M$$
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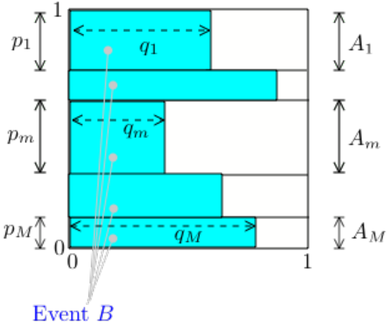
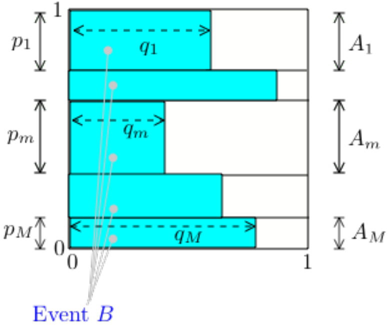
Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$

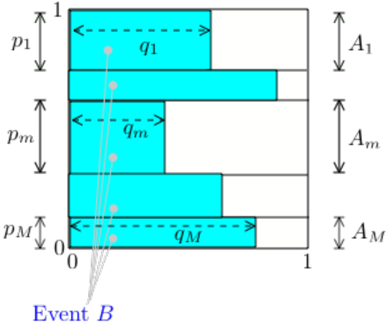
Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$
$$Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots p_M q_M}$$

Bayes: General Case



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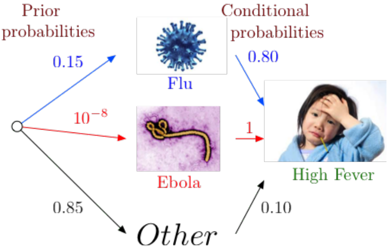
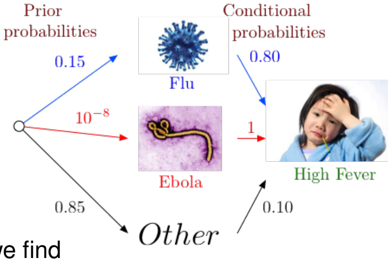
$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$
$$Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots p_M q_M} = \text{ fraction of } B \text{ inside } A_m.$$

Why do you have a fever?

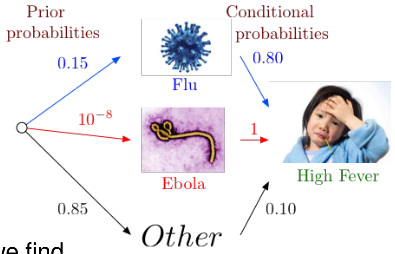


Why do you have a fever?



Using Bayes' rule, we find

Why do you have a fever?



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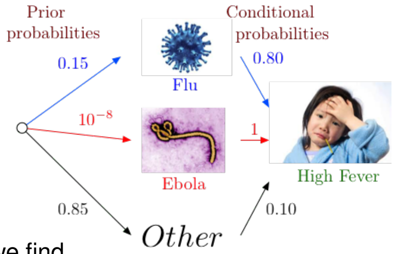

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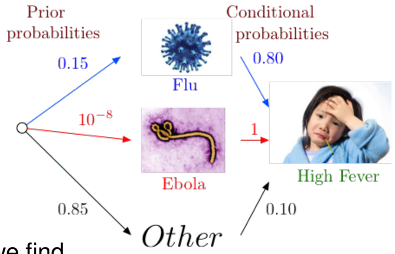

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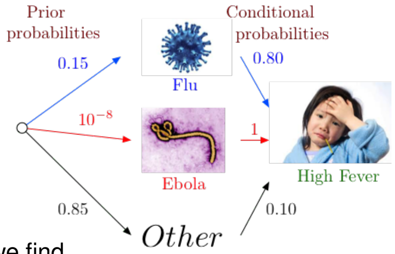

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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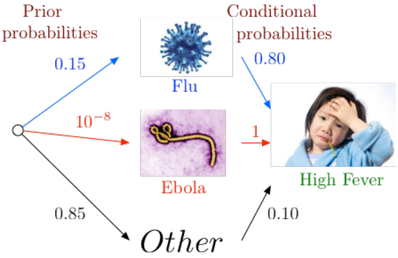

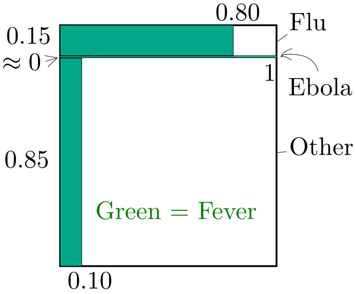

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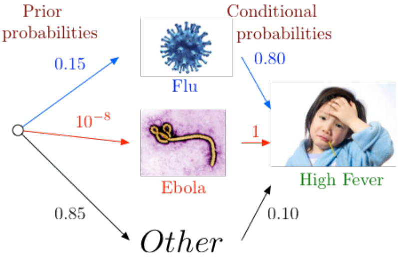

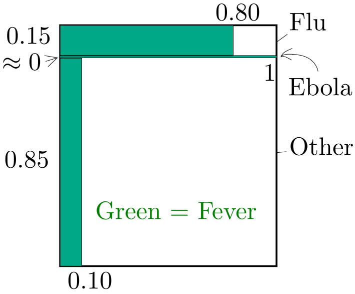

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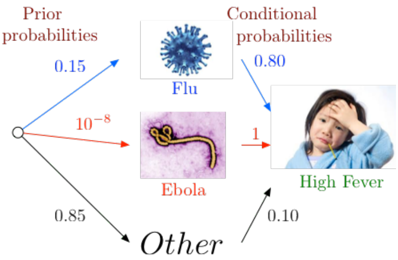

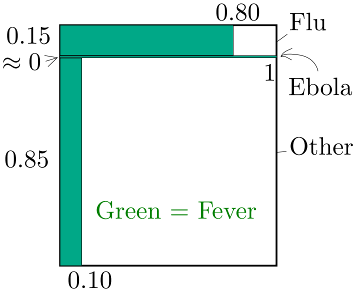

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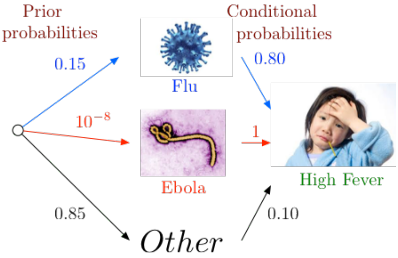

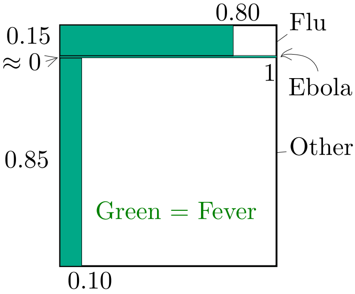

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This example shows the importance of the prior probabilities.

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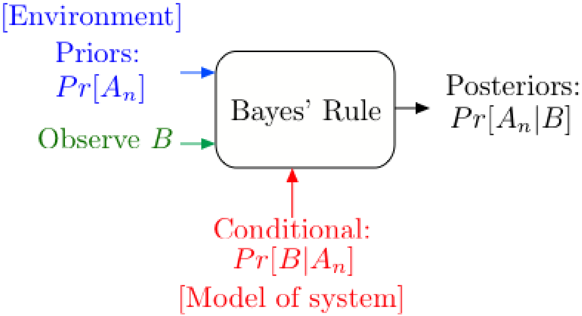

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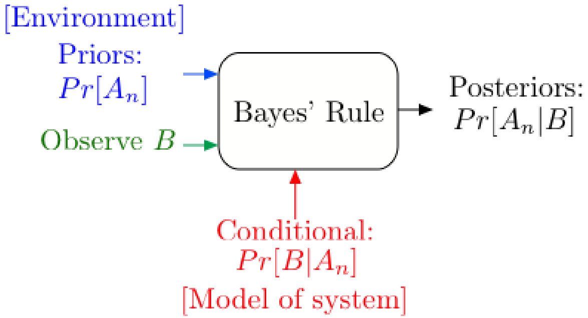

Bayes' Rule is the canonical example of how information changes our opinions.

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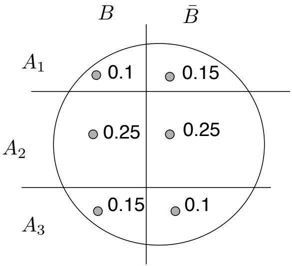

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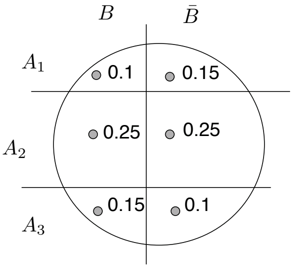

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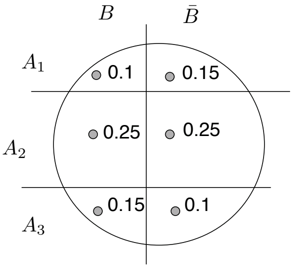

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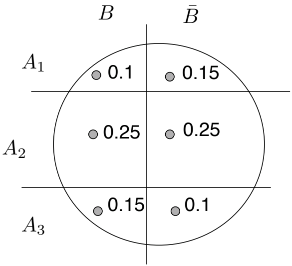

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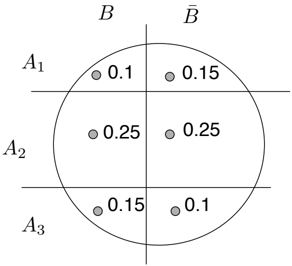

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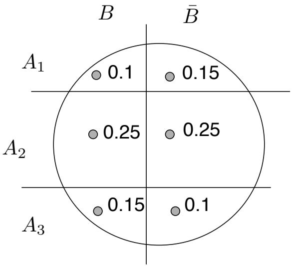

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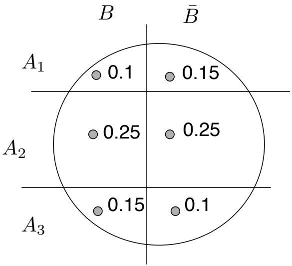

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Pairwise Independence

Flip two fair coins. Let

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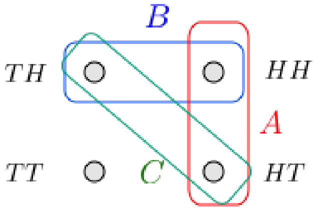

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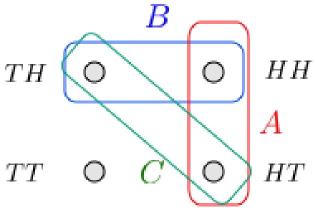

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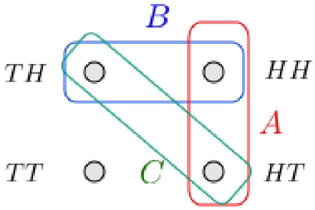

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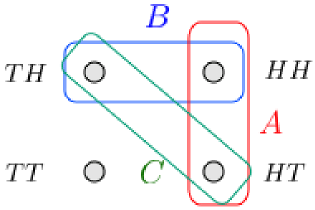

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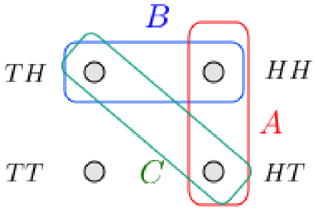

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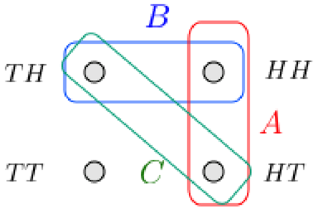

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This leads to a definition ....

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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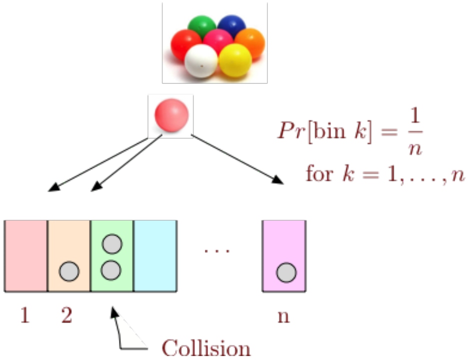

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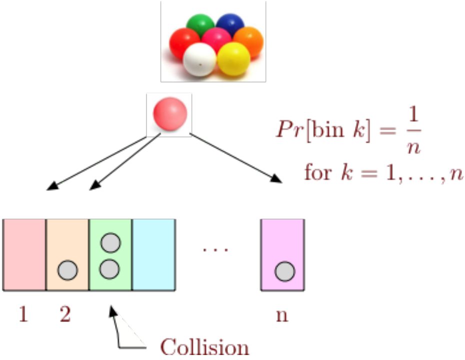

Theorem:
$Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\}$, for large enough n.

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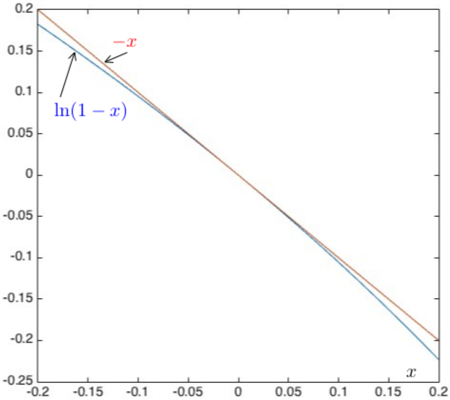
$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



$$\exp\{-x\} = 1 - x + \frac{1}{2!}x^2 + \cdots \approx 1 - x, \text{ for } |x| \ll 1.$$

Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Theorem:

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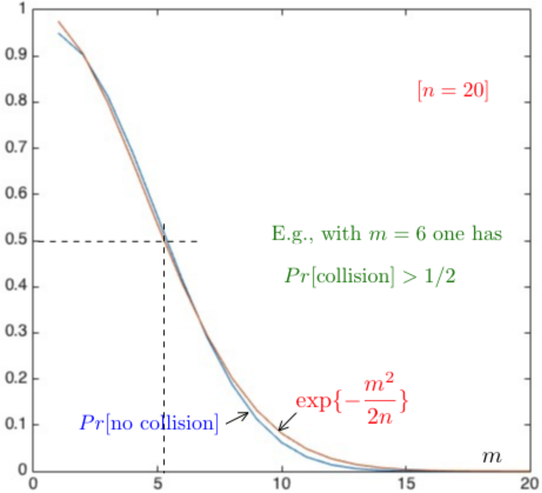

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The birthday paradox

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If $m = 366$, then $Pr[\text{no collision}] = 0$. (No approximation here!)

The birthday paradox

n	$p(n)$
1	0.0%
5	2.7%
10	11.7%
20	41.1%
23	50.7%
30	70.6%
40	89.1%
50	97.0%
60	99.4%
70	99.9%
100	99.99997%
200	99.9999999999999999999999999998%
300	$(100 - (6\times10^{-80}))$%
350	$(100 - (3\times10^{-129}))$%
365	$(100 - (1.45\times10^{-155}))$%
366	100%
367	100%

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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(b) Pr[miss any one of the items] $\leq ne^{-\frac{m}{n}}$.

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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