Algebraic Structures and Polynomials

CS70 Summer 2016 - Lecture 7C

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UC Berkeley

Today

Review: Chinese Remainder Theorem and Blum Coin Flipping

Algebraic Structures: Groups, Rings, and Fields

Galois Fields

Polynomials

Applications: Secret Sharing and Erasure Codes

Motivation

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Can we express turn numbers and congruences in modular arithmetic into their own “number system”?

Define *algebraic structures* through axioms that define how they behave.

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Notice that there no commutativity requirement. "$\cdot$" may be non-commutative! If it is commutative, we refer to the group as *abelian*. Formally, Abelian groups must satisfy requires another axiom:

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Also, note that $+$ doesn't necessarily have to represent addition in the normal sense. Elements of G may not even be numbers!

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Examples: With addition and multiplication defined in the usual sense $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is a commutative ring but not a field.

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Definition: For prime p, the field $(\mathbb{Z}_p, +, \cdot)$, with $+$ and $\cdot$ defined as modular arithmetic $\pmod{p}$, is known as the prime field[1] of order p, denoted $GF(p)$.

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A polynomial is said to contain a point (x, y) if $p(x) = y$.

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$$\begin{aligned} y_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_d x_1^d \\ &\vdots \\ y_{d+1} &= a_0 + a_1 x_{d+1} + a_2 x_{d+1}^2 + \cdots + a_d x_{d+1}^d \end{aligned}$$

Or in matrix form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^d \\ 1 & x_2 & x_2^2 & \ldots & x_2^d \\ 1 & x_3 & x_3^2 & \ldots & x_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{d+1} & x_{d+1}^2 & \ldots & x_{d+1}^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{d+1} \end{bmatrix}$$

(This matrix is called the *Vandermonde matrix*.)

Lagrange Interpolation (1/2)

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$$\Delta_1(x) := y_1 \frac{(x-x_2)(x-x_3)\dots(x-x_{d+1})}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_{d+1})}$$

Value at x_1?

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Value at x_1? y_1. Value at $x_2, \dots, x_{d+1}$? 0. General idea behind interpolation: make these polynomials for all i and add them together.

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Now we have a polynomial passing through a collection of $d+1$ points. Is it the *only* polynomial passing through these points? Or: we know the system given by the Vandermonde matrix has a solution. Is it a unique solution or is the system underdetermined?

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We already know there is such a polynomial (we constructed one). Remains to show uniqueness.

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Polynomial Division

Given a degree-d polynomial $f(x)$ and a polynomial $g(x)$ of degree at most d, we can use long division to write $f(x) = g(x)q(x) + r(x)$ for some polynomials $q(x), r(x)$ such that the degree of $r(x)$ is strictly smaller than the degree of $f(x)$. Method: same as elementary-school long division for numbers!

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$$
\begin{array}{r|l}
 & x^2 + x + 3 \\
\hline
x - 3 & x^3 - 2x^2 + 0x - 4 \\
 & \underline{x^3 - 3x^2} \\
 & \quad +x^2 + 0x \\
 & \quad \underline{+x^2 - 3x} \\
 & \qquad +3x - 4 \\
 & \qquad \underline{+3x - 9} \\
 & \qquad\quad +5
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So $x^3 - 2x^2 - 4 = (x-3)(x^2 + x + 3) + 5$.

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Lemma 2: If a degree-d polynomial $p(x)$ has d distinct roots $a_1, \ldots, a_d$, then it can be written as $p(x) = c(x - a_1)\ldots(x - a_d)$ for some constant c.

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It immediately follows that a nonzero polynomial of degree d has at most d roots. Why? Suppose for contradiction that it has more than d. Take first d roots and write the polynomial as $c(x - a_1)\ldots(x - a_d)$. Plug in the $d+1$st root, a_{d+1}. Since it's distinct from $a_1, \ldots, a_d$ this polynomial must be nonzero, contradicting our assertion that a_{d+1} was a root. Therefore, we've proven Theorem 1.

Up next...

Counting polynomials.

Applications: Shamir's secret sharing and error-correcting codes.

Polynomial identity testing and the Schwartz-Zippel lemma