Week 2: Solving System of Linear Equations: Elimination

Manipulating Equations:

constant $\neq 0$

1. Multiplying by constants (or Divide)
2. Adding two equations (or Subtract)
3. Swaping two equations

Example:

$$\begin{aligned} 5a + b &= 17 \\ 4a - 3b &= 6 \end{aligned} \quad \begin{aligned} &\xrightarrow{(\times 4)} \\ &\xrightarrow{(\times 5)} \end{aligned} \quad \begin{aligned} 20a + 4b &= 68 \\ 20a - 15b &= 30 \end{aligned} \quad \text{(subtract)}$$

solution 1

$$19b = 38 \quad \text{(get b)}$$

$$b = \frac{38}{19}$$

$$\boxed{b = 2}$$

substitute with b, to get a

$$5a + (2) = 17$$

$$a = \frac{17 - 2}{5}$$

$$\boxed{a = 3}$$

Solution 2

$$\begin{aligned} 5a + b &= 17 \\ 4a - 3b &= 6 \end{aligned} \quad \begin{aligned} &\xrightarrow{(\div 5)} \\ &\xrightarrow{(\div 4)} \end{aligned} \quad \begin{aligned} a + 0.2b &= 3.4 \\ a - 0.75b &= 1.5 \end{aligned} \quad \text{(subtract)}$$

$$0.95b = 1.9 \quad \text{(get b)}$$

$$\boxed{b = 2}$$

$$a + 0.2(2) = 3.4$$

$$a = 3.4 - 0.4$$

$$\boxed{a = 3}$$

If the system is singular, we can't eliminate one variable from the equation: [Redundant]

$$\begin{aligned} a + b &= 10 \\ 2a + 2b &= 20 \end{aligned} \quad \begin{aligned} &\xrightarrow{(\div 1)} \\ &\xrightarrow{(\div 2)} \end{aligned} \quad \begin{aligned} a + b &= 10 \\ a + b &= 10 \end{aligned} \; -$$

$$0 = 0$$

→ Infinite solutions

if we let $\boxed{a = x}$, then $\boxed{b = 10 - x}$; where $x \in \mathbb{R}$.

Degree of Freedom x.

If the sigulor equations' system is contradictory, we would have gotten $(0 \neq 0)$ after subtraction. Therefore, it can't be solved.

The steps two solve a system of more variables are similar to the steps of two variables:

1. Divide by the coefficient of the first variable in all equations.
2. Subtract them from the first equation (1^{st} variable eliminated from all equations, but the first)

, Then remove (eliminate) the second variable from the equations of the system but the first and the second equation, and so on, till you get one variable in the last equation.

If we apply the same principles to a matrix it is called Matrix Row-Reduction or Gauss Elimination:

Example on a 3X3 Matrix

→ we get a matrix in its original form $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Non-Sigular

→ we transform it to upper-diagonal matrix $\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_1 \\ 0 & 0 & 1 \end{bmatrix}$

Row Echelon Form

→ we transform it then to diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reduced Row Echelon Form

Upper-digonal matrix

If the matrix was singular the Row Echelon Form is going to be: $\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_1 \\ 0 & 0 & 0 \end{bmatrix}$

(one or more zero rows)

upper-digonal

If you have (0) in the diagonal of the matrix, then all coefficients on the right are zeros, and all elements after in the diagonal are zeros.

check it using det. # Row manipulation preserve the singularity state of a matrix and linear equations' system.

Rank:

Method ①

Rank is equal to the number of linearly independent equations in a system of linear equations.

System 1

$a + b + c = 0$
$a + 2b + c = 0$
$a + 2b + 2c = 0$

3 Equations
3 pieces of information

Rank 3

System 2

$a + b + c = 0$
$\xrightarrow{\frac{R_1+R_3}{2}} a + b + 2c = 0$
$a + b + 3c = 0$

3 Equations
2 pieces of information

Rank 2

System 3

$a + b + c = 0$
$2a + 2b + 2c = 0$
$3a + 3b + 3c = 0$

3 Equations
1 piece of information

Rank 1

System 4

$0a + 0b + 0c = 0$
$0a + 0b + 0c = 0$
$0a + 0b + 0c = 0$

3 Equation
0 Pieces of information

Rank 0

Method ②

Row Echelon Form is an easier way to calculate the rank:
→ Get the upper-diagonal (row echelon) form of the matrix
→ Rank = no. of ones in the main diagonal (pivots)

in (2X2)
in general

solution	D
Point	0D
Line	1D
Plane	2D
Volume (space)	3D
↓	↓

If the rank = no. of rows, then the matrix is non-singular. Otherwise, the matrix is singular.

Rank = no. of rows − no. of dimensions of solution
Rank = no. of pivots
Rank = no. of pieces of information

Pivot:

→ Number of points in the matrix with all the values on the left equal zero.
→ From the first to the final row, pivots are on the right side of the upper-row's pivot.

Example

$$\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \xrightarrow{\div 3} \\ \xrightarrow{\div 1} \\ \xrightarrow{\div (-4)} \end{array} \begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Both are in row echelon form.
Both have three pivots and Rank = 3.

Reduced Row Echelon Form:

It requires one more step after getting the row echelon form:
→ Using lower row to eliminate upper coefficients of the same pivot.

continue example

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$