Linear Algebra for Machine Learning and Data Science

Week 1: System of Linear Equations

System of sentences behaves a lot like systems of equations. If a system has redundancies, contradictions or both, the system is called Singular, otherwise, the system is complete and Non-Singular.

System 1:
* You bought 1 apple and 1 banana for $10. Day 1
* You bought 1 apple and 2 bananas for $12. Day 2

System 2:
* You bought 1 apple and 1 banana for $10. Day 1
* You bought 2 apples and 2 bananas for $20. Day 2

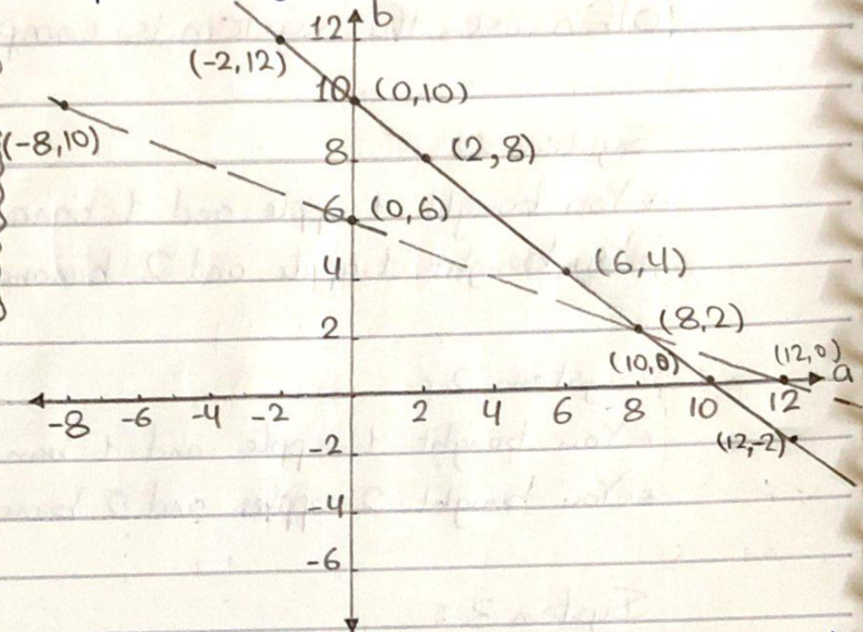
System 3:
* You bought 1 apple and 1 banana for $10. Day 1
* You bought 2 apples and 2 bananas for $24. Day 2

System 1	System 2	System 3
$a + b = 10$	$a + b = 10$	$a + b = 10$
$a + 2b = 12$	$2a + 2b = 20$	$2a + 2b = 24$
Unique Solution	Infinite Solutions	No Solution
$a = 8$	$a = 8, 7, 6$	
$b = 2$	$b = 2, 3, 4 \dots$	
Complete	Redundant	Contradictory
Non-Singular	Singular	Singular

Linear Equation	Non-Linear Equation
$a+b=10$	$a^2+b^2=10$
$2a+3b=15$	$\sin(a)+b^5=15$
$3.4a+48.99b-2C=122.5$	$2^a-3^b=0$
	$ab^2+\frac{b}{a}-\frac{3}{b}-\log(c)=4^a$

Linear equations can be represented as lines on the coordinate plane systems.

This is a system of 2 linear equations
— $a+b=10$ (1)
-- $a+2b=12$ (2)



Slope: is the value that describes the direction and the steepness of the line.
Y-intercept: the value of the vertical axis at which the line or the curve passes.

(1) slope $=-1$
Y-intercept $=10$

(2) slope $=-0.5$
Y-intercept $=6$

Because (1) & (2) has a unique solution, the intersect at one point. For system 2, the two lines will overlap each other, giving infinite intersection points (solutions). For system (3), there will be no intersection points (solutions); two parallel lines.

Applies for system of 2 linear equations	# For line systems, putting the Y-intercept = 0, serves as geometric notion of singularity: 1. If the two lines become one line (single line), then the system is singular (redundant or contradictory). 2. If the two lines still two different lines and intersect at the origin, then the system is singular (complete).

Matrices have lots of very important properties and they arise from many different place in math. In a system of linear equations, they arise from the coefficients in the system.

Singular matrices are from singular systems, and non-singular matrices arise from non-singular systems.

2X2 Matrices

System (1)		System (2)	
$a + b = 0$ $a + 2b = 0$	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	$a + b = 0$ $2a + 2b = 0$	$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
Non-Singular System (Unique Solution)	Non-Singular Matrix	Singular System (Infinitely Many Solutions)	Singular Matrix

second equation in system (2) is a multiple of the first equation, so the system is singular. The same thing applies to the matrix rows, therefore the Rows of matrix 2 are linearly dependent.

The rows of the first system are linearly independent, which makes the matrix non-singular.

Matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if: $\begin{bmatrix} a & b \end{bmatrix} * k = \begin{bmatrix} c & d \end{bmatrix}$

Constant (Real Number)

$$\begin{aligned} ak &= c \\ bk &= d \end{aligned} \rightsquigarrow \frac{c}{a} = \frac{d}{b} = k \rightsquigarrow ad = cb \rightsquigarrow \underbrace{\boxed{ad - cb}}_{\text{Determinant}} = 0$$

The matrix is singular, if its determinant = 0. If the determinant ≠ 0, the matrix is non-singular.

3 equations Systems Examples

System 1

$a + b + c = 10$

$a + 2b + c = 15$

$a + b + 2c = 12$

Unique Solution

$a = 3$, $b = 5$ & $c = 2$

Complete & Non-Singular

System 2

$a + b + c = 10$

$a + b + 2c = 15$

$a + b + 3c = 20$

Infinitely Many Solutions

$c = 5$ & $a + b = 5$

(0,5,5), (1,4,5), (2,3,5), ...etc.

Redundant & Singular

System 3

$a + b + c = 10$

$a + b + 2c = 15$

$a + b + 3c = 18$

No Solution

1st & 2nd → $c = 5$

2nd & 3rd → $c = 3$

1st & 3rd → $c = 4$

Contradictory & Singular

System 4

$a + b + c = 10$

$2a + 2b + 2c = 20$

$3a + 3b + 3c = 30$

Infinitely Many Solutions

Any 3 numbers that add to 10 work:

(0,0,10), (2,7,1), (3,2,5), ...etc.

Redundant & Singular

If we replaced the constants at the end of the linear equations, we can build 3x3 matrices from the previous four systems, and get the singularity with solutions.

3X3 Matrices

System 1	System 2 & System 3	System 4
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$
Complete Non-Singular	Redundant Singular	Redundant Singular

matrix of System one in graph: each line represents a plane. The three planes intersect at one point (unique solution)

matrix of Systems 2&3 in graph are three planes intersecting at a line. Solutions are all the points on that line.

Matrix of system 4 in graph are three identical planes. The solution can be any point on that plane.

Linear Independence in 3x3 Matrices:

Rows of the Matrix are

Matrix is dependent, if:

→ One row's sum equal to a multiple of the sum of the two other rows. (Matrix of systems 2&3)

→ One row is the multiple of the other. (Matrix of system 4)

Otherwise, the matrix rows are independant. (Matrix of system 1)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 2 & 4 & 10 \end{bmatrix}$$

$3R_1+2R_2=R_3$	$R_1-R_2=R_3$	No Relation	$2R_1=R_3$
Dependant (Singular) Determinant=0	Dependent (Singular) Determinant=0	Independent (Non-Singular) Determinant=6	Dependent (Singular) Determinant=0

a	b	c
d	e	f
g	h	i

The Determinant (3X3):

$$\text{Determinant} = \begin{pmatrix} a & & \\ & e & \\ & & i \end{pmatrix} + \begin{pmatrix} b & & \\ & f & \\ & & g \end{pmatrix} + \begin{pmatrix} c & & \\ & d & \\ & & h \end{pmatrix}$$

$$- \begin{pmatrix} & & c \\ & e & \\ g & & \end{pmatrix} - \begin{pmatrix} & & a \\ & f & \\ h & & \end{pmatrix} - \begin{pmatrix} & & b \\ & d & \\ i & & \end{pmatrix}$$

use any of the forms. All are the same.

$$= (a\,e\,i) + (b\,f\,g) + (c\,d\,h) - (c\,e\,g) - (a\,f\,h) - (b\,d\,i)$$

$$= a\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

If the matrix has a row of zeros the determinant will be zero for sure.