Alex Psomas: Lecture 15.

Bayes' Rule, Mutual Independence, Collisions and Collecting

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1. Conditional Probability
2. Independence
3. Bayes' Rule
4. Balls and Bins
5. Coupons

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Question: What should you do in order to maximize the probability of winning?

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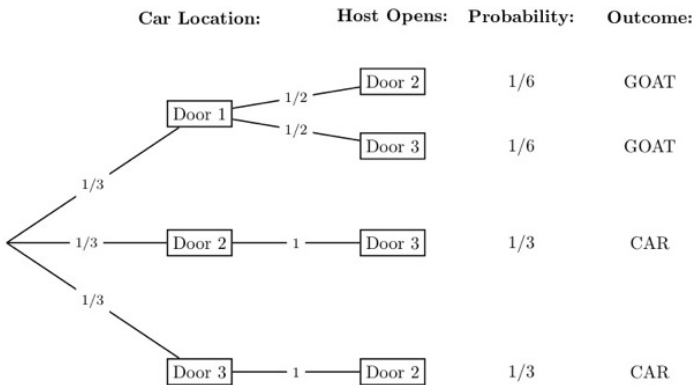
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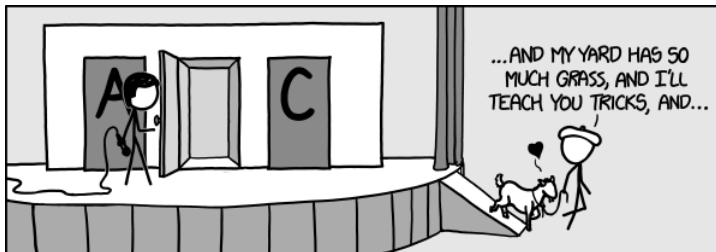
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Door 1 — 1/2 — Door 2   1/6   GOAT

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Door 2 — 1 — Door 3   1/3   CAR

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1/3

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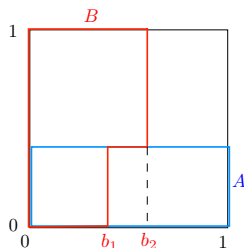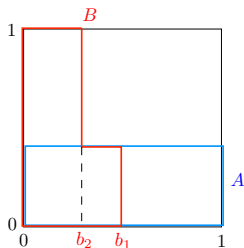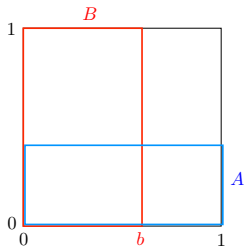
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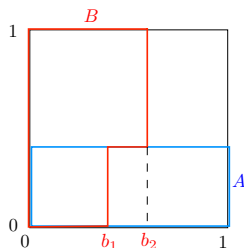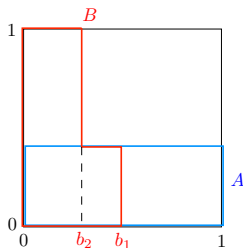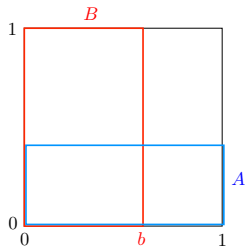
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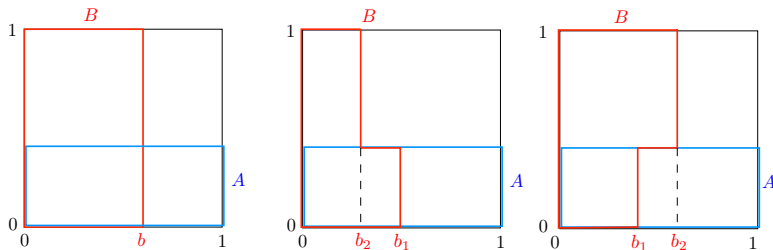
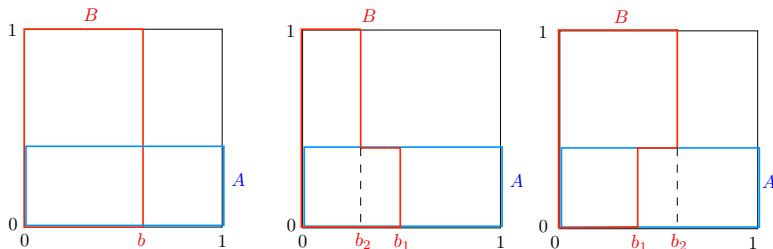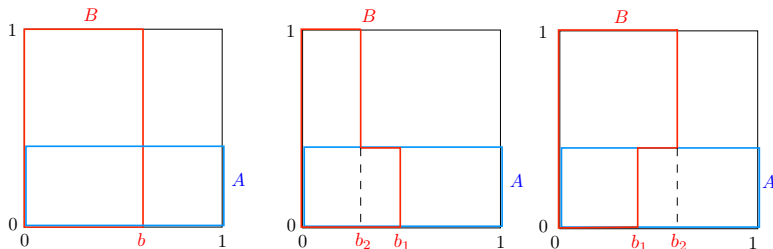- ▶ Left: *A* and *B* are independent.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
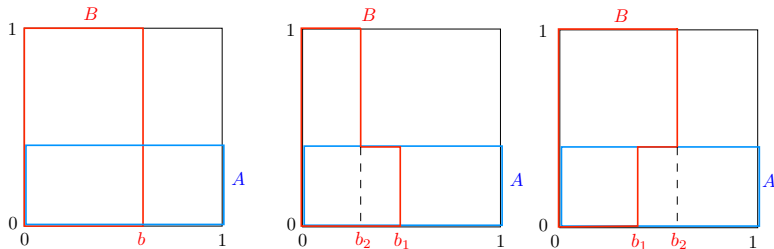
- Left: A and B are independent. $Pr[B] =$

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
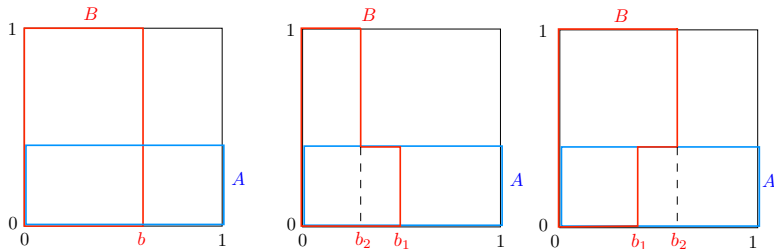
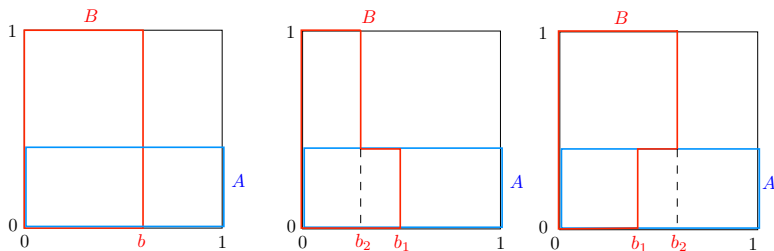- ► Left: A and B are independent. $Pr[B] = b$;

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
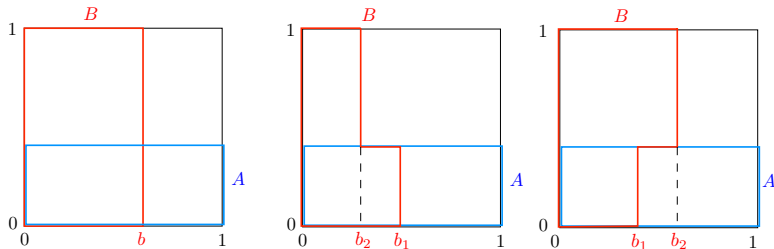
- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] =$

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
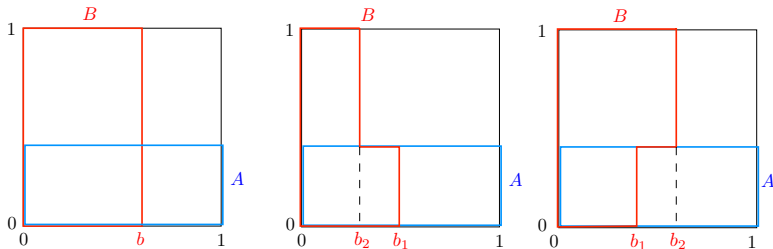
- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
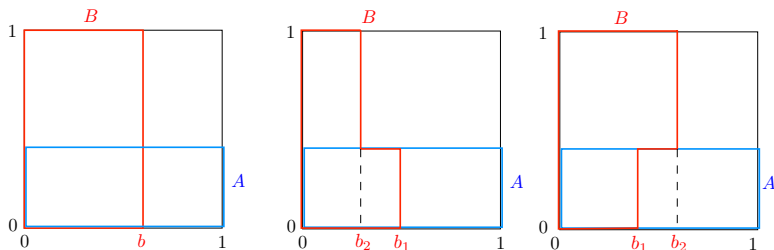
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- Left: A and B are independent. $Pr[B] = b$; $Pr[B|A] = b$.
- Middle: A and B are

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b$; $Pr[B|A] = b$.
- ▶ Middle: A and B are positively correlated.

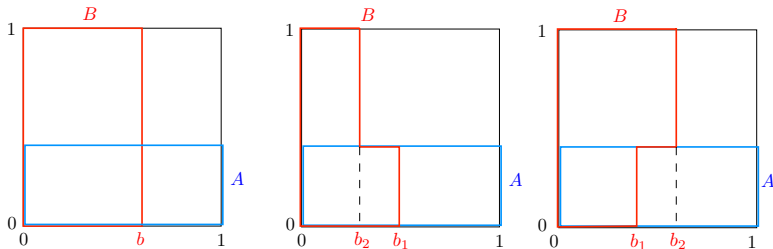# Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
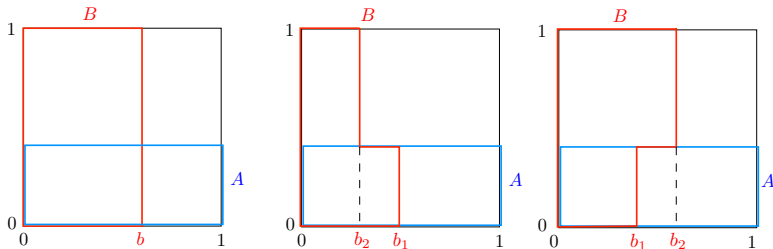
- Left: A and B are independent. $Pr[B] = b$; $Pr[B|A] = b$.
- Middle: A and B are positively correlated.
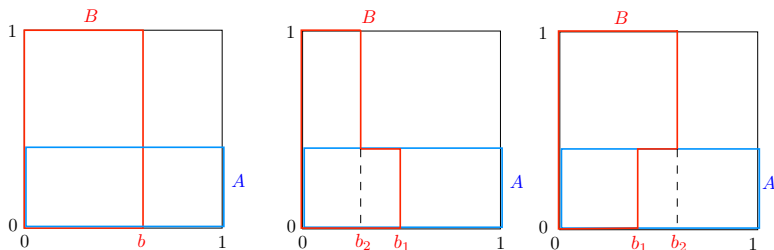 $Pr[B|A] =$

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square
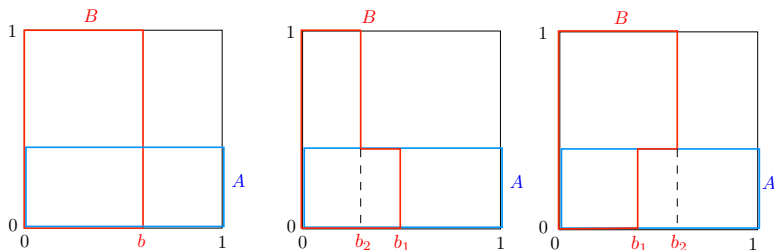
- ► Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- ► Middle: A and B are positively correlated.
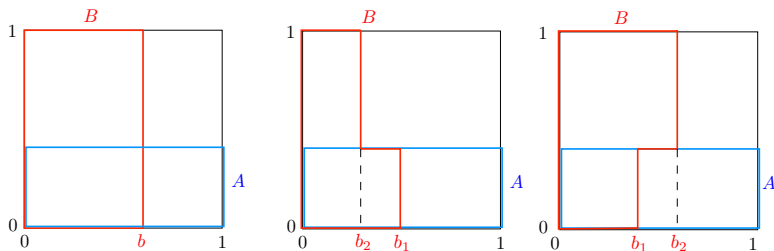 $Pr[B|A] = b_1 > Pr[B|\bar{A}] =$

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- ▶ Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- ▶ Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
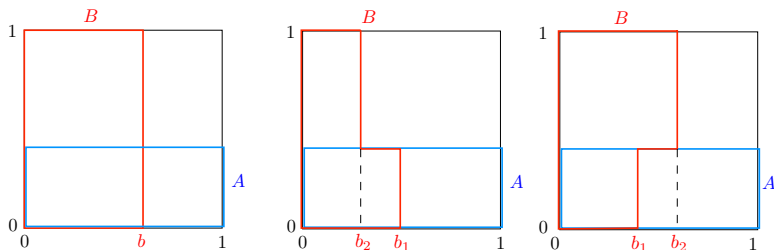
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
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- ▶ Right: A and B are

Conditional Probability: Pictures

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- ▶ Left: A and B are independent. $Pr[B] = b$; $Pr[B|A] = b$.

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- ▶ Right: A and B are negatively correlated.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.

- ▶ Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- ▶ Right: A and B are negatively correlated.
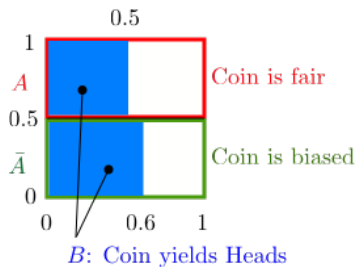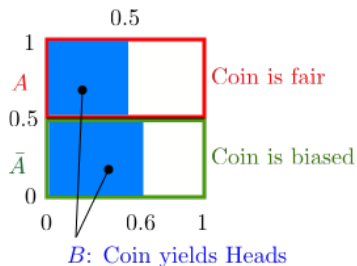 $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- ▶ Left: A and B are independent. $Pr[B] = b$; $Pr[B|A] = b$.

- ▶ Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- ▶ Right: A and B are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$.

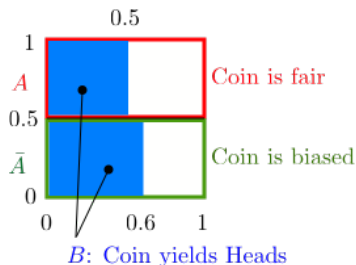# Bayes and Biased Coin

Bayes and Biased Coin

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then
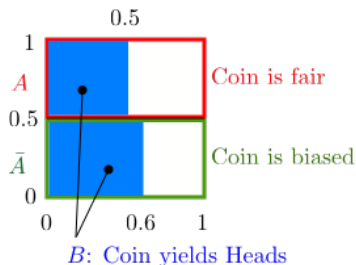
Bayes and Biased Coin

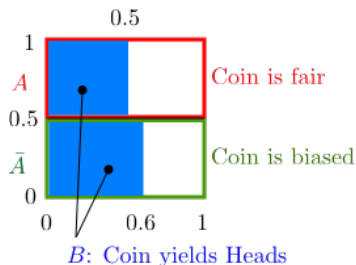Pick a point uniformly at random in the unit square. Then

$$Pr[A] =$$

Bayes and Biased Coin

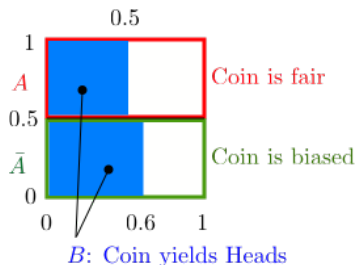Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5;$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then
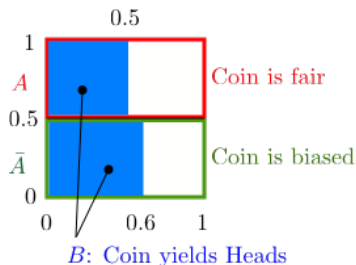
$$Pr[A] = 0.5; Pr[\bar{A}] =$$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

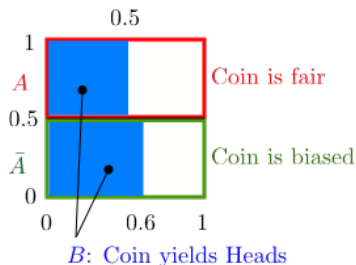$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

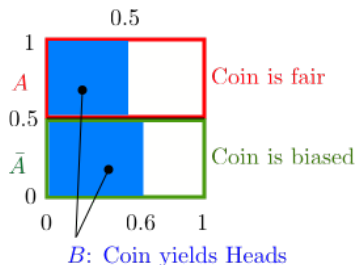$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

$Pr[B|A] =$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$
$Pr[B|A] = 0.5;$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$
$Pr[B|A] = 0.5; Pr[B|\bar{A}] =$

Bayes and Biased Coin

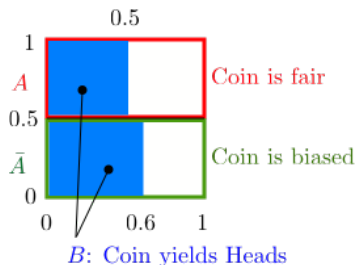Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$
$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6;$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] =$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

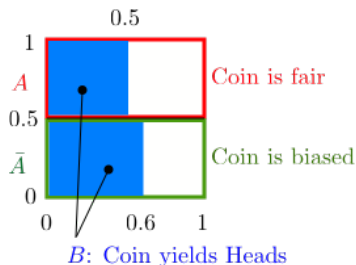$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$
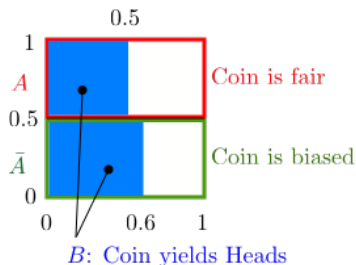
$Pr[B] =$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6$$

Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
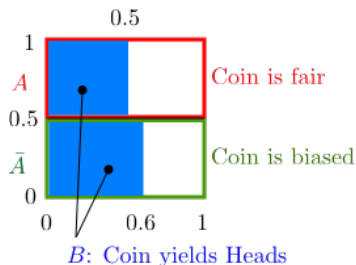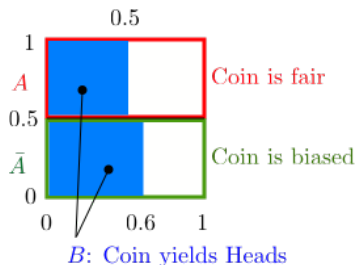
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6}$$
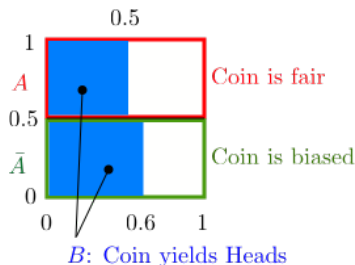
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$

Bayes and Biased Coin
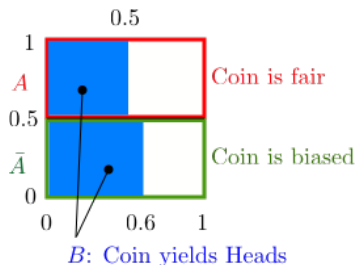

B: Coin yields Heads

Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46$$

Bayes and Biased Coin

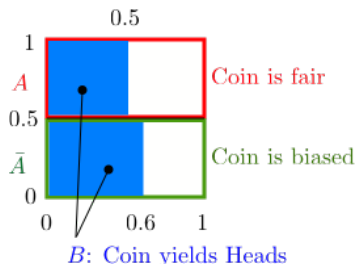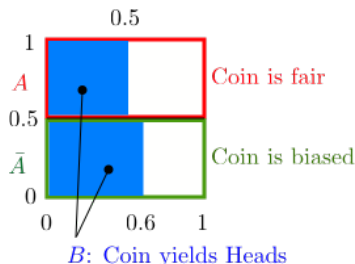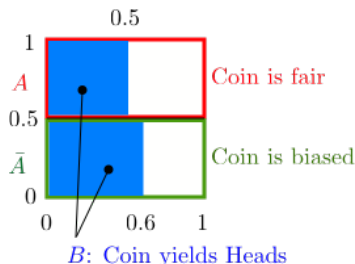Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case
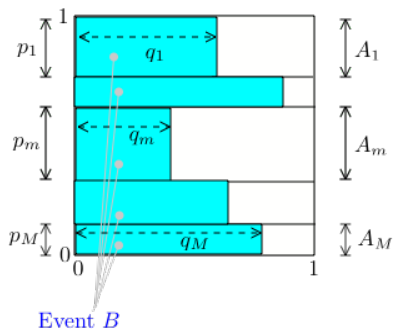
Bayes: General Case
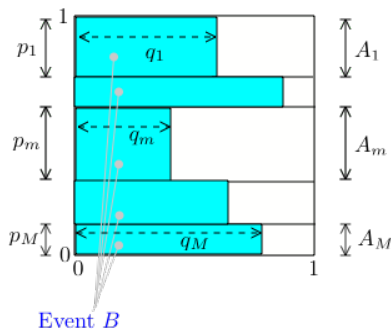
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

Bayes: General Case
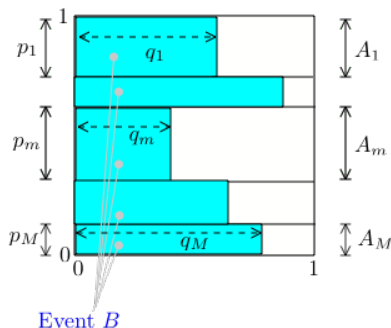
Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$

Bayes: General Case
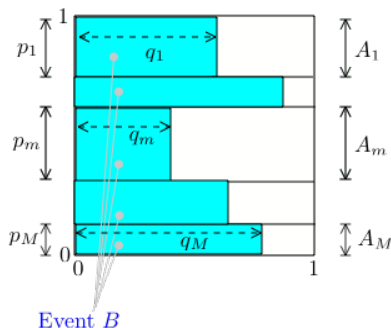
Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M;$$

Bayes: General Case
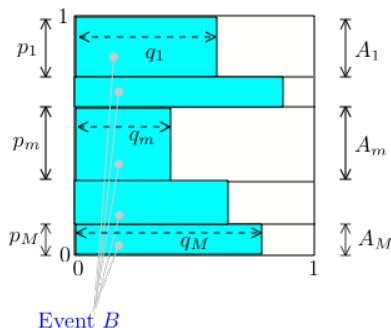
Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] =$$

Bayes: General Case
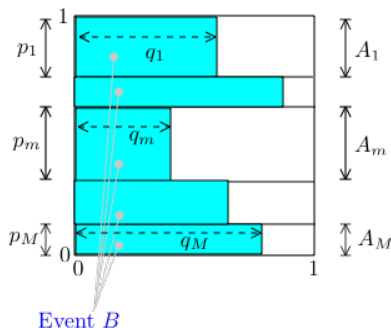
Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$

Bayes: General Case
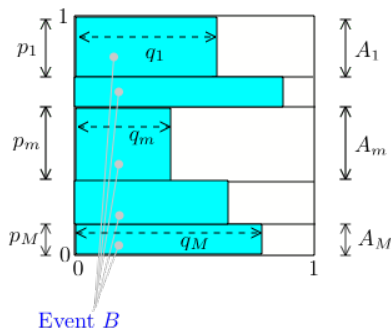
Event B

Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$

Bayes: General Case
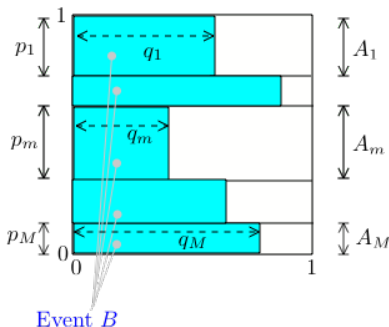
Event B

Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$
$$Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots p_M q_M}$$

Bayes: General Case
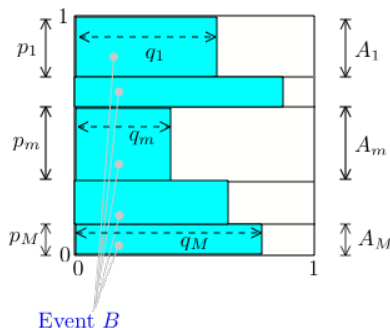
Event B

Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, m = 1, \ldots, M; Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots p_M q_M$$
$$Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots p_M q_M} = \text{ fraction of } B \text{ inside } A_m.$$

Why do you have a fever?

Why do you have a fever?

Prior probabilities

Conditional probabilities

0.15 Flu 0.80

10^{-8} Ebola 1

0.85 *Other* 0.10
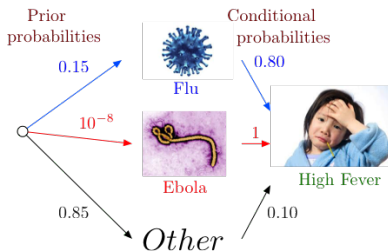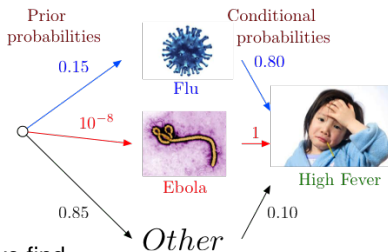
High Fever

Using Bayes' rule, we find

Why do you have a fever?
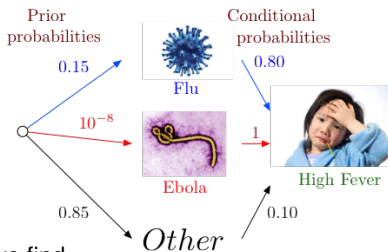
Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

Why do you have a fever?
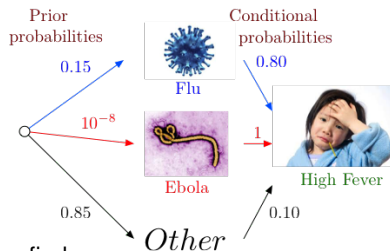
Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

$$Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

Why do you have a fever?
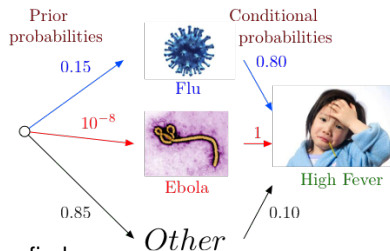
Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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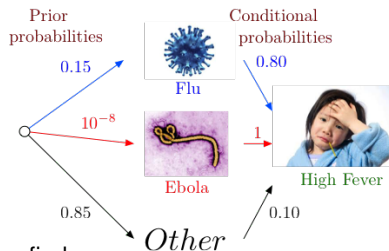
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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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Our "Bayes' Square" picture:

Why do you have a fever?

Our "Bayes' Square" picture:

Prior probabilities

0.15 → Flu
10^{-8} → Ebola
0.85 → Other

Conditional probabilities

Flu → 0.80 → High Fever
Ebola → 1 → High Fever
Other → 0.10 → High Fever

0.15
≈ 0
0.80 Flu
1 Ebola
Other
0.85
Green = Fever
0.10
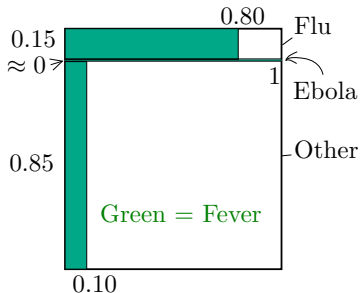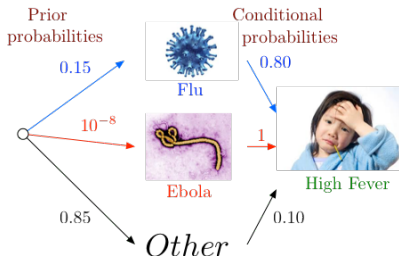
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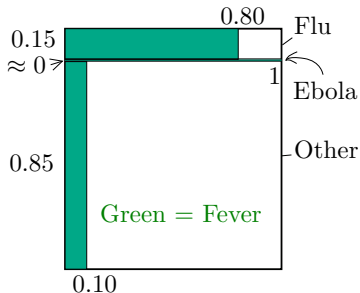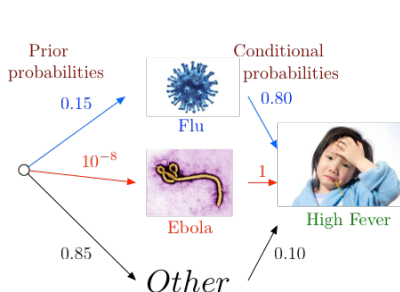
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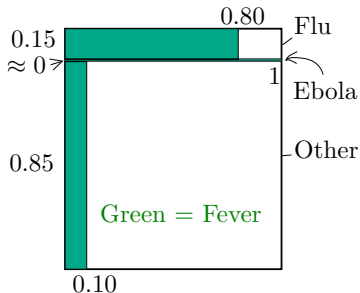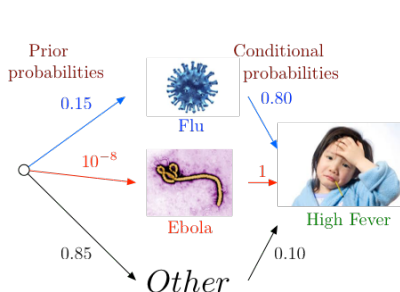
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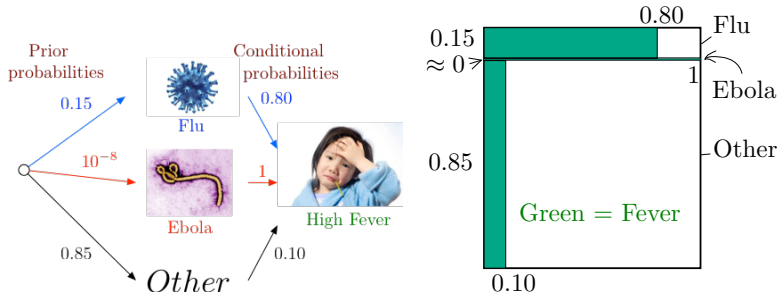
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This example shows the importance of the prior probabilities.

Bayes' Rule Operations

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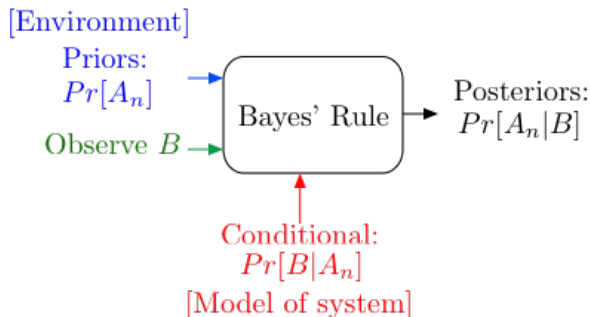
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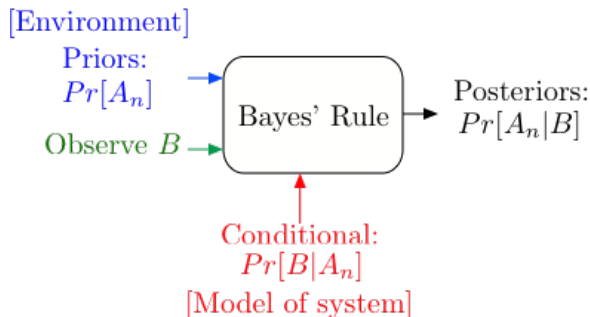
Bayes' Rule is the canonical example of how information changes our opinions.

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Pairwise Independence

Flip two fair coins. Let

- $A =$ 'first coin is H' $= \{HT, HH\}$;
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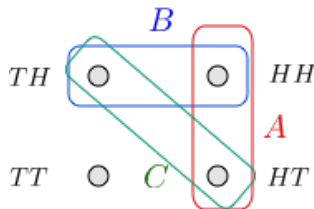
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A did not say anything about C and B did not say anything about C, but $A \cap B$ said something about C!

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This leads to a definition ....

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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1   2   ...   n

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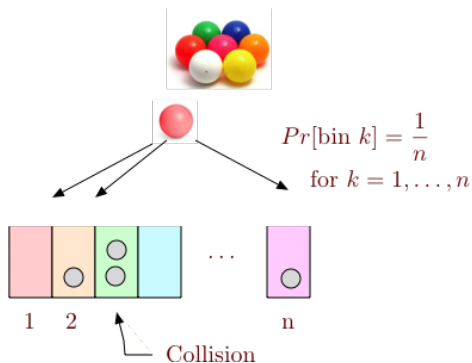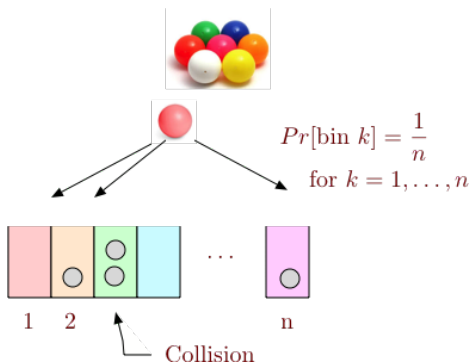
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Theorem:
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$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation
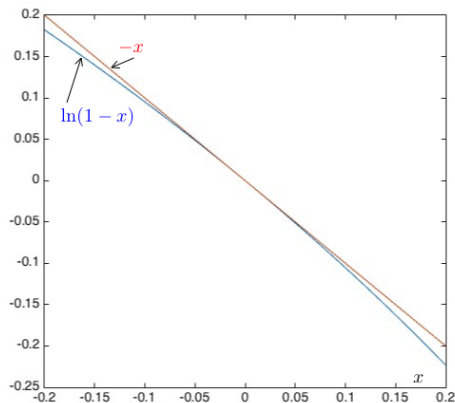
$$\exp\{-x\} = 1 - x + \frac{1}{2!}x^2 + \cdots \approx 1 - x, \text{ for } |x| \ll 1.$$

Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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$[n = 20]$

E.g., with $m = 6$ one has

$Pr[\text{collision}] > 1/2$

$\exp\{-\frac{m^2}{2n}\}$

$Pr[\text{no collision}]$

m
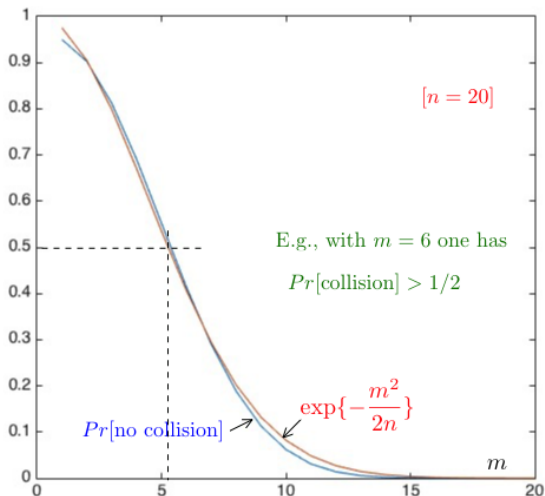
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The birthday paradox

Today's your birthday, it's my birthday too..

Probability that *m* people all have different birthdays?

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If $m = 366$, then $Pr[\text{no collision}] = 0$. (No approximation here!)

The birthday paradox

n	$p(n)$
1	0.0%
5	2.7%
10	11.7%
20	41.1%
23	50.7%
30	70.6%
40	89.1%
50	97.0%
60	99.4%
70	99.9%
100	99.99997%
200	99.9999999999999999999999999998%
300	$(100 - (6 \times 10^{-80}))\%$
350	$(100 - (3 \times 10^{-129}))\%$
365	$(100 - (1.45 \times 10^{-155}))\%$
366	100%
367	100%

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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There are *n* different baseball cards.
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For $p_m = \frac{1}{2}$, we need around $n\ln 2 \approx 0.69n$ boxes.

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Plug in and get

$$p \leq n e^{-\frac{m}{n}}.$$

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Summary.

Bayes' Rule, Mutual Independence, Collisions and Collecting

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  $Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]$.
- Balls in bins: m balls into $n > m$ bins.

$$Pr[\text{no collisions}] \approx \exp\{-\frac{m^2}{2n}\}$$

- Coupon Collection: n items. Buy m cereal boxes.

  $Pr[\text{miss one specific item}] \approx e^{-\frac{m}{n}}$; $Pr[\text{miss any one of the items}] \leq n e^{-\frac{m}{n}}$.

Key Mathematical Fact: $\ln(1 - \varepsilon) \approx -\varepsilon$.