Markov Chains II

CS70 Summer 2016 - Lecture 6C

David Dinh

27 July 2016

UC Berkeley

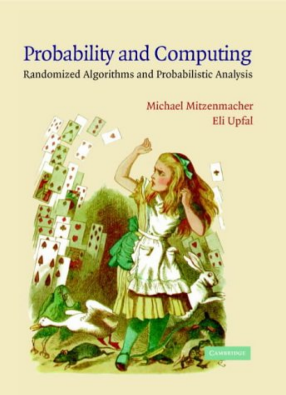
Agenda

Classification of MC states

Aperiodicity, irreducibility, ergodicity

Convergence, limiting and stationary distributions

Reference for this lecture: Ch. 7 of Mitzenmacher and Upfal, ”Probability and Computing”



Markov Chain Properties

Accessibility and Communication

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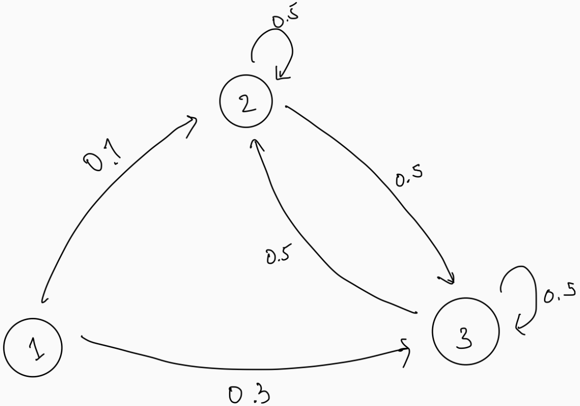
Another way of looking at it: directed connectivity. i communicates with j: exists path from i to j in the graph corresponding to the chain.

Accessibility and Communication: Example



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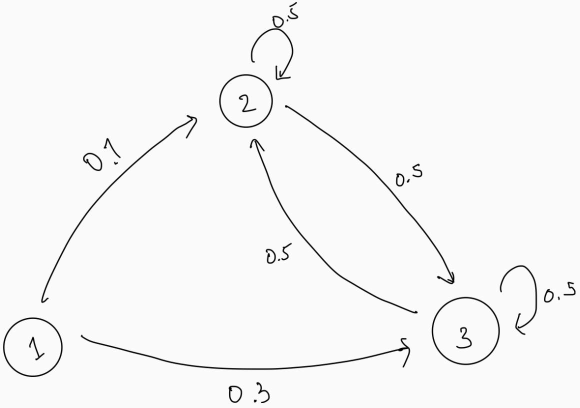

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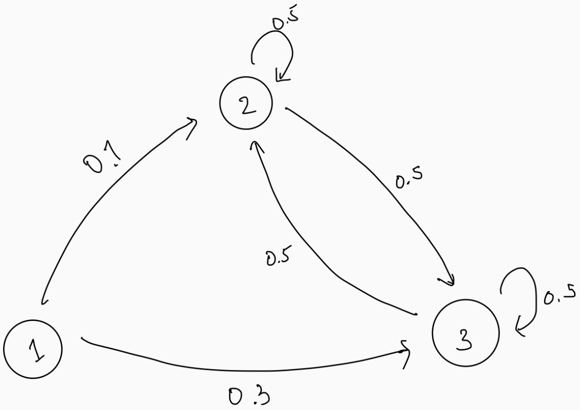

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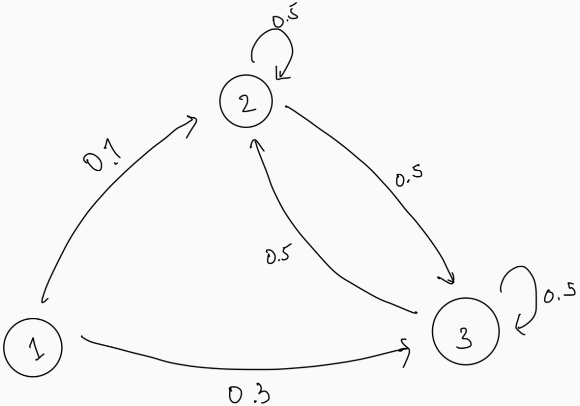

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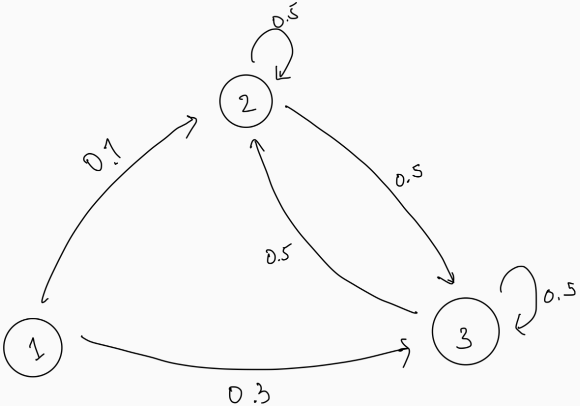

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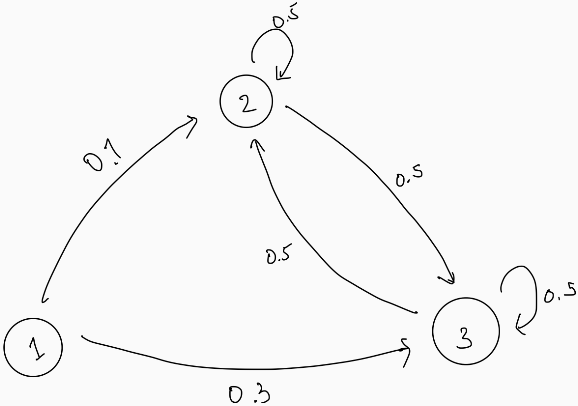

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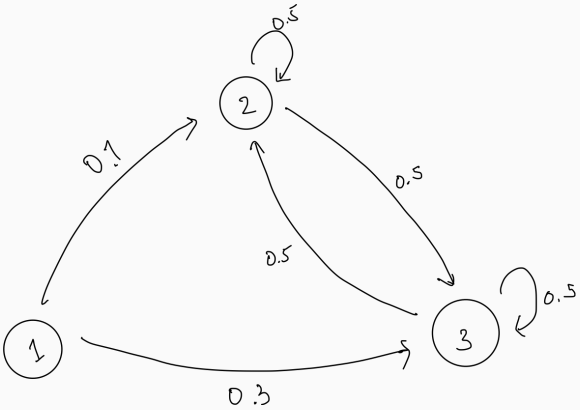

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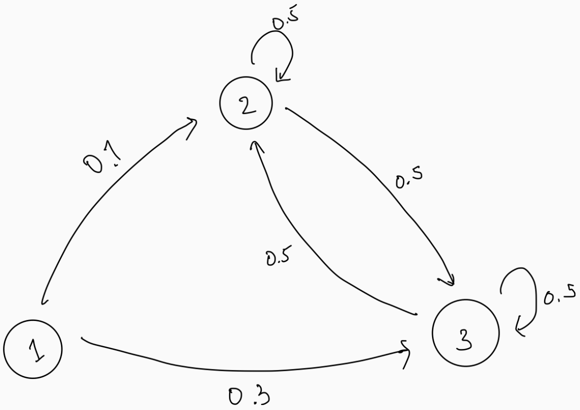

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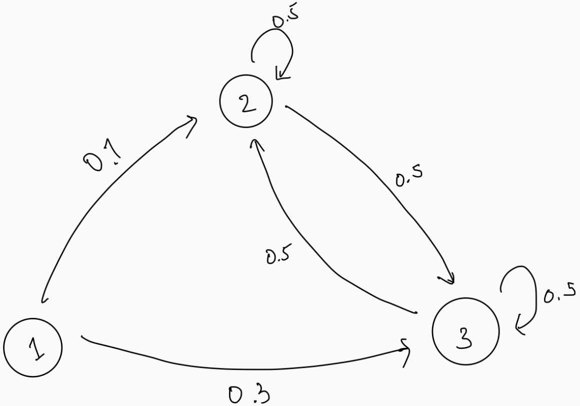

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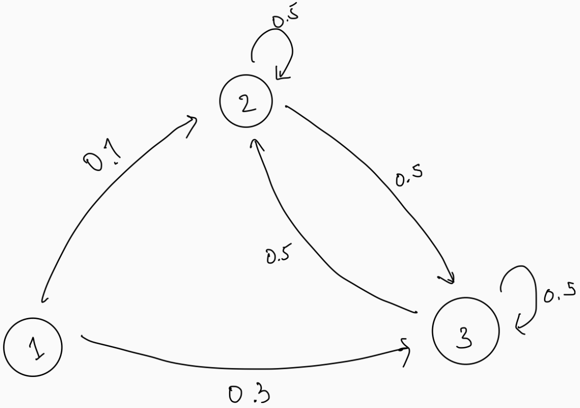

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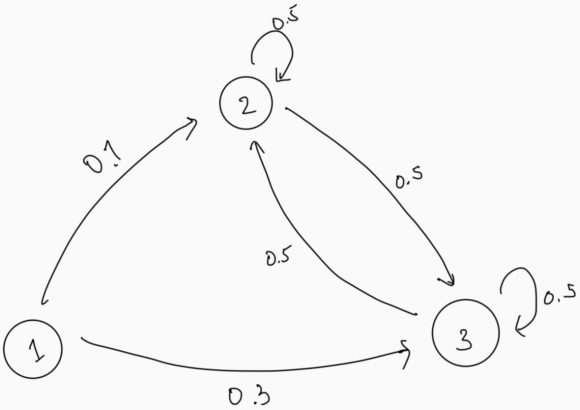

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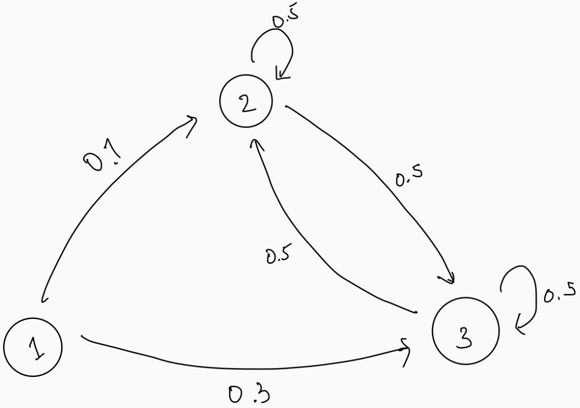

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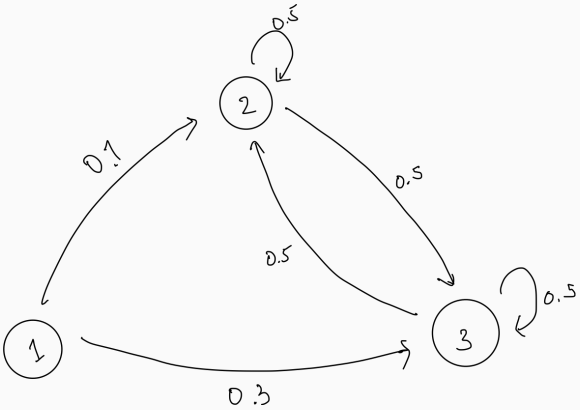

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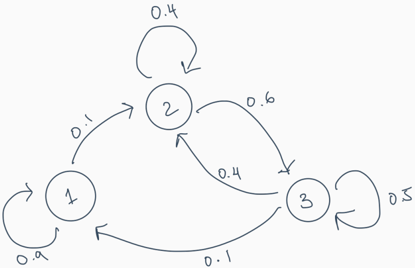

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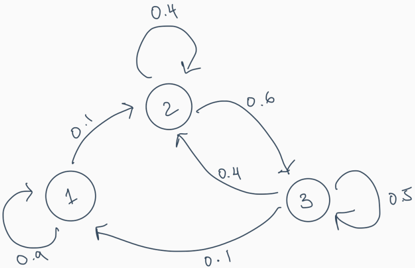

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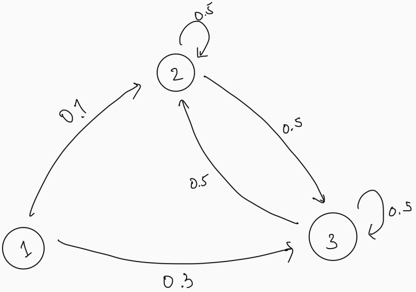

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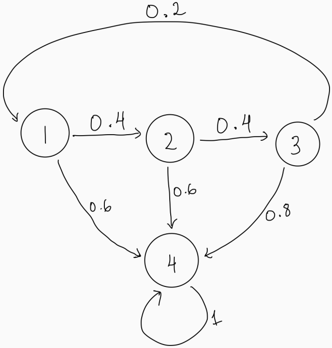

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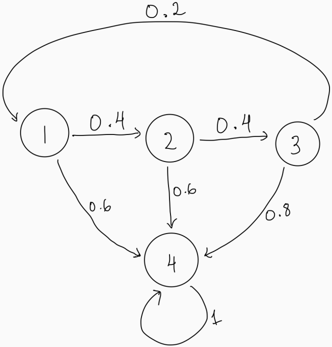

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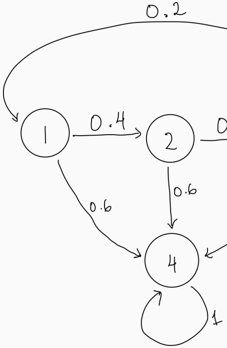

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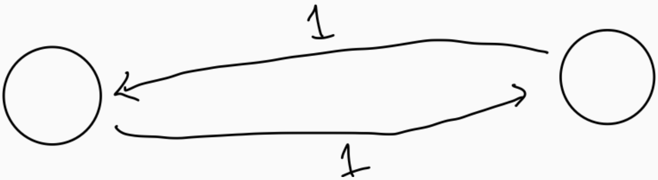

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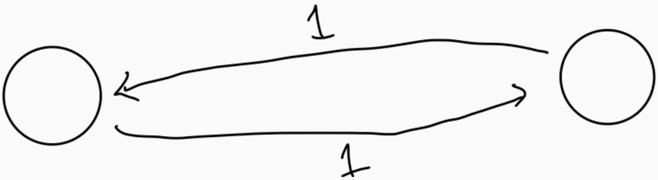

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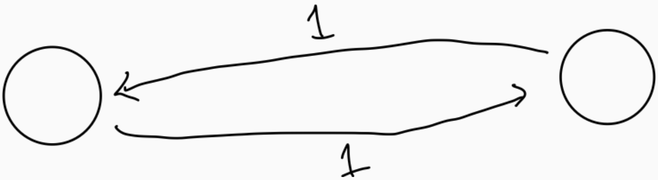

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To capture this intuition: state j is periodic if there exists some integer $\Delta > 1$ such that $P^s_{j,j} = \Pr[X_{t+S} = j | X_t = j] = 0$ unless Δ divides s.

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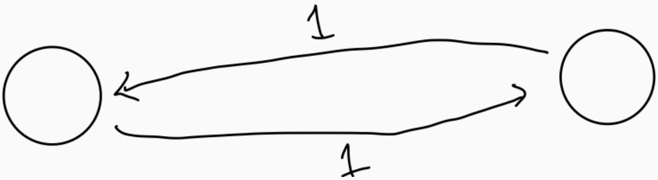

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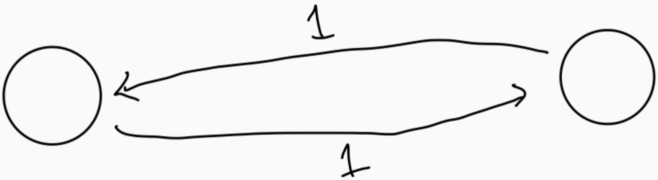

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Opposite of periodic: **aperiodic**.

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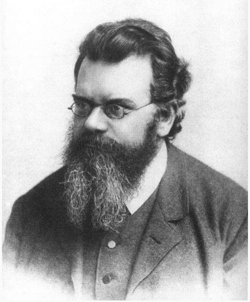
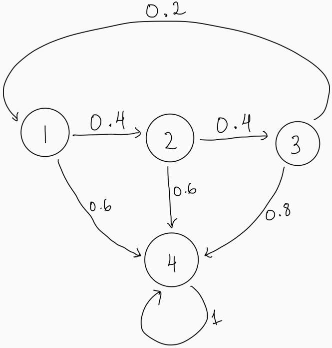


Theorem: A finite, irreducible, aperiodic Markov chain is ergodic.

Stationary and Limiting Distributions

Stationary Distributions: Motivation

Consider the driving exam MC again.



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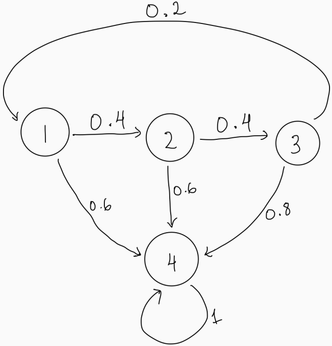

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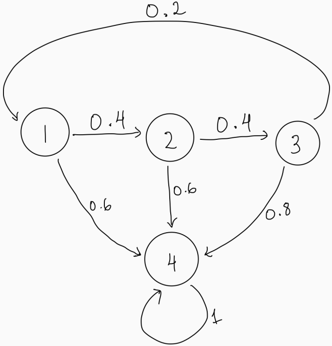

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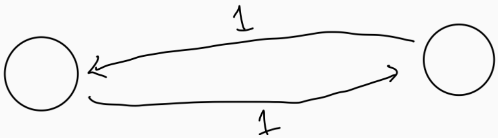
If our distribution is $[0 \;\; 0 \;\; 0 \;\; 1]$: distribution is unchanged over a timestep.

Or how about the two-cycle?



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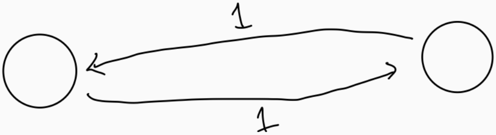

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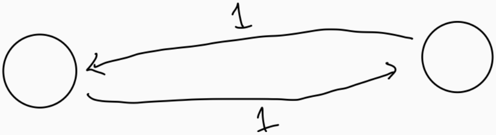

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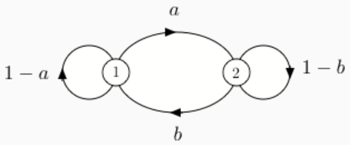
To find stationary distribution: solve $\pi P = \pi$ ("balance equations")

An Example



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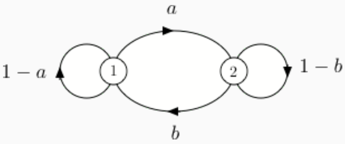

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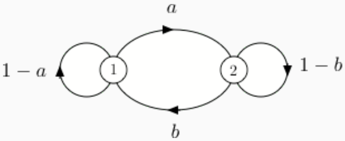

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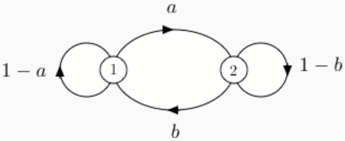

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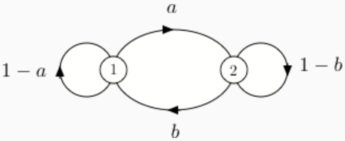

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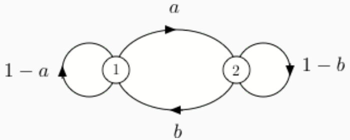

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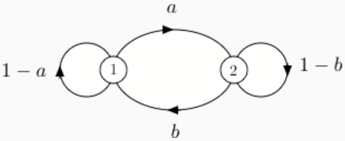

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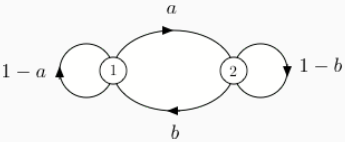

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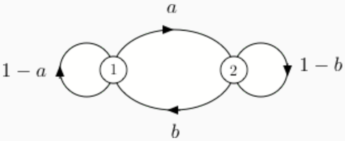

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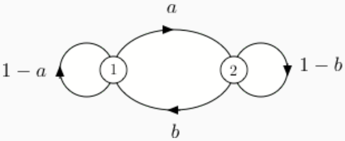

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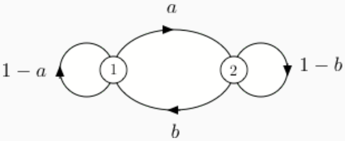

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Proof: really long and messy, see note 18 or Ch. 7 of MU. (we won't expect you to know this).

Connections between Linear Algebra and Markov Chains

It turns out that the convergence of the limiting distribution to the stationary distribution corresponds to a nice result from linear algebra: if you multiply a random vector by a matrix a lot of times, the result will converge towards an eigenvector (specifically, one corresponding to the highest eigenvalue) w.h.p.

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(No, you do not need to know this for the midterms and the homeworks).

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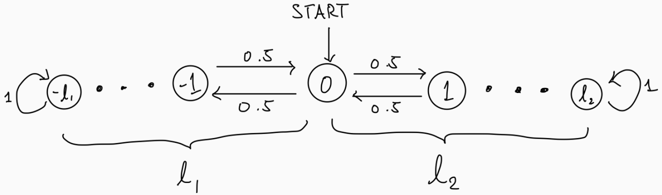

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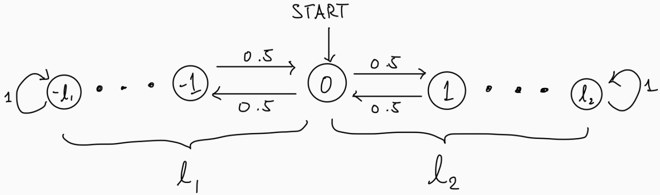

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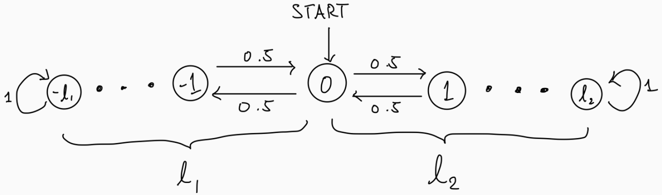

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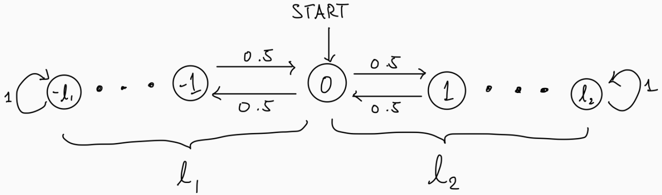

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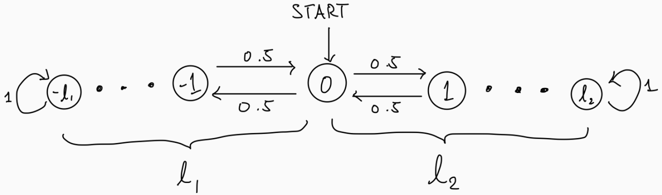

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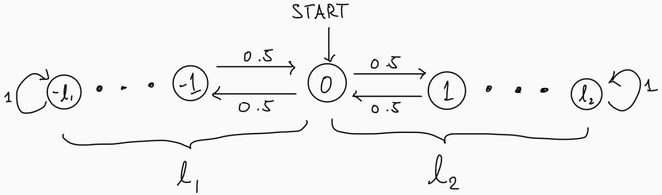

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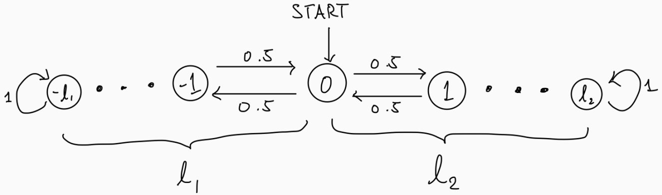

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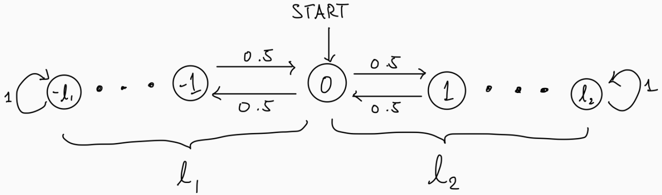

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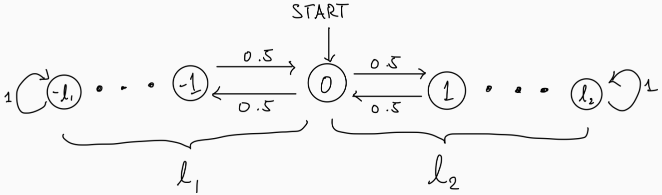

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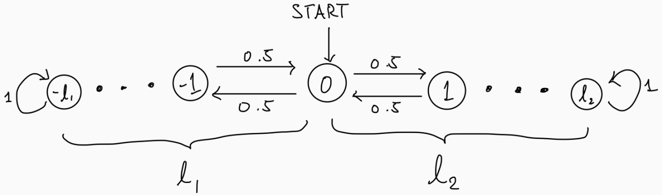

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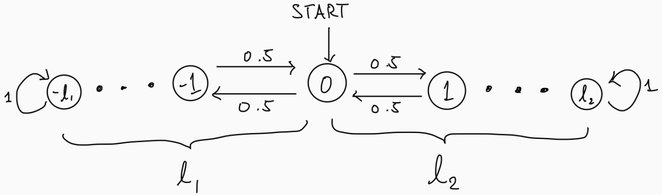

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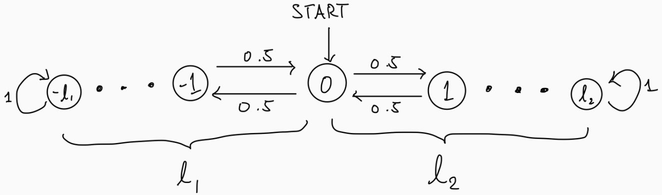

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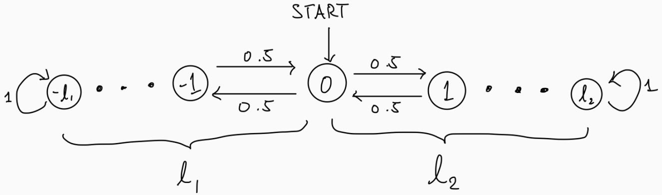

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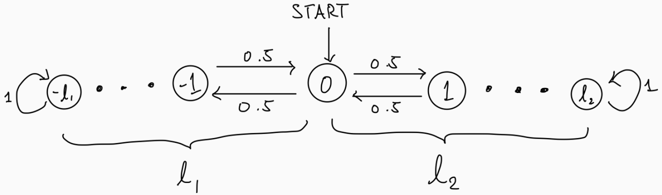

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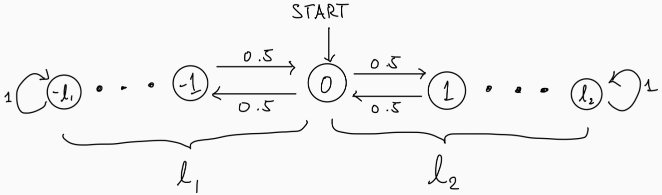

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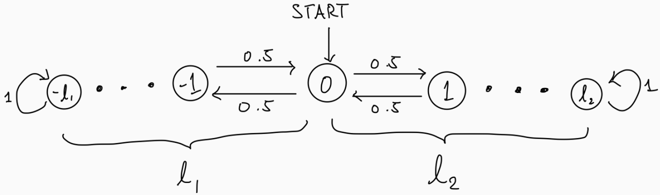

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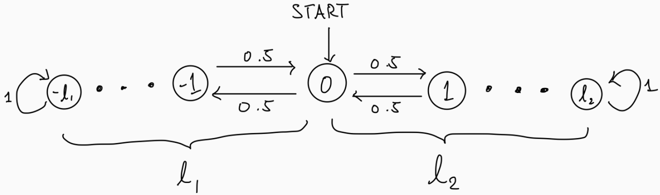

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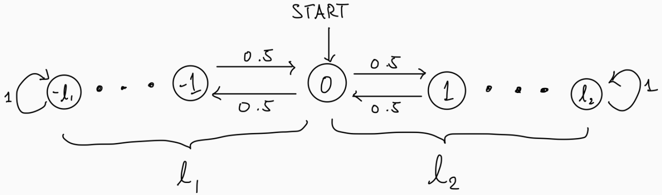
The Gambler's Ruin III



Denote amount of money you have at timestep t as W_t.

What's the expected amount of money you have after a single step? 0. What's the expected gain after t steps, $E[W^t]$? 0, by induction.

So:

$$E[W^t] = \sum_{i \in [-l_1, l_2]} i P_i^t = 0$$

.

$$\lim_{t \to \infty} E[W^t] = l_2 q - l_1(1-q) = 0$$

Solve: $q = l_1/(l_1 + l_2)$. The more money you're willing to bet, the more you win!

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Is it irreducible? Yes, if it's connected.

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$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2|E|} \frac{1}{d(u)} = \frac{d(v)}{2|E|}$$

.

So π solves the balance equations, so it's stationary.

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Since $v \in N(u)$: $h_{v,u} < 2\left|E\right|$

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So: $4|E||V|$ is an upper bound on the cover time.

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Find the limiting distribution by solving an eigenvalue problem! (Math 128B, Math 221)

Gig: Random Text