CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 4C Sol

1. This is Potpourri

1. Out of 1000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.
   (a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.
   (b) What is the probability that the student belongs to a club?
   (c) What is the probability that the student works part time?
   (d) What is the probability that the student belongs to a club AND works part time?
   (e) What is the probability that the student belongs to a club OR works part time?

   (a) The sample space will be Inclusion?exclusion illustrated by a Venn diagram.
   (b) $\Pr[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \boxed{\mathbf{.4}}$
   (c) $\Pr[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \boxed{\mathbf{.5}}$
   (d) $\Pr[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \boxed{\mathbf{.05}}$
   (e) $\Pr[P \cup C] = \Pr[P] + \Pr[C] - \Pr[P \cap C] = \boxed{\mathbf{.85}}$.

2. Suppose you roll an ordinary die 5 times.
   (a) What is the probability of getting at least one six?
   (b) What is the probability of getting exactly two sixes?
   (c) What is the probability of getting a prime number of sixes?

   (a) $\Pr[\text{at least one 6}] = 1 - \Pr[\text{no 6's}] = 1 - \frac{5^5 \text{ ways of getting no 6's}}{6^5 \text{ possible outcomes}} = \boxed{\mathbf{1 - \left(\frac{5}{6}\right)^5}} \approx .598$
   (b) To find the number of ways to roll exactly two 6's, first choose which 2 of the 5 dice will be the 6's ($\binom{5}{2}$ ways of doing this), and then choose what the remaining 3 dice will be (5^3 ways of doing this). So $\Pr[\text{exactly two 6's}] = \boxed{\mathbf{\frac{\binom{5}{2} \cdot 5^3}{6^5}}} \approx .161$
   (c) $\Pr[\text{prime number of 6's}] = \Pr[\text{getting exactly two 6's}] + \Pr[\text{getting exactly three 6's}] + \Pr[\text{getting exactly five 6'}$ The probability of getting i 6's, as follows from part (b), is $\frac{\binom{5}{i} \cdot 5^{5-i}}{6^5}$, because you choose which i dice are the 6's and what the $5-i$ dice left will be. So $\Pr[\text{prime number of 6's}] = \boxed{\mathbf{\frac{\binom{5}{2} \cdot 5^3}{6^5} + \frac{\binom{5}{3} \cdot 5^2}{6^5} + \frac{\binom{5}{5}}{6^5}}} \approx .193$

2. Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in CS70, you are curious to play around with these numbers. Find the probability that

a. A given day is windy and rainy.

Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\Pr(R|W) = 0.3$, $\Pr(R|W^C) = 0.8$ and $\Pr(W) = 0.2$. Then probability that a given day is both rainy and windy is $\Pr(R \cap W) = \Pr(R|W)\Pr(W) = 0.3 \cdot 0.2 = 0.06$

b. It rains on a given day.

Probability that it rains on a given day is $\Pr(R) = \Pr(R|W)\Pr(W) + Pr(R|W^C)Pr(W^C) = 0.3 \cdot 0.2 + 0.8 \cdot 0.8 = 0.7$

c. Exactly one of two days is rainy. (Assume that the two days are independent.)

Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $Pr(R_1) = Pr(R_2) = Pr(R)$. The desired probability is $Pr(R_1)Pr(R_2^C) + Pr(R_1^C)Pr(R_2) = 2 * 0.7 * 0.3 = 0.42$

d. A non-rainy day is also non-windy.

Probability that a non-rainy day is non-windy is $Pr(W^C|R^C) = \frac{Pr(W^C \cap R^C)}{Pr(R^C)} = \frac{Pr(R^C|W^C)(W^C)}{Pr(R^C)} = \frac{0.2 \cdot 0.8}{0.3} = \frac{8}{15}$

3. Monty Hall Again

In the three-door Monty Hall problem, there are two stages to the decision, the initial pick followed by the decision to stick with it or switch to the only other remaining alternative after the host has shown an incorrect door. An extension of the basic problem to multiple stages goes as follow.

Suppose there are four doors, one of which is a winner. The host says: "You point to one of the doors, and then I will open one of the other non-winners. Then you decide whether to stick with your original pick or switch to one of the remaining doors. Then I will open another (other than the current pick) non-winner. You will then make your final decision by sticking with the door picked on the previous decision or by switching to the only other remaining door.

Find the best strategy and compute its probability of winning.

Solution 1

We calculate probability of winning given that we play with a specific strategy. We use RRR to denote picking the right door all 3 times. WRW then means picking the wrong door for the first time, right for the second time, and wrong for the third time. Thus, we win if the third letter is R.

Note the notation of P(WWW ; S1), which means "probability of WWW under strategy S1", instead of P(WWW | S1), i.e. "probability of WWW conditioned on S1", because S1 is not random.

S1: Stick and stick strategy

Case 1 (RRR): Pick the right door at the beginning and stick with it, so we pick the right door all three times. **P(RRR ; S1) = P(pick the right door at the beginning) = 1/4**

Case 2 (WWW): Pick one of the wrong doors at the beginning. **P(WWW ; S1) = 3/4**

Notice that the sum of the two cases is 1, so we do not miss any case.

The reason that we only need to consider 2 cases is because we stick to the same door throughout, so we either always picked the right door, or always picked the wrong door.

P(win ; S1) = P(RRR;S1) = 1/4

S2: Stick and switch strategy

Case 1 (RRW): pick the right one at the beginning. **P(RRW ; S2) = 1/4**

Case 2 (WWR): Pick one of the wrong doors at the beginning. Then, the host will open another wrong one. We first stick with our door, so the host will open yet another wrong door. Thus, we are left with the right door to switch to. In short, if we pick a wrong one at the beginning, it is guaranteed that we will pick the right one at the end with this strategy. **P(WWR; S2) = P(pick a wrong door at the beginning) = 3/4**.

P(win ; S2) = P(WWR;S2) = 3/4

As an example, we will formally show the derivation of P(RRW; S2):

- P(RR ; S2) = P(R ; S2) = P(pick the right door at the beginning) = 1/4 since we are sticking the first round (so picking the right door at first = picking the right door the first 2 times), and the probability is just 1/4.

- P(RRW | RR; S2) = 1, because conditioned on us picking the right door, the only other door that remains closed must be wrong, so we are guaranteed to pick the wrong door if we switch.

- P(RRW; S2) = P(RR; S2)*P(RRW|RR; S2) = 1/4 * 1 = 1/4 by bayes rule.

For P(WWR; S2): - P(WW; S2) = P(W; S2) = 3/4, once again because we stick the first - P(RRW | RR; S2) = 1, because conditioned on us picking the right door, the only other door that remains closed must be right, so we are guaranteed to pick the right door if we switch.

- P(WWR; S2) = P(WWR|WW; S2) * P(WW; S2) = 1 * 3/4 = 3/4 by bayes rule.

Similar reasoning applies to the other cases.

S3: Switch and stick strategy

Case 1: **P(RWW ; S3) = 1/4**

Case 2: **P(WWW ; S3) = 3/4 * 1/2**. This comes from 3/4 probability of picking one of the wrong doors at first. Host opens another wrong door. Now there are 2 doors left, so the probability of picking the wrong door the second time is 1/2, then we stick with the wrong door.

Case 3: **P(WRR ; S3) = 3/4 * 1/2**. Follow the similar logic as case 2.

P(win ; S3) = P(WRR ; S3) = 3/8

S4: Switch and Switch strategy

Case 1: **P(RWR ; S4) = 1/4**, which is the probability of picking the right door at the first time. If we pick the right one the first time, host opens one of the wrong doors. We then switch to another wrong door. Host opens the last remaining wrong door. Finally, we can only switch to the original door, which is the right door.

Case 2: **P(WWR ; S4) = 3/4 * 1/2**. This come from 3/4 probability of picking one of the wrong doors at first. Host opens another wrong door. There are 2 doors left, so the probability of picking the wrong door the second time is 1/2. Since our second pick is wrong, the host must open the door we pick the first time, so we will switch to the only one door left, which is the right door.

Case 3: **P(WRW ; S4) = 3/4 * 1/2**. Follow the similar logic as case 2.

P(win ; S4) = P(RWR ; S4) + P(WWR ; S4) = 5/8

Thus, stick-and-switch strategy is the best.

Solution 2

Let Alice be a player and Bob be a host. There are four doors: 1, 2, 3, and 4. We let the first door Alice picks be door 1 without loss of generality.

Assume Alice always sticks to door 1. She has a probability $1/4$ of winning.

Assume Alice sticks to door 1 and then switches the second time. After Alice has picked door 1 twice, Bob will open two other doors that do not have a prize. Since the probability of the prize not behind door 1 is

$3/4$, and we know that the other two doors do not have a prize for sure, the other door left thus has $3/4$ probability of having the prize behind it. Therefore, Alice has $3/4$ probability of winning.

Assume Alice switches to one of the two other doors that Bob did not open. The probability that she picks the door that has the prize is $(3/4) \times (1/2)$. Indeed, this happens if the prize is not behind door 1 (probability $3/4$) and if she also picks the one door out of two that has the prize (probability $1/2$).

If Alice sticks to her second choice, she then has a probability $(3/4) \times (1/2) = 3/8$ of winning.

Assume she switches again after Bob shows her a second door that has no prize. The claim is that she wins with probability $5/8$. To see this, note that she always wins if her first choice was the door with the prize. Indeed, say the prize was behind door 1 that Alice picks. Bob opens door 4, Alice switches to door 2, Bob opens door 3, Alice switches back to door 1. Also, we claim that if Alice had not picked the correct door, then she wins with probability $1/2$. Indeed, say the prize is not behind door 1 that Alice first picks. Bob opens one of the doors (2, 3, 4), say door 4. Alice picks door 2. With probability $1/2$, the prize is behind door 2 and Bob opens door 3, Alice switches to door 1 and loses. With probability $1/2$, the prize is behind door 3, Bob opens door 1, Alice switches to door 3 and wins. Thus, if she switches twice, Alice wins with probability $1/4 + (3/4)(1/2) = 5/8$.

Among these four strategies, stick and switch strategy has the highest probability of winning.