Alex Psomas: Lecture 18.

Random Variables: Variance

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1. Variance
2. Distributions

Variance

Flip a coin:

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Any other measures???

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Any other measures???
What else that's informative can we say?

Variance

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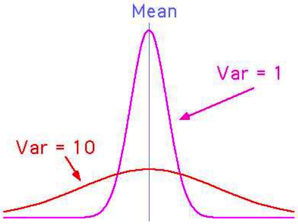

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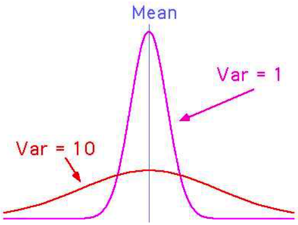

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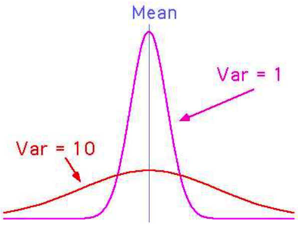

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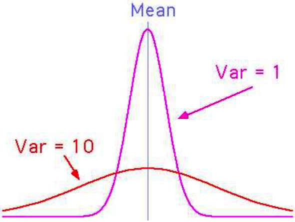

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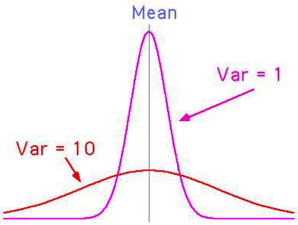

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$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of X.

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Fact:

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Consider X with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

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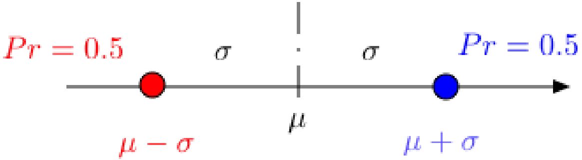

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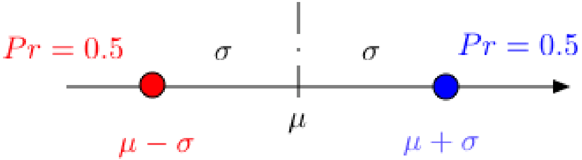

Consider the random variable X such that

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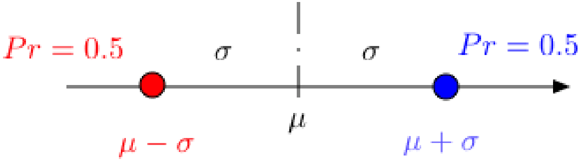

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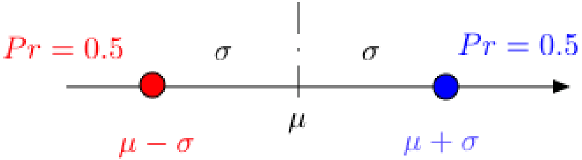

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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$\square$

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&= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots) \\
&= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0 \\
&= var(X) + var(Y) + var(Z) + \cdots.
\end{aligned}$$

$\square$

Distributions

- Bernoulli
- Binomial
- Uniform
- Geometric

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Flip a coin, with heads probability p.

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Jacob Bernoulli

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$$Pr[X = i] = \binom{n}{i} p^i(1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

Expectation of Binomial Distribution

Indicator for the i-th coin:

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$$E[X]=\sum_{m=1}^{n} mPr[X=m]=\sum_{m=1}^{n} m\times\frac{1}{n}=\frac{1}{n}\frac{n(n+1)}{2}=\frac{n+1}{2}.$$

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This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

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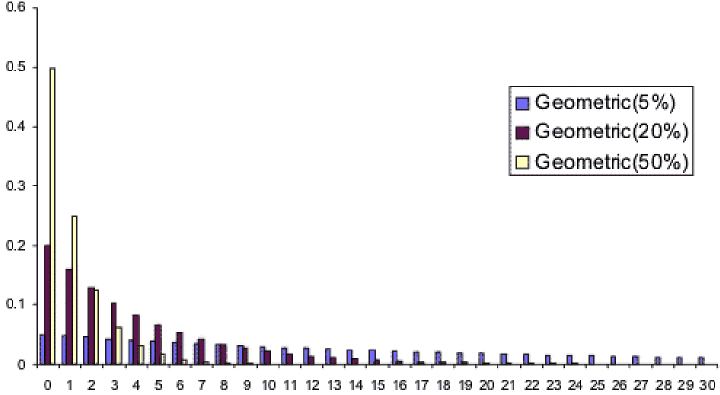

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X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second (distinct) coupon after getting first.

Pr[“get second distinct coupon”|“got first coupon$''$] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

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Review: Harmonic sum

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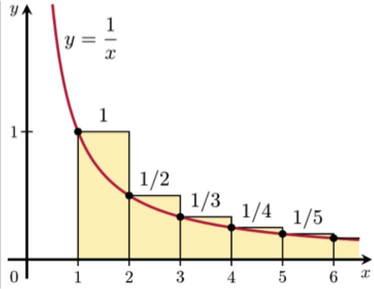

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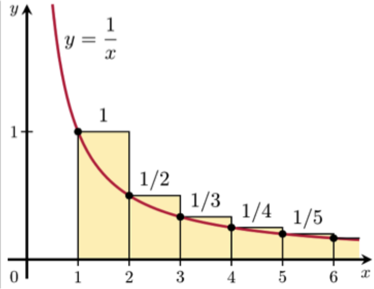

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

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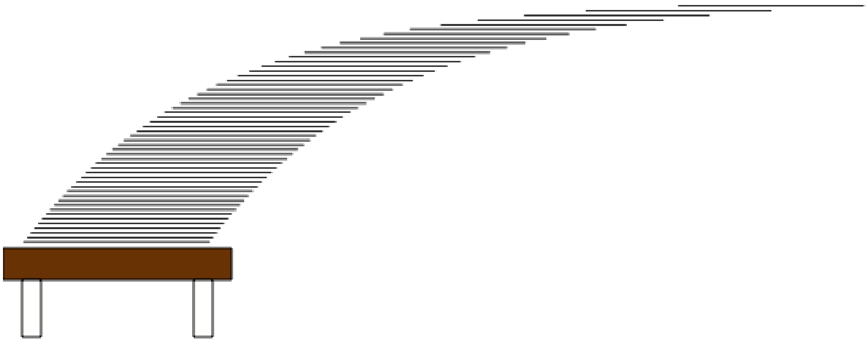

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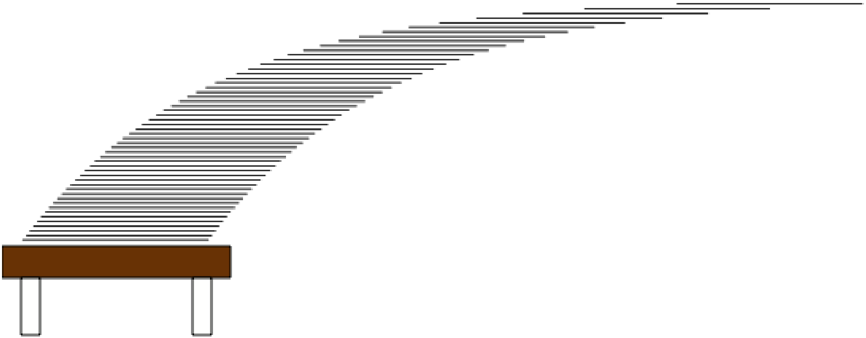

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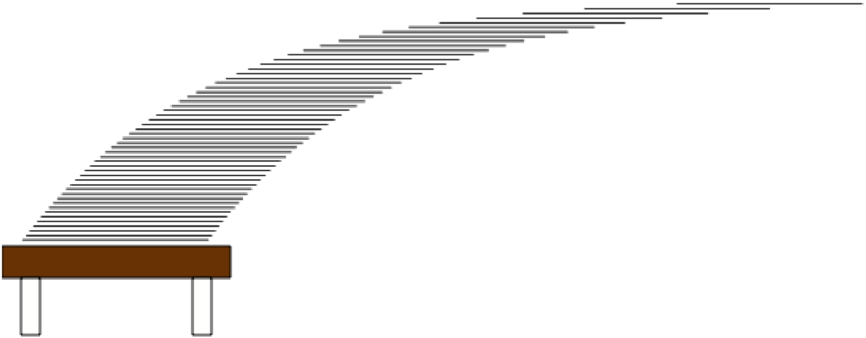
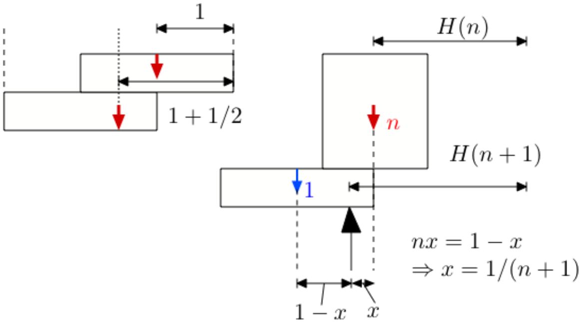


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Stacking



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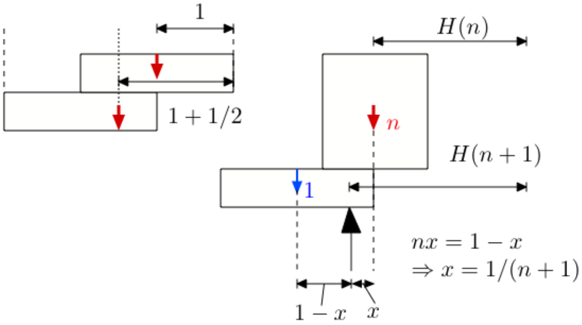

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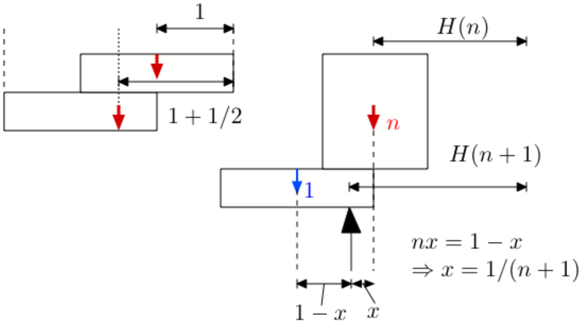

The cards have width 2. Induction shows that the center of gravity after *n* cards is $H(n)$ away from the right-most edge.

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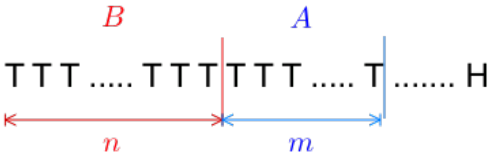

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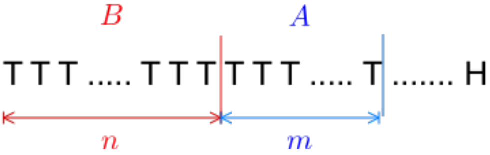

The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

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[See later for a proof.]

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$\square$

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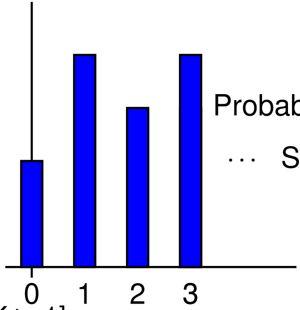

Probability mass at i, counted i times.

$\cdots$ Same as $\sum_{i=1}^{\infty} i \times Pr[X = i]$.

$Pr[X \geq 1] \longrightarrow$

$Pr[X \geq 2] \longrightarrow$

$Pr[X \geq 3] \longrightarrow$

$\vdots$

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